Lecture 8: Estimation and Confidence Intervals

STAT 324

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Estimation

The art of coming up with our "best guess" for the truth. This is called *an estimate*.

An *estimator* is a function that takes values from a sample and provides an estimate ("best guess").

Estimation

In order to come up with a good estimator, it is important to know how the sample was gathered. Three important definitions:

- a sample is called a **simple random sample** (SRS) if every possible element is equally likely to be sampled
  - unless otherwise stated, **all samples in this class are SRS**
- a sample is drawn **with replacement** if an element is replaced to the population before the next element is drawn. Otherwise, we say it is drawn **without replacement**.
  - without replacement = each element can only be sampled once.
- a collection of RVs $X_1, X_2, \ldots, X_n$ are said to be **independent and identically distributed** (iid) if
  1. they are all independent of each other
  2. they all follow the same distribution.

Technically, SRS **ONLY** if done with replacement. However, if population is "big enough", sample without repleacement is approximately the same as with replacement. (Recall last weeks discussion.)

Estimation: Population Mean

- A car manufacturer uses an automatic device to apply paint to engine blocks.
- engine blocks get very hot, so the paint must be heat-resistant,
- important that the amount applied is of a minimum thickness
- warehouse contains thousands of blocks painted by the automatic device
- he manufacturer wants to know the average amount of paint applied by the device

16 blocks will be selected at random, and the paint thickness measured in mm. Let $X_1, \ldots, X_{16}$ be RVs indicating the thickness of the 16 blocks.

Let's assume these RVs are iid -- i.e. independent and identically distributed. There exists some true expected value of these: $E(X_i) = \mu$. There also exists some true variance, $\mathrm{Var}(X_i) = \sigma^2$.

Estimation: Population Mean

Now we actually observe 16 realizations of these RVs:

```
paint_thickness <- data.frame(
  thickness = c(1.29, 1.12, 0.88, 1.65, 1.48, 1.59, 1.04, 0.83,
                1.76, 1.31, 0.88, 1.71, 1.83, 1.09, 1.62, 1.49)
)
```

Using these data, what would be your "best guess" for the true mean μ?

I would use the sample average:

```
paint_thickness %>%
  summarize(Mean = mean(thickness))
```

```
##       Mean
## 1 1.348125
```

This **estimATE** comes from the **estimatOR**: $\hat{\mu} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$.

Notice: the **estimator** is a **RV** while the **estimate** is a **realization** of that RV.

Estimation: Population Mean

Since $\bar{X}$ is an RV, we can talk about the expected value and variance of it:

$$
\begin{aligned}
E(\bar{X}) &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} E(X_i) \\
&= \mu,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(\bar{X}) &= \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) \\
&= \frac{\sigma^2}{n}.
\end{aligned}
$$

$$\mathrm{SD}(\bar{X}) = \sqrt{\mathrm{Var}(\bar{X})} = \frac{\sigma}{\sqrt{n}}.$$

$\mathrm{SD}(\bar{X})$ is often called SE (Standard Error)

Estimation: Good estimators

What makes an estimator a good estimator? We will consider two properties of estimators:

- Unbiasedness:
    - an estimator is said to be **unbiased** if $E(\hat{\theta}) = \theta$.
    - I.e. the estimator gives us the correct value *on average*
    - the **bias** of an estimator is $\mathrm{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$
- Shrinking/small variance.
    - unbiased estimator with huge variance is not very reliable
    - if choosing between multiple unbiased estimators, choose smallest variance!

Estimation: Good estimators

Example of good estimator: $\hat{\mu} = \bar{X}$.

- Unibased.
- Can be shown to have smallest variance of ALL unbiased estimators...

Other examples of good estimators:

- $\hat{\sigma}^2 = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.
    - Actually unbiased
        - if divided by n, would not be!
- $\hat{\sigma} = S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$.
    - Biased!
        - if divided by n, would be more biased!
        - hard to find a better candidate, though, so we still use this.

Remember, we have

- sample variance: $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$
- population variance: $\sum_{i=1}^{n} P(X = x_i)(x_i - E(X))^2$

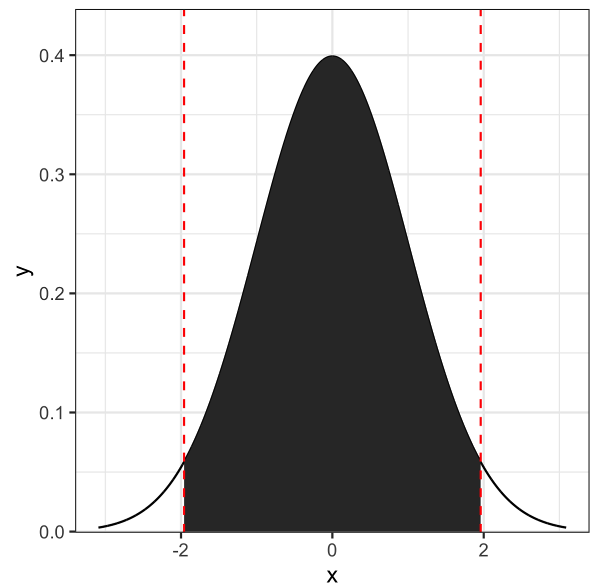
Estimation: Confidence Intervals

If X_i's are iid $N(\mu, \sigma^2)$, what is the distribution of $\bar{X}$?

$\bar{X} \sim N(\mu, \sigma^2/n)$. So what is $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$?

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - E(\bar{X})}{\mathrm{SD}(\bar{X})} = Z \sim N(0,1).$$

Now, we can find values x_1, x_2 such that $P(x_1 \leq Z \leq x_2) = 1 - \alpha$. Let's for simplicity use $\alpha = 0.05$. I.e. we want to find x_1, x_2 such that this area is 0.95.



Estimation: Confidence Intervals

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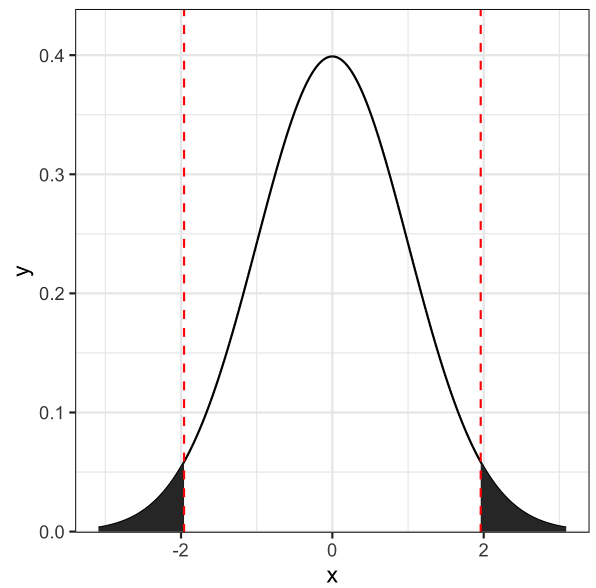
If we decide the two areas in the tails are the same, $x_1 = -x_2$.

x_2 is by definition the $\alpha/2$ (0.025 in this case) critical value, $z_{\alpha/2}$ - it cuts off $\alpha/2$ (0.025) to the right!



So,

$$
\begin{aligned}
1-\alpha &= P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) \\
&= P\left(-z_{\alpha/2} \leq \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right) \\
&= P\left(-z_{\alpha/2}\sigma/\sqrt{n} \leq \bar{X}-\mu \leq z_{\alpha/2}\sigma/\sqrt{n}\right) \\
&= P\left(-\bar{X}-z_{\alpha/2}\sigma/\sqrt{n} \leq -\mu \leq -\bar{X}+z_{\alpha/2}\sigma/\sqrt{n}\right) \\
&= P\left(\bar{X}+z_{\alpha/2}\sigma/\sqrt{n} \geq \mu \geq \bar{X}-z_{\alpha/2}\sigma/\sqrt{n}\right) \\
&= P\left(\bar{X}-z_{\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \bar{X}+z_{\alpha/2}\sigma/\sqrt{n}\right)
\end{aligned}
$$

Estimation: Confidence Intervals

The interval $[\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}]$ is called a $(1-\alpha)\cdot 100\%$ Confidence Interval:

We are $(1-\alpha)\cdot 100\%$ confident that an interval constructed this way will contain the true value μ!

Example

Through years, and years of experience, the car manufacturer has learned that the true standard deviation of the paint thickness is 0.34. We can now construct a 95% confidence interval for the true mean.

First, we find $z_{\alpha/2}$. When looking for a 95% CI, $\alpha = 0.05$. $z_{0.025}$ cuts off 0.025 on the right hand side. I.e. it cuts off $1 - 0.025 = 0.975$ to the left. So $P(X \leq z_{0.025}) = 0.975$, which makes $z_{0.025}$ the 0.975 quantile:

```
Z <- Normal()

z_crit <- quantile(Z, 0.975)
z_crit
```

```
## [1] 1.959964
```

Estimation: Confidence Intervals

We can find the confidence interval as

```
paint_thickness %>%
  summarize(mean = mean(thickness),
            LL = mean - z_crit*0.34/sqrt(n()),
            UL = mean + z_crit*0.34/sqrt(n()))
```

```
##       mean       LL       UL
## 1 1.348125 1.181528 1.514722
```

or

```
xbar <- mean(paint_thickness$thickness)
n <- nrow(paint_thickness)
```

```
xbar - z_crit*0.34/sqrt(n)
```

```
## [1] 1.181528
```

```
xbar + z_crit*0.34/sqrt(n)
```

```
## [1] 1.514722
```

Estimation: Confidence Intervals

We are 95% confident that the true mean thickness is between 1.18 and 1.51.

What is $P(1.18 \leq \mu \leq 1.51)$? 0 or 1. We do not know which, but those are the only possible values.

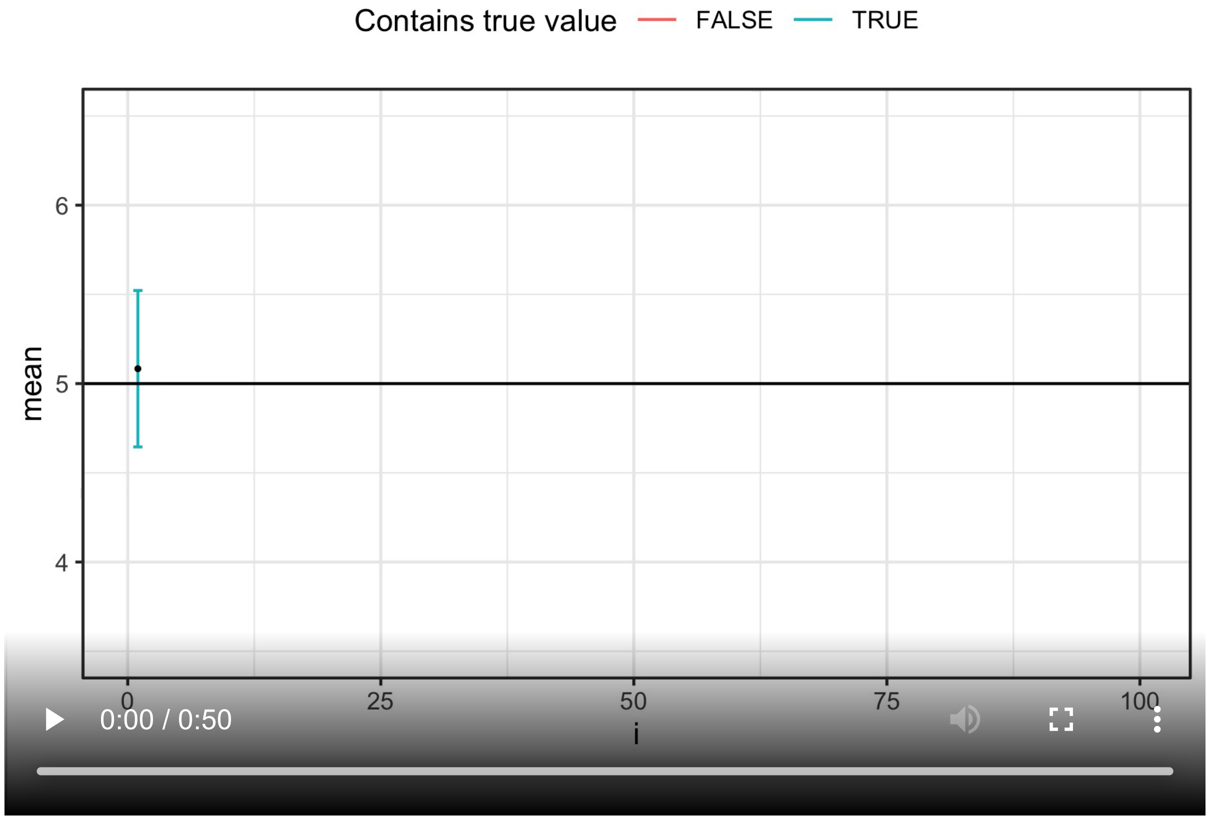
We can think of LL and UL as random variables: new sample -> new CI. So makes sense to say $P(LL \leq \mu \leq UL) = 1 - \alpha$.

But as soon as we get a sample, observe values, and find the realization of the RVs, no more randomness. Hence, not meaningful to talk about probabilities anymore.

What does it mean that we are "95% confident the true value is in the interval?"

Estimation: Confidence Intervals

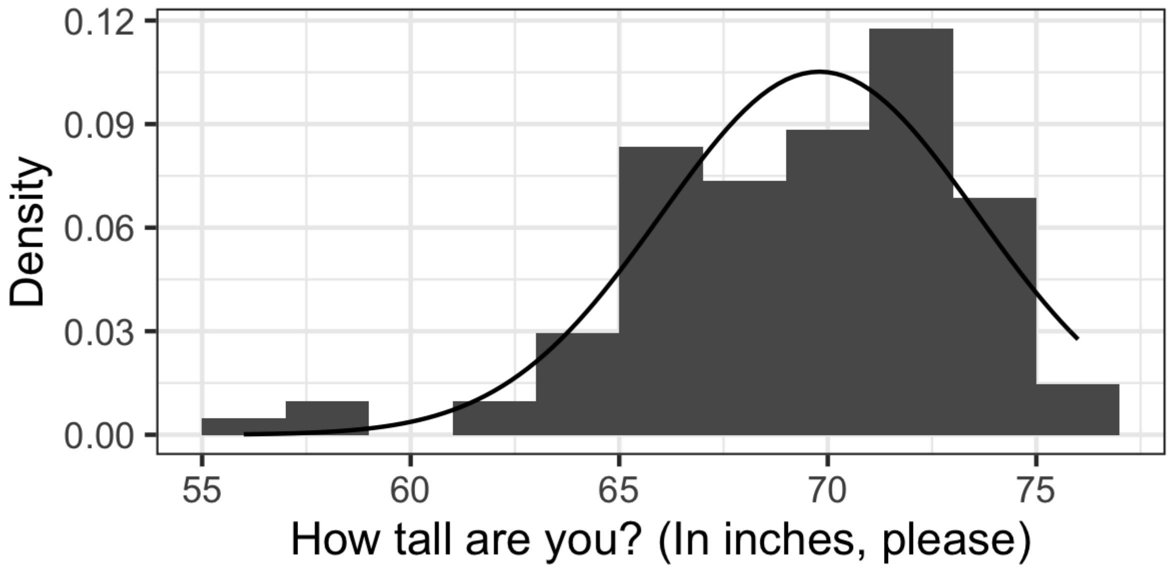
If we repeat the process, the interval we get will contain the true value 95% of the time.



Estimation: Confidence Intervals

Example: student heights

The true population:



Estimation: Confidence Intervals

Example: student heights

Take a sample of 15 students, calculate 95% CI, repeat 100 times.

Estimation: Confidence Intervals

Example: Framingham Heart Study

Total cholesterol in adults.

```
fram <- read_csv(here::here("csv_data/framingham.csv")) %>%
  filter(!is.na(totChol))
```

Remember, so far we need the data to be normal.

Histogram:

```
ggplot(fram,
       aes(x = totChol)) +
  geom_histogram(bins = 35)
```

QQ-plot:

```
ggplot(fram, aes(sample = totChol)) +
  geom_qq() +
  geom_abline(aes(slope = sd(totChol), i
```

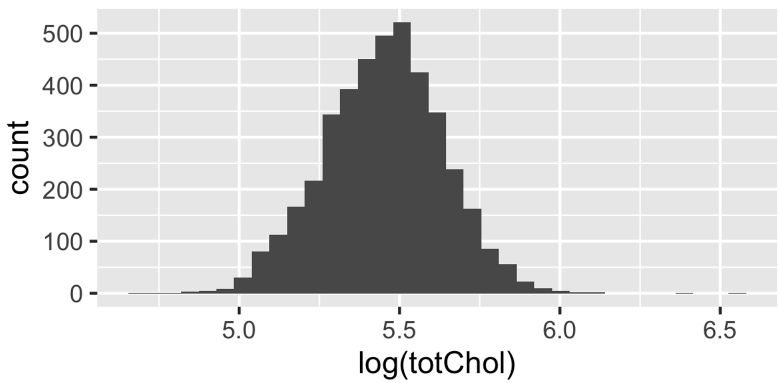
Estimation: Confidence Intervals

Example: Framingham Heart Study

Not convincing, but if we log transform:

Histogram:

```
ggplot(fram,
       aes(x = log(totChol))) +
  geom_histogram(bins = 35)
```



QQ-plot:

```
ggplot(fram, aes(sample = log(totChol)))
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```
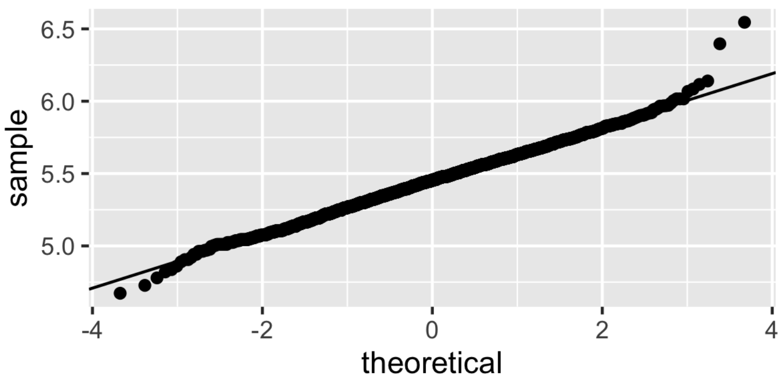



Example: Framingham Heart Study

So we can construct a 95% confidence interval for the log(totChol)... only we do not know the true value of σ?! Let's assume for know that we do know it, and it is 0.19.

```
mean(log(fram$totChol))
```

```
## [1] 5.449583
```

Lower limit:

$$\begin{aligned} \bar{x} - 1.96 \cdot \frac{\sigma}{\sqrt{n}} &= 5.45 - 1.96 \frac{0.19}{\sqrt{4190}} \\ &= 5.444 \end{aligned}$$

Upper limit:

$$\begin{aligned} \bar{x} + 1.96 \cdot \frac{\sigma}{\sqrt{n}} &= 5.45 + 1.96 \frac{0.19}{\sqrt{4190}} \\ &= 5.456 \end{aligned}$$

We are 95% confident that the true *population mean* of the log total cholesterol is in this interval.

Estimation: Confidence Intervals

All of this build on some key assumptions:

1. $\bar{X}$ normally distributed
2. Know the true standard deviation.

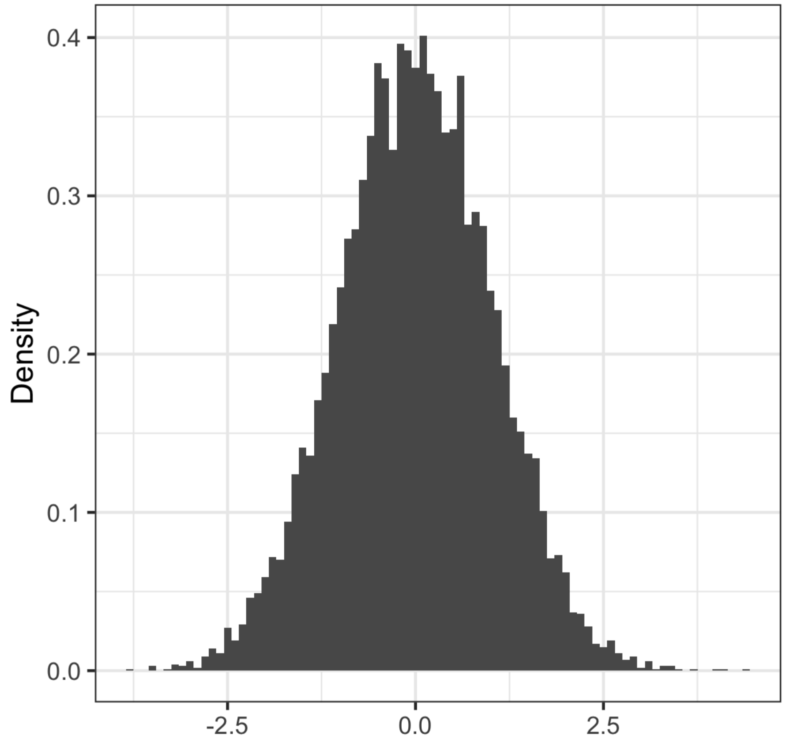
When both satisfied, $\frac{\bar{X}-E(\bar{X})}{\text{SD}(\bar{X})} \sim N(0,1)$.

We said "if $X_1, \ldots, X_n$ are normal, then $\bar{X}$ normal". Let's check. Actually, let's check that if $X_1, \ldots, X_n$ are normal, then $\frac{\bar{X}-\mu}{\text{SD}(\bar{X})} \sim N(0,1)$.

How would we go about that? Get many, many samples, calculate the mean for each, and plot a histogram. We'll use $\mu = 5$, $\sigma = 10$, and $n = 5$.

Estimation: Confidence Intervals

Histogram of $\frac{\bar{x}-5}{10/\sqrt{5}}$ for 10000 samples:



Estimation: Confidence Intervals

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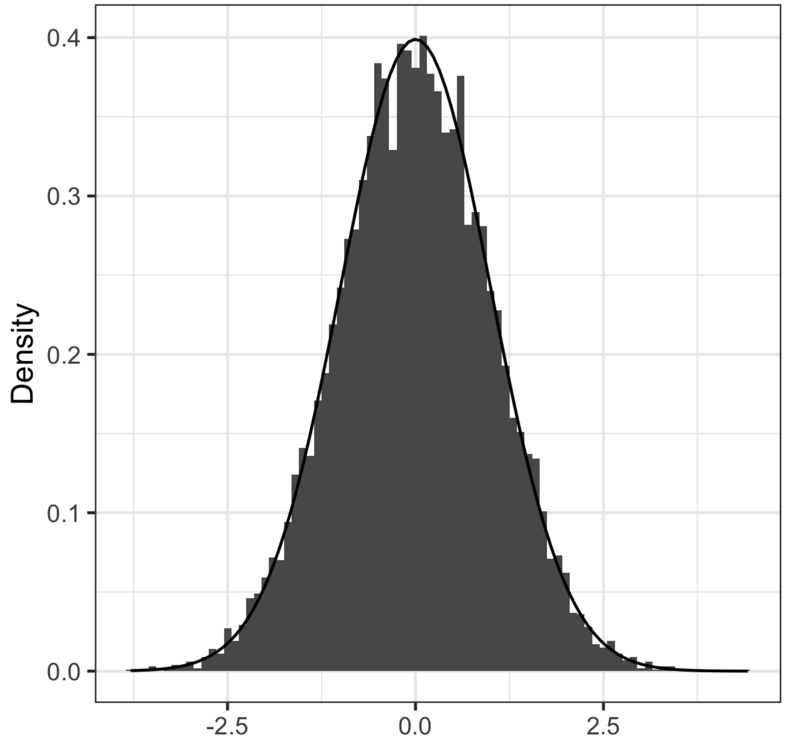
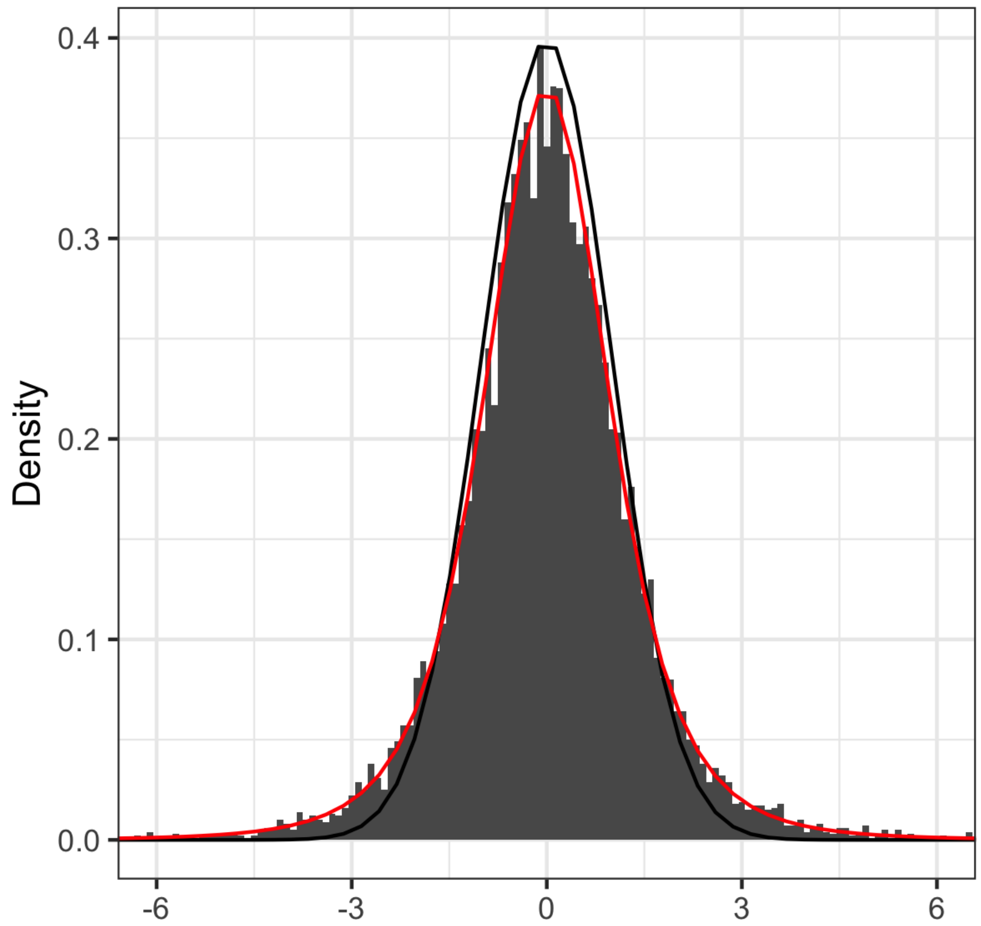


Estimation: Confidence Intervals

What if we do not know the true value of σ? We would use $\hat{\sigma} = S$... but the distribution of $\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}}$ is NOT $N(0,1)$:

Estimation: Confidence Intervals

The distribution of $\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}}$ is a so-called t-distribution with $n-1$ degrees of freedom:

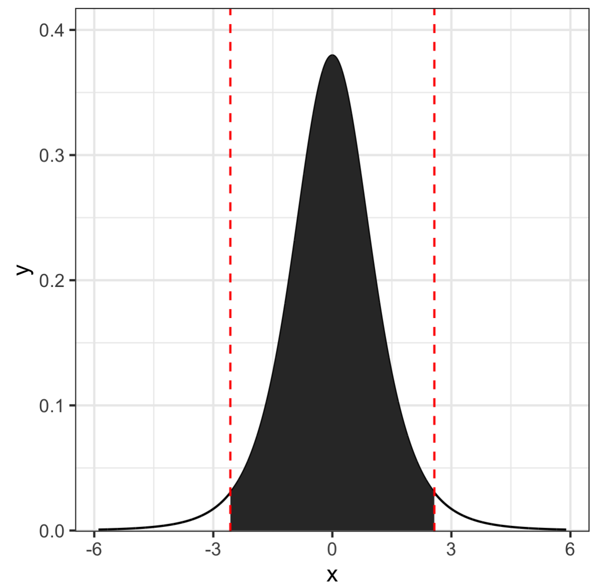


Estimation: Confidence Intervals

Does this mean all is lost? No, of course not! We just need to adjust a bit. We now have

$$\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} = \frac{\bar{X}-E(\bar{X})}{\widehat{\mathrm{SD}}(\bar{X})}$$
$$= T_{n-1} \sim \ t_{n-1}$$

Now, we can find values x_1, x_2 such that $P\left(x_1 \leq T_{n-1} \leq x_2\right) = 1-\alpha$. Let's for simplicity use $\alpha = 0.05$. I.e. we want to find x_1, x_2 such that this area is 0.95.



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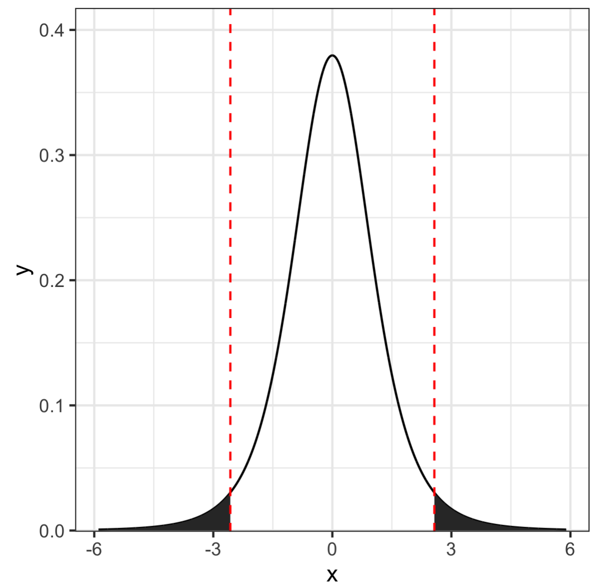
If we decide the two areas in the tails are the same, $x_1 = -x_2$.

x_2 is by definition the $\alpha/2$ (0.025 in this case) critical value *in the t-distribution*. We call it $t_{n-1,\alpha/2}$ - it cuts off $\alpha/2$ (0.025) to the right!



Estimation: Confidence Intervals

So,

$$
\begin{aligned}
1-\alpha &= P(-t_{n-1,\alpha/2} \le T_{n-1} \le t_{n-1,\alpha/2}) \\
&= P\left(-t_{n-1,\alpha/2} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le t_{n-1,\alpha/2}\right) \\
&= P\left(-t_{n-1,\alpha/2}\sigma/\sqrt{n} \le \bar{X}-\mu \le t_{n-1,\alpha/2}\sigma/\sqrt{n}\right) \\
&= P\left(-\bar{X}-t_{n-1,\alpha/2}\sigma/\sqrt{n} \le -\mu \le -\bar{X}+t_{n-1,\alpha/2}\sigma/\sqrt{n}\right) \\
&= P\left(\bar{X}+t_{n-1,\alpha/2}\sigma/\sqrt{n} \ge \mu \ge \bar{X}-t_{n-1,\alpha/2}\sigma/\sqrt{n}\right) \\
&= P\left(\bar{X}-t_{n-1,\alpha/2}\sigma/\sqrt{n} \le \mu \le \bar{X}+t_{n-1,\alpha/2}\sigma/\sqrt{n}\right)
\end{aligned}
$$

Estimation: Confidence Intervals

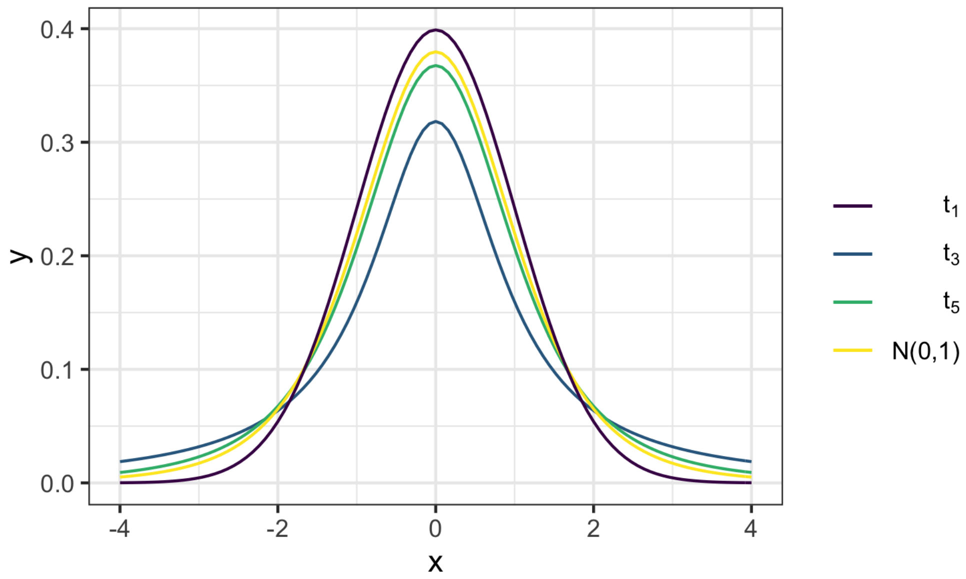
When $\bar{X}$ is normal, but the true value of σ is unknown, the interval $[\bar{X} - t_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}}, \bar{X} + t_{n-1,\alpha/2}\frac{\hat{\sigma}}{\sqrt{n}}]$ is called a $(1-\alpha)\cdot 100\%$ Confidence Interval:

We are $(1-\alpha)\cdot 100\%$ confident that an interval constructed this way will contain the true value μ!

New Distribution: t-distribution

The t-distribution is very similar to the standard normal.

It is defined by a single parameter called the *degrees of freedom* (denoted df). We will use T_{df} as notation for a random variable that follows the t-distribution with df degrees of freedom, i.e. $T_{df} \sim t_{df}$.



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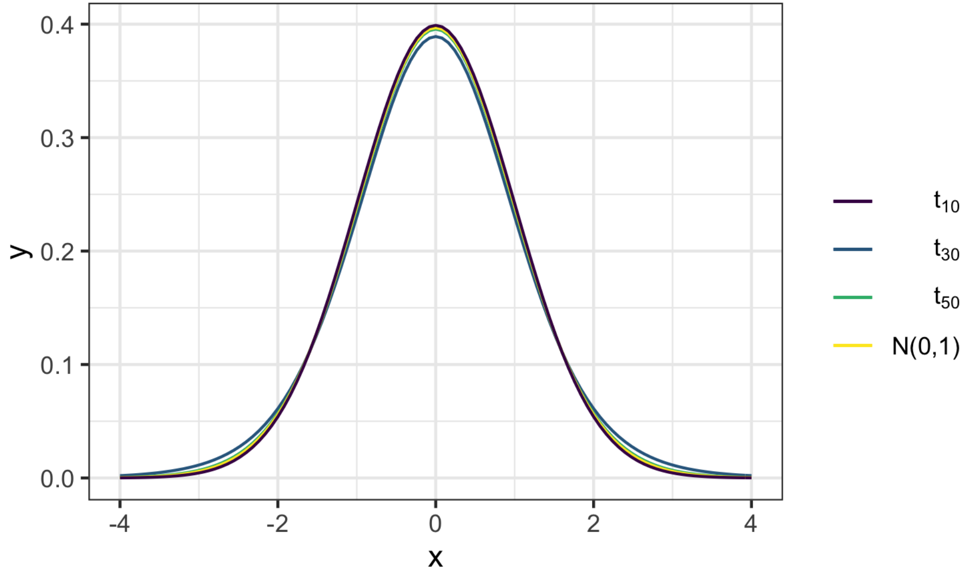

Actually, if " df $= \infty$ ", the t-distribution is *exactly* the standard normal.