Nonlinear Optimization Lecture 3

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Last time

- Convex functions and Convex sets ==> convex optimization
- ϵ-neighborhood (open ball)
- open sets
- closed sets

Some vocabulary

Interior of a set S

The interior of a set, $\text{Int}(S) = \{x \in S \colon \exists \epsilon > 0, N_\epsilon(x) \subset S\}$. The interior of a closed set is just the open set having removed the boundary.

Observation: S *is open* $\Leftrightarrow S = \text{Int}(S)$.

Notes: $\Leftrightarrow$ means Iff. or equivalent. $A \Rightarrow B$ is A implies B, $A \Leftarrow B$ is B implies A.

$\mathbb{R}$ is open by definition of open sets, and $\emptyset$ is closed because $\emptyset^C = \mathbb{R}$ is open. But $\text{Int}(\emptyset) = \emptyset$ so $\emptyset$ is open. And then $\mathbb{R}^C$ is open so $\mathbb{R}$ is closed by the same logic. Thus $\mathbb{R}$ and $\emptyset$ are both open and closed (so neither are *well-defined*: *clopen set*).

A set that has a partial boundary is neither closed nor open (See **Fig. 3.1**).

Figure 1: Fig 3.1

Fig. 3.2: venn diagram of sets

Boundary points

A point, x, is a **boundary point** of S if for each $\epsilon > 0$, the ϵ neighborhood $N_\epsilon(x)$ contains a point in S and a point not in S.

Figure 2: Fig 3.2

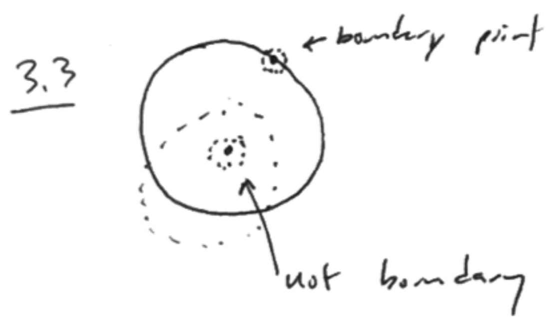

Figure 3: Fig 3.3

Boundary of S

The **boundary** of $S = \delta S =$ set of all boundary points.

Closure of S

The closure of $S = C\ell(S) = S \cup \delta S$. S is closed iff. $S = C\ell(S)$.

Bounded sets

- Let $S \subset \mathbb{R}^n$
- S is **bounded** if $\exists m > 0, m \in \mathbb{R}$
- such that $\|x\| \leq m, \forall x \in S$
- (norm: $\sqrt{x_1^2 + x_2^2 + x_3^2 + \dots}$)

Compact sets

A set $S \subset \mathbb{R}^n$ is **compact** if it is closed and bounded.

Weirstrass Theorem

Weirstrass Theorem

A continuous function defined on a non-empty compact set, attains a minimum on the set.

(compact = closed and bounded)

This is the typical form, but let's consider the form most useful to this class:

$$\begin{aligned} &\min f(x) \\ &\text{s.t} x \in X \end{aligned}$$

If f is continuous and X is compact, then *there exists* an optimal solution.

Fig 3.4: Compact and unbounded sets that cause problems and break things.

In LP, *continuous* is less of a worry because *linear* functions are continuous, but in NLP both *compact* and *continuous* are a big deal. (Although depending on the way in which a function is not continuous, it may not matter.)

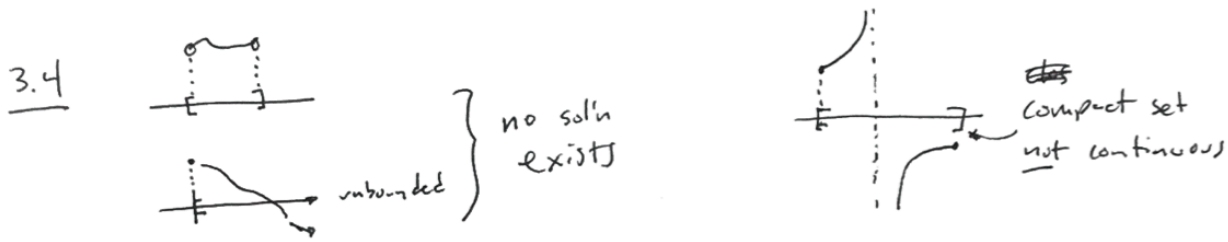

Figure 4: Fig 3.4

Minimum Distance point

Theorem Let S be a non-empty closed, convex subset of $\mathbb{R}^n$. Let $y \notin S$. Then there exists the unique point $\bar{x} \in S$ that is closest to y.

Furthermore $(y - \bar{x})^T (x - \bar{x}) \leq 0$, $\forall x \in S$.

Fig. 3.5: Example of $S, \bar{x}, y$.

Figure 5: Fig 3.5

Proof

$\bar{x}$ is a solution of

$$\min_x \|y - x\|$$
$$\text{s.t} x \in S$$

Existence. We haven't assumed that S is bounded. Let $T = \{x \in \mathbb{R}^n \colon \|y - x\| \leq \|y - x_0\|\}$, see **Fig. 3.6**.

By doing this, we can change this problem to

$$\min \|y - x\|$$
$$\text{s.t} x \in S \cap T$$

Then, by the Weirstrass Theorem, $\bar{x}$ exists, because $S \cap T$ is compact.

Uniqueness.

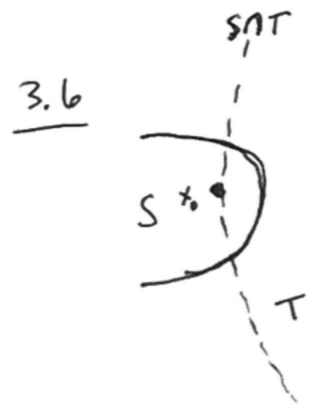

Figure 6: Fig 3.6

This proves existence, but we need to now prove uniqueness, which is often done via contradiction. Also notice that *convexity* in the theorem must play a key role in the proof (or it wouldn't have been in the theorem.)

First, suppose that $\bar{x}$ is not unique, i.e. $\exists \hat{x} \in S, \hat{x} \neq \bar{x}$ such that $\|y - \bar{x}\| = \|y - \hat{x}\|$.

(There's another point $\hat{x}$ that is the same distance from y as $\bar{x}$.)

Since S is convex, consider $\frac{1}{2}\hat{x} + \frac{1}{2}\bar{x} \in S$.

Then

$$\|y - (\frac{1}{2}\bar{x} + \frac{1}{2}\hat{x})\| \leq \frac{1}{2}\|y - \bar{x}\| + \frac{1}{2}\|y - \hat{x}\|$$

by the triangle inequality, but this is the same as

$$\|y - \bar{x}\|$$

by the above definition. But then notice that if the $<$ holds, that $\bar{x}$ is not a solution.

$$\|y - (\frac{1}{2}\bar{x} + \frac{1}{2}\hat{x})\| = \|y - \bar{x}\|$$

See **Fig. 3.7** that under these conditions, y must be on a line between $\hat{x}$ and $\bar{x}$ if they are not the same point.

Figure 7: Fig 3.7

Let $(y-\bar{x}) = \lambda(y-\hat{x})$ for some λ. From $\|y-\bar{x}\| = \|y-\hat{x}\|$ we know that $\lambda = +1$ or -1.

If $\lambda = +1$, then

$$\begin{aligned} y-\bar{x} &= y-\hat{x} \\ \Rightarrow \bar{x} &= \hat{x} \end{aligned}$$

If $\lambda = -1$, then

$$\begin{aligned} y-\bar{x} &= -(y-\hat{x}) \\ \Rightarrow y &= \frac{\bar{x}-\hat{x}}{2} \in S \end{aligned}$$

Contradition! $\bar{x}$ must be unique.

Futhermore... S is convex $\Rightarrow \bar{x} + \lambda(x-\bar{x}) \in S$, $\forall \lambda \in [0,1]$.

And also, we know that

$$\begin{aligned} \|y-(\bar{x}+\lambda(x-\bar{x}))\| &\geq \|y-\bar{x}\|, \ \forall \lambda \in [0,1] \\ \|y-(\bar{x}+\lambda(x-\bar{x}))\|^2 &= \|y-\bar{x}\|^2 + \lambda^2\|x-\bar{x}\|^2 - 2\lambda(y-\bar{x})^T(x-\bar{x}) \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda^2\|x-\bar{x}\|^2 - 2\lambda(y-\bar{x})^T(x-\bar{x}) &\geq 0 \\ \Rightarrow (y-\bar{x})T(x-\bar{x}) &\leq \frac{\lambda}{2}\|x-\bar{x}\|^2, \ \forall \lambda \in [0,1], x \in S \end{aligned}$$

Pick $\lambda = 0 \Rightarrow (y-\bar{x})^T(x-\bar{x}) \leq 0$, $\forall x \in S$.