> ### Nonlinear Optimization Lecture 11
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KKT Necessary Conditions

$$\min \quad f(x)$$
$$\text{s.t} \quad g_i(x) \le 0 \quad i = 1, \ldots, m$$

$\bar{x}$ feasible with constrain qualifications – $\nabla g_i(\bar{x})$, $i \in I$ are linearly independent.

If $\bar{x}$ is a local minimum, then KKT conditions state that $\exists u$ such that

$$\nabla f(\bar{x}) + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x}) = 0$$
$$u_i g_i(\bar{x}) = 0 \quad \forall i = 1, \ldots, m$$
$$u_i \ge 0$$

or in vector notation

$$\nabla f(\bar{x}) + \nabla g(\bar{x})^T u = 0$$
$$g(\bar{x})^T u = 0$$
$$u \ge 0$$

where u is a vector of each u_i.

Definition. u_i is called a *Lagrangian Multiplier* or a *dual variable.*

$$\bar{x} \text{ feasible} \rightarrow \text{ Primal Feasibility}$$
$$\left. \begin{array}{l} \nabla f(\bar{x}) + \nabla g(\bar{x})^T u = 0 \\ u \ge 0 \end{array} \right\} \rightarrow \text{ Dual Feasibility}$$
$$g(\bar{x})^T u = 0 \rightarrow \text{ Complimentary Slackness}$$

Figure 1: Example 1
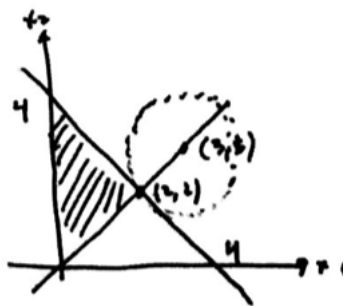
Example 1

$$\min \quad (x_1 - 3)^2 + (x_2 - 3)^2$$

$$\text{s.t} \qquad\qquad x_1^2 - x_2^2 \qquad \le 0 (u_1)$$
$$x_1 + x_2 - 4 \quad \le 0 (u_2)$$
$$-x_1 \qquad \le 0 (u_3)$$
$$-x_2 \qquad \le 0 (u_4)$$

Fritz-John points: $\bar{x} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $I = \{1, 2\}$.

$$u_0 + \nabla f(\bar{x}) + u_1 \nabla g_1(\bar{x}) + u_2 \nabla g_2(\bar{x}) = 0$$

$$\nabla f(x) = \begin{bmatrix} 2(x_1 - 3) \\ 2(x_2 - 3) \end{bmatrix}$$

$$\nabla g_1(\bar{x}) = \begin{bmatrix} 2x_1 \\ -2x_2 \end{bmatrix}$$

$$\nabla g_2(\bar{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_0 \begin{bmatrix} -2 \\ -2 \end{bmatrix} + u_1 \begin{bmatrix} 4 \\ -4 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_0, u_1, u_2 \ge 0$$

$$(u_0, u_1, u_2) \ne 0$$

$$u_0 = 1$$

$$u_1 = 0$$

$$u_2 = 2$$

$$u_3 = 0$$

$$u_4 = 0$$

Then $\bar{x}$ is a Fritz-John point.

What about $\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$? This is not a local minimum, but is it a Fritz-John point? In this case, $I = \{1, 3, 4\}$, then

$$u_0 \begin{bmatrix} -6 \\ -6 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_0 = u_3 = u_4 = 0$$

$$u_1 = 1$$

$$u_2 \in \mathbb{R}$$

$\bar{x}$ is Fritz-John point, but it is not a local minimum (but still satisfies Fritz-John)
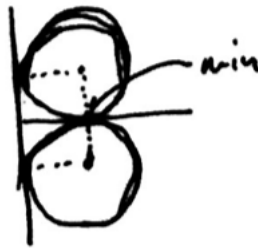
Example 2

Figure 2: Example 2

$$\begin{aligned}
\min \quad & x_1 \\
\text{s.t} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 - 1 \leq 0 \quad (u_1) \\
& (x_1 - 1)^2 + (x_2 + 1)^2 - 1 \leq 0 \quad (u_2)
\end{aligned}$$

This is a convex function and a convex set.

$$\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{x} \text{ local min} \Rightarrow \bar{x} \text{ F-J point}$$

$$\nabla f(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\nabla g_1(x) = \begin{bmatrix} 2(x_1 - 1) \\ x(x_2 - 1) \end{bmatrix}$$

$$\nabla g_2(x) = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 + 1) \end{bmatrix}$$

$$u_0 \nabla f(\bar{x}) + u_1 \nabla g_1(\bar{x}) + u_2 \nabla g_2(\bar{x}) = 0$$

$$u_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ -2 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u_0 = 0, u_1 = u_2 = 1$$

Compare with KKT Conditions

$$\nabla f(\bar{x}) + u_1 \nabla g_1(\bar{x}) + u_2 \nabla g_2(\bar{x}) = 0$$
$$u_1 g_1(\bar{x}) = 0$$
$$u_2 g_2(\bar{x}) = 0$$
$$u_1, u_2 \geq 0$$

We know that $g_1(\bar{x}) = 0$, $g_2(\bar{x}) = 0$.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ -2 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Where there is no solution. So $\bar{x}$ is a Fritz-John point, but *not* a KKT point. What's wrong is that $\begin{bmatrix} 0 & -2 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 2 \end{bmatrix}^T$ are not linearly independent, meaning that the constraint qualifications are not satisfied.

In this case, $S = \{(1,0)\}$ and $Int(S) = \emptyset$.

Some constraint qualifications

1. $\nabla g_i(\bar{x})$, $i \in I$ are linearly independent
2. Slater's CQ for convex optimization problems: $Int(S) \neq \emptyset$
3. Abadie's CQ: constraints are all linear.

Inequality and Equality

$$\begin{aligned}
\min \quad & f(x) \\
\text{s.t} \quad & g_i(x) \leq 0 \quad i = 1, \ldots, m \\
& h_j(x) = 0 \quad j = 1, \ldots, l
\end{aligned}$$

CQ: $\nabla g_i(\bar{x})$, $\forall i \in I$ and $\nabla h_j(\bar{x})$, $\forall j = 1, \ldots, l$ are linearly independent.

If $\bar{x}$ is a local minimum, then $\exists u, v$ such that

$$\begin{aligned}
\nabla f(\bar{x}) + \nabla g(\bar{x})^T u + \nabla h(\bar{x})^T v &= 0 \\
g(\bar{x})^T u &= 0 \\
u &\geq 0
\end{aligned}$$

Necessary Conditions. $\bar{x}$ a local min $\Rightarrow_{CQ}$ KKT conditions.

Sufficient Conditions. $\bar{x}$ local min $\Leftarrow_?$ KKT conditions (under some conditions). So we are interested in: *under what conditions do the KKT conditions prove local minimum?*

- f is pseudoconvex at $\bar{x}$
- g_i is quasiconvex at $\bar{x}$, $\forall i \in I$ and $g_i(\bar{x}) = 0$
- h_j is quasiconvex at $\bar{x}$, $\forall j$ such that $v_j > 0$
- h_j is quasiconcave at $\bar{x}$, $\forall j$ such that $v_j < 0$

Sufficient conditions are satisfied when the above hold – these are the minimum requirements.