Nonlinear Optimization Lecture 5

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Taylor Expansion

First-order mean value theorem

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ and f is differentiable, then there exists $\hat{x} = \lambda x' + (1-\lambda)x^2$ for some $\lambda \in (0,1)$ such that

$$f'(\hat{x}) = \frac{f(x') - f(x^2)}{x' - x^2}$$

$$\Rightarrow f(x') = f(x^2) + (x' - x^2) f'(\hat{x})$$

or $x' = x$ and $x^2 = \bar{x}$, then

$$f(x) = f(\bar{x}) + (x - \bar{x}) f'(\hat{x})$$

(2) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and f is differentiable, then there exists $\hat{x} = \lambda x + (1-\lambda)\bar{x}$ for some $\lambda \in (0,1)$ such that

$$f(x) = f(\bar{x}) + (x - \bar{x})^T \nabla f(\hat{x})$$

Second-order Mean Value Theorem

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and f is twice-differentiable, $f \in C^2$,[1] then there exists $\hat{x} = \lambda x + (1-\lambda)\bar{x}$ for some $\lambda \in (0,1)$ such that

Class notes:

$$\begin{aligned} f(x) = f(\bar{x}) \quad & + (x-\bar{x})^T \nabla f(\bar{x}) + (x-\bar{x})^T H(\hat{x})(x-\bar{x}) \\ f(x) = f(\bar{x}) \quad & + (x-\bar{x})^T \nabla f(\bar{x}) + (x-\bar{x})^T H(\bar{x})(x-\bar{x}) \\ = \cdots \quad & \qquad\qquad + \bar{x} + \cdots \end{aligned}$$

As stated in the book, Appendix 1.

The *second-order form* of Taylor's Theorem is stated as for every $x, \bar{x} \in S$ we must have

[1] C^0 is the set of continuous functions, C^1 is the set of differentiable functions, and C^2 is the set of twice-differentiable functions.

$$f(x) = f(\bar{x}) + (x - \bar{x})^T \nabla f(\bar{x}) + \frac{1}{2}(x - \bar{x})^T \mathbf{H}(\hat{x})(x - \bar{x})$$

where $\mathbf{H}(\hat{x})$ is the Hessian of f at $\hat{x}$ and where $\hat{x} = \lambda x + (1 - \lambda)\bar{x}$ for some $\lambda \in (0, 1)$.

Subgradient

Let $S \subset \mathbb{R}^n$ be convex, $S \neq \emptyset$ and $f : S \to \mathbb{R}$ be convex.

Definition: A vector $\xi \in \mathbb{R}^n$ is a *subgradient* of f at $\bar{x} \in S$ if $f(x) \geq f(\bar{x}) + \xi^T(x - \bar{x})$, $\forall x \in S$.

Theorem. For $S \subset \mathbb{R}^n$, $S \neq \emptyset$ and $f : S \to \mathbb{R}$ (convex).

For $\bar{x} \in \text{Int}S$, there exists a vector ξ such that the hyperplane

$$H = \{(x, y) : y = f(\bar{x})\xi^T(x - \bar{x})\}$$

supports the epigraph of f – epi$f$ – at $(\bar{x}, f(\bar{x}))$.

In particular,

$$f(x) \geq f(\bar{x}) + \xi^T(x - \bar{x}) \; \forall x \in S$$

that is, ξ is a subgradient of f at $\bar{x}$.

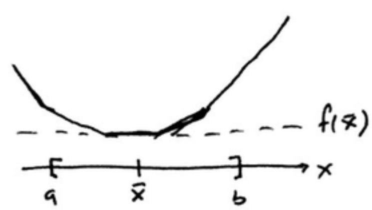

Figure 1:

Note that $\bar{x}$ is in the interior of S and we can always find a supporting hyperplane for the epigraph of f, as long as f is convex, but that if you have a differentiable function, you can find only one supporting hyperplane.

Theorem. $S \subset \mathbb{R}^n$ is a convex, nonempty set. $f : S \to \mathbb{R}$ is convex, differentiable. Then $\nabla f(\bar{x})$ is the unique subgradient for all $\bar{x} \in \text{Int}S$.

Proof. (Proof by contradiction). Suppose that ξ is another subgradient at $\bar{x} \in \text{Int}S$ and $\xi \neq \nabla f(\bar{x})$.

$f(x) \geq f(\bar{x}) + \xi^T(x - \bar{x}) \; \forall x \in S$

$x = \bar{x} + \lambda d$ for a certain vector d and a small constant λ.

Side note: Many algorithms look like this: start a point, choose a direction, move in a step size. From the new point, choose another direction, move again in a given step size (d and λ).

$\Rightarrow f(\bar{x} + \lambda d) \geq f(\bar{x}) + \xi^T(\lambda d)$ for all $d \in \mathbb{R}^n$ and sufficiently small $\lambda > 0$.

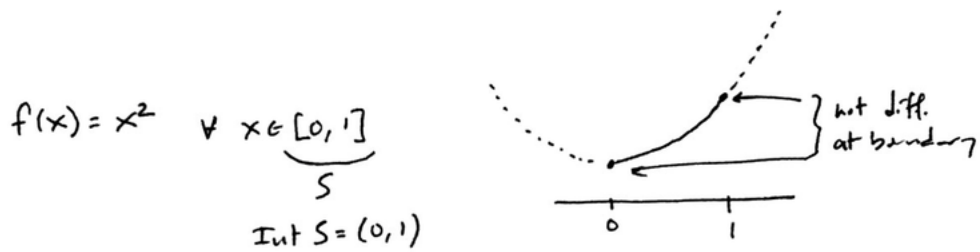

Figure 2: Demonstration of why this theorem is limited to Int$S$. Because otherwise f may not be differentiable at the boundary points.

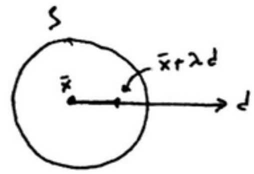

Figure 3: Note that λ must be sufficiently small to stay inside S.

Let's look at the **Taylor Expansion** (which gives equality and then we subtract it from the inequality above):

$$
\begin{aligned}
f(\bar{x}+\lambda d) &= f(\bar{x}) + \lambda \nabla f(\bar{x})^T d + \lambda \|d\| \alpha(\bar{x}; \lambda d) \\
\Rightarrow f(\bar{x}+\lambda d) - f(\bar{x}+\lambda d) & \\
0 &\geq \lambda \left[\xi - \nabla f(\bar{x})\right]^T d - \lambda \|d\| \alpha(\bar{x}; \lambda d)
\end{aligned}
$$

Then let $\lambda \to 0^+$ and pick $d = \xi - \nabla f(\bar{x})$:

$$
\begin{aligned}
\left[\xi - \nabla f(\bar{x})\right]^T \left[\xi - \nabla f(\bar{x})\right] &\leq 0 \\
\Rightarrow \|\xi - \nabla f(\bar{x})\|^2 &\leq 0
\end{aligned}
$$

The result is that if the function is smooth and differentiable, then the subgradient is unique.

Example. Find the set of subgradients at $\bar{x} = 2$, where $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \max\{x^2, x+2\}$$

The set of subgradients $\partial f(\bar{x})$ at $x = 2 \to \partial f(2)$

$$
\begin{aligned}
&\partial f(2) \\
&\qquad = \{\xi \in \mathbb{R}\colon f(x) \geq f(2) + \xi(x-2),\ \forall x \in \mathbb{R}\} \\
&\qquad = \{\xi \in \mathbb{R}\colon 1 \leq \xi \leq 4\}
\end{aligned}
$$

Note: the subgradient must support the epigraph, that is the main thing we are discussing here.

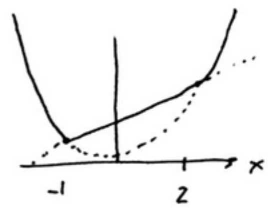

Figure 4: Illustration of $f(x)$ for example

Some characteristics of convex functions

The idea is to list some properties of convex functions that we can use to demonstrate optimality.

1

$f\colon \mathbb{R}^n \to \mathbb{R}$. f is **convex** on S if and only if

$$f(\lambda\bar{x} + (1-\lambda)\hat{x}) \leq \lambda f(\bar{x}) + (1-\lambda) f(\hat{x})$$

for all $\bar{x}, \hat{x} \in S$ and $\lambda \in [0,1]$.

f is **strictly convex** on S if and only if

$$f(\lambda\bar{x} + (1-\lambda)\hat{x}) < \lambda f(\bar{x}) + (1-\lambda) f(\hat{x})$$

where we have simply removed the inequality, but we also need to limit $\lambda \in (0,1)$ and $\bar{x} \neq \hat{x}$.

2

When $f\colon S \to \mathbb{R}$, and $S \subset \mathbb{R}^n,\ S \neq \emptyset$ is convex.

Then f is convex on S if and only if epi$f$ is convex.

3

When $f\colon S \to \mathbb{R}$, $S \subset \mathbb{R}^n$, S is open convex, then f is differentiable on S[2].

f is convex on S[3] if and only if for all $\bar{x} \in S$

$$f(x) \geq f(\bar{x}) + (x-\bar{x})^T \nabla f(\bar{x})\ \forall x \in S$$

Proof (⇒). If f is convex, then second condition is true.

Proof (⇐). If second condition is true, then f must be convex.

[2] *Have to say: f is differentiable on open S.*
[3] Note: some people use this definition for convex functions if the function is differentiable.

4

For $f\colon S \to \mathbb{R}$ and $S \subset \mathbb{R}^n$ open, convex, nonepty, $f \in C^1(S)$.

Then f is convex if and only iff

$$\left[\nabla f(x) - \nabla f(\bar{x})\right]^T (x - \bar{x}) \geq 0$$

$\forall x, \bar{x} \in S$ (or $\nabla f(x)$ is monotone[4] on S.)

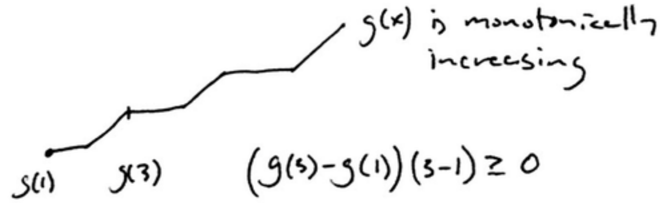

Figure 5: Demonstration with $\mathbb{R}^2$ function.

Proof (⇒). $\bar{x}, \hat{x} \in S$, f is convex means that

$$\begin{aligned} f(\hat{x}) &\geq f(\bar{x}) + \left[\nabla f(\bar{x})\right]^T (\hat{x} - \bar{x}) \\ f(\bar{x}) &\geq f(\hat{x}) + \left[\nabla f(\hat{x})\right]^T (\bar{x} - \hat{x}) \\ \text{Sum these two...} & \\ 0 &\geq \left[\nabla(f(\bar{x})) - \nabla f(\hat{x})\right]^T (\hat{x} - \bar{x}) \\ \Rightarrow \left[\nabla(f(\hat{x})) - \nabla f(\bar{x})\right]^T (\bar{x} - \hat{x}) \geq 0 & \end{aligned}$$

Proof (⇐). $\bar{x}, \hat{x} \in S$, invoking the FOMVT tells us that there exists an $\tilde{x} = \lambda\bar{x} + (1-\lambda)\hat{x}$ for $\lambda \in (0,1)$ such that $f(\bar{x}) = f(\hat{x}) + [\nabla f(\tilde{x})]^T(\bar{x} - \tilde{x})$.

We know that $\tilde{x} \in S$.

$$\begin{aligned} &\left[\nabla f(\tilde{x}) - \nabla f(\hat{x})\right]^T (\bar{x} - \hat{x}) \geq 0 \\ &\quad \Rightarrow [\nabla f(\tilde{x})]^T (\bar{x} - \hat{x}) \geq [\nabla f(\bar{x})]^T (\bar{x} - \hat{x}) \\ &\qquad \Rightarrow f(\tilde{x}) - f(\hat{x}) \geq [\nabla f(\bar{x})]^T (\bar{x} - \hat{x}) \\ &\qquad\quad \Rightarrow f(\tilde{x}) \geq f(\hat{x}) + [\nabla f(\hat{x})]^T (\bar{x} - \hat{x}) \\ &\qquad\qquad \Rightarrow f \text{is convex.} \end{aligned}$$

[4] In 2D we say *monotonically increasing*, but in vector form we can only really say that the function is monotone – how could we define *increasing*?