# *Nonlinear Optimization Lecture 14*

Garrick Aden-Buie

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## Homework 3 Review

### Problem 3

> Consider $\min f(x)$ s.t. $g_i(x) \leq 0$ for i = 1, ..., m\$. Let $\bar{x}$ be a feasilbe point and $I = \{i \colon g_i(\bar{x}) = 0\}$. Suppose $f$ is differentiable at $\bar{x}$ and $g_i$ for $i \in i$ differentiable and concave at $\bar{x}$ and $g_i$ for $i \notin I$ is continuous at $\bar{x}$. Then consider

$$\begin{aligned} \min \quad & \nabla f(\bar{x})^T d \\ \text{s.t} \quad & \nabla g_i(\bar{x})^T d \leq 0 \quad && \forall i \in I \\ & -1 \leq d \leq 1 && \forall j = 1, \ldots, n \end{aligned}$$

> Let $\bar{d}$ be an optimal solution with objective function value $\bar{z}$.

#### 3.b

> Show that if $\bar{z} < 0$, then there exists $\delta > 0$ such that $\bar{x} + \lambda\bar{d}$ is feasible, $f(\bar{x} + \lambda\bar{d}) < f(\bar{x})$ for each $\lambda \in (0, \delta)$.

We know that $\nabla f(\bar{x})^T\bar{d} < 0$ and $\nabla g_i(\bar{x})^T\bar{d}_i \leq 0$ for all $i \in I$. We also know that $g_i$ is concave $\forall i \in I$, which implies that

$$g_i(y) \leq g_i(\bar{y}) + \nabla g_i(\bar{y})^T(y - \bar{y}), \ \forall y, \bar{y}$$

Let $y = \bar{x} + \lambda\bar{d}$ and $\bar{y} = \bar{x}$. Then

$$g_i(\bar{x} + \lambda\bar{d}) \leq g_i(\bar{x}) + \nabla g_i(\bar{x})^T\lambda\bar{d}, \ \forall\lambda \geq 0, \ \forall i \in I$$

Then (1) $g_i(\bar{x} + \lambda\bar{d}) \leq 0, \ \forall\lambda \geq 0, \ \forall i \in I$. And (2) $g_i(\bar{x} + \lambda\bar{d}) \leq 0, \ \forall i \notin I$. Then there exists $\delta_1 > 0$ such that $g_i(\bar{x} + \lambda\bar{d}) \leq 0$ for all $\lambda \in [0, \delta_1)$. (1) + (2) implies that $\bar{x} + \lambda\bar{d}$ feasible for all $\lambda \in [0, \delta_1)$ and knowing that $\nabla f(\bar{x})^T\bar{d} < 0$ gives us that $\exists\delta_2 > 0 \colon f(\bar{x} + \lambda\bar{d}) < f(\bar{x}), \ \forall\lambda \in (0, \delta_2)$.

Then if we let $\delta = \min\{\delta_1, \delta_2\}$ we have the proof.

**3.c**

> Show that if $\bar{z} = 0$ then $\bar{x}$ is a KKT point.

Rewrite the above problem as

$$\begin{aligned} \min \quad & c^T d \\ \text{s.t} \quad & Ad \leq b \end{aligned}$$

Where

$$\begin{aligned} c &= \nabla f(\bar{x}) \\ b &= \begin{bmatrix} \mathbf{0}_I \\ \mathbf{1}_m \\ \mathbf{1}_m \end{bmatrix} \\ A &= \begin{bmatrix} \left.\begin{matrix} \nabla g_i(\bar{x})^T \\ \vdots \end{matrix}\right\} i \in I \\ I_m \\ -I_m \end{bmatrix} \\ d &= \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix} \end{aligned}$$

And the dual problem is

$$\begin{aligned} \max_w \quad & b^T w \\ \text{s.t} \quad & A^T w = c \\ & w \leq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max_w \quad & \sum_{j=1}^m \mu_j + \sum_{j=1}^m \eta_j \\ \text{s.t} \quad & \begin{bmatrix} \nabla g_i(\bar{x}) & \dots & | & I & | & -I \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \\ \eta \end{bmatrix} = \nabla f(\bar{x}) \\ & \lambda, \mu, \eta \leq 0 \end{aligned}$$

If $\bar{z} = 0$, then $\mu = 0$ and $\eta = 0$.

Then the constraint becomes

$$\begin{aligned} \sum_{i \in I} \nabla g_i(\bar{x}) \lambda_i &= \nabla f(\bar{x}), \qquad \lambda_i \leq 0 \\ \Rightarrow \nabla f(\bar{x}) - \sum_{i \in I} \nabla g_i(\bar{x}) \lambda_i &= 0 \end{aligned}$$

Let $\rho_i = -\lambda_i \geq 0$. Then

$$\nabla f(\bar{x}) + \sum_{i \in I} \nabla g_i(\bar{x}) \rho_i = 0$$

# Review

## Unconstrained Optimization

- Line Search
  - Single dimension problem: $\min f(x),\ a \leq x \leq b$, we know $x^* \in [a, b]$ and $f(x)$ is strictly quasiconvex.
- Dichotomous Search
- Golden Section Search
- (No derivatives used in the above…)

# Line Search with Derivative

We are still minimizing $f(x),\ x \in \mathbb{R}$, but we are going to use $f'(x)$ and $f$ is pseudoconvex, hence differentiable.

- Bisection Method
  - Will cover next lecture
  - Evaluate $f'(\frac{a+b}{2})$ and depending on the direction decide which section of $[a, b]$ to explore.