Nonlinear Optimization Lecture 21

Garrick Aden-Buie

Thursday, April 7, 2016

Review

- $VI(F, \Omega)$
  - $y \in \Omega, F(y)^T(x - y) \geq 0 \, \forall x \in \Omega$
- $NCP(F)$
  - $F(y)^T y = 0$
  - $F(y) \geq 0$
  - $y \geq 0$
- $FPP_{\min}(F, \Omega)$
  - $y \in P_\Omega[y - F(y)]$
  - $y \in \Omega$

Theorem

If $\Omega = \mathbb{R}_+^n[\hat{}1]$ then $VI(F, \Omega) \Leftrightarrow NCP(F)$.

Proof (⇒). If y is a solution of $VI(F, \Omega)$, then y is a solution of $NCP(F)$.

$$F(y)^T(x - y) \geq 0 \, \forall x \in \mathbb{R}_+^n$$

i. $x = 0 \in \mathbb{R}_+^n$: $F(y)^T(-y) \geq 0$

ii. $x = 2y \in \mathbb{R}_+^n$: $F(y)^T y \geq 0$

iii. (i) & (ii) $\Rightarrow F(y)^T y = 0$. Now we need to show that $F(y) \geq 0$.

   Suppose not: $F(y) \not\geq 0 \Rightarrow \exists j \in 1, \ldots, n$ such that $F_j(y) < 0$, which is to say that $F(y)$ is not non-negative.

$$F(y)^T(x-y) = \sum_{i=1}^{n} F_i(y)(x_i - y_i) \geq 0$$

$$\text{Pick } x = \begin{bmatrix} y_1 \\ \vdots \\ x_j = \infty \\ \vdots \\ y_n \end{bmatrix}$$

Now, then $x_i - y_i = 0$ for all elements that are not x_j, but then x_j forces $F_j(y)(x_j - y_j) < 0$, so contradiction.

Proof ($\Leftarrow$). If y is a solution to $NCP(F)$, then y is also a solution to $VI(F, \Omega)$. $F(y) \geq 0, y \geq 0, F(y)^T y = 0$. Then we choose an $x \in \Omega$ which will by definition be non-negative and observe the following

$$\begin{aligned} F(y)^T x \geq 0 \quad & \forall x \in \Omega \\ \Rightarrow F(y)^T x - F(y)^T y \geq 0 \quad & \forall x \in \Omega \\ \Rightarrow F(y)^T (x - y) \geq 0 \quad & \forall x \in \Omega \end{aligned}$$

Theorem

$\Omega \subset \mathbb{R}^n$ convex, nonempty, closed. Then $FPP_{\min}(F, \Omega) \Leftrightarrow VI(F, \Omega)$.

Proof. $P_\Omega[y - F(y)]$ requires a solution to

$$\begin{aligned} \min_x \quad & \|(y - F(y)) - x\| \\ \text{s.t} \quad & x \in \Omega \end{aligned}$$

Without loss of generality we can rewrite as

$$\begin{aligned} \min_x \quad & \frac{1}{2}\|(y - F(y)) - x\|^2 \\ \text{s.t} \quad & x \in \Omega \end{aligned}$$

Which is also the same as

$$\begin{aligned} \min_x \quad & \frac{1}{2}(y - F(y) - x)^T (y - F(y) - x) = Z(x; y) \\ \text{s.t} \quad & x \in \Omega \end{aligned}$$

Now, we can rewrite this as the KKT conditions, which are sufficient conditions given the convexity of Ω, although in this case we'll use the VI-type optmiality:

$$\Leftrightarrow \nabla_x Z(x^*; y)^T (x - x^*) \geq 0 \; \forall x \in \Omega$$

But note that x, x^*, and y are all the same at the end, because we are solving the Fixed-Point Problem.

Note that in the simple case if we have Ω convex, then $\min f(x)$ s.t. $x \in \Omega \Leftrightarrow \nabla f(x^*)^T (x - x^*) \geq 0 \; \forall x \in \Omega$. So in this case, note that

$$\begin{aligned} \nabla_x Z(x^*; y) &= -(y - F(y) - x^*) \\ (\text{Note that } \nabla_x Z(x^*; y) &= \nabla_x Z(x; y)|_{x=x^*}) \\ \Leftrightarrow -(y - F(y) - x^*)^T (x - x^*) &\geq 0 \qquad\qquad \forall x \in \Omega \end{aligned}$$

At this point we have shown that $FPP_{\min}(F\Omega) \Rightarrow VI(F, \Omega)$, and we have that $y = x^*$.

The next step is to show that $VI(F, \Omega) \Rightarrow FPP_{\min}(F, \Omega)$. Start from the condition $F(y)^T (x - y) \geq 0 \; \forall x \in \Omega$ (1). We also have $-(y - F(y) - x^*)^T (x - x^*) \geq 0$ (2) which gives x^* as a result of the projection.

So from the side of (1), y is a VI solution and x^* is a projection result from a given y. So we pick $x = x^*$. This gives $F(y)^T (x^* - y) \geq 0$.

Then from (2), pick $x = y \Rightarrow -(y - x^*)^T (y - x^*) + F(y)^T (y - x^*) \geq 0 \Rightarrow 0 \geq F(y)^T (y - x^*) \geq (y - x^*)^T (y - x^*) \geq 0$. Notice that $(y - x^*)^T (y - x^*)$ is sandwiched between 0 and 0, so y needs to be x^*.

Overall, the proof starts with knowing that at the end x, y, and x^* are all the same, but starting from the assumption that they may be different.

Existence (of Fixed-Point)

Brower's Fixed-Point Theorem

If Ω is a convex, nonempty and compact set and $F(x)$ is continuous on Ω, then the fixed-point problem

$$\begin{aligned} &x = F(x) \\ &x \in \Omega \end{aligned}$$

has a solution.

Theorem

- Ω convex, nonempty, compact
- $F(x)$ continuous on Ω
- Then $VI(F, \Omega)$ has a solution

Proof. $VI(F, \Omega) \Leftrightarrow FPP_{\min}(F, \Omega)$. Then $y = P_\Omega(y - F(y))$, or equivalently $y = G(y)$. If we can show that $G(y)$ is continuous, then we are done. From advanced linear algebra (and commonly taught in advance linear algebra courses), the projection operator is continuous, so $G(y)$ is continuous.

Theorem: Nash Game

$\max u_i(x^i; x^{-i})$

If

- u_i is pseudo-concave with respect to x^i for all x^{-i}.
- $\Omega = \prod_{i=1}^{N} \Omega_i$ is convex, compact, nonempty

Then there exists a Nash equilibrium.

Uniqueness

- Optimization: strictly pseudoconvex
- Nash equilibrium: strictly monotone

Definition

- $F\colon \mathbb{R}^n \to \mathbb{R}^n$
- F is (strictly) monotone on Ω if

$$\begin{aligned}\left[F(y^1) - F(y^2))\right]^T (y^1 - y^2) \geq (>) 0 \\ \forall y^1, y^2 \in \Omega \;\; (y^1 \neq y^2)\end{aligned}$$

Theorem

If $y \in \Omega \subset \mathbb{R}^n$ is a solution of $VI(F, \Omega)$ and $F(\cdot)$ is strictly monotone

Then y is the unique solution.