Nonlinear Optimization Lecture 4

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Last time

Theorem S is a non-empty, closed convex set and $y \notin S$, then $\bar{x} \in S$ is the min. distance point. And $(y - \bar{x})^T(x - \bar{x}) \leq 0$, $\forall x \in S$.

Separation

Let $S_1, S_2 \subset \mathbb{R}^n$ and $H = \{x \in \mathbb{R}^n : p^T x = \alpha\}$.

Definition. H *separates* S_1 and S_2 if

$$p^T x \geq \alpha \quad \forall x \in S_1$$
$$p^T x \leq \alpha \quad \forall x \in S_2$$

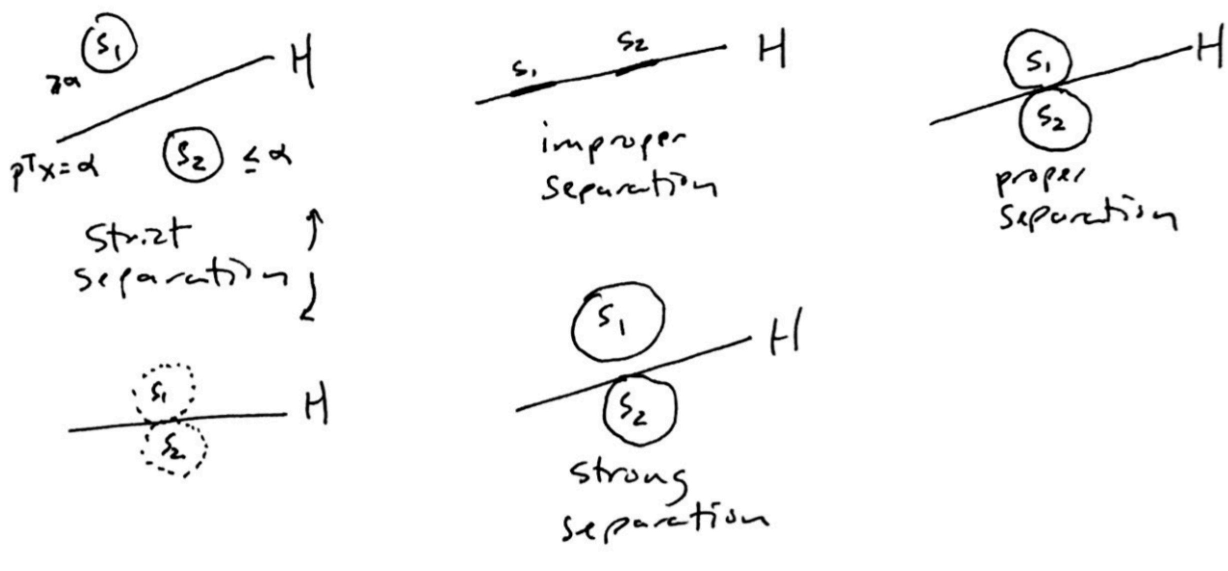

Figure 1: Types of separation

H *strictly* separates S_1 and S_2 if

$$p^T x > \alpha \quad \forall x \in S_1$$
$$p^T x < \alpha \quad \forall x \in S_2$$

H *strongly* separates S_1 and S_2 if

$$\begin{aligned} p^T x &\geq \alpha + \epsilon \quad \forall x \in S_1 \\ p^T x &\leq \alpha \quad \forall x \in S_2 \end{aligned}$$

for some $\epsilon > 0$.

Theorem If S is a non-empty, closed, convex set and $y \notin S$. Then there exists a hyperplane that strongly separates S and $\{y\}$ (singleton).

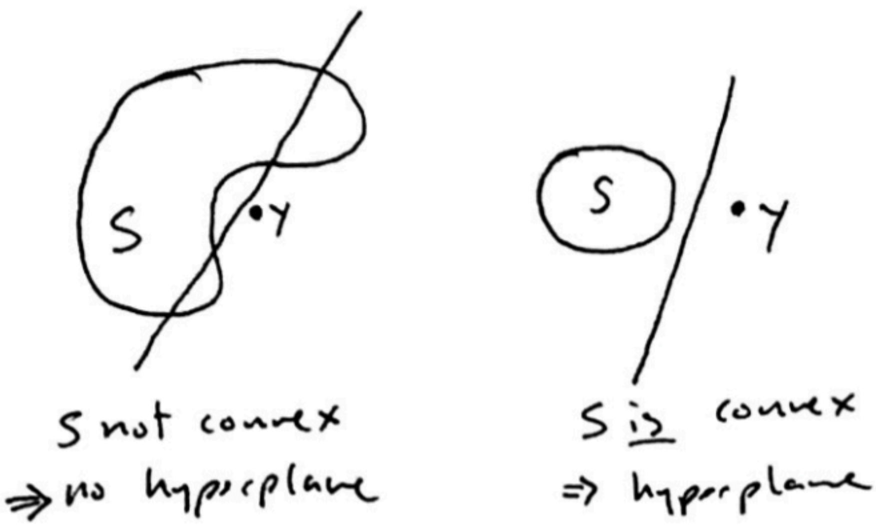

Figure 2: Requirement of convexity for theorem

Proof. There exists an $\bar{x} \in S$ such that $(y - \bar{x})^T (x - \bar{x}) \leq 0$, $\forall x \in S$.

Let

$$\begin{aligned} p &= y - \bar{x} \\ \Rightarrow p^T (x - \bar{x}) &\leq 0 \quad \forall x \in S \\ p^T x &\leq p^T \bar{x} \quad \forall x \in S \\ p^T x &\leq \alpha \quad \forall x \in S \end{aligned}$$

where $\alpha = p^T \bar{x}$.

Now we want to show that

$$p^T x \geq \alpha + \epsilon \;\; \forall x = y$$

for some $\epsilon > 0$, or

$$p^T y > \alpha$$

.

$$
\begin{aligned}
& p^T y - \alpha \ (> 0) \\
&= (y - \bar{x})^T y - (y - \bar{x})^T \bar{x} \\
&= (y - \bar{x})^T (y - \bar{x}) \\
&= \|y - \bar{x}\|^2 > 0
\end{aligned}
$$

(Note that all norms satisfy $\|a\| = 0$ iff $a = 0$.)

This proves the two conditions listed above for *strong separation*.

Supporting Hyperplane

Let $S \in \mathbb{R}^n$, $S \neq \emptyset$, $\bar{x} \in \delta S$, and $H = \{x \in \mathbb{R}^n : p^T(x - \bar{x}) = 0\}$. Note that H is a hyperplane that passes through $\bar{x}$ because setting $x = \bar{x}$ satisfies $p^T(x - \bar{x}) = 0$.

$$
\begin{aligned}
H^+ &= \{x \in \mathbb{R}^n : p^T(x - \bar{x}) \geq 0\} \\
H^- &= \{x \in \mathbb{R}^n : p^T(x - \bar{x}) \leq 0\}
\end{aligned}
$$

H is a *supporting hyperplane* for S at $\bar{x}$ if $S \subset H^+$ or $S \subset H^-$.

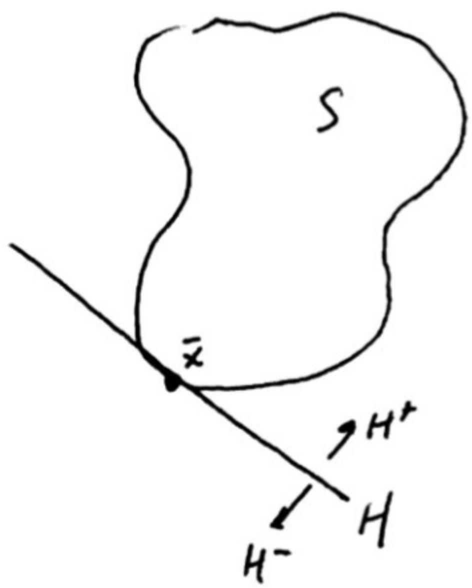

Figure 3:

Theorem Let $S \subset \mathbb{R}^n$ be a closed, non-empty convex set, $y \notin S$ and $\bar{x} \in S$ is the closest point to y.

Then there exists a supporting hyperplane at $\bar{x}$.

Intuition: Consider y and $\bar{x}$. If you move y closer and closer to $\bar{x}$, then eventually you move the hyperplane up to the boundary of S. This "process" would converge to the supporting hyperplane.

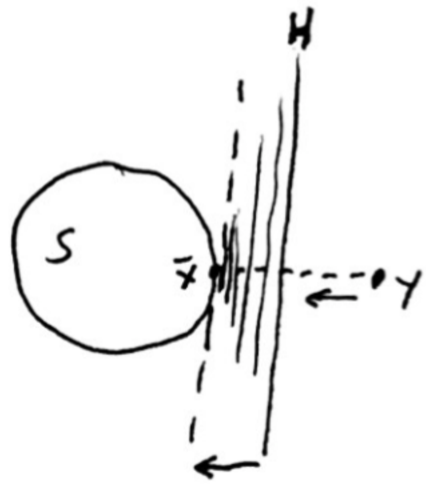

Figure 4: Intuition illustrated

Convex Functions

f is convex on S if

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2)$$
$$\forall x_1, x_2 \in S, \quad \lambda \in [0,1]$$

Level Set

Let $\alpha \in \mathbb{R}$, then $S_\alpha = \{x \in S \colon f(x) \le \alpha\}$ is the level set.

Theorem If f is convex on S, then S_alpha is a convex set for all $\alpha \in \mathbb{R}$.

Proof. Let $x_1, x_2 \in S_alpha$ and $\lambda \in [0,1]$. We need to show that $\lambda x_1 + (1-\lambda)x_2 \in S_alpha$ for all $\alpha \in \mathbb{R}$.

We know that

$$f(x_1) \le \alpha$$
$$f(x_2) \le \alpha$$
$$\Rightarrow \lambda f(x_1) \le \lambda\alpha$$
$$(1-\lambda)f(x_2) \le (1-\lambda)\alpha$$

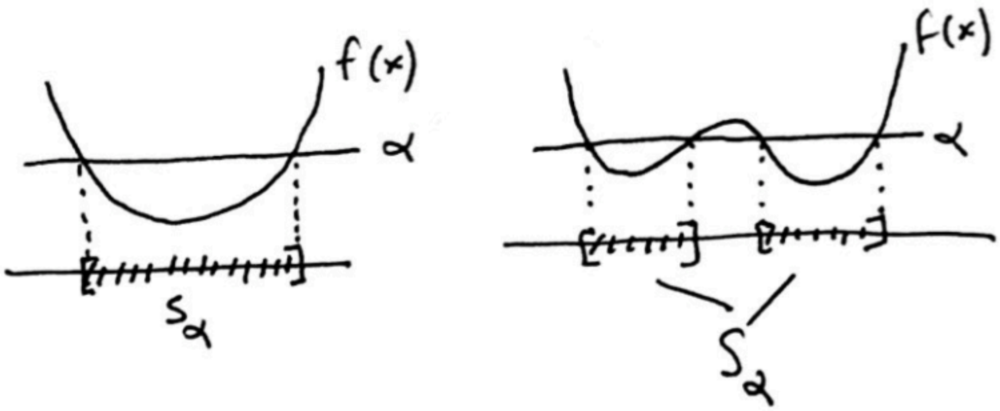

Figure 5: Level sets

f is convex, so

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &\leq \lambda f(x_1) + (1-\lambda) f(x_2) \\ &\leq \lambda\alpha + (1-\lambda)\alpha \\ &= \alpha \end{aligned}$$

Epigraph

Definition. An *epigraph* of $f = \text{epi} f = \{(x, y) \colon x \in \mathbb{R}^n, y \in \mathbb{R}^1, y \geq f(x)\}$.

Thus, $f \colon \mathbb{R}^n \to \mathbb{R}$ and $\text{epi} f \subset \mathbb{R}^{n+1}$.

Figure 6: Epigraph

Theorem $f \colon \mathbb{R}^n \to \mathbb{R}$. f is a convex function $\Leftrightarrow$ $\text{epi} f$ is a convex set.

Proof. **Homework assignment**. Sketch: show $\Rightarrow$ and then show $\Leftarrow$. Intuition, if you have a non-convex function, then clearly $\text{epi} f$ is going to be a non-convex set.

Derivatives

When we talk about minimizing functions, $\min f(x)$, then the most obvious thing to do is to take the first derivative of f and setting equal to zero, $f'(x) = 0$.

Partial Derivative and Gradient

Consider

$$\lim_{\lambda \to 0} \frac{f(x + \lambda e_i) - f(x)}{\lambda}$$

where e_i is the i^{th} unit vector $= [0 \dots 1 \dots 0]^T$ with 1 at i^{th} element.

If the above limit exists

$$\frac{\partial f}{\partial x_i}(x) = \lim_{\lambda \to 0} \frac{f(x + \lambda e_i) - f(x)}{\lambda}$$

(which is the partial derivative of $f(x)$ with respect to $x_i), then the *gradient* of f at x is

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix} \in \mathbb{R}^n$$

(Note that in the single-variable case, we would typically have Δx in the denominator. But the multi-directional case, this term would be a vector. So we need to rely on the partial derivative instead.)

Directional Derivative

For any $d \in \mathbb{R}^n$ the *directional derivative* of f in the direction d is

$$f'(x; d) = \lim_{\lambda \to 0^+} \frac{f(x + \lambda d) - f(x)}{\lambda}$$

Def. f is *differentiable* at x if and only if $\nabla f(x)$ exists and $\nabla f(x)^T d = f'(x; d)$ for all $d \in \mathbb{R}^n$. This is *Gateaux-differentiability.*

We will be using this definition many times in discussing optimality conditions.

Theorem Given $f \colon \mathbb{R}^n \to \mathbb{R}$ is convex and $\bar{x} \in \mathbb{R}^n, d \in \mathbb{R}^n$ a non-zero direction.

Then the directional derivative

$$f'(\bar{x}; d)$$

exists.

Figure 7:

Remark. Can a convex function be discontinuous on its own domain? (Answer is no.)

Remark. Can a convex function be non-differentiable? (Answer: yes, consider $y = |x|$. Notice that in this case, y is not differentiable at one point, but the *directional* derivative does exist. The directional derivative is a more "general" derivative.)

Taylor Expansion

Given $f(x)\colon \mathbb{R}^n \to \mathbb{R}$:

1. $f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{1}{2!}f^{''}(x)(x-\bar{x})^2 + \cdots$
2. $f(x) = f(\bar{x}) + (x-\bar{x})^T\nabla f(\bar{x}) + \frac{1}{2!}(x-\bar{x})^T H(x)(x-\bar{x}) + \cdots$
   - Where $\nabla f(\bar{x})$ is the gradient and $H(x)$ is the Hessian (look up!)

Precisely the $(\cdots)$ terms in (2) is a function $\|x-\bar{x}\|^2 \cdot \alpha(\bar{x}; x-\bar{x})$ and $lim_{x\to\bar{x}}\alpha(\bar{x}; x-\bar{x}) = 0$.