Nonlinear Optimization Lecture 20

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Game Theory Intro

There are two purposes to game theory: descriptive and predictive. In engineering, the primary use is *predictive*.

Noncooperative N-Player Game

Theorem. Nash Equilibrium Problem

For each player i:

$$\begin{aligned} \max \quad & u_i(x^i, x^{-i}) \\ \text{s.t} \quad & x^i \in X_i \subset \mathbb{R}^{m_i} \\ & X = \prod_{i=1}^{N} X_i = X_1 \times X_2 \times \cdots \times X_N \end{aligned}$$

where $u_i \colon X \to \mathbb{R}$ is continuously differentiable with respect to x^i or pseudo-concave with respect to x^i.

$x^* \in X$ is[1] a solution to $NE(X, u)$ if and only if $x^* \in X$ satisfies

$$\sum_{i=1}^{N} \left[\nabla_{x^i} u_i(x^*)\right]^T (x^i - x^{i*}) \leq 0 \;\forall x \in X$$

This kind of problem is called *variational inequality*, because the form of the inequality changes with $x^i - x^{i*}$, as in the number of inequalities that x^* must satisfy is infinite, as the above must be satisfied $\forall x$.

Recall. For $\min f(x)$ s.t. $x \in X$, $x^* \in X$ is optimal if and only if $\nabla f(x^*)^T (x - x^*) \geq 0 \;\forall x \in X$.

Remark. In the following formulation, the optimum is for the global "system" maximum. Note the difference with the Nash equilibrium.

[1] $x^* = (x^{1*}, x^{2*}, \ldots, x^{N*})$

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} u_i(x) \\ \text{s.t} \quad & x \in X \\ \Leftrightarrow \quad & \sum_{i=1}^{N} (\nabla_x u_i)^T (x - x^*) \leq 0 \quad \forall x \in X \end{aligned}$$

Proof (⇒). For each i: given x^{i*}, x^{i*} maximizes u_i.

$$\left[\nabla_{x^i} u_i(x^{i*}, x^{-i*})\right]^T (x^i - x^{i*}) \leq 0 \; \forall x^i \in X_i$$

This then simply leads to the Variational Inequality, so proven.

Proof (⇐). Assume that x^* satisfies the VI, and show that it is a nash equilibrium point.

Fix $j \in 1, 2, \ldots, N$ and let $y = \left[x^{1*}, x^{2*}, \ldots, y^j, \ldots, x^{N*}\right]$ for some $y^j \in X^j$. Essentially: take the optimal for all players not j and let one player's strategy vary. Note also that $y \in X$.

From here, when taking $x^i - x^{i*}$, all terms cancel except y^j.

$$\left[\nabla_{x^j} u_j(x^*)\right]^T (y^j - x^{j*}) \leq 0$$

Because our choice of y^j was arbitrary, we can do the same thing for all $y^j \in X_j$. Then x^{j*} maximizes u_j given x^{j*} (because pseudo-convex).

Scalar-based version of Nash Equilibrium Variational Inequality

$$\sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\partial u_i(x^*)}{\partial x_j^i} (x_j^i x_j^{i*}) \leq 0 \; \forall x \in X$$

Variational Inequality

Definition.

$$\begin{aligned} VI(F, \Omega) \\ \Omega \subset \mathbb{R}^n \text{ non-empty} \\ F\colon \Omega \to \mathbb{R}^n \\ VI(F, \Omega) \text{ is to find a vector } y \\ \text{such that } y \in \Omega \\ [F(y)]^T (x - y) \geq 0 \; \forall x \in \Omega \\ \langle F(y), x - y \rangle \geq 0 \; \forall x \in \Omega \end{aligned}$$

Nonlinear Complimentary Problem

$F: \mathbb{R}^n \to \mathbb{R}^n$. $NCP(F)$ is to find a vector y such that

$$\begin{aligned} F(y)^T y &= 0 \\ F(y) &\geq 0 \\ y &\geq 0 \end{aligned}$$

Note that this is similar to the KKT conditions, but more general. Sometimes these three conditions are written in the following form:

$$0 \leq F(y) \perp y \geq 0$$

Fixed-Point Problem

- $\Omega in \mathbb{R}^n$ non-empty
- $F: \Omega \to \Omega$
- $FPP(F, \Omega)$ is to find a vector y such that $y \in \Omega, y = F(y)$.
- This is related to the notion of equilibrium.

Minimum Norm Projection

$$\begin{aligned} y &= P_\Omega[v] \qquad\qquad v \in \mathbb{R}^n, \Omega \subset \mathbb{R}^n \quad (v \text{ vector}, \Omega \text{ set}) \\ &= \arg\min_{x \in \Omega} \|v - x\| \\ \Rightarrow y &\in \Omega \end{aligned}$$

Fixed-Point Problem based on Min Norm Projection

An extension of the general form.

- $\Omega \subset \mathbb{R}^n$ non-empty
- $F: \Omega \to \Omega$
- $FPP_{\min}(F, \Omega)$ is to find a vector y such that $y \in \Omega$ and $y = P_\Omega[y - F(y)]$.

Here P_Ω is the projection onto feasible space and $F(y)$ is like the gradient of vector function. But FPP implies you stay at some point $\Rightarrow$ optimal solution.

$$y = P_\Omega[y - F(y)] \Rightarrow y = \arg\min_{x \in \Omega} \|y - F(y) - x\|$$

Connection between VI, NCP, FPP_{min}

So far we have been discussing VI, NCP, FPP_{min} and we are going to now show that these three are all related.

Consider the following optimization problem:

$$\left.\begin{array}{rl} \min & f(x) \\ \text{s.t} & x \in X \\ & f \text{ pseudoconvex} \\ & X \text{ convex} \end{array}\right\} \Rightarrow \nabla f(x^*)^T(x - x^*) \geq 0 \; \forall x \in X$$

Whenever you have an optimization problem in this situation, you can convert to VI problem.

What is we have VI problem $VI(F, \Omega)\colon F(y)^T(x-y) \geq 0 \; \forall x \in \Omega$? We can go from the optimization problem to the VI using gradient. Can we go VI $\to$ optimization? The following theorem shows this.

Theorem

Suppose $\Omega \subset \mathbb{R}^n$ and $F\colon \Omega \to \mathbb{R}^n$.

Then $VI(F, \Omega)$ is equivalent to $\min \oint_0^x F(z)dz$ s.t. $x \in \Omega$.

Note. x, z, dz are all vectors with dimension n, so the integral is a line integral $\oint$.

The theorem holds if $\oint_0^x F(z)dz$ is *single-valued*, where single-valued means that $\oint_0^x F(z)dz = c \; \forall$ paths from $\mathbf{0} \to \mathbf{x}$.