Nonlinear Optimization Lecture 20

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Game Theory Intro

There are two purposes to game theory: descriptive and predictive. In engineering, the primary use is *predictive*.

Noncooperative N-Player Game

Theorem. Nash Equilibrium Problem

For each player i:

$$\begin{aligned} \max \quad & u_i(x^i, x^{-i}) \\ \text{s.t} \quad & x^i \in X_i \subset \mathbb{R}^{m_i} \\ & X = \prod_{i=1}^{N} X_i = X_1 \times X_2 \times \cdots \times X_N \end{aligned}$$

where $u_i \colon X \to \mathbb{R}$ is continuously differentiable with respect to x^i or pseudo-concave with respect to x^i.

$x^* \in X$ is[1] a solution to $NE(X, u)$ if and only if $x^* \in X$ satisfies

$$\sum_{i=1}^{N} \left[\nabla_{x^i} u_i(x^*)\right]^T (x^i - x^{i*}) \leq 0 \ \forall x \in X$$

This kind of problem is called *variational inequality*, because the form of the inequality changes with $x^i - x^{i*}$, as in the number of inequalities that x^* must satisfy is infinite, as the above must be satisfied $\forall x$.

Recall. For min $f(x)$ s.t. $x \in X$, $x^* \in X$ is optimal if and only if $\nabla f(x^*)^T(x - x^*) \geq 0 \ \forall x \in X$.

Remark. In the following formulation, the optimum is for the global "system" maximum. Note the difference with the Nash equilibrium.

[1] $x^* = (x^{1*}, x^{2*}, \ldots, x^{N*})$

$$\begin{aligned} \max \quad & \sum_{i=1}^{N} u_i(x) \\ \text{s.t} \quad & x \in X \\ \Leftrightarrow \quad & \sum_{i=1}^{N} (\nabla_x u_i)^T (x - x^*) \leq 0 \quad \forall x \in X \end{aligned}$$

Proof (⇒). For each i: given x^{i*}, x^{i*} maximizes u_i.

$$\left[\nabla_{x^i} u_i(x^{i*}, x^{-i*})\right]^T (x^i - x^{i*}) \leq 0 \ \forall x^i \in X_i$$

This then simply leads to the Variational Inequality, so proven.

Proof (⇐). Assume that x^* satisfies the VI, and show that it is a nash equilibrium point.

Fix $j \in 1, 2, \dots, N$ and let $y = \left[x^{1*}, x^{2*}, \dots, y^j, \dots, x^{N*}\right]$ for some $y^j \in X^j$. Essentially: take the optimal for all players not j and let one player's strategy vary. Note also that $y \in X$.

From here, when taking $x^i - x^{i*}$, all terms cancel except y^j.

$$\left[\nabla_{x^j} u_j(x^*)\right]^T (y^j - x^{j*}) \leq 0$$

Because our choice of y^j was arbitrary, we can do the same thing for all $y^j \in X_j$. Then x^{j*} maximizes u_j given x^{j*} (because pseudo-convex).

Scalar-based version of Nash Equilibrium Variational Inequality

$$\sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{\partial u_i(x^*)}{\partial x_j^i} (x_j^i x_j^{i*}) \leq 0 \ \forall x \in X$$

Variational Inequality

Definition.

$$\begin{aligned} & VI(F, \Omega) \\ & \Omega \subset \mathbb{R}^n \text{ non-empty} \\ & F : \Omega \to \mathbb{R}^n \\ & VI(F, \Omega) \text{ is to find a vector } y \\ & \text{such that } y \in \Omega \\ & [F(y)]^T (x - y) \geq 0 \ \forall x \in \Omega \\ & \langle F(y), x - y \rangle \geq 0 \ \forall x \in \Omega \end{aligned}$$