Complex Analysis
Revision Exercises

1. Write the following complex numbers in polar form $z = r\left(\cos(\theta) + i\sin(\theta)\right)$ (or equivalently, $z = r\exp(i\theta)$):
   (a) $1 + i$
   (b) $-1 + i$
   (c) $1 + i\sqrt{3}$
   (d) $\dfrac{(1+i)^7}{(1+i\sqrt{3})^2}$

Solution:

(a) $1 + i = \sqrt{2}\left(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right)$

(b) $-1 + i = \sqrt{2}\left(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\right)$

(c) $1 + i\sqrt{3} = 2\left(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})\right)$

(d) While we could expand the numerator and denominator and then simplify, it is much easier to use De Moivre's Theorem:

$$r\left(\cos(\theta) + i\sin(\theta)\right)^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right) \quad (n \in \mathbb{Z}).$$

$$\begin{aligned}(1+i)^7 &= 2^{7/2}\left(\cos(7\pi/4) + i\sin(7\pi/4)\right),\\ (1+i\sqrt{3})^{-2} &= 2^{-2}\left(\cos(-2\pi/3) + i\sin(-2\pi/3)\right).\end{aligned}$$

Then we get

$$\begin{aligned}\frac{(1+i)^7}{(1+i\sqrt{3})^2} &= (1+i)^7(1+i\sqrt{3})^{-2}\\ &= 2^{3/2}\left(\cos(13\pi/12) + i\sin(13\pi/12)\right).\end{aligned}$$

This is fine, however, if we wish to use the principal value of the argument, we must subtract an appropriate integer multiple of 2π from $13\pi/12$ (to ensure that the argument lies in $(-\pi, \pi]$). This is given by $13\pi/12 - 2\pi = -11\pi/12$, thus

$$\frac{(1+i)^7}{(1+i\sqrt{3})^2} = 2^{3/2}\left(\cos(-11\pi/12) + i\sin(-11\pi/12)\right).$$

An equally valid way of answering this is using the exponential polar form:

$$(1+i)^7 = 2^{7/2}\exp(7\pi/4) \qquad (1+i\sqrt{3})^{-2} = 2^{-2}\exp(-2\pi/3).$$

The properties $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$ and $\exp(z + 2\pi i) = \exp(z)$ give:

$$\frac{(1+i)^7}{(1+i\sqrt{3})^2} = 2^{7/2}\exp(7\pi/4)\,2^{-2}\exp(-2\pi/3) = 2^{3/2}\exp(-11\pi/12).$$

2. (a) For $z = a + ib$, express z^{-1} (i.e, $\dfrac{1}{z}$) in Cartesian form

(b) Write down the modulus and Principal Argument of z^{-1} in terms of $|z|$ and $\mathrm{Arg}(z)$.

Solution:

(a) We use the fact that $\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2}$ to invert any nonzero complex number:

$$\frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.$$

(b) Writing $z = r\left(\cos(\theta) + i\sin(\theta)\right)$, De Moivre's Theorem gives

$$z^{-1} = \left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^{-1} = \frac{1}{r}\left(\cos(-\theta) + i\sin(-\theta)\right).$$

Hence $\left|z^{-1}\right| = 1/|z|$ and $\mathrm{Arg}(z^{-1}) = -\mathrm{Arg}(z)$.

3. Express the following complex numbers in Cartesian form (that is to say, as $a + ib$ for $a, b \in \mathbb{R}$)

$$\frac{i-1}{1-i} \qquad \frac{1}{1+i} \qquad \frac{3+4i}{1-2i}.$$

Solution: In each case, we make the denominator of the quotient $\frac{z_1}{z_2}$ real by multiplying by $\overline{z_2}/\overline{z_2}$, which gives

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{|z_2|^2}.$$

Thus for example

$$\begin{aligned}\frac{i-1}{1-i} &= \frac{i-1}{1-i} \cdot \frac{1+i}{1+i}\\ &= \frac{1}{2}\left[i(1+i) - 1(1+i)\right]\\ &= \frac{1}{2}\left[-2\right]\\ &= -1\ (+i0).\end{aligned}$$

Similarly

$$\begin{aligned}\frac{1}{1+i} &= \frac{1}{2} - \frac{i}{2}\\ \frac{3+4i}{1-2i} &= -1 + 2i.\end{aligned}$$

4. Find the principal argument of the four points $\pm 1 \pm i\sqrt{3}$.

Solution:

$$\mathrm{Arg}(1 + i\sqrt{3}) = \frac{\pi}{3}$$

$$\begin{aligned}\mathrm{Arg}(-1+i\sqrt{3}) &= \frac{2\pi}{3}\\ \mathrm{Arg}(-1-i\sqrt{3}) &= -\frac{2\pi}{3}\\ \mathrm{Arg}(1-i\sqrt{3}) &= -\frac{\pi}{3}.\end{aligned}$$

5. Calculate

$$\mathrm{Arg}\left(\frac{1}{2}+\frac{1}{z^2}\right)\bigg|_{z=1+i}.$$

(The notation $f(z)|_{z=w}$ means $f(z)$ evaluated at $z=w$, or in other words, $f(w)$.)

Solution: We have

$$\frac{1}{2}+\frac{1}{(1+i)^2}=\frac{1}{2}-\frac{i}{2},$$

with principal argument $-\frac{\pi}{4}$.

6. Show that

$$2\left(\frac{z}{z+i}\right)\frac{(z+i-z)}{(z+i)^2}\bigg|_{z=i}=\frac{-i}{4}.$$

Solution:

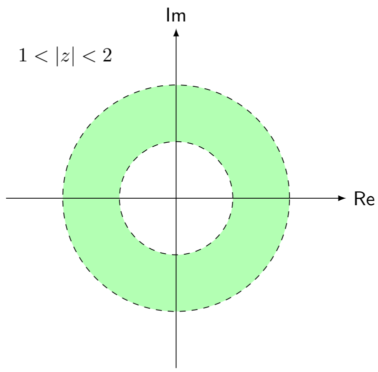
$$\begin{aligned}2\left(\frac{z}{z+i}\right)\frac{(z+i-z)}{(z+i)^2}\bigg|_{z=i} &= 2\left(\frac{i}{2i}\right)\left(\frac{i+i-i}{(2i)^2}\right)\\ &= \frac{2i}{2i}\frac{i}{(-4)}=-\frac{i}{4}.\end{aligned}$$

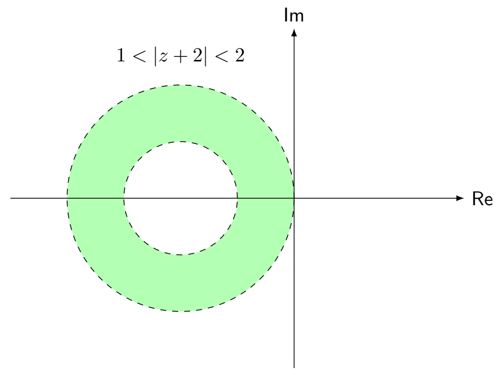
7. Sketch the following regions of $\mathbb{C}$:
   (a) $1<|z|<2$ (This notation is shorthand for the set $\{z\in\mathbb{C}:1<|z|<2\}$.)
   (b) $1<|z+2|<2$
   (c) $1<\mathsf{Im}(z-i)<2$

Solution:

(a),(b) Note that for $\alpha\in\mathbb{C}$ and $r>0$, the set $\{z\in\mathbb{C}:|z-\alpha|=r\}$ is precisely the circle of radius r centred at α. The sets $0<|z-\alpha|<r$ and $|z-\alpha|>r$ respectively are the regions inside and outside the circle.

Hence $1<z<2$ is the region outside the circle of radius 1 centred at 0, and inside the circle of radius 2 centred at 0. The set $1<|z+2|<2$ is the same but with circles centred at -2. A set of this type is called an *annulus*.

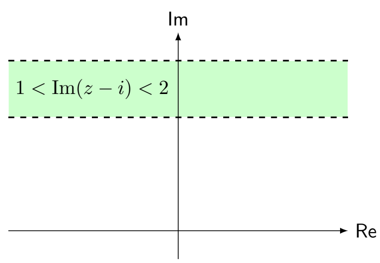




(c) Write $z = x + iy$, then $z - i = x + i(y-1)$ and so

$$\begin{aligned}\{z \in \mathbb{C} : 1 < \mathsf{Im}(z-i) < 2\} &= \{x + iy \in \mathbb{C} : 1 < y - 1 < 2\}\\ &= \{x + iy \in \mathbb{C} : 2 < y < 3\}\,,\end{aligned}$$

which is the infinite horizontal band bounded by the lines $y = 2$ and $y = 3$.



8. (a) Prove that for two nonzero complex numbers z_1 and z_2 we have

$$|z_1 z_2| = |z_1| \cdot |z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg(z_1)\arg(z_2)$$

(hint: write z_1 and z_2 in polar form). Is it always true that $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_2) + \mathrm{Arg}(z_2)$?

Solution: Writing

$$z_1 = r_1\left(\cos(\theta_1) + i\sin(\theta_1)\right) \quad \text{and} \quad z_2 = r_2\left(\cos(\theta_2) + i\sin(\theta_2)\right)$$

we see that

$$\begin{aligned} z_1 z_2 &= r_1 r_2\left(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) + i\left[\cos(\theta_1)\sin(\theta_2) + \cos(\theta_2)\sin(\theta_1)\right]\right)\\ &= r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right).\end{aligned}$$

Thus

$$|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2| \quad \text{and} \quad \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2).$$

It is not always true that $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$. Indeed, if $z_1 = z_2 = -1$ we have $\mathrm{Arg}(-1) = \pi$ but

$$\mathrm{Arg}((-1)(-1)) = \mathrm{Arg}(1) = 0.$$

(b) Show that $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$.

Solution: With $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ a similar calculation to part (a) shows that

$$\begin{aligned}\exp(z_1)\exp(z_2) &= e^{x_1}\left(\cos(y_1) + i\sin(y_1)\right)e^{x_2}\left(\cos(y_2) + i\sin(y_2)\right)\\ &= e^{x_1}e^{x_2}\left(\cos(y_1+y_2) + i\sin(y_1+y_2)\right)\\ &= e^{x_1+x_2}\left(\cos(y_1+y_2) + i\sin(y_1+y_2)\right) = \exp(z_1+z_2).\end{aligned}$$

9. Write down the 3^{rd} roots of -8 in Cartesian form.

Solution: Writing $-8 = 8\left(\cos(\pi) + i\sin(\pi)\right)$, the three cubic roots are of the form $\sqrt[3]{8}\left(\cos((\pi+2k\pi)/3) + i\sin((\pi+2k\pi)/3)\right)$ for $k = 0, 1, 2$. In polar form, these are

$$2\left(\cos(\pi/3) + i\sin(\pi/3)\right), \quad 2\left(\cos(\pi) + i\sin(\pi)\right) \quad \text{and} \quad 2\left(\cos(5\pi/3) + i\sin(5\pi/3)\right),$$

or in Cartesian form $1+\sqrt{3}$, -2 and $1-\sqrt{3}$ respectively.

10. Find the values of z for which $z^2 + 4iz - 1 = 0$. Which of these values lies inside the circle $C = \{z \in \mathbb{C} : |z| = 1\}$.

Solution: This is simply the usual quadratic formula: the roots of this polynomial are given by

$$\frac{-4i \pm \sqrt{(4i)^2 - 4(1)(-1)}}{2} = \frac{-4i \pm i\sqrt{12}}{2} = i(-2 \pm \sqrt{3}).$$

To see which of these lies inside the circle $|z| = 1$, we examine the modulus and see that $\left|i(-2+\sqrt{3})\right| \approx 0.071 < 1$ and $\left|i(-2-\sqrt{3})\right| \approx 1.93 > 1$. Thus only $i(-2+\sqrt{3})$ lies inside this circle.

11. Show that $\mathsf{Re}(z) \leq |\mathsf{Re}(z)| \leq |z|$ and $|\mathsf{Re}(z)| + |\mathsf{Im}(z)| \leq \sqrt{2}\,|z|$.

Solution: The fact that $\mathsf{Re}(z) \leq |\mathsf{Re}(z)|$ is trivial. Moreover, writing $z = x + iy$ we see that

$$|\mathsf{Re}(z)| = \sqrt{x^2} \leq \sqrt{x^2 + y^2}$$

since $x^2 \leq x^2 + y^2$ (and the square root function is increasing).
For the second inequality, note that with $z = x + iy$,

$$\begin{aligned}\left(|\mathsf{Re}(z)| + |\mathsf{Im}(z)|\right)^2 &= \left(|x| + |y|\right)^2\\ &\leq \left(|x| + |y|\right)^2 + \overbrace{\left(|x| - |y|\right)^2}^{\geq 0}\\ &= x^2 + y^2 + 2\,|xy| + x^2 + y^2 - 2\,|xy|\\ &= 2(x^2 + y^2) = 2\,|z|^2.\end{aligned}$$

Taking square roots of both sides yields the required inequality.