MA2003 Complex Analysis
Exercise Sheet 4

1. Express each of the following complex numbers in Cartesian form $a + ib$:

$$(a)\ \text{Log}(i), \quad (b)\ \text{Log}(ie) \quad (c)\ \text{Log}(-1 - i\sqrt{3}).$$

2. Express each of the following complex numbers in Cartesian form $a + ib$:

$$(a)\ (1+i)^i, \quad (b)\ (ie)^{i\pi}, \quad (c)\ (-1 - i\sqrt{3})^{1+i}.$$

3. (a) Use the definition $z^\alpha = \exp(\alpha \operatorname{Log}(z))$ to show that $z^3 = zzz$.
   (b) Show that $\operatorname{Log}(i^3) \neq 3 \operatorname{Log}(i)$
   (c) Define $\sqrt{z} = z^{1/2} (= \exp(\frac{1}{2} \operatorname{Log}(z)))$ for $z \in \mathbb{C} \setminus \{0\}$. Where is the mistake in

$$-1 = i^2 = ii = \sqrt{-1}\sqrt{-1} = \sqrt{-1 \times -1} = \sqrt{1} = 1?$$

   (d) Show that for all $\alpha, \beta \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{0\}$ we have $z^\alpha z^\beta = z^{\alpha+\beta}$. Is it true that $\operatorname{Log}(\alpha\beta) = \operatorname{Log}(\alpha) + \operatorname{Log}(\beta)$?

4. Recall that the Principal Logarithm function Log is holomorphic on the region $\mathbb{C}_\pi$, where $\mathbb{C}_\pi = \{z \in \mathbb{C} : z \neq 0 \text{ and } \operatorname{Arg}(z) \neq \pi\}$. Let F be the function defined by

$$F(z) = \frac{1}{2i} \left(\operatorname{Log}(z+i) - \operatorname{Log}(z-i) \right).$$

   (a) Describe (or sketch) the region $\mathcal{R}$ on which the function F is holomorphic.
   (b) Show that F is an antiderivative for the function $f : \mathcal{R} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z^2+1} \quad \text{for all} \quad z \in \mathcal{R}.$$

5. Let U be a starlit region with star centre $z_* \in U$ and let $g : U \to \mathbb{C}$ be a holomorphic function.
   (a) Prove that if $g(z) \neq 0$ for all $z \in U$, then the function $\dfrac{g'}{g}$ has an antiderivative on U, stating any results used (you may assume that g holomorphic on U implies g' holomorphic on U).
   (b) Prove that if in addition $g(z) \in \mathbb{C}_\pi$ for all $z \in U$ then

$$\int_{[z_*, z]} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta = \operatorname{Log}(g(z)) + \alpha$$

   for some constant α.

6. Evaluate the integral

$$\int_{\mathcal{C}} \frac{\exp(2z)}{4z + i\pi}\, dz$$

where $\mathcal{C}$ is (i) the anticlockwise contour whose points lie on the circle $\{z : |z| = 1\}$, and (ii) when $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z - 2i| = 2\}$. The use of any Theorems made to obtain the value of these integrals should be justified.

7. Evaluate the integral

$$\int_{\mathcal{C}} \frac{\cos(z^2)}{3i + 2z}\, dz,$$

where (i) $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z| = 1\}$, and (ii) $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z| = 5\}$. The use of any Theorems made to obtain the value of these integrals should be justified.

8. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 6x + 25}\, dx$$

in the following way (compare Example 5.8 in the notes):

(a) Define the complex function f by $f(z) = \dfrac{1}{z^2 + 6z + 25}$ and find z_0 and z_1 so that $f(z) = \dfrac{1}{(z - z_0)(z - z_1)}$ (where z_0 lies in the upper half-plane and z_1 in the lower half-plane).

(b) Choose a suitable function g, holomorphic on the simply connected region $\mathcal{R} = \left\{z \in \mathbb{C} : \mathsf{Im}(z) > \frac{1}{2}\mathsf{Im}(z_1)\right\}$, so that

$$f(z) = \frac{g(z)}{(z - z_0)}.$$

(c) Justify the use of Cauchy's Integral Formula to find

$$\int_{\mathcal{C}_R} f = \int_{\mathcal{C}_R} \frac{g(z)}{(z - z_0)}\, dz,$$

where $\mathcal{C}_R = L_R + S_R$ with L_R the straight line path from $-R$ to R and S_R a suitable semicircular contour from R to $-R$, with R sufficiently large to apply the Theorem.

(d) Show that for large R and $z \in S_R$, we have $\left|z^2 + 6z + 25\right| \geq R^2 - 6R - 25$.

(e) Use the Estimation Lemma to show that

$$\left|\int_{S_R} f\right| \to 0 \text{ as } R \to \infty.$$

(f) Deduce the value of

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 6x + 25}\, dx.$$

9. (Liouville's Theorem) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic everywhere, and suppose that f is bounded, i.e. there exists $M > 0$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Show that f is constant on $\mathbb{C}$, in the following way:

(a) Let $z_1, z_2 \in \mathbb{C}$, and let $R > 0$ be sufficiently large so that z_1 and z_2 are enclosed by the countour $\mathcal{C}_R$ consisting of the anticlockwise circle with centre 0 and radius R. Use Cauchy's Integral Formula to write $f(z_1) - f(z_2)$ as a single integral along $\mathcal{C}_R$.

(b) Use the Estimation Lemma (and the backwards triangle inequality) to show that

$$|f(z_1) - f(z_2)| \leq M \frac{|z_1 - z_2|}{(R - |z_1|)(R - |z_2|)} \cdot 2\pi R$$

for all (sufficiently large) R.

(c) Deduce that $f(z_1) = f(z_2)$.