MA2003 Complex Analysis
Exercise Sheet 1

1. Use the triangle inequality $|z_1 - z_2| \leq |z_1| + |z_2|$ to prove the reverse triangle inequality:

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

Solution: The triangle inequality gives

$$\begin{aligned} |z_1| &= |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2| \\ |z_2| &= |z_2 - z_1 + z_1| \leq |z_2 - z_1| + |z_1| \;\; (= |z_1 - z_2| + |z_1|). \end{aligned}$$

Rearranging gives the two inequalities

$$\begin{aligned} |z_1 - z_2| &\geq |z_1| - |z_2| \\ |z_1 - z_2| &\geq |z_2| - |z_1|, \end{aligned}$$

or in other words

$$|z_1 - z_2| \geq \max\left(|z_1| - |z_2|, |z_2| - |z_1|\right) = ||z_1| - |z_2||.$$

2. Use the triangle and reverse triangle inequalities to show that for all z on the circle $|z| = 2$, we have

$$|z + 2| \leq 4 \text{ and } |z - 3 + 4i| \geq 3.$$

Describe these inequalities geometrically.

Solution: If $|z| = 2$ then the triangle inequality ensures that

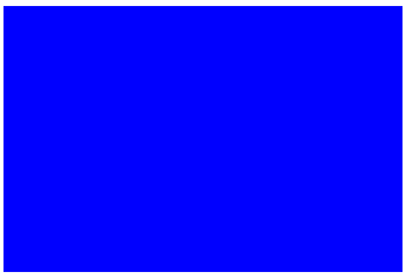
$$|z + 2| \leq |z| + 2 = 4,$$

and the backwards triangle inequality gives

$$\begin{aligned} |z - 3 + 4i| &= |z - (3 - 4i)| \\ &\geq ||z| - |3 - 4i|| \\ &= |2 - 5| = 3. \end{aligned}$$

Geometrically, these inequalities show respectively that the circle $|z| = 2$ is

- Contained in the disc with centre -2 and radius 4 (i.e., the disc $|z + 2| \leq 4$), and
- Outside of the disc with centre $3 - 4i$ and radius 3.



3. Use the triangle and reverse triangle inequalities to show that for all z on the circle $|z+3i| = 3$ we have

$$|z-4| \le 8, \quad |z+5i| \ge 1 \text{ and } \left|\frac{z-4}{z+5i}\right| \le 8.$$

Solution:

As with the previous question, the triangle inequality yields

$$\begin{aligned} |z-4| &= |z+3i-(3i+4)| \\ &\le |z+3i| + |3i+4| \\ &= 3+5 = 8. \end{aligned}$$

and the backwards triangle inequality yields

$$\begin{aligned} |z+5i| &= |z+3i+2i| \\ &\ge ||z+3i| - |2i|| \\ &= 3-2 = 1. \end{aligned}$$

The third inequality is then obvious.

4. Let L be the line segment $[0,h]$ where $h \in \mathbb{C}$ and $|h| < r$. Show that if $\beta \in \mathbb{C}$ with $|\beta| > 2r$ and $z \in L$ then

$$\left|\frac{h-z}{\beta - z}\right| < \frac{|h|}{r}.$$

Do this using the reverse triangle inequality. It can also be seen as follows. Draw L and two circles, both with centre 0, C_1 with radius r and C_2 with radius $2r$. Why does L lie inside C_1? Where is β on your diagram? Why is $|\beta - z| > r$? If you can answer these three questions then the inequality should follow easily.

Solution:

Since z lies on L, it is clear that $|h-z| \le |h|$ (more formally, we can write $z = th$ for some t with $0 \le t \le 1$, so that $|h-z| = |(1-t)h| = (1-t)\,|h| \le |h|$).

The backwards triangle inequality, together with the fact that $|z| < r < 2r < |\beta|$ gives

$$\begin{aligned} |\beta - z| &\ge ||\beta| - |z|| \\ &= |\beta| - |z| \\ &> 2r - r = r. \end{aligned}$$

Combining these two inequalities we see that

$$\begin{aligned}\left|\frac{h-z}{\beta-z}\right| &= \frac{|h-z|}{|\beta-z|}\\ &\le \frac{|h|}{|\beta-z|}\\ &< \frac{|h|}{r}.\end{aligned}$$

5. The function $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^2$ is defined by $\mathbf{f}(x,y)=(0,2y)$. Show that the corresponding complex function $f:\mathbb{C}\to\mathbb{C}$ is $f(z)=z-\overline{z}$, and that

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$

does not exist at any point $z_0\in\mathbb{C}$.

Solution: The corresponding function is

$$f(z)=f(x+iy)=i(2y)=i(2\mathsf{Im}(z))=2i\cdot\frac{z-\overline{z}}{2i}=z-\overline{z}.$$

If $z_0\in\mathbb{C}$ and $h\in\mathbb{C}\setminus\{0\}$ then

$$\frac{f(z_0+h)-f(z_0)}{h}=\frac{(z_0+h)-\overline{(z_0+h)}-(z-\overline{z})}{h}=\frac{h-\overline{h}}{h}=1-\frac{\overline{h}}{h}.$$

Looking at restricted limits along the real and imaginary axes:

$$1-\frac{\overline{h}}{h}\to\begin{cases}1-1=0 & \text{as } h\to 0, h\in\mathbb{R}\\ 1-(-1)=2 & \text{as } h\to 0, h\in i\mathbb{R}.\end{cases}$$

Since the restricted limits are not equal the (unrestricted) limit does not exist for any $z_0\in\mathbb{C}$.

6. Same as question 5 but with

$$\mathbf{f}(x,y)=(x^2-y^2-x,2xy+y+1)\quad\text{and}\quad f(z)=z^2-\overline{z}+i.$$

Solution: This time, it is easier to start with the complex function f and substitute $z=x+iy$:

$$\begin{aligned}f(z)&=z^2-\overline{z}+i\\ &=(x+iy)^2-(x-iy)+i\\ &=x^2-y^2+i2xy-x+iy+i\\ &=x^2-y^2-x+i\,(2xy+y+1)\,,\end{aligned}$$

which shows that f corresponds to the function $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}^2$ where

$$\mathbf{f}(x,y)=\left(x^2-y^2-x,2xy+y+1\right).$$

For $z_0\in\mathbb{C}$ and $h\in\mathbb{C}\setminus\{0\}$ the difference quotient is

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h}&=\frac{(z_0+h)^2-\overline{(z_0+h)}+i-(z_0^2-\overline{z_0}+i)}{h}\\ &=h+2z_0-\frac{\overline{h}}{h}.\end{aligned}$$

Looking at some restricted limits:

$$\lim_{\substack{h\to 0,\\ h\in\mathbb{R}\setminus\{0\}}} \frac{f(z_0+h)-f(z_0)}{h} = 2z_0 - 1$$
$$\lim_{\substack{h\to 0,\\ h\in i\mathbb{R}\setminus\{0\}}} \frac{f(z_0+h)-f(z_0)}{h} = 2z_0 + 1.$$

These are not equal for any $z_0 \in \mathbb{C}$, so the unrestricted limit

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

does not exist at any $z_0 \in \mathbb{C}$.

7. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = |z|^2$. Show that f is differentiable at $z = 0$ and nowhere else.

Solution: Since $|z|^2 = z\overline{z}$,

$$\begin{aligned}
\frac{f(z_0+h)-f(z_0)}{h} &= \frac{(z_0+h)\overline{(z_0+h)} - z_0\overline{z_0}}{h} \\
&= \frac{z_0\overline{h} + h\overline{z_0} + h\overline{h}}{h} \\
&= z_0\frac{\overline{h}}{h} + \overline{z_0} + \overline{h} \\
&\longrightarrow \begin{cases} 2z_0 & \text{as } h\to 0,\ \ h\in\mathbb{R}\setminus\{0\} \\ 0 & \text{as } h\to 0, h\in i\mathbb{R}\setminus\{0\}. \end{cases}
\end{aligned}$$

For $z_0 \neq 0$, the restricted limits are not equal, hence the unrestricted limit does not exist and so f is not differentiable at any point $z_0 \in \mathbb{C}\setminus\{0\}$.

At $z_0 = 0$, we have

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\overline{h} = 0,$$

so that f is differentiable at 0 with $f'(0) = 0$.

8. Use the rules of differentiation to find the derivatives of the following functions:

(a) $f(z) = \left(z^2+4\right)^3$

(b) $g(z) = \dfrac{z+i}{z-i}$.

Find the values of $f'(i)$ and $g'(1)$.

Solution:

(a) Since f is the composition of holomorphic functions, we can use the chain rule to find $f'(z)$:

$$f'(z) = 3(z^2+4)^2(2z) = 6z(z^2+4)^2 \quad \text{for all } z\in\mathbb{C},$$

and

$$f'(i) = 6i(i^2+4)^2 = 6i(3)^2 = 54i.$$

(b) Since the functions $z \mapsto z \pm i$ are holomorphic on $\mathbb{C}$, g is holomorphic on $\mathbb{C} \setminus \{i\}$, and the quotient rule gives

$$g'(z) = \frac{(z-i)(1) - (z+i)(1)}{(z-i)^2} = \frac{-2i}{(z-i)^2}$$

for all $z \in \mathbb{C} \setminus \{i\}$. Hence

$$g'(1) = \frac{-2i}{(1-i)^2} = \frac{-2i}{-2i} = 1.$$

MA2003 Complex Analysis
Exercise Sheet 2

1. For each function f below, write f in the form

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

and determine whether or not the Cauchy-Riemann equations are satisfied:

$$(a)\ f(z) = \exp(i\ \overline{z}) \quad (b)\ f(z) = z + \frac{1}{z} \quad (c)\ f(z) = z^3.$$

In the cases where f is differentiable, find the derivative of f both using the rules of differentiation and using the Cauchy-Riemann equations.

Solution:

(a) We have

$$f(x+iy) = \exp(i(x-iy)) = \exp(y+ix) = \underbrace{e^y\cos(x)}_{u(x,y)} + i\,\underbrace{e^y\sin(x)}_{v(x,y)}.$$

The corresponding partial derivatives are

$$\frac{\partial u}{\partial x} = -e^y\sin(x) \qquad \frac{\partial v}{\partial y} = e^y\sin(x)$$
$$\frac{\partial u}{\partial y} = e^y\cos(x) \qquad \frac{\partial v}{\partial x} = e^y\cos(x).$$

The Cauchy Riemann Equations are satisfied at a point $x+iy \in \mathbb{C}$ if and only if both

$$e^y\sin(x) = -e^y\sin(x) \quad \text{and} \quad e^y\cos(x) = -e^y\cos(x).$$

Since e^y is never zero, this can only occur if $\sin(x) = \cos(x) = 0$, which is impossible.

(b) For all $x+iy \in \mathbb{C}\setminus\{0\}$,

$$\begin{aligned} f(x+iy) &= (x+iy) + \frac{1}{x+iy} \\ &= (x+iy) + \frac{x-iy}{x^2+y^2} \\ &= \underbrace{\left(x + \frac{x}{x^2+y^2}\right)}_{u(x,y)} + i\underbrace{\left(y - \frac{y}{x^2+y^2}\right)}_{v(x,y)} \end{aligned}$$

and the corresponding partial derivatives are

$$\frac{\partial u}{\partial x} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$
$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = 1 - \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Hence the Cauchy Riemann equations are satisfied at every $x+iy \in \mathbb{C}\setminus\{0\}$. Using the Cauchy Riemann equations to find the derivative of f, we get

$$\begin{aligned} f'(x+iy) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \left(1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right) + i\left(\frac{2xy}{(x^2+y^2)^2}\right) \\ &= 1 + \frac{y^2 - x^2 + i2xy}{(x^2+y^2)^2} \\ &= 1 - \frac{x^2 - y^2 - i2xy}{(x^2+y^2)^2} \\ &= 1 - \frac{(x-iy)^2}{[(x+iy)(x-iy)]^2} \\ &= 1 - \frac{1}{(x+iy)^2} = 1 - \frac{1}{z^2}, \end{aligned}$$

which agrees with the derivative of f obtained from the rules of differentiation.

(c) This time

$$u(x,y) = x^3 - 3xy^2 \quad \text{and} \quad v(x,y) = 3x^2y - y^3$$

hence for all $x+iy \in \mathbb{C}$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}.$$

Thus

$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial y}(x,y) = 3x^2 - 3y^2 + i(6xy) = 3\left((x^2-y^2) + i(2xy)\right) = 3(x+iy)^2$$

which agrees with the derivative obtained from the Chain Rule: $f'(z) = 3z^2$.

2. Show that the Cauchy-Riemann equations are satisfied by the function f defined on the open upper half plane $H_+ = \{z \in \mathbb{C} : \mathsf{Im}(z) > 0\}$ by

$$f(x+iy) = u(x,y) + iv(x,y) = \log\left(\sqrt{x^2+y^2}\right) + i\left(\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right)\right).$$

Assuming that f is indeed holomorphic on H_+, show that

$$f'(x+iy) = \frac{1}{x+iy} \quad \text{i.e., that} \quad f'(z) = \frac{1}{z}.$$

Solution: We have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \log'\left(\sqrt{x^2+y^2}\right) \cdot \frac{\partial}{\partial x}\left[\sqrt{x^2+y^2}\right] \\ &= \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x) \\ &= \frac{x}{x^2+y^2} \end{aligned}$$

and similarly

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}.$$

For the partial derivatives of v, we have

$$\begin{aligned}
\frac{\partial v}{\partial x} &= -\arctan'\left(\frac{x}{y}\right) \cdot \frac{\partial}{\partial x}\left[\frac{x}{y}\right] \\
&= \frac{-1}{(1 + (\frac{x}{y})^2)} \cdot \frac{1}{y} \\
&= \frac{-1}{y + \frac{x^2}{y}} = \frac{-y}{x^2 + y^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial v}{\partial y} &= -\arctan'\left(\frac{x}{y}\right) \cdot \frac{\partial}{\partial y}\left[\frac{x}{y}\right] \\
&= \frac{-1}{(1 + (\frac{x}{y})^2} \cdot \frac{-x}{y^2} \\
&= \frac{x}{x^2 + y^2}.
\end{aligned}$$

Thus the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied everywhere in H_+. Assuming moreover that f is holomorphic on H_+, the derivative of f is thus

$$\begin{aligned}
f'(x + iy) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} \\
&= \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy}
\end{aligned}$$

for all $x + iy \in H_+$ as required.

3. Describe the geometric effect of applying the functions:

   (a) $f(z) = \dfrac{1}{z}$ to a small disc centred at $1 - i$, and

   (b) $g(z) = \exp(2iz)$ to a small disc centred at $\frac{\pi}{4} + i$.

   Solution: For a function f that is holomorphic at a point z_0, we know that a small disc centred at z_0 is approximately mapped to a small disc centred at $f(z_0)$, and is scaled by a factor of $|f'(z_0)|$ and rotated by an angle of $\arg(f'(z_0))$ about the point $f(z_0)$.

   (a) In this example, $f(1 - i) = \dfrac{1}{1 - i} = \dfrac{1}{2} + \dfrac{i}{2}$, so a small disc centred at $1 - i$ is approximately mapped to a small disc centred at $\dfrac{1}{2} + \dfrac{i}{2}$. Since $f'(z) = -\dfrac{1}{z^2}$ for all $z \in \mathbb{C} \setminus \{0\}$, we have $f'(i) = -\dfrac{1}{(1 - i)^2} = \frac{i}{2}$. Thus the disc is scaled by a factor of $\frac{1}{2}$ and rotated by an angle of $\frac{\pi}{2}$ about $\dfrac{1}{2} - \dfrac{i}{2}$.

(b) Mapped to a disc centred at $g(\frac{\pi}{4}+i) = e^{-2}i$, scaled by a factor of $2e^{-2}$ and rotated by an angle of π about this point (clockwise or anticlockwise; it doesn't matter this time).

4. The *set* of points $L = [0, 1-2i]$ is a line segment. It is also a *path* because we have a parametrisation given by $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = (1-2i)t$. Use this parametrisation to evaluate the integral

$$\int_L (\mathsf{Im}(z) + 3i)\ dz.$$

Solution:

We have

$$\begin{aligned}\int_\Gamma (\mathsf{Im}(z) + 3i)\ dz &= \int_0^1 (\mathsf{Im}(\gamma(t)) + 3i)\,\gamma'(t)\ dt \\ &= \int_0^1 (-2t+3i)\,(1-2i)\ dt \\ &= (1-2i)\int_0^1 (-2t+3i)\ dt \\ &= (1-2i)\left[-t^2+3it\right]_0^1 \\ &= (1-2i)(-1+3i) = 5+5i.\end{aligned}$$

5. Find the value of

$$\int_{\Gamma_1} f(z)\ dz \text{ and } \int_{\Gamma_2} f(z)\ dz,$$

where $f(z) = 3\overline{z}$, Γ_1 is the straight line path from 0 to $-i$ and Γ_2 is the straight line path from $1-i$ to $1+i$.

Solution: For the path Γ_1 we use the parametrisation $\gamma_1 : [0,1] \to \mathbb{C}$, $\gamma_1(t) = -it$. Then $\gamma_1'(t) = -i$ and $f(\gamma_1(t)) = 3it$. Hence

$$\int_{\Gamma_1} f = \int_0^1 (3it)(-i)dt = \int_0^1 3t\ dt = \frac{3}{2}.$$

Parametrise Γ_2 with $\gamma_2 : [0,1] \to \mathbb{C}$, where

$$\gamma_2(t) = (1-i) + t\,[1+i-(1-i)] = 1 + i(2t-1).$$

We get

$$\int_{\Gamma_2} f = 6i.$$

6. Fix a point $z_0 \in \mathbb{C}$ and define a complex function f via

$$f(z) = (z-z_0)^n$$

where $n \in \mathbb{Z}$. Find the value of

$$\int_\Gamma f(z)\ dz,$$

where Γ is the circle with centre z_0 and radius $r > 0$, traversed in the anticlockwise direction (use the parametrisation $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + r(\cos(t) + i\sin(t))$). Do this separately for the cases $n = -1$ and $n \neq -1$.

(Hint: for the case $n \neq -1$, you need to show that

$$\frac{d}{dt}\left[(\cos(t) + i\sin(t))^{n+1}\right] = i(n+1)(\cos(t) + i\sin(t))^{n+1}$$

and then use the (real) Fundamental Theorem of Calculus).

Solution: Following the hint, we first note that

$$\begin{aligned}\frac{d}{dt}\left[(\cos(t) + i\sin(t))^{n+1}\right] &= (n+1)(\cos(t) + i\sin(t))^{n}(-\sin(t) + i\cos(t))\\ &= (n+1)(\cos(t) + i\sin(t))^{n}\, i(\cos(t) + i\sin(t))\\ &= i(n+1)(\cos(t) + i\sin(t))^{n+1}.\end{aligned}$$

We have

$$f(\gamma(t)) = (\gamma(t) - z_0)^n = r^n(\cos(t) + i\sin(t))^n, \quad \gamma'(t) = ir(\cos(t) + i\sin(t)).$$

Hence for $n \neq -1$,

$$\begin{aligned}\int_\Gamma (z - z_0)^n &= \int_0^{2\pi} ir^{n+1}(\cos(t) + i\sin(t))^{n+1}\; dt\\ &= \frac{r^{n+1}}{n+1}\int_0^{2\pi} \frac{d}{dt}\left[(\cos(t) + i\sin(t))^{n+1}\right]\; dt\\ &= \frac{r^{n+1}}{n+1}\left[(\cos(t) + i\sin(t))^{n+1}\right]_0^{2\pi} && \text{by FTC}\\ &= \frac{r^{n+1}}{n+1}\left[1 + 0i - (1 + 0i)\right] = 0 && .\end{aligned}$$

If $n = -1$ then

$$\int_\Gamma (z - z_0)^{-1} = \int_0^{2\pi} \frac{1}{r(\cos(t) + i\sin(t))} \cdot ir(\cos(t) + i\sin(t))\; dt = \int_0^{2\pi} i\; dt = i2\pi.$$

7. Let $f, g : U \to \mathbb{C}$ be continuous, and let Γ be a smooth path contained in U parametrised by $\gamma : [a, b] \to \mathbb{C}$. Prove that

   (a) for every constant $\alpha \in \mathbb{C}$ we have $\int_\Gamma (f + \alpha g) = \int_\Gamma f + \alpha \int_\Gamma g$, and

   (b) if $\tilde{\Gamma}$ denotes the reverse of Γ, we have $\int_{\tilde{\Gamma}} f = -\int_\Gamma f$. As a hint, parametrise $\tilde{\Gamma}$ using $\tilde{\gamma} : [a, b] \to \mathbb{C}$, $\tilde{\gamma}(t) = (a + b - t)$, and use the substitution $s = a + b - t$.

   Solution:

   (a)

$$\begin{aligned}\int_\Gamma (f + \alpha g) &= \int_a^b (f + \alpha g)(\gamma(t))\gamma'(t)\; dt\\ &= \int_a^b (f(\gamma(t)) + \alpha g(\gamma(t)))\,\gamma'(t)\; dt\\ &= \int_a^b \big(f(\gamma(t))\gamma'(t) + \alpha g(\gamma(t))\gamma'(t)\big)\; dt\end{aligned}$$

Then using linearity of the real integral this becomes

$$\int_a^b f(\gamma(t))\gamma'(t)\ dt + \alpha \int_a^b g(\gamma(t))\gamma'(t)\ dt = \int_\Gamma f + \alpha \int_\Gamma g.$$

(b) Following the hint,

$$\begin{aligned}\int_{\tilde{\Gamma}} f &= \int_a^b f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\ dt \\ &= \int_a^b f(\gamma(a+b-t))\left(-\gamma'(a+b-t)\right)\ dt\end{aligned}$$

and using the substitution $s = a+b-t$, we have $ds = -dt$ and the limits are reversed, so the above becomes

$$\begin{aligned}\int_b^a f(\gamma(s))\left(-\gamma'(s)\right)(-ds) &= \int_b^a f(\gamma(s))\gamma'(s)\ ds \\ &= -\int_a^b f(\gamma(s))\gamma'(s)\ ds \\ &= -\int_\Gamma f.\end{aligned}$$