MA2003 Complex Analysis
Solutions to Exercise Sheet 1

1. Use the triangle inequality $|z_1 - z_2| \le |z_1| + |z_2|$ to prove the reverse triangle inequality:

$$|z_1 - z_2| \ge ||z_1| - |z_2||$$

Solution: The triangle inequality gives

$$\begin{aligned} |z_1| &= |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2| \\ |z_2| &= |z_2 - z_1 + z_1| \le |z_2 - z_1| + |z_1| \ \ (= |z_1 - z_2| + |z_1|). \end{aligned}$$

Rearranging gives the two inequalities

$$\begin{aligned} |z_1 - z_2| &\ge |z_1| - |z_2| \\ |z_1 - z_2| &\ge |z_2| - |z_1|, \end{aligned}$$

or in other words

$$|z_1 - z_2| \ge \max\left(|z_1| - |z_2|, |z_2| - |z_1|\right) = ||z_1| - |z_2||.$$

2. Use the triangle and reverse triangle inequalities to show that for all z on the circle $|z| = 2$, we have

$$|z + 2| \le 4 \text{ and } |z - 3 + 4i| \ge 3.$$

Describe these inequalities geometrically.

Solution: If $|z| = 2$ then the triangle inequality ensures that

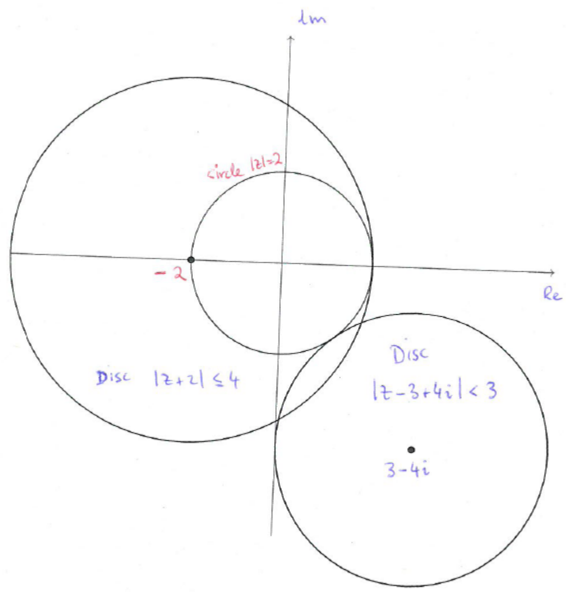
$$|z + 2| \le |z| + 2 = 4,$$

and the backwards triangle inequality gives

$$\begin{aligned} |z - 3 + 4i| &= |z - (3 - 4i)| \\ &\ge ||z| - |3 - 4i|| \\ &= |2 - 5| = 3. \end{aligned}$$

Geometrically, these inequalities show respectively that the circle $|z| = 2$ is

- Contained in the disc with centre -2 and radius 4 (i.e., the disc $|z + 2| \le 4$), and
- Outside of the disc with centre $3 - 4i$ and radius 3.



3. Use the triangle and reverse triangle inequalities to show that for all z on the circle $|z+3i| = 3$ we have

$$|z-4| \leq 8, \quad |z+5i| \geq 1 \text{ and } \left|\frac{z-4}{z+5i}\right| \leq 8.$$

Solution:
As with the previous question, the triangle inequality yields

$$\begin{aligned} |z-4| &= |z+3i-(3i+4)| \\ &\leq |z+3i| + |3i+4| \\ &= 3+5 = 8. \end{aligned}$$

and the backwards triangle inequality yields

$$\begin{aligned} |z+5i| &= |z+3i+2i| \\ &\geq ||z+3i| - |2i|| \\ &= 3-2 = 1. \end{aligned}$$

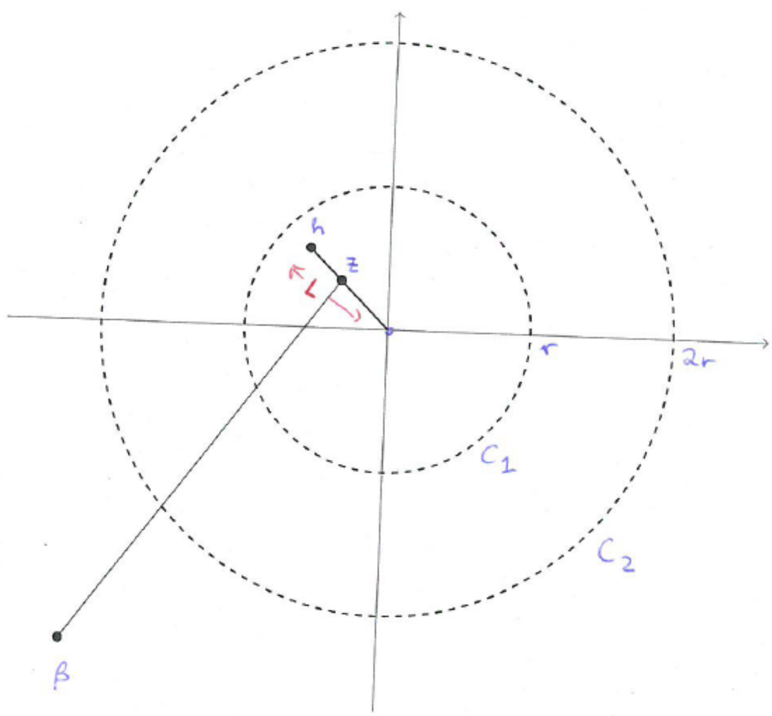
The third inequality is then obvious.

4. Let L be the line segment $[0,h]$ where $h \in \mathbb{C}$ and $|h| < r$. Show that if $\beta \in \mathbb{C}$ with $|\beta| > 2r$ and $z \in L$ then

$$\left|\frac{h-z}{\beta-z}\right| < \frac{|h|}{r}.$$

Do this using the reverse triangle inequality. It can also be seen as follows. Draw L and two circles, both with centre 0, C_1 with radius r and C_2 with radius $2r$. Why does L lie inside C_1? Where is β on your diagram? Why is $|\beta - z| > r$? If you can answer these three questions then the inequality should follow easily.

Solution:



Since z lies on L, it is clear that $|h-z| \leq |h|$ (more formally, we can write $z = th$ for some t with $0 \leq t \leq 1$, so that $|h-z| = |(1-t)h| = (1-t)\,|h| \leq |h|$).
The backwards triangle inequality, together with the fact that $|z| < r < 2r < |\beta|$ gives

$$\begin{aligned} |\beta - z| &\geq ||\beta| - |z|| \\ &= |\beta| - |z| \\ &> 2r - r = r. \end{aligned}$$

Combining these two inequalities we see that

$$\begin{aligned} \left|\frac{h-z}{\beta-z}\right| &= \frac{|h-z|}{|\beta-z|} \\ &\leq \frac{|h|}{|\beta-z|} \\ &< \frac{|h|}{r}. \end{aligned}$$

5. The function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\mathbf{f}(x,y) = (0, 2y)$. Show that the corresponding complex function $f : \mathbb{C} \to \mathbb{C}$ is $f(z) = z - \overline{z}$, and that

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

does not exist at any point $z_0 \in \mathbb{C}$.

Solution: The corresponding function is

$$f(z) = f(x+iy) = i(2y) = i(2\mathsf{Im}(z)) = 2i \cdot \frac{z - \overline{z}}{2i} = z - \overline{z}.$$

If $z_0 \in \mathbb{C}$ and $h \in \mathbb{C}\setminus\{0\}$ then

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{(z_0+h) - \overline{(z_0+h)} - (z - \overline{z})}{h} = \frac{h - \overline{h}}{h} = 1 - \frac{\overline{h}}{h}.$$

Looking at restricted limits along the real and imaginary axes:

$$1 - \frac{\overline{h}}{h} \to \begin{cases} 1 - 1 = 0 & \text{as } h \to 0, h \in \mathbb{R} \\ 1 - (-1) = 2 & \text{as } h \to 0, h \in i\mathbb{R}. \end{cases}$$

Since the restricted limits are not equal the (unrestricted) limit does not exist for any $z_0 \in \mathbb{C}$.

6. Same as question 5 but with

$$\mathbf{f}(x,y) = (x^2 - y^2 - x, 2xy + y + 1) \quad \text{and} \quad f(z) = z^2 - \overline{z} + i.$$

Solution: This time, it is easier to start with the complex function f and substitute $z = x + iy$:

$$\begin{aligned} f(z) &= z^2 - \overline{z} + i \\ &= (x+iy)^2 - (x - iy) + i \\ &= x^2 - y^2 + i2xy - x + iy + i \\ &= x^2 - y^2 - x + i\,(2xy + y + 1)\,, \end{aligned}$$

which shows that f corresponds to the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\mathbf{f}(x,y) = \left(x^2 - y^2 - x, 2xy + y + 1\right).$$

For $z_0 \in \mathbb{C}$ and $h \in \mathbb{C}\setminus\{0\}$ the difference quotient is

$$\begin{aligned} \frac{f(z_0+h) - f(z_0)}{h} &= \frac{(z_0+h)^2 - \overline{(z_0+h)} + i - (z_0^2 - \overline{z_0} + i)}{h} \\ &= h + 2z_0 - \frac{\overline{h}}{h}. \end{aligned}$$

Looking at some restricted limits:

$$\lim_{\substack{h\to 0,\\ h\in\mathbb{R}\setminus\{0\}}} \frac{f(z_0+h) - f(z_0)}{h} = 2z_0 - 1$$

$$\lim_{\substack{h\to 0,\\ h\in i\mathbb{R}\setminus\{0\}}} \frac{f(z_0+h) - f(z_0)}{h} = 2z_0 + 1.$$

These are not equal for any $z_0 \in \mathbb{C}$, so the unrestricted limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

does not exist at any $z_0 \in \mathbb{C}$.

7. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = |z|^2$. Show that f is differentiable at $z = 0$ and nowhere else.

Solution: Since $|z|^2 = z\overline{z}$,

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} &= \frac{(z_0 + h)\overline{(z_0 + h)} - z_0\overline{z_0}}{h} \\ &= \frac{z_0\overline{h} + h\overline{z_0} + h\overline{h}}{h} \\ &= z_0\frac{\overline{h}}{h} + \overline{z_0} + \overline{h} \\ &\longrightarrow \begin{cases} 2z_0 & \text{as } h \to 0,\ h \in \mathbb{R}\setminus\{0\} \\ 0 & \text{as } h \to 0, h \in i\mathbb{R}\setminus\{0\}\,. \end{cases} \end{aligned}$$

For $z_0 \neq 0$, the restricted limits are not equal, hence the unrestricted limit does not exist and so f is not differentiable at any point $z_0 \in \mathbb{C}\setminus\{0\}$.
At $z_0 = 0$, we have

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \overline{h} = 0,$$

so that f is differentiable at 0 with $f'(0) = 0$.

8. Use the rules of differentiation to find the derivatives of the following functions:
   (a) $f(z) = \left(z^2 + 4\right)^3$
   (b) $g(z) = \dfrac{z + i}{z - i}$.

   Find the values of $f'(i)$ and $g'(1)$.

Solution:

(a) Since f is the composition of holomorphic functions, we can use the chain rule to find $f'(z)$:

$$f'(z) = 3(z^2 + 4)^2(2z) = 6z(z^2 + 4)^2 \quad \text{for all } z \in \mathbb{C},$$

and

$$f'(i) = 6i(i^2 + 4)^2 = 6i(3)^2 = 54i.$$

(b) Since the functions $z \mapsto z \pm i$ are holomorphic on $\mathbb{C}$, g is holomorphic on $\mathbb{C}\setminus\{i\}$, and the quotient rule gives

$$g'(z) = \frac{(z - i)(1) - (z + i)(1)}{(z - i)^2} = \frac{-2i}{(z - i)^2}$$

for all $z \in \mathbb{C} \setminus \{i\}$. Hence

$$g'(1) = \frac{-2i}{(1-i)^2} = \frac{-2i}{-2i} = 1.$$