MA2003 Complex Analysis
Exercise Sheet 2

1. For each function f below, write f in the form

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

and determine whether or not the Cauchy-Riemann equations are satisfied:

$$(a)\ f(z) = \exp(i\,\overline{z}) \quad (b)\ f(z) = z + \frac{1}{z} \quad (c)\ f(z) = z^3.$$

In the cases where f is differentiable, find the derivative of f both using the rules of differentiation and using the Cauchy-Riemann equations.

2. Show that the Cauchy-Riemann equations are satisfied by the function f defined on the open upper half plane $H_+ = \{z \in \mathbb{C} : \mathsf{Im}(z) > 0\}$ by

$$f(x+iy) = u(x,y) + iv(x,y) = \log\left(\sqrt{x^2+y^2}\right) + i\left(\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right)\right).$$

Assuming that f is indeed holomorphic on H_+, show that

$$f'(x+iy) = \frac{1}{x+iy} \quad \text{i.e., that} \quad f'(z) = \frac{1}{z}.$$

3. Describe the geometric effect of applying the functions:
   (a) $f(z) = \dfrac{1}{z}$ to a small disc centred at $1-i$, and
   (b) $g(z) = \exp(2iz)$ to a small disc centred at $\frac{\pi}{4} + i$.

4. The *set* of points $L = [0, 1-2i]$ is a line segment. It is also a *path* because we have a parametrisation given by $\gamma : [0,1] \to \mathbb{C},\ \gamma(t) = (1-2i)t$. Use this parametrisation to evaluate the integral

$$\int_L (\mathsf{Im}(z) + 3i)\ dz.$$

5. Find the value of

$$\int_{\Gamma_1} f(z)\ dz \text{ and } \int_{\Gamma_2} f(z)\ dz,$$

where $f(z) = 3\overline{z}$, Γ_1 is the straight line path from 0 to $-i$ and Γ_2 is the straight line path from $1-i$ to $1+i$.

6. Fix a point $z_0 \in \mathbb{C}$ and define a complex function f via

$$f(z) = (z - z_0)^n$$

where $n \in \mathbb{Z}$. Find the value of

$$\int_\Gamma f(z)\ dz,$$

where Γ is the circle with centre z_0 and radius $r > 0$, traversed in the anticlockwise direction (use the parametrisation $\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) = z_0 + r(\cos(t) + i\sin(t))$). Do this separately for the cases $n = -1$ and $n \neq -1$.

(Hint: for the case $n \neq -1$, you need to show that

$$\frac{d}{dt}\left[(\cos(t) + i\sin(t))^{n+1}\right] = i(n+1)(\cos(t) + i\sin(t))^{n+1}$$

and then use the (real) Fundamental Theorem of Calculus).

7. Let $f, g : U \to \mathbb{C}$ be continuous, and let Γ be a smooth path contained in U parametrised by $\gamma : [a, b] \to \mathbb{C}$. Prove that

   (a) for every constant $\alpha \in \mathbb{C}$ we have $\displaystyle\int_\Gamma (f + \alpha g) = \int_\Gamma f + \alpha \int_\Gamma g$, and

   (b) if $\tilde{\Gamma}$ denotes the reverse of Γ, we have $\displaystyle\int_{\tilde{\Gamma}} f = -\int_\Gamma f$. As a hint, parametrise $\tilde{\Gamma}$ using $\tilde{\gamma} : [a, b] \to \mathbb{C}$, $\tilde{\gamma}(t) = (a + b - t)$, and use the substitution $s = a + b - t$.