Level 4 - MA1001- Elementary Differential Equations

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Chapter 1 Introductory Examples

Example 1.1
Radioactive isotope decays according to the differential

$$\frac{dm}{dt} = -km$$

where $k > 0$ is constant. m is the mass present at time t. Find the half life such that $m(\tau) = \frac{1}{2}m(0)$

Example 1.2
A student goes home for Christmas leaving a mouldy burger under the sofa. The mould grows according to

$$\frac{dm}{dt} = km$$

where $k > 0$ is constant. Find the time $\tau = 0$ at which $m(\tau) > M$, assuming $m(0)$ is known. Here M is the mass of the sofa.

Example 1.3
The previous model is changed to

$$\frac{dm}{dt} = km(M - m)$$

where $k > 0$, $M > 0$ are constants. Show how the solution behaves.

Example 1.4
Show that the differential equation

$$\frac{dy}{dt} = 2\sqrt{y}$$

has more than one solution with :

$$y_1(t) = t^2 \text{ for t} \geq 0$$

$$y_2(t) = \ 0 \text{ for all t}$$

A third solution :

$$y_3(t) = \begin{cases} y_2(t) \text{ for t } \leq 0 \\ y_1(t) \text{ for t } > 0 \end{cases}$$

Example 1.5
Show that the solutions of the equation :

$$\frac{dy}{dt} = 1 + y^2$$

blows up in finite time.

Chapter 2 First Order Differentials

A first order (ordinary differential equations) is an equation of the form :

$$\frac{dy}{dt} = f(t, y).$$

Here any solution y is a real or complex valued function of a single real variable and f is a function of 2 variables.
In Newtonian notation the same equation would be written as :

$$y'(t) = f(t, y(t)).$$

2.1 Separable Equations

A separable equation is an equation of the form :

$$\frac{dy}{dt} = g(t)h(y).$$

Suppose that h never takes the value 0. Then we can write the equation as :

$$\frac{1}{h(y)}\frac{dy}{dt} = g(t).$$

We can also (at least on principle) find a function F whose derivative is $\frac{1}{h}$:

$$F' = \frac{1}{h},$$

hence : $F'(y(t))\frac{dy}{dt} = g(t) \implies \frac{d}{dt}[F(y(t))] = g(t)$.

Example 2.6
Solve

$$\frac{dy}{dt} = (1 + y^2)(1 + \alpha y)$$

where $\alpha \geq 0$ is constant.

Solution: Write the equation as

$$\frac{1}{1 + y^2}\frac{dy}{dt} = 1 + \alpha t.$$

Here we need $F'(y) = \frac{1}{1+y^2}$ so we choose :

$$F(y) = tan^{-1}(y)$$

and get :

$$\frac{d}{dt}(tan^{-1}(y(t))) = 1 + \alpha t,$$

and so by integration

$$tan^{-1}(y(t)) = t + \alpha\frac{t^2}{2} + \beta$$

where β is constant. Hence : $y(t) = tan(t + \alpha\frac{t^2}{2} + \beta)$. Remark : The solution of this equation always blows up in finite time even when $\alpha = \beta = 0$. In this case the solution blows up at $t = \frac{\pi}{2}$.

Example 2.7
Find the solution of

$$\frac{dy}{dt} = 2\sqrt{y}.$$

Solution: Divide by $2\sqrt{y}$ to get

$$\frac{1}{2\sqrt{y}}\frac{dy}{dt} = 1.$$

For this case :

$$F'(y) = \frac{1}{2\sqrt{y}}$$

and so we choose $F(y) + \sqrt{y}$ and get

$$\frac{d}{dt}(\sqrt{y(t)} = 1.$$

So, on integration

$$\sqrt{y(t)} = t + c$$

where c is a constant. This gives $y(t) = (t + c)^2$. Suppose we want $y(0) = 0$ we get :

$$0 = y(0) = (0 + c)^2 = c^2.$$

Giving $c = 0$. Thus $y(t) = t^2$.
Warning : This is not the only solution of $\frac{dy}{dt} = 2\sqrt{y}$ which satisfies $y(0) = 0$, as we said previously. We could also have $y(t) = 0$ for all t.

Example 2.8
Solve the equation :

$$\frac{dy}{dt} = m(1 - m).$$

Solution: Suppose $m \neq 0$, $m \neq 1$ Then

$$\frac{1}{m(1-m)}\frac{dm}{dt} = 1.$$

We need to find a function F such that :

$$F'(m) = \frac{1}{m(1-m)} = \frac{1}{m} + \frac{1}{1-m}.$$

A suitable F is:

$$F(m) = \log(m) - \log(1-m) = \log(\frac{m}{1-m})$$

and so our equation is :

$$\frac{d}{dt}[\log(\frac{m}{1-m})] = 1 \implies \log(\frac{m}{1-m}) = t + c$$

where c is a constant, thus

$$\frac{m(t)}{1-m(t)} = \exp(t+c) = \alpha \exp(t)$$

where $\alpha = \exp(c)$ is also a constant. Suppose that $m(0) = \mu$ then :

$$\frac{\mu}{1-\mu} = \alpha.$$

The equation for $m(t)$ can be rearranged as

$$\mu(t) = \frac{\alpha \exp(t)}{1 + \alpha \exp(t)} = \frac{1}{\frac{1}{\alpha}\exp(-t) + 1} = \frac{1}{(\frac{1-\mu}{\mu})\exp(-t) + 1} = \frac{\mu}{(1-\mu)\exp(-t) = \mu}.$$

Remark 2.9

- If we take $m(0) = \mu = 0$ we get $m(t) = 0$ for all t then :
  If $0 < \mu, 1$ then $(1-\mu)\exp(-t) > 0$ and hence
  $$0, m(t) < \frac{\mu}{0+\mu} = 1$$
  Thus
  $$\frac{dm}{dt} = m(1-m) > 0 \text{ also } \lim m(t) = 1.$$
- If we take $m(0) = \mu = 1$ we get $m(t) = 1$ for all t then :
  If $\mu > 1$ then the expression
  $$m(t) = \frac{\mu}{\mu(1-\exp(-t)) + \exp(-t)}$$
  tells us that $m(t) > 0$ whilst the expression
  $$m(t) = \frac{\mu}{(1-m)\exp(-t) + \mu}$$
  tells us that $m(t) > 1$ for all $t \geq 0$. Also $\frac{dm}{dt} = m(1-m) < 0$, so m is strictly decreasing. Also $\lim m(t) = 1$.

2.2 Homogeneous Equations

A homogeneous equation is an equation of the form :

$$\frac{dy}{dt} = f(\frac{y}{t}).$$

We introduce a new function $z = \frac{y}{t}$; more precisely $z(t) = \frac{y(t)}{t}$. Hence $y(t) = tz(t)$ which gives

$$f(z) = \frac{dy}{dt} = z(t) + t\frac{dz}{dt}.$$

Rearranging yields:

$$\frac{dz}{dt} = \frac{f(z) - z}{t} = \frac{1}{z}(f(z) - z)$$

which is now separable.

Example 2.10
Solve the equation :

$$t^2\frac{dy}{dt} = ty + t^2 + y^2$$

and examine the behavior of the solution as t tends to 0.

Solution: Divide by t^2 to get

$$\frac{dy}{dt} = \frac{y}{t} + 1 + (\frac{y}{t})^2 = f(\frac{y}{t})$$

where $f(z) = z + 1 + z^2$. We let $y = tz$ and get

$$\frac{dz}{dt} = \frac{1}{t}(f(z) - z) = \frac{1}{t}(1 + z^2) \implies \frac{1}{1+z^2}\frac{dz}{dt} = \frac{1}{z},$$

so

$$\frac{1}{1+z^2}dz = \frac{1}{t}dt.$$

Integrate both sides to obtain $tan^{-1} = \log(t) + c$ where c is a constant of integration. This yields

$$z = tan(c + \log(t)) \implies y - t * tan(c + \log(t)).$$

Put $c = \log \alpha$ where $\alpha = \exp(c)$ so that

$$y = t * tan(\log \alpha + \log t) = t * tan(\log(\alpha t)).$$

This blows up whenever $\log(\alpha t) = (2k+1)\frac{\pi}{2}$, $k \in \mathbb{Z}$ i.e

$$\alpha t = \exp((2k+1)\frac{\pi}{2}), \text{ or } t = \frac{1}{\alpha}\exp((2k+1)\frac{\pi}{2}).$$

Letting k decrease to $-\infty$ through the negative integers we see that these are infinitely many points at which y blows up, in any neighborhood of zero.

Example 2.11
Solve the homogeneous equation :

$$\frac{dy}{dt} = \frac{y^{\frac{1}{3}}}{t}.$$

Solution: Put $y = tz, f(z) = z^{\frac{1}{3}}$, and obtain

$$\frac{dz}{dt} = \frac{1}{t}(f(z) - z) = \frac{1}{t}(z^{\frac{1}{3}} - z).$$

This yields

$$\frac{1}{z^{\frac{1}{3}} - z}dz = \frac{1}{t}dt.$$

Alternatively, the original equation is already separable, as $y^{\frac{-1}{3}}dy = t^{\frac{-1}{3}}dt$. By integration :

$$y^{\frac{2}{3}} = t^{\frac{2}{3}} + c$$

where c is a constant of integration, thus

$$y = (c + t^{\frac{2}{3}})^{\frac{3}{2}}.$$

2.3 Linear Equations

A first-order differential linear equation is an equation of the form :

$$\frac{dy}{dt} = a(t)y + b(t)$$

where $a(t)$ and $b(t)$ are given functions of the dependant variable t. For linear equations we can always write the solution y explicitly as a function.

Solution Procedure

1) Choose a function $p(t)$ such that

$$\frac{dp}{dt} = a(t)$$

2) Observe that

$$\frac{d}{dt}(ye^{-p(t)}) = \frac{dy}{dt}e^{-p(t)} + ye^{-p(t)} = (\frac{dy}{dt} - a(t)y)e^{-p(t)} = b(t)e{-}p(t)$$

3) Integrate the equation

$$\frac{d}{dt}(ye^{-p(t)}) = b(t)e^{-p(t)}$$

This integrates to

$$ye^{-p(t)} = \int b(t)e{-}p(s)ds + c$$

where c is a constant By doing this we get :

$$y = e^p(t)(c + \int b(s)e^{-p(s)}ds$$

Example 2.12
Solve the differential equation

$$\frac{dy}{dt} = ky + \sin(t)$$

subject to the condition $y(0) = 0$.

Solution: Choose $p = \int k dt = kt$ where $a(t) = k$ and $b(t) = \sin(t)$. Therefore

$$\frac{d}{dt}(e^{-kt}y) = e^{-kt}\sin(t).$$

There are at least two ways of integrating the right hand side of the equation. One is to integrate by parts and the other is to use complex exponentials, therefore :

$$\sin(t) = Im(e^{it}).$$

Hence :

$$\int e^{-kt}\sin(t)dt \implies \int e^{-kt}Im(e^{-it}dt = Im(\int e^{-kt+it}dt) \implies Im(\frac{e^{(i-k)t}}{(i-k)}$$

$$= Im(\frac{(-i-k)(\cos(t)+\sin(t)) * e^{-kt}}{1+k^2} = Im(\frac{(-k\cos(t)+\sin(t)+i(-\cos(t)-k\sin(t)))e^{-kt}}{1+k^2})$$

$$= \frac{-(\cos(t)+k\sin(t))e^{-kt}}{1+k^2}.$$

Therefore :

$$e^{-kt}y = -(\frac{-(\cos(t)+k\sin(t))e^{-kt}}{1+k^2})$$

where c is a constant and when $t = 0$ and $y = 0$ so :

$$c - \frac{1}{1+k^2} = 0 \implies c = \frac{1}{1+k^2}$$

Thus the solution is

$$y = \frac{e^{kt} - (\cos(t) + k\sin(t))}{1+k^2}.$$

Example 2.13
Solve the linear differential equation

$$t^2\frac{dy}{dt} + 2ty = e^t$$

subject to $y(1) = 1$. Is there a solution subject to the condition $y(0) = 1$?

Solution: We can write the equation in the form

$$\frac{dy}{dt} = \frac{-2}{t}y + \frac{e^t}{t^2}.$$

Then we find the integrating factors as before however in this case we can simplify the equation by inspection :

$$t^2\frac{dy}{dt} + 2ty = \frac{d}{dt}(t^2 y) = e^t.$$

Then by integration :

$$t^2 y = e^t + c \implies y = \frac{e^t + c}{t^2}$$

where c is a constant. The initial condition $y(1) = 1$ gives you

$$1 = \frac{e + c}{1} \implies c = 1 - e.$$

Thus

$$y = \frac{e^t + 1 - e}{t^2}.$$

There is no solution which takes the value 1 when $t = 0$! This is because when we write the equation as $\frac{dy}{dt} = f(t, y)$ we have

$$\frac{dy}{dt} = \frac{-2y}{t} + \frac{e^t}{t^2}$$

and this function is badly behaved as $t \to 0$. The initial conditions can not be imposed at a point where the right hand side of the differential equation blows up.

Example 2.14
Solve the equation

$$(t\log(t))\frac{dy}{dt} + y = 3t^3.$$

Solution: First we must find an integrating factor which is

$$\exp(\int \frac{1}{t\log(t)}dt).$$

Then by integration by substitution where $u = \log(t)$ and $\frac{du}{dt} = \frac{1}{t}$ we have that

$$\int \frac{1}{t\log(t)}dt \implies \int \frac{1}{u}du = \log(u) \implies \log(\log(t)).$$

Now the integrating factor is $\exp(\log(\log(t))$ which is $\log(t)$ and therefore :

$$\log(t)\frac{dy}{dt} + \frac{y}{t} = 3t^2 \implies \frac{d}{dt}[\log(t)y] = 3t^2.$$

Then by integration you get

$$\log(t)y = \int 3t^2 dt = t^3 + c$$

where c is a constant. Thus by rearranging

$$y = \frac{t^3 + c}{\log(t)}.$$

2.4 Bernoulli Equations

A Bernoulli equation is an equation in the form

$$\frac{dy}{dt} + a(t)y = b(t)y^n.$$

Where when $n = 0$ we have a linear equation and when $n = 1$ we have linear and separable equation. We choose an integration factor $p(t)$ such that

$$\frac{p'(t)}{p(t)} = a(t) \text{ so that } \frac{dy}{dt} + \frac{p'(t)}{p(t)}y = b(t)y^n.$$

Now we multiply both sides by $p(t)$ to get

$$p(t)\frac{dy}{dt} + \frac{dp}{dt}y = p(t)b(t)y^n.$$

Then by differentiation

$$\frac{d}{dt}(p(t)y(t0 = \frac{b(t)}{p(t)^{n-1}}(p(t)y(t))^n.$$

We introduce a new function z by the formula $z(t) = p(t)y(t)$ which satisfies

$$\frac{dz}{dt} = q(t)z^n \text{ where } q(t) = \frac{b(t)}{p(t)^{n-1}}.$$

Example 2.15
Solve the equation

$$\frac{dy}{dt} + \frac{2}{t}y = \exp(t)y^2.$$

Solution: Multiply both sides by t^2 to obtain

$$t^2\frac{dy}{dt} + 2ty = t^2\exp(t)y^2 \implies \frac{d}{dt}(t^2y) = \frac{1}{t^2}\exp(t)(t^2y)^2.$$

Let $z = t^2y$ such that

$$\frac{dz}{dt} = \frac{1}{t^2}\exp(t)z^2$$

which is now a separable equation. This separates as

$$\frac{1}{z^2}\frac{dz}{dt} = \frac{1}{t^2}\exp(t) \text{ or } \frac{d}{dt}(\frac{-1}{z}) = \frac{1}{t^2}\exp(t)$$

Any further progress depends on integrating the right hand side, probably with an incomplete gamma-function.

Example 2.16
Solve the equation

$$\frac{dy}{dt} = t\exp(t^2 - y).$$

Solution: Thanks to the property $\exp(t^2 - y) = \frac{\exp(t^2)}{\exp(y)}$ this is a separable equation. We rearrange it as

$$\exp(y)\frac{dy}{dt} = t\exp(t^2) \implies \frac{d}{dt}\exp(y)) = \frac{d}{dt}(\frac{1}{2}\exp(t^2.)$$

Thus

$$\exp(y) = \frac{1}{2}\exp(t^2) + c$$

where c is a constant. Finally

$$y = \log(c + \frac{1}{2}\exp(t^2)).$$

For the special case when $c = 0$ we get

$$y = \log(\frac{1}{2}\exp(t^2)) \implies \log(\frac{1}{2} + t^2 = t^2 - \log(2).$$

Chapter 3 Second Order Equations

A second order linear differential equation is an equation of the following form :

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t).$$

The functins $a_1(*)$, $a_0(*)$ and $f(*)$ are supposed to be known and we want to find all of the solutions to y.

Example 3.17
The Legendre equation is

$$\frac{d^2y}{dt^2} - \frac{2t}{1-t^2}\frac{dy}{dt} + \frac{n(n+1)}{1-t^2}Y = 0.$$

The equation has a solution $P_n(t)$, the Legendre polynomial of degree n.

- For $n = 0$ we can take $P_0(t)$ for all t
- For $n = 1$ we let $P_1(t) = at + b$. We get

  $$\frac{-2t}{1-t^2}a + \frac{2}{1-t^2}(at+b) = 0$$

  for all $t \in \mathbb{R}\backslash\{\pm 1\}$ This gives $b = 0$ and we can take $a = 1$ to get $P_1(t) = t$

Example 3.18
Bessel's equation of order n is :

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + (1 - \frac{n^2}{t^2})y = 0.$$

The solution are Bessel functions, $J_n(t)$ and $Y_n(t)$.

3.1 Equations with Constant Coefficients

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt}a_0y = f(t)$$

In order to solve the above equation we adopt a two step solution strategy:

1) Find all the solution of the equation with $f = 0$. The equation when $f = 0$ is called the homogeneous equation

2) Develop a formula called the variation of parameters formula for the case $f \neq 0$, using the solutions of the homogeneous equation

Solution Method for $\mathbf{f = 0}$

We look for a solution of the form $y(t) = A\exp(\mu t)$, where A and μ are constants. We have:

$$\frac{dy}{dt} = \mu A\exp(\mu t) \quad \text{and} \quad \frac{d^2y}{dt^2} = \mu^2 A\exp(\mu t).$$

Substituting into the differential equation gives:

$$(\mu^2 + a_1\mu + a_0) * A\exp(\mu t) = 0.$$

Since $\exp(\mu t) \neq 0$ and we do not want the trivial solution which comes from choosing $A = 0$, we have $\mu^2 + a_1\mu + a_0 = 0$ this is called the auxiliary quadratic. The possibilities are:

- Two distinct roots
- One double root

If a_1 and a_0 are real there is an alternative dichotomy:

- Two real roots (which may or may not coincide)
- A complex conjugate pair of distinct roots

Case 1 : Distinct Roots $\boldsymbol{\mu_1 \neq \mu_2}$

$$\mu^2 + a_1\mu + a_0 = (\mu - \mu_1)(\mu - \mu_2).$$

We have the solutions

$$y_j(t) = A_j\exp(\mu_j t) \quad j = 1, 2.$$

For completely arbitary constants A_1, A_2. Suppose $y(t) = y_1(t) + y_2(t)$ then

$$\frac{dy}{dt} = \frac{dy_1}{dt} + \frac{dy_2}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2y_1}{dt^2} + \frac{d^2y_2}{dt^2}.$$

So,

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = [\frac{d^2y_1}{dt^2} + a_1\frac{dy_1}{dt} + a_0y_1] + [\frac{d^2y_2}{dt^2} + a_1\frac{dy_1}{dt} + a_0y_2].$$

Theorem 3.19
Suppose that the auxiliary quadratic has a distinct root $\mu_1 \neq \mu_2$. Then every solution of the homogeneous equation has the form

$$y(t) = A_1\exp(\mu_1 t) + A_2\exp(\mu_2 t)$$

for some A_1, $A_2 \in \mathbb{C}$.

Example 3.20

$$\frac{d^2y}{dt^2} - (\mu_1 + \mu_2)\frac{dy}{dt} + \mu_1\mu_2 y = 0$$

such that $y(0) = y(0)$ and $y'(0) = v_0$.

Solution:
We know $y(t) = A_1 \exp(\mu_1 t) + A_2 \exp(\mu_2 t)$ and we choose A_1, A_2 so that:

$$\begin{cases} y_0 = y(0) = A_1 + A_2 \\ v_0 = y'(0) = A_1\mu_1 + A_2\mu_2 \end{cases}$$

Hence

$$\mu_2 y_0 - v_0 = A_1(\mu_2 - \mu_1) \implies$$
$$\begin{cases} A_1 = \frac{\mu_2 y_0 - v_0}{\mu_2 - \mu_1} \\ A_2 = y_0 - A_1 = \frac{v_0 - \mu_1 y_0}{\mu_2 - \mu_1} \end{cases}$$

Thus

$$y_1(t) = \frac{\mu_2 y_0 - v_0}{\mu_2 - \mu_1} \exp(\mu_1 t) + \frac{v_0 - \mu_1 y_0}{\mu_2 - \mu_1} \exp(\mu_2 t) = y_0 \left(\frac{\mu_2 \exp(\mu_1 t) - \mu_1 \exp(\mu_2 t)}{\mu_2 - \mu 1} \right) + v_0 \left(\frac{\exp(\mu_2 t) - \exp(}{\mu_2 - \mu_1} \right.$$

Example 3.21
In the previous example let μ_1 be fixed and find $\lim_{\mu_2 \to \mu_1} y(t)$.

Solution: First we compute

$$\lim_{\mu_2 \to \mu_1} \left[\frac{\exp(\mu_2 t) - \exp(\mu_1 t)}{\mu_2 - \mu_1} \right] = t \exp(\mu t) \lim_{\mu_2 \to \mu_1} \left[\frac{\exp((\mu_2 - \mu_1)t - 1}{(\mu_2 - \mu_1)t} \right].$$

Now set $h = \mu_2 - \mu_1$. Thus :

$$t \exp(\mu_1 t) \lim_{h \to 0} \left[\frac{\exp(h) - 1}{h} \right]$$

$$= t \exp(\mu_1) \lim_{h \to 0} \left[\frac{\exp(h) - \exp(0)}{h} \right] = t \exp(\mu_1 t) \exp'(0).$$

Where $\exp'(0) = 0$. Hence

$$\lim_{\mu_2 \to \mu_1} \left(\frac{\exp(\mu_2 t) - \exp(\mu - 1t)}{\mu_2 - \mu_1} \right) = t \exp(\mu_1 t).$$

Next we compute:

$$\lim_{\mu_2 \to \mu_1} \left(\frac{\mu_2 \exp(\mu_1 t) - \mu_1 \exp(\mu - 2t)}{\mu_2 - \mu_1} \right).$$

We observe that:

$$\frac{\mu_2 \exp(\mu_1 t) - \mu_1 \exp(\mu - 2t)}{\mu_2 - \mu_1} = \left[\frac{\mu_1(\exp(\mu_2 t) - \exp(\mu_1 t)) + (\mu_2 - \mu_1) \exp(\mu_1 t)}{\mu_2 - \mu_1} \right]$$

$$= \exp(\mu_1 t) - \mu_1 \left[\frac{\exp(\mu_2 t - \exp(\mu_1 t)}{\mu_2 - \mu_1} \right].$$

Thus

$$\lim_{\mu_2 \to \mu_1} \left(\frac{\mu_2 \exp(\mu_1 t) - \mu_1 \exp(\mu - 2t)}{\mu_2 - \mu_1} \right) = \exp(\mu_1 t) - \mu_1 t \exp(\mu_1 t).$$

Hence
$$\lim_{\mu_2 \to \mu_1} y(t) = y_0(1-\mu_1)\exp(\mu_1 t) + v_t \exp(\mu_1 t).$$
Let
$$z(t) = y_0(1-\mu_1 t)\exp(\mu_1 t) + v_0 t \exp(\mu_1 t).$$
Then
$$\begin{cases} z(0) = y_0 \\ z'(t) = y_0(-\mu_1 + \mu_1(1-\mu_1 t))\exp(\mu_1 t) + v_0(1+\mu_1 t)\exp(\mu_1 t) \end{cases}$$
giving $z'(0) = v_0$. Let $z(t) = (At+B)\exp(\mu_1 t)$, where A and B are constants. We want to check that
$$\frac{d^2 z}{dt^2} - 2\mu_1 \frac{dz}{dt} + {\mu^2}_1 z = 0.$$
We have
$$z(t) = (At+B)\exp(\mu_1 t) \implies \frac{dz}{dt} = (A+\mu_1(At+B))\exp(\mu_1 t) = (\mu_1 At + (A+\mu_1 B))\exp(\mu_1 t)$$
$$\implies \frac{d^2 z}{dt2} = (\mu_1 At + (\mu_1 A + \mu_1(A+\mu_1 B))\exp(\mu_1).$$
Hence
$$\frac{d^2 z}{dt^2} - 2\mu_1 \frac{dz}{dt} + {\mu^2}_1 z = (({\mu^2}_1 A - 2\mu_1 * \mu_1 A + {\mu^2}_1 A)t + 2\mu_1 A + {\mu^2}_1 B - 2\mu_1(A+\mu_1 B) + \mu^2 - 1B)\exp(\mu_1 t) = 0 \text{ as r}$$

Theorem 3.22
Every solution of the equation
$$\frac{d^2 y}{dt^2} - 2y\frac{dy}{dt} + \mu^2 y = 0$$
has the form
$$y(t) = (At+B)\exp(\mu t)$$
for some appropriate constants A, B.

Example 3.23
Find the general solution of
$$\frac{d^2 y}{dt^2} + \omega^2 y = 0$$
where $\omega > 0$ is constant.

Solution: We look for a solution of the form
$$y(t) = A\exp(\mu t)$$
and obtain the auxiliary quadratic $\mu^2 + \omega^2 = 0$, which has roots $\mu_1 = i\omega$, $\mu_2 = -i\omega$. Thus the general solution has the form
$$y(t) = A\exp(i\omega t) + B\exp(-i\omega t)$$
$$= A(\cos(\omega t) + i\sin(\omega t)) + B(\cos(\omega t) - i\sin(\omega t))$$
$$= (A+B)\cos(\omega t) + i(A-B)\sin(\omega t).$$

Suppose $y(0) = y_0 \in \mathbb{R}$ and $y'(0) = v_0 \in \mathbb{R}$. Then

$$\begin{cases} y(0) = A + B = y_0 \\ y'(0) = i\omega(A - B) = V_0 \end{cases}$$

Thus $y(t) = y_0 \cos(\omega t) + v_) \frac{\sin(\omega t)}{\omega}$.

Remark 3.24
Observe that $\lim_{\omega \to 0} \cos(\omega t) = \cos(0) = 1$ and $\lim_{\omega \to 0} \frac{\sin(\omega t)}{\omega} = t$. When $\omega = 0$, we get

$$y(t) = y_0 + v_0 t$$

which does indeed solve $\frac{d^2y}{dt^2} = 0$.

Definition 3.25
Let y_1, y_2 be solutions of

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = 0.$$

We say that y_1 and y_2 are linearly independent if and only if neither solution can be written as a constant multiple of the other.

Example 3.26
In the previous example,

$$y_1(t) = \cos(\omega t) \text{ and } y_2 = \frac{\sin(\omega t)}{\omega}$$

are linearly independent: so are

$$y_1(t) = \exp(i\omega t) \text{ and } y_2(t) = \exp(-i\omega t)$$

$$y_1(t) = \cos(\omega t) \text{ and } y_2(t) = \exp(i\omega t).$$

However if $y_1 t = \cos(\omega t)$ and $y_2(t) = 17\cos(\omega t)$, are not linearly independent.

Definition 3.27
Let y_1 and y_2 be two differentiable functions. The Wronskian (determinant) of y_1 and y_2 is the function

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2$$

Objective

We want to find a formula for the general solution of

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = f$$

where the function f on the right hand side is non-trivial. We assume that we know two lineralry independent solutions y_1 and y_2 of the homogeneous equation

$$\frac{d^2y}{dt^2}j + a_1\frac{dy}{dt}j + a_0 yj = 0 \text{ for } j = 1, 2.$$

We can look for a solution in the form

$$y(t) = A_1(t) + A_2(t)y_2(t).$$

Differentiating,

$$y'(t) = A_1(t)y_1'(x) + A_2(t)y_2'(t) + A_1'(t)y_1(t) + A_2'(t)y_2(t).$$

We impose the condition $A_1'y_1 + A_2'y_2 = 0$. Then we obtain by the second derivative

$$y''(t) = A_1(t)y_1''(t) + A_2y_2''(t) + A_1'(t)y_1'(t) + A_2'(t)y_2'(t).$$

Combining the expressions for y, y' and y'', we obtain

$$y'' + a_1y + a_0y = A_1(y_1'' + a_1y_1' + a_0y1) + A_2(y_2'' + a_1y_2' + a_0y2) + A_1'y_1' + A_2'y_2' = A_1'y_1' + A_2'y_2'$$

because y_1 and y_2 solve the homogeneous equation. Using the original formula we see that our second equation for A_1 and A_2 is

$$A_1'y_1' + A_2'y_2' = f.$$

We write the equation for A_1 and A_2 as

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which yields

$$\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix} = \frac{1}{W} \begin{bmatrix} -y_2f \\ y_1f \end{bmatrix}$$

This means

$$\begin{cases} A_1' = \frac{-y_2f}{W} \\ A_2' = \frac{y_1f}{W} \end{cases}$$

giving

$$\begin{cases} A_1(t) = \int_{t_0}^{t} \frac{-y_2(x)f(x)}{W(x)}dx + \alpha_1 \\ A_2(t) = \int_{t_0}^{t} \frac{y-1(x)f(x)}{W(x)}dx + \alpha_2 \\ \end{cases}$$

Where α_1 and α_2 are constants of integration and t_0 can be chosen. Recalling that

$$y(t) = A_1(t)y_1(t) + A_2(t)y_2(t)$$

we obtain

$$y(t) = \int_{t_0}^{t} \left(\frac{y_1(x)y_2(t) - y_1(t)y_2(x)}{W(x)} \right) f(x)dx + \alpha_1y_1(t) + \alpha_2y_2(t).$$

This formula is called the variation of parameters or variation of constants formula.

- y is the general solution of
$$y'' + a_1y' + a_0y = f$$
- a_1, a_0 and f are given functions.
- y_1, y_2 are the solutions of the homogeneous equation
$$y_j'' + a_1y_j' + a_0y_j = 0, \;\; j = 1, 2$$

- W is the Wronskian

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2$$

Example 3.28
Find the general solution of the equation

$$y'' - y = 1.$$

Solution: The corresponding homogeneous equation is $y'' - y = 0$, with auxiliary quadratic $\lambda^2 - 1 = 0$ having roots $\lambda = \pm 1$. Thus the functions

$$\begin{cases} y_1(x) = \exp(+1x) \\ y_2(x) = \exp(-1x) \end{cases}$$

are two linearly independent solutions of the homogeneous equation. Hence

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} \exp(x) & \exp(-x) \\ \exp(x) & -\exp(-x) \end{bmatrix} = -2.$$

For this example $f(x) = 1$. The variation of parameters formula gives

$$y(t) = \int_t^{t_0} \frac{\exp(x)\exp(-t) - \exp(t)exp(-x)}{(-2)} * 1dx + \alpha_1 \exp(t) + \alpha_2 \exp(-t)$$

and we chose $t_0 = 0$ for convenience. Thus

$$y(t) = \frac{-1}{2}\exp(-t)\int_0^t exp(x)dx + \frac{1}{2}\exp(t)\int_0^t \exp(-x)dx + \alpha_1 \exp(t) + \alpha_2 \exp(-t)$$

which yields

$$y(t) = \frac{-1}{2}\exp(-t)[exp(t) - 1] + \frac{1}{2}\exp(t)[1 - exp(-t)] + \alpha_1 \exp(t) + \alpha_2 \exp(-t)$$

$$= -1 + (\alpha_1 + \frac{1}{2})\exp(t) + (\alpha_2 + \frac{1}{2})\exp(-t).$$

Hence

$$y(t) = -1 + A_1 \exp(t) + A_2 \exp(-t).$$

Where A_1 and A_2 are constants.

Example 3.29
Find the general solution of

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = t\exp(2t).$$

Solution: Our auxiliary quadratic is

$$\lambda^2 - 4\lambda + 4 = 0 \text{ or } (\lambda - 2)^2 = 0.$$

One solution of the homogeneous equation is $y_1(t) = \exp(2t)$ and the second is $y_2(t) = t\exp(2t)$. Thus

$$W = \begin{bmatrix} \exp(2t) & t\exp(2t) \\ 2\exp(2t) & (1+2t)\exp(2t) \end{bmatrix} = \exp(4t)$$

Also $f(t) = t\exp(2t)$. The variation of parameters formula gives

$$y(t) = \int_0^t \frac{\exp(2X)t\exp(2t) - \exp(2x)x\exp(2t)}{\exp(4(x)} x\exp(2x)dx + (\alpha_1 + \alpha_2 t)\exp(2t)$$

$$= \int_0^t (tx\exp(2t) - x^2\exp(2t))dx + (\alpha_1 + \alpha_2 t)\exp(2t)$$

$$= t\exp(2t)\left[\frac{x^2}{2}\right]_0^t - \left[\frac{x^3}{3}\right]_0^t \exp(2t) + (\alpha_1 + \alpha_2 t)\exp(2t).$$

Finally

$$y(t) = \frac{1}{6}t^3\exp(2t) + (\alpha_1 + \alpha_2 t)\exp(2t).$$

Theorem 3.30
Let y_1 and y_2 be linearly independent solutions of

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = 0$$

and let y_p be any solution of the equation

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = f,$$

then every solution y of this equation has the form

$$y(t) = y_p(t) + A_1 y_1(t) + A_2 y_2(t)$$

for the appropriate constants A_1 and A_2, furthermore for every pair of constants A_1 A_2 the function y defined above and solves the homogeneous equation.The Proof for this will be given in year 2.

Guidelines for Guessing $\mathbf{y_p}$

1) If $f(t) = \alpha\exp(\beta t)$, where β is not a root of the auxiliary quadratic, then there exists a particular solution y_p of the form

$$y_p(t) = \gamma\exp(\beta t).$$

This is proven below.

$$\frac{d^2y_p}{dt^t} = \beta^2 y_p$$

and so

$$(\beta^2 + a_1\beta + a_0)y_p = \alpha\exp(\beta t)$$

by canceling the common factor $\exp(\beta t)$ we get

$$(\beta^2 + a_1\beta + a_0)\gamma = \alpha \neq 0$$

since β is not a root of the auxiliary quadratic. Hence,

$$\gamma = \alpha(\beta^2 + a_1\beta + a_0).$$

2) Suppose that the auxiliary quadratic has distinct root $\lambda_1 \neq \lambda_2$ and that

$$f(t) = \alpha \exp(\lambda_1 t)$$

then there exists a solution of the form

$$y_p(t)\gamma t \exp(\lambda_1 t).$$

This again is proven below.

$$\frac{dy_p}{dt} = \gamma \exp(\lambda_1 t)[1 + \lambda_1 t]$$
$$\frac{d^2y_p}{dt^2} = \gamma \exp(\lambda_1 t)[2\lambda_1 + \lambda_1^2 t]$$

Hence,

$$\frac{d^2y_p}{dt^2} + a_1\frac{dy_p}{dt} + a_0 y_p$$
$$= \gamma \exp(\lambda_1 t)[2\lambda_1 + \lambda_1^2 t + a_1(\lambda_1 t + 1) + a_0 t]$$
$$= \gamma \exp(\lambda_1 t)[t(\lambda_1^2 + a_1\lambda_1 + a_0) + 2\lambda + 1 + a_1]$$

we want this to be equal to

$$f(t) = \alpha \exp(\lambda_1 t)$$

which as

$$(\lambda_1^2 + a_1\lambda_1 + a_0) = 0$$

this holds precisely when

$$\gamma(2\lambda_1 + a_1) = \alpha$$

as $a_1 = -\lambda_1 - \lambda_2$ and as a result

$$\gamma(\lambda_1 - \lambda_2) = \alpha$$

so

$$\gamma = \frac{\alpha}{(\lambda_1 - \lambda_2}.$$

3) This deals with the case when the auxiliary quadratic has roots $\lambda_1 = \lambda_2$ and

$$f(t) = \alpha t + \mu) \exp(\lambda_1 t)$$

this particular solution has the form

$$y_p(t) = q(t) \exp(\lambda_1 t)$$

of the form

$$q(t) = at^3 + bt^2$$

4) If

$$f(t) = \alpha_1 \sin(\beta_1 t) + \alpha_2 \cos(\beta_2 t)$$

where β_1 and β_2 are real numbers and $i\beta_1$ and $i\beta_2$ are not solutions of the auxiliary quadratic. Then there exists a solution

$$y_p(t) = [a_1 \sin(\beta_1 t) + c_1 \cos(\beta_1 t) + a_2 \sin(\beta_2 t) + c_2 \cos(\beta_2 t)]$$

where a_1, c_1, a_2, c_2 are to be found.

3.2 Coupled Linear Systems

We are interested in equations of the form

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

in which a, b, c and d are constant.

Solution Method 1

Reduce to a single, second order equation. Differentiating the second equation you get

$$\frac{d^2y}{dt^2} = c\frac{dx}{dt} + d\frac{dy}{dt} = c(ax + by) + d\frac{dy}{dt}$$

$$= acx + bcy + d\frac{dy}{dt}.$$

From our second equation, if $c \neq 0$ then

$$x = \frac{1}{c}\frac{dy}{dt} - \frac{d}{c}y$$

and so

$$\frac{d^2y}{dt^2} = a\frac{dy}{dt} - ady + bcy + d\frac{dy}{dt}$$

or equivalently

$$\frac{d^2y}{dt^2} - (a + d)\frac{dy}{dt} + (ad - bc)y = 0.$$

If we write this system as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} =: A$$

Then

$$\frac{d^2y}{dt^2} - trace(A)\frac{dy}{dt} + \det(A)y = 0.$$

If $c \neq 0$ then x is obtained from

$$x = \frac{1}{c}\frac{dy}{dt} - \frac{d}{c}y.$$

If $c = 0$ and $b \neq 0$ then elimiate y and obtain

$$\frac{d^2x}{dt^2} - trace(A)\frac{dx}{dt} + \det(A)x = 0.$$

Then if $c = 0 = b$ and

$$\frac{dx}{dt} = ax$$

$$\frac{dy}{dt} = dy$$

therefore they are no longer coupled.

Solution Method 2

Let $z = \begin{bmatrix} x \\ y \end{bmatrix}$ so that

$$\frac{dz}{dt} = Az, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We look for a solution $z(t) = v \exp(\lambda t)$ where v is a constant vector and $\lambda \in \mathbb{C}$. Then

$$\frac{dz}{dt} = v\lambda \exp(\lambda t) \implies Az = Av \exp(\lambda t)$$

which is equivalent to

$$Av = \lambda v.$$

This means that v must be an eigenvector of A with eigenvalue λ. The eigenvalues of a matrix A satisfy

$$\det(A - \lambda I) = 0$$

or

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

this is the same as the auxiliary quadratic of second order equation arising from method one.

Example 3.31
Solve the coupled linear equations;

$$\frac{dz}{dt} = Az, \qquad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Solution: The eigenvalues of A solve,

$$\lambda^2 + 4\lambda + 3 = 0$$

this factorises to give $\lambda = -3$ and -1 then we need the eigenvector for each, first $\lambda = -1$ this eigenvector satisfies

$$\begin{bmatrix} -2 - (-1) & 1 \\ 1 & 2 - (-1) \end{bmatrix} v = 0$$

so we take

$$v = \alpha \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \alpha \in \mathbb{C}.$$

Some true for $\lambda - 3$ that gives

$$v = \alpha \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \alpha \in \mathbb{C}$$

Chapter 4 Conservative Equations

Definition 4.32
A conservative equation is an equation of the form

$$\frac{d^2x}{dt^2} + V'(x) = 0.$$

Remark 4.33
This means that we can consider any equation

$$\frac{d^2x}{dt^2} + F(x) = 0$$

in which we can find an indefinite integral of F.

Trick for Conservative Equation

Multiply the equation by $\frac{dx}{dt}$ to obtain

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + V'(x)\frac{dx}{dt} = 0$$

or by the chain rule,

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right] = 0.$$

Hence

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = E$$

where E is a constant. We could rearrange this as

$$\frac{dx}{dt} = \pm\sqrt{2E - 2V(x)}.$$

This is an infinite family (parametrized by E) of separable first order equations. Unfortunately this usually does not help.

Example 4.34
Consider a pendulum of length l with all the concentrated at the end. The equation governing the motion of the pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0$$

where g is the acceleration due to gravity. Prove that there exisits $T > 0$ such that $\theta(t+T) = \theta(t)$ for all t. The least such T is called the period of θ. So

$$V'(\theta) = \frac{g}{l}\sin(\theta)$$

and without loss of generality we take

$$V(\theta) = \frac{-g}{l}\cos(\theta)$$

and the energy conservation equation becomes

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{g}{l}\cos(\theta) = E.$$

In the special case where $E = \frac{g}{l}$ we get

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{g}{l}(1 + \cos(\theta) = \frac{2g}{l}\cos^2\left(\frac{\theta}{2}\right).$$

However this is the only cases where an explicit solution is possible. For the other cases we shall plot the curves in $\mathbb{R}^2$ given by $t \mapsto (\theta(t), \frac{d\theta}{dt})$.

Definition 4.35
Consider a conservative equation

$$\frac{d^2x}{dt^2} + V'(x) = 0.$$

The phase diagram for this equation is the set of parametric curves

$$t \to (x(t), x'(t))$$

where x solves the differential equation.

Example 4.36 (Linearized Pendulum)
From the previous example

$$\frac{d^2x}{dt^2} + \frac{g}{l}x = 0$$

has an auxiliary quadratic

$$\lambda^2 + \frac{g}{l} = 0$$

with roots

$$\lambda = \pm i\omega, \qquad \omega = \sqrt{\frac{g}{l}}.$$

The general solution is $x(t) = A\sin(\omega t) + B\cos(\omega t)$ where A, B are constants. We can also write this as $x(t) = R\cos(\omega t - \delta)$. Then

$$R\cos(\omega t - \delta) = R\cos(\omega t)\cos(\delta) + R\sin(\omega t)sin(\delta)$$

and thus

$$R\cos(\delta) = B, \qquad R\sin(\delta) = A.$$

Hence

$$x'(t) = -\omega R\sin(\omega t - \delta)$$

and so

$$(x(t), x'(t) = (R\cos(\omega t - \delta), -\omega R\sin(\omega t - delta)).$$

These are ellipses because

$$\left(\frac{x(t)}{R}\right)^2 + \left(\frac{x'(t)}{\omega R}\right)^2 = 1$$

The fact that the solutions are periodic is revealed by the fact that the curves in the phase diagram are closed.

Example 4.37 (Pendulum Phase Diagram)
For the lineraized pendulum
$$\frac{d^2x}{dt^2} + \frac{g}{l}x = 0.$$
We have
$$V'(x) = \frac{g}{l}x = \omega^2 x.$$
We can take
$$V(x) = \frac{1}{2}\omega^2 x^2$$
and the energy conservation equation is
$$\frac{1}{2}(x'(t))^2 + V(x) = E$$
where E is a constant i.e
$$\frac{1}{2}\omega"(x(t))^2 + \frac{1}{2}(x'(t))^2 = E.$$
Then by multiplying with 2 and dividing by $2E$ to get
$$\left(\frac{x(t)\omega}{\sqrt{2E}}\right)^2 + \left(\frac{x'(t)}{\sqrt{2E}}\right)^2 = 1$$
which is the same as the previous example with
$$R = \frac{\sqrt{2E}}{\omega}.$$
For the full pendulum equation
$$V'(x) = \frac{g}{l}\sin(x) = \omega^2\sin(x)$$
and we take
$$V(x) = \omega^2 - \omega^2\cos(x).$$
The phase curves are
$$\frac{1}{2}(x'(t))^2 + \omega^2(1-\cos(x)) = E.$$
We want to plot the curves, and we already have the case
$$V(x) = \frac{1}{2}x^2$$
or generally
$$V(x) = kx^2 \qquad k > 0$$
when the phase curves are ellipses. In both of these examples, $V(x) \geq 0$ everywhere so we need $E \leq 0$. When $E = 0$ we need both $y = 0$ and $v(x) = 0$. For the case $V(x) = \frac{g}{l}(1-\cos(x))$ this gives the points $(2n\pi, 0), n \in \mathbb{Z}$. For the case $V(x) = kx^2$ we just get (0,0) when $E > 0$ is small. Thus we draw a line at height E across the graph $V(x)$. For the pendulum equation when $E \geq \frac{2g}{l}$ then $E \geq V(x)$ for all x.

Suppose we now wish to approximate the phase curves, either for the pendulum equation or any other, in a small neighborhood of a point $(x_0, 0)$ at which V has a local maximum e.g $x_0 = \pi$ for the pendulum equation with $V(x) = \frac{g}{l}(1-\cos(x))$. We use the Taylor expansion
$$V(x) = V(x_0) + (x-x_0) + V'(x_0) + \frac{1}{2!}(x-x_0)^2V''(x_0) + ...$$

Since V has a local maximum at x_0

$$V'(x_0) = 0$$

$$V''(x_0) \leq 0.$$

Assume $V''(x_0) < 0$ and so the energy conservation equation is

$$E = \frac{1}{2}(x')^2 + V(x) = \frac{1}{2}(x')^2 + V(x_0) + \frac{1}{2|}(x - x_0)^2 V''(x_0) + ...$$

$$\approx \frac{1}{2}(x')^2 - \frac{1}{2}\omega^2(x - x_0)^2$$

where $\omega^2 = -V''(x_0) > 0$. The curves $\frac{1}{2}y^2 - \frac{1}{2}\omega^2(x - x_0)^2 = E$ are hyperbola.

Example 4.38
Draw the phase diagram for the equation

$$\frac{d^2x}{dt^2} + 2x - 3x^2 = 0$$

show that there exists periodic solutions and obtain an integral expression for their periods.

Solution: Here $V'(x) = 2x - 3x^2$ so $V(x) = x^2 - x^3$ is a suitable choice for v. The energy equation is

$$\frac{1}{2}(x')^2 + x^2 + x^3 = E$$

Integral Expression for the Period of a Periodic Solution

Observe that since $\frac{1}{2}(x')^2 + V(x)$ is a constant and since $x' = 0$ when $x = \alpha$ and $x = \beta$ we have

$$V(\alpha) = v(\beta).$$

The period of the solution is the time it takes from α to β plus the time to return. By symmetry these times are equal so the period is twice the transit from α to β. Thus the period is

$$T = \int_0^T dt = 2\int_\alpha^\beta \frac{dt}{dx}dx = 2\int_\alpha^\beta \frac{dx}{x'}$$

Now

$$\frac{1}{2}(x')^2 + V(x) = V(\alpha) = v(\beta)$$

giving

$$x' = \pm\sqrt{2(V(\beta) - V(x))}.$$

From α to β, x increases so

$$x' = \sqrt{2(V(\beta) - V(x))}.$$

Thus

$$T = \sqrt{2}\int_\alpha^\beta \frac{dx}{\sqrt{V(\beta) - V(x)}}.$$