MA2003 Complex Analysis
Exercise Sheet 0

1. Write the following complex numbers in polar form $z = r\left(\cos(\theta) + i\sin(\theta)\right)$ (or equivalently, $z = r\exp\left(i\theta\right)$):

   (a) $1 + i$

   (b) $-1 + i$

   (c) $1 + i\sqrt{3}$

   (d) $\dfrac{(1+i)^7}{(1+i\sqrt{3})^2}$

2. (a) For $z = a + ib$, express z^{-1} (i.e, $\dfrac{1}{z}$) in Cartesian form

   (b) Write down the modulus and Principal Argument of z^{-1} in terms of $|z|$ and $\mathrm{Arg}(z)$.

3. Express the following complex numbers in Cartesian form (that is to say, as $a + ib$ for $a, b \in \mathbb{R}$)
$$\frac{i-1}{1-i} \qquad \frac{1}{1+i} \qquad \frac{3+4i}{1-2i}.$$

4. Find the principal argument of the four points $\pm 1 \pm i\sqrt{3}$.

5. Calculate
$$\mathrm{Arg}\left(\frac{1}{2} + \frac{1}{z^2}\right)\bigg|_{z=1+i}.$$
(The notation $f(z)|_{z=w}$ means $f(z)$ evaluated at $z = w$, or in other words, $f(w)$.)

6. Show that
$$2\left(\frac{z}{z+i}\right)\frac{(z+i-z)}{(z+i)^2}\bigg|_{z=i} = \frac{-i}{4}.$$

7. Sketch the following regions of $\mathbb{C}$:

   (a) $1 < |z| < 2$ (This notation is shorthand for the set $\{z \in \mathbb{C} : 1 < |z| < 2\}$.)

   (b) $1 < |z + 2| < 2$

   (c) $1 < \mathsf{Im}(z - i) < 2$

8. (a) Prove that for two nonzero complex numbers z_1 and z_2 we have

$$|z_1 z_2| = |z_1| \cdot |z_2| \quad \text{and} \quad \arg(z_1 z_2) = \arg(z_1)\arg(z_2)$$

(hint: write z_1 and z_2 in polar form). Is it always true that $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_2) + \mathrm{Arg}(z_2)$?

(b) Show that $\exp(z_1)\exp(z_2) = \exp(z_1 + z_2)$.

9. Write down the 3^{rd} roots of -8 in Cartesian form.

10. Find the values of z for which $z^2 + 4iz - 1 = 0$. Which of these values lies inside the circle $C = \{z \in \mathbb{C} : |z| = 1\}$.

11. Show that $\mathsf{Re}(z) \le |\mathsf{Re}(z)| \le |z|$ and $|\mathsf{Re}(z)| + |\mathsf{Im}(z)| \le \sqrt{2}\,|z|$.

MA2003 Complex Analysis
Exercise Sheet 1

1. Use the triangle inequality $|z_1 - z_2| \leq |z_1| + |z_2|$ to prove the reverse triangle inequality:
$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

2. Use the triangle and reverse triangle inequalities to show that for all z on the circle $|z| = 2$, we have
$$|z + 2| \leq 4 \text{ and } |z - 3 + 4i| \geq 3.$$
Describe these inequalities geometrically.

3. Use the triangle and reverse triangle inequalities to show that for all z on the circle $|z + 3i| = 3$ we have
$$|z - 4| \leq 8, \quad |z + 5i| \geq 1 \text{ and } \left|\frac{z-4}{z+5i}\right| \leq 8.$$

4. Let L be the line segment $[0, h]$ where $h \in \mathbb{C}$ and $|h| < r$. Show that if $\beta \in \mathbb{C}$ with $|\beta| > 2r$ and $z \in L$ then
$$\left|\frac{h-z}{\beta-z}\right| < \frac{|h|}{r}.$$
Do this using the reverse triangle inequality. It can also be seen as follows. Draw L and two circles, both with centre 0, C_1 with radius r and C_2 with radius $2r$. Why does L lie inside C_1? Where is β on your diagram? Why is $|\beta - z| > r$? If you can answer these three questions then the inequality should follow easily.

5. The function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\mathbf{f}(x, y) = (0, 2y)$. Show that the corresponding complex function $f : \mathbb{C} \to \mathbb{C}$ is $f(z) = z - \overline{z}$, and that
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$
does not exist at any point $z_0 \in \mathbb{C}$.

6. Same as question 5 but with
$$\mathbf{f}(x, y) = (x^2 - y^2 - x, 2xy + y + 1) \quad \text{and} \quad f(z) = z^2 - \overline{z} + i.$$

7. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = |z|^2$. Show that f is differentiable at $z = 0$ and nowhere else.

8. Use the rules of differentiation to find the derivatives of the following functions:

(a) $f(z) = \left(z^2 + 4\right)^3$

(b) $g(z) = \dfrac{z + i}{z - i}$.

Find the values of $f'(i)$ and $g'(1)$.

MA2003 Complex Analysis
Exercise Sheet 2

1. For each function f below, write f in the form
$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$
and determine whether or not the Cauchy-Riemann equations are satisfied:
$$(a)\ f(z) = \exp(i\,\overline{z}) \quad (b)\ f(z) = z + \frac{1}{z} \quad (c)\ f(z) = z^3.$$
In the cases where f is differentiable, find the derivative of f both using the rules of differentiation and using the Cauchy-Riemann equations.

2. Show that the Cauchy-Riemann equations are satisfied by the function f defined on the open upper half plane $H_+ = \{z \in \mathbb{C} : \mathsf{Im}(z) > 0\}$ by
$$f(x+iy) = u(x,y) + iv(x,y) = \log\left(\sqrt{x^2+y^2}\right) + i\left(\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right)\right).$$
Assuming that f is indeed holomorphic on H_+, show that
$$f'(x+iy) = \frac{1}{x+iy} \quad \text{i.e., that} \quad f'(z) = \frac{1}{z}.$$

3. Describe the geometric effect of applying the functions:
   (a) $f(z) = \dfrac{1}{z}$ to a small disc centred at $1-i$, and
   (b) $g(z) = \exp(2iz)$ to a small disc centred at $\frac{\pi}{4} + i$.

4. The *set* of points $L = [0, 1-2i]$ is a line segment. It is also a *path* because we have a parametrisation given by $\gamma : [0,1] \to \mathbb{C},\ \gamma(t) = (1-2i)t$. Use this parametrisation to evaluate the integral
$$\int_L (\mathsf{Im}(z) + 3i)\ dz.$$

5. Find the value of
$$\int_{\Gamma_1} f(z)\ dz \text{ and } \int_{\Gamma_2} f(z)\ dz,$$
where $f(z) = 3\overline{z}$, Γ_1 is the straight line path from 0 to $-i$ and Γ_2 is the straight line path from $1-i$ to $1+i$.

6. Fix a point $z_0 \in \mathbb{C}$ and define a complex function f via

$$f(z) = (z - z_0)^n$$

where $n \in \mathbb{Z}$. Find the value of

$$\int_\Gamma f(z)\ dz,$$

where Γ is the circle with centre z_0 and radius $r > 0$, traversed in the anticlockwise direction (use the parametrisation $\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) = z_0 + r(\cos(t) + i\sin(t))$). Do this separately for the cases $n = -1$ and $n \neq -1$.

(Hint: for the case $n \neq -1$, you need to show that

$$\frac{d}{dt}\left[(\cos(t) + i\sin(t))^{n+1}\right] = i(n+1)\,(\cos(t) + i\sin(t))^{n+1}$$

and then use the (real) Fundamental Theorem of Calculus).

7. Let $f, g : U \to \mathbb{C}$ be continuous, and let Γ be a smooth path contained in U parametrised by $\gamma : [a, b] \to \mathbb{C}$. Prove that

   (a) for every constant $\alpha \in \mathbb{C}$ we have $\displaystyle\int_\Gamma (f + \alpha g) = \int_\Gamma f + \alpha \int_\Gamma g$, and

   (b) if $\tilde{\Gamma}$ denotes the reverse of Γ, we have $\displaystyle\int_{\tilde{\Gamma}} f = -\int_\Gamma f$. As a hint, parametrise $\tilde{\Gamma}$ using $\tilde{\gamma} : [a, b] \to \mathbb{C}$, $\tilde{\gamma}(t) = (a + b - t)$, and use the substitution $s = a + b - t$.

MA2003 Complex Analysis
Exercise Sheet 3

1. Find antiderivatives for the following functions:

   (a) $f(z) = \alpha + \beta(z - z_0)$,

   (b) $f(z) = (z - z_0)^n$,

   where α, β and $z_0 \in \mathbb{C}$ are constants and n is an integer, $n \neq -1$. Does $g(z) = (z - z_0)^{-1}$ have an antiderivative on $\mathbb{C} \setminus \{z_0\}$? Question 6 on Exercise Sheet 2 may help here.

2. Evaluate the following contour integrals:

   $$\int_{\mathcal{C}} z^3 \quad \text{and} \quad \int_{\mathcal{C}} \frac{1}{z^2}$$

   along $\mathcal{C}$ where $\mathcal{C}$ is

   (a) any contour from i to -2, and

   (b) any closed contour.

   For the second integral, you may assume that $\mathcal{C}$ does not contain 0.

3. Let U be a region in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be holomorphic on U with $f(z)$ real-valued for all $z \in U$. Prove that f is constant.

4. Find an upper estimate for

   $$\int_{\mathcal{C}} \frac{1}{1 + z^4},$$

   where $\mathcal{C}$ is the upper semicircular contour from R to $-R$ given by $\gamma : [0, \pi] \to \mathbb{C}$, $\gamma(t) = R\cos(t) + iR\sin(t)$.

5. Show that for all points z on the circle $\{z : |z| = 5\}$ we have

   $$|z - 7| \leq 12 \quad \text{and} \quad |\overline{z} + 8| \geq 3,$$

   and use this to find an upper estimate for the integral

   $$\int_S \frac{z - 7}{(\overline{z} + 8)^2}\, dz$$

   where S is the same circle oriented anticlockwise.

6. Let S_a be the anticlockwise square contour with corners at $\pm a(1+i), \pm a(1-i)$ where $a > 0$. Show that if $z \in S_a$ then
$$\frac{1}{|z|} \leq \frac{1}{a}$$
and hence
$$\left| \int_{S_a} \frac{1}{z} dz \right| \leq 8,$$
for all $a > 0$.

7. Prove each of the following:

   (a) For z_0 and h in $\mathbb{C}$ we have $\displaystyle\int_{[z_0, z_0+h]} 1\ dz = h$.

   (b) For $f : U \to \mathbb{C}$ and $z_0 \in U$, $f(z_0) = \displaystyle\frac{1}{h} \int_{[z_1, z_1+h]} f(z_0)\ dz$.

   (c) If α is a complex number and M a fixed real number with $|\alpha| \leq \epsilon M$ for all $\epsilon > 0$ then $\alpha = 0$.

MA2003 Complex Analysis
Exercise Sheet 4

1. Express each of the following complex numbers in Cartesian form $a + ib$:

$$(a)\ \operatorname{Log}(i), \quad (b)\ \operatorname{Log}(ie) \quad (c)\ \operatorname{Log}(-1 - i\sqrt{3}).$$

2. Express each of the following complex numbers in Cartesian form $a + ib$:

$$(a)\ (1+i)^{i}, \quad (b)\ (ie)^{i\pi}, \quad (c)\ (-1 - i\sqrt{3})^{1+i}.$$

3. (a) Use the definition $z^{\alpha} = \exp(\alpha \operatorname{Log}(z))$ to show that $z^3 = zzz$.
   (b) Show that $\operatorname{Log}(i^3) \neq 3 \operatorname{Log}(i)$
   (c) Define $\sqrt{z} = z^{1/2} (= \exp(\frac{1}{2} \operatorname{Log}(z)))$ for $z \in \mathbb{C} \setminus \{0\}$. Where is the mistake in

$$-1 = i^2 = ii = \sqrt{-1}\sqrt{-1} = \sqrt{-1 \times -1} = \sqrt{1} = 1?$$

   (d) Show that for all $\alpha, \beta \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{0\}$ we have $z^{\alpha} z^{\beta} = z^{\alpha+\beta}$. Is it true that $\operatorname{Log}(\alpha\beta) = \operatorname{Log}(\alpha) + \operatorname{Log}(\beta)$?

4. Recall that the Principal Logarithm function Log is holomorphic on the region $\mathbb{C}_{\pi}$, where $\mathbb{C}_{\pi} = \{z \in \mathbb{C} : z \neq 0 \text{ and } \operatorname{Arg}(z) \neq \pi\}$. Let F be the function defined by

$$F(z) = \frac{1}{2i} \left(\operatorname{Log}(z+i) - \operatorname{Log}(z-i) \right).$$

   (a) Describe (or sketch) the region $\mathcal{R}$ on which the function F is holomorphic.
   (b) Show that F is an antiderivative for the function $f : \mathcal{R} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z^2+1} \quad \text{for all} \quad z \in \mathcal{R}.$$

5. Let U be a starlit region with star centre $z_* \in U$ and let $g : U \to \mathbb{C}$ be a holomorphic function.

(a) Prove that if $g(z) \neq 0$ for all $z \in U$, then the function $\dfrac{g'}{g}$ has an antiderivative on U, stating any results used (you may assume that g holomorphic on U implies g' holomorphic on U).

(b) Prove that if in addition $g(z) \in \mathbb{C}_\pi$ for all $z \in U$ then

$$\int_{[z_*, z]} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta = \mathrm{Log}(g(z)) + \alpha$$

for some constant α.

6. Evaluate the integral

$$\int_{\mathcal{C}} \frac{\exp(2z)}{4z + i\pi}\, dz$$

where $\mathcal{C}$ is (i) the anticlockwise contour whose points lie on the circle $\{z : |z| = 1\}$, and (ii) when $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z - 2i| = 2\}$. The use of any Theorems made to obtain the value of these integrals should be justified.

7. Evaluate the integral

$$\int_{\mathcal{C}} \frac{\cos(z^2)}{3i + 2z}\, dz,$$

where (i) $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z| = 1\}$, and (ii) $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z| = 5\}$. The use of any Theorems made to obtain the value of these integrals should be justified.

8. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 6x + 25}\, dx$$

in the following way (compare Example 5.8 in the notes):

(a) Define the complex function f by $f(z) = \dfrac{1}{z^2 + 6z + 25}$ and find z_0 and z_1 so that $f(z) = \dfrac{1}{(z - z_0)(z - z_1)}$ (where z_0 lies in the upper half-plane and z_1 in the lower half-plane).

(b) Choose a suitable function g, holomorphic on the simply connected region $\mathcal{R} = \left\{z \in \mathbb{C} : \mathsf{Im}(z) > \frac{1}{2}\mathsf{Im}(z_1)\right\}$, so that

$$f(z) = \frac{g(z)}{(z - z_0)}.$$

(c) Justify the use of Cauchy's Integral Formula to find

$$\int_{\mathcal{C}_R} f = \int_{\mathcal{C}_R} \frac{g(z)}{(z-z_0)}\, dz,$$

where $\mathcal{C}_R = L_R + S_R$ with L_R the straight line path from $-R$ to R and S_R a suitable semicircular contour from R to $-R$, with R sufficiently large to apply the Theorem.

(d) Show that for large R and $z \in S_R$, we have $\left|z^2 + 6z + 25\right| \geq R^2 - 6R - 25$.

(e) Use the Estimation Lemma to show that

$$\left|\int_{S_R} f\right| \to 0 \text{ as } R \to \infty.$$

(f) Deduce the value of

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 6x + 25}\, dx.$$

9. (Liouville's Theorem) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic everywhere, and suppose that f is bounded, i.e. there exists $M > 0$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Show that f is constant on $\mathbb{C}$, in the following way:

(a) Let $z_1, z_2 \in \mathbb{C}$, and let $R > 0$ be sufficiently large so that z_1 and z_2 are enclosed by the countour $\mathcal{C}_R$ consisting of the anticlockwise circle with centre 0 and radius R. Use Cauchy's Integral Formula to write $f(z_1) - f(z_2)$ as a single integral along $\mathcal{C}_R$.

(b) Use the Estimation Lemma (and the backwards triangle inequality) to show that

$$|f(z_1) - f(z_2)| \leq M \frac{|z_1 - z_2|}{(R - |z_1|)(R - |z_2|)} \cdot 2\pi R$$

for all (sufficiently large) R.

(c) Deduce that $f(z_1) = f(z_2)$.

MA2003 Complex Analysis
Exercise Sheet 5

1. Locate the poles of each of the following functions, and calculate the residues at these poles:

   (a) $f(z) = \dfrac{1}{z(i-z)^3}$

   (b) $f(z) = \dfrac{z^2}{(z^2+1)^2}$

   (c) $f(z) = \dfrac{\mathrm{Log}(z)}{(4z-i)^2}$

   (d) $f(z) = \dfrac{1}{\exp(z)-1}$.

2. Evaluate
$$\int_{\mathcal{C}} \frac{1}{z(z-1)(z+2)}\, dz,$$
where $\mathcal{C}$ is the anticlockwise circle with centre 0 and radius 3/2.

3. Evaluate
$$\int_{\mathcal{C}} \frac{1}{(z^2+1)^3}\, dz$$
where $\mathcal{C}$ is the anticlockwise square with vertices $1,\ 1+2i, -1+2i$ and -1.

4. Use contour integration to evaluate each of the following real integrals:

   (a) $$\int_0^{2\pi} \frac{1}{5+4\sin\theta}\, d\theta$$

   (b) $$\int_0^{\infty} \frac{1}{x^4+1}\, dx$$

5. Use contour integration to evaluate the following real integrals:

   (a) $$\int_0^{2\pi} \frac{1}{16\cos^2(t)+25\sin^2(t)}\, dt.$$

   (b) $$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\, dx.$$

(c)

$$\int_0^{\infty} \frac{\cos(5x)}{x^2+4}$$

(Hint: first use the usual method to evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{i5x}}{x^2+4}\, dx$.)

6. (a) Let N be a natural number and let α_j be constants for $-N \leq j \leq N$. If $f(z) = \displaystyle\sum_{j=-N}^{N} \alpha_j z^j$, write down the value of $\mathrm{Res}(f; 0)$.

   (b) Write $\displaystyle\int_0^{2\pi} [\cos(t)]^8 \; dt$ as a contour integral, and use part (a) to evaluate it.