MA 2003 Complex Analysis
Exercise Sheet 5

1. Locate the poles of each of the following functions, and calculate the residues at these poles:
   (a) $f(z) = \dfrac{1}{z(i-z)^3}$
   (b) $f(z) = \dfrac{z^2}{(z^2+1)^2}$
   (c) $f(z) = \dfrac{\mathrm{Log}(z)}{(4z-i)^2}$
   (d) $f(z) = \dfrac{1}{\exp(z)-1}$.

2. Evaluate
$$\int_{\mathcal{C}} \frac{1}{z(z-1)(z+2)}\, dz,$$
where $\mathcal{C}$ is the anticlockwise circle with centre 0 and radius $3/2$.

3. Evaluate
$$\int_{\mathcal{C}} \frac{1}{(z^2+1)^3}\, dz$$
where $\mathcal{C}$ is the anticlockwise square with vertices $1,\ 1+2i, -1+2i$ and -1.

4. Use contour integration to evaluate each of the following real integrals:
   (a) $$\int_0^{2\pi} \frac{1}{5+4\sin\theta}\, d\theta$$
   (b) $$\int_0^{\infty} \frac{1}{x^4+1}\, dx$$

5. Use contour integration to evaluate the following real integrals:
   (a)
$$\int_0^{2\pi} \frac{1}{16\cos^2(t)+25\sin^2(t)}\, dt.$$
   (b)
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\, dx.$$
   (c)
$$\int_0^{\infty} \frac{\cos(5x)}{x^2+4}$$
   (Hint: first use the usual method to evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{i5x}}{x^2+4}\, dx$.)

6. (a) Let N be a natural number and let α_j be constants for $-N \le j \le N$. If $f(z) = \sum_{j=-N}^{N} a_j z^j$, write down the value of $\mathrm{Res}(f;0)$.
   (b) Write $\displaystyle\int_0^{2\pi} [\cos(t)]^8\ dt$ as a contour integral, and use part (a) to evaluate it.