MA1001

Elementary Differential Equations

Examples

MM

Version: 16th January 2018

Contents

Chapter 1 Introductory Examples

Example 1.1. Radioactive isotope decays according to the differential

$$\frac{dm}{dt} = -km$$

where $k > 0$ is constant. m is the mass present at time t. Find the half life such that $m(\tau) = \frac{1}{2}m(0)$

Example 1.2. A student goes home for Christmas leaving a mouldy burger under the sofa. The mould grows according to

$$\frac{dm}{dt} = km$$

where $k > 0$ is constant. Find the time $\tau = 0$ at which $m(\tau) > M$, assuming $m(0)$ is known. Here M is the mass of the sofa.

Example 1.3. The previous model is changed to

$$\frac{dm}{dt} = km(M - m)$$

where $k > 0$, $M > 0$ are constants. Show how the solution behaves.

Example 1.4. Show that the differential equation

$$\frac{dy}{dt} = 2\sqrt{y}$$

has more than one solution with :

$$y_1(t) = t^2 \text{ for t} \geq 0$$
$$y_2(t) = \ 0 \text{ for all t}$$

A third solution :

$$y_3(t) = \begin{cases} y_2(t) \text{ for t } \leq \ 0 \\ y_1(t) \text{ for t } > \ 0 \end{cases}$$

Example 1.5. Show that the solutions of the equation :

$$\frac{dy}{dt} = 1 + y^2$$

blows up in finite time.

Chapter 2 First Order Differentials

A first order (ordinary differential equations) is an equation of the form :

$$\frac{dy}{dt} = f(t, y).$$

Here any solution y is a real or complex valued function of a single real variable and f is a function of 2 variables.

In Newtonian notation the same equation would be written as :

$$y'(t) = f(t, y(t)).$$

2.1 Separable Equations

A separable equation is an equation of the form :

$$\frac{dy}{dt} = g(t)h(y).$$

Suppose that h never takes the value 0. Then we can write the equation as :

$$\frac{1}{h(y)}\frac{dy}{dt} = g(t).$$

We can also (at least on principle) find a function F whose derivative is $\frac{1}{h}$:

$$F' = \frac{1}{h},$$

hence : $F'(y(t))\frac{dy}{dt} = g(t) \implies \frac{d}{dt}[F(y(t))] = g(t)$.

Example 2.1. Solve

$$\frac{dy}{dt} = (1 + y^2)(1 + \alpha y)$$

where $\alpha \geq 0$ is constant.

Solution

Example 2.2. Find the solution of

$$\frac{dy}{dt} = 2\sqrt{y}.$$

Solution

Example 2.3. Solve the equation :

$$\frac{dy}{dt} = m(1-m).$$

Solution

Remark. • If we take $m(0) = \mu = 0$ we get $m(t) = 0$ for all t then :
If $0 < \mu, 1$ then $(1 - \mu)\exp(-t) > 0$ and hence

$$0, m(t) < \frac{\mu}{0 + \mu} = 1$$

Thus

$$\frac{dm}{dt} = m(1 - m) > 0 \text{ also } \lim m(t) = 1.$$

- If we take $m(0) = \mu = 1$ we get $m(t) = 1$ for all t then :
If $\mu > 1$ then the expression

$$m(t) = \frac{\mu}{\mu(1 - \exp(-t)) + \exp(-t)}$$

tells us that $m(t) > 0$ whilst the expression

$$m(t) = \frac{\mu}{(1 - m)\exp(-t) + \mu}$$

tells us that $m(t) > 1$ for all $t \geq 0$. Also $\frac{dm}{dt} = m(1 - m) < 0$, so m is strictly decreasing. Also $\lim m(t) = 1$.

2.2 Homogeneous Equations

A homogeneous equation is an equation of the form :

$$\frac{dy}{dt} = f(\frac{y}{t}).$$

We introduce a new function $z = \frac{y}{t}$; more precisely $z(t) = \frac{y(t)}{t}$. Hence $y(t) = tz(t)$ which gives

$$f(z) = \frac{dy}{dt} = z(t) + t\frac{dz}{dt}.$$

Rearranging yields:

$$\frac{dz}{dt} = \frac{f(z) - z}{t} = \frac{1}{z}(f(z) - z)$$

which is now separable.

Example 2.4. Solve the equation :

$$t^2\frac{dy}{dt} = ty + t^2 + y^2$$

and examine the behavior of the solution as t tends to 0.

Solution

Example 2.5. Solve the homogeneous equation :

$$\frac{dy}{dt} = \frac{y}{t}^{\frac{1}{3}}.$$

Solution

2.3 Linear Equations

A first-order differential linear equation is an equation of the form :

$$\frac{dy}{dt} = a(t)y + b(t)$$

where $a(t)$ and $b(t)$ are given functions of the dependant variable t. For linear equations we can always write the solution y explicitly as a function.

Solution Procedure

1. Choose a function $p(t)$ such that

$$\frac{dp}{dt} = a(t)$$

2. Observe that

$$\frac{d}{dt}(ye^{-p(t)}) = \frac{dy}{dt}e^{-p(t)} + ye^{-p(t)} = (\frac{dy}{dt} - a(t)y)e^{-p(t)} = b(t)e{-}p(t)$$

3. Integrate the equation

$$\frac{d}{dt}(ye^{-p(t)}) = b(t)e^{-p(t)}$$

   This integrates to

$$ye^{-p(t)} = \int b(t)e{-}p(s)ds + c$$

   where c is a constant By doing this we get :

$$y = e^{p}(t)(c + \int b(s)e^{-p(s)}ds$$

Example 2.6. Solve the differential equation

$$\frac{dy}{dt} = ky + \sin(t)$$

subject to the condition $y(0) = 0$.

Solution

Example 2.7. Solve the linear differential equation

$$t^2 \frac{dy}{dt} + 2ty = e^t$$

subject to $y(1) = 1$. Is there a solution subject to the condition $y(0) = 1$?

Solution

Example 2.8. Solve the equation

$$(t \log(t)) \frac{dy}{dt} + y = 3t^3.$$

Solution

2.4 Bernoulli Equations

A Bernoulli equation is an equation in the form

$$\frac{dy}{dt} + a(t)y = b(t)y^n.$$

Where when $n = 0$ we have a linear equation and when $n = 1$ we have linear and separable equation. We choose an integration factor $p(t)$ such that

$$\frac{p'(t)}{p(t)} = a(t) \text{ so that } \frac{dy}{dt} + \frac{p'(t)}{p(t)}y = b(t)y^n.$$

Now we multiply both sides by $p(t)$ to get

$$p(t)\frac{dy}{dt} + \frac{dp}{dt}y = p(t)b(t)y^n.$$

Then by differentiation

$$\frac{d}{dt}(p(t)y(t0 = \frac{b(t)}{p(t)^{n-1}}(p(t)y(t))^n.$$

We introduce a new function z by the formula $z(t) = p(t)y(t)$ which satisfies

$$\frac{dz}{dt} = q(t)z^n \text{ where } q(t) = \frac{b(t)}{p(t)^{n-1}}.$$

Example 2.9. Solve the equation

$$\frac{dy}{dt} + \frac{2}{t}y = \exp(t)y^2.$$

Solution

Example 2.10. Solve the equation

$$\frac{dy}{dt} = t\exp(t^2 - y).$$

Solution

Chapter 3 Second Order Equations

A second order linear differential equation is an equation of the following form :

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t).$$

The functins $a_1(*)$, $a_0(*)$ and $f(*)$ are supposed to be known and we want to find all of the solutions to y.

Example 3.1. The Legendre equation is

$$\frac{d^2y}{dt^2} - \frac{2t}{1-t^2}\frac{dy}{dt} + \frac{n(n+1)}{1-t^2}Y = 0.$$

The equation has a solution $P_n(t)$, the Legendre polynomial of degree n.

- For $n = 0$ we can take $P_0(t)$ for all t
- For $n = 1$ we let $P_1(t) = at + b$. We get

  $$\frac{-2t}{1-t^2}a + \frac{2}{1-t^2}(at + b) = 0$$

  for all $t \in \mathbb{R}\backslash\{\pm 1\}$ This gives $b = 0$ and we can take $a = 1$ to get $P_1(t) = t$

Example 3.2. Bessel's equation of order n is :

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + (1 - \frac{n^2}{t^2})y = 0.$$

The solution are Bessel functions, $J_n(t)$ and $Y_n(t)$.

3.1 Equations with Constant Coefficients

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt}a_0y = f(t)$$

In order to solve the above equation we adopt a two step solution strategy:

1. Find all the solution of the equation with $f = 0$. The equation when $f = 0$ is called the homogeneous equation
2. Develop a formula called the variation of parameters formula for the case $f \neq 0$, using the solutions of the homogeneous equation

Solution Method for $\mathbf{f} = \mathbf{0}$

We look for a solution of the form $y(t) = A\exp(\mu t)$, where A and μ are constants. We have:

$$\frac{dy}{dt} = \mu A\exp(\mu t) \quad \text{and} \quad \frac{d^2y}{dt^2} = \mu^2 A\exp(\mu t).$$

Substituting into the differential equation gives:

$$(\mu^2 + a_1\mu + a_0) * A\exp(\mu t) = 0.$$

Since $\exp(\mu t) \neq 0$ and we do not want the trivial solution which comes from choosing $A = 0$, we have $\mu^2 + a_1\mu + a_0 = 0$ this is called the auxiliary quadratic. The possibilities are:

- Two distinct roots
- One double root

If a_1 and a_0 are real there is an alternative dichotomy:

- Two real roots (which may or may not coincide)
- A complex conjugate pair of distinct roots

Case 1 : Distinct Roots $\boldsymbol{\mu_1 \neq \mu_2}$

$$\mu^2 + a_1\mu + a_0 = (\mu - \mu_1)(\mu - \mu_2).$$

We have the solutions

$$y_j(t) = A_j\exp(\mu_j t) \quad j = 1, 2.$$

For completely arbitary constants A_1, A_2. Suppose $y(t) = y_1(t) + y_2(t)$ then

$$\frac{dy}{dt} = \frac{dy_1}{dt} + \frac{dy_2}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2y_1}{dt^2} + \frac{d^2y_2}{dt^2}.$$

So,

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = [\frac{d^2y_1}{dt^2} + a_1\frac{dy_1}{dt} + a_0y_1] + [\frac{d^2y_2}{dt^2} + a_1\frac{dy_1}{dt} + a_0y_2].$$

Theorem 3.3. Suppose that the auxiliary quadratic has a distinct root $\mu_1 \neq \mu_2$. Then every solution of the homogeneous equation has the form

$$y(t) = A_1\exp(\mu_1 t) + A_2\exp(\mu_2 t)$$

for some A_1, $A_2 \in \mathbb{C}$.

Example 3.4.

$$\frac{d^2y}{dt^2} - (\mu_1 + \mu_2)\frac{dy}{dt} + \mu_1\mu_2 y = 0$$

such that $y(0) = y(0)$ and $y'(0) = v_0$.

Solution

Example 3.5. In the previous example let μ_1 be fixed and find $\lim_{\mu_2 \to \mu_1} y(t)$.

Solution

$$y(t) = (At + B)\exp(\mu t)$$

Theorem 3.6. Every solution of the equation

$$\frac{d^2y}{dt^2} - 2y\frac{dy}{dt} + \mu^2 y = 0$$

has the form

$$y(t) = (At + B)\exp(\mu t)$$

for some appropriate constants A, B.

Example 3.7. Find the general solution of

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

where $\omega > 0$ is constant.

Solution

Remark. Observe that $\lim_{\omega\to 0}\cos(\omega t) = \cos(0) = 1$ and $\lim_{\omega\to 0}\frac{\sin(\omega t)}{\omega} = t$. When $\omega = 0$, we get

$$y(t) = y_0 + v_0 t$$

which does indeed solve $\frac{d^2y}{dt^2} = 0$.

Definition 3.8. Let y_1, y_2 be solutions of

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = 0.$$

We say that y_1 and y_2 are linearly independent if and only if neither solution can be written as a constant multiple of the other.

Example 3.9. In the previous example,

$$y_1(t) = \cos(\omega t) \text{ and } y_2 = \frac{\sin(\omega t)}{\omega}$$

are linearly independent: so are

$$y_1(t) = \exp(i\omega t) \text{ and } y_2(t) = \exp(-i\omega t)$$

$$y_1(t) = \cos(\omega t) \text{ and } y_2(t) = \exp(i\omega t).$$

However if $y_1 t = \cos(\omega t)$ and $y_2(t) = 17\cos(\omega t)$, are not linearly independent.

Definition 3.10. Let y_1 and y_2 be two differentiable functions. The Wronskian (determinant) of y_1 and y_2 is the function

$$W = \det \begin{bmatrix} y_1 \; y_2 \\ y_1' \; y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2$$

Objective

We want to find a formula for the general solution of

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = f$$

where the function f on the right hand side is non-trivial. We assume that we know two lineralry independent solutions y_1 and y_2 of the homogeneous equation

$$\frac{d^2y}{dt^2} j + a_1 \frac{dy}{dt} j + a_0 y j = 0 \text{ for } j = 1, 2.$$

We can look for a solution in the form

$$y(t) = A_1(t) + A_2(t) y_2(t).$$

Differentiating,

$$y'(t) = A_1(t) y_1'(x) + A_2(t) y_2'(t) + A_1'(t) y_1(t) + A_2'(t) y_2(t).$$

We impose the condition $A_1' y_1 + A_2' y_2 = 0$. Then we obtain by the second derivative

$$y''(t) = A_1(t) y_1''(t) + A_2 y_2''(t) + A_1'(t) y_1'(t) + A_2'(t) y_2'(t).$$

Combining the expressions for y, y' and y'', we obtain

$$y'' + a_1 y + a_0 y = A_1(y_1'' + a_1 y_1' + a_0 y1) + A_2(y_2'' + a_1 y_2' + a_0 y2) + A_1' y_1' + A_2' y_2' = A_1' y_1' + A_2' y_2'$$

because y_1 and y_2 solve the homogeneous equation. Using the original formula we see that our second equation for A_1 and A_2 is

$$A_1' y_1' + A_2' y_2' = f.$$

We write the equation for A_1 and A_2 as

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which yields

$$\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix} = \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix}$$

This means

$$\begin{cases} A_1' = \frac{-y_2 f}{W} \\ A_2' = \frac{y_1 f}{W} \end{cases}$$

giving

$$\begin{cases} A_1(t) = \int_{t_0}^{t} \frac{-y_2(x)f(x)}{W(x)} dx + \alpha_1 \\ A_2(t) = \int_{t_0}^{t} \frac{y-1(x)f(x)}{W(x)} dx + \alpha_2 \\ \end{cases}$$

Where α_1 and α_2 are constants of integration and t_0 can be chosen. Recalling that

$$y(t) = A_1(t)y_1(t) + A_2(t)y_2(t)$$

we obtain

$$y(t) = \int_{t_0}^{t} \left(\frac{y_1(x)y_2(t) - y_1(t)y_2(x)}{W(x)} \right) f(x)dx + \alpha_1 y_1(t) + \alpha_2 y_2(t).$$

This formula is called the variation of parameters or variation of constants formula.

- y is the general solution of

$$y'' + a_1 y' + a_0 y = f$$

- a_1, a_0 and f are given functions.
- y_1, y_2 are the solutions of the homogeneous equation

$$y_j'' + a_1 y_j' + a_0 y_j = 0, \quad j = 1, 2$$

- W is the Wronskian

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2$$

Example 3.11. Find the general solution of the equation

$$y'' - y = 1.$$

Solution

Example 3.12. Find the general solution of

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = t\exp(2t).$$

Solution

Theorem 3.13. Let y_1 and y_2 be linearly independent solutions of

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = 0$$

and let y_p be any solution of the equation

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = f,$$

then every solution y of this equation has the form

$$y(t) = y_p(t) + A_1y_1(t) + A_2y_2(t)$$

for the appropriate constants A_1 and A_2, furthermore for every pair of constants A_1 A_2 the function y defined above and solves the homogeneous equation.The Proof for this will be given in year 2.

Guidelines for Guessing $\mathbf{y_p}$

1. If $f(t) = \alpha\exp(\beta t)$, where β is not a root of the auxiliary quadratic, then there exists a particular solution y_p of the form

   $$y_p(t) = \gamma\exp(\beta t).$$

   This is proven below.

   $$\frac{d^2y_p}{dt^t} = \beta^2y_p$$

   and so

   $$(\beta^2 + a_1\beta + a_0)y_p = \alpha\exp(\beta t)$$

   by canceling the common factor $\exp(\beta t)$ we get

   $$(\beta^2 + a_1\beta + a_0)\gamma = \alpha \neq 0$$

   since β is not a root of the auxiliary quadratic. Hence,

   $$\gamma = \alpha(\beta^2 + a_1\beta + a_0).$$

2. Suppose that the auxiliary quadratic has distinct root $\lambda_1 \neq \lambda_2$ and that

$$f(t) = \alpha \exp(\lambda_1 t)$$

then there exists a solution of the form

$$y_p(t)\gamma t \exp(\lambda_1 t).$$

This again is proven below.

$$\frac{dy_p}{dt} = \gamma \exp(\lambda_1 t)[1 + \lambda_1 t]$$

$$\frac{d^2 y_p}{dt^2} = \gamma \exp(\lambda_1 t)[2\lambda_1 + \lambda_1^2 t]$$

Hence,

$$\frac{d^2 y_p}{dt^2} + a_1 \frac{dy_p}{dt} + a_0 y_p$$

$$= \gamma \exp(\lambda_1 t)[2\lambda_1 + \lambda_1^2 t + a_1(\lambda_1 t + 1) + a_0 t]$$

$$= \gamma \exp(\lambda_1 t)[t(\lambda_1^2 + a_1\lambda_1 + a_0) + 2\lambda + 1 + a_1]$$

we want this to be equal to

$$f(t) = \alpha \exp(\lambda_1 t)$$

which as

$$(\lambda_1^2 + a_1\lambda_1 + a_0) = 0$$

this holds precisely when

$$\gamma(2\lambda_1 + a_1) = \alpha$$

as $a_1 = -\lambda_1 - \lambda_2$ and as a result

$$\gamma(\lambda_1 - \lambda_2) = \alpha$$

so

$$\gamma = \frac{\alpha}{(\lambda_1 - \lambda_2}.$$

3. This deals with the case when the auxiliary quadratic has roots $\lambda_1 = \lambda_2$ and

$$f(t) = \alpha t + \mu) \exp(\lambda_1 t)$$

this particular solution has the form

$$y_p(t) = q(t) \exp(\lambda_1 t)$$

of the form

$$q(t) = at^3 + bt^2$$

4. If

$$f(t) = \alpha_1 \sin(\beta_1 t) + \alpha_2 \cos(\beta_2 t)$$

where β_1 and β_2 are real numbers and $i\beta_1$ and $i\beta_2$ are not solutions of the auxiliary quadratic. Then there exists a solution

$$y_p(t) = [a_1 \sin(\beta_1 t) + c_1 \cos(\beta_1 t) + a_2 \sin(\beta_2 t) + c_2 \cos(\beta_2 t)]$$

where a_1, c_1, a_2, c_2 are to be found.

3.2 Coupled Linear Systems

We are interested in equations of the form

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

in which a, b, c and d are constant.

Solution Method 1

Reduce to a single, second order equation. Differentiating the second equation you get

$$\begin{aligned}\frac{d^2y}{dt^2} &= c\frac{dx}{dt} + d\frac{dy}{dt} = c(ax+by) + d\frac{dy}{dt} \\ &= acx + bcy + d\frac{dy}{dt}.\end{aligned}$$

From our second equation, if $c \neq 0$ then

$$x = \frac{1}{c}\frac{dy}{dt} - \frac{d}{c}y$$

and so

$$\frac{d^2y}{dt^2} = a\frac{dy}{dt} - ady + bcy + d\frac{dy}{dt}$$

or equivalently

$$\frac{d^2y}{dt^2} - (a+d)\frac{dy}{dt} + (ad-bc)y = 0.$$

If we write this system as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} =: A$$

Then

$$\frac{d^2y}{dt^2} - trace(A)\frac{dy}{dt} + \det(A)y = 0.$$

If $c \neq 0$ then x is obtained from

$$x = \frac{1}{c}\frac{dy}{dt} - \frac{d}{c}y.$$

If $c = 0$ and $b \neq 0$ then elimiate y and obtain

$$\frac{d^2x}{dt^2} - trace(A)\frac{dx}{dt} + \det(A)x = 0.$$

Then if $c = 0 = b$ and

$$\frac{dx}{dt} = ax$$
$$\frac{dy}{dt} = dy$$

therefore they are no longer coupled.

Solution Method 2

Let $z = \begin{bmatrix} x \\ y \end{bmatrix}$ so that

$$\frac{dz}{dt} = Az, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We look for a solution $z(t) = v\exp(\lambda t)$ where v is a constant vector and $\lambda \in \mathbb{C}$. Then

$$\frac{dz}{dt} = v\lambda\exp(\lambda t) \implies Az = Av\exp(\lambda t)$$

which is equivalent to

$$Av = \lambda v.$$

This means that v must be an eigenvector of A with eigenvalue λ. The eigenvalues of a matrix A satisfy

$$\det(A - \lambda I) = 0$$

or

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

this is the same as the auxiliary quadratic of second order equation arising from method one.

Example 3.14. Solve the coupled linear equations;

$$\frac{dz}{dt} = Az, \qquad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Solution

Chapter 4 Conservative Equations

Definition 4.1. A conservative equation is an equation of the form

$$\frac{d^2x}{dt^2} + V'(x) = 0.$$

Remark. This means that we can consider any equation

$$\frac{d^2x}{dt^2} + F(x) = 0$$

in which we can find an indefinite integral of F.

Trick for Conservative Equation

Multiply the equation by $\frac{dx}{dt}$ to obtain

$$\frac{dx}{dt}\frac{d^2x}{dt^2} + V'(x)\frac{dx}{dt} = 0$$

or by the chain rule,

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x)\right] = 0.$$

Hence

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = E$$

where E is a constant. We could rearrange this as

$$\frac{dx}{dt} = \pm\sqrt{2E - 2V(x)}.$$

This is an infinite family (parametrized by E) of separable first order equations. Unfortunately this usually does not help.

Example 4.2. Consider a pendulum of length l with all the concentrated at the end. The equation governing the motion of the pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0$$

where g is the acceleration due to gravity. Prove that there exisits $T > 0$ such that $\theta(t+T) = \theta(t)$ for all t. The least such T is called the period of θ. So

$$V'(\theta) = \frac{g}{l}\sin(\theta)$$

and without loss of generality we take

$$V(\theta) = \frac{-g}{l}\cos(\theta)$$

and the energy conservation equation becomes

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 - \frac{g}{l}\cos(\theta) = E.$$

In the special case where $E = \frac{g}{l}$ we get

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{g}{l}(1+\cos(\theta) = \frac{2g}{l}\cos^2\left(\frac{\theta}{2}\right).$$

However this is the only cases where an explicit solution is possible. For the other cases we shall plot the curves in $\mathbb{R}^2$ given by $t \mapsto (\theta(t), \frac{d\theta}{dt})$.

Definition 4.3. Consider a conservative equation

$$\frac{d^2x}{dt^2} + V'(x) = 0.$$

The phase diagram for this equation is the set of parametric curves

$$t \to (x(t), x'(t))$$

where x solves the differential equation.

Example 4.4 (Linearized Pendulum)**.** From the previous example

$$\frac{d^2x}{dt^2} + \frac{g}{l}x = 0$$

has an auxiliary quadratic

$$\lambda^2 + \frac{g}{l} = 0$$

with roots

$$\lambda = \pm i\omega, \quad \omega = \sqrt{\frac{g}{l}}.$$

The general solution is $x(t) = A\sin(\omega t) + B\cos(\omega t)$ where A, B are constants. We can also write this as $x(t) = R\cos(\omega t - \delta)$. Then

$$R\cos(\omega t - \delta) = R\cos(\omega t)\cos(\delta) + R\sin(\omega t)sin(\delta)$$

and thus

$$R\cos(\delta) = B, \quad R\sin(\delta) = A.$$

Hence
$$x'(t) = -\omega R \sin(\omega t - \delta)$$
and so
$$(x(t), x'(t) = (R\cos(\omega t - \delta), -\omega R \sin(\omega t - delta)).$$
These are ellipses because
$$\left(\frac{x(t)}{R}\right)^2 + \left(\frac{x'(t)}{\omega R}\right)^2 = 1$$
The fact that the solutions are periodic is revealed by the fact that the curves in the phase diagram are closed.

Example 4.5 (Pendulum Phase Diagram)**.** For the lineraized pendulum
$$\frac{d^2x}{dt^2} + \frac{g}{l}x = 0.$$
We have
$$V'(x) = \frac{g}{l}x = \omega^2 x.$$
We can take
$$V(x) = \frac{1}{2}\omega^2 x^2$$
and the energy conservation equation is
$$\frac{1}{2}(x'(t))^2 + V(x) = E$$
where E is a constant i.e
$$\frac{1}{2}\omega^{"}(x(t))^2 + \frac{1}{2}(x'(t))^2 = E.$$
Then by multiplying with 2 and dividing by $2E$ to get
$$\left(\frac{x(t)\omega}{\sqrt{2E}}\right)^2 + \left(\frac{x'(t)}{\sqrt{2E}}\right)^2 = 1$$
which is the same as the previous example with
$$R = \frac{\sqrt{2E}}{\omega}.$$
For the full pendulum equation
$$V'(x) = \frac{g}{l}\sin(x) = \omega^2 \sin(x)$$
and we take
$$V(x) = \omega^2 - \omega^2\cos(x).$$
The phase curves are
$$\frac{1}{2}(x'(t))^2 + \omega^2(1 - \cos(x)) = E.$$

We want to plot the curves, and we already have the case

$$V(x) = \frac{1}{2}x^2$$

or generally

$$V(x) = kx^2 \qquad k > 0$$

when the phase curves are ellipses. In both of these examples, $V(x) \geq 0$ everywhere so we need $E \leq 0$. When $E = 0$ we need both $y = 0$ and $v(x) = 0$. For the case $V(x) = \frac{g}{l}(1 - \cos(x))$ this gives the points $(2n\pi, 0), n \in \mathbb{Z}$. For the case $V(x) = kx^2$ we just get (0,0) when $E > 0$ is small. Thus we draw a line at height E across the graph $V(x)$. For the pendulum equation when $E \geq \frac{2g}{l}$ then $E \geq V(x)$ for all x.

Suppose we now wish to approximate the phase curves, either for the pendulum equation or any other, in a small neighborhood of a point $(x_0, 0)$ at which V has a local maximum e.g $x_0 = \pi$ for the pendulum equation with $V(x) = \frac{g}{l}(1 - \cos(x))$. We use the Taylor expansion

$$V(x) = V(x_0) + (x - x_0) + V'(x_0) + \frac{1}{2!}(x - x_0)^2 V''(x_0) + ...$$

Since V has a local maximum at x_0

$$V'(x_0) = 0$$

$$V''(x_0) \leq 0.$$

Assume $V''(x_0) < 0$ and so the energy conservation equation is

$$E = \frac{1}{2}(x')^2 + V(x) = \frac{1}{2}(x')^2 + V(x_0) + \frac{1}{2|}(x - x_0)^2 V''(x_0) + ...$$

$$\approx \frac{1}{2}(x')^2 - \frac{1}{2}\omega^2(x - x_0)^2$$

where $\omega^2 = -V''(x_0) > 0$. The curves $\frac{1}{2}y^2 - \frac{1}{2}\omega^2(x - x_0)^2 = E$ are hyperbola.

Example 4.6. Draw the phase diagram for the equation

$$\frac{d^2x}{dt^2} + 2x - 3x^2 = 0$$

show that there exists periodic solutions and obtain an integral expression for their periods.

Solution

Integral Expression for the Period of a Periodic Solution

Observe that since $\frac{1}{2}(x')^2 + V(x)$ is a constant and since $x' = 0$ when $x = \alpha$ and $x = \beta$ we have

$$V(\alpha) = v(\beta).$$

The period of the solution is the time it takes from α to β plus the time to return. By symmetry these times are equal so the period is twice the transit from α to β. Thus the period is

$$T = \int_0^T dt = 2\int_\alpha^\beta \frac{dt}{dx}dx = 2\int_\alpha^\beta \frac{dx}{x'}$$

Now

$$\frac{1}{2}(x')^2 + V(x) = V(\alpha) = v(\beta)$$

giving

$$x' = \pm\sqrt{2(V(\beta) - V(x))}.$$

From α to β, x increases so

$$x' = \sqrt{2(V(\beta) - V(x))}.$$

Thus

$$T = \sqrt{2}\int_\alpha^\beta \frac{dx}{\sqrt{V(\beta) - V(x)}}.$$