

MA1001

ELEMENTARY DIFFERENTIAL  
EQUATIONS

EXAMPLES



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# Chapter 1    Introductory Examples

**Example 1.1.** Radioactive isotope decays according to the differential

$$\frac{dm}{dt} = -km$$

where  $k > 0$  is constant.  $m$  is the mass present at time  $t$ . Find the half life such that  $m(\tau) = \frac{1}{2}m(0)$

**Example 1.2.** A student goes home for Christmas leaving a mouldy burger under the sofa. The mould grows according to

$$\frac{dm}{dt} = km$$

where  $k > 0$  is constant. Find the time  $\tau = 0$  at which  $m(\tau) > M$ , assuming  $m(0)$  is known. Here  $M$  is the mass of the sofa.

**Example 1.3.** The previous model is changed to

$$\frac{dm}{dt} = km(M - m)$$

where  $k > 0$ ,  $M > 0$  are constants. Show how the solution behaves.

**Example 1.4.** Show that the differential equation

$$\frac{dy}{dt} = 2\sqrt{y}$$

has more than one solution with :

$$y_1(t) = t^2 \text{ for } t \geq 0$$

$$y_2(t) = 0 \text{ for all } t$$

A third solution :

$$y_3(t) = \begin{cases} y_2(t) & \text{for } t \leq 0 \\ y_1(t) & \text{for } t > 0 \end{cases}$$

**Example 1.5.** Show that the solutions of the equation :

$$\frac{dy}{dt} = 1 + y^2$$

blows up in finite time.



# Chapter 2    First Order Differentials

A first order (ordinary differential equations) is an equation of the form :

$$\frac{dy}{dt} = f(t, y).$$

Here any solution  $y$  is a real or complex valued function of a single real variable and  $f$  is a function of 2 variables.

In Newtonian notation the same equation would be written as :

$$y'(t) = f(t, y(t)).$$

## 2.1    Separable Equations

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A separable equation is an equation of the form :

$$\frac{dy}{dt} = g(t)h(y).$$

Suppose that  $h$  never takes the value 0. Then we can write the equation as :

$$\frac{1}{h(y)} \frac{dy}{dt} = g(t).$$

We can also (at least on principle) find a function  $F$  whose derivative is  $\frac{1}{h}$  :

$$F' = \frac{1}{h},$$

hence :  $F'(y(t)) \frac{dy}{dt} = g(t) \implies \frac{d}{dt}[F(y(t))] = g(t).$

**Example 2.1.** Solve

$$\frac{dy}{dt} = (1 + y^2)(1 + \alpha y)$$

where  $\alpha \geq 0$  is constant.

**Solution**

Write the equation as

$$\frac{1}{1+y^2} \frac{dy}{dt} = 1 + \alpha t.$$

Here we need  $F'(y) = \frac{1}{1+y^2}$  so we choose :

$$F(y) = \tan^{-1}(y)$$

and get :

$$\frac{d}{dt}(\tan^{-1}(y(t))) = 1 + \alpha t,$$

and so by integration

$$\tan^{-1}(y(t)) = t + \alpha \frac{t^2}{2} + \beta$$

where  $\beta$  is constant. Hence :  $y(t) = \tan(t + \alpha \frac{t^2}{2} + \beta)$ . Remark : The solution of this equation always blows up in finite time even when  $\alpha = \beta = 0$ . In this case the solution blows up at  $t = \frac{\pi}{2}$ .

**Example 2.2.** Find the solution of

$$\frac{dy}{dt} = 2\sqrt{y}.$$

**Solution**

Divide by  $2\sqrt{y}$  to get

$$\frac{1}{2\sqrt{y}} \frac{dy}{dt} = 1.$$

For this case :

$$F'(y) = \frac{1}{2\sqrt{y}}$$

and so we choose  $F(y) = \sqrt{y}$  and get

$$\frac{d}{dt}(\sqrt{y(t)}) = 1.$$

So, on integration

$$\sqrt{y(t)} = t + c$$

where  $c$  is a constant. This gives  $y(t) = (t + c)^2$ . Suppose we want  $y(0) = 0$  we get :

$$0 = y(0) = (0 + c)^2 = c^2.$$



Giving  $c = 0$ . Thus  $y(t) = t^2$ .

Warning : This is not the only solution of  $\frac{dy}{dt} = 2\sqrt{y}$  which satisfies  $y(0) = 0$ , as we said previously. We could also have  $y(t) = 0$  for all  $t$ .

**Example 2.3.** Solve the equation :

$$\frac{dy}{dt} = m(1 - m).$$

**Solution**

Suppose  $m \neq 0$ ,  $m \neq 1$  Then

$$\frac{1}{m(1 - m)} \frac{dm}{dt} = 1.$$

We need to find a function  $F$  such that :

$$F'(m) = \frac{1}{m(1 - m)} = \frac{1}{m} + \frac{1}{1 - m}.$$

A suitable  $F$  is:

$$F(m) = \log(m) - \log(1 - m) = \log\left(\frac{m}{1 - m}\right)$$

and so our equation is :

$$\frac{d}{dt}[\log\left(\frac{m}{1 - m}\right)] = 1 \implies \log\left(\frac{m}{1 - m}\right) = t + c$$

where  $c$  is a constant, thus

$$\frac{m(t)}{1 - m(t)} = \exp(t + c) = \alpha \exp(t)$$

where  $\alpha = \exp(c)$  is also a constant. Suppose that  $m(0) = \mu$  then :

$$\frac{\mu}{1 - \mu} = \alpha.$$

The equation for  $m(t)$  can be rearranged as

$$\mu(t) = \frac{\alpha \exp(t)}{1 + \alpha \exp(t)} = \frac{1}{\frac{1}{\alpha} \exp(-t) + 1} = \frac{1}{\left(\frac{1 - \mu}{\mu}\right) \exp(-t) + 1} = \frac{\mu}{(1 - \mu) \exp(-t) + \mu}.$$

**Remark.** • If we take  $m(0) = \mu = 0$  we get  $m(t) = 0$  for all  $t$  then :  
 If  $0 < \mu, 1$  then  $(1 - \mu) \exp(-t) > 0$  and hence

$$0, m(t) < \frac{\mu}{0 + \mu} = 1$$

Thus

$$\frac{dm}{dt} = m(1 - m) > 0 \text{ also } \lim m(t) = 1.$$

- If we take  $m(0) = \mu = 1$  we get  $m(t) = 1$  for all  $t$  then :  
 If  $\mu > 1$  then the expression

$$m(t) = \frac{\mu}{\mu(1 - \exp(-t)) + \exp(-t)}$$

tells us that  $m(t) > 0$  whilst the expression

$$m(t) = \frac{\mu}{(1 - m) \exp(-t) + \mu}$$

tells us that  $m(t) > 1$  for all  $t \geq 0$ . Also  $\frac{dm}{dt} = m(1 - m) < 0$ , so  $m$  is strictly decreasing. Also  $\lim m(t) = 1$ .

## 2.2 Homogeneous Equations

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A homogeneous equation is an equation of the form :

$$\frac{dy}{dt} = f\left(\frac{y}{t}\right).$$

We introduce a new function  $z = \frac{y}{t}$ ; more precisely  $z(t) = \frac{y(t)}{t}$ . Hence  $y(t) = tz(t)$  which gives

$$f(z) = \frac{dy}{dt} = z(t) + t \frac{dz}{dt}.$$

Rearranging yields:

$$\frac{dz}{dt} = \frac{f(z) - z}{t} = \frac{1}{z}(f(z) - z)$$

which is now separable.

**Example 2.4.** Solve the equation :

$$t^2 \frac{dy}{dt} = ty + t^2 + y^2$$

and examine the behavior of the solution as  $t$  tends to 0.

**Solution**

Divide by  $t^2$  to get

$$\frac{dy}{dt} = \frac{y}{t} + 1 + \left(\frac{y}{t}\right)^2 = f\left(\frac{y}{t}\right)$$

where  $f(z) = z + 1 + z^2$ . We let  $y = tz$  and get

$$\frac{dz}{dt} = \frac{1}{t}(f(z) - z) = \frac{1}{t}(1 + z^2) \implies \frac{1}{1 + z^2} \frac{dz}{dt} = \frac{1}{z},$$

so

$$\frac{1}{1 + z^2} dz = \frac{1}{t} dt.$$

Integrate both sides to obtain  $\tan^{-1} = \log(t) + c$  where  $c$  is a constant of integration. This yields

$$z = \tan(c + \log(t)) \implies y = t * \tan(c + \log(t)).$$

Put  $c = \log \alpha$  where  $\alpha = \exp(c)$  so that

$$y = t * \tan(\log \alpha + \log t) = t * \tan(\log(\alpha t)).$$

This blows up whenever  $\log(\alpha t) = (2k + 1)\frac{\pi}{2}$ ,  $k \in \mathbb{Z}$  i.e

$$\alpha t = \exp((2k + 1)\frac{\pi}{2}), \text{ or } t = \frac{1}{\alpha} \exp((2k + 1)\frac{\pi}{2}).$$

Letting  $k$  decrease to  $-\infty$  through the negative integers we see that these are infinitely many points at which  $y$  blows up, in any neighborhood of zero.

**Example 2.5.** Solve the homogeneous equation :

$$\frac{dy}{dt} = \frac{y^{\frac{1}{3}}}{t}.$$

**Solution**

Put  $y = tz$ ,  $f(z) = z^{\frac{1}{3}}$ , and obtain

$$\frac{dz}{dt} = \frac{1}{t}(f(z) - z) = \frac{1}{t}(z^{\frac{1}{3}} - z).$$

This yields

$$\frac{1}{z^{\frac{1}{3}} - z} dz = \frac{1}{t} dt.$$

Alternatively, the original equation is already separable, as  $y^{-\frac{1}{3}} dy = t^{-\frac{1}{3}} dt$ .

By integration :

$$y^{\frac{2}{3}} = t^{\frac{2}{3}} + c$$

where  $c$  is a constant of integration, thus

$$y = (c + t^{\frac{2}{3}})^{\frac{3}{2}}.$$

## 2.3 Linear Equations

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A first-order differential linear equation is an equation of the form :

$$\frac{dy}{dt} = a(t)y + b(t)$$

where  $a(t)$  and  $b(t)$  are given functions of the dependant variable  $t$ . For linear equations we can always write the solution  $y$  explicitly as a function.

### Solution Procedure

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1. Choose a function  $p(t)$  such that

$$\frac{dp}{dt} = a(t)$$

2. Observe that

$$\frac{d}{dt}(ye^{-p(t)}) = \frac{dy}{dt}e^{-p(t)} + ye^{-p(t)} = (\frac{dy}{dt} - a(t)y)e^{-p(t)} = b(t)e^{-p(t)}$$

3. Integrate the equation

$$\frac{d}{dt}(ye^{-p(t)}) = b(t)e^{-p(t)}$$

This integrates to

$$ye^{-p(t)} = \int b(t)e^{-p(s)}ds + c$$

where  $c$  is a constant By doing this we get :

$$y = e^p(t)(c + \int b(s)e^{-p(s)}ds$$

**Example 2.6.** Solve the differential equation

$$\frac{dy}{dt} = ky + \sin(t)$$

subject to the condition  $y(0) = 0$ .

**Solution**

Choose  $p = \int k dt = kt$  where  $a(t) = k$  and  $b(t) = \sin(t)$ . Therefore

$$\frac{d}{dt}(e^{-kt}y) = e^{-kt} \sin(t).$$

There are at least two ways of integrating the right hand side of the equation. One is to integrate by parts and the other is to use complex exponentials, therefore :

$$\sin(t) = \text{Im}(e^{it}).$$

Hence :

$$\begin{aligned} \int e^{-kt} \sin(t) dt &\Rightarrow \int e^{-kt} \text{Im}(e^{-it} dt) = \text{Im}(\int e^{-kt+it} dt) \Rightarrow \text{Im}(\frac{e^{(i-k)t}}{(i-k)}) \\ &= \text{Im}(\frac{(-i-k)(\cos(t) + \sin(t)) * e^{-kt}}{1+k^2}) = \text{Im}(\frac{(-k \cos(t) + \sin(t) + i(-\cos(t) - k \sin(t)))e^{-kt}}{1+k^2}) \\ &= \frac{-(\cos(t) + k \sin(t))e^{-kt}}{1+k^2}. \end{aligned}$$

Therefore :

$$e^{-kt}y = -(\frac{-(\cos(t) + k \sin(t))e^{-kt}}{1+k^2})$$

where  $c$  is a constant and when  $t = 0$  and  $y = 0$  so :

$$c - \frac{1}{1+k^2} = 0 \Rightarrow c = \frac{1}{1+k^2}$$

Thus the solution is

$$y = \frac{e^{kt} - (\cos(t) + k \sin(t))}{1+k^2}.$$

**Example 2.7.** Solve the linear differential equation

$$t^2 \frac{dy}{dt} + 2ty = e^t$$

subject to  $y(1) = 1$ . Is there a solution subject to the condition  $y(0) = 1$ ?

**Solution**

We can write the equation in the form

$$\frac{dy}{dt} = \frac{-2}{t}y + \frac{e^t}{t^2}.$$

Then we find the integrating factors as before however in this case we can

simplify the equation by inspection :

$$t^2 \frac{dy}{dt} + 2ty = \frac{d}{dt}(t^2 y) = e^t.$$

Then by integration :

$$t^2 y = e^t + c \implies y = \frac{e^t + c}{t^2}$$

where  $c$  is a constant. The initial condition  $y(1) = 1$  gives you

$$1 = \frac{e + c}{1} \implies c = 1 - e.$$

Thus

$$y = \frac{e^t + 1 - e}{t^2}.$$

There is no solution which takes the value 1 when  $t = 0$ ! This is because when we write the equation as  $\frac{dy}{dt} = f(t, y)$  we have

$$\frac{dy}{dt} = \frac{-2y}{t} + \frac{e^t}{t^2}$$

and this function is badly behaved as  $t \rightarrow 0$ . The initial conditions can not be imposed at a point where the right hand side of the differential equation blows up.

**Example 2.8.** Solve the equation

$$(t \log(t)) \frac{dy}{dt} + y = 3t^3.$$

**Solution**

First we must find an integrating factor which is

$$\exp\left(\int \frac{1}{t \log(t)} dt\right).$$

Then by integration by substitution where  $u = \log(t)$  and  $\frac{du}{dt} = \frac{1}{t}$  we have that

$$\int \frac{1}{t \log(t)} dt \implies \int \frac{1}{u} du = \log(u) \implies \log(\log(t)).$$

Now the integrating factor is  $\exp(\log(\log(t)))$  which is  $\log(t)$  and therefore :

$$\log(t) \frac{dy}{dt} + \frac{y}{t} = 3t^2 \implies \frac{d}{dt}[\log(t)y] = 3t^2.$$

Then by integration you get

$$\log(t)y = \int 3t^2 dt = t^3 + c$$

where  $c$  is a constant. Thus by rearranging

$$y = \frac{t^3 + c}{\log(t)}.$$

## 2.4 Bernoulli Equations

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A Bernoulli equation is an equation in the form

$$\frac{dy}{dt} + a(t)y = b(t)y^n.$$

Where when  $n = 0$  we have a linear equation and when  $n = 1$  we have linear and separable equation. We choose an integration factor  $p(t)$  such that

$$\frac{p'(t)}{p(t)} = a(t) \text{ so that } \frac{dy}{dt} + \frac{p'(t)}{p(t)}y = b(t)y^n.$$

Now we multiply both sides by  $p(t)$  to get

$$p(t)\frac{dy}{dt} + \frac{dp}{dt}y = p(t)b(t)y^n.$$

Then by differentiation

$$\frac{d}{dt}(p(t)y(t)) = \frac{b(t)}{p(t)^{n-1}}(p(t)y(t))^n.$$

We introduce a new function  $z$  by the formula  $z(t) = p(t)y(t)$  which satisfies

$$\frac{dz}{dt} = q(t)z^n \text{ where } q(t) = \frac{b(t)}{p(t)^{n-1}}.$$

**Example 2.9.** Solve the equation

$$\frac{dy}{dt} + \frac{2}{t}y = \exp(t)y^2.$$

**Solution**

Multiply both sides by  $t^2$  to obtain

$$t^2 \frac{dy}{dt} + 2ty = t^2 \exp(t)y^2 \implies \frac{d}{dt}(t^2 y) = \frac{1}{t^2} \exp(t)(t^2 y)^2.$$

Let  $z = t^2 y$  such that

$$\frac{dz}{dt} = \frac{1}{t^2} \exp(t) z^2$$

which is now a separable equation. This separates as

$$\frac{1}{z^2} \frac{dz}{dt} = \frac{1}{t^2} \exp(t) \text{ or } \frac{d}{dt} \left( \frac{-1}{z} \right) = \frac{1}{t^2} \exp(t)$$

Any further progress depends on integrating the right hand side, probably with an incomplete gamma-function.

**Example 2.10.** Solve the equation

$$\frac{dy}{dt} = t \exp(t^2 - y).$$

**Solution**

Thanks to the property  $\exp(t^2 - y) = \frac{\exp(t^2)}{\exp(y)}$  this is a separable equation. We rearrange it as

$$\exp(y) \frac{dy}{dt} = t \exp(t^2) \implies \frac{d}{dt} \exp(y) = \frac{d}{dt} \left( \frac{1}{2} \exp(t^2) \right)$$

Thus

$$\exp(y) = \frac{1}{2} \exp(t^2) + c$$

where  $c$  is a constant. Finally

$$y = \log\left(c + \frac{1}{2} \exp(t^2)\right).$$

For the special case when  $c = 0$  we get

$$y = \log\left(\frac{1}{2} \exp(t^2)\right) \implies \log\left(\frac{1}{2}\right) + t^2 = t^2 - \log(2).$$



# Chapter 3 Second Order Equations

A second order linear differential equation is an equation of the following form :

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t).$$

The functions  $a_1(*)$ ,  $a_0(*)$  and  $f(*)$  are supposed to be known and we want to find all of the solutions to  $y$ .

**Example 3.1.** The Legendre equation is

$$\frac{d^2y}{dt^2} - \frac{2t}{1-t^2}\frac{dy}{dt} + \frac{n(n+1)}{1-t^2}Y = 0.$$

The equation has a solution  $P_n(t)$ , the Legendre polynomial of degree  $n$ .

- For  $n = 0$  we can take  $P_0(t)$  for all  $t$
- For  $n = 1$  we let  $P_1(t) = at + b$ . We get

$$\frac{-2t}{1-t^2}a + \frac{2}{1-t^2}(at + b) = 0$$

for all  $t \in \mathbb{R} \setminus \{\pm 1\}$  This gives  $b = 0$  and we can take  $a = 1$  to get  $P_1(t) = t$

**Example 3.2.** Bessel's equation of order  $n$  is :

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(1 - \frac{n^2}{t^2}\right)y = 0.$$

The solution are Bessel functions,  $J_n(t)$  and  $Y_n(t)$ .

## 3.1 Equations with Constant Coefficients

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$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = f(t)$$

In order to solve the above equation we adopt a two step solution strategy:

1. Find all the solution of the equation with  $f = 0$ . The equation when  $f = 0$  is called the homogeneous equation
2. Develop a formula called the variation of parameters formula for the case  $f \neq 0$ , using the solutions of the homogeneous equation

### Solution Method for $f = 0$

We look for a solution of the form  $y(t) = A \exp(\mu t)$ , where  $A$  and  $\mu$  are constants. We have:

$$\frac{dy}{dt} = \mu A \exp(\mu t) \quad \text{and} \quad \frac{d^2y}{dt^2} = \mu^2 A \exp(\mu t).$$

Substituting into the differential equation gives:

$$(\mu^2 + a_1\mu + a_0) * A \exp(\mu t) = 0.$$

Since  $\exp(\mu t) \neq 0$  and we do not want the trivial solution which comes from choosing  $A = 0$ , we have  $\mu^2 + a_1\mu + a_0 = 0$  this is called the auxiliary quadratic. The possibilities are:

- Two distinct roots
- One double root

If  $a_1$  and  $a_0$  are real there is an alternative dichotomy:

- Two real roots (which may or may not coincide)
- A complex conjugate pair of distinct roots

#### Case 1 : Distinct Roots $\mu_1 \neq \mu_2$

$$\mu^2 + a_1\mu + a_0 = (\mu - \mu_1)(\mu - \mu_2).$$

We have the solutions

$$y_j(t) = A_j \exp(\mu_j t) \quad j = 1, 2.$$

For completely arbitrary constants  $A_1, A_2$ . Suppose  $y(t) = y_1(t) + y_2(t)$  then

$$\frac{dy}{dt} = \frac{dy_1}{dt} + \frac{dy_2}{dt} \quad \text{and} \quad \frac{d^2y}{dt^2} = \frac{d^2y_1}{dt^2} + \frac{d^2y_2}{dt^2}.$$

So,

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = \left[ \frac{d^2y_1}{dt^2} + a_1 \frac{dy_1}{dt} + a_0 y_1 \right] + \left[ \frac{d^2y_2}{dt^2} + a_1 \frac{dy_2}{dt} + a_0 y_2 \right].$$

**Theorem 3.3.** Suppose that the auxiliary quadratic has a distinct root  $\mu_1 \neq \mu_2$ . Then every solution of the homogeneous equation has the form

$$y(t) = A_1 \exp(\mu_1 t) + A_2 \exp(\mu_2 t)$$

for some  $A_1, A_2 \in \mathbb{C}$ .

#### Example 3.4.

$$\frac{d^2y}{dt^2} - (\mu_1 + \mu_2) \frac{dy}{dt} + \mu_1 \mu_2 y = 0$$

such that  $y(0) = y_0$  and  $y'(0) = v_0$ .

**Solution**

We know  $y(t) = A_1 \exp(\mu_1 t) + A_2 \exp(\mu_2 t)$  and we choose  $A_1, A_2$  so that:

$$\begin{cases} y_0 = y(0) = A_1 + A_2 \\ v_0 = y'(0) = A_1 \mu_1 + A_2 \mu_2 \end{cases}$$

Hence

$$\begin{aligned} \mu_2 y_0 - v_0 &= A_1(\mu_2 - \mu_1) \implies \\ \begin{cases} A_1 = \frac{\mu_2 y_0 - v_0}{\mu_2 - \mu_1} \\ A_2 = y_0 - A_1 = \frac{v_0 - \mu_1 y_0}{\mu_2 - \mu_1} \end{cases} \end{aligned}$$

Thus

$$y_1(t) = \frac{\mu_2 y_0 - v_0}{\mu_2 - \mu_1} \exp(\mu_1 t) + \frac{v_0 - \mu_1 y_0}{\mu_2 - \mu_1} \exp(\mu_2 t) = y_0 \left( \frac{\mu_2 \exp(\mu_1 t) - \mu_1 \exp(\mu_2 t)}{\mu_2 - \mu_1} \right) + v_0 \left( \frac{\exp(\mu_2 t) - \exp(\mu_1 t)}{\mu_2 - \mu_1} \right)$$

**Example 3.5.** In the previous example let  $\mu_1$  be fixed and find  $\lim_{\mu_2 \rightarrow \mu_1} y(t)$ .

**Solution**

First we compute

$$\lim_{\mu_2 \rightarrow \mu_1} \left[ \frac{\exp(\mu_2 t) - \exp(\mu_1 t)}{\mu_2 - \mu_1} \right] = t \exp(\mu_1 t) \lim_{\mu_2 \rightarrow \mu_1} \left[ \frac{\exp((\mu_2 - \mu_1)t) - 1}{(\mu_2 - \mu_1)t} \right].$$

Now set  $h = \mu_2 - \mu_1$ . Thus :

$$\begin{aligned} & t \exp(\mu_1 t) \lim_{h \rightarrow 0} \left[ \frac{\exp(h) - 1}{h} \right] \\ &= t \exp(\mu_1 t) \lim_{h \rightarrow 0} \left[ \frac{\exp(h) - \exp(0)}{h} \right] = t \exp(\mu_1 t) \exp'(0). \end{aligned}$$

Where  $\exp'(0) = 1$ . Hence

$$\lim_{\mu_2 \rightarrow \mu_1} \left( \frac{\exp(\mu_2 t) - \exp(\mu_1 t)}{\mu_2 - \mu_1} \right) = t \exp(\mu_1 t).$$

Next we compute:

$$\lim_{\mu_2 \rightarrow \mu_1} \left( \frac{\mu_2 \exp(\mu_1 t) - \mu_1 \exp(\mu_2 t)}{\mu_2 - \mu_1} \right).$$

We observe that:

$$\frac{\mu_2 \exp(\mu_1 t) - \mu_1 \exp(\mu_2 t)}{\mu_2 - \mu_1} = \left[ \frac{\mu_1 (\exp(\mu_2 t) - \exp(\mu_1 t)) + (\mu_2 - \mu_1) \exp(\mu_1 t)}{\mu_2 - \mu_1} \right]$$

$$= \exp(\mu_1 t) - \mu_1 \left[ \frac{\exp(\mu_2 t) - \exp(\mu_1 t)}{\mu_2 - \mu_1} \right].$$

Thus

$$\lim_{\mu_2 \rightarrow \mu_1} \left( \frac{\mu_2 \exp(\mu_1 t) - \mu_1 \exp(\mu_1 t)}{\mu_2 - \mu_1} \right) = \exp(\mu_1 t) - \mu_1 t \exp(\mu_1 t).$$

Hence

$$\lim_{\mu_2 \rightarrow \mu_1} y(t) = y_0(1 - \mu_1) \exp(\mu_1 t) + v_0 t \exp(\mu_1 t).$$

Let

$$z(t) = y_0(1 - \mu_1 t) \exp(\mu_1 t) + v_0 t \exp(\mu_1 t).$$

Then

$$\begin{cases} z(0) = y_0 \\ z'(t) = y_0(-\mu_1 + \mu_1(1 - \mu_1 t)) \exp(\mu_1 t) + v_0(1 + \mu_1 t) \exp(\mu_1 t) \end{cases}$$

giving  $z'(0) = v_0$ . Let  $z(t) = (At + B) \exp(\mu_1 t)$ , where  $A$  and  $B$  are constants. We want to check that

$$\frac{d^2 z}{dt^2} - 2\mu_1 \frac{dz}{dt} + \mu_1^2 z = 0.$$

We have

$$\begin{aligned} z(t) = (At + B) \exp(\mu_1 t) &\implies \frac{dz}{dt} = (A + \mu_1(At + B)) \exp(\mu_1 t) = (\mu_1 At + (A + \mu_1 B)) \exp(\mu_1 t) \\ &\implies \frac{d^2 z}{dt^2} = (\mu_1 A + (\mu_1 A + \mu_1(A + \mu_1 B)) \exp(\mu_1 t). \end{aligned}$$

Hence

$$\frac{d^2 z}{dt^2} - 2\mu_1 \frac{dz}{dt} + \mu_1^2 z = ((\mu_1^2 A - 2\mu_1 * \mu_1 A + \mu_1^2 A)t + 2\mu_1 A + \mu_1^2 B - 2\mu_1(A + \mu_1 B) + \mu_1^2 - 1B) \exp(\mu_1 t) = 0$$

**Theorem 3.6.** Every solution of the equation

$$\frac{d^2 y}{dt^2} - 2\omega \frac{dy}{dt} + \omega^2 y = 0$$

has the form

$$y(t) = (At + B) \exp(\omega t)$$

for some appropriate constants  $A, B$ .

**Example 3.7.** Find the general solution of

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0$$

where  $\omega > 0$  is constant.

**Solution**

We look for a solution of the form

$$y(t) = A \exp(\mu t)$$

and obtain the auxiliary quadratic  $\mu^2 + \omega^2 = 0$ , which has roots  $\mu_1 = i\omega$ ,  $\mu_2 = -i\omega$ . Thus the general solution has the form

$$\begin{aligned} y(t) &= A \exp(i\omega t) + B \exp(-i\omega t) \\ &= A(\cos(\omega t) + i \sin(\omega t)) + B(\cos(\omega t) - i \sin(\omega t)) \\ &= (A + B) \cos(\omega t) + i(A - B) \sin(\omega t). \end{aligned}$$

Suppose  $y(0) = y_0 \in \mathbb{R}$  and  $y'(0) = v_0 \in \mathbb{R}$ . Then

$$\begin{cases} y(0) = A + B = y_0 \\ y'(0) = i\omega(A - B) = v_0 \end{cases}$$

Thus  $y(t) = y_0 \cos(\omega t) + v_0 \frac{\sin(\omega t)}{\omega}$ .

**Remark.** Observe that  $\lim_{\omega \rightarrow 0} \cos(\omega t) = \cos(0) = 1$  and  $\lim_{\omega \rightarrow 0} \frac{\sin(\omega t)}{\omega} = t$ . When  $\omega = 0$ , we get

$$y(t) = y_0 + v_0 t$$

which does indeed solve  $\frac{d^2 y}{dt^2} = 0$ .

**Definition 3.8.** Let  $y_1, y_2$  be solutions of

$$\frac{d^2 y}{dt^2} + a_1(t) \frac{dy}{dt} + a_0(t)y = 0.$$

We say that  $y_1$  and  $y_2$  are linearly independent if and only if neither solution can be written as a constant multiple of the other.

**Example 3.9.** In the previous example,

$$y_1(t) = \cos(\omega t) \text{ and } y_2 = \frac{\sin(\omega t)}{\omega}$$

are linearly independent: so are

$$\begin{aligned} y_1(t) &= \exp(i\omega t) \text{ and } y_2(t) = \exp(-i\omega t) \\ y_1(t) &= \cos(\omega t) \text{ and } y_2(t) = \exp(i\omega t). \end{aligned}$$

However if  $y_1(t) = \cos(\omega t)$  and  $y_2(t) = 17 \cos(\omega t)$ , are not linearly independent.

**Definition 3.10.** Let  $y_1$  and  $y_2$  be two differentiable functions. The Wronskian (determinant) of  $y_1$  and  $y_2$  is the function

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2$$

## Objective

We want to find a formula for the general solution of

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = f$$

where the function  $f$  on the right hand side is non-trivial. We assume that we know two linearly independent solutions  $y_1$  and  $y_2$  of the homogeneous equation

$$\frac{d^2y}{dt^2} j + a_1 \frac{dy}{dt} j + a_0 y j = 0 \text{ for } j = 1, 2.$$

We can look for a solution in the form

$$y(t) = A_1(t) + A_2(t)y_2(t).$$

Differentiating,

$$y'(t) = A_1(t)y_1'(t) + A_2(t)y_2'(t) + A_1'(t)y_1(t) + A_2'(t)y_2(t).$$

We impose the condition  $A_1'y_1 + A_2'y_2 = 0$ . Then we obtain by the second derivative

$$y''(t) = A_1(t)y_1''(t) + A_2(t)y_2''(t) + A_1'(t)y_1'(t) + A_2'(t)y_2'(t).$$

Combining the expressions for  $y$ ,  $y'$  and  $y''$ , we obtain

$$y'' + a_1 y' + a_0 y = A_1(y_1'' + a_1 y_1' + a_0 y_1) + A_2(y_2'' + a_1 y_2' + a_0 y_2) + A_1'y_1' + A_2'y_2' = A_1'y_1' + A_2'y_2'$$

because  $y_1$  and  $y_2$  solve the homogeneous equation. Using the original formula we see that our second equation for  $A_1$  and  $A_2$  is

$$A_1'y_1' + A_2'y_2' = f.$$

We write the equation for  $A_1$  and  $A_2$  as

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which yields

$$\begin{bmatrix} A_1' \\ A_2' \end{bmatrix} = \frac{1}{W} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix} = \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix}$$

This means

$$\begin{cases} A_1' = \frac{-y_2 f}{W} \\ A_2' = \frac{y_1 f}{W} \end{cases}$$

giving

$$\begin{cases} A_1(t) = \int_{t_0}^t \frac{-y_2(x)f(x)}{W(x)} dx + \alpha_1 \\ A_2(t) = \int_{t_0}^t \frac{y_1(x)f(x)}{W(x)} dx + \alpha_2 \end{cases}$$

Where  $\alpha_1$  and  $\alpha_2$  are constants of integration and  $t_0$  can be chosen. Recalling that

$$y(t) = A_1(t)y_1(t) + A_2(t)y_2(t)$$

we obtain

$$y(t) = \int_{t_0}^t \left( \frac{y_1(x)y_2(t) - y_1(t)y_2(x)}{W(x)} \right) f(x)dx + \alpha_1 y_1(t) + \alpha_2 y_2(t).$$

This formula is called the variation of parameters or variation of constants formula.

- $y$  is the general solution of

$$y'' + a_1 y' + a_0 y = f$$

- $a_1, a_0$  and  $f$  are given functions.
- $y_1, y_2$  are the solutions of the homogeneous equation

$$y_j'' + a_1 y_j' + a_0 y_j = 0, \quad j = 1, 2$$

- $W$  is the Wronskian

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_1' y_2$$

**Example 3.11.** Find the general solution of the equation

$$y'' - y = 1.$$

### Solution

The corresponding homogeneous equation is  $y'' - y = 0$ , with auxiliary quadratic  $\lambda^2 - 1 = 0$  having roots  $\lambda = \pm 1$ . Thus the functions

$$\begin{cases} y_1(x) = \exp(+1x) \\ y_2(x) = \exp(-1x) \end{cases}$$

are two linearly independent solutions of the homogeneous equation. Hence

$$W = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} \exp(x) & \exp(-x) \\ \exp(x) & -\exp(-x) \end{bmatrix} = -2.$$

For this example  $f(x) = 1$ . The variation of parameters formula gives

$$y(t) = \int_t^{t_0} \frac{\exp(x) \exp(-t) - \exp(t) \exp(-x)}{(-2)} * 1 dx + \alpha_1 \exp(t) + \alpha_2 \exp(-t)$$

and we chose  $t_0 = 0$  for convenience. Thus

$$y(t) = \frac{-1}{2} \exp(-t) \int_0^t \exp(x) dx + \frac{1}{2} \exp(t) \int_0^t \exp(-x) dx + \alpha_1 \exp(t) + \alpha_2 \exp(-t)$$

which yields

$$\begin{aligned} y(t) &= \frac{-1}{2} \exp(-t) [\exp(t) - 1] + \frac{1}{2} \exp(t) [1 - \exp(-t)] + \alpha_1 \exp(t) + \alpha_2 \exp(-t) \\ &= -1 + \left(\alpha_1 + \frac{1}{2}\right) \exp(t) + \left(\alpha_2 + \frac{1}{2}\right) \exp(-t). \end{aligned}$$

Hence

$$y(t) = -1 + A_1 \exp(t) + A_2 \exp(-t).$$

Where  $A_1$  and  $A_2$  are constants.

**Example 3.12.** Find the general solution of

$$\frac{d^2 y}{dt^2} - 4 \frac{dy}{dt} + 4y = t \exp(2t).$$

**Solution**

Our auxiliary quadratic is

$$\lambda^2 - 4\lambda + 4 = 0 \text{ or } (\lambda - 2)^2 = 0.$$

One solution of the homogeneous equation is  $y_1(t) = \exp(2t)$  and the second is  $y_2(t) = t \exp(2t)$ . Thus

$$W = \begin{bmatrix} \exp(2t) & t \exp(2t) \\ 2 \exp(2t) & (1 + 2t) \exp(2t) \end{bmatrix} = \exp(4t)$$

Also  $f(t) = t \exp(2t)$ . The variation of parameters formula gives

$$\begin{aligned} y(t) &= \int_0^t \frac{\exp(2X) t \exp(2t) - \exp(2x) x \exp(2t)}{\exp(4(x))} x \exp(2x) dx + (\alpha_1 + \alpha_2 t) \exp(2t) \\ &= \int_0^t (tx \exp(2t) - x^2 \exp(2t)) dx + (\alpha_1 + \alpha_2 t) \exp(2t) \\ &= t \exp(2t) \left[ \frac{x^2}{2} \right]_0^t - \left[ \frac{x^3}{3} \right]_0^t \exp(2t) + (\alpha_1 + \alpha_2 t) \exp(2t). \end{aligned}$$

Finally

$$y(t) = \frac{1}{6} t^3 \exp(2t) + (\alpha_1 + \alpha_2 t) \exp(2t).$$



**Theorem 3.13.** Let  $y_1$  and  $y_2$  be linearly independent solutions of

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0$$

and let  $y_p$  be any solution of the equation

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = f,$$

then every solution  $y$  of this equation has the form

$$y(t) = y_p(t) + A_1 y_1(t) + A_2 y_2(t)$$

for the appropriate constants  $A_1$  and  $A_2$ , furthermore for every pair of constants  $A_1$   $A_2$  the function  $y$  defined above and solves the homogeneous equation. The Proof for this will be given in year 2.

### Guidelines for Guessing $y_p$

1. If  $f(t) = \alpha \exp(\beta t)$ , where  $\beta$  is not a root of the auxiliary quadratic, then there exists a particular solution  $y_p$  of the form

$$y_p(t) = \gamma \exp(\beta t).$$

This is proven below.

$$\frac{d^2 y_p}{dt^2} = \beta^2 y_p$$

and so

$$(\beta^2 + a_1 \beta + a_0) y_p = \alpha \exp(\beta t)$$

by canceling the common factor  $\exp(\beta t)$  we get

$$(\beta^2 + a_1 \beta + a_0) \gamma = \alpha \neq 0$$

since  $\beta$  is not a root of the auxiliary quadratic. Hence,

$$\gamma = \alpha(\beta^2 + a_1 \beta + a_0).$$

2. Suppose that the auxiliary quadratic has distinct root  $\lambda_1 \neq \lambda_2$  and that

$$f(t) = \alpha \exp(\lambda_1 t)$$

then there exists a solution of the form

$$y_p(t) = \gamma t \exp(\lambda_1 t).$$

This again is proven below.

$$\frac{dy_p}{dt} = \gamma \exp(\lambda_1 t) [1 + \lambda_1 t]$$

$$\frac{d^2 y_p}{dt^2} = \gamma \exp(\lambda_1 t) [2\lambda_1 + \lambda_1^2 t]$$

Hence,

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + a_1 \frac{dy_p}{dt} + a_0 y_p &= \gamma \exp(\lambda_1 t) [2\lambda_1 + \lambda_1^2 t + a_1(\lambda_1 t + 1) + a_0 t] \\ &= \gamma \exp(\lambda_1 t) [t(\lambda_1^2 + a_1 \lambda_1 + a_0) + 2\lambda_1 + 1 + a_1] \end{aligned}$$

we want this to be equal to

$$f(t) = \alpha \exp(\lambda_1 t)$$

which as

$$(\lambda_1^2 + a_1 \lambda_1 + a_0) = 0$$

this holds precisely when

$$\gamma(2\lambda_1 + a_1) = \alpha$$

as  $a_1 = -\lambda_1 - \lambda_2$  and as a result

$$\gamma(\lambda_1 - \lambda_2) = \alpha$$

so

$$\gamma = \frac{\alpha}{(\lambda_1 - \lambda_2)}.$$

3. This deals with the case when the auxiliary quadratic has roots  $\lambda_1 = \lambda_2$  and

$$f(t) = \alpha t + \mu \exp(\lambda_1 t)$$

this particular solution has the form

$$y_p(t) = q(t) \exp(\lambda_1 t)$$

of the form

$$q(t) = at^3 + bt^2$$

4. If

$$f(t) = \alpha_1 \sin(\beta_1 t) + \alpha_2 \cos(\beta_2 t)$$

where  $\beta_1$  and  $\beta_2$  are real numbers and  $i\beta_1$  and  $i\beta_2$  are not solutions of the auxiliary quadratic. Then there exists a solution

$$y_p(t) = [a_1 \sin(\beta_1 t) + c_1 \cos(\beta_1 t) + a_2 \sin(\beta_2 t) + c_2 \cos(\beta_2 t)]$$

where  $a_1, c_1, a_2, c_2$  are to be found.

## 3.2 Coupled Linear Systems

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We are interested in equations of the form

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

in which  $a$ ,  $b$ ,  $c$  and  $d$  are constant.

### Solution Method 1

Reduce to a single, second order equation. Differentiating the second equation you get

$$\begin{aligned}\frac{d^2y}{dt^2} &= c \frac{dx}{dt} + d \frac{dy}{dt} = c(ax + by) + d \frac{dy}{dt} \\ &= acx + bcy + d \frac{dy}{dt}.\end{aligned}$$

From our second equation, if  $c \neq 0$  then

$$x = \frac{1}{c} \frac{dy}{dt} - \frac{d}{c} y$$

and so

$$\frac{d^2y}{dt^2} = a \frac{dy}{dt} - a d y + b c y + d \frac{dy}{dt}$$

or equivalently

$$\frac{d^2y}{dt^2} - (a + d) \frac{dy}{dt} + (ad - bc)y = 0.$$

If we write this system as

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} =: A$$

Then

$$\frac{d^2y}{dt^2} - \text{trace}(A) \frac{dy}{dt} + \det(A)y = 0.$$

If  $c \neq 0$  then  $x$  is obtained from

$$x = \frac{1}{c} \frac{dy}{dt} - \frac{d}{c} y.$$

If  $c = 0$  and  $b \neq 0$  then eliminate  $y$  and obtain

$$\frac{d^2x}{dt^2} - \text{trace}(A) \frac{dx}{dt} + \det(A)x = 0.$$

Then if  $c = 0 = b$  and

$$\begin{aligned}\frac{dx}{dt} &= ax \\ \frac{dy}{dt} &= dy\end{aligned}$$

therefore they are no longer coupled.

## Solution Method 2

Let  $z = \begin{bmatrix} x \\ y \end{bmatrix}$  so that

$$\frac{dz}{dt} = Az, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We look for a solution  $z(t) = v \exp(\lambda t)$  where  $v$  is a constant vector and  $\lambda \in \mathbb{C}$ . Then

$$\frac{dz}{dt} = v\lambda \exp(\lambda t) \implies Az = Av \exp(\lambda t)$$

which is equivalent to

$$Av = \lambda v.$$

This means that  $v$  must be an eigenvector of  $A$  with eigenvalue  $\lambda$ . The eigenvalues of a matrix  $A$  satisfy

$$\det(A - \lambda I) = 0$$

or

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

this is the same as the auxiliary quadratic of second order equation arising from method one.

**Example 3.14.** Solve the coupled linear equations;

$$\frac{dz}{dt} = Az, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

### Solution

The eigenvalues of  $A$  solve,

$$\lambda^2 + 4\lambda + 3 = 0$$

this factorises to give  $\lambda = -3$  and  $-1$  then we need the eigenvector for each, first  $\lambda = -1$  this eigenvector satisfies

$$\begin{bmatrix} -2 - (-1) & 1 \\ 1 & 2 - (-1) \end{bmatrix} v = 0$$

so we take

$$v = \alpha \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \alpha \in \mathbb{C}.$$

Some true for  $\lambda = 3$  that gives

$$v = \alpha \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \alpha \in \mathbb{C}$$



# Chapter 4    Conservative Equations

**Definition 4.1.** A conservative equation is an equation of the form

$$\frac{d^2x}{dt^2} + V'(x) = 0.$$

**Remark.** This means that we can consider any equation

$$\frac{d^2x}{dt^2} + F(x) = 0$$

in which we can find an indefinite integral of  $F$ .

## Trick for Conservative Equation

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Multiply the equation by  $\frac{dx}{dt}$  to obtain

$$\frac{dx}{dt} \frac{d^2x}{dt^2} + V'(x) \frac{dx}{dt} = 0$$

or by the chain rule,

$$\frac{d}{dt} \left[ \frac{1}{2} \left( \frac{dx}{dt} \right)^2 + V(x) \right] = 0.$$

Hence

$$\frac{1}{2} \left( \frac{dx}{dt} \right)^2 + V(x) = E$$

where  $E$  is a constant. We could rearrange this as

$$\frac{dx}{dt} = \pm \sqrt{2E - 2V(x)}.$$

This is an infinite family (parametrized by  $E$ ) of separable first order equations. Unfortunately this usually does not help.

**Example 4.2.** Consider a pendulum of length  $l$  with all the concentrated at the end. The equation governing the motion of the pendulum is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin(\theta) = 0$$

where  $g$  is the acceleration due to gravity. Prove that there exists  $T > 0$  such that  $\theta(t + T) = \theta(t)$  for all  $t$ . The least such  $T$  is called the period of  $\theta$ . So

$$V'(\theta) = \frac{g}{l} \sin(\theta)$$

and without loss of generality we take

$$V(\theta) = \frac{-g}{l} \cos(\theta)$$

and the energy conservation equation becomes

$$\frac{1}{2} \left( \frac{dx}{dt} \right)^2 - \frac{g}{l} \cos(\theta) = E.$$

In the special case where  $E = \frac{g}{l}$  we get

$$\frac{1}{2} \left( \frac{dx}{dt} \right)^2 = \frac{g}{l} (1 + \cos(\theta)) = \frac{2g}{l} \cos^2 \left( \frac{\theta}{2} \right).$$

However this is the only cases where an explicit solution is possible. For the other cases we shall plot the curves in  $\mathbb{R}^2$  given by  $t \mapsto (\theta(t), \frac{d\theta}{dt})$ .

**Definition 4.3.** Consider a conservative equation

$$\frac{d^2x}{dt^2} + V'(x) = 0.$$

The phase diagram for this equation is the set of parametric curves

$$t \rightarrow (x(t), x'(t))$$

where  $x$  solves the differential equation.

**Example 4.4** (Linearized Pendulum). From the previous example

$$\frac{d^2x}{dt^2} + \frac{g}{l}x = 0$$

has an auxiliary quadratic

$$\lambda^2 + \frac{g}{l} = 0$$

with roots

$$\lambda = \pm i\omega, \quad \omega = \sqrt{\frac{g}{l}}.$$

The general solution is  $x(t) = A \sin(\omega t) + B \cos(\omega t)$  where  $A, B$  are constants. We can also write this as  $x(t) = R \cos(\omega t - \delta)$ . Then

$$R \cos(\omega t - \delta) = R \cos(\omega t) \cos(\delta) + R \sin(\omega t) \sin(\delta)$$

and thus

$$R \cos(\delta) = B, \quad R \sin(\delta) = A.$$



Hence

$$x'(t) = -\omega R \sin(\omega t - \delta)$$

and so

$$(x(t), x'(t)) = (R \cos(\omega t - \delta), -\omega R \sin(\omega t - \delta)).$$

These are ellipses because

$$\left(\frac{x(t)}{R}\right)^2 + \left(\frac{x'(t)}{\omega R}\right)^2 = 1$$

The fact that the solutions are periodic is revealed by the fact that the curves in the phase diagram are closed.

**Example 4.5** (Pendulum Phase Diagram). For the linearized pendulum

$$\frac{d^2x}{dt^2} + \frac{g}{l}x = 0.$$

We have

$$V'(x) = \frac{g}{l}x = \omega^2 x.$$

We can take

$$V(x) = \frac{1}{2}\omega^2 x^2$$

and the energy conservation equation is

$$\frac{1}{2}(x'(t))^2 + V(x) = E$$

where  $E$  is a constant i.e

$$\frac{1}{2}\omega^2 (x(t))^2 + \frac{1}{2}(x'(t))^2 = E.$$

Then by multiplying with 2 and dividing by  $2E$  to get

$$\left(\frac{x(t)\omega}{\sqrt{2E}}\right)^2 + \left(\frac{x'(t)}{\sqrt{2E}}\right)^2 = 1$$

which is the same as the previous example with

$$R = \frac{\sqrt{2E}}{\omega}.$$

For the full pendulum equation

$$V'(x) = \frac{g}{l} \sin(x) = \omega^2 \sin(x)$$

and we take

$$V(x) = \omega^2 - \omega^2 \cos(x).$$

The phase curves are

$$\frac{1}{2}(x'(t))^2 + \omega^2(1 - \cos(x)) = E.$$

We want to plot the curves, and we already have the case

$$V(x) = \frac{1}{2}x^2$$

or generally

$$V(x) = kx^2 \quad k > 0$$

when the phase curves are ellipses. In both of these examples,  $V(x) \geq 0$  everywhere so we need  $E \leq 0$ . When  $E = 0$  we need both  $y = 0$  and  $v(x) = 0$ . For the case  $V(x) = \frac{g}{l}(1 - \cos(x))$  this gives the points  $(2n\pi, 0), n \in \mathbb{Z}$ . For the case  $V(x) = kx^2$  we just get  $(0,0)$  when  $E > 0$  is small. Thus we draw a line at height  $E$  across the graph  $V(x)$ . For the pendulum equation when  $E \geq \frac{2g}{l}$  then  $E \geq V(x)$  for all  $x$ .

Suppose we now wish to approximate the phase curves, either for the pendulum equation or any other, in a small neighborhood of a point  $(x_0, 0)$  at which  $V$  has a local maximum e.g  $x_0 = \pi$  for the pendulum equation with  $V(x) = \frac{g}{l}(1 - \cos(x))$ . We use the Taylor expansion

$$V(x) = V(x_0) + (x - x_0)V'(x_0) + \frac{1}{2!}(x - x_0)^2V''(x_0) + \dots$$

Since  $V$  has a local maximum at  $x_0$

$$V'(x_0) = 0$$

$$V''(x_0) \leq 0.$$

Assume  $V''(x_0) < 0$  and so the energy conservation equation is

$$\begin{aligned} E &= \frac{1}{2}(x')^2 + V(x) = \frac{1}{2}(x')^2 + V(x_0) + \frac{1}{2!}(x - x_0)^2V''(x_0) + \dots \\ &\approx \frac{1}{2}(x')^2 - \frac{1}{2}\omega^2(x - x_0)^2 \end{aligned}$$

where  $\omega^2 = -V''(x_0) > 0$ . The curves  $\frac{1}{2}y^2 - \frac{1}{2}\omega^2(x - x_0)^2 = E$  are hyperbola.

**Example 4.6.** Draw the phase diagram for the equation

$$\frac{d^2x}{dt^2} + 2x - 3x^2 = 0$$

show that there exists periodic solutions and obtain an integral expression for their periods.

**Solution**

Here  $V'(x) = 2x - 3x^2$  so  $V(x) = x^2 - x^3$  is a suitable choice for  $v$ . The energy equation is

$$\frac{1}{2}(x')^2 + x^2 - x^3 = E$$

## Integral Expression for the Period of a Periodic Solution

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Observe that since  $\frac{1}{2}(x')^2 + V(x)$  is a constant and since  $x' = 0$  when  $x = \alpha$  and  $x = \beta$  we have

$$V(\alpha) = V(\beta).$$

The period of the solution is the time it takes from  $\alpha$  to  $\beta$  plus the time to return. By symmetry these times are equal so the period is twice the transit from  $\alpha$  to  $\beta$ . Thus the period is

$$T = \int_0^T dt = 2 \int_{\alpha}^{\beta} \frac{dt}{dx} dx = 2 \int_{\alpha}^{\beta} \frac{dx}{x'}$$

Now

$$\frac{1}{2}(x')^2 + V(x) = V(\alpha) = V(\beta)$$

giving

$$x' = \pm \sqrt{2(V(\beta) - V(x))}.$$

From  $\alpha$  to  $\beta$ ,  $x$  increases so

$$x' = \sqrt{2(V(\beta) - V(x))}.$$

Thus

$$T = \sqrt{2} \int_{\alpha}^{\beta} \frac{dx}{\sqrt{V(\beta) - V(x)}}.$$