MA2003 Complex Analysis
Solutions to Exercise Sheet 3

1. Find antiderivatives for the following functions:

   (a) $f(z) = \alpha + \beta(z - z_0)$,

   (b) $f(z) = (z - z_0)^n$,

   where α, β and $z_0 \in \mathbb{C}$ are constants and n is an integer, $n \neq -1$. Does $g(z) = (z - z_0)^{-1}$ have an antiderivative on $\mathbb{C}\setminus\{z_0\}$? Question 6 on Exercise Sheet 2 may help here.

Solution:

(a) An antiderivative for f on $\mathbb{C}$ is given by $F : \mathbb{C} \to \mathbb{C}$ where

$$F(z) = \alpha z + \frac{1}{2}\beta z^2 - \beta z_0 z,$$

since

$$F'(z) = \alpha + (\frac{1}{2}\beta)(2z) - \beta z_0 = \alpha + \beta(z - z_0) = f(z)$$

for all $z \in \mathbb{C}$.

(b) For $n \geq 0$ (respectively $n < -1$), and antiderivative for f on $\mathbb{C}$ (respectively $\mathbb{C}\setminus\{z_0\}$) is given by F where

$$F(z) = \frac{1}{n+1}(z - z_0)^{n+1}.$$

The function g does not have an antiderivative on $\mathbb{C}\setminus\{0\}$, since if it did, the Fundamental Theorem of Calculus would imply that

$$\int_\Gamma g = 0$$

where Γ is the anticlockwise circular contour with centre 0 and radius 1. We know from Exercise Sheet 2 that this integral has the value $2\pi i \neq 0$.

2. Evaluate the following contour integrals:

$$\int_\mathcal{C} z^3 \quad \text{and} \quad \int_\mathcal{C} \frac{1}{z^2}$$

   along $\mathcal{C}$ where $\mathcal{C}$ is

   (a) any contour from i to -2, and

   (b) any closed contour.

   For the second integral, you may assume that $\mathcal{C}$ does not contain 0.

Solution: An antiderivative for $f(z) = z^3$ on $\mathbb{C}$ is given by the function F with $F(z) = \frac{1}{4}z^4$. Hence by the Fundamental Theorem of Calculus we have

$$\int_\mathcal{C} z^3 = F(-2) - F(i) = \frac{1}{4}(-2)^4 - \frac{1}{4}(i)^4 = \frac{15}{4}$$

where $\mathcal{C}$ is any contour from i to -2, and

$$\int_{\mathcal{C}} z^3 = 0$$

when $\mathcal{C}$ is any closed contour.
Similarly $G(z) = \frac{-1}{z}$ is an antiderivative for $g(z) = \frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$, and $\mathbb{C} \setminus \{0\}$ is a region in $\mathbb{C}$ (it is open and connected). Hence by the Fundamental Theorem of Calculus we have

$$\int_{\mathcal{C}} \frac{1}{z} = G(-2) - G(i) = \frac{-1}{-2} - \frac{-1}{i} = \frac{1}{2} - i$$

for any contour $\mathcal{C}$ in $\mathbb{C} \setminus \{0\}$ from i to -2, and

$$\int_{\mathcal{C}} \frac{1}{z} = 0$$

for any closed contour $\mathcal{C}$ in $\mathbb{C} \setminus \{0\}$.

3. Let U be a region in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be holomorphic on U with $f(z)$ real-valued for all $z \in U$. Prove that f is constant.

Solution: Write f in the form

$$f(x + iy) = u(x, y) + iv(x, y)$$

where u and v are real-valued functions of two real variables. If $f(z)$ is real valued then $v(x, y) = 0$ on U, hence $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$ on U.
Since f is holomorphic, u and v satisfy the Cauchy-Riemann equations, it follows that $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 0$ on U, hence

$$f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$$

on U.
Since U is connected, this implies that f is constant by the Fundamental Theorem of Calculus.

4. Find an upper estimate for

$$\int_{\mathcal{C}} \frac{1}{1 + z^4},$$

where $\mathcal{C}$ is the upper semicircular contour from R to $-R$ given by $\gamma : [0, \pi] \to \mathbb{C}$, $\gamma(t) = R\cos(t) + iR\sin(t)$.

Solution: We shall do this using the Estimation Lemma. In order to apply the Lemma, we need to find $\ell\Gamma$ and an upper bound for $\left|\frac{1}{1+z^4}\right|$ along $\mathcal{C}$. Now, we know that

$$\gamma'(t) = -R\sin(t) + iR\cos(t)$$

with modulus

$$\left|\gamma'(t)\right| = \sqrt{(-R\sin(t))^2 + (R\cos(t))^2} = \sqrt{R^2(\sin^2(t) + \cos^2(t))} = R$$

for all $t \in [0, \pi]$. Hence the length of $\mathcal{C}$ is given by

$$\ell(\mathcal{C}) = \int_0^\pi \left|\gamma'(t)\right| dt = \int_0^\pi R\, dt = \pi R.$$

Moreover, if z lies in $\mathcal{C}$ then $z = \gamma(t)$ for some $t \in [0, \pi]$, hence

$$|z| = |\gamma(t)| = \sqrt{(R\cos(t))^2 + (R\sin(t))^2} = R.$$

Using the backwards triangle inequality, for all $z \in \mathcal{C}$ we have

$$\left|1 + z^4\right| \geq \left|1 - \left|-z^4\right|\right| = \left|1 - R^4\right| = R^4 - 1$$

(since $R > 1$), and so

$$\left|\frac{1}{1+z^4}\right| \leq \frac{1}{R^4 - 1}$$

whenever $z \in \mathcal{C}$. Thus the Estimation Lemma tells us that

$$\left|\int_{\mathcal{C}} \frac{1}{1+z^4}\right| \leq \frac{1}{R^4-1} \cdot (\pi R) = \frac{\pi R}{R^4 - 1},$$

so that $\pi R/(R^4 - 1)$ is an upper estimate for the integral $\int_{\mathcal{C}} 1/(1+z^4)$.

5. Show that for all points z on the circle $\{z : |z| = 5\}$ we have

$$|z - 7| \leq 12 \quad \text{and} \quad |\overline{z} + 8| \geq 3,$$

and use this to find an upper estimate for the integral

$$\int_S \frac{z-7}{(\overline{z}+8)^2}\, dz$$

where S is the same circle oriented anticlockwise.

Solution: The first two inequalities can be shown using the triangle and backwards triangle inequalities respectively. Using these, we have

$$\left|\frac{z-7}{(\overline{z}+8)^2}\right| = \frac{|z-7|}{|\overline{z}+8|^2} \leq \frac{12}{3^2} = \frac{4}{3}$$

for all $z \in S$. The length of S is 10π, and so by the Estimation Lemma, we get the upper estimate

$$\left|\int_S \frac{z-7}{(\overline{z}+8)^2}\, dz\right| \leq \frac{4}{3} \cdot 10\pi = \frac{40\pi}{3}.$$

6. Let S_a be the anticlockwise square contour with corners at $\pm a(1+i), \pm a(1-i)$ where $a > 0$. Show that if $z \in S_a$ then

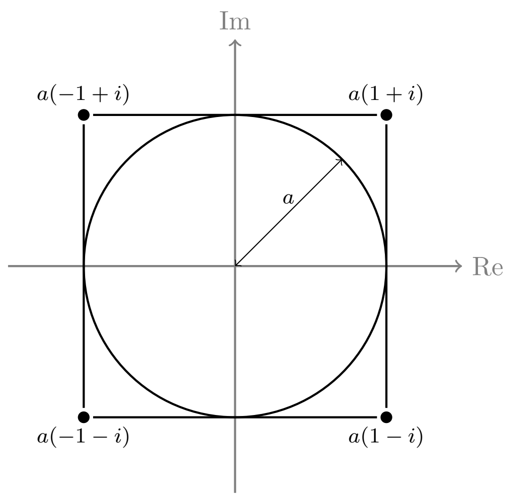
$$\frac{1}{|z|} \leq \frac{1}{a}$$

and hence

$$\left| \int_{S_a} \frac{1}{z} dz \right| \leq 8,$$

for all $a > 0$.

Solution: Note that this square lies in the region outside the circle $\{z : \mathbb{C} : |z| = a\}$.



Any point z outside of this circle has modulus $|z| \geq a$; in particular, this is true for any $z \in S_a$. Hence for all such z, $\left|\frac{1}{z}\right| \leq \frac{1}{a}$.
The length of each straight edge of S_a is $2a$, and hence $\ell(S_a) = 8a$. The Estimation Lemma then gives the required inequality.

7. Prove each of the following:
   (a) For z_0 and h in $\mathbb{C}$ we have $\displaystyle\int_{[z_0, z_0+h]} 1 \; dz = h$.
   (b) For $f : U \to \mathbb{C}$ and $z_0 \in U$, $f(z_0) = \displaystyle\frac{1}{h} \int_{[z_1, z_1+h]} f(z_0) \; dz$.
   (c) If α is a complex number and M a fixed real number with $|\alpha| \leq \epsilon M$ for all $\epsilon > 0$ then $\alpha = 0$.

Solution:
(a) Using the parametrisation $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = z_0 + t(z_0 + h - z_0) = z_0 + th$, we have $\gamma'(t) = h$ and $f(\gamma(t)) = 1$. Hence

$$\int_{[z_0, z_0+h]} 1 dz = \int_0^1 f(\gamma(t))\gamma'(t) \; dt = \int_0^1 1(h) \; dt = h.$$

Alternatively, using the antiderivative $F(z) = z$ for f, we have (by the Fundamental Theorem of Calculus)

$$\int_{[z_0, z_0+h]} 1 dz = F(z_0 + h) - F(z_0) = z_0 + h - z_0 = h.$$

(b) Since $f(z_0)$ is a constant we have

$$\frac{1}{h} \int_{[z_1, z_1+h]} f(z_0) dz = \frac{1}{h} f(z_0) \int_{[z_1, z_1+h]} 1 dz = \frac{1}{h} f(z_0) h = f(z_0)$$

by the previous part.

(c) Suppose that $\alpha \neq 0$. Then $|\alpha| > 0$ (since for $z \in \mathbb{C}$ we have $|z| = 0$ if and only if $z = 0$). Setting $\epsilon = |\alpha| / 2M$ we have $\epsilon > 0$, which implies that

$$|\alpha| \leq \epsilon M = \frac{|\alpha|}{2M} \cdot M = \frac{|\alpha|}{2}.$$

But this is only possible if $|\alpha| = 0$, a contradiction. Hence $|\alpha| = 0$ and so $\alpha = 0$.