MA2003 Complex Analysis
Exercise Sheet 3

1. Find antiderivatives for the following functions:
   (a) $f(z) = \alpha + \beta(z - z_0)$,
   (b) $f(z) = (z - z_0)^n$,

   where α, β and $z_0 \in \mathbb{C}$ are constants and $n \neq -1$. Does $g(z) = (z - z_0)^{-1}$ have an antiderivative on $\mathbb{C} \setminus \{z_0\}$?

2. Evaluate the following contour integrals:
$$\int_{\mathcal{C}} z^3 \quad \text{and} \quad \int_{\mathcal{C}} \frac{1}{z^2}$$
   along $\mathcal{C}$ where $\mathcal{C}$ is
   (a) any contour from i to -2, and
   (b) any closed contour.

   For the second integral, you may assume that $\mathcal{C}$ does not contain 0.

3. Let U be a region in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be holomorphic on U with $f(z)$ real-valued for all $z \in U$. Prove that f is constant.

4. Find an upper estimate for
$$\int_{\mathcal{C}} \frac{1}{1 + z^4},$$
   where $\mathcal{C}$ is the upper semicircular contour from R to $-R$ given by $\gamma : [0, \pi] \to \mathbb{C}$, $\gamma(t) = R\cos(t) + iR\sin(t)$.

5. Show that for all points z on the circle $\{z : |z| = 5\}$ we have
$$|z - 7| \leq 12 \quad \text{and} \quad |\overline{z} + 8| \geq 3,$$
   and use this to find an upper estimate for the integral
$$\int_S \frac{z - 7}{(\overline{z} + 8)^2}\, dz$$
   where S is the same circle oriented anticlockwise.

6. Let S_a be the anticlockwise square contour with corners at $\pm a(1 + i), \pm a(1 - i)$ where $a > 0$. Show that if $z \in S_a$ then
$$\frac{1}{|z|} \leq \frac{1}{a}$$
   and hence
$$\left| \int_{S_a} \frac{1}{z} dz \right| \leq 8,$$
   for all $a > 0$.

7. Prove each of the following:
   (a) For z_0 and h in $\mathbb{C}$ we have $\displaystyle\int_{[z_0, z_0+h]} 1\, dz = h$.
   (b) For $f : U \to \mathbb{C}$ and $z_0 \in U$, $f(z_0) = \displaystyle\frac{1}{h} \int_{[z_0, z_0+h]} f(z_0)\, dz$.
   (c) If α is a complex number and M a fixed real number with $|\alpha| \leq \epsilon M$ for all $\epsilon > 0$ then $\alpha = 0$.