MA2003

Complex Analysis I

Lecture Notes

DM

Version: 16th January 2018

Contents

Chapter 1 Preliminaries on Complex Numbers and Functions

1.1 Basic Definitions and Algebraic Operations

Formally, a *complex number* is an ordered pair (x, y) of real numbers. We denote the set of all complex numbers by $\mathbb{C}$, and define addition and multiplication on $\mathbb{C}$ as follows:

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$,

(ii) $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$.

With this definition, $\mathbb{C}$ and $\mathbb{R}^2$ are equal *as sets*, however, we have also defined an operation of multiplication on $\mathbb{C}$.

The subset of $\mathbb{C}$ given by

$$\{(x, 0) : x \in \mathbb{R}\}$$

is called the *real axis.* For complex numbers in this subset we have

$$\begin{aligned} (x_1, 0) + (x_2, 0) &= (x_1 + x_2, 0) \\ (x_1, 0)(x_2, 0) &= (x_1x_2, 0), \end{aligned}$$

so these complex numbers behave exactly like real numbers. For this reason, we shall use $\mathbb{R}$ to refer to the real axis and denote the complex number $(x, 0)$ by x.

Writing $i = (0, 1)$, the definition of multiplication in $\mathbb{C}$ gives $i^2 = -1$. With this notation, we get a more familiar 'definition': a complex number $z = (x, y)$ can be written as

$$\begin{aligned} z = (x, y) &= (x, 0) + (y, 0)(0, 1) \\ &= x + iy. \end{aligned}$$

The real numbers x and y are the *real* and *imaginary* parts of z respectively, and we often write

$$\mathsf{Re}(z) = x, \quad \mathsf{Im}(z) = y.$$

Note that the imaginary part of z is *real.*

With this more familiar notation, the algebraic operations of addition and multiplication on $\mathbb{C}$ can be expressed as follows: for $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \in \mathbb{C}$, we have

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2) \\ z_1z_2 &= (x_1 + iy_1)(x_2 + iy_2) \end{aligned}$$

$$= x_1x_2 + i(x_1y_2) + i(y_1x_2) + (i)^2(y_1y_2)$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Note in particular that for $r \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$ we have

$$rz = r(x + iy) = rx + iry.$$

It is convenient to identify the complex number $z \in \mathbb{C}$ with the point (or sometimes, the vector) $(\mathsf{Re}(z), \mathsf{Im}(z)) \in \mathbb{R}^2$. Obviously the map $z \mapsto (\mathsf{Re}(z), \mathsf{Im}(z))$ is a bijection $\mathbb{C} \to \mathbb{R}^2$, and thus geometrically, we view $\mathbb{C}$ as $\mathbb{R}^2$.

Definition 1.1 (Complex Conjugate)**.** Given $z \in \mathbb{C}$, the *complex conjugate* $\overline{z}$ of z is defined as

$$\overline{z} := \mathsf{Re}(z) - i\ \mathsf{Im}(z).$$

Definition 1.2 (Modulus)**.** Given $z \in \mathbb{C}$, we define the modulus of z to be

$$|z| := \sqrt{\mathsf{Re}(z)^2 + \mathsf{Im}(z)^2}.$$

In other words, for $z = x + iy$, $\overline{z} = x - iy$ and $|z| = \sqrt{x^2 + y^2}$.

Geometrically, the complex conjugate of z describes the reflection of z through the real axis, and the modulus of z is the distance of z from the origin.

Proposition 1.3. *Let $z, z_1, z_2 \in \mathbb{C}$ be given. Then:*

(i) $\overline{\overline{z}} = z$ *and* $|\overline{z}| = |z|$.

(ii) $|z| = \sqrt{\overline{z}z}$

(iii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ *and* $\overline{z_1z_2} = \overline{z_1}\ \overline{z_2}$.

(iv) $\overline{z} = z$ *if and only if z is real, i.e., if and only if* $\mathsf{Im}(z) = 0$.

(v) $\mathsf{Re}(z) = \dfrac{z + \overline{z}}{2}$ *and* $\mathsf{Im}(z) = \dfrac{z - \overline{z}}{2i}$.

(vi) $|z| = 0$ *if and only if* $z = 0$.

(vii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$ (Triangle Inequality).

(viii) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$ (Reverse Triangle Inequality).

(ix) $|z_1z_2| = |z_1|\,|z_2|$.

You really should be familiar with all of these properties. I would suggest trying to prove at least some of them, or verifying them for some specific choices of z_1 and z_2. Most of the proofs are short.

Question. The rules of multiplication allow us to divide a complex number z by a nonzero real number r; i.e.

$$\frac{z}{r} = \left(\frac{1}{r}\right) z.$$

How do we define division of complex numbers?

Answer

For a complex number $z \neq 0$, we first define the multiplicative inverse of z, denoted z^{-1} or $\frac{1}{z}$. Note that by Proposition 1.3 (ii), we have $|z|^2 = z\overline{z}$. Since $|z|$ is real, we may define the complex number

$$\frac{\overline{z}}{|z|^2}$$

which must satisfy

$$z\left(\frac{\overline{z}}{|z|^2}\right) = \frac{z\overline{z}}{z\overline{z}} = \frac{|z|^2}{|z|^2} = 1.$$

It follows that the multiplicative inverse of z is

$$z^{-1} = \frac{\overline{z}}{|z|^2}.$$

With this in mind, we may divide any complex number z_1 by any nonzero complex number z_2 as follows:

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1\overline{z_2}}{|z_2|^2}.$$

Definition 1.4 (Argument)**.** The *argument* of a complex number $z \neq 0$ is the angle $\arg(z)$ from the positive real axis to the vector representing z in the anticlockwise direction.

Of course negative values of $arg(z)$ are allowed: these values represent angles taken in the clockwise direction. Note that there are typically many ways in which we can represent $\arg(z)$, since we identify any two angles that differ by an integer multiple of 2π with one another. For instance, for (the complex number) $z = 1$ any of the angles

$$\ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$$

are valid choices for $\arg(z)$. Similarly, if we look at $z = -1 - i$: We can write $\arg(z) = \frac{5\pi}{4}$ or $\arg(z) = -\frac{3\pi}{4}$ (or indeed an integer multiple of 2π added to either of these angles). By convention we usually take $\arg(z) \in (-\pi, \pi]$.

Definition 1.5. For $z \in \mathbb{C}$, $z \neq 0$, we define the *Principal argument of* z (or the *Principal value of* $\arg(z)$) to be the value of $\arg(z)$ that lies in $(-\pi, \pi]$. We shall denote this value by $\operatorname{Arg}(z)$.

So for $z = -1 - i$ we have $\operatorname{Arg}(z) = -\frac{3\pi}{4}$, while $\arg(z)$ can be taken to be any of the values

$$\ldots, -\frac{11\pi}{4}, -\frac{3\pi}{4}, \frac{5\pi}{4}, \frac{13\pi}{4}, \ldots.$$

Definition 1.6. The *polar form* of a complex number z is given by writing z in the form

$$z = r\left(\cos(\theta) + i\sin(\theta)\right),$$

where $r, \theta \in \mathbb{R}$ and $r \geq 0$.

The polar form of z is found by setting $r = |z|$ and $\theta = \arg(z)$ (any value of $\arg(z)$ will of course suffice). Equivalently, we may write

$$z = re^{i\theta}$$

(though we have yet to define the complex exponential function). For completeness, we should specify that the polar form of 0 is simply 0, since $\arg(0)$ is not defined.

Definition 1.7. Let $w = r(\cos(\theta) + i\sin(\theta))$ be (the polar form of) a complex number. Then the n^{th} *complex roots* of w are defined to be the n solutions $z_0, z_1, \ldots, z_{n-1}$ of the equation $z^n = w$. These roots are given by

$$z_k = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right),$$

for $k = 0, 1, \ldots, n-1$, where $\sqrt[n]{r}$ is the (positive) n^{th} real root of r.

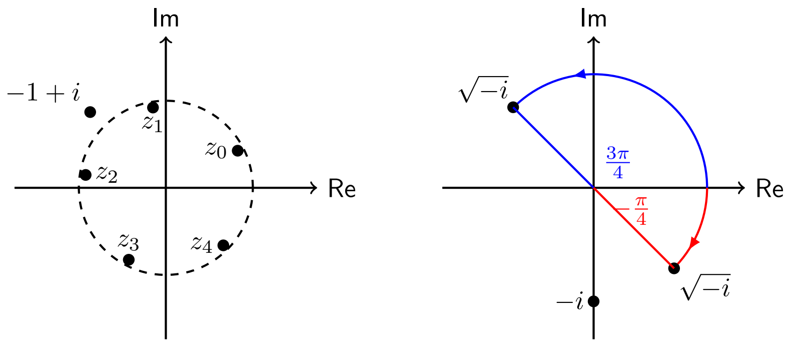

Figure 1.1: The 5^{th} roots of $z = -1 + i$ (left), and the two square roots of $-i$ (right).
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Theorem 1.8. *Let z_1 and z_2 be nonzero complex numbers. Then*

(i) $|z_1 z_2| = |z_1|\ |z_2|$, *and*

(i) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Proof: Exercise.

1.2 Complex Functions

We now start our investigation of functions from the complex plane to itself. First of all, note that a function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ can be written in the form

$$\mathbf{f}(x, y) = (u(x, y), v(x, y))$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$. For example, if

$$\mathbf{f}(x, y) = \left(x^2 + y^2 - 2x, 2y - 6\right),$$

we have

$$u(x, y) = x^2 + y^2 - 2x \text{ and } v(x, y) = 2y - 6.$$

Since we have a formal definition of $\mathbb{C}$ as $\mathbb{R}^2$, we can express a function $f : \mathbb{C} \to \mathbb{C}$ as

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

for all $z = x + iy \in \mathbb{C}$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$. In this way, we can extend our definition of real and imaginary parts from complex numbers to functions $f : \mathbb{C} \to \mathbb{C}$, writing $\mathsf{Re}(f) = u$ and $\mathsf{Im}(f) = v$.

Going in the other direction, given two functions $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$, we can create a function $f : \mathbb{C} \to \mathbb{C}$ by *defining* $f(x + iy) := u(x, y) + iv(x, y)$. So,

There is a correspondence between functions from the complex plane to itself, and pairs of functions from the real plane to the real line.

Example 1.9. The function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \overline{z}$$

corresponds to a function

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathbf{f}(x, y) = (u(x, y), v(x, y))$$

$$u(x, y) = x \qquad \text{and} \qquad v(x, y) = -y$$

since

$$\overline{z} = \overline{x + iy} = x - iy = \underbrace{x}_{u(x,y)} + i \underbrace{(-y)}_{v(x,y)}.$$

Example 1.10. Let us determine the functions $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ corresponding to

$$h : \mathbb{C} \to \mathbb{C}, z \mapsto z^2.$$

Solution

With $z = x + iy$ we have

$$z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$$

Thus we identify $h : \mathbb{C} \to \mathbb{C}$ with $\mathbf{h} : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{h}(x, y) = (u(x, y), v(x, y))$ where

$$u : \mathbb{R}^2 \to \mathbb{R}, \quad u(x, y) = x^2 - y^2,$$
$$v : \mathbb{R}^2 \to \mathbb{R}, \quad v(x, y) = 2xy.$$

Example 1.11. Let find the complex function f corresponding to the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ where

(i) $\mathbf{f}(x, y) = (-2y + 3, 2x)$, and

(ii) $\mathbf{f}(x, y) = (x^2 + y^2 - 2x, 2y - 6)$.

Solution

We could use the fact that if $z = x + iy$ then

$$x = \mathsf{Re}(z) = \frac{1}{2}(z + \overline{z})$$
$$y = \mathsf{Im}(z) = \frac{1}{2i}(z - \overline{z}).$$

Substituting these expressions for x and y, we could then simplify and find $f(z)$. However, this is time consuming, and it is sometimes easier to look for familiar expressions in the definition of f.

In part (i), $\mathbf{f}$ corresponds to $f : \mathbb{C} \to \mathbb{C}$ defined by

$$\begin{aligned} f(z) = f(x + iy) &= -2y + 3 + i(2x) \\ &= 2(ix - y) + 3 \\ &= 2(ix + i^2 y) + 3 \\ &= 2i(x + iy) + 3 \\ &= 2iz + 3. \end{aligned}$$

In part (ii), $\mathbf{f}$ corresponds to $f : \mathbb{C} \to \mathbb{C}$ where

$$f(x + iy) = x^2 + y^2 - 2x + i2y - 6i.$$

With $z = x + iy$, notice that

$$x^2 + y^2 = |z|^2 = z\overline{z},$$

and that

$$-2x + i2y = -2(x - iy) = -2\overline{z},$$

hence

$$x^2 + y^2 - 2x + i2y - 6i = z\overline{z} - 2\overline{z} - 6i.$$

Hence $f(z) = z\overline{z} - 2\overline{z} - 6i$ is the corresponding complex function.

It is worth recalling that some familiar functions defined on $\mathbb{R}$ extend to $\mathbb{C}$.

Definition 1.12. The *exponential function* is the function $\exp : \mathbb{C} \to \mathbb{C}$ defined by

$$\exp(x + iy) = \exp(x)\,(\cos(y) + i\sin(y))$$

for all complex numbers $x + iy \in \mathbb{C}$, where $\exp(x)(= e^x)$ is the usual (real) exponential of x.

We shall often write e^z in place of $\exp(z)$. The trigonometric functions also extend to $\mathbb{C}$ via

$$\cos(z) = \frac{1}{2}(\exp(iz) + \exp(-iz)) \tag{1.1}$$
$$\sin(z) = \frac{1}{2i}(\exp(iz) - \exp(-iz)). \tag{1.2}$$

Remark. We conclude this section with some remarks about how you might go about 'visualising' a complex function. For functions $f : \mathbb{R} \to \mathbb{R}$, we usually do so by drawing the graph of f, that is, the subset $\{(x, f(x)) : x \in \mathbb{R}\}$ of $\mathbb{R}^2$.

We cannot draw the 'graph' of a function $f : \mathbb{C} \to \mathbb{C}$, as to do so would require four coordinates

$$\{\mathsf{Re}(z), \mathsf{Im}(z), \mathsf{Re}\,(f(z))\,, \mathsf{Im}\,(f(z))\}$$

and thus four dimensions. We can however view f as a 'transformation' of the complex plane, and examine the effect of applying f to

- Regions (subsets) of $\mathbb{C}$
- Curves in $\mathbb{C}$ (lines, circles etc)
- A combination of the two.

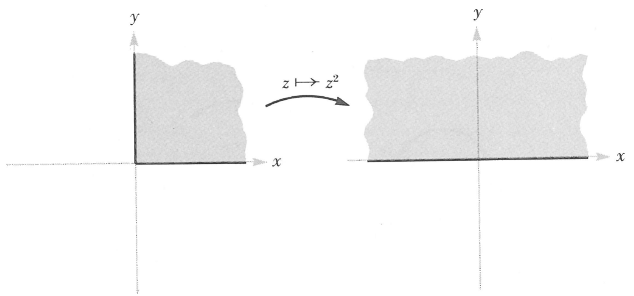


Figure 1.2: The image of the first quadrant under $z \mapsto z^2$.

Figure 1.2 indicates that the image of the first quadrant under the map $z \mapsto z^2$ is the region consisting of the first and second quadrants. This follows from the fact that for any $z_1, z_2 \in \mathbb{C}$,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

Hence $\arg(z^2) = 2\arg(z)$, and so if $0 \le \arg(z) \le \pi/2$ (i.e. z is in the first quadrant), we have $0 \le \arg(z^2) \le \pi$ (z^2 lies in either the first or second quadrant). Of course, Figure 1.2 does not tell us anything about the modulus of z^2.

Figure 1.3 illustrates how $z \mapsto z^2$ also squares the modulus; the circle of radius $r > 0$ and centre 0 is sent to the circle of radius r^2 and centre 0. Moreover, the anticlockwise arrows on the circles indicate that if z_2 lies (a small distance) anticlockwise of z_1, then z_2^2 lies anticlockwise of z_1^2.

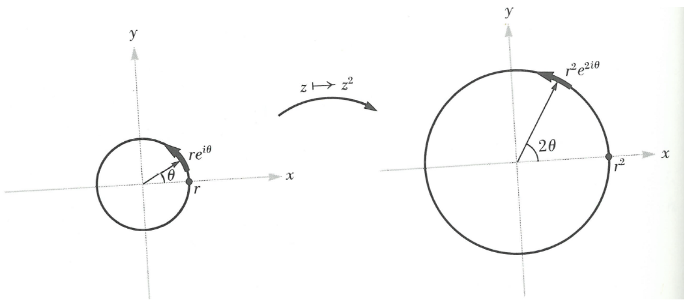


Figure 1.3: The effect of the map $z \mapsto z^2$ on a circle of radius r centred at the origin.

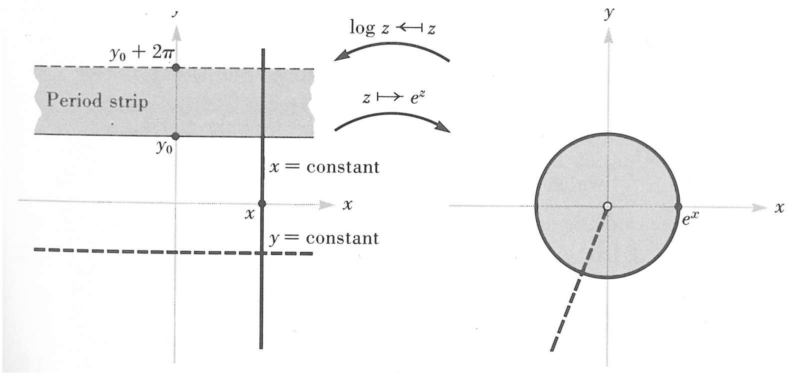


Figure 1.4: The geometric effect of the exponential map.

Figure 1.4 combines the approaches of the previous examples. Here we see that the shaded region,

$$\{z \in \mathbb{C} : \mathsf{Re}(z) \leq x \text{ and } y_0 \leq \mathsf{Im}(z) < y_0 + 2\pi\}$$

is sent to the shaded disc of radius e^x and centre 0. The vertical line $\mathsf{Re}(z) = x$ is sent to the circle of radius e^x and centre 0, while the dashed horizontal line is sent to the 'infinite ray' from 0 indicated with another dashed line.

Strictly speaking this diagram is inaccurate, as some of the region to the right of the line $\mathsf{Re}(z) = x$ is also shaded.

I will not ask you to produce diagrams like this in the exam, but it may be helpful to have the idea of complex functions as transformations in your mind throughout this module.

Figures 1.2, 1.3 and 1.4 are from *Basic Complex Analysis*, (Jerrold. E. Marsden, published by W.H. Freeman and Company, 1973).

1.3 Open Sets in $\mathbb{C}$

Definition 1.13 (Open Disc)**.** Let $z_0 \in \mathbb{C}$ and let $r > 0$ be some real number. The open disc of radius r centred at z_0 is defined to be the set

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\} .$$

In other words, $D(z_0, r)$ is the set of all points that lie (strictly) inside the circle or radius r with centre z_0.

Definition 1.14. The set of points

$$D'(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\} ,$$

where $r > 0$ and $z_0 \in \mathbb{C}$, is called the *punctured open disc* of radius r, centre z_0.

Note that $D'(z_0, r)$ is the set obtained by removing the centre z_0 from the open disc $D(z_0, r)$.

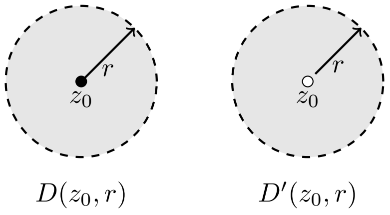


Figure 1.5: The open disc $D(z_0, r)$ and punctured open disc $D'(z_0, r)$.

Notation. This is a good place to introduce some conventions for sketching sets. The broken circle in Figure 1.5 around the disc indicates that the boundary is not included in this set. The filled point $\bullet$ beside z_0 indicates that the point z_0 is included in $D(z_0, r)$, while the 'hollow' point $\circ$ beside z_0 indicates that z_0 is not included in the punctured disc $D'(z_0, r)$.

Definition 1.15. Let $U \subseteq \mathbb{C}$, then we say that U is an *open set* if given any $z \in U$ there is some $r_z > 0$ with $D(z, r_z) \subseteq U$.

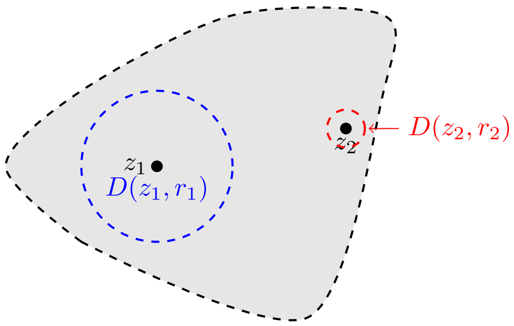
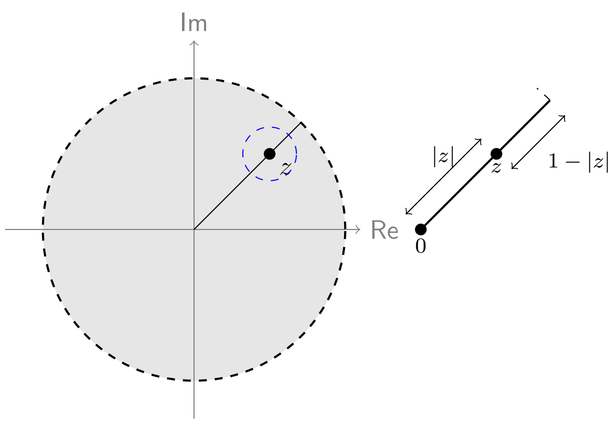


Figure 1.6: An open set U with two examples of open discs around points of U. Note that the radius typically depends on the point z; points nearer the 'edge' of U will need smaller discs.

Informally, we think of an open set as a set that does not include its 'boundary.' Thus a set U is open if, given any $z \in U$, we can move a small distance in *any* direction without leaving U.

Example 1.16. (i) The open disc $D(0, 1)$ is an open set.



Let $z \in D(0, 1)$. Our goal is to find some $r > 0$ so that $D(z, r) \subseteq D(0, 1)$. To this end, set $r = (1 - |z|)/2$. Then if $w \in D(z, r)$,

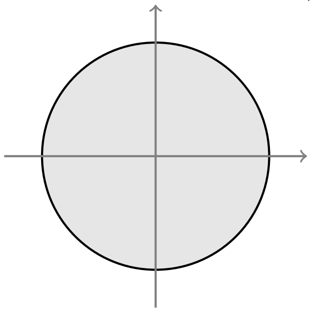
$$\begin{aligned} |w - 0| &\leq |w - z| + |z| \\ &< \frac{1 - |z|}{2} + |z| \end{aligned}$$

$$= \frac{1+|z|}{2} < \frac{1+1}{2} = 1,$$

which shows that $w \in D(0,1)$.

Hence $D(z,r) \subseteq D(0,1)$ and so $D(0,1)$ is open.

(ii) The closed disc $\overline{D}(0,1) := \{z \in \mathbb{C} : |z| \leq 1\}$ is not an open set (note: *closed* does not necessarily mean not open).



The solid circle in this image indicated that the boundary is included.

Given any point z on the boundary (e.g. $z = 1$) and any $r > 0$, the open disc $D(z,r)$ contains points that do not belong to $\overline{D}(0,1)$. For example, $w = 1 + r/2$ belongs to $D(1,r)$, since

$$|w - z| = |(1 + r/2) - 1| = r/2 < r$$

but not to $\overline{D}(0,1)$, since

$$|w - 0| = |1 + r/2| = 1 + r/2 > 1.$$

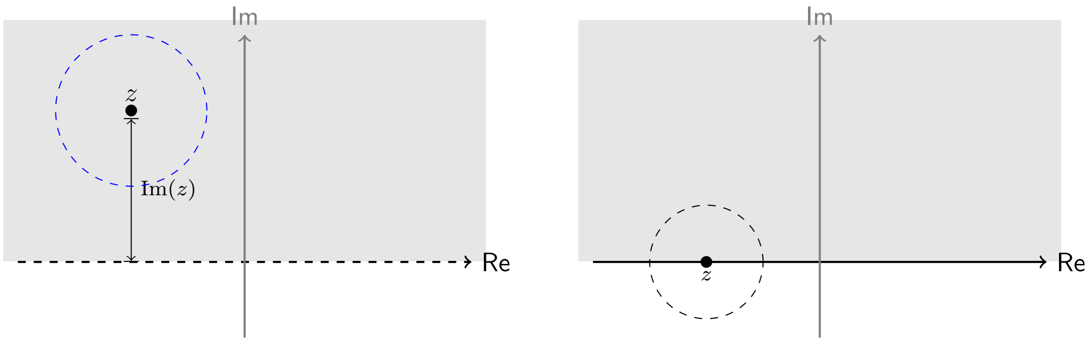
Example 1.17. The upper-half plane H_+, where

$$H_+ := \{z \in \mathbb{C} : \mathsf{Im}(z) > 0\}$$

is an open set, while the set K_+ defined by

$$K_+ := \{z \in \mathbb{C} : \mathsf{Im}(z) \geq 0\}$$

is not.



If $z \in H_+$ then $\mathsf{Im}(z) > 0$, so we set $r = \frac{1}{2}\mathsf{Im}(z)$. Then $D(z,r) \subseteq H_+$, or in other words, given any $w \in D(z,r)$, we must have $\mathsf{Im}(w) > 0$.

For the set K_+, if we choose z on the real axis (e.g $z = -1$), then any open disc centred at z, no matter how small, contains points below the real axis. Thus there is no $r > 0$ with $D(z,r) \subset K_+$, so K_+ is not open.

Note 1.18. Some more examples of open sets include $\mathbb{C}\setminus\{0\}$, or indeed $\mathbb{C}\setminus F$ where F is any finite set. Many regions determined by *strict* inequalities of real numbers are also open, for example, sets of the form $\{z \in \mathbb{C} : c_1 < |z| < c_2\}$, $\{z \in \mathbb{C} : 0 < \operatorname{Arg}(z) < \pi/4\}$ or $\{z \in \mathbb{C} : c_1 < \mathsf{Re}(z) < c_2\}$, where $c_1 < c_2$ are real numbers.

More examples of sets that are not open include circles, lines, curves, single points or finite sets. Do not use *closed* to mean *not open*. In the context of analysis, closed has a different meaning.

Throughout this module, we shall be mostly concerned with functions $f : U \to \mathbb{C}$, where U is an open subset of $\mathbb{C}$.

1.4 Limits

Limits in $\mathbb{C}$ are defined in an analogous way to those in $\mathbb{R}$ (and almost identically to those in $\mathbb{R}^2$).

First, fix some complex number $z_0 \in \mathbb{C}$. For any $z \in \mathbb{C}$, the modulus $|z - z_0|$ measures the distance between z and z_0. Note that if we write $z = x + iy$ and $z_0 = x_0 + iy_0$, then $|z - z_0|$ is exactly the same as the Euclidean distance between (x, y) and (x_0, y_0) in $\mathbb{R}^2$.

For a complex function f, what does it mean to say that $f(z)$ approaches $L \in \mathbb{C}$ as z approaches z_0? Intuitively, we want

$$|f(z) - L| \text{ is small whenever } |z - z_0| \text{ is small.}$$

More formally, this can be written as

$$\text{Given any } \epsilon > 0 \text{ there is a } \delta > 0 \text{ such that}$$

$$|f(z) - L| < \epsilon \text{ whenever } |z - z_0| < \delta.$$

This definition works whenever f is defined on all of $\mathbb{C}$, but it is not so clear how it should work when $f(z)$ is not defined near z_0. In other words, we want to be able to exclude points z_0 where $|z - z_0|$ small implies $f(z)$ does not exist.

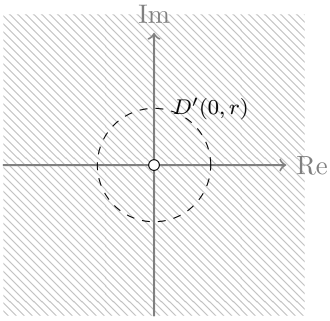
Definition 1.19. A point z_0 is a *limit point* of of a set $S \subseteq \mathbb{C}$ if for any $\delta > 0$, we have

$$D'(z_0, \delta) \cap S \neq \emptyset.$$

In other words, any punctured disc centred at z_0, no matter how small, contains at least one point of S.

A limit point of a set S may or may not belong to S. Moreover, a point $z_0 \in S$ may or may not be a limit point of S. If S is an open set however, then any $z_0 \in S$ is necessarily a limit point of S.

Example 1.20. (i) The point 0 is a limit point of the punctured plane $\mathbb{C}\setminus\{0\} = \{z \in \mathbb{C} : z \neq 0\}$.



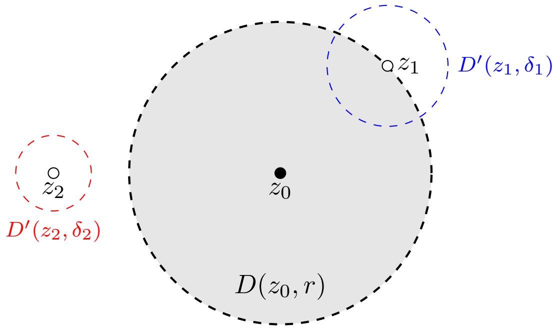
Indeed, it is clear that every punctured disc $D'(0,r)$ contains points of $\mathbb{C}\setminus\{0\}$.

In fact, every point z of $\mathbb{C}$ is a limit point of $\mathbb{C}\setminus\{0\}$, since every disc $D'(z,r)$ must contain a point of $\mathbb{C}\setminus\{0\}$.

(ii) If $S=\{z_0\}$ is a one-point set, then there are no limit points of S.

(iii) The set of limit points of the open disc $S=D(z_0,r)$ is precisely the closed disc

$$\{z\in\mathbb{C} : |z-z_0|\le r\}\,.$$

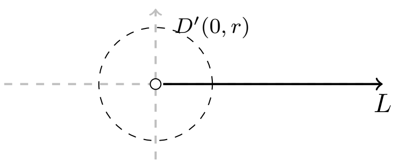


For a point z_1 on the boundary, any punctured disc $D'(z_1,r)$ intersects S, while for a point z_2 outside of the boundary, there are punctured discs $D'(z_2,r_2)$ that do not.

(iv) Let L be the strictly positive real axis, regarded as a subset of $\mathbb{C}$, that is

$$L=\{x+iy\in\mathbb{C} : x>0 \text{ and } y=0\}\,.$$

Then $z=0$ is a limit point of L.



Note 1.21. To confuse things further, some authors require only allow points $z_0\in\mathbb{C}\setminus S$ to be limit points of S. When reading textbooks or lecture notes, take care as to which definition is being used.

You need to be familiar with the concept of a limit point, though I will not ask you to prove that a given point z_0 is a limit point of a set S.

We are now in a position to define limits of complex functions.

Definition 1.22. Let f be a complex function, $S \subseteq \mathbb{C}$ the domain of f, and let $z_0 \in \mathbb{C}$ be a limit point of S. Then we say that

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \alpha \in \mathbb{C}$$

if given any $\epsilon > 0$ there is some $\delta > 0$ such that

$$\text{if } z \in S \text{ and } 0 < |z - z_0| < \delta \text{ then } |f(z) - \alpha| < \epsilon.$$

Note 1.23. (i) Note that this definition allows us to examine limits of functions at point that lie on the boundary of their domains. For example, we can sensibly speak about things like

$$\lim_{z \to 0} \frac{\sin(z)}{z}$$

since this function is defined on $\mathbb{C} \setminus \{0\}$ and 0 is a limit point of this set.

(ii) Definition 1.22 may equivalently be written in the language of open discs:

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \alpha$$

if given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$z \in D'(z_0, \delta) \cap S \text{ implies } f(z) \in D(\alpha, \epsilon).$$

(iii) When

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \alpha$$

we sometimes write

$$f(z) \to \alpha \text{ as } z \to z_0.$$

Question. Is it possible to find a function f for which $f(z) \to \alpha_1$ and $f(z) \to \alpha_2$ as $z \to z_0$, where $\alpha_1 \neq \alpha_2$?

Answer

The answer, unsurprisingly, is no. Indeed, set $\epsilon = \frac{1}{3} |\alpha_1 - \alpha_2| > 0$. If $f(z) \to \alpha_1$ and $f(z) \to \alpha_2$ as $z \to z_0$, then there would be some $\delta > 0$ such that

$$z \in D'(z_0, \delta) \implies f(z) \in D'(\alpha_1, \epsilon) \cap D'(\alpha_2, \epsilon).$$

But this is impossible since clearly $D'(\alpha_1, \epsilon) \cap D'(\alpha_2, \epsilon) = \emptyset$.

Proposition 1.24 (Algebra of limits; proof non-examinable)**.** *Let $S \subseteq \mathbb{C}$ and consider functions $f, g : S \to \mathbb{C}$. Suppose that z_0 is a limit point of S, and that $\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \alpha$ and $\lim_{\substack{z \to z_0 \\ z \in S}} g(z) = \beta$. Then* *(i)* $\lim\limits_{\substack{z \to z_0 \\ z \in S}} (f(z) + g(z)) = \alpha + \beta$,

(ii) $\lim\limits_{\substack{z \to z_0 \\ z \in S}} (f(z)g(z)) = \alpha\beta$,

(iii) *If in addition* $\beta \neq 0$, *and* z_0 *is a limit point of the set* $T = \{z \in S : g(z) \neq 0\}$, *then*

$$\lim_{\substack{z \to z_0 \\ z \in T}} \frac{f(z)}{g(z)} = \frac{\alpha}{\beta}$$

(The proof of this proposition is almost identical to the corresponding proof for real functions, except that $|\cdot|$ refers to the modulus and not the absolute value, and is thus omitted.)

Question. Suppose that T is a subset of S, z_0 is a limit point of both T and S, and that $\lim\limits_{\substack{z \to z_0 \\ z \in S}} f(z) = \alpha$. What can we say about $\lim\limits_{\substack{z \to z_0 \\ z \in T}} f(z)$?

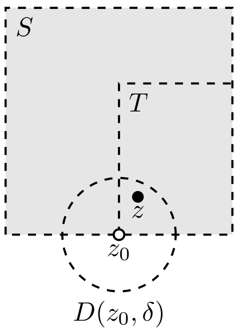


Figure 1.7: A subset T of S and a point z_0 that is a limit point of both T and S.

Answer

Let $\epsilon > 0$ be given, then we know that there exists $\delta > 0$ such that

$$z \in S,\ 0 < |z - z_0| < \delta \ \Rightarrow |f(z) - \alpha| < \epsilon.$$

Note that

$$z \in D'(z_0, \delta) \cap T \text{ implies } z \in D'(z_0, \delta) \cap S,$$

or in other words,

$$z \in T \ \text{ and } 0 < |z - z_0| < \delta \Rightarrow z \in S \text{ and } 0 < |z - z_0| < \delta.$$

It follows that

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \alpha \text{ implies } \lim_{\substack{z \to z_0 \\ z \in T}} f(z) = \alpha$$

The second limit, where f is restricted to the subset T of S, is called a *restricted limit.* If a function has a limit at a point z_0 then all restricted limits of that function at z_0 must be the same. In particular,

If we have two subsets $T_1, T_2 \subseteq S$ *such that*

$$\lim_{\substack{z \to z_0 \\ z \in T_1}} f(z) = \alpha_1 \textit{ and } \lim_{\substack{z \to z_0 \\ z \in T_2}} f(z) = \alpha_2,$$

with $\alpha_1 \neq \alpha_2$, then

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z)$$

does not exist.

Let f be a complex function with domain S and let z_0 be a limit point of S. In what follows, we shall simply write $\lim\limits_{z \to z_0} f(z)$ in place of

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z),$$

and we shall reserve the second subscript for restricted limits.

Example 1.25. Consider the function

$$f : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad f(z) = \frac{\overline{z}}{z}.$$

Does $\lim\limits_{z \to 0} f(z)$ exist?

Note 1.26. In future, we may simply write things like "Let $f(z) = \dfrac{\overline{z}}{z}$" without specifying the domain, since it is clear that this function f is defined on $\mathbb{C} \setminus \{0\}$.

Solution

We shall look at two restricted limits of $f(z)$ as z approaches 0, namely the limits when z is restricted to the subsets

- the nonzero real axis $\mathbb{R} \setminus \{0\}$, and
- the nonzero imaginary axis $i\mathbb{R} \setminus \{0\}$ (Since every point on the imaginary axis is of the form iy for some $y \in \mathbb{R}$, it makes sense to use the notation $i\mathbb{R}$ for this set.)

of the domain $\mathbb{C} \setminus \{0\}$. Note that 0 is a limit point of both $\mathbb{R} \setminus \{0\}$ and $i\mathbb{R} \setminus \{0\}$. If $z \in \mathbb{R} \setminus \{0\}$, then $\overline{z} = z$ and so $f(z) = \frac{\overline{z}}{z} = \frac{z}{z} = 1$ on $\mathbb{R} \setminus \{0\}$. Hence

$$\lim_{\substack{z \to 0 \\ z \in \mathbb{R} \setminus \{0\}}} f(z) = 1.$$

For $z \in i\mathbb{R} \setminus \{0\}$, we have $\overline{z} = -z$, so that $f(z) = -1$ for all such z, giving

$$\lim_{\substack{z \to 0 \\ z \in i\mathbb{R} \setminus \{0\}}} = -1.$$

Since these limits are not equal, it follows that the (unrestricted) limit $\lim_{z \to 0} f(z)$ does not exist.

1.5 Continuity

Definition 1.27. Let $S \subseteq \mathbb{C}$ and let $f : S \to \mathbb{C}$ be given. For a point $z_0 \in S$, we say that f is *continuous at* z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$. If f is continuous at all points $z_0 \in S$ then we say that f is *continuous on* S.

Note 1.28. Note that the definition of continuity at z_0 only makes sense when z_0 belongs to the domain of f.

Example 1.29. The functions

(i) $f(z) = \mathsf{Re}(z)$,

(ii) $f(z) = \mathsf{Im}(z)$, and

(iii) $f(z) = \overline{z}$

are all continuous on $\mathbb{C}$.

Solution

Fix $z_0 = x_0 + iy_0 \in \mathbb{C}$ and let $\epsilon > 0$ be given. Note that for any $z = x + iy \in \mathbb{C}$ we have the following inequalities:

$$\begin{aligned} |\mathsf{Re}(z) - \mathsf{Re}(z_0)| = |\mathsf{Re}(z - z_0)| &= \sqrt{(x - x_0)^2} \\ &\leq \sqrt{(x - x_0)^2 + (y - y_0)^2} = |z - w| \\ |\mathsf{Im}(z) - \mathsf{Im}(z_0)| = |\mathsf{Im}(z - z_0)| &= \sqrt{(y - y_0)^2} \\ &\leq \sqrt{(x - x_0)^2 + (y - y_0)^2} = |z - z_0| \\ |\overline{z} - \overline{z_0}| = |\overline{z - z_0}| &= |z - z_0|. \end{aligned}$$

Thus in all three cases, setting $\delta = \epsilon$, we get

$$0 < |z - z_0| < \delta \Longrightarrow |f(z) - f(z_0)| < \epsilon.$$

Proposition 1.30. *Let $S \subseteq \mathbb{C}$ and let $f, g : S \to \mathbb{C}$ be functions that are continuous on S. Then*

(i) The function $f + g$, where $(f + g)(z) = f(z) + g(z)$, is continuous on S,

(ii) The function fg, where $(fg)(z) := f(z)g(z)$, is continuous on S,

(iii) For any $\alpha \in \mathbb{C}$, the function αf, where $(\alpha f)(z) = \alpha (f(z))$, is continuous on S,

(iv) If $T = \{z \in S : g(z) \neq 0\}$ then the function f/g, where $\left(\frac{f}{g}\right)(z) = \frac{f(z)}{g(z)}$, is continuous on T.

Proof: Immediate from Proposition 1.24.

Note 1.31. If we have

$$f(x + iy) = u(x, y) + iv(x, y),$$

and $z_0 = x_0 + iy_0$ is a point in the domain of f, then

$$\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x, y) + i \lim_{(x,y) \to (x_0,y_0)} v(x, y),$$

provided these limits exist. Hence

$$f \text{ continuous at } z_0 \Leftrightarrow u \text{ and } v \text{ continuous at } (x_0, y_0).$$

This is useful when the real and imaginary parts of f are familiar functions that we know to be continuous.

Chapter 2 Differentiability and Holomorphic Functions

2.1 The Derivative of a Complex Function

Throughout this section, let U be an open subset of $\mathbb{C}$. The definition of differentiability for a function $f : U \to \mathbb{C}$ looks almost identical to the corresponding definition for real functions.

Definition 2.1. Let $f : U \to \mathbb{C}$ be a function, and let $z \in U$. We say that f is *differentiable at* z if the limit

$$\lim_{\substack{h\to 0 \\ h\in\mathbb{C}\setminus\{0\}}} \frac{f(z+h)-f(z)}{h} \tag{2.1}$$

exists. When it does, we denote its value by $f'(z)$.

Since U is open, for a fixed $z \in U$, the quantity (the *difference quotient*)

$$\frac{f(z+h)-f(z)}{h}$$

is defined whenever h is 'sufficiently small' (to ensure that $z + h \in U$ and so $f(z + h)$ is defined) but not zero (to avoid division by zero). In other words, 0 is a limit point of the domain of the difference quotient (regarded as a function of h).

For this reason, we can again omit the second subscript and write (2.1) as

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

without ambiguity.

Example 2.2. Let us investigate whether or not the function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = -i\overline{z}$$

is differentiable at any points $z \in \mathbb{C}$.

Solution

Fix $z \in \mathbb{C}$. To evaluate the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h},$$

we first try to simplify the expression

$$\frac{f(z+h)-f(z)}{h}.$$

For any $z \in \mathbb{C}\setminus\{0\}$,

$$\frac{f(z+h)-f(z)}{h} = \frac{-i\,\overline{(z+h)}-(-i)\overline{z}}{h} = -i\frac{\overline{h}}{h}.$$

But then the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} -i\frac{\overline{h}}{h},$$

is the same as the limit that we considered in Example 1.25, except scaled by a factor of $-i$. Hence this limit does not exist for the same reason; that is

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} -i\frac{\overline{h}}{h} = -i, \text{ while } \lim_{\substack{h\to 0\\ h\in i\mathbb{R}\setminus\{0\}}} -i\frac{\overline{h}}{h} = i.$$

So f is not differentiable at any point $z \in \mathbb{C}$.

Example 2.3. The function $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\mathbf{g}(x,y) = (1, -x^2-y^2) \quad (x,y) \in \mathbb{R}^2.$$

Find the corresponding complex function $g : \mathbb{C} \to \mathbb{C}$ and investigate where it is differentiable.

Solution

The corresponding function $g : \mathbb{C} \to \mathbb{C}$ is

$$g(x+iy) = 1 + i(-x^2-y^2),$$

or equivalently,

$$g(z) = 1 - iz\overline{z}.$$

Again, we fix $z \in \mathbb{C}$ and look at the difference quotient

$$\frac{g(z+h)-g(z)}{h} = -i(z\frac{\overline{h}}{h} + \overline{z} + \overline{h})$$

for $h \in \mathbb{C}\setminus\{0\}$.

As with the previous example, we have $\frac{\overline{h}}{h}$ appearing in the difference quotient, so it looks like our limit will not exist. Again, we shall check by evaluating the restricted limits along the real and imaginary axes.

$$\lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} -i\left(z\frac{\overline{h}}{h} + \overline{z} + \overline{h}\right) = -i(z+\overline{z})$$

$$\lim_{\substack{h\to 0\\ h\in i\mathbb{R}\setminus\{0\}}} -i\left(z\frac{\overline{h}}{h}+\overline{z}+\overline{h}\right) = -i(-z+\overline{z}).$$

These restricted limits are equal if and only if $z+\overline{z} = -z+\overline{z}$, which occurs if and only if $z = 0$. It follows that the unrestricted limit

$$\lim_{h\to 0}\frac{g(z+h)-g(z)}{h}$$

does not exist at any $z \in \mathbb{C}\setminus\{0\}$, hence g is not differentiable at any of these points.

What about when $z = 0$? While the restricted limits along the real and imaginary axes are equal, this is *not* enough to conclude that the unrestricted limit exists. Indeed, we must verify this directly:

$$\lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} -i\overline{h} = 0.$$

(Here we have used the fact that $z \mapsto \overline{z}$ is continuous (Example 1.29), and so limit can be found by substitution of $z = 0$.) In other words, g is differentiable at 0 (and nowhere else), with $g'(0) = 0$.

Note 2.4. In Example 2.3, both the real and the imaginary parts of g are differentiable with respect to both x and y everywhere in $\mathbb{R}^2$.

Example 2.5. Consider the function f defined by $f(z) = \dfrac{1}{z}$. At which points $z \in \mathbb{C}$ is f differentiable?

Solution

Since the domain of f is the (open) set $\mathbb{C}\setminus\{0\}$, we must necessarily restrict our attention to points z in this set. Then with $h \neq 0$, we have

$$\frac{f(z+h)-f(z)}{h} = \frac{\left(\frac{1}{z+h}\right)-\frac{1}{z}}{h} = \frac{-1}{z(z+h)}.$$

Since by assumption $z \neq 0$, the algebra of limits (Proposition 1.24) tells us that $-\dfrac{1}{z(z+h)} \to -\dfrac{1}{z^2}$ as $h \to 0$. Thus f is differentiable at every $z \in \mathbb{C}\setminus\{0\}$, with $f'(z) = -\dfrac{1}{z^2}$.

Example 2.6. Fix $\alpha \in \mathbb{C}$ and let f be the constant function $f(z) = \alpha$ for all $z \in \mathbb{C}$. Then f is differentiable at every point $z \in \mathbb{C}$, and satisfies $f'(z) = 0$ for all $z \in \mathbb{C}$.

Solution

For any $z \in \mathbb{C}$, and $h \in \mathbb{C}\setminus\{0\}$,

$$\frac{f(z+h)-f(z)}{h} = \frac{\alpha-\alpha}{h} = 0 \to 0 \text{ as } h \to 0.$$

Question. If $f : U \to \mathbb{C}$ is differentiable at every point z in U, and has $f'(z) = 0$ for all z, is f necessarily constant?

Answer

Not necessarily. If $U = U_1 \cup U_2$ where U_1 and U_2 are disjoint open subsets of $\mathbb{C}$, and $f : U \to \mathbb{C}$ is defined by

$$f(z) = \begin{cases} 0 & \text{if } z \in U_1 \\ 1 & \text{if } z \in U_2, \end{cases}$$

then $f'(z)$ exists and is equal to 0 at every $z \in U$, but f is non-constant.

Example 2.7. Fix $\beta \in \mathbb{C}$ and let $f(z) = \beta z$. Then $f'(z_0) = \beta$ for all $z_0 \in \mathbb{C}$.

2.2 Holomorphic Functions

Definition 2.8. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a complex function. If f is differentiable at z for all $z \in U$, we say that f is *holomorphic on* U.

Why *holomorphic* and not simply *differentiable* on U? One reason for this is that there are many functions f that are differentiable on the real line, but fail to be differentiable throughout any open subset $U \subseteq \mathbb{C}$.

Definition 2.9. Let $f : U \to \mathbb{C}$ be a complex function and $z \in U$. If there is some $r > 0$ such that f is differentiable at every point in the open disc $D(z, r)$, then f is said to be *holomorphic at* z.

If a function f is holomorphic at z, then it is differentiable at z, but the converse is not necessarily true.

Example 2.10. Let us examine whether or not the functions from Examples 2.2, 2.3 and 2.5 are holomorphic on any subsets of $\mathbb{C}$.

- $f(z) = -i\bar{z}$ is not differentiable at any $z \in \mathbb{C}$ and therefore not holomorphic on any $U \subseteq \mathbb{C}$.
- $f(z) = 1 - iz\bar{z}$ is differentiable at 0 and nowhere else. Since the one-point set $\{0\}$ is not an open set, there is no open subset $U \subseteq \mathbb{C}$ with f holomorphic on U.
- $f(z) = \dfrac{1}{z}$ is differentiable at every point of the (open) set $\mathbb{C}\setminus\{0\}$ and is therefore holomorphic on this set.

If $f : U \to \mathbb{C}$ is holomorphic on U, then we get a new function $f' : U \to \mathbb{C}$, $z \mapsto f'(z)$, called the *derivative of* f. In other words, for $z \in U$, $f'(z)$ is defined as the limit

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}\setminus\{0\}}} \frac{f(z+h) - f(z)}{h}.$$

As for differentiable functions in $\mathbb{R}$, we have sum, product, chain and quotient rules for holomorphic functions defined on open subsets of $\mathbb{C}$.

Theorem 2.11 (Rules of Differentiation; proof non-examinable)**.** *Let U and V be open subsets of $\mathbb{C}$ and let $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ be holomorphic on U and V respectively. Then*

1. *(Sum rule) $f + g$ is homomorphic on $U \cap V$ and $(f+g)'(z) = f'(z) + g'(z)$*
2. *(Scalar Multiples) For any $\alpha \in \mathbb{C}$, (αf) is holomorphic on U and $(\alpha f)'(z) = \alpha f'(z)$*
3. *(Product Rule) fg is holomorphic on $U \cap V$ and $(fg)'(z) = f'(z)g(z) + g'(z)f(z)$*
4. *(Quotient Rule) The quotient f/g is holomorphic on $U \cap \{z \in V : g(z) \neq 0\}$ and*
$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$
5. *(Chain Rule) $f \circ g$ is holomophic on $V \cap g^{-1}(U)$ and $(f \circ g)'(z) = f'(g(z))g'(z)$.*

Example 2.12. Let $p : \mathbb{C} \to \mathbb{C}$ be a complex polynomial

$$p(z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_n z^n.$$

Then p is holomorphic on $\mathbb{C}$.

Solution

We saw in Examples 2.6 and 2.7 that the functions $f(z) = \beta z$ and $g(z) = \alpha$ (where $\alpha, \beta \in \mathbb{C}$ are fixed), are holomorphic on $\mathbb{C}$ with derivatives $f'(z) = \beta$ and $g'(z) = 0$ for all $z \in \mathbb{C}$. Together with Theorem 2.11, this shows that $p(z)$ is holomorphic on $\mathbb{C}$ with derivative

$$p'(z) = \alpha_1 + 2\alpha_2 z + \ldots + n\alpha_n z^{n-1}.$$

Thus complex polynomials can be differentiated using exactly the same rules as for real polynomials

Note 2.13. A similar argument shows that if $g(z) = \dfrac{1}{z^n}$ where $n > 0$, then $g'(z) = -\dfrac{n}{z^{n+1}}$.

Example 2.14. The complex functions $\exp$, $\sin$ and $\cos$ defined in Chapter 1 are also holomorphic on $\mathbb{C}$, with derivatives

$$(\exp(z))' = \exp(z) \quad (\sin(z))' = \cos(z) \quad \text{and} \quad (\cos(z))' = -\sin(z).$$

Proposition 2.15 (Proof non-examinable)**.** *Let $f : U \to \mathbb{C}$ be differentiable at a point $z \in U$. Then f is continuous at z.*

Question. If f is holomophic on U, is f' holomorphic on U?

Example 2.16. In real analysis, consider $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0. \end{cases}$$

It is easily shown that g is differentiable on $\mathbb{R}$ with $g'(x) = 2\,|x|$ for all $x \in \mathbb{R}$. However, we know that $x \mapsto 2\,|x|$ fails to be differentiable at 0.

In fact for holomorphic functions, the answer is yes. This gives us an even stronger result: if $f : U \to \mathbb{C}$ is holomorphic on U, then f is infinitely differentiable on U. We shall return to this later on in the module.

2.3 The Cauchy Riemann Equations

Again, let $U \subseteq \mathbb{C}$ be open. Suppose we are given $f : U \to \mathbb{C}$, then we have seen how to write f as a sum of its real and imaginary parts:

$$f(x + iy) = u(x, y) + iv(x, y)$$

for all $z = x + iy \in U$.

Question. If u and v are differentiable with respect to both x and y, does it follow that f is holomorphic?

Answer

No. We saw in Example 2.2 that

$$f(x + iy) = 1 - i(x^2 + y^2),$$

whose real and imaginary parts are differentiable everywhere in $\mathbb{R}^2$ with respect to both x and y, is not differentiable at any point $z \in \mathbb{C} \setminus \{0\}$.

For clarity, let us recall the definition of partial derivatives.

Definition 2.17. Suppose that $u : \mathbb{R}^2 \to \mathbb{R}$ is a function. Then the *partial derivatives* of u with respect to x and y are the functions $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ respectively, defined via

$$\frac{\partial u}{\partial x}(x_0, y_0) := \lim_{\substack{h \to 0 \\ h \in \mathbb{R} \setminus \{0\}}} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h}$$
$$\frac{\partial u}{\partial y}(x_0, y_0) := \lim_{\substack{h \to 0 \\ h \in \mathbb{R} \setminus \{0\}}} \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h}$$

at the points (x_0, y_0) where the limits exist.

Theorem 2.18 (Differentiability Implies the Cauchy-Riemann Equations)**.** *Let f be a complex-valued function defined on some open set U and let $z_0 = x_0 + iy_0 \in U$. Write*

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then if f is differentiable at the point z_0 we have the following:

(i) The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist at the point $(x_0, y_0) \in \mathbb{R}^2$ corresponding to z_0.

(ii) The partial derivatives satisfy the Cauchy-Riemann Equations at (x_0, y_0):

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \tag{2.2}$$

(iii) At this point the derivative of f satisfies

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Before proving Theorem 2.18, let us look at some examples that demonstrate how it is a powerful result.

Example 2.19. Verify that the Cauchy-Riemann equations are satisfied by the function $f(z) = z^2$.

Solution

Here we have

$$f(x + iy) = \underbrace{x^2 - y^2}_{u(x,y)} + i \underbrace{2xy}_{v(x,y)},$$

and hence

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

Thus (2.2) holds at every $(x, y) \in \mathbb{R}^2$. Moreover, we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y = 2(x + iy) = 2z = f'(z),$$

as expected.

Example 2.20. Use the Cauchy-Riemann equations to investigate the differentiability of $f(z) = \overline{z}$.

Solution

This time

$$f(x + iy) = \underbrace{x}_{u(x,y)} + i \underbrace{-y}_{v(x,y)},$$

so that $\dfrac{\partial u}{\partial x} = 1$ while $\dfrac{\partial v}{\partial y} = -1$ and (2.2) does not hold at any point $(x, y) \in \mathbb{R}^2$ (note that we need both of the equations in (2.2) to hold). Thus we conclude that $f(z) = \overline{z}$ is not differentiable at any point $z \in \mathbb{C}$.

Proof

[Proof of Theorem 2.18] Since f is differentiable at z_0 we know that

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}\setminus\{0\}}} \frac{f(z_0+h)-f(z_0)}{h}$$

exists and is equal to $f'(z_0)$. In particular, all of corresponding restricted limits exist, and are also equal to $f'(z_0)$. As before, we shall examine the restricted limits along the real and imaginary axes.

If we restrict h to the nonzero real axis, then

$$z_0 + h = (x_0 + h) + iy_0,$$

which corresponds to the point $(x_0 + h, y_0) \in \mathbb{R}^2$. Thus the restricted limit satisfies:

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} \frac{f(z_0+h)-f(z_0)}{h}\\
&= \lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} \frac{[u(x_0+h,y_0)+iv(x_0+h,y_0)]-[u(x_0,y_0)+iv(x_0,y_0)]}{h}\\
&= \left(\lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} \frac{u(x_0+h,y_0)-u(x_0,y_0)}{h}\right) + i\left(\lim_{\substack{h\to 0\\ h\in\mathbb{R}\setminus\{0\}}} \frac{v(x_0+h,y_0)-v(x_0,y_0)}{h}\right)\\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).
\end{aligned}$$

Hence both $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ exist at the point (x_0, y_0), and the derivative of f at z_0 satisfies

$$f'(z_0) = f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0). \qquad (\dagger)$$

We shall now examine the restricted limit along the nonzero imaginary axis. This limit must also satisfy

$$f'(z_0) = \lim_{\substack{h\to 0\\ h\in i\mathbb{R}\setminus\{0\}}} \frac{f(z_0+h)-f(z_0)}{h}.$$

If $h \in i\mathbb{R}\setminus\{0\}$, then $h = 0 + ik$ for some $k \in \mathbb{R}\setminus\{0\}$, and so

$$z_0 + h = x_0 + i(y_0 + k),$$

which corresponds to the point $(x_0, y_0 + k) \in \mathbb{R}^2$.

Thus

$$\begin{aligned}
f'(z_0) &= \lim_{\substack{h\to 0\\ h\in i\mathbb{R}\setminus\{0\}}} \frac{f(z_0+h)-f(z_0)}{h}\\
&= \lim_{\substack{k\to 0\\ k\in \mathbb{R}\setminus\{0\}}} \frac{[u(x_0,y_0+k)+iv(x_0,y_0+k)]-[u(x_0,y_0)+iv(x_0,y_0)]}{ik}\\
&= \lim_{\substack{k\to 0\\ k\in \mathbb{R}\setminus\{0\}}} \frac{i\ [v(x_0,y_0+k)-v(x_0,y_0)]}{ik} + \lim_{\substack{k\to 0\\ k\in \mathbb{R}\setminus\{0\}}} \frac{u(x_0,y_0+k)-u(x_0,y_0)}{ik}\\
&= \lim_{\substack{k\to 0\\ k\in \mathbb{R}\setminus\{0\}}} \frac{v(x_0,y_0+k)-v(x_0,y_0)}{k} - i\left(\lim_{\substack{k\to 0\\ k\in \mathbb{R}\setminus\{0\}}} \frac{u(x_0,y_0+k)-u(x_0,y_0)}{k}\right)\\
&= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0).
\end{aligned}$$

So $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial y}$ also exist at (x_0, y_0), and the derivative of f at z_0 also satisfies

$$f'(z_0) = f'(x_0+iy_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0). \tag{‡}$$

Equating the real and imaginary parts of the two expressions (†) and (‡) for $f'(z_0)$ gives

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \text{ and } \frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0),$$

which completes the proof. The Cauchy Riemann Equations provide a very useful way of showing that a function is *not* holomorphic. We cannot use Theorem 2.18 to show that a function is holomorphic.

Example 2.21. For the function $f : \mathbb{C} \to \mathbb{C}$ defined by

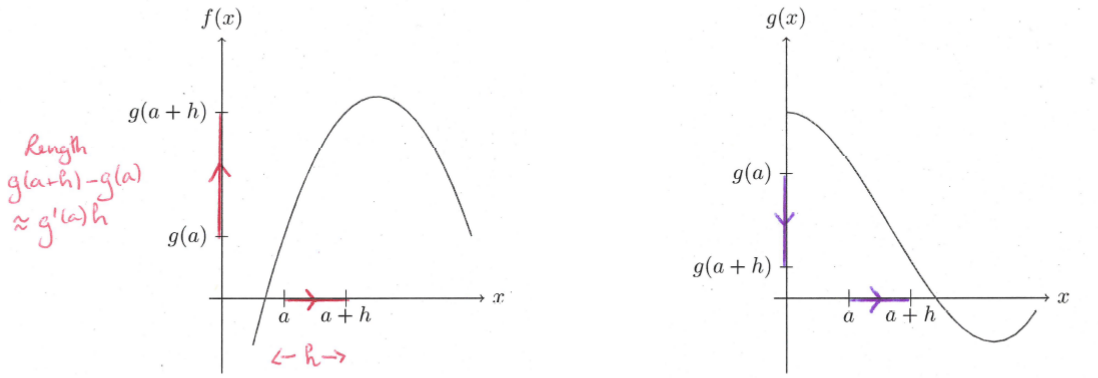
$$f(z) = \begin{cases} \exp(-z^{-4}) & z \neq 0 \\ 0 & z = 0 \end{cases}$$

the partial derivatives of u and v satisfy the Cauchy-Riemann equations everywhere, but f is not differentiable (nor even continuous) at $z = 0$. (This is difficult to prove).

2.4 Geometry of Derivatives for Complex-Valued Functions

When working with differentiable functions in $\mathbb{R}$, it is useful to have the geometric picture of the derivative of a function $g : \mathbb{R} \to \mathbb{R}$ - for example, by considering $g'(x)$ as the slope of the tangent to the graph of g at the point $(x, g(x))$. In this section we will try to give a geometric description of the derivative of a holomorphic function $f : U \to \mathbb{C}$ at a point $z_0 \in U$. Since we cannot draw the graph of such a function, some care is needed.

Returning to the real case, consider the following graph of a differentiable function $g : \mathbb{R} \to \mathbb{R}$:



For sufficiently small h we have

$$\frac{g(a+h)-g(a)}{h} \approx g'(a).$$

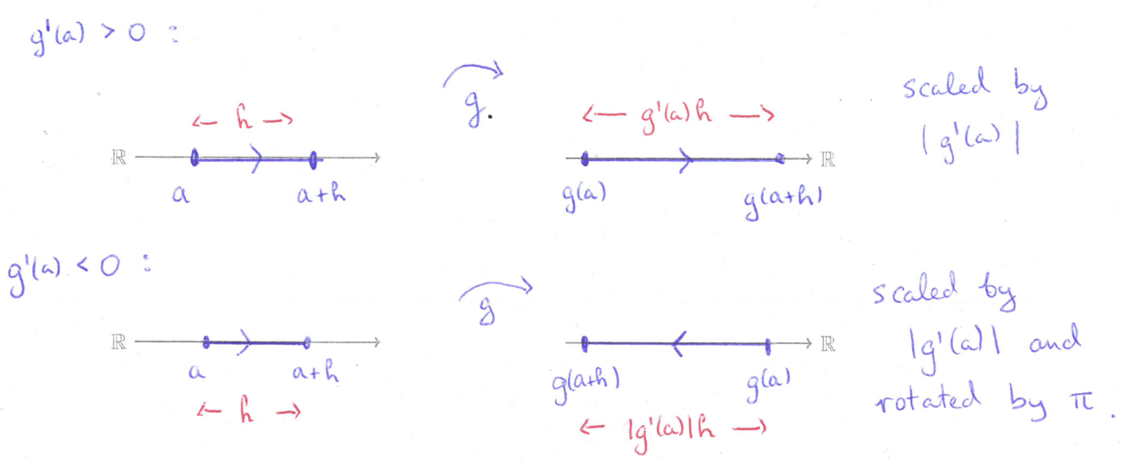
We can rewrite this approximation as

$$g(a+h)-g(a) \approx g'(a)h.$$

Now, at least when $g'(a) \neq 0$, we know that g maps the interval $[a, a+h]$, of length h, to the interval $[g(a), g(a+h)]$ of length $g(a+h)-g(a)$ i.e. length approximately $g'(a)h$.

In other words, g moves $[a, a+h]$ to an interval from $g(a)$, and (approximately) scales it by a factor of $g'(a)$. If $g'(a)$ is negative, then $[a, a+h]$ is approximately sent to $[g(a+h), g(a)]$.

Then the mapping reverses the direction of the interval, i.e., sends $[a, a+h]$ to $[g(a+h), g(a)]$. Again, the interval is scaled by a factor of $|g'(a)|$, but this time, also rotated by an angle of π. We can represent these mappings using one-dimensional figures as shown.



For a complex function $f : U \to \mathbb{C}$, its graph is the set of points

$$\{(z, f(z)) : z \in U\},$$

as subset of $\mathbb{C}^2$. We would need 4 coordinates to draw such a graph, which is impossible.

Question. How do we describe the geometry of derivatives of complex functions?

Answer

To describe the derivative of $f'(z_0)$ for a point $z_0 \in U$ geometrically, we will look at an analogy of the real case we have described above. Indeed, rather than looking at a small interval of the form $[a, a+h]$, we look at a small disk $D(z_0, r)$ centred at z_0. Again, let us restrict to the case where $g'(z_0) \neq 0$.
If r is 'small' and $z = z_0 + h \in D(z_0, r)$ then again

$$\frac{f(z_0+h) - f(z_0)}{h} \approx f'(z_0).$$

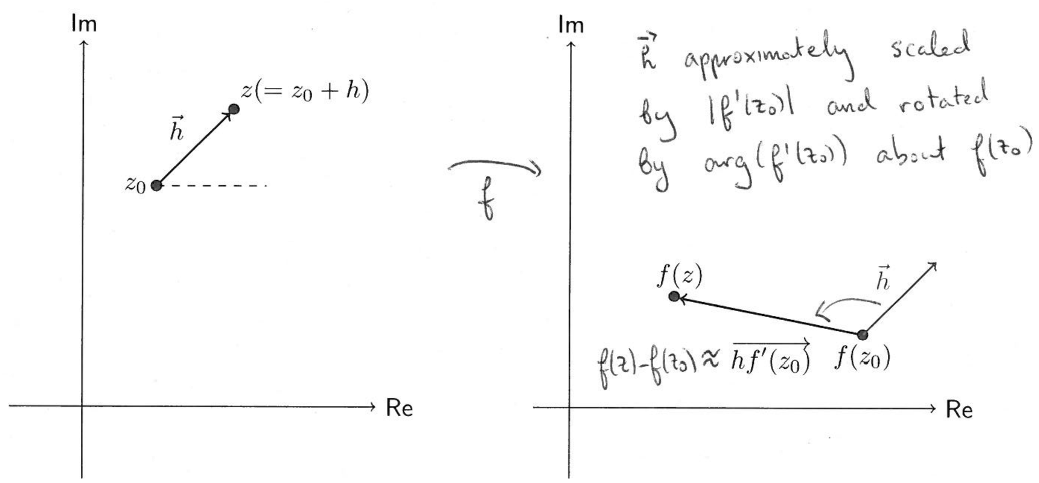
This can be rewritten as

$$f(z_0+h) - f(z_0) \approx hf'(z_0), \quad \text{i.e.} \quad f(z) - f(z_0) \approx hf'(z_0).$$

If we think of h as the vector from z_0 to z, and $f(z) - f(z_0)$ as the vector from $f(z_0)$ to $f(z)$, then roughly speaking, f sends h to $hf'(z_0)$. In other words, the vector h gets moved to a vector from $f(z_0)$, and approximately gets multiplied by $f'(z_0)$, i.e.

- scaled by a factor of $|f'(z_0)|$ and
- rotated by angle $\arg(f'(z_0))$ anticlockwise about $f(z_0)$.

Again, we are using the fact that for $z_1, z_2 \in \mathbb{C}$, $|z_1 z_2| = |z_1|\,|z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.



Since the above remarks hold for any $z \in D(z_0, r)$, the mapping f approximately transforms all points in $D(z_0, r)$ in the same way.

Summary 2.22. Let f be differentiable at a point $z_0 \in \mathbb{C}$ with $f'(z_0) \neq 0$, then f approximately maps small disks centred at z_0 to small disks centred at $f(z_0)$ as follows:

scaling by a factor of $|f'(z_0)|$ *and rotating by an angle of* $\arg(f'(z_0))$ *anticlockwise about*

$$f(z_0).$$

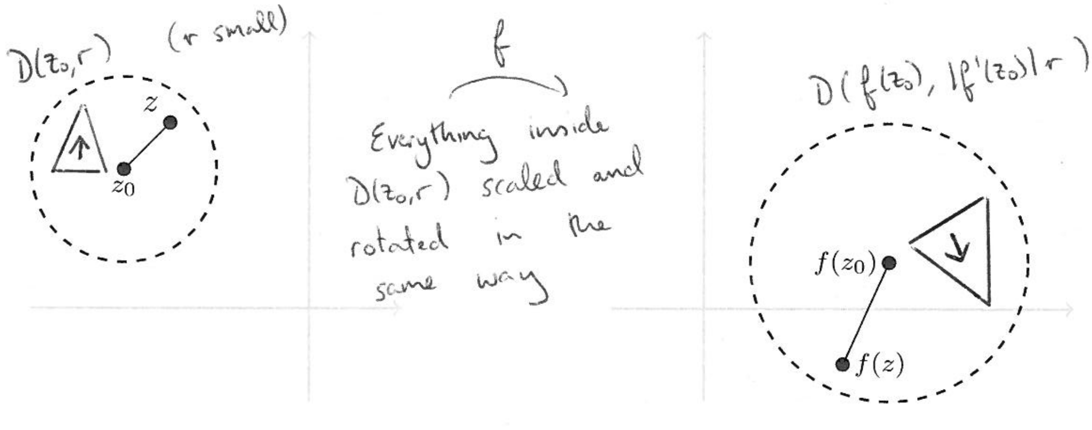


Figure 2.1: The image under f of a small disc centred at z_0.

Example 2.23. Find the geometric effect of applying the function

$$f(z) = z^2 - \frac{i}{z^2}$$

to a small disk centred at i.

Solution

Since f is differentiable at i, the previous remarks indicate that a small disc centred at i gets sent to a small disc centred at $f(i)$, where

$$f(i) = i^2 - \frac{i}{i^2} = -i + 1.$$

To determine the geometric effect of f applied to this disc, we need to calculate $|f'(i)|$ and $\arg(f'(i))$. The rules of differentiation tell us that

$$f'(z) = 2z - i\left(\frac{-2}{z^3}\right) = 2z + \frac{2i}{z^3},$$

and hence $f'(i) = 2i - 2$, which has modulus $|2i - 2| = 2\sqrt{2}$ and argument $3\pi/4$. Hence a small disc at i is approximately mapped to a small disc at $-1 + i$, and is scaled by a factor of $2\sqrt{2}$ and rotated by an angle of $3\pi/4$ in the anticlockwise direction about $-1 + i$.

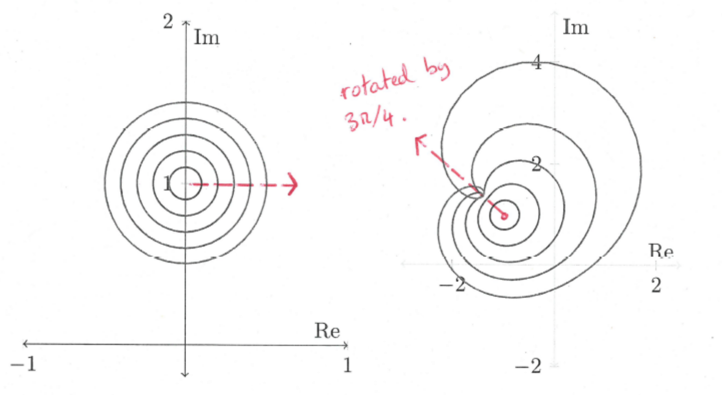


Figure 2.2: The geometric effect of applying $f(z) = z^2 - \dfrac{i}{z^2}$ to small circles centred at i. The transformed circles are centred at $f(i)$. As Figure 2.2 shows, the images of the smaller circles are almost circular, but the larger ones less so. The horizontal line from i is rotated by $\arg(f'(i))$.

Example 2.24. Determine the geometric effect of applying the function f, where $f(z) = z^3$, to a small disc centred at $z_0 = \sqrt{3} - i$.

Solution

This time, the disc gets sent to a disc centred at $f(z_0) = (\sqrt{3} - i)^3 = -4i$. Since $f'(z) = 3z^2$, we have

$$f'(\sqrt{3} - i) = 3(\sqrt{3} - i)^2 = 3(2 - i2\sqrt{3}) = 6 - i6\sqrt{3}.$$

Thus

$$\left|f'(\sqrt{3} - i)\right| = \sqrt{(6)^2 + \left(6\sqrt{3}\right)^2} = 12,$$

and

$$\arg\left(f'(\sqrt{3} - i)\right) = \arg\left(6 - i6\sqrt{3}\right) = -\pi/3.$$

Hence a small disc at $\sqrt{3} - i$ is approximately mapped to a small disc at $-4i$, and is scaled by a factor of 12 and rotated by an angle of $-\pi/3$ in the *clockwise* direction about $-4i$ (note: this time the rotation is clockwise since $\arg(f'(z_0))$ is negative).

Chapter 3 Path Integrals in the Complex Plane

3.1 Paths

In real analysis, we often consider the definite integral of a function $f : \mathbb{R} \to \mathbb{R}$ from a real number a to another real number b. Of course, on the real line, there is only one 'natural' route from a to b, namely, along the real line itself.

In order to make a sensible definition of integrating 'between' two complex numbers z_1 and z_2, we need to take into account the fact that there are typically many routes from z_1 to z_2. An interesting case arises when we consider paths with the same start and end-points.

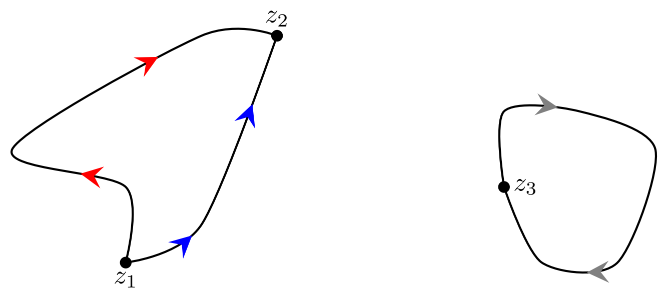


Figure 3.1: Some examples of paths between complex numbers.

Example 3.1. Consider the line segment $L = [z_1, z_2]$, that is, the straight line segment joining two complex numbers $z_1, z_2 \in \mathbb{C}$. For an interval $[a, b] \subseteq \mathbb{R}$, define

$$\gamma : [a, b] \to \mathbb{C}, \quad \gamma(t) = z_1 + \left(\frac{t-a}{b-a}\right)(z_2 - z_1).$$

Then the line segment L is precisely the range $\gamma\left([a, b]\right)$ of γ, with $\gamma(a) = z_1$ and $\gamma(b) = z_2$.

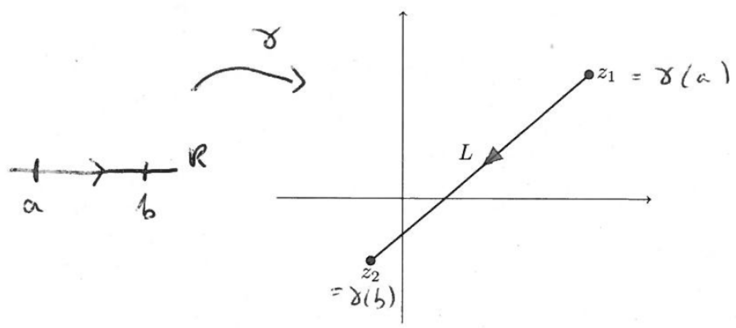


Figure 3.2: The *function* $\gamma : [a, b] \to \mathbb{C}$ describes the *set* L, and gives it a direction (i.e. from $\gamma(a) = z_1$ to $\gamma(b) = z_2$).

The process of finding $\gamma : [a, b] \to \mathbb{C}$ such that each point on L is of the form $\gamma(t)$, for some $t \in [a, b]$, is called a *parametrisation of* L, and we call t the *parameter.*

If we think of t as 'time,' then as t increases from a to b, $\gamma(t)$ moves from $\gamma(a) = z_1$ to $\gamma(b) = z_2$. It does so with 'velocity'

$$\begin{aligned}\gamma'(t) &= \frac{\text{displacement}}{\text{time}} \\ &= \frac{\gamma(b) - \gamma(a)}{b - a} = \frac{z_2 - z_1}{b - a}.\end{aligned}$$

If we think about complex numbers as points or vectors (in $\mathbb{R}^2$), this tells us that $\gamma(t)$ moves in the direction $z_2 - z_1$ with constant speed, as you might expect.

We should clarify the definition of the derivative $\gamma'(t)$ of $\gamma(t)$ before we continue.

Definition 3.2. Let $\gamma : [a, b] \to \mathbb{C}$, where $[a, b] \subseteq \mathbb{R}$, be a complex valued function of a real variable. Then for $t \in [a, b]$, γ is said to be differentiable at t if the limit

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}\setminus\{0\}}} \frac{\gamma(t + h) - \gamma(t)}{h},$$

exists. When it does exist, we denote its value by $\gamma'(t)$, called the *derivative* of γ at t.

Note that if γ is written in terms of its real and imaginary parts

$$\gamma(t) = u(t) + iv(t),$$

where $u, v : [a, b] \to \mathbb{R}$, then we have

$$\gamma'(t) = u'(t) + iv'(t),$$

at points $t \in [a, b]$ where these derivatives exist.

Definition 3.3. A *path* is a subset Γ of $\mathbb{C}$ for which there is a continuous function $\gamma : [a, b] \to \mathbb{C}$ with

$$\Gamma = \{\gamma(t) : t \in [a, b]\}.$$

The function γ is called a *parametrisation* of Γ, and we call the points $\gamma(a)$ and $\gamma(b)$ the *start-* and *end-points* of Γ respectively.

Note that Γ and γ are distinct mathematical objects: Γ is a set and γ is a function. The function γ gives a 'direction' to the path Γ ; Γ is a path *from* $\gamma(a)$ *to* $\gamma(b)$. Thus when we define a path, it is usually necessary to specify a parametrisation, or at least clarify its direction, in order to avoid ambiguity.

There are typically many functions that can be used to paramterise Γ. Having said this, it is perfectly acceptable to define Γ by specifying a parametrisation γ. Thus if we say "Consider the path defined by the function $\gamma : [a, b] \to \mathbb{C}$," then it is understood that $\Gamma = \{\gamma(t) : t \in [a, b]\}$.

Example 3.4. For the line segment $L = [z_1, z_2]$, there are many different choices of function γ that describe L.

- Taking $a = 0$ and $b = 1$, L is parametrised by

$$\gamma : [0, 1] \to \mathbb{C}, \qquad \gamma(t) = z_1 + t(z_2 - z_1).$$

  Substituting the relevant values of t we see that this path starts at $\gamma(0) = z_1$ and ends at $\gamma(1) = z_2$. Here we have $\gamma'(t) = z_2 - z_1$.

- We could also use the function

$$\gamma : [-2, 2] \to \mathbb{C},\ \gamma(t) = z_1 + \left(\frac{t - (-2)}{(2 - (-2))}\right)(z_2 - z_1) = \tfrac{1}{2}(z_1 + z_2) + t\left(\frac{z_2 - z_1}{4}\right).$$

  This time, $\gamma'(t) = \frac{1}{4}(z_2 - z_1)$, which makes sense since it takes 4 units of time for $\gamma(t)$ to move from z_1 to z_2.

- Yet another option would be the function

$$\gamma : [0, 1] \to \mathbb{C}, \qquad \gamma(t) = z_1 + t^2(z_2 - z_1),$$

  which has non-constant velocity $\gamma'(t) = 2t(z_2 - z_1)$.

  In all three cases, the *set* L is unchanged, but the *function* γ is different.

Example 3.5. Fix $R > 0$ and consider the function

$$\gamma : [0, \pi] \to \mathbb{C}, \gamma(t) = R\cos(t) + iR\sin(t).$$

Let us examine the path described by γ.

For all t,

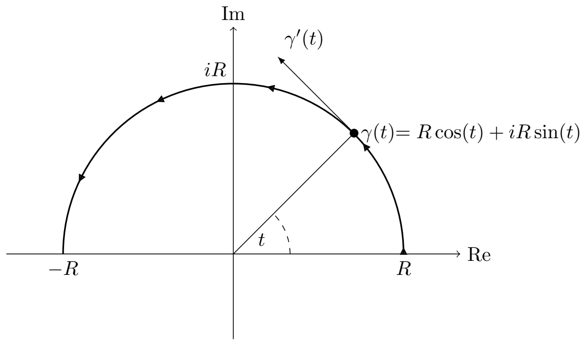
$$|R\cos(t) + iR\sin(t)| = \sqrt{R^2\cos^2(t) + R^2\sin^2(t)} = R,$$

and thus every point $\gamma(t)$ lies on the circle with centre 0 and radius R. Thus the given path consists of part of this circle.

To determine which part of the circle, we note that γ

$$\begin{aligned}
&\text{starts at } \gamma(0) = R\cos(0) + iR\sin(0) = R,\\
&\text{`visits' } \gamma(\pi/2) = R(\cos(\pi/2) + i\sin(\pi/2)) = iR\\
&\text{end at } \gamma(\pi) = R(\cos(\pi) + i\sin(\pi)) = -R.
\end{aligned}$$

Thus γ describes the upper semicircle, centre 0 and radius R, traversed in the anticlockwise direction.



The tangent vector to this path at the point $\gamma(t)$ is given by

$$\begin{aligned}\gamma'(t) &= -R\sin(t) + iR\cos(t)\\ &= i^2 R\sin(t) + iR\cos(t)\\ &= i\gamma(t).\end{aligned}$$

Thus the tangent vector to the path at the point $\gamma(t)$ is perpendicular to the position vector $\gamma(t)$ (since multiplication by i corresponds to anticlockwise rotation by $\pi/2$).

Note that in examples 3.1, 3.4 and 3.5, it is necessary to specify both $\gamma(t)$ and the domain of γ in order to describe the path completely.

Definition 3.6. We say that a parametrisation $\gamma : [a, b] \to \mathbb{C}$ of a path Γ is *smooth* if

(i) γ is differentiable on $[a, b]$,

(ii) γ' is continuous on $[a, b]$, and

(iii) γ' is nonzero on $[a, b]$.

A path Γ is called smooth if there exists a smooth parametrisation of Γ.

Informally, a smooth path is one with no corners or sharp turns. The derivative $\gamma'(t)$, regarded as a vector, is the tangent vector to the path Γ at the point $\gamma(t)$.

The paths in Examples 3.1, and 3.5 are all smooth.

Definition 3.7. Let Γ be a path with smooth parametrisation $\gamma : [a, b] \to \mathbb{C}$. Then the *reverse* of Γ is the path $\tilde{\Gamma}$ consisting of the same set of points as Γ, but traversed in the opposite direction. The path $\tilde{\Gamma}$ may be parametrised by the function $\tilde{\gamma} : [a, b] \to \mathbb{C}$ where

$$\tilde{\gamma}(t) = \gamma(a + b - t) \quad \text{for all } t \in [a, b].$$

Note that if we define the start- and end-points of Γ to be

$$\gamma(a) = z_1, \quad \gamma(b) = z_2$$

then

$$\begin{aligned}\tilde{\gamma}(a) &= \gamma(a + b - a) = \gamma(b) = z_2\\ \tilde{\gamma}(b) &= \gamma(a + b - b) = \gamma(a) = z_1.\end{aligned}$$

Thus $\tilde{\Gamma}$ starts at z_2 and ends at z_1. Moreover, if $t \in [a, b]$ then $a \leq a + b - t \leq b$, and so $\tilde{\gamma}(t)$ describes a point on the original path Γ (i.e. $\tilde{\Gamma}$ is the same set as Γ).

The tangent vector is given by

$$\tilde{\gamma}'(t) = \frac{d}{dt}\gamma(a + b - t) = \gamma'(a + b - t)\frac{d}{dt}(a + b - t) = -\gamma'(a + b - t).$$

In other words, at any point $z = \tilde{\gamma}(t) = \gamma(a + b - t)$ on the path Γ, the tangent vector to this path at z, when travelling in the reverse direction, points in the opposite direction to the tangent vector obtained when travelling in the original direction, as expected.

Example 3.8. Let us find the reverse of the paths considered in Examples 3.4 and 3.5.

Solution

- $L = [z_1, z_2]$, parametrised by $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(t) = z_1 + t(z_2 - z_1)$. It is clear that the reverse of L is the line segment $\tilde{L} = [z_2, z_1]$, i.e. the same line segment taken from z_2 to z_1. We may parametrise $\tilde{L}$ with the function $\tilde{\gamma} : [0, 1] \to \mathbb{C}$, where

$$\begin{aligned}\tilde{\gamma}(t) &= \gamma(0 + 1 - t)\\ &= z_1 + (1 - t)(z_2 - z_1)\\ &= z_2 + t(z_1 - z_2).\end{aligned}$$

  On examining the equation for $\tilde{\gamma}(t)$, it is clear that this function describes the path that starts at z_2 and travels in the direction $(z_1 - z_2)$.

- Γ the semicircle parametrised by $\gamma : [0, \pi] \to \mathbb{C}$, $\gamma(t) = R\cos(t) + iR\sin(t)$.

  In this case $\tilde{\Gamma}$ is the clockwise semicircular arc from $-R$ to R via iR. We may parametrise $\tilde{\Gamma}$ using $\tilde{\gamma} : [0, \pi] \to \mathbb{C}$, where

$$\begin{aligned}\tilde{\gamma}(t) &= \gamma(\pi - t)\\ &= R\cos(\pi - t) + iR\sin(\pi - t)\\ &= -R\cos(t) + iR\sin(t).\end{aligned}$$

  Note that $\tilde{\gamma}(t)$ is $\gamma(t)$ reflected through the imaginary axis; and thus the path $\tilde{\Gamma}$ is the reflection of the path Γ in the imaginary axis.

We shall often need to consider the paths that we obtain from joining smooth paths together. Note that the resulting path may fail to be smooth, e.g., if there is a 'corner' at the point where they meet.

Suppose we have two or more (smooth) paths Γ_1, Γ_2 etc., parametrised by $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$, and suppose that the end-point of Γ_1 is the same as the start-point of Γ_2, i.e. $\gamma_1(b_1) = \gamma_2(a_2)$.

The curve obtained from $\Gamma_1 \cup \Gamma_2$ certainly looks like a path, but how do we make this precise? In other words, can we describe this curve as the image of some continuous $\gamma : [a, b] \to \mathbb{C}$?

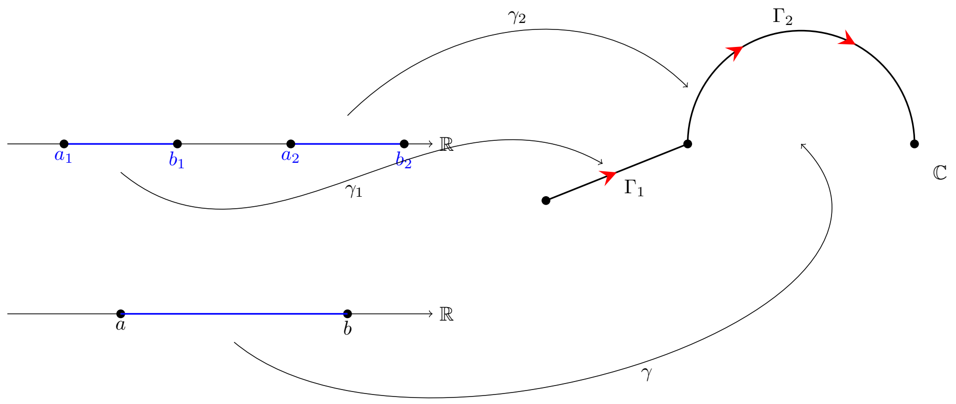
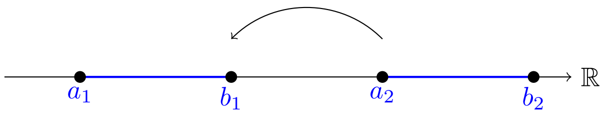


Figure 3.3: We have two functions $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ parameterising Γ_1 and Γ_2 respectively. We want a single continuous function $\gamma : [a, b] \to \mathbb{C}$ parameterising $\Gamma_1 \cup \Gamma_2$

.

To do this, we first move $[a_2, b_2]$ to an interval from b_1 and then reparametrise Γ_2. For $t \in \mathbb{R}$, let $\alpha(t) = t - (a_2 - b_1)$, so that α maps $[a_2, b_2]$ bijectively to $[b_1, b_1 + b_2 - a_2]$.



Now, parametrise Γ_2 using the function $\gamma_{1+2} : [b_1, b_1 + b_2 - a_2] \to \mathbb{C}$, where

$$\gamma_{1+2}(t) = \gamma_2(\alpha^{-1}(t)) = \gamma_2(t + a_2 - b_1)$$

for all $t \in [b_1, b_1 + b_2 - a_2]$.

Definition 3.9. Suppose that Γ_1 and Γ_2 are two paths parametrised by $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ respectively, such that $\gamma_1(b_1) = \gamma_2(a_2)$. Then we define the *join* of Γ_1 and Γ_2 to be the path $\Gamma_1 + \Gamma_2$ parametrised by $\gamma_{1+2} : [a_1, b_1 + b_2 - a_2] \to \mathbb{C}$, where

$$\gamma_{1+2}(t) = \begin{cases} \gamma_1(t) & t \in [a_1, b_1] \\ \gamma_2(t - b_1 + a_2) & t \in [b_1, b_1 + b_2 - a_2]. \end{cases}$$

If we have another path Γ_3 we can join Γ_3 to $\Gamma_1 + \Gamma_2$ to get $\Gamma_1 + \Gamma_2 + \Gamma_3$ and so on.

Definition 3.10. A *contour* is a path which is the join of finitely many smooth paths.

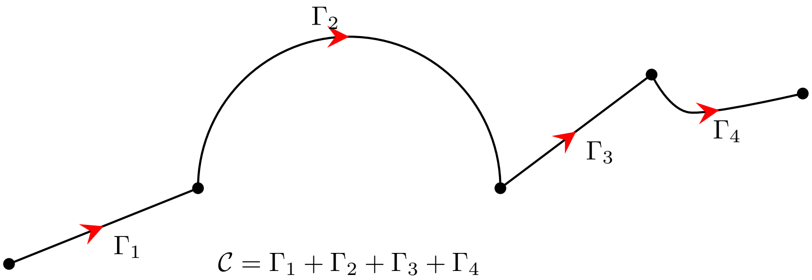


Figure 3.4: A contour $\mathcal{C}$ constructed from the join of smooth paths $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$. Note that $\mathcal{C}$, considered as a path in its own right, need not be smooth.

Definition 3.11. Let Γ be a smooth path parametrised by $\gamma : [a, b] \to \mathbb{C}$. Then the *length* $\ell(\Gamma)$ of Γ is defined to be

$$\ell(\Gamma) := \int_a^b \left|\gamma'(t)\right| \, dt.$$

For a contour $\mathcal{C} = \Gamma_1 + \Gamma_2 + \ldots + \Gamma_n$, where $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are smooth, we define the *length* $\ell(\mathcal{C})$ of $\mathcal{C}$ to be

$$\ell(\mathcal{C}) = \ell(\Gamma_1) + \ell(\Gamma_2) + \ldots + \ell(\Gamma_n).$$

Example 3.12. Let us compute the length of the line segment $L = [0, 3 + 4i]$ using the parametrisation $\gamma : [0, 1] \to \mathbb{C}$ where

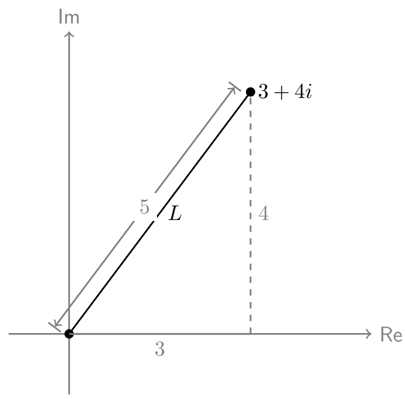
$$\gamma(t) = (3 + 4i)t, \quad t \in [0, 1].$$

Solution

We have $\gamma'(t) = 3 + 4i$, with modulus $|\gamma'(t)| = |3 + 4i| = 5$ for all t. Hence

$$\ell(L) = \int_0^1 \left|\gamma'(t)\right| \, dt = \int_0^1 5 \, dt = 5,$$

which is what we would expect given the geometry of this line segment.



Example 3.13. Now we shall compute the length of the semicircular path Γ described by $\gamma : [0, \pi] \to \mathbb{C}$

$$\gamma(t) = R\cos(t) + iR\sin(t), \quad t \in [0, \pi].$$

Solution

We have shown already that for all $t \in [0, \pi]$, $\gamma'(t) = i\gamma(t)$, and so

$$\left|\gamma'(t)\right| = |i\gamma(t)| = |i|\,|R\cos(t) + iR\sin(t)| = R$$

for all such t. It follows that

$$\ell(\Gamma) = \int_0^{\pi} R\ dt = \pi R,$$

which is again what we would expect for a semicircle with radius R.

3.2 The Integral along a path in $\mathbb{C}$

In this section we will define the integral of a complex function along a smooth path Γ. We first recall the following results about integrals of real functions from Foundations.

Theorem 3.14. *Let $f, g : [a, b] \to \mathbb{R}$ be two* continuous *real valued functions defined on some interval $[a, b] \subset \mathbb{R}$. Then the integrals $\int_a^b f(t)\ dt$ and $\int_a^b g(t)\ dt$ both exist and satisfy the following properties:*

(i) (Linearity) For any $c \in \mathbb{R}$ we have

$$\int_a^b (cf(t) + g(t))\ \ dt = c\int_a^b f(t)\ dt + \int_a^b g(t)\ dt.$$

(ii) (Fundamental Theorem of Calculus) If $F : [a, b]$ is an antiderivative for f on $[a, b]$ (that is, $F'(t) = f(t)$ for all $t \in [a, b]$), then

$$\int_a^b f(t)\ dt = F(b) - F(a),$$

(iii) (Monotonicity) If $f(t) \leq g(t)$ for all $t \in [a, b]$ then

$$\int_a^b f(t)\ dt \leq \int_a^b g(t)\ dt.$$

We shall frequently use the results of Theorem 3.14 without reference.

Now let us define the integral of a complex-valued function of a real variable, $g : [a, b] \to \mathbb{C}$ defined on some interval $[a, b] \subset \mathbb{R}$.

Definition 3.15. Let $g : [a, b] \to \mathbb{C}$ be a complex valued function defined on the (real) interval $[a, b]$, and assume that the real and imaginary parts $\mathsf{Re}(g)$ and $\mathsf{Im}(g)$ are both continuous. Then we define the integral of g from a to b via

$$\int_a^b g(t)\ dt = \int_a^b \mathsf{Re}(g)(t)\ dt + i\int_a^b \mathsf{Im}(g)(t)\ dt.$$

Note that the functions $\mathsf{Re}(g)$ and $\mathsf{Im}(g)$ are both real-valued and continuous, and so we have defined the integral of g in terms of integrals of real functions. So for example,

$$\int_0^1 t + i2t\ dt = \int_0^1 t\ dt + i\int_0^1 2t\ dt = \left[\frac{t^2}{2}\right]_0^1 + i\left[2\frac{t^2}{2}\right]_0^1 = \frac{1}{2} + i.$$

Using the definition of the integral of a complex-valued function of a real variable, together with Theorem 3.14, the results of Theorem 3.16 follow easily.

Theorem 3.16. *Let $f, g : [a, b] \to \mathbb{C}$ be complex valued functions of a real variable defined on the interval $[a, b] \subset \mathbb{R}$, and assume that the real and imaginary parts of both f and g are all continuous. Then the integrals $\int_a^b f(t)\ dt$ and $\int_a^b g(t)\ dt$ both exist and satisfy the following properties:*

(i) (Linearity) For any $\alpha \in \mathbb{C}$ we have

$$\int_a^b (\alpha f(t) + g(t))\ dt = \alpha\int_a^b f(t)\ dt + \int_a^b g(t)\ dt.$$

(ii) (Fundamental Theorem of Calculus) If $F : [a, b] \to \mathbb{C}$ is an antiderivative for f on $[a, b]$ (that is, $F'(t) = f(t)$ for all $t \in [a, b]$), then

$$\int_a^b f(t)\ dt = F(b) - F(a).$$

Note that there is no way to extend Theorem 3.14(iii) to complex valued functions, as there is no sensible way to interpret the expression $\alpha \leq \beta$ for $\alpha, \beta \in \mathbb{C}$.

Definition 3.17. Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ a continuous function and let Γ be a smooth path contained in U, parametrised by $\gamma : [a, b] \to \mathbb{C}$. Then the *integral of f along γ*, which we write as

$$\int_\Gamma f,$$

is defined via

$$\int_\Gamma f = \int_a^b f(\gamma(t))\,\gamma'(t)\ dt. \tag{3.1}$$

Note 3.18. (i) The composition $f(\gamma(t))$ is a complex valued function of a real variable, hence so is $f(\gamma(t))\,\gamma'(t)$. It follows that the integral

$$\int_a^b f(\gamma(t))\,\gamma'(t)\ dt$$

on the right hand side of (3.1) is of the type defined in Definition 3.15.

(ii) It is sometimes convenient to write

$$\int_\Gamma f \text{ as } \int_\Gamma f(z)\ dz \text{ or } \int_\Gamma f(\zeta)\ d\zeta.$$

(iii) The value of the integral depends on both the function f and the path Γ. It looks like it should also depend on our choice of parametrisation γ, but this is not the case.

Example 3.19. Fix α and $\beta \in \mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \alpha z + \beta \overline{z}.$$

Let us compute the value of $\int_\Gamma f$ along each of the three paths

$$\begin{aligned} \Gamma_1 &= [0, 2] \\ \Gamma_2 &= [2, 2 + 2i] \\ \Gamma_3 &= [2 + 2i, 0]. \end{aligned}$$

Solution

Following the method of Example 3.4, parametrise each Γ_j by the function $\gamma_j : [0, 1] \to \mathbb{C}$, where

$$\begin{aligned} \gamma_1(t) &= 2t \\ \gamma_2(t) &= 2 + i2t \\ \gamma_3(t) &= (1 - t)(2 + 2i) \end{aligned}$$

for $t \in [0, 1]$.
For the path Γ_1, we have

$$f(\gamma_1(t)) = \alpha(2t) + \beta(\overline{2t}) = (\alpha + \beta)(2t) \text{ and } \gamma_1'(t) = 2$$

for all $t \in [0, 1]$. Thus

$$\begin{aligned} \int_{\Gamma_1} f &= \int_0^1 f(\gamma_1(t))\gamma_1'(t)\ dt \\ &= \int_0^1 (\alpha + \beta)(2t)(2)\ dt \\ &= 4(\alpha + \beta) \int_0^1 t\ dt = 2(\alpha + \beta). \end{aligned}$$

For Γ_2, we have

$$\begin{aligned} f(\gamma_2(t)) &= \alpha(2 + i2t) + \beta(\overline{2 + i2t}) \\ &= \alpha(2 + i2t) + \beta(2 - 2it) \end{aligned}$$

and

$$\gamma_2'(t) = 2i.$$

hence

$$\begin{aligned} f(\gamma_2(t))\gamma_2'(t) &= \alpha(2 + i2t) + \beta(2 - 2it)2i \\ &= -4(\alpha - \beta)t + 4i(\alpha + \beta), \end{aligned}$$

and so the required path integral is

$$\begin{aligned} \int_{\Gamma_2} f &= \int_0^1 [-4(\alpha - \beta)t + 4i(\alpha + \beta)]\ dt \\ &= \left(-4(\alpha - \beta) \int_0^1 t\ dt\right) + i \left(4(\alpha + \beta) \int_0^1 1\ dt\right) \\ &= -2(\alpha - \beta) + 4i(\alpha + \beta). \end{aligned}$$

Finally, we have

$$\begin{aligned} f(\gamma_3(t)) &= 2(\alpha+\beta)(1-t) + i2(\alpha-\beta)(1-t) \\ \gamma_3'(t) &= -2-2i \end{aligned}$$

so that

$$f(\gamma_3(t))\gamma_3'(t) = [\alpha(2+2i) + \beta(2-2i)]\,(-2-2i)(1-t)$$

giving

$$\begin{aligned} \int_{\Gamma_3} f &= [\alpha(2+2i) + \beta(2-2i)]\,(-2-2i)\int_0^1 (1-t)\ dt \\ &= -4\beta - 4\alpha i. \end{aligned}$$

Example 3.20. Find

$$\int_\Gamma f,$$

where f is the complex function $f(z) = \overline{z}$ and Γ is the semicircular path joining 1 and -1 defined by $\gamma : [0, \pi] \to \mathbb{C}$, $\gamma(t) = \cos(t) + i\sin(t)$.

Solution

Here we have

$$f(\gamma(t)) = \overline{\cos(t) + i\sin(t)} = \cos(t) - i\sin(t),$$

and $\gamma'(t) = -\sin(t) + i\cos(t)$, so that

$$\begin{aligned} f(\gamma(t))\gamma'(t) &= (-\sin(t) + i\cos(t))\,(\cos(t) - i\sin(t)) \\ &= -\cos(t)\sin(t) + \sin(t)\cos(t) + i\,[(-\sin(t))(-\sin(t)) + \cos(t)\cos(t)] \\ &= i. \end{aligned}$$

Hence

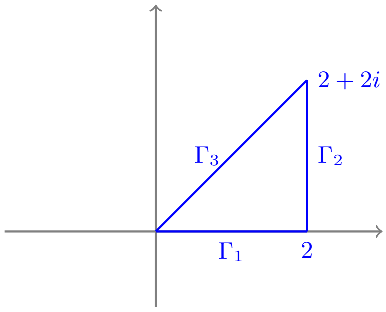
$$\int_\Gamma \overline{z}\ dz = \int_0^\pi i\ dt = \pi i.$$

We now extend the definition of the integral along a smooth path to the integral along a contour $\mathcal{C}$ which is the join of a finite number of smooth paths $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ as follows:

$$\int_{\mathcal{C}} f = \int_{\Gamma_1} f + \int_{\Gamma_2} f + \ldots + \int_{\Gamma_n} f$$

Example 3.21. Compute the value of $\int_{\mathcal{C}} f$ where $f = \alpha z + \beta\overline{z}$ and $\mathcal{C} = \Gamma_1 + \Gamma_2 + \Gamma_3$ from Example 3.19.

Solution



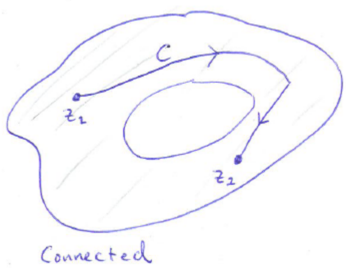
Here we have

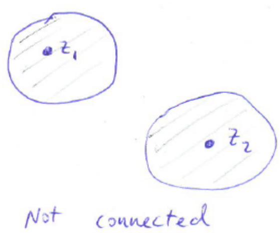
$$\begin{aligned}\int_{\mathcal{C}} f &= \int_{\Gamma_1} f + \int_{\Gamma_2} f + \int_{\Gamma_3} f \\ &= 2(\alpha+\beta) - 2(\alpha-\beta) + 4i(\alpha+\beta) \\ &- 4\beta - 4\alpha i \\ &= i4\beta.\end{aligned}$$

Note that the function $f(z) = \alpha z + \beta\overline{z}$ is not differentiable anywhere in $\mathbb{C}$ unless $\beta = 0$. Moreover, if $\beta = 0$ then $f(z) = \alpha z$ is holomorphic on $\mathbb{C}$ and $\int_{\mathcal{C}} f = 0$ for the contour $\mathcal{C} = \Gamma_1 + \Gamma_2 + \Gamma_3$ in the preceding example, while $\int_{\mathcal{C}} f \neq 0$ when $\beta \neq 0$. This is not a coincidence, as we shall see in subsequent sections.

3.3 The Fundamental Theorem of Complex Calculus

Definition 3.22. A set $S \subseteq \mathbb{C}$ is *connected* if given any pair of points $z_1, z_2 \in S$, there is a contour contained in S that starts at z_1 and ends at z_2. A *region* $\mathcal{R}$ is a non-empty, open, connected subset of $\mathbb{C}$.





Definition 3.23. Let $\mathcal{R}$ be a region and $f : \mathcal{R} \to \mathbb{C}$ a function defined on $\mathcal{R}$. A function $F : \mathcal{R} \to \mathbb{C}$ is called an *antiderivative for* f *on* $\mathcal{R}$ if

(i) F is holomorphic on $\mathcal{R}$ and

(ii) $F'(z) = f(z)$ for all $z \in \mathcal{R}$.

Example 3.24. Find antiderivatives for the functions

(i) $f(z) = \alpha z + \beta$ (where $\alpha, \beta \in \mathbb{C}$ are fixed) on the region $\mathbb{C}$.

(ii) $f(z) = \dfrac{1}{(1+iz)^2}$ on the region $\mathbb{C}\setminus\{i\}$.

Solution

(i) The function F defined by $F(z) = \frac{\alpha}{2}z^2 + \beta z$ is an antiderivative for f on $\mathbb{C}$, as F is holomorphic on $\mathbb{C}$ with $F'(z) = f(z)$ for all z.

(ii) The function $F : \mathbb{C}\setminus\{i\} \to \mathbb{C}$ defined by

$$F(z) = \frac{i}{1+iz}$$

is an antiderivative for f on $\mathbb{C}\setminus\{i\}$ by the quotient rule.

Question. Does the function $f(z) = \overline{z}$ have an antiderivative on $\mathbb{C}$? We will answer this question shortly.

We know already that $f(z) = \overline{z}$ cannot be the antiderivative of any function $g : \mathbb{C} \to \mathbb{C}$ as f is not differentiable anywhere.

Theorem 3.25 (Fundamental Theorem of Complex Calculus)**.** *Let f be continuous on a region $\mathcal{R}$ suppose that F is an antiderivative for f on $\mathcal{R}$. If $\mathcal{C}$ is a contour contained in $\mathcal{R}$, then we have*

$$\int_{\mathcal{C}} f = F(z_2) - F(z_1)$$

where z_1 and z_2 are the start- and end-points of $\mathcal{C}$ respectively.

Proof

Let us first consider the case where $\mathcal{C}$ consists of a single smooth path Γ, parametrised by $\gamma : [a,b] \to \mathbb{C}$. Since F is an antiderivative for f on $\mathcal{R}$ and γ is smooth, the Chain Rule gives

$$\frac{d}{dt}[F(\gamma(t))] = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t).$$

Hence

$$\begin{aligned}
\int_{\Gamma} f &= \int_a^b f(\gamma(t))\gamma'(t)\ dt \\
&= \int_a^b \frac{d}{dt}[F(\gamma(t))]\ dt \\
&= F(\gamma(b)) - F(\gamma(a)) && \text{by Theorem 3.16(ii)} \\
&= F(z_2) - F(z_1).
\end{aligned}$$

Now, if $\mathcal{C}$ is the join of n smooth paths $\mathcal{C} = \Gamma_1 + \ldots + \Gamma_n$, let w_{j-1} and w_j denote the start- and end-points of Γ_j respectively, so that $w_0 = z_1$ and $w_n = z_2$. Then

$$\int_{\mathcal{C}} f = \int_{\Gamma_1} f + \int_{\Gamma_2} f + \ldots + \int_{\Gamma_n} f$$

$$\begin{aligned} &= (F(w_1) - F(w_0)) + (F(w_2) - F(w_1)) + \ldots + (F(w_n) - F(w_{n-1})) \\ &= F(w_n) - F(w_0) = F(z_2) - F(z_1). \end{aligned}$$

The Fundamental Theorem of Complex Calculus tells us that if an antiderivative F for f on $\mathcal{R}$ is known, then a potentially complicated contour integral $\int_{\mathcal{C}} f$ may be evaluated by simply computing the values of $F(z_1)$ and $F(z_2)$.

Example 3.26. Let $\alpha \in \mathbb{C}$ be fixed and let $f(z) = \alpha z$ for all $z \in \mathbb{C}$. We shall evaluate $\int_{\mathcal{C}} f$ along the contour $\mathcal{C} = \Gamma_1 + \Gamma_2$ where $\Gamma_1 = [0, 2]$ and $\Gamma_2 = [2, 2 + 2i]$ using Theorem 3.25.

Solution

The function F defined by $F(z) = \frac{\alpha}{2} z^2$ is an antiderivative for f on $\mathbb{C}$. The start- and end-points of $\mathcal{C}$ are 0 and $2 + 2i$ respectively. Hence by the Fundamental Theorem of Complex Calculus,

$$\begin{aligned} \int_{\mathcal{C}} f &= F(2 + 2i) - F(0) \\ &= \frac{\alpha(2 + 2i)^2}{2} - 0 \\ &= i4\alpha. \end{aligned}$$

Example 3.27. With f, Γ_1 and Γ_2 as in Example 3.26 and let $\Gamma_3 = [2 + 2i, 0]$. We will calculate $\int_{\mathcal{C}} f$ where $\mathcal{C} = \Gamma_1 + \Gamma_2 + \Gamma_3$.

Solution

This time $\mathcal{C}$ starts and ends at 0, so that

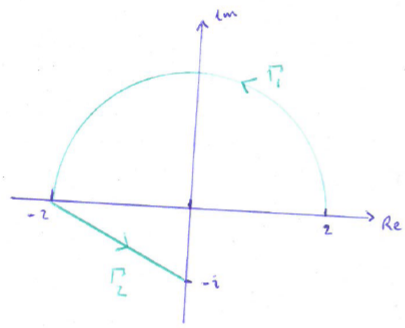
$$\int_{\mathcal{C}} f = F(0) - F(0) = 0.$$

Example 3.28. Let Γ_1 be the path consisting of the arc of the circle of radius 2, centre 0, traversed in the anticlockwise direction from 2 to -2 and let Γ_2 be the line segment $[-2, -i]$. Calculate

$$\int_{\mathcal{C}} f,$$

where $f(z) = \dfrac{1}{(1 + iz)^2}$ and $\mathcal{C} = \Gamma_1 + \Gamma_2$.

Solution



The contour $\mathcal{C}$ is contained in the region $\mathbb{C}\setminus\{i\}$, and $F(z) = \frac{i}{(1+iz)}$ is an antiderivative for f on this region. Since $\mathcal{C}$ starts at 2 and ends at $-i$, we have

$$\begin{aligned}\int_{\mathcal{C}} f &= F(-i) - F(2)\\ &= \frac{i}{2} - \frac{i}{1+2i}\\ &= -\frac{2}{5} + i\frac{3}{10}.\end{aligned}$$

Theorem 3.29 (Contour Independence)**.** *Let f be continuous on $\mathcal{R}$ and let F be an antiderivative for f on $\mathcal{R}$. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two contours inside $\mathcal{R}$ with the same start- and end-points, we have*

$$\int_{\mathcal{C}_1} f = \int_{\mathcal{C}_2} f.$$

Proof

Let z_1 be common the start-point of both $\mathcal{C}_1$ and $\mathcal{C}_2$, and z_2 the end-point. Then by Theorem 3.25,

$$\int_{\mathcal{C}_1} f = F(z_2) - F(z_1) = \int_{\mathcal{C}_2} f.$$

One application of Theorem 3.29 is that it allows us to replace a potentially complicated contour integral along $\mathcal{C}_1$ with an easier one alone $\mathcal{C}_2$.

Definition 3.30. A contour $\mathcal{C}$ is called a *closed contour* if its end point is the same as its start point.

Theorem 3.31 (Antiderivatives and Closed Contours)**.** *Let f be continuous on $\mathcal{R}$ and let F be an antiderivative for f on $\mathcal{R}$. If $\mathcal{C}$ is any closed contour inside $\mathcal{R}$ then*

$$\int_{\mathcal{C}} f = 0.$$

Proof

This time $\mathcal{C}$ has the same start- and end-point z_1. Hence

$$\int_{\mathcal{C}} f = F(z_1) - F(z_1) = 0.$$

Question. Now, do we know whether or not $f(z) = \overline{z}$ has an antiderivative on $\mathbb{C}$? Does Example 3.21 tell you anything?

Answer

The function $f(z) = \overline{z}$ is a special case of the function considered in Example 3.21, with $\alpha = 0$ and $\beta = 1$. If $\mathcal{C}$ is the closed triangular contour of that example, then we have seen that

$$\int_{\mathcal{C}} \overline{z}\, dz = 4i \neq 0.$$

Thus as a consequence of Theorem 3.31, we see that $f(z) = \overline{z}$ cannot have an antidervative on $\mathbb{C}$.

Theorem 3.32 (Zero Derivative Theorem). *Let F be holomorphic on a region $\mathcal{R}$ and suppose that $F'(z) = 0$ for all $z \in \mathcal{R}$. Then F is constant on $\mathcal{R}$.*

Proof

Let $z_1, z_2 \in \mathcal{R}$. We will show that $F(z_1) = F(z_2)$.

Since $\mathcal{R}$ is connected there is a contour $\mathcal{C}$ in $\mathcal{R}$ that starts at z_1 and ends at z_2. Hence by Theorem 3.25,

$$F(z_2) - F(z_1) = \int_{\mathcal{C}} F'(z)\, dz = \int_{\mathcal{C}} 0\, dz = 0.$$

In other words $F(z_1) = F(z_2)$. Since this is true for all $z_1, z_2 \in \mathcal{R}$, F must be constant on $\mathcal{R}$.

Chapter 4 Cauchy's Theorem

4.1 The Estimation Lemma

The Estimation Lemma is an extremely important result that will be used to prove a number of important Theorems in subsequent sections. For this reason, it has been promoted to the rank of 'Theorem.'

Theorem 4.1 (The Estimation Lemma). *Let f be a function which is continuous along a smooth path Γ given by the function $\gamma : [a, b] \to \mathbb{C}$, then*

$$\left| \int_\Gamma f \right| \leq \int_a^b |f(\gamma(t))| \left|\gamma'(t)\right| \, dt,$$

and if there exists a real number $M > 0$ with $|f(z)| \leq M$ for all $z \in \Gamma$,

$$\left| \int_\Gamma f \right| \leq ML,$$

where L is the length of Γ.

Proof

By definition of the integral of f along Γ, we have

$$\left| \int_\Gamma f \right| = \left| \int_a^b f\left(\gamma(t)\right) \gamma'(t) \, dt \right|,$$

and thus we need to show that

$$\left| \int_a^b f\left(\gamma(t)\right) \gamma'(t) \, dt \right| \leq \int_a^b \left|f\left(\gamma(t)\right)\right| \, \left|\gamma'(t)\right| \, dt.$$

Let $g(t) = f(\gamma(t))\gamma'(t)$, so that we need to show

$$\left| \int_a^b g(t) \, dt \right| \leq \int_a^b |g(t)| \, dt.$$

Note that if $\int_a^b g(t) = 0$, then the inequality holds trivially. If not, let

$$\lambda = \frac{\left| \int_a^b g(t) \, dt \right|}{\int_a^b g(t) \, dt},$$

and note that $|\lambda| = 1$ and

$$\lambda \int_a^b g(t) \, dt = \left| \int_a^b g(t) \, dt \right|.$$

It follows that

$$\begin{aligned}\left|\int_a^b g(t)\ dt\right| &= \int_a^b \lambda g(t)\ dt \\ &= \int_a^b \mathsf{Re}\left(\lambda g(t)\right)\ dt + i\int_a^b \mathsf{Im}\left(\lambda g(t)\right)\ dt.\end{aligned}$$

Since the modulus is always real, we must have

$$\int_a^b \mathsf{Im}\left(\lambda g(t)\right)\ dt = 0$$

and so

$$\left|\int_a^b g(t)\ dt\right| = \int_a^b \mathsf{Re}\left(\lambda g(t)\right)\ dt.$$

Now, since $\mathsf{Re}(z) \leq |z|$ for all $z \in \mathbb{C}$, we have

$$\mathsf{Re}\left(\lambda g(t)\right) \leq |\lambda g(t)| = |\lambda|\,|g(t)| = |g(t)|\,.$$

Together with Monotonicity of the real integral (That is to say, if $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ with $\phi_1(t) \leq \phi_2(t)$ for all t, we have $\int_a^b \phi_1(t)\ dt \leq \int_a^b \phi_2(t)\ dt$ (Theorem 3.14)), we have

$$\begin{aligned}\left|\int_a^b g(t)\ dt\right| &= \int_a^b \mathsf{Re}\left(\lambda g(t)\right)\ dt \\ &\leq \int_a^b |g(t)|\ dt.\end{aligned}$$

In other words, we have shown that

$$\left|\int_\Gamma f\right| = \left|\int_a^b f\left(\gamma(t)\right)\gamma'(t)\ dt\right| \leq \int_a^b \left|f\left(\gamma(t)\right)\right|\ \left|\gamma'(t)\right|\ dt.$$

For the second part, if there is some $M > 0$ with $|f(z)| \leq M$ for all $z \in \Gamma$, then monotonicity and linearity of the real integral gives us

$$\begin{aligned}\left|\int_\Gamma f\right| &\leq \int_a^b M\left|\gamma'(t)\right|\ dt \\ &= M\int_a^b \left|\gamma'(t)\right|\ dt \\ &= ML,\end{aligned}$$

where the final equality follows from the definition of the length of a smooth path. It follows easily from the definition of the integral of f along a contour $\mathcal{C} = \Gamma_1 + \ldots + \Gamma_n$ that

$$\left|\int_{\mathcal{C}} f\right| \leq ML$$

where $|f(z)| \leq M$ for all $z \in \mathcal{C}$ and L is the length of $\mathcal{C}$.

Note 4.2. If we can show that

$$\left|\int_{\mathcal{C}} f\right| \leq K,$$

then K is called an *upper estimate* for $\displaystyle\int_{\mathcal{C}} f$

Example 4.3. The function f is defined by

$$f(z) = \frac{1}{1+z^2}$$

and Γ is described by

$$\gamma : [0, \pi] \to \mathbb{C}, \quad \gamma(t) = r\cos(t) + ir\sin(t),$$

where $r > 1$. We shall find an upper estimate for $\int_\Gamma f$ and show that

$$\int_\Gamma f \to 0 \text{ as } r \to \infty.$$

Solution

The domain of f is $\mathbb{C}\setminus\{i, -i\}$, as f is defined and continuous everywhere except where $1 + z^2 = 0$. As Γ consists of part of the circle with centre 0 and radius $r > 1$, Γ does not contain i or $-i$.

Now, for any $z \in \Gamma$, the Backwards Triangle Inequality gives

$$\left|1 + z^2\right| \geq \left||1| - \left|z^2\right|\right| = r^2 - 1$$

since $|z| = r > 1$ for all $z \in \Gamma$. Thus

$$|f(z)| = \left|\frac{1}{1+z^2}\right| \leq \frac{1}{r^2-1}$$

for all $z \in \Gamma$.

Setting $M = \frac{1}{r^2-1}$, we have $|f(z)| \leq M$ for all $z \in \Gamma$, and since Γ is a semicircle, the length L of Γ is equal to πr. Thus the estimation Lemma gives the upper estimate

$$\left|\int_\Gamma f\right| \leq ML = \frac{\pi r}{r^2-1}$$

for this integral.

If we rewrite the above inequality as

$$\left|\int_\Gamma f\right| \leq \frac{\pi}{r - \frac{1}{r}},$$

we see that

$$\left|\int_\Gamma f\right| \to 0 \text{ as } r \to \infty$$

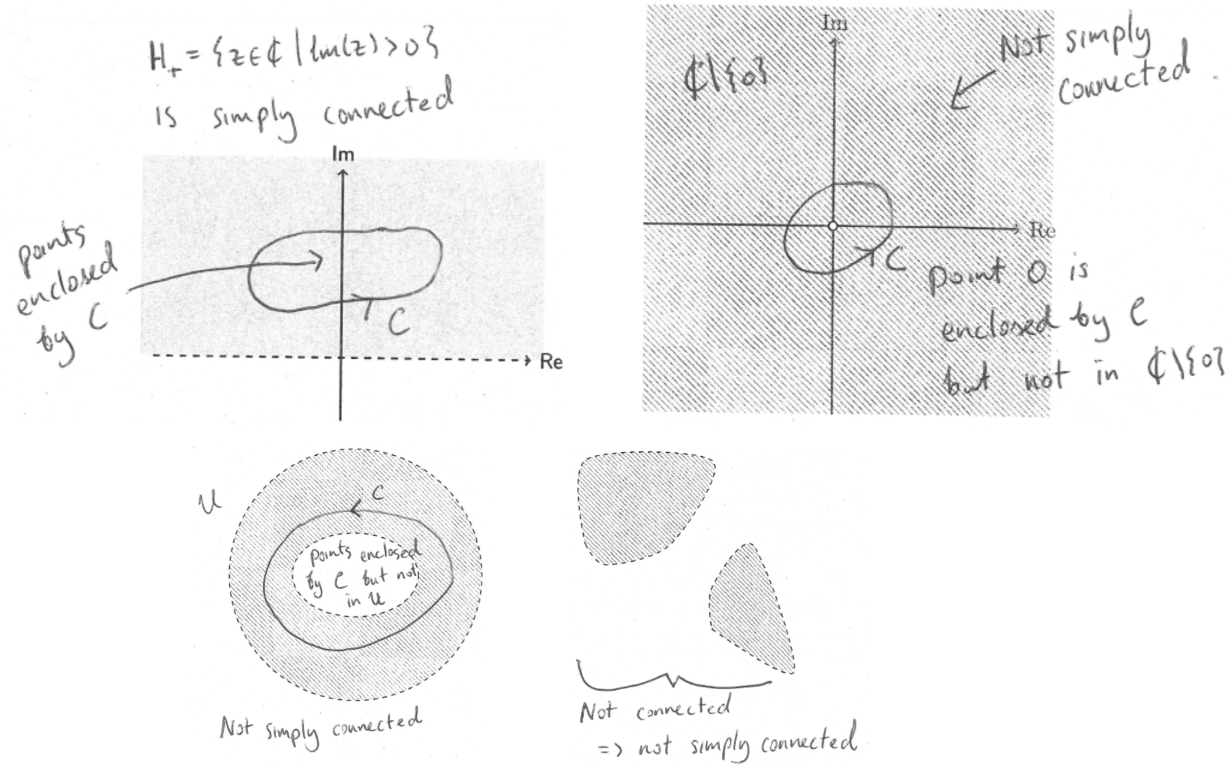
and hence

$$\int_\Gamma f \to 0 \text{ as } r \to \infty.$$

4.2 Cauchy's Theorem for a Triangle

Definition 4.4. A region $\mathcal{R}$ is called *simply connected* if given any closed contour $\mathcal{C}$ in $\mathcal{R}$, all points enclosed by $\mathcal{C}$ also belong to $\mathcal{R}$.

It is surprisingly difficult to write down the precise definition of a point being enclosed by a contour $\mathcal{C}$, thus we shall treat this notion informally and avoid complicated cases. Essentially, a simply connected region cannot have any ‘holes.’



In the proof of Cauchy’s Theorem we will need the following fact about the integral of a continuous function f along a smooth path Γ: if $\tilde{\Gamma}$ denotes the reverse of Γ then

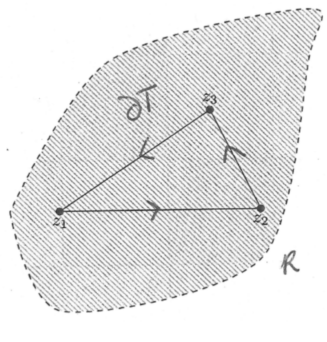
$$\int_{\tilde{\Gamma}} f = -\int_{\Gamma} f.$$

We shall also fix the following notation: $T(z_1, z_2, z_3)$ denotes the triangle with vertices z_1, z_2, z_3, and hence with edges given by the line segments $[z_1, z_2], [z_2, z_3], [z_3, z_1]$. The boundary of T is denoted by ∂T, which defines a closed contour

$$\partial T = [z_1, z_2] + [z_2, z_3] + [z_3, z_1].$$

Theorem 4.5 (Cauchy’s Theorem for a Triangle)**.** *Let f be a function that is holomorphic on a simply connected region $\mathcal{R}$ and let $T(z_1, z_2, z_3)$ be a triangle in $\mathcal{R}$ with boundary ∂T. Then*

$$\int_{\partial T} f = 0.$$



Note that if we knew that f were to have an antiderivative F on $\mathcal{R}$, then Theorem 3.31 would show that

$$\int_{\partial T} f = 0.$$

Later we will see that in fact f does have an antiderivative on $\mathcal{R}$, but the proof will rely Cauchy's Theorem for a triangle - thus we need to prove Theorem 4.5 without this assumption.

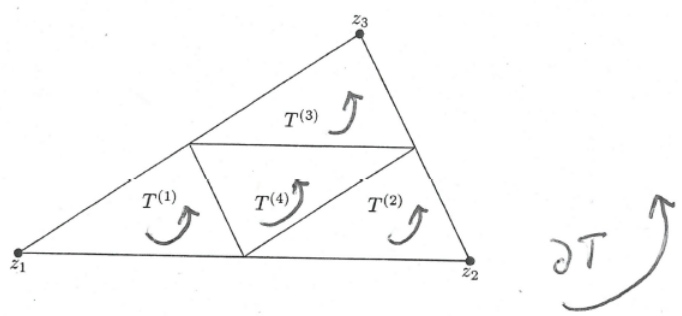
Before proving Theorem 4.5, we need the following lemma:

Lemma 4.6. *Let f be a function that is holomorphic on a simply connected region $\mathcal{R}$ and let $T(z_1, z_2, z_3)$ be a triangle in $\mathcal{R}$. Then there exists a nested sequence of triangles $T \supseteq T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n \supseteq \ldots$ with the property that*

$$\frac{1}{4^n}\left|\int_{\partial T} f\right| \leq \left|\int_{\partial T_n} f\right| \text{ and } \ell(\partial T_n) = \frac{1}{2^n}\ell(\partial T)$$

for all n. Moreover, there exists a point $z_0 \in \mathcal{R}$ with $z_0 \in T_n$ for all n.

Proof Using the midpoints of the sides of T, construct four similar triangles $T^{(1)}, T^{(2)}, T^{(3)}$ and $T^{(4)}$ as shown.

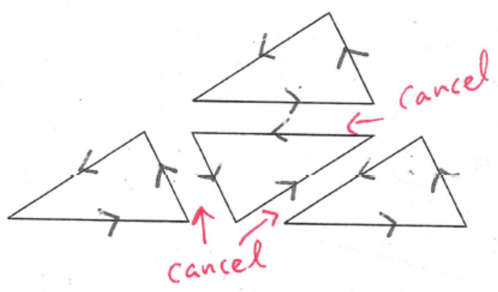


Then for $j = 1, 2, 3, 4$ we have $\ell(\partial T^{(j)}) = \frac{1}{2}\ell(\partial T)$ by construction.

Note that with all contours taken anticlockwise,

$$\int_{\partial T^{(1)}} f + \int_{\partial T^{(2)}} f + \int_{\partial T^{(3)}} f + \int_{\partial T^{(4)}} f = \int_{\partial T} f,$$

as the integrals along the 'interior' edges of the $T^{(j)}$ cancel in pairs (this uses $\int_{\tilde{\Gamma}} f = -\int_{\Gamma} f$, where $\tilde{\Gamma}$ is the reverse of Γ):



Moreover, by the triangle inequality

$$\left|\int_{\partial T} f\right| \leq \left|\int_{\partial T^{(1)}} f\right| + \left|\int_{\partial T^{(2)}} f\right| + \left|\int_{\partial T^{(3)}} f\right| + \left|\int_{\partial T^{(4)}} f\right|.$$

At least one of these four integrals has modulus greater than or equal to the other three; call the corresponding triangle T_1. Then

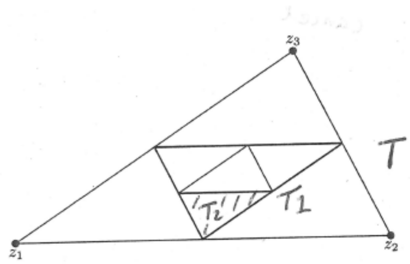
$$\left|\int_{\partial T} f\right| \leq \left|\int_{\partial T^{(1)}} f\right| + \left|\int_{\partial T^{(2)}} f\right| + \left|\int_{\partial T^{(3)}} f\right| + \left|\int_{\partial T^{(4)}} f\right| \leq 4\left|\int_{\partial T_1} f\right|,$$

and so

$$\frac{1}{4}\left|\int_{\partial T} f\right| \leq \left|\int_{\partial T_1} f\right|.$$

Now, use the midpoints of the edges of T_1 to get four more triangles, one of which is a triangle T_2 satisfying

$$\frac{1}{4}\left|\int_{\partial T_1} f\right| \leq \left|\int_{\partial T_2} f\right| \quad \text{and} \quad \ell(\partial T_2) = \frac{1}{2}\ell(\partial T_1) = \frac{1}{2^2}\ell(\partial T),$$



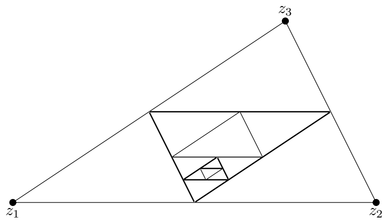
Thus

$$\frac{1}{4^2}\left|\int_{\partial T} f\right| \leq \left|\int_{\partial T_2} f\right|.$$

Continuing in this way we get $T \supseteq T_1 \supseteq T_2 \supseteq \ldots \supseteq T_n \supseteq \ldots$ with

$$\frac{1}{4^n}\left|\int_{\partial T} f\right| \leq \left|\int_{\partial T_n} f\right| \quad \text{and} \quad \ell(\partial T_n) = \frac{1}{2^n}\ell(\partial T)$$

for each n.



As the sequence is nested, there exists a point z_0 with $z_0 \in T_n$ for all n, and since $\mathcal{R}$ is simply connected, $z_0 \in \mathcal{R}$. $\square$

We are now in a position to prove Cauchy's Theorem for a Triangle.

Proof

[of Cauchy's Theorem for a Triangle (Theorem 4.5)] We are going to show that given any $\epsilon > 0$ we have
$$\left|\int_{\partial T} f\right| \leq \epsilon[\ell(\partial T)]^2$$
where $\ell(\partial T)$ is the length of the boundary of the triangle T.

Let $\{T_n\}$ be the nested sequence of triangles constructed in Lemma 4.6 and $z_0 \in \mathcal{R}$ such that $z_0 \in T_n$ for all n.

Step 1: Let $\epsilon > 0$ be given. We will show that there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Longrightarrow \left|f(z) - f(z_0) - (z - z_0)f'(z_0)\right| \leq \epsilon\,|z - z_0|\,.$$

Since f is holomorphic on $\mathcal{R}$ it is differentiable at z_0 with derivative $f'(z_0)$, or in other words
$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0).$$

This means that there is some $\delta > 0$ such that

$$0 < |h| < \delta \Longrightarrow \left|\frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0)\right| < \epsilon$$

or in other words, writing $z = z_0 + h$,
$$\begin{aligned} 0 < |z - z_0| < \delta &\Longrightarrow \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon \\ &\Longrightarrow \left|\frac{f(z) - f(z_0) - (z - z_0)f'(z_0)}{z - z_0}\right| < \epsilon. \end{aligned}$$

Thus
$$0 < |z - z_0| < \delta \Longrightarrow \left|f(z) - f(z_0) - (z - z_0)f'(z_0)\right| < \epsilon\,|z - z_0|\,.$$

If we allow $z = z_0$, this becomes

$$|z - z_0| < \delta \Longrightarrow \left|f(z) - f(z_0) - (z - z_0)f'(z_0)\right| \leq \epsilon\,|z - z_0|$$

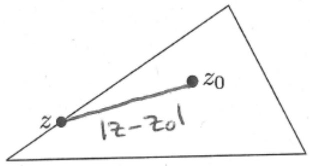
(since when $z = z_0$, both sides are zero).

Step 2: With $\epsilon > 0$ as before, and $\delta > 0$ as chosen in the previous step, we choose n large enough so that $\ell(\partial T_n) < \delta$, which can always be done as $\ell(\partial T_n) = \frac{1}{2^n}\ell(\partial T)$ (note that n depends on δ, which in turn depends on ϵ).

We will show that for $z \in \partial T_n$ we have

$$\left|f(z) - f(z_0) - (z - z_0)f'(z_0)\right| < \epsilon \ell(\partial T_n).$$

If $z \in \partial T_n$, then the distance from z to z_0 can be no more than the length of the longest edge of T_n, which is less than $\ell(\partial T_n)$.



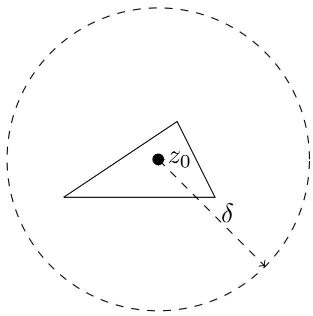
Thus

$$z \in \partial T_n \implies |z - z_0| \underbrace{< \ell(\partial T_n)}_{\text{geometry}} \underbrace{< \delta}_{\text{choice of } n},$$

and hence $T_n \subset D(z_0, \delta)$.

Thus for any n large enough so that $\ell(\partial T_n) < \delta$, we have

$$\begin{aligned} z \in \partial T_n \implies \left|f(z) - f(z_0) - (z - z_0)f'(z_0)\right| &\leq \epsilon |z - z_0| \\ &< \epsilon \ell(\partial T_n). \end{aligned}$$



Step 3: We use the Estimation Lemma to find an upper estimate for $\int_{\partial T_n} f$, where n was chosen in the previous step.

Note that

$$\int_{\partial T_n} (\alpha + \beta(z - z_0))dz = 0 \text{ for } \alpha, \beta \in \mathbb{C},$$

because $z \mapsto \alpha + \beta(z - z_0)$ has an antiderivative $z \mapsto \alpha z + \frac{\beta(z-z_0)^2}{2}$.

This means that

$$\int_{\partial T_n} \Big(\underbrace{f(z_0)}_{\alpha} + \underbrace{f'(z_0)}_{\beta}(z - z_0)\Big)dz = 0,$$

and so

$$\int_{\partial T_n} f = \int_{\partial T_n} \Big(f(z) - \underbrace{f(z_0) - f'(z_0)(z - z_0)}_{\text{integral 0 along } \partial T_n}\Big)dz$$

We shall use the Estimation Lemma to find an upper estimate for this integral. Indeed, we have

$$\left|\int_{\partial T_n} f\right| = \left|\int_{\partial T_n} \left[f(z) - f(z_0) - f'(z_0)(z - z_0)\right]\ dz\right|$$

$$
\begin{aligned}
&\leq \epsilon \ell(\partial T_n)\ell(\partial T_n) && \text{By step 3}\\
&= \epsilon\left(\frac{1}{2^n}\ell(\partial T)\right)\left(\frac{1}{2^n}\ell(\partial T)\right) && \text{as } \ell(\partial T_n) = \frac{1}{2^n}\ell(\partial T)\\
&= \epsilon \frac{1}{4^n}\left[\ell(\partial T)\right]^2.
\end{aligned}
$$

Step 4: Completing the proof.

By Step 3, given any $\epsilon > 0$, there is n such that

$$\left|\int_{\partial T_n} f\right| \leq \epsilon \frac{1}{4^n}\left[\ell(\partial T)\right]^2.$$

In Lemma 4.6, we showed that

$$\frac{1}{4^n}\left|\int_{\partial T} f\right| \leq \left|\int_{\partial T_n} f\right|,$$

and so combining these facts, we have

$$\left|\int_{\partial T} f\right| \leq \epsilon\left[\ell(\partial T)\right]^2$$

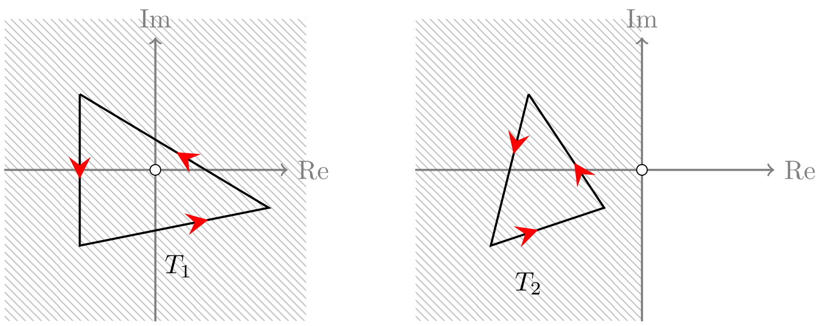
for all $\epsilon > 0$. Because $\epsilon > 0$ was arbitrary and $\ell(\partial T)$ is fixed, we get

$$\left|\int_{\partial T} f\right| = 0, \text{ hence } \int_{\partial T} f = 0.$$

Example 4.7. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \sin\left(\exp\left(\cos\left(z^3 - 4z^2 + i\right)\right)\right)$, which is holomorphic on $\mathbb{C}$. There is no obvious antiderivative for f, but nonetheless, Cauchy's Theorem shows us that for any triangle T we have

$$\int_{\partial T} f = 0.$$

Example 4.8. Let $f(z) = \dfrac{1}{z}$, which is holomorphic on $\mathbb{C}\setminus\{0\}$. We shall evaluate $\int_{\partial T} f$ for different triangles T: let T_1 be a triangle enclosing the origin, and T_2 a triangle lying to the left of the imaginary axis as shown.



Solution

Since $\mathbb{C}\setminus\{0\}$ is not simply connected, Cauchy's Theorem for a triangle tells us nothing about

$$\int_{\partial T_1} \frac{1}{z}\, dz.$$

However, $\mathbb{C}\setminus\{0\}$ has simply connected subregions, for example, the left-half plane

$$\mathcal{L} = \{z \in \mathbb{C} : \mathsf{Re}(z) < 0\}.$$

The triangle T_2 is contained in $\mathcal{L}$, so Cauchy's Theorem for a triangle gives

$$\int_{\partial T_2} \frac{1}{z}\, dz = 0.$$

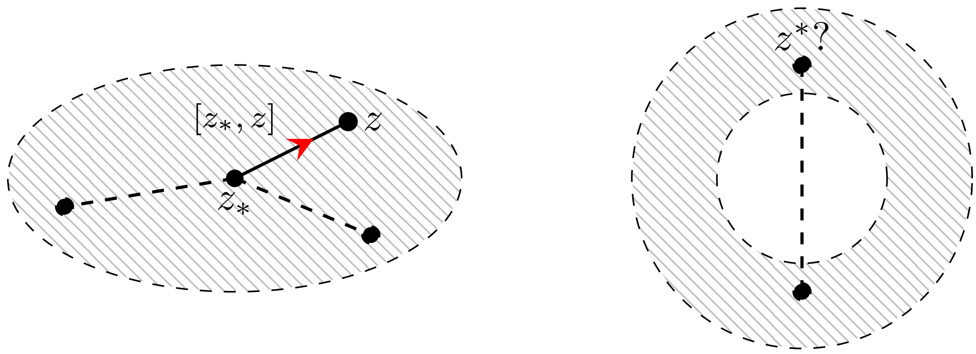
In fact, if T is any triangle in $\mathbb{C}\setminus\{0\}$ that does not enclose the origin, we can find a simply connected subregion of $\mathbb{C}\setminus\{0\}$ that contains T, and conclude that

$$\int_{\partial T} \frac{1}{z}\, dz = 0.$$

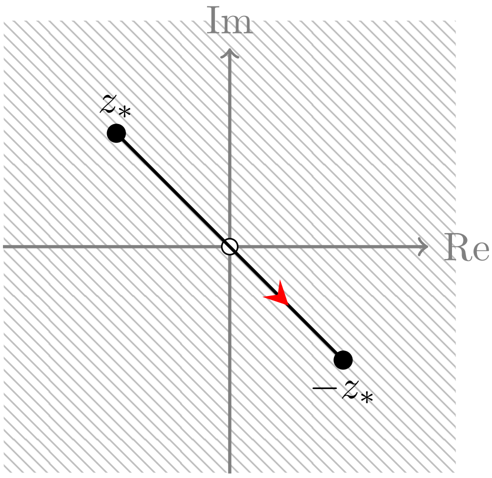
4.3 Cauchy's Theorem for Starlit Regions

Definition 4.9. A region $\mathcal{R}$ is called *starlit* if there is a point $z_* \in \mathcal{R}$ such that for any $z \in \mathcal{R}$ the line segment $[z_*, z]$ lies inside $\mathcal{R}$. The point z_* is called a *star centre* for $\mathcal{R}$.

The name starlit comes from the idea that the 'rays of light' from the 'star' z_* fall on every point of $\mathcal{R}$.



The region $\mathcal{R}_1$ on the left is starlit, as any point $z \in \mathcal{R}_1$ can be connected to z_* using the line segment $[z_*, z]$. The region $\mathcal{R}_2$ on the right is not, since no matter which point we choose as z_*, the line segment joining z_* to the point opposite it leaves $\mathcal{R}_2$.

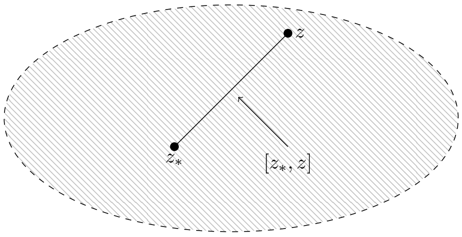


The region $\mathbb{C}\setminus\{0\}$ is not starlit; indeed if z_* were a star centre, then the line segment $[z_*, -z_*]$ would be contained in $\mathbb{C}\setminus\{0\}$, which is not the case for any choice of z_*.

Theorem 4.10 (The Existence of Antiderivatives on Starlit Regions)**.** *Let f be holomorphic on a starlit region $\mathcal{R}$ with star centre $z_* \in \mathcal{R}$. Then the function $F : \mathcal{R} \to \mathbb{C}$ defined by*

$$F(z) = \int_{[z_*, z]} f$$

is an antiderivative for f on $\mathcal{R}$.



Since $\mathcal{R}$ is starlit, $[z_*, z]$ is contained in $\mathcal{R}$ and so $\int_{[z_*,z]} f$ is well-defined for all $z \in \mathcal{R}$.

Proof

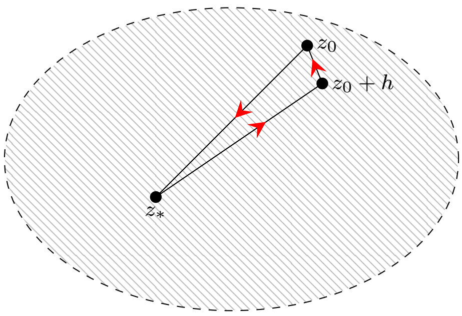
We need to show that for any $z_0 \in \mathcal{R}$, $F'(z_0)$ exists and is equal to $f(z_0)$, or in other words

$$\lim_{h \to 0} \frac{F(z_0 + h) - F(z_0)}{h} = f(z_0).$$

Write

$$\frac{F(z_0 + h) - F(z_0)}{h} = \frac{1}{h} \int_{[z_*, z_0+h]} f - \frac{1}{h} \int_{[z_*, z_0]} f.$$

Since $\mathcal{R}$ is open, for sufficiently small h, the triangle $T(z_*, z_0 + h, z_0)$ is contained in $\mathcal{R}$.



Cauchy's Theorem for a Triangle gives

$$\int_{[z_*,z_0+h]} f + \int_{[z_0+h,z_0]} f + \int_{[z_0,z_*]} f = 0,$$

hence

$$\int_{[z_*,z_0+h]} f - \int_{[z_*,z_0]} f = -\int_{[z_0+h,z_0]} f = \int_{[z_0,z_0+h]} f.$$

This gives

$$\frac{F(z_0+h)-F(z_0)}{h} = \frac{1}{h}\int_{[z_0,z_0+h]} f.$$

We now use the fact that

$$\int_{[z_0,z_0+h]} 1dz = h,$$

which implies that

$$f(z_0) = f(z_0)\cdot\frac{1}{h}\cdot h = f(z_0)\frac{1}{h}\int_{[z_0,z_0+h]} 1dz = \frac{1}{h}\int_{[z_0,z_0+h]} f(z_0)\; dz$$

(since $f(z_0)$ is a constant). Combining this with the previous step, we see that

$$\begin{aligned}\frac{F(z_0+h)-F(z_0)}{h} - f(z_0) &= \frac{1}{h}\left[\int_{[z_0,z_0+h]} f(z)\; dz - \int_{[z_0,z_0+h]} f(z_0)\; dz\right]\\ &= \frac{1}{h}\int_{[z_0,z_0+h]} (f(z)-f(z_0))\; dz\end{aligned}$$

(since both integrals are along the same path, we can combine them). Hence

$$\left|\frac{F(z_0+h)-F(z_0)}{h} - f(z_0)\right| = \frac{1}{|h|}\left|\int_{[z_0,z_0+h]} (f(z)-f(z_0))dz\right|$$

Now we shall use the Estimation Lemma to obtain an upper estimate for this integral. It is easy to see that $\ell([z_0,z_0+h]) = |h|$, thus we must find an upper bound for $|f(z)-f(z_0)|$ when $z \in [z_0,z_0+h]$.

We use the fact that f holomorphic on $\mathcal{R}$ implies f continuous on $\mathcal{R}$, and in particular, continuous at z_0. Hence given any $\epsilon > 0$ there is some $\delta > 0$ such that

$$0 < |z-z_0| < \delta \Longrightarrow |f(z)-f(z_0)| < \epsilon.$$

Thus if $|h| < \delta$ then

$$\begin{aligned} z \in [z_0,z_0+h] &\Longrightarrow |z-z_0| < |h| < \delta \\ &\Longrightarrow |f(z)-f(z_0)| < \epsilon.\end{aligned}$$

In other words, once $|h| < \delta$, ϵ is an upper bound for $|f(z)-f(z_0)|$ for $z \in [z_0,z_0+h]$. Thus the Estimation Lemma gives the upper estimate

$$\left|\int_{[z_0,z_0+h]} (f(z)-f(z_0))\; dz\right| \le \epsilon\,|h|$$

whenever $|h| < \delta$.

Thus given any $\epsilon > 0$ there is $\delta > 0$ such that

$$0 < |h| < \delta \Longrightarrow \left| \frac{F(z_0 + h) - F(z_0)}{h} - f(z_0) \right| = \frac{1}{|h|} \left| \int_{[z_0, z_0+h]} (f(z) - f(z_0) dz \right|$$
$$\leq \frac{1}{|h|} \epsilon |h| = \epsilon.$$

But this is equivalent to saying

$$\lim_{h \to 0} \frac{F(z_0 + h) - F(z_0)}{h} = f(z_0),$$

or in other words, $F'(z_0) = f(z_0)$, as required.

Theorem 4.11 (Cauchy's Theorem for Starlit Regions)**.** *Let f be a function that is holomorphic in a starlit region $\mathcal{R}$ and let $\mathcal{C}$ be a closed contour in $\mathcal{R}$. Then*

$$\int_{\mathcal{C}} f = 0.$$

Proof By Theorem 4.10, f has an antiderivative in $\mathcal{R}$. Thus by Theorem 3.31

$$\int_{\mathcal{C}} f = 0.$$

□

In fact, Cauchy's Theorem extends to more general regions, as we shall see later.

4.4 Application: The Complex Logarithm and Power Functions

In this section we will define the Complex Logarithm and Power functions.

Consider the Real function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$. Since f is continuous on $(0, +\infty)$, the Fundamental Theorem of Calculus ensures that f has an antiderivative on $(0, +\infty)$. One way of defining the *natural logarithm* function, is to define it as an antiderivative of $\dfrac{1}{x}$ on $(0, +\infty)$:

$$\log(x) = \int_1^x \frac{1}{t}\, dt \qquad \text{for all } x \in (0, +\infty). \tag{4.1}$$

Note 4.12. (i) There are of course other ways of defining $\log(x)$ for $x \in (0, +\infty)$, most commonly as the inverse of the exponential function $\exp : \mathbb{R} \to (0, +\infty)$. This definition is equivalent.

(ii) The Fundamental Theorem of calculus actually tells us that for *any* choice of $a \in (0, +\infty)$, the function $F : (0, +\infty) \to \mathbb{R}$ defined by

$$F(x) = \int_a^x \frac{1}{t}\, dt$$

is an antiderivative for $x \mapsto \frac{1}{x}$. However, we must take $a = 1$ to ensure that F is indeed the inverse of $x \mapsto e^x$: since $e^0 = 1$ we want $\log(1) = 0$.

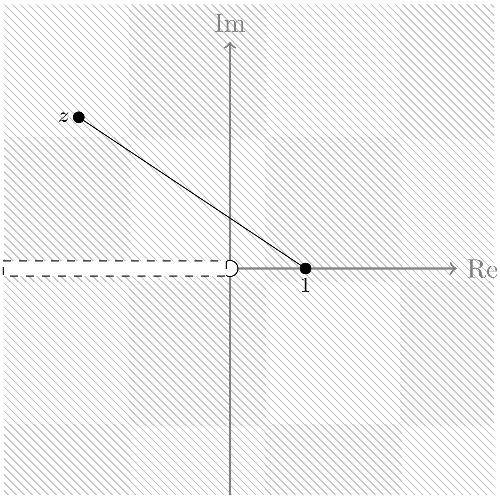
We shall generalise (4.1) to define the Complex Logarithm function, using Theorem 4.10 (the existence of antiderivatives on starlit regions). The function

$$f(z) = \frac{1}{z} \quad (z \in \mathbb{C}\setminus\{0\})$$

is holomorphic on $\mathbb{C}\setminus\{0\}$. However, this set is neither simply connected nor starlit. Therefore, we will restrict our attention to the subset

$$\mathbb{C}_\pi = \{z \in \mathbb{C} : z \neq 0 \text{ and } \mathrm{Arg}(z) \neq \pi\},$$

which is both starlit and simply connected.



The point 1 on the real axis is a star centre for $\mathbb{C}_\pi$ - in particular, this allows us to apply Theorem 4.10 to conclude that

$$z \mapsto \int_{[1,z]} \frac{1}{\zeta}\, d\zeta$$

is an antiderivative for $z \mapsto \frac{1}{z}$ on $\mathbb{C}_\pi$. We use the variable ζ (zeta) as the variable of integration because z has been used already.

Definition 4.13. The *Complex Logarithm* function $\mathrm{Log} : \mathbb{C}_\pi \to \mathbb{C}$ is defined by

$$\mathrm{Log}(z) = \int_{[1,z]} \frac{1}{\zeta}\, d\zeta \qquad \text{for } z \in \mathbb{C}_\pi.$$

It is holomorphic on $\mathbb{C}_\pi$.

In other words, to find the value of $\mathrm{Log}(z)$ we need to evaluate the integral

$$\int_{[1,z]} \frac{1}{\zeta}\, d\zeta.$$

We can do this directly by choosing a parametrisation of the path $[1, z]$, however, it is much easier to obtain an alternative definition using the following example.

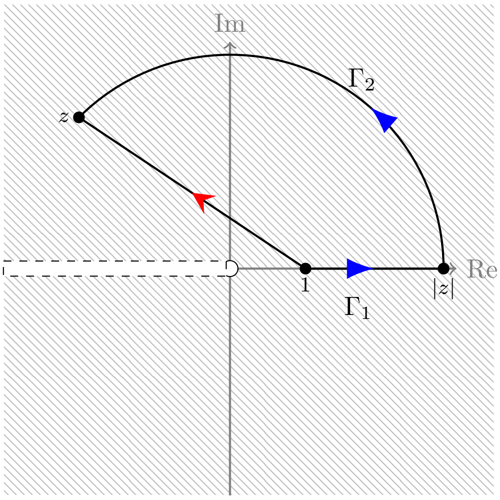
Example 4.14. We shall show that for all $z \in \mathbb{C}_\pi$ we have

$$\mathrm{Log}(z) = \log(|z|) + i\mathrm{Arg}(z)$$

where $\log : (0, +\infty) \to \mathbb{R}$ denotes the real (natural) logarithm and Arg is the principal value of the argument.

Solution

Since the function f defined by $f(z) = \frac{1}{z}$ has an antiderivative on $\mathbb{C}_\pi$, the Contour Independence Theorem (Theorem 3.29) tells us that $\displaystyle\int_{\mathcal{C}} f = \int_{[1,z]} f$ for any contour $\mathcal{C}$ in $\mathbb{C}_\pi$ that starts at 1 and ends at z.



We shall use the contour $\mathcal{C} = \Gamma_1 + \Gamma_2$, where $\Gamma_1 = [1, |z|]$ is the straight line segment along the real axis from 1 to $|z|$, and Γ_2 is the arc of the circle with centre 0 and radius $|z|$ from the point $|z|$ on the positive real axis to z.
Parametrise Γ_1 with $\gamma_1 : [1, |z|] \to \mathbb{C}$, $\gamma_1(t) = t$, and Γ_2 with $\gamma_2 : [0, \mathrm{Arg}(z)] \to \mathbb{C}$,

$$\gamma_2(t) = |z| \left(\cos(t) + i \sin(t)\right)$$

Note that Γ_2 is a path from $\gamma_2(0) = |z|$ to $\gamma_2(\mathrm{Arg}(z)) = z$, hence is traversed

- anticlockwise if $\mathrm{Arg}(z) \geq 0$, and
- clockwise if $\mathrm{Arg}(z) < 0$,

which ensures that we stay within $\mathbb{C}_\pi$.
We have $\gamma_1'(t) = 1$ and $\gamma_2'(t) = i\gamma_2(t)$, and so

$$\begin{aligned}
\mathrm{Log}(z) &= \int_{\Gamma_1} \frac{1}{\zeta}\, d\zeta + \int_{\Gamma_2} \frac{1}{\zeta}\, d\zeta \\
&= \int_1^{|z|} \frac{1}{t}\, dt + \int_0^{\mathrm{Arg}(z)} \frac{1}{\gamma_2(t)} \gamma_2'(t)\, dt \\
&= \left[\log|t|\right]_1^{|z|} + \int_0^{\mathrm{Arg}(z)} i\, dt \\
&= \log|z| + i\mathrm{Arg}(z).
\end{aligned}$$

This yields the following alternative definition of the Complex Logarithm function:

$$\mathrm{Log}(z) = \log(|z|) + i\mathrm{Arg}(z) \quad (z \in \mathbb{C}_\pi). \tag{4.2}$$

Note 4.15. This definition of $\mathrm{Log}(z)$ uses the principal value of the argument - that is to say, our definition depends on us taking $\arg(z) \in (-\pi, \pi]$. For this reason it is sometimes called the *principal branch* of the (complex) Logarithm function. Other 'branches' (i.e. other definitions) of $\mathrm{Log}(z)$ are possible by taking different values of the argument.

Example 4.16. Compute the following principal logarithms: $\mathrm{Log}(7)$, $\mathrm{Log}(-2i)$ and $\mathrm{Log}(1+i)$.

Solution

$$\begin{aligned}
\mathrm{Log}(7) &= \log|7| + i\mathrm{Arg}(7) = \log(7) \\
\mathrm{Log}(-2i) &= \log|-2i| + i\mathrm{Arg}(-2i) \\
&= \log(2) - i\frac{\pi}{2} \\
\mathrm{Log}(1+i) &= \log|1+i| + i\mathrm{Arg}(1+i) \\
&= \log(\sqrt{2}) + i\frac{\pi}{4} \\
&= \frac{1}{2}\log(2) + i\frac{\pi}{4.}
\end{aligned}$$

So far, $\mathrm{Log}(z)$ is defined (and is holomorphic) on $\mathbb{C}_\pi$. We can extend the domain of Log to $\mathbb{C}\setminus\{0\}$ by defining it on the negative real axis, where points have principal argument π, as follows:

$$\mathrm{Log}(z) = \log(|z|) + i\pi, \text{ for } z \neq 0 \text{ on the negative real axis.}$$

Question. Is $\mathrm{Log}(z)$ continuous on $\mathbb{C}\setminus\{0\}$?

Answer

For t on the negative real axis,

$$\lim_{\substack{z\to t \\ \mathrm{Im}(z)>0}} \mathrm{Arg}(z) = \pi, \text{ while } \lim_{\substack{z\to t \\ \mathrm{Im}(z)<0}} \mathrm{Arg}(z) = -\pi.$$

Thus $z \mapsto \mathrm{Arg}(z)$ is discontinuous on the negative real axis. Since $\mathrm{Arg}(z)$ is the imaginary part of $\mathrm{Log}(z)$, it follows that Log is also discontinuous on the negative real axis.

Question. Is Log the inverse of exp?

Answer

Not exactly. For any $z \in \mathbb{C}\setminus\{0\}$, we have

$$\begin{aligned}
\exp(\mathrm{Log}(z)) &= \exp\left(\log(|z|) + i\mathrm{Arg}(z)\right) \\
&= e^{\log|z|}\left(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))\right)
\end{aligned}$$

$$= \underbrace{|z|\left(\cos(\mathrm{Arg}(z)) + i\sin(\mathrm{Arg}(z))\right)}_{\text{Polar form of } z}$$
$$= z.$$

However, $\exp : \mathbb{C} \to \mathbb{C}$ is not injective (e.g. $\exp(0) = \exp(i2\pi) = 1$), thus we cannot have $\mathrm{Log}(\exp(z)) = z$ for all z. We do have

$$\begin{aligned} \mathrm{Log}(\exp(x+iy)) &= \log|\exp(x+iy)| + i\mathrm{Arg}(\exp(x+iy)) \\ &= \log(e^x) + i(y - 2k\pi) \qquad \text{for some } k \in \mathbb{Z} \end{aligned}$$

Hence $\mathrm{Log}(\exp(z)) = z$ if and only if $-\pi < \mathsf{Im}(z) \leq \pi$.

We have seen already how to take roots of complex numbers. We will now extend this definition to *complex* powers of complex numbers, allowing us to compute expressions such as

$$2^i, i^i, (1+3i)^{1-i}$$

and so on.

The following observation about powers of real numbers may be useful: suppose we want to find the value of x^a for some $a, x \in \mathbb{R}$ with $x > 0$. We know that

$$\log(x^a) = a\log(x),$$

and since $e^{\log(y)} = y$ for all $y \in (0, +\infty)$, we have

$$x^a = e^{\log(x^a)} = e^{a\log(x)}.$$

Thus we may take $e^{a\log(x)}$ to be the definition of x^a.

Definition 4.17. For $\alpha \in \mathbb{C}$, the *Principal α^{th} Power Function* is defined by

$$z^\alpha = \exp\left(\alpha\,\mathrm{Log}(z)\right)$$

for all $z \in \mathbb{C}\backslash\{0\}$.

Because Log is holomorphic on $\mathbb{C}_\pi$ and exp is holomorphic on $\mathbb{C}$, the Principal Power function is holomorphic on $\mathbb{C}_\pi$. 'Principal' refers to the fact that we have used the principal branch of the Logarithm function to define the power function - taking different branches of the Logarithm function will give different power functions.

Example 4.18. Let us verify that $i^2 = -1$ agrees with Definition 4.17.

Solution

Indeed, we have

$$\begin{aligned} i^2 = \exp\left(2\,\mathrm{Log}(i)\right) &= \exp\left(2\left[\log|i| + i\mathrm{Arg}(i)\right]\right) \\ &= \exp\left(2\left[\log(1) + i\frac{\pi}{2}\right]\right) \\ &= \exp(i\pi) = -1. \end{aligned}$$

Example 4.19. Calculate 2^i and $(1+i)^{-i}$.

Solution

This time, we have

$$\begin{aligned} 2^i = \exp(i \operatorname{Log}(2)) &= \exp(i \log(2)) \\ &= \cos(\log(2)) + i \sin(\log(2)) \end{aligned}$$

and

$$\begin{aligned} (1+i)^{-i} &= \exp\left(-i \operatorname{Log}(1+i)\right) \\ &= \exp\left(-i\left(\tfrac{1}{2}\log(2) + i\frac{\pi}{2}\right)\right) \\ &= \exp\left(\tfrac{\pi}{4} - i\tfrac{1}{2}\log(2)\right) \\ &= e^{\pi/4}\left(\cos\left(\tfrac{1}{2}\log(2)\right) + i\sin\left(\tfrac{1}{2}\log(2)\right)\right). \end{aligned}$$

Chapter 5 Cauchy's Integral Formula and its Applications

5.1 Cauchy's Integral Formula

We have seen so far that if $f : \mathcal{R} \to \mathbb{C}$ is holomorphic on a starlit region $\mathcal{R}$ then

$$\int_{\mathcal{C}} f = 0$$

for any closed contour $\mathcal{C}$ in $\mathcal{R}$. What happens when $\mathcal{R}$ is not starlit?

It may happen that while $\mathcal{R}$ itself is not starlit, the contour $\mathcal{C}$ is contained in a starlit subregion $\mathcal{S}$ of $\mathcal{R}$.

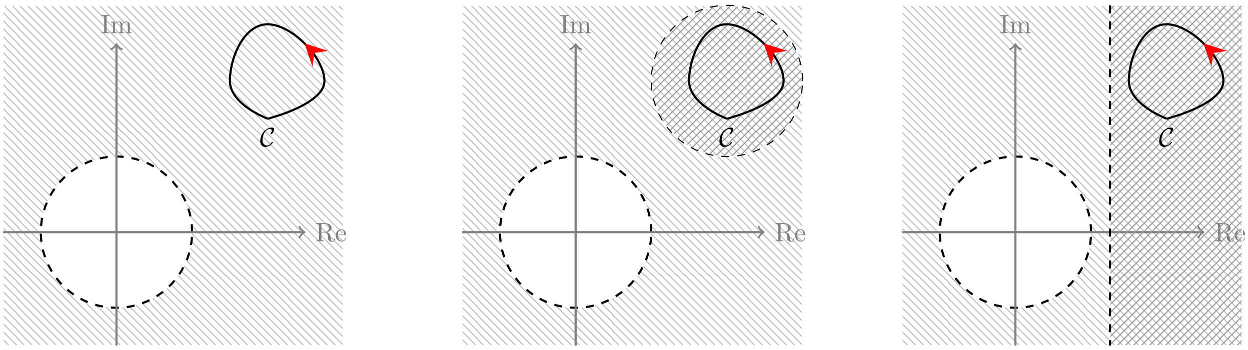


Figure 5.1: The region $\mathcal{R}$ consists of the complex plane with a disc centred at the origin removed. This region is not starlit, but there are starlit subregions of $\mathcal{R}$ containing the contour $\mathcal{C}$.

Thus if f is holomorphic on $\mathcal{R}$, then f is also holomorphic on $\mathcal{S}$ and so Cauchy's Theorem for Starlit Regions (Theorem 4.11) gives

$$\int_{\mathcal{C}} f = 0.$$

If $\mathcal{R}$ is not simply connected, then it may not be possible to do this; in particular, when $\mathcal{C}$ encloses a point z_0 at which f is not holomorphic. For example, suppose $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by $f(z) = \dfrac{1}{z}$, and $\mathcal{C}$ is the contour shown in Figure 5.2.

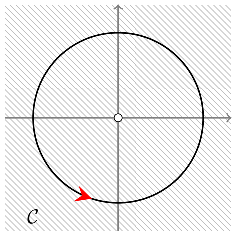


Figure 5.2: A contour $\mathcal{C}$ contained in the non-simply connected region $\mathbb{C}\setminus\{0\}$.

Here $\mathbb{C}\setminus\{0\}$ is not starlit, and there is no starlit subregion of $\mathbb{C}\setminus\{0\}$ containing $\mathcal{C}$. Nonetheless, in certain circumstances, it is at least possible to replace $\mathcal{C}$ something easier in order to calculate the integral.

Definition 5.1. A contour $\mathcal{C}$, that is not closed, is said to be *simple* if $\mathcal{C}$ does not intersect itself. A *simple closed contour* $\mathcal{C}$ is one that does not intersect itself except that its start point is the same as its end point.

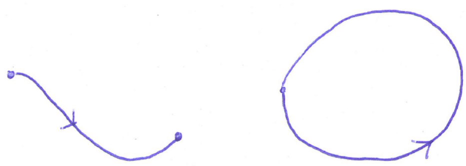


Figure 5.3: Simple contours.

Figure 5.4: Non-simple contours. Note that the closed, non-simple contour on the right has no obvious orientation - it is neither clockwise nor anticlockwise.

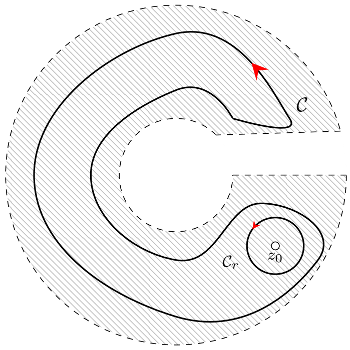
For a simple closed contour $\mathcal{C}$, when we speak of $\mathcal{C}$ being clockwise or anticlockwise, we mean relative to the points enclosed by $\mathcal{C}$. In other words, a contour $\mathcal{C}$ is anticlockwise if, as we travel along $\mathcal{C}$, the region enclosed by $\mathcal{C}$ always lies to our left. Again, we shall avoid giving a precise definition, and treat this notion informally.

The following Theorem shows us how in some cases, a potentially complicated contour integral may be reduced to an easier one.

Theorem 5.2 (Shrinking Contour Theorem/Deformation Theorem)**.** *Let $\mathcal{R}$ be a simply connected region, $\mathcal{C}$ an anticlockwise, simple closed contour in $\mathcal{R}$, z_0 a point enclosed by $\mathcal{C}$ and g a function which is holomorphic on $\mathcal{R}\setminus\{z_0\}$. Then*

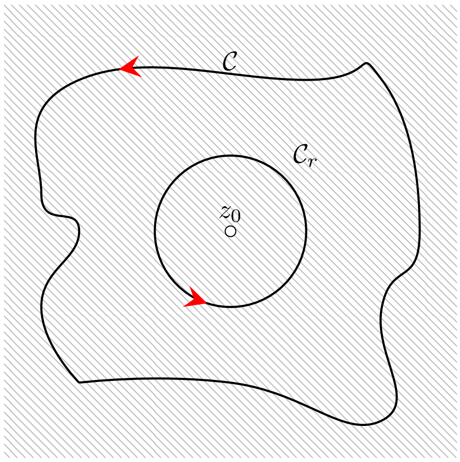
$$\int_{\mathcal{C}} g = \int_{\mathcal{C}_r} g$$

where $\mathcal{C}_r$ is an anticlockwise circular contour with centre z_0 and radius r, which is small enough so that $\mathcal{C}_r$ is contained in $\mathcal{R}$.



Sketch of Proof

Let us first consider the case where $\mathcal{R} = \mathbb{C}$ as shown:



Draw a straight line through z_0 to get two starlit subregions $\mathcal{R}_1$ and $\mathcal{R}_2$, with g holomorphic on both $\mathcal{R}_1$ and $\mathcal{R}_2$. Join $\mathcal{C}$ and $\mathcal{C}_r$ along this straight line to get two new (simple, closed, anticlockwise) contours $\mathcal{C}_1$ and $\mathcal{C}_2$ contained in $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively.

By Cauchy's Theorem for Starlit Regions,

$$\int_{\mathcal{C}_1} g = \int_{\mathcal{C}_2} g = 0.$$

Moreover,

$$0 = \int_{\mathcal{C}_1} f + \int_{\mathcal{C}_2} f = \int_{\mathcal{C}} f - \int_{\mathcal{C}_r} f,$$

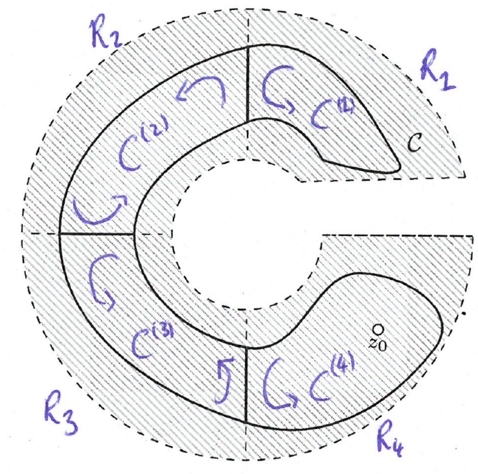
as the integrals along the straight line segments cancel, leaving us with

- an anticlockwise copy of $\mathcal{C}$, and
- a clockwise copy of $\mathcal{C}_r$, i.e., the reverse of $\mathcal{C}_r$.

Thus

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}_r} f.$$

For more general simply connected regions $\mathcal{R}$, we may need to partition $\mathcal{R}$ into starlit subregions $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_n$, numbered so that $z_0 \in \mathcal{R}_n$. By joining points on $\mathcal{C}$, we get anticlockwise closed contours $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \ldots, \mathcal{C}^{(n)}$, with $\mathcal{C}^{(j)}$ contained in $\mathcal{R}_j$ for each j. Label the regions $\mathcal{R}_j$ so that $z_0 \in \mathcal{R}_n$.



As before,

$$\int_{\mathcal{C}} g = \int_{\mathcal{C}^{(1)}} g + \int_{\mathcal{C}^{(2)}} g + \ldots + \int_{\mathcal{C}^{(n)}} g,$$

as the integrals along the connecting edges cancel in pairs.

For each $j \neq n$, g is holomorphic on $\mathcal{R}_j$, and $\mathcal{C}^{(j)}$ is a closed contour contained in the starlit region $\mathcal{R}_j$. Thus by Cauchy's Theorem for Starlit regions, we have

$$\int_{\mathcal{C}^{(j)}} g = 0 \quad \text{for } j = 1, 2, \ldots, n-1.$$

Combining these two observations it follows that

$$\int_{\mathcal{C}} g = \int_{\mathcal{C}^{(n)}} g,$$

and thus we need to show that

$$\int_{\mathcal{C}^{(n)}} g = \int_{\mathcal{C}_r} g.$$

The proof of this is almost identical to the case $\mathcal{R} = \mathbb{C}$.

□

A very similar argument is used to prove Cauchy's Theorem for Simply Connected Regions.

Theorem 5.3 (Cauchy's Theorem for Simply Connected Regions)**.** *Let f be a function that is holomorphic in a simply connected region $\mathcal{R}$, and let $\mathcal{C}$ be a closed contour in $\mathcal{R}$. Then*

$$\int_{\mathcal{C}} f = 0.$$

Proof This is identical to the Proof of Theorem 5.2, except that now, f is holomorphic on the starlit region $\mathcal{R}_n$, so that

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}_r} f = 0.$$

□

Theorem 5.4 (Cauchy's Integral Formula)**.** *Let $\mathcal{R}$ be a simply connected region, $\mathcal{C}$ an anticlockwise simple closed contour in $\mathcal{R}$, z_0 a point enclosed by $\mathcal{C}$ and f a function that is holomorphic on $\mathcal{R}$. Then*

$$\int_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

Proof

Define a new function g by the formula

$$g(z) = \frac{f(z)}{z - z_0}$$

so that g is holomorphic on $\mathcal{R} \backslash \{z_0\}$, i.e., where f is holomorphic and $z - z_0 \neq 0$. Thus we are trying to show that

$$\int_{\mathcal{C}} g = 2\pi i f(z_0).$$

Using the Shrinking Contour Theorem 5.2, we may replace $\mathcal{C}$ with an anticlockwise circular contour $\mathcal{C}_r$ with centre z_0 and radius r to get

$$\int_{\mathcal{C}} g = \int_{\mathcal{C}_r} g$$

whenever $r > 0$ is small enough so that $\mathcal{C}_r$ is contained in $\mathcal{R}$.

If we define I by

$$I = \left(\int_{\mathcal{C}_r} \frac{f(z)}{z - z_0}\, dz \right) - 2\pi i f(z_0),$$

then we want to show that $I = 0$. To do this, we will write I as a single integral along $\mathcal{C}_r$ and use the Estimation Lemma (together with the fact that our definition of I does not depend on our choice of sufficiently small r).

Parametrising $\mathcal{C}_r$ with $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + r\exp(it)$, we have

$$\int_{\mathcal{C}_r} \frac{1}{z - z_0}\, dz = \int_0^{2\pi} \frac{1}{z_0 + r\exp(it) - z_0} \cdot ir\exp(it)\, dt = \int_0^{2\pi} i\, dt = 2\pi i$$

(which we saw in Exercise Sheet 2). Therefore, since $f(z_0)$ is a constant, we get

$$2\pi i f(z_0) = \int_{\mathcal{C}_r} \frac{f(z_0)}{z - z_0}\, dz.$$

Thus

$$\begin{aligned} I &= \int_{\mathcal{C}_r} \frac{f(z)}{z - z_0}\, dz - \int_{\mathcal{C}_r} \frac{f(z_0)}{z - z_0}\, dz \\ &= \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz. \end{aligned}$$

To apply the Estimation Lemma, we need to find an upper bound for

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| \qquad \text{where } z \in \mathcal{C}_r.$$

Since $\mathcal{C}_r$ is a circle with centre z_0 and radius r, it is clear that $|z - z_0| = r$ for all $z \in \mathcal{C}_r$, and thus we look at $|f(z) - f(z_0)|$.

Since f is holomorphic on $\mathcal{R}$, it is continuous on $\mathcal{R}$ and in particular, continuous at z_0. Thus given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon.$$

In other words, given any $\epsilon > 0$, if we choose r so that $0 < r < \delta$, we have

$$z \in \mathcal{C}_r \Rightarrow |z - z_0| = r < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon.$$

For any such r we thus have

$$z \in \mathcal{C}_r \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} \right| = \frac{|f(z) - f(z_0)|}{|z - z_0|} < \frac{\epsilon}{r}.$$

By the Estimation Lemma (using the fact that $\ell(\mathcal{C}_r) = 2\pi r$), we see that

$$\left| \int_{\mathcal{C}_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leq \frac{\epsilon}{r} 2\pi r = 2\pi\epsilon.$$

However, since the definition of I does not depend on our choice of (small) r, it follows that

$$|I| < 2\pi\epsilon$$

for all $\epsilon > 0$. Hence $|I| = 0$ which implies $I = 0$, or in other words,

$$\int_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

Note 5.5. Cauchy's Integral Formula, and its proof, can be easily remembered using the following approximation: if r is small and $z \in \mathcal{C}_r$ then $f(z) \approx f(z_0)$, hence

$$\int_{\mathcal{C}_r} \frac{f(z)}{z - z_0}\, dz \approx \int_{\mathcal{C}_r} \frac{f(z_0)}{z - z_0}\, dz = f(z_0) \int_{\mathcal{C}_r} \frac{1}{z - z_0}\, dz = f(z_0)(2\pi i).$$

Suppose we wish to evaluate the integral $\int_{\mathcal{C}} g$ of a continuous function g along some closed contour $\mathcal{C}$. If we can find some point z_0 enclosed by $\mathcal{C}$ and a function f holomorphic on $\mathcal{R}$ with

$$g(z) = \frac{f(z)}{z - z_0} \quad \text{for all } z \in \mathcal{C}$$

then our integral may be evaluated using

$$\int_{\mathcal{C}} g = \int_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

In other words, it is enough to know the value of $f(z_0)$ in order to calculate $\int_{\mathcal{C}} g$.

Example 5.6. Evaluate the integral

$$\int_{\mathcal{C}} \frac{\mathrm{Log}(z)}{z^2 + 9}$$

where Log is the Principal Logarithm function and $\mathcal{C}$ is the anticlockwise circle with centre $4i$ and radius 3.

Solution

We know that Log is holomorphic on $\mathbb{C}_\pi$, and moreover, we have

$$z^2 + 9 = (z + 3i)(z - 3i).$$

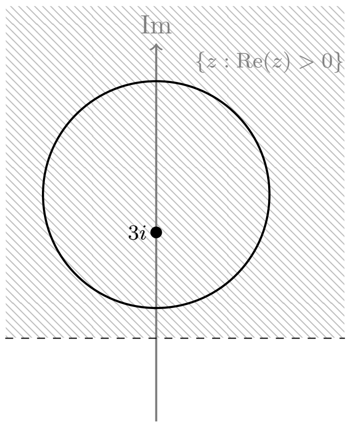
If we write

$$\frac{\mathrm{Log}(z)}{z^2 + 9} = \frac{\mathrm{Log}(z)}{(z + 3i)(z - 3i)} = \frac{f(z)}{z - z_0}$$

where $z_0 = 3i$ and

$$f(z) = \frac{\mathrm{Log}(z)}{z + 3i}$$

then z_0 is enclosed by $\mathcal{C}$ and f is holomorphic on $\mathbb{C}_\pi \setminus \{-3i\}$, which is a region containing $\mathcal{C}$. However, it is clear that the region $\mathbb{C}_\pi \setminus \{-3i\}$ is not simply connected, so the hypotheses of Cauchy's Integral Formula are not fully satisfied. We deal with this by replacing $\mathbb{C}_\pi \setminus \{-3i\}$ with a subregion that is simply connected.



Indeed, if we consider the region $\mathcal{R} = \{z \in \mathbb{C} : \mathsf{Im}(z) > 0\}$, then we see that

- $\mathcal{R}$ is simply connected,
- f is holomorphic on $\mathcal{R}$, and
- $\mathcal{C}$ is a simple, closed anticlockwise contour that is contained in $\mathcal{R}$ and encloses the point $z_0 = 3i$.

Therefore, we can apply Cauchy's Integral Formula, which gives

$$\begin{aligned}
\int_{\mathcal{C}} \frac{\mathrm{Log}(z)}{z^2 + 9}\, dz &= \int_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz \\
&= 2\pi i f(z_0) \\
&= 2\pi i \left(\frac{\mathrm{Log}(3i)}{3i + 3i} \right) \\
&= \frac{\pi}{6} \left(\log(3) + i\frac{\pi}{2} \right).
\end{aligned}$$

Theorem 5.7 (Cauchy's Integral Formula for Derivatives; proof non examinable)**.** *Let $\mathcal{R}$ be a simply connected region, $\mathcal{C}$ a simple, closed anticlockwise contour contained in $\mathcal{R}$ and f a function that is holomorphic on $\mathcal{R}$. Then*

1. *f is infinitely differentiable on $\mathcal{R}$, and*
2. *At every point z in the region enclosed by $\mathcal{C}$, the k^{th} derivative $f^{(k)}$ of f satisfies*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta.$$

Theorem 5.8 (Liouville's Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic (i.e. f is holomorphic on the entire complex plane) and suppose that f is bounded, i.e., there exists $M > 0$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then f is constant.*

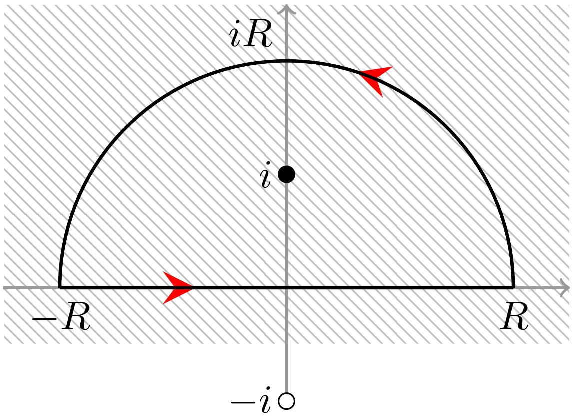
Proof: Exercise Sheet 4.

5.2 Evaluating Real Integrals Using Cauchy's Integral Formula

An important application of Cauchy's Integral Formula (Theorem 5.4) is that it allows us to evaluate certain *real* integrals that would be difficult, if not impossible, to evaluate otherwise. The following example illustrates the general method for doing so, which we shall develop in the next few sections.

Example 5.9. Let $R > 1$, and define a contour $\mathcal{C}_R$ by joining the line segment $[-R, R]$ and the upper semicircle $\mathcal{S}_R$ from R to $-R$ via iR. Use Cauchy's Integral Formula to evaluate

$$\int_{\mathcal{C}_R} \frac{1}{1+z^2}\, dz.$$



Solution

The function f is holomorphic on $\mathbb{C} \setminus \{i, -i\}$. Write

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)} = \frac{g(z)}{z - z_0}$$

where

$$g(z) = \frac{1}{z+i} \quad \text{and } z_0 = i.$$

Then g is holomorphic on $\mathbb{C} \setminus \{-i\}$ and in particular, on the simply connected region $\mathcal{R} = \left\{z \in \mathbb{C} : \mathsf{Im}(z) > -\frac{1}{2}\right\}$. Moreover, $\mathcal{C}_R$ is a closed, simple anticlockwise contour in $\mathcal{R}$ and $z_0 = i$ is a point enclosed by $\mathcal{C}_R$. Hence by Cauchy's Integral Formula,

$$\int_{\mathcal{C}_R} \frac{1}{1+z^2}\, dz = \int_{\mathcal{C}_R} \frac{g(z)}{z-i}\, dz$$

$$= 2\pi i g(i)$$
$$= 2\pi i \frac{1}{i+i} = \pi.$$

(Note that this integral does not depend on the value of R once $R > 1$).

Example 5.10. Now, let's use Example 5.9 to evaluate the real integral

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx.$$

Solution

Having evaluated the integral using Cauchy's Integral Formula, we now look at the integrals along the two paths L_R and S_R;

$$\int_{\mathcal{C}_R} \frac{1}{z^2+1}\, dz = \int_{L_R} \frac{1}{z^2+1}\, dz + \int_{S_R} \frac{1}{z^2+1}\, dz.$$

parametrise L_R with $\gamma_L : [-R, R] \to \mathbb{C}$, $\gamma(t) = t$, so that $\gamma'(t) = 1$ and

$$\int_{L_R} \frac{1}{1+z^2}\, dz = \int_{-R}^{R} \frac{1}{1+t^2}\, dt.$$

We have seen already (Example 4.3) that

$$\lim_{R\to\infty} \int_{S_R} \frac{1}{1+z^2}\, dz = 0.$$

Hence

$$\begin{aligned}
\pi &= \lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{1}{1+z^2}\, dz \\
&= \left(\lim_{R\to\infty} \int_{L_R} \frac{1}{1+z^2}\, dz\right) + \left(\lim_{R\to\infty} \int_{S_R} \frac{1}{1+z^2}\, dz\right) \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+t^2}\, dt + 0 \\
&= \int_{-\infty}^{\infty} \frac{1}{1+t^2}\, dt.
\end{aligned}$$

So we have used contour integration to show that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \pi.$$

Example 5.11. We shall evaluate the integral

$$\int_{\Gamma} \frac{2i}{3z^2 - 10z + 3}\, dz,$$

where Γ is the anticlockwise unit circle $\{z : |z| = 1\}$, and use it to evaluate the real integral

$$\int_0^{2\pi} \frac{1}{5 - 3\cos(t)}\, dt.$$

Solution

If we factorise the denominator, we can write

$$\frac{2i}{3z^2 - 10z + 3} = \frac{2i}{(3z-1)(z-\frac{1}{3}}.$$

The point $z_0 = \frac{1}{3}$ is the only point enclosed by Γ at which this function is not holomorphic. If we let

$$g(z) = \frac{2i}{3(z-3)}$$

we have

$$\frac{g(z)}{z - \frac{1}{3}} = \frac{2i}{(3z-1)(z-3)}$$

and g is holomorphic on the simply connected region $\{z \in \mathbb{C} : \mathsf{Re}(z) < 2\}$, which contains $\mathcal{C}$. Thus

$$\begin{aligned}\int_\Gamma \frac{2i}{(3z-1)(z-3)}\, dz &= \int_\Gamma \frac{g(z)}{z - \frac{1}{3}}\, dz \\ &= 2\pi i g(\tfrac{1}{3}) \\ &= 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}.\end{aligned}$$

For the second part, we parametrise Γ using $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$ and use

$$\begin{aligned}\int_\Gamma f &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)\, dt \\ &= \int_0^{2\pi} \frac{2i}{3e^{2it} - 10e^{it} + 3} \cdot ie^{it}\, dt \\ &= \int_0^{2\pi} \frac{-2}{-10 + 3(e^{it} - e^{-it})}\, dt.\end{aligned}$$

Since $\cos(t) = \frac{1}{2}\left(e^{it} - e^{-it}\right)$, it follows that

$$\frac{-2}{-10 + 3(e^{it} + e^{-it})} = \frac{-2}{-10 + 6\cos(t)} = \frac{1}{5 - 3\cos(t)}.$$

Hence

$$\int_0^{2\pi} \frac{1}{5 - 3\cos(t)}\, dt = \int_\Gamma \frac{2i}{3z^2 - 10z + 3} = \frac{\pi}{2}.$$

We can generalise the method of Example 5.11 to evaluate other real trigonometric integrals.

Definition 5.12. A *Rational Function* $R : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^2$, is a function of the form

$$R(x,y) = \frac{f(x,y)}{g(x,y)},$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ are two polynomials in x and y with real coefficients.

We will now describe how to use contour integration to evaluate integrals of the form

$$\int_0^{2\pi} R(\cos(t), \sin(t))\ dt,$$

where R is a rational function of two real variables. For example, if R is defined by $R(x,y) = \dfrac{1}{16x^2 + 25y^2}$, then we are looking at the integral

$$\int_0^{2\pi} R(\cos(t), \sin(t))\ dt = \int_0^{2\pi} \frac{1}{16\,(\cos(t))^2 + 25\,(\sin(t))^2}\ dt.$$

Note that cos and sin are periodic with period 2π, and that the above integral has limits 0 and 2π.

Example 5.13. For any rational function R of two real variables, with $R(\cos(t), \sin(t))$ defined for every $t \in [0, 2\pi]$, we can find a suitable closed contour $\mathcal{C}$ and complex function f so that

$$\int_0^{2\pi} R(\cos(t), \sin(t)) = \int_{\mathcal{C}} f.$$

Let $\mathcal{C}$ be the anticlockwise circle $\{z \in \mathbb{C} : |z| = 1\}$, parametrised using $\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) = \exp(it)$. Let f denote the complex function we are looking for, so we want

$$\int_0^{2\pi} R(\cos(t), \sin(t))\ dt = \int_{\mathcal{C}} f = \int_0^{2\pi} f\left(\gamma(t)\right)\gamma'(t)\ dt,$$

or in other words,

$$f(\gamma(t)) = \frac{1}{\gamma'(t)} \cdot R(\cos(t), \sin(t)). \tag{R1}$$

For each $z \in \mathcal{C}$, $z = \exp(it)$ for some $t \in [0, 2\pi]$ and so

$$\cos(t) = \frac{\exp(it) + \exp(-it)}{2} = \frac{z + z^{-1}}{2}$$
$$\sin(t) = \frac{\exp(it) - \exp(-it)}{2i} = \frac{z - z^{-1}}{2i}$$

Hence for $z = \exp(it) \in \mathcal{C}$, we have

$$R(\cos(t), \sin(t)) = R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right), \tag{R2}$$

(this is not true for all z, only for z lying on the contour $\mathcal{C}$). Moreover, $\gamma'(t) = i\exp(it) = iz$; so combining (R1) and (R2), we define

$$f(z) = R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \cdot \frac{1}{iz}.$$

Then

$$\begin{aligned}
\int_{\mathcal{C}} f &= \int_0^{2\pi} f(\gamma(t))\gamma'(t) \\
&= \int_0^{2\pi} R\left(\tfrac{1}{2}(\exp(it)+\exp(-it)), \tfrac{1}{2i}(\exp(it)-\exp(-it))\right) \cdot \frac{1}{i\exp(it)} \cdot i\exp(it)\ dt \\
&= \int_0^{2\pi} R(\cos(t), \sin(t))\ dt.
\end{aligned}$$

Example 5.14. Use a suitable contour integral to evaluate

$$\int_0^{2\pi} \frac{1}{2+\sin\theta}\ d\theta.$$

Solution

With $\mathcal{C}$ the anticlockwise unit circle, we have

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{2+\sin\theta}\ d\theta &= \int_{\mathcal{C}} \frac{1}{2+\frac{z-z^{-1}}{2i}} \cdot \frac{1}{iz}\ dz \\
&= \int_{\mathcal{C}} \frac{2}{z^2+4iz-1}\ dz \\
&= \int_{\mathcal{C}} \frac{2}{(z-(-2+\sqrt{3})i)(z-(-2-\sqrt{3})i)}\ dz.
\end{aligned}$$

By Cauchy's Integral Formula, this integral has the value

$$2\pi i \cdot \frac{2}{(-2+\sqrt{3})i-(-2-\sqrt{3})i} = \frac{2\pi}{\sqrt{3}}.$$

Example 5.15. Set up a contour integral that could be used to evaluate

$$\int_0^{2\pi} \frac{1}{16\cos^2(t)+25\sin^2(t)}\ dt.$$

Solution

Here $R(x,y) = \dfrac{1}{16x^2+25y^2}$, so the required integral (with $\mathcal{C}$ the anticlockwise circle $\{z \in \mathbb{C} : |z| = 1\}$) is

$$\begin{aligned}
\int_{\mathcal{C}} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \cdot \frac{1}{iz}\ dz &= \int_{\mathcal{C}} \frac{1}{16\left(\frac{1}{2}(z+z^{-1})\right)^2 + 25\left(\frac{1}{2i}(z-z^{-1})\right)^2} \cdot \frac{1}{iz}\ dz \\
&= \int_{\mathcal{C}} \frac{1}{4(z+2+z^{-2}) - \frac{25}{4}(z^2-2+z^{-2})} \cdot \frac{1}{iz}\ dz \\
&= \int_{\mathcal{C}} \frac{4i}{9z^4-82z^2+9}\ dz.
\end{aligned}$$

We do not yet know how to evaluate this integral, we shall do this in Chapter 6.

5.3 Series Representations of Holomorphic Functions

Definition 5.16. A sequence z_n of complex numbers is said to converge to a complex number z if, for all $\epsilon > 0$ there is a natural number N such that $n \geq N$ implies that $|z_n - z| < \epsilon$.

An infinite series $\sum_{k=1}^{\infty} a_k$ is said to converge to a complex number s if the sequence of partial sums $s_n := \sum_{k=1}^{n} a_k$ converges to s. When this occurs we write $\sum_{k=1}^{\infty} a_k = s$.

For a fixed $z_0 \in \mathbb{C}$, a *power series* (centred at z_0) is a series of the form $\displaystyle\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where a_n are complex numbers. As with power series in $\mathbb{R}$, a complex power series may converge for some values of z and diverge for others. Note that if $\mathcal{S}$ denotes the subset

$$\mathcal{S} = \left\{ z \in \mathbb{C} : \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges} \right\},$$

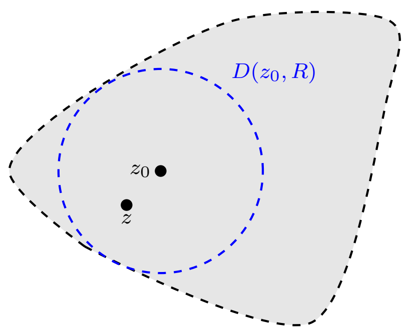
then we get a function $f : \mathcal{S} \to \mathbb{C}$ defined by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Theorem 5.17 (Taylor's Theorem)**.** *Let $\mathcal{R}$ be a region, $f : \mathcal{R} \to \mathbb{C}$ holomorphic and z_0 a point in $\mathcal{R}$. Then f can be represented by the power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

for all $z \in D(z_0; R)$, whenever $R > 0$ is small enough so that $D(z_0; R) \subseteq \mathcal{R}$.

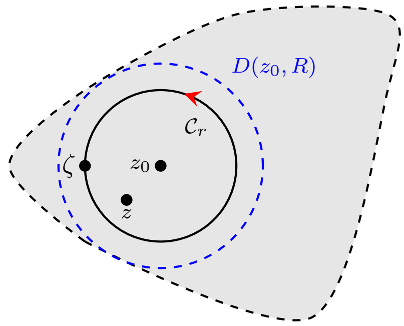


Before proving Taylor's Theorem, we need the following lemma:

Lemma 5.18. *Let $f : \mathcal{R} \to \mathbb{C}$ be a function, $z_0 \in \mathcal{R}$ and $R > 0$ be such that $D(z_0, R) \subseteq \mathcal{R}$. Then for any $0 < r < R$, $z \in D(z_0, r)$ and $\zeta \in \mathcal{C}_r$ we have*

$$\frac{f(\zeta)}{\zeta - z} = \left(\sum_{j=0}^{n} \frac{(z - z_0)^j f(\zeta)}{(\zeta - z_0)^{j+1}} \right) + \frac{(z - z_0)^{n+1} f(\zeta)}{(\zeta - z_0)^{n+1} (\zeta - z)},$$

where $\mathcal{C}_r$ is the anticlockwise circular contour with centre z_0 and radius r.



Proof

Note that we can write
$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \frac{1}{\left(1 - \frac{z-z_0}{\zeta - z_0}\right)}. \tag{T1}$$

Rearranging the formula for a finite geometric sum,

$$\sum_{j=0}^{n} w^j = \left(\frac{1 - w^{n+1}}{1 - w}\right), \quad \text{gives} \quad \frac{1}{1-w} = \left(\sum_{j=0}^{n} w^j\right) + \frac{w^{n+1}}{1-w},$$

and so substituting $w = \dfrac{z - z_0}{\zeta - z_0}$ we get

$$\begin{aligned}\frac{1}{\left(1 - \frac{z-z_0}{\zeta-z_0}\right)} &= \left(\sum_{j=0}^{n} \left(\frac{z - z_0}{\zeta - z_0}\right)^j\right) + \frac{(z - z_0)^{n+1}}{(\zeta - z_0)^{n+1}\left(1 - \frac{z-z_0}{\zeta-z_0}\right)} \\ &= \left(\sum_{j=0}^{n} \left(\frac{z - z_0}{\zeta - z_0}\right)^j\right) + \frac{(z - z_0)^{n+1}}{(\zeta - z_0)^n (\zeta - z)}.\end{aligned}$$

Combining this with (T1), and multiplying both sides by $f(\zeta)$ we get

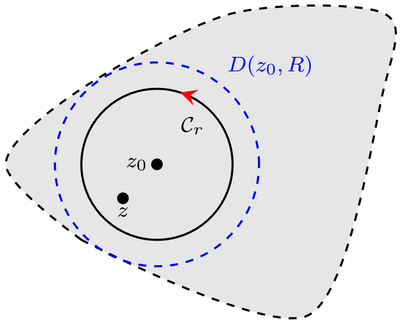
$$\frac{f(\zeta)}{\zeta - z} = \left(\sum_{j=0}^{n} (z - z_0)^j \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\right) + \frac{(z - z_0)^{n+1} f(\zeta)}{(\zeta - z_0)^{n+1} (\zeta - z)}$$

Proof

[of Taylor's Theorem 5.17] Choose $r > 0$ with $0 < r < R$ and let $\mathcal{C}_r$ be the anticlockwise circle with centre z_0 and radius r. Then by Cauchy's Integral Formula,

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all z with $|z - z_0| < r$.



Together with Lemma 5.18 we see that

$$2\pi i f(z) = \int_{\mathcal{C}_r} \left[\left(\sum_{j=0}^{n} (z - z_0)^j \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \right) + \frac{(z - z_0)^{n+1} f(\zeta)}{(\zeta - z_0)^{n+1} (\zeta - z)} \right] d\zeta.$$

Using linearity of contour integrals, we first get

$$2\pi i f(z) = \left(\sum_{j=0}^{n} \int_{\mathcal{C}_r} (z - z_0)^j \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta \right) + \int_{\mathcal{C}_r} \frac{(z - z_0)^{n+1} f(\zeta)}{(\zeta - z_0)^{n+1} (\zeta - z)} \, d\zeta.$$

and noting that since we are integrating with respect to ζ, the $(z - z_0)$ terms are constant, this becomes

$$2\pi i f(z) = \left(\sum_{j=0}^{n} (z - z_0)^j \int_{\mathcal{C}_r} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \, d\zeta \right) + \int_{\mathcal{C}_r} \frac{(z - z_0)^{n+1} f(\zeta)}{(\zeta - z_0)^{n+1} (\zeta - z)} \, d\zeta$$

(we leave the last term as it is for now).

Using Cauchy's Integral Formula For Derivatives (Theorem 5.7), and dividing both sides by $2\pi i$, this becomes

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2 f''(z_0)}{2!} + \ldots + \frac{(z - z_0)^n f^{(n)}(z_0)}{n!} + I_n,$$

where

$$I_n = \frac{1}{2\pi i} \int_{\mathcal{C}_r} \frac{(z - z_0)^{n+1} f(\zeta)}{(\zeta - z_0)^{n+1} (\zeta - z)} \, d\zeta.$$

To complete the proof, we must show that $I_n \to 0$ as $n \to \infty$. We shall use the Estimation Lemma to do this.

- For $\zeta \in \mathcal{C}_r$, we have $|\zeta - z_0| = r$, and since z is enclosed by $\mathcal{C}_r$, $|z - z_0| < r$.
- Since f is continuous on $\mathcal{C}_r$, there is some real number $K \geq 0$ such that $|f(\zeta)| \leq K$ for all ζ in $\mathcal{C}_r$. To see this, note that if $\mathcal{C}_r$ is parametrised by the continuous function $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + re^{it}$, then $t \mapsto |f(\gamma(t))|$ is a continuous, real-valued function on a closed interval, hence is bounded by the Extreme Value Theorem.
- By the backwards triangle inequality,

  $$|\zeta - z| = |\zeta - z_0 + z_0 - z| \geq ||\zeta - z_0| - |z_0 - z|| = r - |z_0 - z|,$$

  a strictly positive constant since z is fixed and $|z_0 - z| < r$.

Hence for all $\zeta \in \mathcal{C}_r$ we have

$$\left|\frac{(z-z_0)^{n+1}\, f(\zeta)}{(\zeta-z_0)^{n+1}\,(\zeta-z)}\right| = \left|\frac{z-z_0}{\zeta-z_0}\right|^{n+1} \cdot \frac{|f(\zeta)|}{|\zeta-z|} \le \left|\frac{z-z_0}{\zeta-z_0}\right|^{n+1} \cdot \frac{K}{r-|z-z_0|}$$

Thus by the Estimation Lemma,

$$|I_n| = \left|\frac{1}{2\pi i}\int_{\mathcal{C}_r} \frac{(z-z_0)^{n+1}\, f(\zeta)}{(\zeta-z_0)^{n+1}\,(\zeta-z)}\, d\zeta\right| \le \frac{1}{2\pi} \cdot \left|\frac{z-z_0}{\zeta-z_0}\right|^{n+1} \cdot \frac{K}{r-|z-z_0|},$$

Since $|z-z_0| < |\zeta - z_0|$, $\left|\frac{z-z_0}{\zeta-z_0}\right| < 1$ and so $\left|\frac{z-z_0}{\zeta-z_0}\right|^{n+1} \to 0$ as $n \to \infty$.

Since all other terms in the estimate are constant, this shows that $I_n \to 0$ as $n \to \infty$.

Many familiar examples of Taylor Series from the real case are also valid for the corresponding complex functions:

Table 3.1 Some Common Expansions

Function	*Taylor Series around 0*	*Where Valid*
e^z	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$	all z
$\sin z$	$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n-1}}{(2n-1)!}$	all z
$\cos z$	$1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$	all z
$\log(1+z)$ (principal branch)	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$	$\|z\| < 1$
$(1+z)^\alpha$ (principal branch with $\alpha \in \mathbb{C}$ fixed)	$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$ (binomial series), where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$, take to be zero if α is an integer $> n$ and $\binom{\alpha}{0} = 1$.	$\|z\| < 1$

These examples can be computed in exactly the same way as the corresponding real series. In fact, we could say a lot more about complex Taylor Series, but we will not have time to do so in this module.

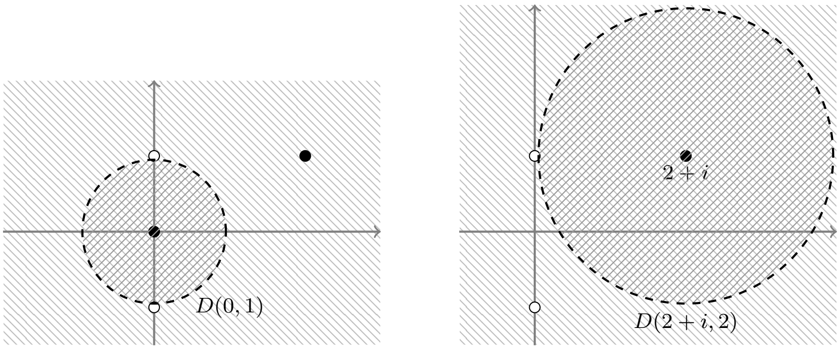
Example 5.19. The Taylor Series expansion of a function $f : \mathcal{R} \to \mathbb{C}$ at a point $z_0 \in \mathcal{R}$ may not be valid everywhere in the domain of f. For example, let us examine the Taylor series at $z = 0$ of the function $f : \mathbb{C}\backslash\{-i, i\} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{1+z^2}.$$

This function is holomorphic on $\mathbb{C}\setminus\{-i,i\}$, and by differentiation we can show that its Taylor series at $z=0$ is

$$\sum_{n=0}^{\infty}(-1)^n z^{2n}.$$

The largest disc centred at 0 that is contained in $\mathbb{C}\setminus\{-i,i\}$ is $D(0,1)$, hence this Taylor Series is valid (i.e. converges to $f(z)$) at every point of this disc by Theorem 5.17. However, Theorem 5.17 tells us nothing about convergence of this series at points of $\mathbb{C}\setminus\{-i,i\}$ that lie outside of this disc (in fact the series diverges at these points).



Similarly, if we compute the Taylor Series for f at $z=2+i$ (which we won't do, but we know it exists), Theorem 5.17 tells us that this series converges for all z inside the disc $D(2+i,2)$.

Remark. It is worth pointing out some of the differences between the real and complex versions of Taylor's Theorem.

(i) In the complex case, once we know that f is differentiable everywhere on an open set $\mathcal{R}$ (i.e. holomorphic on $\mathcal{R}$), we know that f can be represented by a Taylor series. For real functions it is not enough to know that f is differentiable on an open interval; f must be infinitely differentiable.

(ii) There are examples of real functions that are infinitely differentiable, but whose Taylor series have radius on convergence 0, e.g., the function f defined by $f(x)=e^{-1/x^2}$ for $x\neq 0$ and $f(0)=0$. This cannot happen in the complex case; the radius of convergence is always positive.

Chapter 6 Evaluating Contour Integrals Using Residues

6.1 Motivation

Let $\mathcal{C}$ be a simple, closed anticlockwise contour. In the previous section we saw how certain integrals of the form

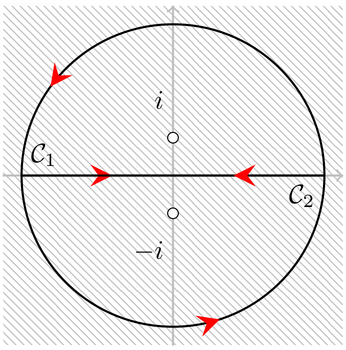
$$\int_{\mathcal{C}} f$$

may be calculated using Cauchy's Integral Formula, when f is a function holomorphic on region $\mathcal{R}\setminus\{z_0\}$ with z_0 a point enclosed by $\mathcal{C}$. However, this does not always work, as the following example illustrates.

Example 6.1. Cauchy's Integral Formula does not allow us to compute $\int_{\mathcal{C}} f$, where $\mathcal{C}$ is the anticlockwise unit circle and f is the function defined by

$$f(z) = \frac{1}{z^2 + \frac{1}{4}} = \frac{1}{(z + \frac{i}{2})(z - \frac{i}{2})},$$

as there are two points enclosed by $\mathcal{C}$ at which f is not holomorphic.



Draw the line $L = [-1, 1]$ to get two new closed anticlockwise contours $\mathcal{C}_1$ and $\mathcal{C}_2$, with $\mathcal{C}_1$ enclosing $i/4$ and $\mathcal{C}_2$ enclosing $-i/4$. Then

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}_1} f + \int_{\mathcal{C}_2} f,$$

since the integral along L in $\mathcal{C}_1$ and the integral along $\tilde{L}$ in $\mathcal{C}_2$ cancel. We can then use Cauchy's Integral Formula to evaluate each of $\displaystyle\int_{\mathcal{C}_1} f$ and $\displaystyle\int_{\mathcal{C}_2} f$ separately.

We will assume without proof that the method we have just described will extend to the case when there is any finite number of points enclosed by $\mathcal{C}$ at which f is not holomorphic:

Theorem 6.2 (Generalised Shrinking Contour/ Deformation Theorem; proof non-examinable)**.** *Let $\mathcal{R}$ be a simply connected region, $\mathcal{C}$ a simple, closed anticlockwise contour in $\mathcal{R}$, and $z_1, z_2, \ldots, z_n$ a finite collection of points that are enclosed by $\mathcal{C}$. If f is holomorphic on $\mathcal{R} \setminus \{z_1, \ldots, z_n\}$, then*

$$\int_{\mathcal{C}} f = \int_{\mathcal{C}_1} f + \ldots + \int_{\mathcal{C}_n} f,$$

where each $\mathcal{C}_j$ is a closed simple anticlockwise circular contour enclosing z_j and no other z_k.

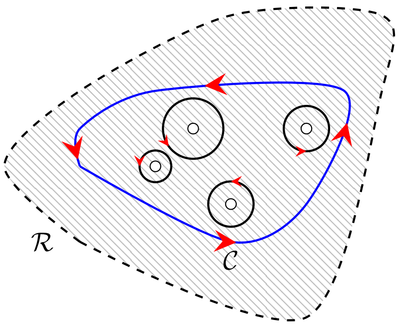


Figure 6.1: A contour $\mathcal{C}$ enclosing points z_1, z_2, z_3 and z_4 at which f is not holomorphic. The integral of f along $\mathcal{C}$ is determined by the integrals along the circular contours enclosing these points.

Theorem 6.2 allows us, as least in principle, to evaluate each $\int_{\mathcal{C}_j} f$ separately to obtain $\int_{\mathcal{C}} f$. Thus we have essentially reduced the problem to that of evaluating integrals where there is one point enclosed by the given contour at which f is not holomorphic.

This approach will not always work, however.

Example 6.3. Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be defined by

$$f(z) = \frac{\sin(z)}{z^2},$$

and let $\mathcal{C}$ be any anticlockwise closed circular contour centred at 0. We cannot write

$$\frac{\sin(z)}{z^2} = \frac{g(z)}{(z - z_0)}$$

for any z_0 enclosed by $\mathcal{C}$ and g holomorphic on a simply connected region containing $\mathcal{C}$. Indeed the obvious choice would be to take $z_0 = 0$, but then we would have to define $g(z) = \dfrac{\sin(z)}{z}$, which still fails to be holomorphic on a suitable region.

6.2 Singulairities of complex functions

Theorem 6.2 shows that when calculating the integral of f along a simple, anticlockwise, closed contour $\mathcal{C}$, the value $\int_{\mathcal{C}} f$ is in some sense depends only on those points at which f is not holomorphic. Therefore, we shall study these points in more detail.

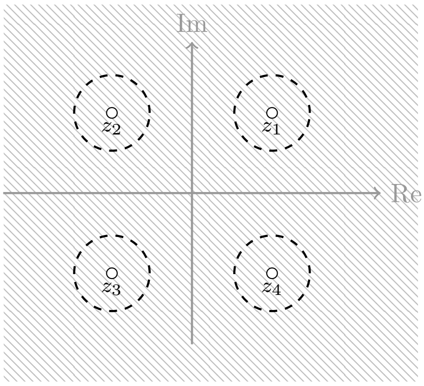
Definition 6.4. A function f has an *isolated singularity* at the point z_0 if for some $r > 0$, f is holomorphic on a punctured disc $D'(z_0, r)$ but not on the (unpunctured) open disc $D(z_0, r)$.

Note that if f has an isolated singularity at z_0, it may be the case that f is not defined at z_0, or f is defined at z_0 but not differentiable there.

Example 6.5. The function f defined by

$$f(z) = \frac{1}{z^4 + 1}$$

has isolated singularities at the points $e^{i\pi/4}$, $e^{3i\pi/4}$, $e^{5i\pi/4}$, $e^{7i\pi/4}$; in each case, take $r = \frac{1}{4}$ for example.



Example 6.6. The function f defined by

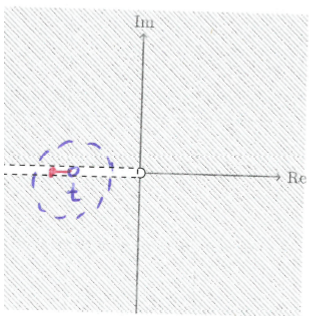
$$f(z) = \frac{\sin(z)}{z^2}$$

has an isolated singularity at 0.

Example 6.7. The Principal Logarithm function Log : $\mathbb{C} \setminus \{0\}$ defined by

$$\text{Log}(z) = \log(|z|) + i\text{Arg}(z),$$

is not holomorphic at any point on the negative real axis. No such point is an isolated singularity of Log



Indeed, if $t \leq 0$ is a point on the negative real axis and $r > 0$, then $D'(t, r)$ contains points at which Log is not holomorphic; $z = t - \frac{r}{2}$ for example.

Definition 6.8. Let f be continuous on $\mathcal{R} \setminus \{z_0\}$. We say that $\lim_{z \to z_0} f(z) = \infty$ if given any $M > 0$ there is some $r > 0$ such that

$$|f(z)| > M \text{ for all } z \in D'(z_0, r).$$

Example 6.9. Consider the function

$$f(z) = \frac{1}{(z - 2i)^3}$$

which has an isolated singularity at $2i$. We will show that $\lim_{z \to 2i} f(z) = \infty$.

Solution

Indeed, given $M > 0$, let $r = M^{-1/3}$. Then

$$\begin{aligned} 0 < |z - 2i| < r \Rightarrow |f(z)| &= \left| \frac{1}{(z - 2i)^3} \right| \\ &= \frac{1}{|z - 2i|^3} \\ &> \frac{1}{(M^{-1/3})^3} = M. \end{aligned}$$

In most of the examples that we consider, $\lim_{z \to z_0} f(z) = \infty$ will occur whenever evaluating f at z_0 would involve division by 0 (except when $\frac{0}{0}$ would appear). We shall make this more precise shortly.

Definition 6.10. A function f with an isolated singularity z_0 is said to have a *pole at* z_0 if

$$\lim_{z \to z_0} f(z) = \infty.$$

Moreover, for $n \geq 1$, f is said to have a *pole of order* n at z_0 if for some $r > 0$, f can be represented in the form

$$f(z) = \frac{g(z)}{(z - z_0)^n} \qquad \text{for all } z \in D'(z_0, r)$$

where g is holomorphic on $D(z_0, r)$ and $g(z_0) \neq 0$.

Note that the representation

$$f(z) = \frac{g(z)}{(z - z_0)^n}$$

need not be valid everywhere on the domain of f, only inside the 'small' disc $D(z_0, r)$. The function g, unlike f, is both defined and differentiable at z_0.

Example 6.11. Let us return to the example of

$$f(z) = \frac{1}{1 + z^4}$$

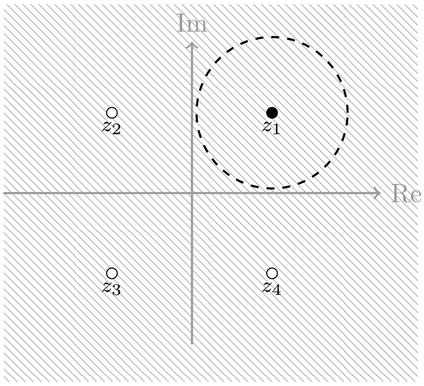
and investigate the pole $z_1 = e^{i\pi/4}$ of f.

Solution

Denote by z_2, z_3 and z_4 the other complex 4^{th} roots of -1, and let g be the function defined by

$$g(z) = \frac{1}{(z - z_2)(z - z_3)(z - z_4)}.$$

Then g is holomorphic on $\mathbb{C} \setminus \{z_2, z_3, z_4\}$, and in particular, holomorphic on $D(z_1, \frac{1}{2})$ for example (the precise value of $r > 0$ is not important).



Moreover, $g(z_1) \neq 0$ and

$$f(z) = \frac{g(z)}{z - z_1} \quad \text{for all} \quad z \in D'(z_1, \tfrac{1}{2}).$$

This shows that f has a pole of order 1 at z_1.

Example 6.12. Let us investigate the poles of the function

$$f(z) = \frac{1}{(z^2 + 9)^2}.$$

Solution

Using the factorisation

$$z^2 + 9 = (z + 3i)(z - 3i)$$

we have

$$f(z) = \frac{1}{(z + 3i)^2 (z - 3i)^2}$$

so that f has isolated singularities at $\pm 3i$.

If we define the function g_1 by

$$g_1(z) = \frac{1}{(z + 3i)^2}$$

then g_1 is holomorphic on $\mathbb{C} \setminus \{-3i\}$ and in particular, holomorphic at $3i$ (that is, holomorphic on some open disc centred at $3i$, for example, $D(3i, 1)$). Since

$$g_1(3i) = -\frac{1}{36} \neq 0 \quad \text{and} \quad f(z) = \frac{g_1(z)}{(z - 3i)^2} \quad \text{for } z \in D'(3i, 1)$$

we see that f has a pole of order 2 at $z = 3i$.
Similarly, by considering the function g_2 defined by

$$g_2(z) = \frac{1}{(z - 3i)^2}$$

we see that g_2 is holomorphic and nonzero at $z = -3i$ and

$$f(z) = \frac{g_2(z)}{(z + 3i)^2} \qquad \text{for} \qquad z \in D'(-3i, 1)$$

so that f has a pole of order 2 at $z = -3i$ also.

6.3 The Residue Theorem

Definition 6.13. Let f have an isolated singularity at z_0, then we define the *residue of f at z_0*, denoted by $\mathrm{Res}(f; z_0)$, to be

$$\mathrm{Res}(f; z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} f$$

where $\mathcal{C}$ is an anticlockwise simple closed contour which contains z_0 and no other singularities of f, and lies inside a region in which f is holomorphic.

If f has an isolated singularity at z_0, then such a contour $\mathcal{C}$ can always be found. Indeed, we know that f is holomorphic on the punctured disc $D'(z_0, r)$ for some $r > 0$. Thus we may take $\mathcal{C}$ to be the anticlockwise circle with centre z_0 and radius $r/2$.

For example, we have seen before that with $\mathcal{C}$ the anticlockwise unit circle,

$$\int_{\mathcal{C}} \frac{1}{z}\, dz = 2\pi i \qquad \text{and} \qquad \int_{\mathcal{C}} \frac{1}{z^2}\, dz = 0.$$

Thus for the functions f and g defined by $f(z) = \dfrac{1}{z}$ and $g(z) = \dfrac{1}{z^2}$ we have

$$\mathrm{Res}(f; 0) = 1 \qquad \text{and} \qquad \mathrm{Res}(g; 0) = 0.$$

Theorem 6.14 is essentially a reformulation of the Generalised Deformation Theorem (Theorem 6.2), and generalises Cauchy's Integral Formula.

Theorem 6.14 (The Residue Theorem)**.** *Let $\mathcal{R}$ be a simply connected region and let f be a function that is holomorphic on the region $\mathcal{R}\setminus\{z_1, z_2, \dots, z_n\}$ and has isolated singularities at the points $z_1, z_2, \dots, z_n$. If $\mathcal{C}$ is an anticlockwise simple closed contour that lies in $\mathcal{R}\setminus\{z_1, z_2, \dots, z_n\}$ and encloses the points $z_1, z_2, \dots, z_n$, then*

$$\int_{\mathcal{C}} f = 2\pi i \left(\mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2) + \ldots + \mathrm{Res}(f; z_n)\right).$$

In other words

$$\int_{\mathcal{C}} f = 2\pi i \,(\text{ sum of residues of } f \text{ at isolated singularities enclosed by } \mathcal{C}\ )$$

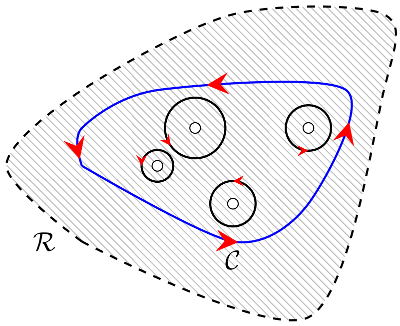


Figure 6.2: At each singularity z_j, the residue of f at z_j is given by the the integral of f along the small circular contour surrounding z_j (divided by $2\pi i$). The integral of f along $\mathcal{C}$ is then determined by these residues.

We want to use the Residue Theorem to evaluate contour integrals. The theorem may not look particularly useful yet, since residues themselves are defined to be integrals along circular contours. However, the following two theorems show us that certain residues may be calculated without performing any integration.

We will need the following Lemma, the proof of which is omitted.

Lemma 6.15 (Proof non-examinable)**.** *Suppose that $z_0 \in \mathbb{C}$, h is holomorphic at the point z_0 with $h(z_0) = 0$ and $h'(z_0) \neq 0$. Then there is some $\delta > 0$ and a holomophic function $k : D(z_0, \delta) \to \mathbb{C}$ with $k(z) \neq 0$ and $h(z) = (z - z_0)k(z)$ for all $z \in D(z_0, \delta)$.*

Theorem 6.16 (The g/h rule)**.** *Let $z_0 \in \mathbb{C}$, $r > 0$ and let f be holomorphic on $D'(z_0, r)$. Suppose that f can be represented by*

$$f(z) = \frac{g(z)}{h(z)} \qquad \text{for } z \in D'(z_0, r)$$

where g and h are holomorphic on $D(z_0, r)$, and $g(z_0) \neq 0$, $h(z_0) = 0$ and $h'(z_0) \neq 0$. Then

(i) The function f has an isolated singularity, and in particular, a pole of order one at z_0.

(iii) The residue of f at z_0 is given by

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}.$$

Proof

By Lemma 6.15 we obtain $\delta > 0$ (making $\delta < r$ if necessary) and $k : D(z_0, \delta) \to \mathbb{C}$, with k holomorphic and nonzero on $D(z_0, \delta)$ and

$$f(z) = \frac{g(z)}{k(z)} \cdot \frac{1}{(z - z_0)} \qquad \text{for all } z \in D'(z_0, \delta).$$

Since $k(z) \neq 0$ for all $z \in D(z_0, \delta)$ the function $z \mapsto \frac{g(z)}{k(z)}$ is holomorphic on $D(z_0, \delta)$, and since $g(z_0) \neq 0$, f has a pole of order 1 at z_0.

If $\mathcal{C}$ is the anticlockwise circle with centre z_0 and radius $r/2$, then Cauchy's Integral Formula gives

$$\operatorname{Res}(f; z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} f = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{g(z)}{k(z)} \cdot \frac{1}{z - z_0} \, dz = \frac{g(z_0)}{k(z_0)}.$$

But then by the product rule, $h'(z) = k(z) + (z - z_0)k'(z)$ for all $z \in D(z_0, \delta)$, so that $h'(z_0) = k(z_0)$, thus

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}.$$

Example 6.17. Let us calculate the residues at the poles of the function f defined by

$$f(z) = \frac{1}{1 + z^2}.$$

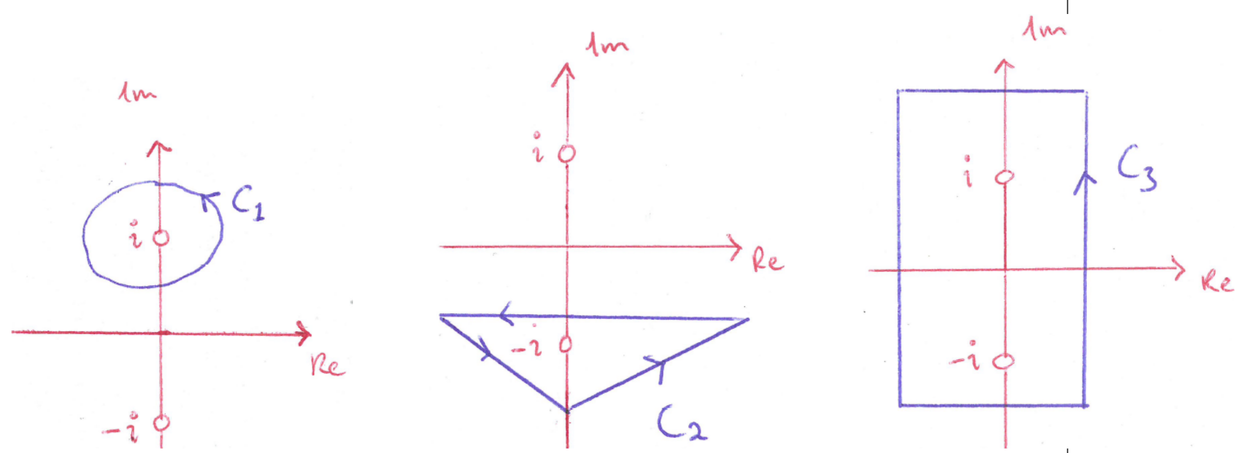
Solution

With $g(z) = 1$ and $h(z) = 1 + z^2$, we have $h(z) = 0$ at $z = \pm i$, and $h'(z) = 2z \neq 0$ at these points. Hence f has poles of order 1 at $z = \pm 1$, with residues

$$\operatorname{Res}(f; i) = \frac{g(i)}{h'(i)} = \frac{1}{2i} = -\frac{i}{2}$$

$$\operatorname{Res}(f; -i) = \frac{g(-i)}{h'(-i)} = \frac{1}{-2i} = \frac{i}{2}.$$

Consider the simple, closed anticlockwise contours $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ shown below.



By the Residue Theorem, we have

$$\int_{\mathcal{C}_1} f = 2\pi i \left[\operatorname{Res}(f; i)\right] = 2\pi i \left[\frac{-i}{2}\right] = \pi$$

$$\int_{\mathcal{C}_2} f = 2\pi i \left[\operatorname{Res}(f; -i)\right] = 2\pi i \left[\frac{i}{2}\right] = -\pi$$

$$\int_{\mathcal{C}_3} f = 2\pi i \left[\operatorname{Res}(f; i) + \operatorname{Res}(f; -i)\right] = 2\pi i \left[-\frac{i}{2} + \frac{i}{2}\right] = 0,$$

since $\mathcal{C}_1$ encloses the singularity $z = i$ and no others, $\mathcal{C}_2$ encloses the singularity $z = -i$ and no others, and $\mathcal{C}_3$ encloses the singularities $z = i$ and $z = -i$.

Example 6.18. Let us do the same for the function

$$f(z) = \frac{-i}{6z^2 + 13z + 6}$$

Solution

Let $g(z) = -i$ and

$$h(z) = 6z^2 + 13z + 6 = (3z + 2)(2z + 3)$$

and note that $h(z) = 0$ for $z = -\frac{2}{3}, -\frac{3}{2}$. Since $h'(z) = 12z + 13$, we have

$$h'(-2/3) = 5 \quad \text{and} \quad h'(-3/2) = -5,$$

both of which are nonzero, so that f has poles of order 1 at $-\frac{2}{3}$ and $-\frac{3}{2}$ by Theorem 6.16. Moreover, by the same result,

$$\begin{aligned} \mathrm{Res}(f; -2/3) &= \frac{g(-2/3)}{h'(-2/3)} = -\frac{i}{5} \\ \mathrm{Res}(f; -3/2) &= \frac{g(-3/2)}{h'(-3/2)} = \frac{i}{5}. \end{aligned}$$

Theorem 6.19 (The Residue at a pole of order n)**.** *Let f have an isolated singularity at z_0 which is a pole of order n, so that for some $r > 0$*

$$f(z) = \frac{g(z)}{(z - z_0)^n} \qquad \textit{for } z \in D'(z_0, r),$$

with g holomorphic on $D(z_0, r)$ and $g(z_0) \neq 0$. Then

$$\mathrm{Res}(f, z_0) = \frac{g^{(n-1)}(z_0)}{(n-1)!}.$$

Proof

By definition,

$$\mathrm{Res}(f; z_0) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{g(z)}{(z - z_0)^n} \, dz,$$

where $\mathcal{C}$ is the anticlockwise circle with centre z_0 and radius $r/2$. By Theorem 5.7 (Cauchy's Integral Formula for Derivatives),

$$g^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i} \int_{\mathcal{C}} \frac{g(z)}{(z - z_0)^n} \, dz,$$

so that

$$g^{(n-1)}(z_0) = (n-1)! \, \mathrm{Res}(f; z_0),$$

or in other words,

$$\mathrm{Res}(f; z_0) = \frac{g^{(n-1)}(z_0)}{(n-1)!}.$$

Example 6.20. Let us consider the function f defined by

$$f(z) = \frac{1}{(z^2+9)^2}.$$

and calculate the residues at the poles of f.

Solution

Following Example 6.12, we define g_1 and g_2 by

$$g_1(z) = \frac{1}{(z+3i)^2} \quad \text{and} \quad g_2(z) = \frac{1}{(z-3i)^2}$$

so that

$$f(z) = \frac{g_1(z)}{(z-3i)^2} = \frac{g_2(z)}{(z+3i)^2},$$

and g_1 is holomorphic and nonzero at $z = 3i$, g_2 is holomorphic and nonzero at $z = -3i$.

Since the poles at $z = \pm 3i$ both have order 2, Theorem 6.19 tells us that

$$\text{Res}(f; 3i) = \frac{g_1'(3i)}{1!} \quad \text{and } \text{Res}(f; -3i) = \frac{g_2'(-3i)}{1!}.$$

The required derivatives are

$$g_1'(z) = -2(z+3i)^{-3} \Rightarrow g_1'(3i) = -2(3i+3i)^{-3} = -\frac{i}{108}$$
$$g_2'(z) = -2(z-3i)^{-3} \Rightarrow g_2'(-3i) = -2(-3i-3i)^{-3} = \frac{i}{108},$$

Hence

$$\text{Res}(f; 3i) = \frac{-i/108}{1!} = -\frac{i}{108} \quad \text{and} \quad \text{Res}(f; -3i) = \frac{i/108}{1!} = \frac{i}{108}.$$

Example 6.21. For the function f defined by

$$f(z) = \frac{\exp(\pi z)}{(z-i)^3}$$

find $\text{Res}(f, i)$. Hence evaluate

$$\int_{\mathcal{C}} \frac{\exp(\pi z)}{(z-i)^3}\, dz,$$

where $\mathcal{C}$ is the anticlockwise triangular contour with vertices 2, $2i$ and -2.

Solution

The only pole of f is a pole of order 3 at $z = i$. With g the holomorphic function defined by $g(z) = \exp(\pi z)$, we have $g'(z) = \pi \exp(\pi z)$ and $g''(z) = \pi^2 \exp(\pi z)$. By

Theorem 6.19,

$$\operatorname{Res}(f;i) = \frac{g''(i)}{2!} = \frac{\pi^2 \exp(\pi i)}{2!} = \frac{\pi^2(-1)}{2} = -\frac{\pi^2}{2}.$$

Since $\mathcal{C}$ is a simple, closed anticlcokwise contour, and i is the only pole of f enclosed by $\mathcal{C}$, the Residue Theorem gives

$$\int_{\mathcal{C}} f = 2\pi i \operatorname{Res}(f;i) = 2\pi i \left(-\frac{\pi^2}{2}\right) = -i\pi^3.$$

6.4 Evaluating Real Integrals using Contour Integration, part 2

Recall how we used contour integration to evaluate

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx.$$

We first considered the complex function f where $f(z) = \dfrac{1}{1+z^2}$, and computed the integral of f along the path $\mathcal{C}_R = L_R + S_R$ consisting of the line segment $[-R, R]$ and the upper semicircle with centre 0 and radius R, from R to $-R$ via iR (where $R > 1$).

We did this using Cauchy's Integral Formula, though we could equally have used the Residue Theorem. We then showed that

$$\lim_{R\to\infty} \int_{S_R} f = 0$$

using The Estimation Lemma. From this we deduced that

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx = \lim_{R\to\infty} \int_{L_R} f = \lim_{R\to\infty} \left(\int_{L_R} f + \int_{S_R} f\right) = \int_{\mathcal{C}_R} f.$$

In fact, the exact same method can be used to evaluate many integrals of the form

$$\int_{-\infty}^{+\infty} f(x)\, dx, \quad \text{for example} \quad \int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\, dx$$

with p and q real polynomials, where q has no real roots and the degree of q is at least the degree of p plus 2.

Example 6.22. Let us evaluate

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+9)^2}\, dx$$

using contour integration.

Solution

We shall follow the approach used in Example 5.9, by considering the complex function f defined via

$$f(z) = \frac{1}{(z^2+9)^2}$$

and the contour $\mathcal{C}_R = L_R + S_R$, where $R > 3$, $L_R = [-R, R]$ and S_R is the upper semicircle with centre 0 and radius R from R to $-R$ via iR.

Using the result of Example 6.20, the only singularity of f enclosed by $\mathcal{C}_R$ is at $z = 3i$, hence by the Residue Theorem

$$\int_{\mathcal{C}_R} f = 2\pi i \operatorname{Res}(f; 3i) = 2\pi i \left(-\frac{i}{108}\right) = \frac{\pi}{54}$$

for all $R > 3$.

If $z \in S_R$, then $|z| = R$ and hence by the reverse triangle inequality

$$\left|z^2+9\right| \geq \left|\left|z^2\right| - 9\right| = \left||z|^2 - 9\right| = R^2 - 9,$$

so that for all such z we have $\left|z^2+9\right|^2 \geq \left(R^2-9\right)^2$. Thus for all $z \in S_R$,

$$\left|\frac{1}{(z^2+9)^2}\right| = \frac{1}{|z^2+9|^2} \leq \frac{1}{(R^2-9)^2}.$$

The Estimation Lemma gives

$$\left|\int_{\mathcal{C}_R} \frac{1}{(z^2+9)^2}\,dz\right| \leq \frac{1}{(R^2-9)^2} \cdot \pi R,$$

hence $\displaystyle\int_{\mathcal{C}_R} f \to 0$ as $R \to \infty$ as before.

Parameterising L_R using $\gamma : [-R, R] \to \mathbb{C}$, $\gamma(t) = t$ we have $\gamma'(t) = 1$ and so

$$\int_{L_R} f = \int_{-R}^{R} \frac{1}{(t^2+9)^2}\,dt.$$

Hence

$$\begin{aligned}
\frac{\pi}{54} &= \int_{\mathcal{C}_R} f \qquad\qquad (\forall\ R > 3)\\
&= \lim_{R\to\infty} \int_{\mathcal{C}_R} f\\
&= \lim_{R\to\infty} \left(\int_{L_R} f\right) + \lim_{R\to\infty} \left(\int_{S_R} f\right)\\
&= \lim_{R\to\infty} \int_{-R}^{+R} \frac{1}{(t^2+9)^2}\,dt\ + 0\\
&= \int_{-\infty}^{+\infty} \frac{1}{(t^2+9)^2}\,dt.
\end{aligned}$$

(Note that it does not matter whether we call the variable of integration x or t).

Example 6.23. Use Example 6.22 to deduce the value of

$$\int_0^\infty \frac{1}{(x^2+9)^2}\,dx.$$

Solution

Since $f(x) = \dfrac{1}{x^2+9}$ is an even function (i.e. $f(-x) = f(x)$) it follows that

$$\int_0^\infty \frac{1}{(x^2+9)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(x^2+9)^2}\,dx = \frac{\pi}{108}.$$

We showed in example 5.13 that for a rational function R of two real variables

$$\int_0^{2\pi} R(\cos(t),\sin(t))\,dt = \int_{\mathcal{C}} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\cdot\frac{1}{iz}\,dz$$

where $\mathcal{C}$ is the anticlockwise circle $\{z\in\mathbb{C} : |z| = 1\}$.

Example 6.24. Evaluate

$$\int_0^{2\pi} \frac{\sin(t)}{5-4\sin(t)}\,dt$$

using contour integration.

Solution

For $z = \exp(it) \in \mathcal{C}$, define f via

$$\begin{aligned}
f(z) = R\left((z+z^{-1})/2, (z-z^{-1})/(2i)\right)\cdot\frac{1}{iz} &= \frac{[z-z^{-1}]/(2i)}{5-4[z-z^{-1}]/(2i)}\cdot\frac{1}{iz}\\
&= \frac{z-z^{-1}}{10i-4z+4z^{-1}}\cdot\frac{1}{iz}\\
&= \frac{z^2-1}{-4z^2+10iz+4}\cdot\frac{1}{iz}\\
&= \frac{i(z^2-1)}{z(4z^2-10iz-4)}\\
&= \frac{1}{2}\cdot\frac{i(z^2-1)}{z(z-2i)(2z-i)}
\end{aligned}$$

so that

$$\int_0^{2\pi} \frac{\sin(t)}{5-4\cos(t)}\,dt = \int_{\mathcal{C}} f\,dz.$$

The function f has simple poles at $0, i/2$ and $2i$, and the first two of these are enclosed by $\mathcal{C}$. Writing

$$g(z) = i(z^2-1) \quad \text{and } h(z) = 4z^3 - 10iz^2 - 4z$$

we have $f = g/h$ and $h'(z) = 12z^2 - 20iz - 4$, hence by the g/h rule

$$\mathrm{Res}(f;0) = \frac{g(0)}{h'(0)} = \frac{-i}{-4} = \frac{i}{4}$$

$$\mathrm{Res}(f;i/2) = \frac{g(i/2)}{h'(i/2)} = \frac{i((\frac{i}{2})^2 - 1)}{12(\frac{i}{2})^2 - 20i(\frac{i}{2}) - 4} = \frac{-5i}{12}.$$

The Residue Theorem gives

$$\int_{\mathcal{C}} \frac{(z^2-1)}{2z(z-2i)(2z-i)}\, dz = 2\pi i \left(\frac{i}{4} - \frac{5i}{12}\right) = 2\pi i \left(\frac{-i}{6}\right) = \frac{\pi}{3}.$$

Hence

$$\int_0^{2\pi} \frac{\sin(t)}{5 - 4\sin(t)}\, dt = \frac{\pi}{3}.$$