MA2003 Complex Analysis
Solutions to Exercise Sheet 2

1. For each function f below, write f in the form

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

and determine whether or not the Cauchy-Riemann equations are satisfied:

$$(a)\ f(z) = \exp(i\,\overline{z}) \quad (b)\ f(z) = z + \frac{1}{z} \quad (c)\ f(z) = z^3.$$

In the cases where f is differentiable, find the derivative of f both using the rules of differentiation and using the Cauchy-Riemann equations.

Solution:

(a) We have

$$f(x+iy) = \exp(i(x-iy)) = \exp(y+ix) = \underbrace{e^y\cos(x)}_{u(x,y)} + i\,\underbrace{e^y\sin(x)}_{v(x,y)}.$$

The corresponding partial derivatives are

$$\frac{\partial u}{\partial x} = -e^y\sin(x) \qquad \frac{\partial v}{\partial y} = e^y\sin(x)$$
$$\frac{\partial u}{\partial y} = e^y\cos(x) \qquad \frac{\partial v}{\partial x} = e^y\cos(x).$$

The Cauchy Riemann Equations are satisfied at a point $x+iy \in \mathbb{C}$ if and only if both

$$e^y\sin(x) = -e^y\sin(x) \quad \text{and} \quad e^y\cos(x) = -e^y\cos(x).$$

Since e^y is never zero, this can only occur if $\sin(x) = \cos(x) = 0$, which is impossible.

(b) For all $x+iy \in \mathbb{C}\setminus\{0\}$,

$$\begin{aligned} f(x+iy) &= (x+iy) + \frac{1}{x+iy} \\ &= (x+iy) + \frac{x-iy}{x^2+y^2} \\ &= \underbrace{\left(x + \frac{x}{x^2+y^2}\right)}_{u(x,y)} + i\underbrace{\left(y - \frac{y}{x^2+y^2}\right)}_{v(x,y)} \end{aligned}$$

and the corresponding partial derivatives are

$$\frac{\partial u}{\partial x} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$
$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = 1 - \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Hence the Cauchy Riemann equations are satisfied at every $x+iy \in \mathbb{C}\setminus\{0\}$. Using the Cauchy

Riemann equations to find the derivative of f, we get

$$
\begin{aligned}
f'(x+iy) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= \left(1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right) + i\left(\frac{2xy}{(x^2+y^2)^2}\right) \\
&= 1 + \frac{y^2 - x^2 + i2xy}{(x^2+y^2)^2} \\
&= 1 - \frac{x^2 - y^2 - i2xy}{(x^2+y^2)^2} \\
&= 1 - \frac{(x-iy)^2}{[(x+iy)(x-iy)]^2} \\
&= 1 - \frac{1}{(x+iy)^2} = 1 - \frac{1}{z^2},
\end{aligned}
$$

which agrees with the derivative of f obtained from the rules of differentiation.

(c) This time

$$u(x,y) = x^3 - 3xy^2 \quad \text{and} \quad v(x,y) = 3x^2y - y^3$$

hence for all $x + iy \in \mathbb{C}$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}.$$

Thus

$$f'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial y}(x,y) = 3x^2 - 3y^2 + i(6xy) = 3\left((x^2 - y^2) + i(2xy)\right) = 3(x+iy)^2$$

which agrees with the derivative obtained from the Chain Rule: $f'(z) = 3z^2$.

2. Show that the Cauchy-Riemann equations are satisfied by the function f defined on the open upper half plane $H_+ = \{z \in \mathbb{C} : \mathsf{Im}(z) > 0\}$ by

$$f(x+iy) = u(x,y) + iv(x,y) = \log\left(\sqrt{x^2+y^2}\right) + i\left(\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right)\right).$$

Assuming that f is indeed holomorphic on H_+, show that

$$f'(x+iy) = \frac{1}{x+iy} \quad \text{i.e., that} \quad f'(z) = \frac{1}{z}.$$

Solution: We have

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \log'\left(\sqrt{x^2+y^2}\right) \cdot \frac{\partial}{\partial x}\left[\sqrt{x^2+y^2}\right] \\
&= \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}(2x) \\
&= \frac{x}{x^2+y^2}
\end{aligned}
$$

and similarly

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}.$$

For the partial derivatives of v, we have

$$\begin{aligned}
\frac{\partial v}{\partial x} &= -\arctan'\left(\frac{x}{y}\right)\cdot\frac{\partial}{\partial x}\left[\frac{x}{y}\right] \\
&= \frac{-1}{(1+(\frac{x}{y})^2)}\cdot\frac{1}{y} \\
&= \frac{-1}{y+\frac{x^2}{y}} = \frac{-y}{x^2+y^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial v}{\partial y} &= -\arctan'\left(\frac{x}{y}\right)\cdot\frac{\partial}{\partial y}\left[\frac{x}{y}\right] \\
&= \frac{-1}{(1+(\frac{x}{y})^2}\cdot\frac{-x}{y^2} \\
&= \frac{x}{x^2+y^2}.
\end{aligned}$$

Thus the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied everywhere in H_+. Assuming moreover that f is holomorphic on H_+, the derivative of f is thus

$$\begin{aligned}
f'(x+iy) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\
&= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} \\
&= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy}
\end{aligned}$$

for all $x+iy \in H_+$ as required.

3. Describe the geometric effect of applying the functions:

   (a) $f(z) = \dfrac{1}{z}$ to a small disc centred at $1-i$, and

   (b) $g(z) = \exp(2iz)$ to a small disc centred at $\frac{\pi}{4}+i$.

Solution: For a function f that is holomorphic at a point z_0, we know that a small disc centred at z_0 is approximately mapped to a small disc centred at $f(z_0)$, and is scaled by a factor of $|f'(z_0)|$ and rotated by an angle of $\arg(f'(z_0))$ about the point $f(z_0)$.

(a) In this example, $f(1-i) = \dfrac{1}{1-i} = \dfrac{1}{2}+\dfrac{i}{2}$, so a small disc centred at $1-i$ is approximately mapped to a small disc centred at $\dfrac{1}{2}+\dfrac{i}{2}$. Since $f'(z) = -\dfrac{1}{z^2}$ for all $z \in \mathbb{C}\setminus\{0\}$, we have

$f'(i) = -\dfrac{1}{(1-i)^2} = \frac{i}{2}$. Thus the disc is scaled by a factor of $\frac{1}{2}$ and rotated by an angle of $\frac{\pi}{2}$ about $\dfrac{1}{2} - \dfrac{i}{2}$.

(b) Mapped to a disc centred at $g(\frac{\pi}{4} + i) = e^{-2}i$, scaled by a factor of $2e^{-2}$ and rotated by an angle of π about this point (clockwise or anticlockwise; it doesn't matter this time).

4. The *set* of points $L = [0, 1-2i]$ is a line segment. It is also a *path* because we have a parametrisation given by $\gamma : [0,1] \to \mathbb{C},\ \gamma(t) = (1-2i)t$. Use this parametrisation to evaluate the integral

$$\int_L (\mathsf{Im}(z) + 3i)\ dz.$$

Solution:
We have

$$\begin{aligned}
\int_\Gamma (\mathsf{Im}(z) + 3i)\ dz &= \int_0^1 (\mathsf{Im}(\gamma(t)) + 3i)\,\gamma'(t)\ dt \\
&= \int_0^1 (-2t + 3i)\,(1-2i)\ dt \\
&= (1-2i)\int_0^1 (-2t + 3i)\ dt \\
&= (1-2i)\left[-t^2 + 3it\right]_0^1 \\
&= (1-2i)(-1+3i) = 5 + 5i.
\end{aligned}$$

5. Find the value of

$$\int_{\Gamma_1} f(z)\ dz \text{ and } \int_{\Gamma_2} f(z)\ dz,$$

where $f(z) = 3\overline{z}$, Γ_1 is the straight line path from 0 to $-i$ and Γ_2 is the straight line path from $1-i$ to $1+i$.

Solution: For the path Γ_1 we use the parametrisation $\gamma_1 : [0,1] \to \mathbb{C},\ \gamma_1(t) = -it$. Then $\gamma_1'(t) = -i$ and $f(\gamma_1(t)) = 3it$. Hence

$$\int_{\Gamma_1} f = \int_0^1 (3it)(-i)dt = \int_0^1 3t\ dt = \frac{3}{2}.$$

Parametrise Γ_2 with $\gamma_2 : [0,1] \to \mathbb{C}$, where

$$\gamma_2(t) = (1-i) + t\,[1 + i - (1-i)] = 1 + i(2t-1).$$

We get

$$\int_{\Gamma_2} f = 6i.$$

6. Fix a point $z_0 \in \mathbb{C}$ and define a complex function f via

$$f(z) = (z - z_0)^n$$

where $n \in \mathbb{Z}$. Find the value of

$$\int_\Gamma f(z)\ dz,$$

where Γ is the circle with centre z_0 and radius $r > 0$, traversed in the anticlockwise direction (use the parametrisation $\gamma : [0, 2\pi] \to \mathbb{C},\ \gamma(t) = z_0 + r(\cos(t) + i\sin(t))$). Do this separately for the cases $n = -1$ and $n \neq -1$.

(Hint: for the case $n \neq -1$, you need to show that

$$\frac{d}{dt}\left[(\cos(t) + i\sin(t))^{n+1}\right] = i(n+1)(\cos(t) + i\sin(t))^{n+1}$$

and then use the (real) Fundamental Theorem of Calculus).

Solution: Following the hint, we first note that

$$\begin{aligned}\frac{d}{dt}\left[(\cos(t) + i\sin(t))^{n+1}\right] &= (n+1)(\cos(t) + i\sin(t))^n(-\sin(t) + i\cos(t)) \\ &= (n+1)(\cos(t) + i\sin(t))^n\, i(\cos(t) + i\sin(t)) \\ &= i(n+1)(\cos(t) + i\sin(t))^{n+1}.\end{aligned}$$

We have

$$f(\gamma(t)) = (\gamma(t) - z_0)^n = r^n(\cos(t) + i\sin(t))^n, \quad \gamma'(t) = ir(\cos(t) + i\sin(t)).$$

Hence for $n \neq -1$,

$$\begin{aligned}\int_\Gamma (z - z_0)^n &= \int_0^{2\pi} ir^{n+1}(\cos(t) + i\sin(t))^{n+1}\ dt \\ &= \frac{r^{n+1}}{n+1}\int_0^{2\pi}\frac{d}{dt}\left[(\cos(t) + i\sin(t))^{n+1}\right]\ dt \\ &= \frac{r^{n+1}}{n+1}\left[(\cos(t) + i\sin(t))^{n+1}\right]_0^{2\pi} && \text{by FTC} \\ &= \frac{r^{n+1}}{n+1}[1 + 0i - (1 + 0i)] = 0 && .\end{aligned}$$

If $n = -1$ then

$$\int_\Gamma (z - \alpha)^{-1} = \int_0^{2\pi}\frac{1}{r(\cos(t) + i\sin(t))}\cdot ir(\cos(t) + i\sin(t))\ dt = \int_0^{2\pi} i\ dt = i2\pi.$$

7. Let $f, g : U \to \mathbb{C}$ be continuous, and let Γ be a smooth path contained in U parametrised by $\gamma : [a, b] \to \mathbb{C}$. Prove that

   (a) for every constant $\alpha \in \mathbb{C}$ we have $\displaystyle\int_\Gamma (f + \alpha g) = \int_\Gamma f + \alpha\int_\Gamma g$, and

(b) if $\tilde{\Gamma}$ denotes the reverse of Γ, we have $\int_{\tilde{\Gamma}} f = -\int_{\Gamma} f$. As a hint, parametrise $\tilde{\Gamma}$ using $\tilde{\gamma} : [a, b] \to \mathbb{C}$, $\tilde{\gamma}(t) = (a + b - t)$, and use the substitution $s = a + b - t$.

Solution:

(a)

$$\begin{aligned}
\int_{\Gamma} (f + \alpha g) &= \int_a^b (f + \alpha g)(\gamma(t))\gamma'(t)\ dt \\
&= \int_a^b \left(f(\gamma(t)) + \alpha g(\gamma(t))\right)\gamma'(t)\ dt \\
&= \int_a^b \big(f(\gamma(t))\gamma'(t) + \alpha g(\gamma(t))\gamma'(t)\big)\ dt
\end{aligned}$$

Then using linearity of the real integral this becomes

$$\int_a^b f(\gamma(t))\gamma'(t)\ dt + \alpha \int_a^b g(\gamma(t))\gamma'(t)\ dt = \int_{\Gamma} f + \alpha \int_{\Gamma} g.$$

(b) Following the hint,

$$\begin{aligned}
\int_{\tilde{\Gamma}} f &= \int_a^b f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\ dt \\
&= \int_a^b f(\gamma(a + b - t))\big(-\gamma'(a + b - t)\big)\ dt
\end{aligned}$$

and using the substitution $s = a + b - t$, we have $ds = -dt$ and the limits are reversed, so the above becomes

$$\begin{aligned}
\int_b^a f(\gamma(s))\big(-\gamma'(s)\big)(-ds) &= \int_b^a f(\gamma(s))\gamma'(s)\ ds \\
&= -\int_a^b f(\gamma(s))\gamma'(s)\ ds \\
&= -\int_{\Gamma} f.
\end{aligned}$$