MA2003 Complex Analysis
Solutions to Exercise Sheet 4

1. Express each of the following complex numbers in Cartesian form $a + ib$:

$$(a)\ \mathrm{Log}(i), \quad (b)\ \mathrm{Log}(ie) \quad (c)\ \mathrm{Log}(-1 - i\sqrt{3}).$$

Solution: Using $\mathrm{Log}(z) = \log|z| + i\mathrm{Arg}(z)$,

$$\mathrm{Log}(i) = \log|i| + i\mathrm{Arg}(i) = \log(1) + i\frac{\pi}{2} = i\frac{\pi}{2}.$$
$$\mathrm{Log}(ie) = \log(e) + i\frac{\pi}{2} = 1 + i\frac{\pi}{2}$$
$$\mathrm{Log}(-1 - i\sqrt{3}) = \log(\sqrt{(-1)^2 + (-\sqrt{3})^2}) + i\frac{-2\pi}{3} = \log(2) - i\frac{2\pi}{3}$$

where $\log : [0, +\infty) \to \mathbb{R}$ denotes the real natural logarithm.

2. Express each of the following complex numbers in Cartesian form $a + ib$:

$$(a)\ (1+i)^i, \quad (b)\ (ie)^{i\pi}, \quad (c)\ (-1 - i\sqrt{3})^{1+i}.$$

Solution:

$$\begin{aligned}
(1+i)^i &= \exp\left(i\,\mathrm{Log}(1+i)\right) = \exp\left(-\frac{\pi}{4} + i\frac{1}{2}\log(2)\right) \\
&= e^{-\frac{\pi}{4}}\cos(\frac{1}{2}\log(2)) + ie^{-\frac{\pi}{4}}\sin(\frac{1}{2}\log(2)). \\
(ie)^{i\pi} &= \exp\left(i\pi\,\mathrm{Log}(ie)\right) \\
&= \exp\left(i\pi - \frac{\pi^2}{2}\right) \\
&= e^{-\frac{\pi^2}{2}}\left(\cos(\pi) + i\sin(\pi)\right) \\
&= -e^{-\frac{\pi^2}{2}} \\
(-1 - i\sqrt{3})^{1+i} &= \exp\left((1+i)\,\mathrm{Log}(-1 - i\sqrt{3})\right) \\
&= \exp\left((1+i)(\log(2) - i\frac{2\pi}{3})\right) \\
&= \exp\left[(\log(2) + \frac{2\pi}{3}) + i(\log(2) - \frac{2\pi}{3})\right] \\
&= e^{\log(2)+2\pi/3}\left(\cos\left(\log(2) - 2\pi/3\right) + i\sin\left(\log(2) - 2\pi/3\right)\right) \\
&= 2e^{2\pi/3}\left(\cos\left(\log(2) - 2\pi/3\right) + i\sin\left(\log(2) - 2\pi/3\right)\right)
\end{aligned}$$

3. (a) Use the definition $z^\alpha = \exp(\alpha \operatorname{Log}(z))$ to show that $z^3 = zzz$.

(b) Show that $\operatorname{Log}(i^3) \neq 3 \operatorname{Log}(i)$

(c) Define $\sqrt{z} = z^{1/2} (= \exp(\frac{1}{2} \operatorname{Log}(z)))$ for $z \in \mathbb{C} \setminus \{0\}$. Where is the mistake in

$$-1 = i^2 = ii = \sqrt{-1}\sqrt{-1} = \sqrt{-1 \times -1} = \sqrt{1} = 1?$$

(d) Show that for all $\alpha, \beta \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{0\}$ we have $z^\alpha z^\beta = z^{\alpha+\beta}$. Is it true that $\operatorname{Log}(\alpha\beta) = \operatorname{Log}(\alpha) + \operatorname{Log}(\beta)$?

Solution:

(a) Using the definition of the Principal 3^{rd} power function

$$\begin{aligned} z^3 &= \exp(3 \operatorname{Log}(z)) \\ &= \exp(\operatorname{Log}(z) + \operatorname{Log}(z) + \operatorname{Log}(z)) \\ &= \exp(\operatorname{Log}(z)) \exp(\operatorname{Log}(z)) \exp(\operatorname{Log}(z)) \\ &= zzz. \end{aligned}$$

(b) We have

$$\operatorname{Log}(i^3) = \operatorname{Log}(-i) = -i\frac{\pi}{2}$$

while

$$3 \operatorname{Log}(i) = 3\left(i\frac{\pi}{2}\right) = i\frac{3\pi}{2}.$$

(c) We do not have

$$\sqrt{z}\sqrt{w} = \sqrt{zw}$$

in general. Indeed

$$\begin{aligned} \sqrt{-1}\sqrt{-1} &= \exp\left(\frac{1}{2}\operatorname{Log}(-1)\right)\exp\left(\frac{1}{2}\operatorname{Log}(-1)\right) \\ &= \exp(\operatorname{Log}(-1)) = -1 \end{aligned}$$

while of course

$$\sqrt{-1 \times -1} = \sqrt{1} = \exp\left(\frac{1}{2}\operatorname{Log}(1)\right) = e^0 = 1.$$

(d)

$$z^\alpha z^\beta = \exp(\alpha \operatorname{Log}(z)) \exp(\beta \operatorname{Log}(z)) = \exp((\alpha+\beta)\operatorname{Log}(z)) = z^{\alpha+\beta}.$$

We do not have $\operatorname{Log}(\alpha\beta) = \operatorname{Log}(\alpha) + \operatorname{Log}(\beta)$ in general; for example if $\alpha = -1$ and $\beta = i$, then

$$\operatorname{Log}(\alpha\beta) = \operatorname{Log}(-i) = -i\frac{\pi}{2}, \text{ while } \operatorname{Log}(\alpha) + \operatorname{Log}(\beta) = i\pi + i\frac{\pi}{2} = i\frac{3\pi}{2}.$$

4. Recall that the Principal Logarithm function Log is holomorphic on the region $\mathbb{C}_\pi$, where $\mathbb{C}_\pi = \{z \in \mathbb{C} : z \neq 0 \text{ and } \operatorname{Arg}(z) \neq \pi\}$. Let F be the function defined by

$$F(z) = \frac{1}{2i}\left(\operatorname{Log}(z+i) - \operatorname{Log}(z-i)\right).$$

(a) Describe (or sketch) the region $\mathcal{R}$ on which the function F is holomorphic.

(b) Show that F is an antiderivative for the function $f : \mathcal{R} \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{z^2+1} \quad \text{for all} \quad z \in \mathcal{R}.$$

Solution: In general, for $z_0 = z_0 + iy_0 \in \mathbb{C}$, the function $z \mapsto \mathrm{Log}(z - z_0)$ is holomorphic on the region $\{z \in \mathbb{C} : z - z_0 \in \mathbb{C}_\pi\}$. For $z = x + iy \in \mathbb{C}$, $z - z_0$ is *not* in $\mathbb{C}_\pi$ if and only if

$$z - z_0 = (x - x_0) + i(y - y_0)$$

lies on the negative real axis. This occurs when

- $y - y_0 = 0$, i.e., when $y = y_0$, and
- $x - x_0 \leq 0$, i.e. $x \leq x_0$.

Hence $z \mapsto \mathrm{Log}(z - z_0)$ is holomorphic on

$$\mathbb{C} \setminus \{z = x + iy \in \mathbb{C} : x \leq x_0 \text{ and } y = y_0\},$$

so in particular, $z \mapsto \mathrm{Log}(z + i)$ and $z \mapsto \mathrm{Log}(z - i)$ are holomorphic on

$$\mathbb{C} \setminus \{x + iy \in \mathbb{C} : x \leq 0 \text{ and } y = -1\} \quad \text{and} \quad \mathbb{C} \setminus \{x : iy \in \mathbb{C} : x \leq 0 \text{ and } y = 1\}$$

respectively.
We know that our function F is holomorphic on the region where both $z \mapsto \mathrm{Log}(z + i)$ and $z \mapsto \mathrm{Log}(z - i)$ are holomorphic; i.e. the intersection of these two sets. This is the set

$$\mathbb{C} \setminus \{x + iy \in \mathbb{C} : x \leq 0 \text{ and } y = \pm 1\}.$$

5. Let U be a starlit region with star centre $z_* \in U$ and let $g : U \to \mathbb{C}$ be a holomorphic function.

(a) Prove that if $g(z) \neq 0$ for all $z \in U$, then the function $\frac{g'}{g}$ has an antiderivative on U, stating any results used (you may assume that g holomorphic on U implies g' holomorphic on U).

(b) Prove that if in addition $g(z) \in \mathbb{C}_\pi$ for all $z \in U$ then

$$\int_{[z_*, z]} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta = \mathrm{Log}(g(z)) + \alpha$$

for some constant α.

Solution: Since g is holomorphic and nonzero on U, $\frac{g'}{g}$ is also holomophic on U.
By The Existence of Antiderivatives on Starlit Regions, the function $G : U \to \mathbb{C}$ defined by

$$G(z) := \int_{[z_*, z]} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta$$

is an antiderivative for $\dfrac{g'}{g}$ on U.

Since Log is holomorphic on $\mathbb{C}_\pi$ and $g(z) \in \mathbb{C}_\pi$ for all $z \in U$, $z \mapsto \operatorname{Log}(g(z))$ is holomorphic on U, with derivative

$$\frac{d}{dz}\left[\operatorname{Log}(g(z))\right] = \frac{g'(z)}{g(z)}.$$

Together with the first part this shows that

$$\frac{d}{dz}\left[\operatorname{Log}(g(z)) - G(z)\right] = 0$$

on U. Since a starlit region is connected, the Fundamental Theorem of Calculus implies that

$$z \mapsto \operatorname{Log}(g(z)) - G(z)$$

is constant.

6. Evaluate the integral

$$\int_{\mathcal{C}} \frac{\exp(2z)}{4z + i\pi}\, dz$$

where $\mathcal{C}$ is (i) the anticlockwise contour whose points lie on the circle $\{z : |z| = 1\}$, and (ii) when $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z - 2i| = 2\}$. The use of any Theorems made to obtain the value of these integrals should be justified.

Answer: $(i) : \frac{\pi}{2}$, $(ii) : 0$.

7. Evaluate the integral

$$\int_{\mathcal{C}} \frac{\cos(z^2)}{3i + 2z}\, dz,$$

where (i) $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z| = 1\}$, and (ii) $\mathcal{C}$ is the anticlockwise contour whose points lie on the circle $\{z : |z| = 5\}$. The use of any Theorems made to obtain the value of these integrals should be justified.

Solution: (i) The given function is holomorphic on $\mathbb{C} \setminus \{z : 3i + 2z = 0\}$, that is to say, on $\mathbb{C} \setminus \left\{-i\frac{3}{2}\right\}$. In particular, it is holomorphic on the simply connected region

$$\left\{z \in \mathbb{C} : \mathsf{Im}(z) > -\tfrac{5}{4}\right\}$$

which contains the (closed) contour $\mathcal{C}$. Thus by Cauchy's Theorem for Starlit regions,

$$\int_{\mathcal{C}} \frac{\cos(z^2)}{3i + 2z}\, dz = 0.$$

(ii) We have

$$\frac{\cos(z^2)}{3i + 2z} = \frac{g(z)}{z - z_0}$$

where

$$z_0 = -i\tfrac{3}{2} \quad \text{and} \quad g(z) = \tfrac{1}{2}\cos(z^2).$$

The function g is holomorphic on $\mathbb{C}$ (which is simply connected), and $\mathcal{C}$ is a closed, simple anticlockwise contour that encloses z_0, so that by Cauchy's Integral Formula

$$\int_{\mathcal{C}} \frac{\cos(z^2)}{3i+2z}\,dz = \int_{\mathcal{C}} \frac{g(z)}{z-(-i\frac{3}{2})}\,dz = 2\pi i g(-i\tfrac{3}{2}) = 2\pi i \tfrac{1}{2}\cos(-\tfrac{9}{4}) = i\pi\cos(\tfrac{9}{4}).$$

8. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+6x+25}\,dx$$

in the following way (compare Example 5.8 in the notes):

(a) Define the complex function f by $f(z) = \dfrac{1}{z^2+6z+25}$ and find z_0 and z_1 so that $f(z) = \dfrac{1}{(z-z_0)(z-z_1)}$ (where z_0 lies in the upper half-plane and z_1 in the lower half-plane).

Solution: The roots of $z^2+6z+25$ can be found using the quadratic formula;

$$\begin{aligned} z &= \frac{-6 \pm \sqrt{6^2 - 4(25)(1)}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} \\ &= \frac{-6+i8}{2} = -3 \pm 4i. \end{aligned}$$

Hence

$$f(z) = \frac{1}{(z-(-3+4i))(z-(-3-4i))}.$$

(b) Choose a suitable function g, holomorphic on the simply connected region $\mathcal{R} = \left\{z \in \mathbb{C} : \mathsf{Im}(z) > \frac{1}{2}\mathsf{Im}(z_1)\right\}$, so that

$$f(z) = \frac{g(z)}{(z-z_0)}.$$

Solution: With $z_0 = -3+4i$ and

$$g(z) = \frac{1}{z-(-3-4i)}$$

then

$$f(z) = \frac{g(z)}{z-(-3+4i)}.$$

Moreover, g is holomorphic on $\mathbb{C}\setminus\{-3-4i\}$, and in particular, on the simply connected region $\mathcal{R} := \{z \in \mathbb{C} : \mathsf{Im}(z) > -2\}$.

(c) Justify the use of Cauchy's Integral Formula to find

$$\int_{\mathcal{C}_R} f = \int_{\mathcal{C}_R} \frac{g(z)}{(z-z_0)}\,dz,$$

where $\mathcal{C}_R = L_R + S_R$ with L_R the straight line path from $-R$ to R and S_R a suitable semicircular contour from R to $-R$, with R sufficiently large to apply the Theorem.

Solution: Once $R > 5$, the contour simple closed anticlockwise contour $\mathcal{C}_R$ encloses z_0. Moreover $\mathcal{C}_R$ is always contained in the simply connected region $\mathcal{R}$ of the previous part, and g is holomorphic on this region. Therefore, we may apply Cauchy's Integral formula:

$$\begin{aligned}\int_{\mathcal{C}_R} f = \int_{\mathcal{C}_R} \frac{g(z)}{z-(-3+4i)}\, dz &= 2\pi i g(-3+4i) \\ &= 2\pi i \cdot \frac{1}{(-3+4i)-(-3-4i)} \\ &= \frac{2\pi i}{8i} = \frac{\pi}{4},\end{aligned}$$

and this is valid for all $R > 5$.

(d) Show that for large R and $z \in S_R$, we have $\left|z^2+6z+25\right| \geq R^2 - 6R - 25$.

Solution: If $z \in S_R$ then $|z| = R$, so that the reverse triangle inequality gives

$$\begin{aligned}\left|z^2+6z+25\right| &\geq \left|\left|z^2\right| - |6z+25|\right| \\ &= \left|\left|z\right|^2 - |6z+25|\right| \\ &= \left|R^2 - |6z+25|\right|.\end{aligned}$$

By the triangle inequality,

$$|6z+25| \leq 6R + 25 \quad \text{for all } z \in S_R.$$

Moreover, if $R > 10$, we have $25 < 2.5R$ so that

$$6R + 25 < 6R + 2.5R < 10R < R^2.$$

Thus for $R > 10$ and $z \in S_R$,

$$\left|z^2+6z+25\right| \geq R^2 - 6R - 25.$$

(e) Use the Estimation Lemma to show that

$$\left|\int_{S_R} f\right| \to 0 \text{ as } R \to \infty.$$

Solution: By the previous part, for $R > 10$ and $z \in S_R$ we have

$$\left|\frac{1}{z^2+6z+25}\right| \leq \frac{1}{R^2-6R-25}.$$

Since the length of S_R is πR, for all $R > 10$ we have

$$\left| \int_{S_R} f \right| \leq \underbrace{\frac{1}{R^2 - 6R - 25}}_{M} \cdot \underbrace{\pi R}_{L} = \frac{\pi}{R - 6 - \frac{25}{R}}$$

by the Estimation Lemma. Hence

$$\left| \int_{S_R} \right| f \to 0 \quad \text{as} \quad R \to \infty.$$

(f) Deduce the value of

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 6x + 25} \, dx.$$

Solution: We have

$$\begin{aligned}
\frac{\pi}{4} &= \int_{\mathcal{C}_R} f && \text{for } R > 5 \\
&= \lim_{R\to\infty} \left(\int_{\mathcal{C}_R} f \right) \\
&= \lim_{R\to\infty} \left(\int_{L_R} f \right) + \lim_{R\to\infty} \left(\int_{S_R} f \right) \\
&= \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{t^2 + 6t + 25} \, dt \right) + 0 && \text{by part (d)} \\
&= \int_{-\infty}^{\infty} \frac{1}{x^2 + 6x + 25} \, dx.
\end{aligned}$$

9. (Liouville's Theorem) Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic everywhere, and suppose that f is bounded, i.e. there exists $M > 0$ with $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Show that f is constant on $\mathbb{C}$, in the following way:

(a) Let $z_1, z_2 \in \mathbb{C}$, and let $R > 0$ be sufficiently large so that z_1 and z_2 are enclosed by the countour $\mathcal{C}_R$ consisting of the anticlockwise circle with centre 0 and radius R. Use Cauchy's Integral Formula to write $f(z_1) - f(z_2)$ as a single integral along $\mathcal{C}_R$.

Solution: As f is holomorphic on $\mathbb{C}$ and $\mathcal{C}_R$ is a simple, closed anticlockwise contour containing both z_1 and z_2, we can apply Cauchy's Integral formula (twice) to get

$$\int_{\mathcal{C}_R} \frac{f(z)}{z - z_1} \, dz = 2\pi i f(z_1) \quad \text{and} \quad \int_{\mathcal{C}_R} \frac{f(z)}{z - z_2} \, dz = 2\pi i f(z_2).$$

Hence (since we may combine integrals along the same path)

$$\begin{aligned} f(z_1) - f(z_2) &= \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{f(z)}{z-z_1}\,dz - \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{f(z)}{z-z_2}\,dz \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_R} \left(\frac{f(z)}{z-z_1} - \frac{f(z)}{z-z_2}\right) dz \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{f(z)(z-z_2) - f(z)(z-z_1)}{(z-z_1)(z-z_2)}\,dz \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{f(z)(z_1-z_2)}{(z-z_1)(z-z_2)}\,dz. \end{aligned}$$

(b) Use the Estimation Lemma (and the backwards triangle inequality) to show that

$$|f(z_1) - f(z_2)| \le M\frac{|z_1 - z_2|}{(R-|z_1|)(R-|z_2|)} \cdot 2\pi R$$

for all (sufficiently large) R.

Solution: If $R > \max(|z_1|, |z_2|)$ and $z \in \mathcal{C}_R$, then by the backwards triangle inequality

$$|z - z_1| \ge ||z| - |z_1|| = R - |z_1|$$

and similarly $|z - z_2| \ge R - |z_2|$. Since $|f(z)| \le M$ we have

$$\begin{aligned} \left|\frac{f(z)(z_1-z_2)}{(z-z_1)(z-z_2)}\right| &= \frac{|f(z)| \cdot |z_1 - z_2|}{|z-z_1| \cdot |z-z_2|} \\ &\le \frac{M\,|z_1-z_2|}{(R-|z_1|)(R-|z_2|)}. \end{aligned}$$

The path $\mathcal{C}_R$ has length $2\pi R$, thus by the Estimation Lemma

$$\left|\int_{\mathcal{C}_R} \frac{f(z)(z_1-z_2)}{(z-z_1)(z-z_2)}\,dz\right| \le M\frac{|z_1-z_2|}{(R-|z_1|)(R-|z_2|)} \cdot 2\pi R$$

(c) Deduce that $f(z_1) = f(z_2)$.

Solution: By parts (b) and (c) we have

$$\begin{aligned} |f(z_1) - f(z_2)| &= \left|\frac{1}{2\pi i}\int_{\mathcal{C}_R} \frac{f(z)(z_1-z_2)}{(z-z_1)(z-z_2)}\,dz\right| \\ &\le \left|\frac{1}{2\pi i}\right| M\frac{|z_1-z_2|}{(R-|z_1|)(R-|z_2|)} \cdot 2\pi R \\ &= \frac{MR\,|z_1-z_2|}{(R-|z_1|)(R-|z_2|)} \\ &= \frac{M\,|z_1-z_2|}{(1-|z_1|/R)(R-|z_2|)} \end{aligned}$$

for all $R > \max(|z_1|, |z_2|)$. Since M and $|z_1 - z_2|$ are constants, it follows that

$$\frac{M|z_1 - z_2|}{(1 - |z_1|/R)(R - |z_2|)} \to 0 \quad \text{as} \quad R \to \infty.$$

Hence $|f(z_1) - f(z_2)| = 0$, or in other words $f(z_1) = f(z_2)$.