MA 2003 Complex Analysis
Exercise Sheet 5

1. Locate the poles of each of the following functions, and calculate the residues at these poles:
   (a) $f(z) = \dfrac{1}{z(i-z)^3}$
   (b) $f(z) = \dfrac{z^2}{(z^2+1)^2}$
   (c) $f(z) = \dfrac{\operatorname{Log}(z)}{(4z-i)^2}$
   (d) $f(z) = \dfrac{1}{\exp(z)-1}$.

Solution:

(a) This time f has a pole of order 1 at 0 and a pole of order 3 at i. With $g_1(z) = \frac{1}{(i-z)^3}$ and $h_1(z) = z$, the g/h rule gives

$$\operatorname{Res}(f;0) = \frac{g(0)}{h'(0)} = \frac{1}{(1)\cdot(i)^3} = i.$$

For the pole of order 3 at i we first rewrite

$$f(z) = \frac{1}{z(-(z-i))^3} = -\frac{1}{z(z-i)^3}.$$

Now, letting $g_2(z) = -z^{-1}$ we can use the formula

$$\operatorname{Res}(f;i) = \frac{g_2''(i)}{2!}.$$

We have $g_2'(z) = z^{-2}$ and $g_2''(z) = -2z^{-3}$, so that $g_2''(i) = -2/(i)^3 = -2i$ and hence

$$\operatorname{Res}(f;i) = \frac{-2i}{2} = -i.$$

(b) Writing

$$f(z) = \frac{z^2}{(z+i)^2(z-i)^2}$$

we see that f has poles of order 2 at $z = \pm i$. If $g_1(z) = \dfrac{z^2}{(z+i)^2}$ then

$$g_1'(z) = \frac{(z+i)^2(2z) - 2(z+i)z^2}{(z+i)^4}, \quad g_1'(i) = \frac{(2i)^3 - 2(2i)i^2}{(2i)^4} = -\frac{i}{4}$$

hence

$$\operatorname{Res}(f;i) = \frac{g_1'(i)}{1!} = -\frac{i}{4}.$$

Similarly, with $g_2(z) = \frac{z^2}{(z-i)^2}$,

$$g_2'(z) = \frac{(z-i)^2(2z) - 2(z-i)z^2}{(z-i)^4}, \quad g_2'(-i) = \frac{(-2i)^2(2i) - 2(-2i)(-2i)^2}{(-2i)^4} = \frac{i}{4}.$$

Hence

$$\operatorname{Res}(f;-i) = \frac{g_2'(i)}{1!} = \frac{i/4}{1} = \frac{i}{4}.$$

(c) Pole of order 2 at $z = i/4$. Write $f(z) = \dfrac{\operatorname{Log}(z)}{16(z-\frac{i}{4})^2}$, then

$$\operatorname{Res}(f;i/4) = \frac{1}{16}\cdot\frac{1}{(i/4)} = -\frac{i}{4}.$$

(d) Use the g/h rule with $g(z) = 1$ and $h(z) = \exp(z) - 1$. Then $h(z) = 0$ whenever $\exp(z) = 1$, i.e. at the points $z_k = 2\pi i k$ where $k \in \mathbb{Z}$, so f has isolated singularities at these points. Since $h'(z) = \exp(z)$, $h'(z_k) = 1$ for all k, and so each z_k is a pole of order 1 for the function f. By the g/h rule,

$$\operatorname{Res}(f;z_k) = \frac{g(z_k)}{h'(z_k)} = 1.$$

2. Evaluate

$$\int_{\mathcal{C}} \frac{1}{z(z-1)(z+2)}\,dz,$$

where $\mathcal{C}$ is the anticlockwise circle with centre 0 and radius $3/2$.

Solution: The function f defined by

$$f(z) = \frac{1}{z(z-1)(z+2)}$$

has isolated singularities at $0, 1$ and 2, and the first two of these are enclosed by $\mathcal{C}$. The residues are

$$\operatorname{Res}(f;0) = \frac{1}{(0-1)(0+2)} = -\frac{1}{2}$$
$$\operatorname{Res}(f;1) = \frac{1}{1(1+2)} = \frac{1}{3}.$$

Hence by the Residue Theorem

$$\int_{\mathcal{C}} \frac{1}{z(z-1)(z+2)}\,dz = 2\pi i\left(-\frac{1}{2}+\frac{1}{3}\right) = -\frac{i\pi}{3}.$$

3. Evaluate

$$\int_{\mathcal{C}} \frac{1}{(z^2+1)^3}\,dz$$

where $\mathcal{C}$ is the anticlockwise square with vertices $1,\ 1+2i, -1+2i$ and -1.

Solution: Using the factorisation $(z^2+1)^3 = (z+i)^3(z-i)^3$, we see that this function, call it f, has poles of order 2 at $z = \pm i$. Of these, only the pole at $z = i$ is enclosed by $\mathcal{C}$, so that

$$\int_{\mathcal{C}} \frac{1}{(z^2+1)^3}\,dz = 2\pi i \operatorname{Res}(f;i).$$

If we let $g(z) = \dfrac{1}{(z+i)^3}$ then

$$f(z) = \frac{g(z)}{(z-i)^3},$$

$g'(z) = -3(z+i)^{-4}$ and $g''(z) = 12(z+i)^{-5}$. Hence

$$\mathrm{Res}(f; i) = \frac{g''(i)}{2!} = \frac{12}{2(2i)^5} = \frac{3}{16i}$$

which gives

$$\int_{\mathcal{C}} f = 2\pi i \left(\frac{3}{16i}\right) = \frac{3\pi}{8}.$$

4. Use contour integration to evaluate each of the following real integrals:

(a)
$$\int_0^{2\pi} \frac{1}{5 + 4\sin\theta}\, d\theta$$

(b)
$$\int_0^{\infty} \frac{1}{x^4 + 1}\, dx$$

Solution:

(a) We use the fact that for a rational function R of two (real) variables,

$$\int_0^{2\pi} R(\cos(t), \sin(t))dt = \int_{\mathcal{C}} f$$

where $\mathcal{C}$ is the anticlockwise unit circle and f is the complex function defined by

$$f(z) = R\left([z + z^{-1}]/2, [z - z^{-1}]/(2i)\right) \cdot \frac{1}{iz}.$$

In this case, we have

$$\begin{aligned} f(z) &= \frac{1}{5 + 4[z - z^{-1}]/2i} \cdot \frac{1}{iz} \\ &= \frac{1}{2z^2 + 5iz - 2} \\ &= \frac{1}{(2z+i)(z+2i)}. \end{aligned}$$

Then with $\mathcal{C}$ the anticlockwise unit circle, the only singularity of f enclosed by $\mathcal{C}$ is at $z = -i/2$, and so

$$\begin{aligned} \int_0^{2\pi} \frac{1}{5 + 4\sin\theta}\, d\theta &= \int_{\mathcal{C}} \frac{1}{(2z+i)(z+2i)}\, dz \\ &= 2\pi i \mathrm{Res}(f; -i/2) \\ &= 2\pi i \cdot \frac{1}{2(-i/2 + 2i)} = \frac{2\pi}{3}. \end{aligned}$$

(b) We shall consider the complex function

$$f(z) = \frac{1}{z^4+1}$$

which is holomorphic everywhere in $\mathbb{C}$ except where $z^4 = -1$; that is to say, f is holomorphic on $\mathbb{C}\setminus\left\{e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4}\right\}$. Let $\mathcal{C}_R = L_R + S_R$ where $R > 1$, L_R is the straight line path $[-R, R]$ and S_R is the upper-semicircle with centre 0 and radius R from R to $-R$ via iR; then the poles of f enclosed by $\mathcal{C}_R$ are at $e^{i\pi/4}$ and $e^{i3\pi/4}$.

With $g(z) = 1$ and $h(z) = z^4 + 1$, we have $h'(z) = 4z^3$ so that by the g/h rule,

$$\operatorname{Res}(f; e^{i\pi/4}) = \frac{1}{4(e^{i\pi/4})^3} = \frac{1}{4e^{3\pi/4}} = \frac{1}{4(-a+ia)}$$
$$\operatorname{Res}(f; e^{i3\pi/4}) = \frac{1}{4(e^{i3\pi/4})^3} = \frac{1}{4e^{9\pi/4}} = \frac{1}{4(a+ia)}$$

where $a = 1/\sqrt{2}$. So (for $R > 1$) we have

$$\begin{aligned}\int_{\mathcal{C}_R} f &= 2\pi i\,(\text{ sum of residues at poles of } f \text{ inside } \mathcal{C}_R)\\ &= 2\pi i\left(\frac{1}{4(-a+ia)} + \frac{1}{4(a+ia)})\right) \qquad = \frac{\pi}{\sqrt{2}}\end{aligned}$$

by the Residue Theorem.

We now show that

$$\lim_{R\to\infty}\int_{S_R} f = 0$$

using the Estimation Lemma. Indeed, for $z \in S_R$, $|z| = R$ and so by the reverse triangle inequality

$$\left|z^4+1\right| \geq \left|\left|z^4\right| - |1|\right| = \left||z|^4 - 1\right| = R^4 - 1.$$

Hence for all such z,

$$|f(z)| = \left|\frac{1}{z^4+1}\right| = \frac{1}{|z^4+1|} \leq \frac{1}{R^4-1}.$$

The Estimation Lemma then gives

$$\left|\int_{S_R} f\right| \leq \frac{1}{R^4-1}\cdot \pi R = \frac{\pi R}{R^4-1}$$

so that

$$\int_{S_R} f \to 0 \quad \text{as} \quad R \to \infty.$$

Using the parametrisation $\gamma_L : [-R, R] \to \mathbb{C}$, $\gamma(t) = t$ of L_R, we have

$$\int_{L_R} f = \int_{-R}^{R} \frac{1}{(t^4+1}\, dt.$$

Hence

$$
\begin{aligned}
\frac{\pi}{\sqrt{2}} &= \int_{\mathcal{C}_R} f \\
&= \lim_{R\to\infty} \int_{\mathcal{C}_R} f \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{t^4+1}\, dt + \lim_{R\to\infty} \int_{S_R} f \\
&= \int_{-\infty}^{\infty} \frac{1}{t^4+1}\, dt.
\end{aligned}
$$

Since the integrand is even, it follows that

$$\int_0^{\infty} \frac{1}{t^4+1}\, dt = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{t^4+1}\, dt = \frac{\pi}{2\sqrt{2}}.$$

5. Use contour integration to evaluate the following real integrals:

   (a)

$$\int_0^{2\pi} \frac{1}{16\cos^2(t) + 25\sin^2(t)}\, dt.$$

Solution: As before, we use the fact that for a rational function R of two (real) variables,

$$\int_0^{2\pi} R(\cos(t), \sin(t))dt = \int_{\mathcal{C}} f$$

where $\mathcal{C}$ is the anticlockwise unit circle and f is the complex function defined by

$$f(z) = R\left([z+z^{-1}]/2, [z-z^{-1}]/(2i)\right) \cdot \frac{1}{iz}.$$

Here f is given by

$$f(z) = \frac{4iz}{9z^4 - 82z^2 + 9},$$

and so

$$\int_{\mathcal{C}} \frac{4iz}{9z^4 - 82z^2 + 9}\, dz$$

where $\mathcal{C}$ is the anticlockwise unit circle. Factorising the denominator gives

$$9z^4 - 82z^2 + 9 = (9z^2 - 1)(z^2 - 9) = (3z - i)(3z + i)(z - 3i)(z + 3i),$$

and so the poles of f enclosed by $\mathcal{C}$ are at $z = \pm i/3$. Hence by the Residue Theorem

$$\int_{\mathcal{C}} f = 2\pi i \left[\mathrm{Res}(f; i/3) + \mathrm{Res}(f; -i/3)\right] = \frac{\pi}{10}$$

and so

$$\int_0^{2\pi} \frac{1}{16\cos^2(t) + 25\sin^2(t)}\, dt = \frac{\pi}{10}.$$

(b)

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx.$$

Solution: With $f(z) = \dfrac{1}{(z^2+1)(z^2+9)} = \dfrac{1}{z^4+10z+9}$, $R > 3$ and $\mathcal{C}_R = L_R + S_R$ as before, the Residue Theorem gives

$$\int_{\mathcal{C}_R} f = 2\pi i \left(\mathrm{Res}(f;i) + \mathrm{Res}(f;3i)\right) = \frac{\pi}{12}.$$

The reverse triangle inequality gives

$$\left|\frac{1}{(z^2+1)(z^2+9)}\right| \le \frac{1}{(R^2-1)(R^2-9)}$$

for all $z \in S_R$ so that by the Estimation Lemma

$$\left|\int_{S_R} f\right| \le \frac{\pi R}{(R^2-1)(R^2-9)} \to 0 \quad \text{as} \quad R \to \infty.$$

A similar argument to the one used before yields

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx = \frac{\pi}{12}.$$

(c)

$$\int_0^{\infty} \frac{\cos(5x)}{x^2+4}$$

(Hint: first use the usual method to evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{i5x}}{x^2+4}\,dx$.)

Solution: Following the hint, we let $f(z) = \dfrac{\exp(5iz)}{z^2+4}$, and consider the contour $\mathcal{C}_R = L_R + S_R$ as before, where $R > 2$. Using the Residue Theorem we get

$$\int_{\mathcal{C}_R} f = 2\pi i \mathrm{Res}(f;2i) = \frac{\pi}{2}e^{-10}.$$

For $z = x + iy \in S_R$ we have

$$\begin{aligned}
|\exp(5iz)| &= |\exp(5i(x+iy))| \\
&= |\exp(-5y+5ix)| \\
&= \left|e^{-5y}\left(\cos(5x) + i\sin(5x)\right)\right| \\
&= \left|e^{-5y}\right| |\cos(5x) + i\sin(5x)| = e^{-5y}.
\end{aligned}$$

Since S_R lies above the real axis, $x+iy \in S_R$ implies $y \ge 0$, so that $-5y \le 0$ and so $e^{-5y} \le e^0 = 1$. Thus $|\exp(5iz)| \le 1$ for all $z \in S_R$.

By the reverse triangle inequality $\left|z^2+4\right| \geq R^2-4$ for all $z \in S_R$, thus for all such z,

$$|f(z)| \leq \frac{1}{R^2-4}.$$

Together with the Estimation Lemma we see that

$$\left|\int_{S_R} f\right| \leq \pi R \cdot \frac{1}{R^2-4} \to 0 \quad \text{as} \quad R \to \infty.$$

Arguing as before, we get

$$\int_{-\infty}^{+\infty} \frac{e^{5ix}}{x^2+4}\, dx = \frac{\pi}{2} e^{-10}.$$

Now, for all $x \in \mathbb{R}$, $e^{5ix} = \cos(5x) + i\sin(5x)$, so that

$$\int_{-\infty}^{+\infty} \frac{e^{5ix}}{x^2+4}\, dx = \int_{-\infty}^{+\infty} \frac{\cos(5x)}{x^2+4}\, dx \ + i \int_{-\infty}^{+\infty} \frac{\sin(5x)}{x^2+4}\, dx$$

or in other words

$$\int_{-\infty}^{+\infty} \frac{\cos(5x)}{x^2+4}\, dx = \mathsf{Re}\left(\int_{-\infty}^{+\infty} \frac{e^{5ix}}{x^2+4}\, dx\right) = \frac{\pi}{2} e^{-10}.$$

Finally, since the integrand is even,

$$\int_0^{\infty} \frac{\cos(5x)}{x^2+4}\, dx = \frac{1}{2}\left(\int_{-\infty}^{+\infty} \frac{\cos(5x)}{x^2+4}\, dx\right) = \frac{\pi}{4} e^{-10}.$$

6. (a) Let N be a natural number and let α_j be constants for $-N \leq j \leq N$. If $f(z) = \sum_{j=-N}^{N} \alpha_j z^j$, write down the value of $\mathrm{Res}(f;0)$.

(b) Write $\int_0^{2\pi} [\cos(t)]^8 \ dt$ as a contour integral, and use part (a) to evaluate it.

Solution: Let $\mathcal{C}$ be the anticlockwise circle with centre 0 and radius 1, so that

$$\mathrm{Res}(f;0) = \frac{1}{2\pi i}\int_{\mathcal{C}} f$$

We know that for $n \neq -1$, the function $z \mapsto z^n$ has an antiderivative on $\mathbb{C} \setminus \{0\}$, hence $\int_{\mathcal{C}} z^n \, dz = 0$ for $n \neq -1$. It follows that

$$\begin{aligned}\mathrm{Res}(f;0) = \frac{1}{2\pi i}\int_{\mathcal{C}} f &= \frac{1}{2\pi i}\int_{\mathcal{C}} \sum_{j=-N}^{N} \alpha_j z^j \ dz \\ &= \frac{1}{2\pi i}\sum_{j=-N}^{N} \alpha_j \int_{\mathcal{C}} z^j \ dz\end{aligned}$$

$$= \frac{1}{2\pi i}\alpha_{-1}\int_{\mathcal{C}} z^{-1}\ dz$$
$$= \frac{1}{2\pi i}\cdot\alpha_{-1}\cdot 2\pi i = \alpha_{-1}.$$

For (b), following previous examples we know that

$$\int_0^{2\pi}(\cos(t))^8\ dt = \int_{\mathcal{C}}\left(\frac{z+z^{-1}}{2}\right)^8\cdot\frac{1}{iz}\ dz$$

The binomial formula allows us to expand

$$(z+z^{-1})^8 = \sum_{k=0}^{8}\binom{8}{k}z^k(z^{-1})^{8-k} = \sum_{k=0}^{8}\binom{8}{k}z^{2k-8},$$

hence

$$\frac{1}{iz}\left(\frac{z+z^{-1}}{2}\right)^8 = \frac{1}{i2^8}\sum_{k=0}^{8}\binom{8}{k}z^{2k-9}.$$

The only isolated singularity of this function is at $z=0$, and by part (a), $\mathrm{Res}(f;0)$ is simply the coefficient of z^{-1}, i.e. $\frac{1}{i2^8}\binom{8}{4} = \frac{35}{128i}$. By the Residue Theorem,

$$\int_{\mathcal{C}} f = 2\pi i\frac{35}{128i} = \frac{35\pi}{64}.$$