Ordinary Least Squares (Linear) Regression

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1 OLS

2 Stata Example

3 R Example

4 Inference

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4 Inference

Uses of Regression

1. Description
2. Prediction
3. Causal Inference

Descriptive Inference

1. We want to understand a *population* of cases
2. We cannot observe them all, so:
    1. Draw a *representative* sample
    2. Perform mathematical procedures on sample data
    3. Use assumptions to make inferences about population
    4. Express uncertainty about those inferences based on assumptions

Parameter Estimation

- We want to observe population *parameter* θ
- If we obtain a representative sample of population units:
  - Our sample statistic $\hat{\theta}$ is an unbiased estimate of θ
  - Our sampling procedure dictates how uncertain we are about the value of θ

An Example

- We want to know $\bar{Y}$ (population mean)
- Our *estimator* is the sample mean formula which produces the sample *estimate* $\bar{y}$:

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{1}$$

- The *sampling variance* is our uncertainty:

$$Var(\bar{y}) = \frac{s^2}{n} \tag{2}$$

where s^2 = sample element variance

Uncertainty

- We never know θ
- Our $\hat{\theta}$ is an estimate that may not equal θ
  - Unbiased due to **Law of Large Numbers**
  - For $\bar{y}$: $N(Y, \sigma^2)$
- The size of sampling variance depends on:
  - Element variance
  - Sample size!
- Note: $SE(\bar{y}) = \sqrt{Var(\bar{y})}$
- We may want to know $\hat{\theta}$ per se, but we are mostly interested in it as an estimate of θ

Causal Inference

Causal Inference

1. Everything that goes into descriptive inference

Causal Inference

1. Everything that goes into descriptive inference
2. Plus, randomization *or* perfectly specified model
   - X comes before Y
   - X measured without error
   - No confounding due to omitted variables

Bivariate Regression I

- Y is continuous
- X is a randomized treatment indicator/dummy $(0, 1)$
- How do we know if the treatment X had an effect on Y?

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- How do we know if the treatment X had an effect on Y?
- Look at mean-difference: $E[Y_i|X_i = 1] - E[Y_i|X_i = 0]$

Three Equations

1. Population: $Y = \beta_0 + \beta_1 X \ \ (+\epsilon)$

2. Sample estimate: $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

3. Unit:

$$\begin{aligned} y_i &= \hat{\beta}_0 + \hat{\beta}_1 x_i + e_i \\ &= \bar{y}_{0i} + (y_{1i} - y_{0i})x_i + (y_{0i} - \bar{y}_{0i}) \end{aligned}$$

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- Mean difference ($E[Y_i|X_i = 1] - E[Y_i|X_i = 0]$) is the regression line slope
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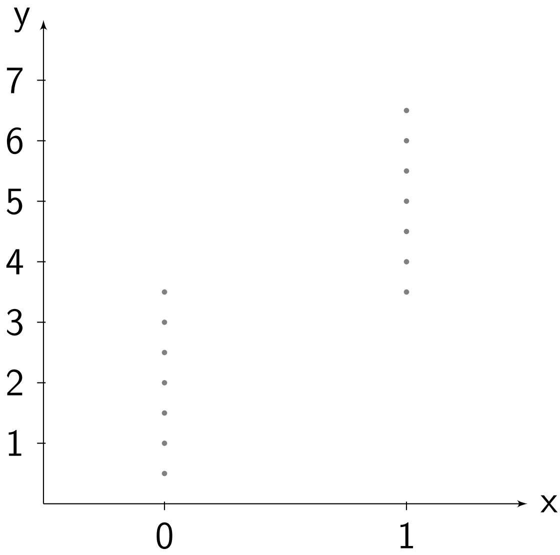
  - $\Delta Y = E[Y_i|X = 1] - E[Y_i|X = 0]$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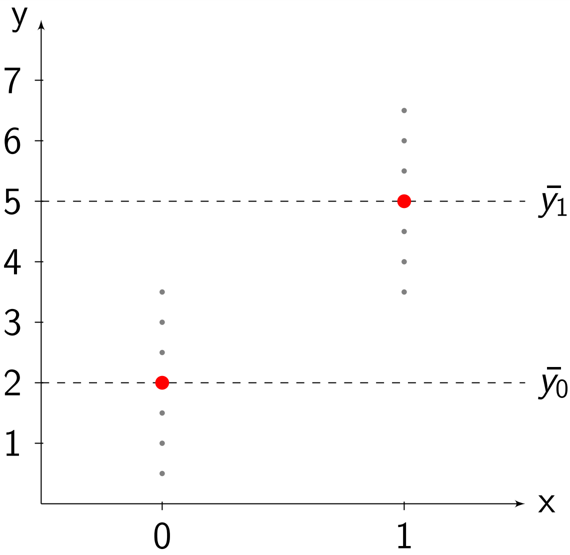
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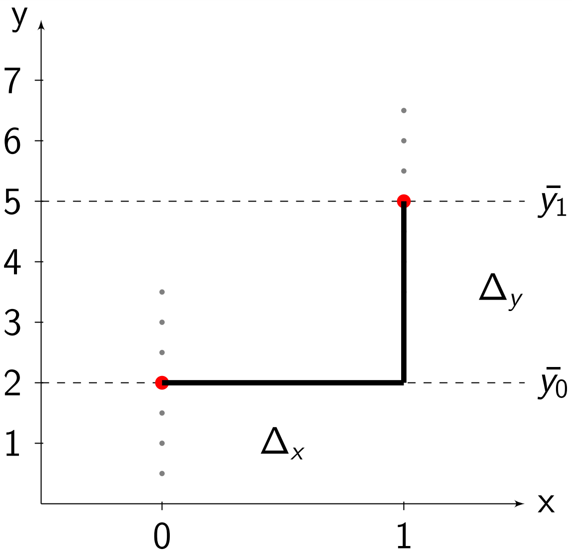

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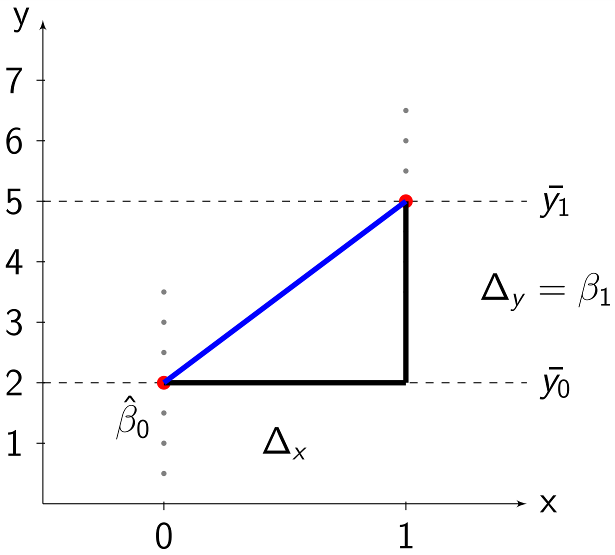
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$\hat{\beta}_0$
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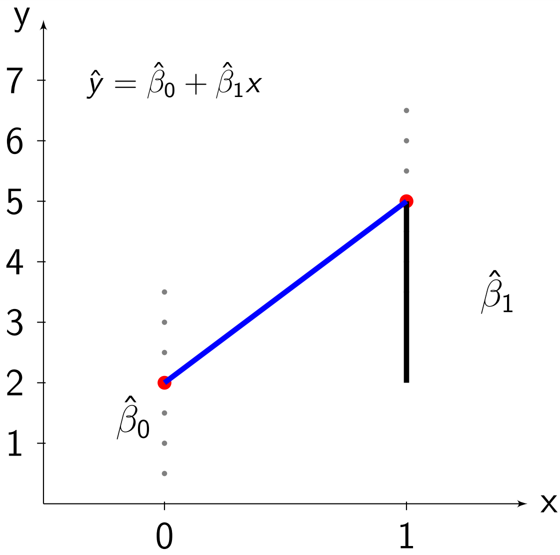

y
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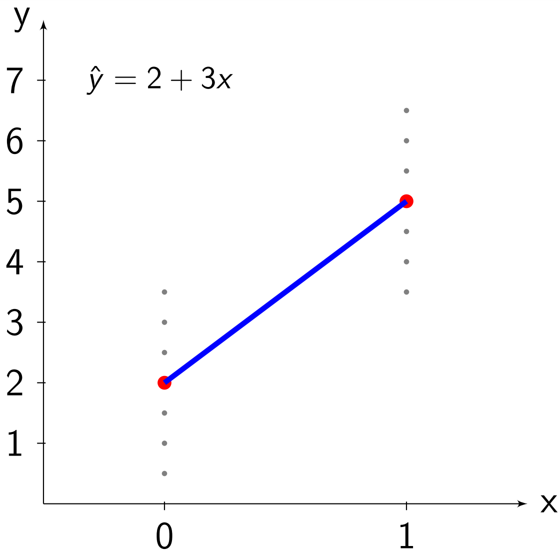
$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$
$\hat{\beta}_1$
$\hat{\beta}_0$
0
1
x



y
x
$\hat{y} = 2 + 3x$
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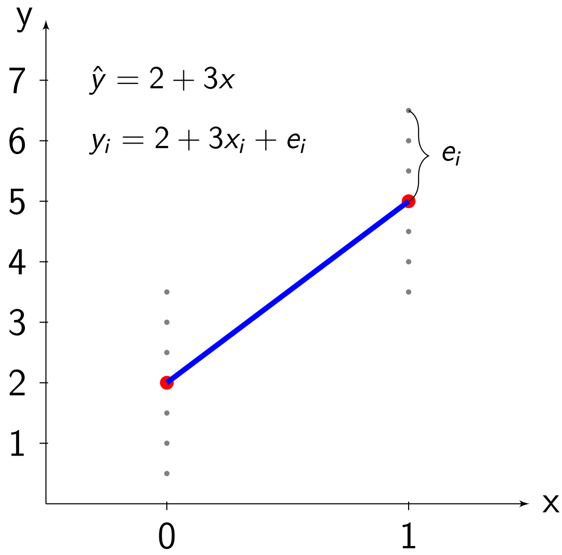

y
x
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$\hat{y} = 2 + 3x$
$y_i = 2 + 3x_i + e_i$
e_i

Systematic versus unsystematic component of the data

- Systematic: Regression line (slope)
  - Linear regression estimates the conditional means of the population data (i.e., $E[Y|X]$)
- Unsystematic: Error term is the deviation of observations from the line
  - The difference between each value y_i and $\hat{y}_i$ is the *residual*: e_i
  - OLS produces an estimate of the relationship between X and Y that minimizes the *residual sum of squares*

Why are there residuals?

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- Measurement error

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- Omitted variables

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- Slope (β) defined as $\frac{\Delta Y}{\Delta X}$
  - $\Delta Y = E[Y_i|X = 1] - E[Y_i|X = 0]$
- How do we know if this is a *significant* difference?
  - Substantive: own judgment
  - Statistical: larger than twice the SE

Bivariate Regression II

- Y is continuous
- X is continuous (and randomized)
- How do we know if the treatment X had an effect on Y?
  - Correlation coefficient (ρ)
  - Regression coefficient (slope; β_1)

Correlation Coefficient (ρ)

- Measures how well a scatterplot is represented by a straight (non-horizontal) line

OLS Coefficient (β_1)[1]

- Measures ΔY given ΔX

[1] Multivariate formula involves matrices; Week 20

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- Formal definition: $\frac{Cov(X,Y)}{Var(X)}$
- As a reminder:
  - $Cov(x, y) = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$
  - $Var(x) = \sum_{i=1}^{n}(x_i - \bar{x})^2$

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- $\hat{\rho}$ and $\hat{\beta}_1$ are just scaled versions of $\widehat{Cov}(x, y)$

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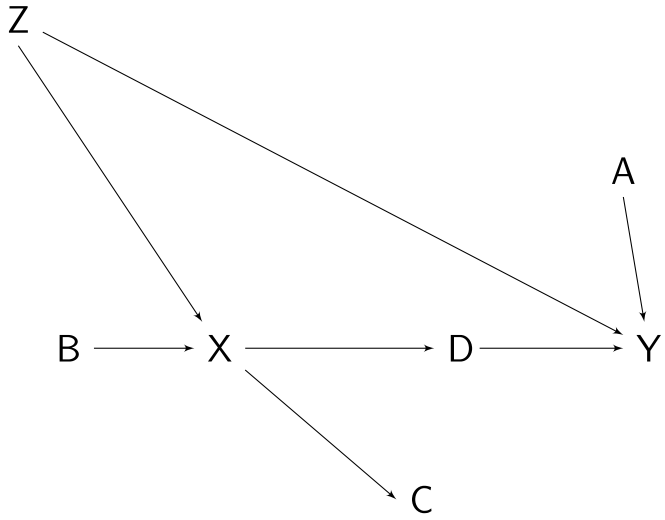
  - Nonlinear regression models (e.g., logit/probit)
- Interpretations are sample-level
  - Sample representativeness determines generalizability
- Remember uncertainty
  - These are *estimates*, not population parameters

Confounding (Selection Bias)

- If x is not randomly assigned, potential outcomes are not independent of x
- Other factors explain why a unit i received their particular value x_i

$$\underbrace{E[Y_i|X_i = 1] - E[Y_i|X_i = 0] =}_{\text{Naive Effect}}$$

$$\underbrace{E[Y_{1i}|X_i = 1] - E[Y_{0i}|X_i = 1]}_{\text{Treatment Effect on Treated (ATT)}} + \underbrace{E[Y_{0i}|X_i = 1] - E[Y_{0i}|X_i = 0]}_{\text{Selection Bias}}$$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Omitted Variable Bias

- We want to estimate:

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + \epsilon$$

- We actually estimate:

$$\begin{aligned}\tilde{y} &= \tilde{\beta}_0 + \tilde{\beta}_1 x + \epsilon \\ &= \tilde{\beta}_0 + \tilde{\beta}_1 x + (0 * z) + \epsilon \\ &= \tilde{\beta}_0 + \tilde{\beta}_1 x + \nu\end{aligned}$$

- Bias: $\tilde{\beta}_1 = \hat{\beta}_1 + \hat{\beta}_2 \tilde{\delta}_1$, where $\tilde{z} = \tilde{\delta}_0 + \tilde{\delta}_1 x$

Size and Direction of Bias

- Bias: $\tilde{\beta}_1 = \hat{\beta}_1 + \hat{\beta}_2 \tilde{\delta}_1$, where $\tilde{z} = \tilde{\delta}_0 + \tilde{\delta}_1 x$

	$Corr(x, z) < 0$	$Corr(x, z) > 0$
$\beta_2 < 0$	Positive	Negative
$\beta_2 > 0$	Negative	Positive

Common Conditioning Strategies

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Which of these are good strategies?

What goes in our regression?

- Use theory to build causal models
    - Often, a causal graph helps
- Some guidance:

What goes in our regression?

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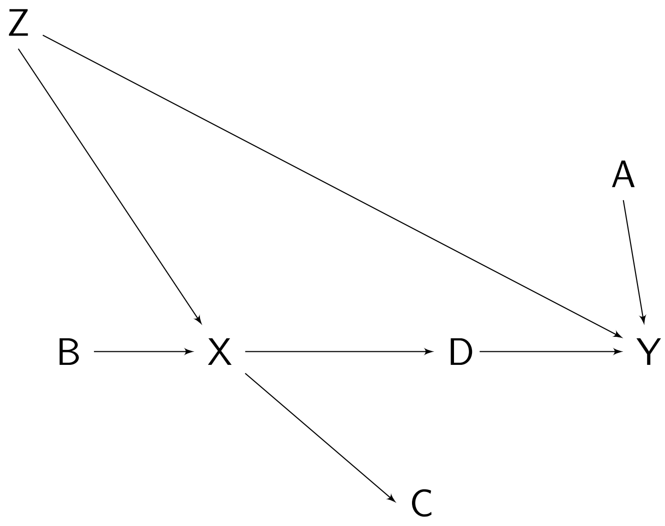
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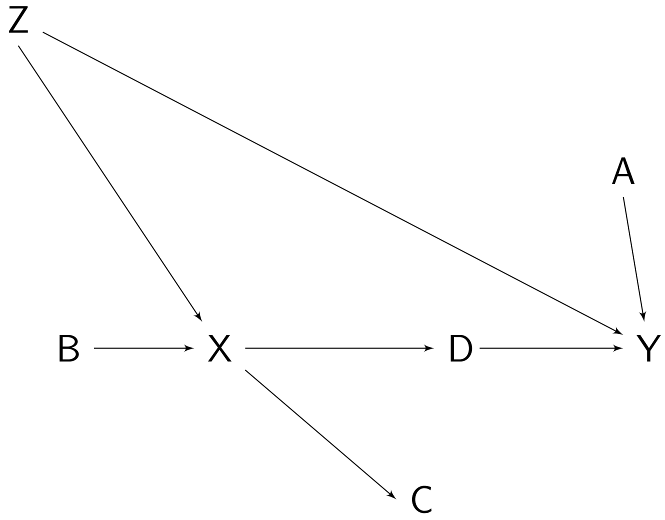
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Post-treatment Bias

- We usually want to know the **total effect** of a cause
- If we include a mediator, D, of the $X \rightarrow Y$ relationship, the coefficient on X:
  - Only reflects the **direct** effect
  - Excludes the **indirect** effect of X through M
- So don't control for mediators!

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    - Do not include *colinear* variables

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4. Can we have highly correlated regressors?

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4. Can we have highly correlated regressors?
   - Generally no (due to multicollinearity)

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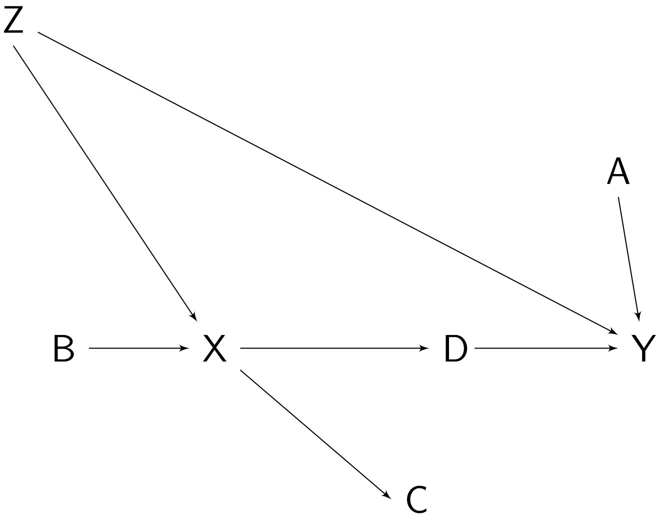
  - Including variables that affect Y alone increases certainty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Questions about specification?

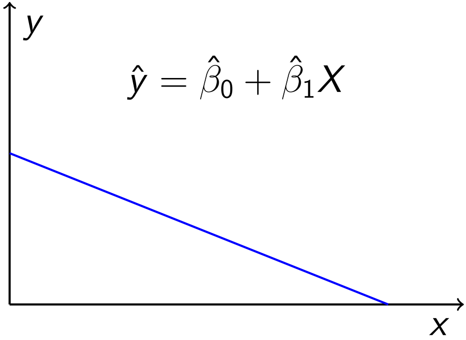
Multivariate Regression Interpretation

- All our interpretation rules from earlier still apply in a multivariate regression
- Now we interpret a coefficient as an effect "all else constant"
- Generally, not good to give all coefficients a causal interpretation
  - Think "forward causal inference"
  - We're interested in the $X \to Y$ effect
  - All other coefficients are there as "controls"

From Line to Surface I

- In simple regression, we estimate a **line**
- In multiple regression, we estimate a **surface**
- Each coefficient is the *marginal effect*, all else constant (at mean)
- This can be hard to picture in your mind

From Line to Surface II



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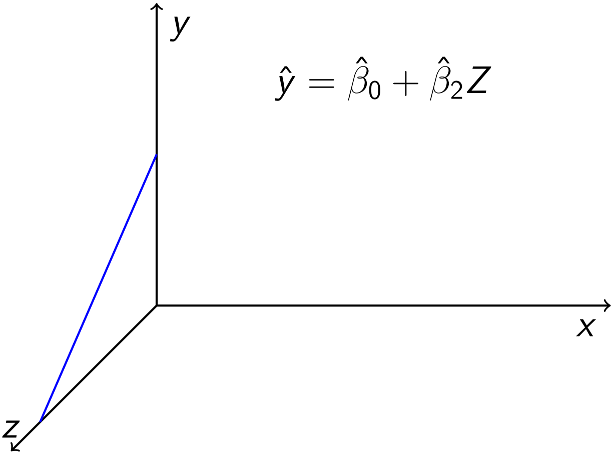
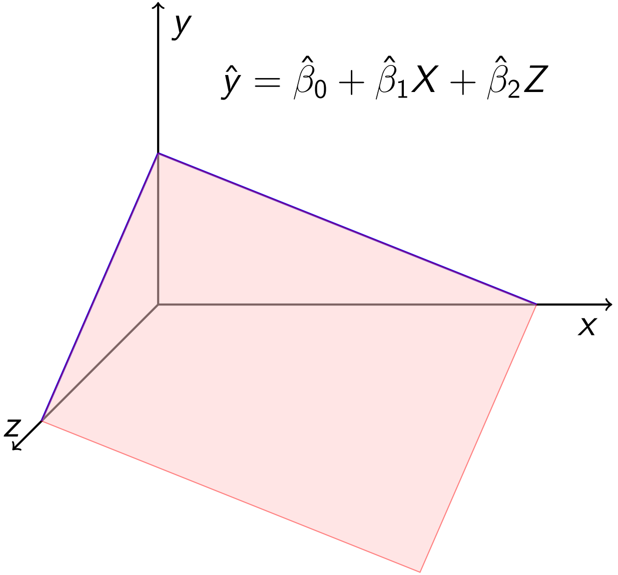


$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_2 Z$$

From Line to Surface II



Goodness-of-Fit

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- We can answer:
  “How well does our model fit the observed data?”

Goodness-of-Fit

- We want to know: “How good is our model?”
- We can answer:
  “How well does our model fit the observed data?”
- Is this what we want to know?

Coefficient of Determination (R^2)

- Definition: $R^2 = \hat{r}^2_{x,y} = \frac{SSE}{SST} = 1 - \frac{SSR}{SST}$
- Interpretation: How much of the total variation in y is explained by the model?
- But, R^2 increases simply by adding more variables
- So, Adjusted-$R^2 = R^2 - (1 - R^2)\frac{k}{n-k-1}$, where k is number of regressors
- Units: none (range 0 to 1)

Standard Error of the Regression (SER)

- "Root mean squared error" or just σ
- Definition: $\hat{\sigma} = \sqrt{\frac{SSR}{n-p}}$, where p is number of parameters estimated
- Interpretation: How far, on average, are the observed y values from their corresponding fitted values $\hat{y}$
  - $sd(y)$ is how far, on average, a given y_i is from $\bar{y}$
  - σ is how far, on average, a given y_i is from $\hat{y}_i$
- Units: same as y (range 0 to $sd(y)$)

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```
use "ESS7GB.dta"
codebook

* outcome (feel close to country)
tab fclcntr
recode fclcntr (7/8=.), gen(feelclose)

* explanatory 1 (country of birth)
tab brncntr
recode brncntr (1=1) (2=0), gen(ukborn)

* explanatory 2 (age)
tab agea
recode agea (999 = .)
```

```
reg feelclose ukborn agea

      Source |       SS           df       MS      Number of obs   =     2,224
-------------+----------------------------------   F(2, 2221)      =    111.58
       Model |  130.792982         2  65.3964908   Prob > F        =    0.0000
    Residual |  1301.74614     2,221  .586108121   R-squared       =    0.0913
-------------+----------------------------------   Adj R-squared   =    0.0905
       Total |  1432.53912     2,223  .644417057   Root MSE        =    .76558

------------------------------------------------------------------------------
   feelclose |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      ukborn |    .282186   .0477851     5.91   0.000     .1884779     .375894
        agea |  -.0128909   .0008924   -14.45   0.000    -.0146409   -.0111409
       _cons |   2.279175   .0597376    38.15   0.000     2.162027    2.396322
------------------------------------------------------------------------------
```

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```r
install.packages("rio")
dat <- rio::import("ESS7GB.dta")
dim(dat)
names(dat)
str(dat[,1:5])

# outcome (feel close to country)
table(dat$fclcntr)
dat$feelclose <- ifelse(dat$fclcntr %in% 7:8, NA, dat$fclcntr)

# explanatory 1 (country of birth)
table(dat$brncntr)
dat$ukborn <- ifelse(dat$brncntr == 1, 1, 0)

# explanatory 2 (age)
table(dat$agea)
dat$agea[dat$agea > 99] <- NA

m <- lm(feelclose ~ ukborn + agea, data = dat)
```

```
summary(m)
## Call:
## lm(formula = feelclose ~ ukborn + agea, data = dat)
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.36800 -0.64610 -0.02709  0.40440  2.50859
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.2791745  0.0597376  38.153  < 2e-16 ***
## ukborn       0.2821860  0.0477851   5.905 4.06e-09 ***
## agea        -0.0128909  0.0008924 -14.445  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' '
##
## Residual standard error: 0.7656 on 2221 degrees of freedom
##   (40 observations deleted due to missingness)
## Multiple R-squared:  0.0913,    Adjusted R-squared:  0.09048
## F-statistic: 111.6 on 2 and 2221 DF,  p-value: < 2.2e-16
```

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Inference from Sample to Population

- We want to know population parameter θ
- We only observe sample estimate $\hat{\theta}$
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- We want to know population parameter θ
- We only observe sample estimate $\hat{\theta}$
- We have a guess but are also uncertain

- What range of values for θ does our $\hat{\theta}$ imply?
- Are values in that range large or meaningful?

How Uncertain Are We?

- Our uncertainty depends on sampling procedures
- Most importantly, *sample size*
  - As $n \to \infty$, uncertainty $\to 0$
- We typically summarize our uncertainty as the *standard error*

Standard Errors (SEs)

- Definition: "The standard error of a sample estimate is the average distance that a sample estimate ($\hat{\theta}$) would be from the population parameter (θ) if we drew many separate random samples and applied our estimator to each."
- In bivariate regression: $Var(\hat{\beta}_1) = \frac{\frac{1}{n-2}SSR}{SST_x}$
- Thus, SE is a ratio of unexplained variance in y (weighted by sample size) and variance in x
- Units: same as coefficient ($\frac{y}{x}$)

What affects size of SEs?

- Larger variance in x means smaller SEs
- More unexplained variance in y means bigger SEs
- More observations reduces the numerator, thus smaller SEs
- Other factors:
    - Homoskedasticity
    - Clustering
- Interpretation:
    - Large SE: Uncertain about population effect size
    - Small SE: Certain about population effect size

Ways to Express Our Uncertainty

1. Standard Error
2. Confidence interval
3. t-statistic
4. p-value

Confidence Interval (CI)

- Definition: Were we to repeat our procedure of sampling, applying our estimator, and calculating a confidence interval *repeatedly* from the population, a fixed percentage of the resulting intervals would include the true population-level slope.
- Interpretation: If the confidence interval overlaps zero, we are uncertain if β differs from zero

Confidence Interval (CI)

- A CI is simply a range, centered on the slope
- Units: Same scale as the coefficient ($\frac{y}{x}$)
- We can calculate different CIs of varying *confidence*
  - Conventionally, $\alpha = 0.05$, so 95% of the CIs will include the β

t-statistic

- A measure of how large a coefficient is relative to our uncertainty about its size
- Typically used to test a formal null hypothesis:
  - No effect null: $t_{\hat{\beta}_1} = \frac{\hat{\beta}_1}{SE_{\hat{\beta}_1}}$
  - Any other null: $\frac{\hat{\beta}_1 - \alpha}{SE_{\hat{\beta}_1}}$, where α is our null hypothesis effect size

t-statistic

- A measure of how large a coefficient is relative to our uncertainty about its size
- Typically used to test a formal null hypothesis:
  - No effect null: $t_{\hat{\beta}_1} = \frac{\hat{\beta}_1}{SE_{\hat{\beta}_1}}$
  - Any other null: $\frac{\hat{\beta}_1 - \alpha}{SE_{\hat{\beta}_1}}$, where α is our null hypothesis effect size
- Note: The *t*-statistic from a *t*-test of mean-difference is the same as the *t*-statistic from a *t*-test on an OLS slope for a dummy covariate

p-value

- A summary measure in a hypothesis test
- General definition: "the probability of a statistic as extreme as the one we observed, if the null hypothesis was true, the statistic is distributed as we assume, and the data are as variable as observed"
- Definition in a regression context: "the probability of a slope as large as the one we observed ..."

The p-value is not:

- The probability that a hypothesis is true or false
- A reflection of our confidence or certainty about the result
- The probability that the true slope is in any particular range of values
- A statement about the importance or substantive size of the effect

Significance

1. Substantive significance
2. Statistical significance

Significance

1. Substantive significance
   - Is the effect size (or range of possible effect sizes) *important* in the real world?

2. Statistical significance

Significance

1. Substantive significance
   - Is the effect size (or range of possible effect sizes) *important* in the real world?
2. Statistical significance
   - Is the effect size (or range of possible effect sizes) larger than a predetermined threshold?
   - Conventionally, $p \leq 0.05$

Questions about inference?