Robust Human Capital Investment under Risk and Ambiguity*

Philipp Eisenhauer

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Abstract

I extend the prototypical life cycle model of human capital investment (Keane & Wolpin, 1994) and study individual decision-making under risk as well as ambiguity. Individuals fear model misspecification and seek robust decisions that work well over a whole range of models about their economic environment. I describe the individual's decision problem as a robust Markov decision process. My Monte Carlo analysis indicates that the empirical finding of large psychic cost of schooling is in part due to model misspecification by econometricians who only analyze individual investment decisions under risk. This changes the mechanisms driving schooling decisions and affects the ex ante evaluation of tuition policies.

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Keywords: life cycle model, human capital, risk, ambiguity, robust Markov decision process

*Department of Economics, University of Bonn. Email: peisenha@uni-bonn.de. I gratefully acknowledge support by the AXA Research Fund. I thank Jörg Stoye, Janoś Gabler, and Rafael Suchy for their comments and suggested improvements. I also thank the Social Sciences Computing Service at the University of Chicago for the permission to use their computational resources.

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1 Introduction

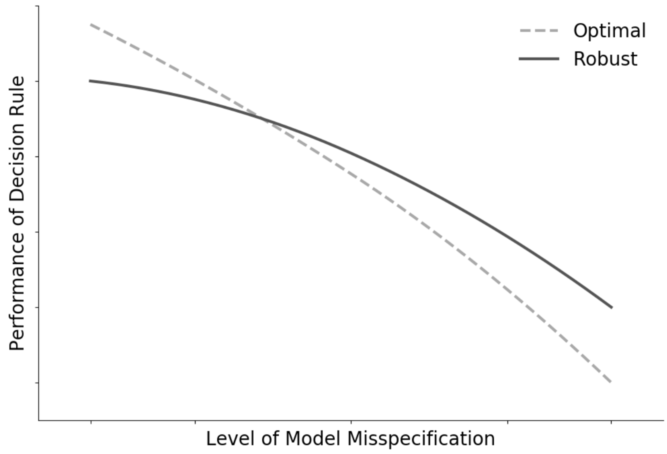
The uncertainties involved in human capital investments are ubiquitous (Marshall, 1890; Becker, 1964). Individuals usually make investments early in life when they are still uncertain about their own abilities and tastes. In addition, returns also depend on demographic, economic, and technological trends that only start to unfold years from now. However, the treatment of uncertainty in life cycle models of human capital investment is very narrow. A model provides individuals with a formalized view about their economic environment and implies unique probabilities for all possible future events. Individuals have no fear of model misspecification.

I address this shortcoming by formulating, implementing, and exploring a life cycle model of robust human capital investment where individuals face risk within a model and ambiguity about the model (Arrow, 1951). Ambiguity arises as individuals simply do not know the true model and consider a whole set of models as reasonable descriptions of their economic environment. Individuals fear model misspecification and thus seek robust decisions, i.e. decisions that perform well over the whole range of models.

Figure 1 clarifies the notion of robust decision-making. It shows the performance of an optimal and a robust decision rule for different levels of model misspecification. The optimal decision rule is designed without any fear of model misspecification, using a single model to inform decisions. It thus performs very well if that model turns out to be true. However, its performance is very sensitive and deteriorates rapidly if the model is misspecified. The robust decision rule explicitly accounts for the possibility of model misspecification and its performance is less affected. At some point it actually outperforms the optimal decision rule.

My approach sheds new light on the puzzle that numerous studies find large monetary returns to attending post-secondary school, but at the same time much fewer than expected individuals continue their education after high school. The previous literature introduced the key distinction between gross and net returns to reconcile these two observations. For example, a simple comparison between the earning streams of a high school and college graduate does not account for the direct cost of a college education such as tuition. However, Carneiro, Hansen, & Heckman (2003) estimate a total cost of schooling that is way too large to be due to tuition cost alone and coin the term psychic cost of effort that further drives a wedge between the gross and net return to schooling. The presence of a large psychic cost is a common finding in the literature on schooling choice (Cunha, Heckman, & Navarro, 2005; Eisenhauer, Heckman, & Mosso, 2015; Hai & Heckman, 2017). For example, Abbott, Gallipoli, Meghir, & Violante (2018) estimate a total psychic cost of about $135,000 for a college degree. However, a casual reference to psychic cost is quite unsatisfactory as they essentially remain an unexplained resid-

Figure 1: Robust decision-making



ual.[1]

Based on a Monte Carlo analysis of an extended version of the prototypical model of human capital investment under risk by Keane & Wolpin (1994), I show that large psychic cost estimates are partly due to model misspecification. If individuals make their schooling choices facing ambiguity, but econometricians view their decisions through the lens of a risk-only model, then this model misspecification shows up as a large psychic cost estimate. This substitutes for the reduction of the net return to schooling actually due to a fear of model misspecification.

Individual preferences are represented by expected utility preferences in the prototypical model of human capital investment under risk. Several approaches have been developed to capture individual preferences in light of risk and ambiguity. I will discuss some of the popular approaches and ultimately rely on maxmin expected utility preferences (Gilboa & Schmeidler, 1989; Epstein & Schneider, 2003). Individuals evaluate their decisions on a set of alternative models and then rank their decisions based on the worst-case outcome. The optimal decision is the one with the least worst outcome. These preferences are closely related to Wald (1950)'s maximin model, which is the most important decision criterion in robust optimization and robust decision-making. This intersection between the economic decision model and robust optimization allows me to draw on the mathematical model of a robust Markov decision process (RMDP) to analyze individual decision-making in light of risk and ambiguity.

My research contributes to a broader effort in economics to incorporate model ambiguity into

[1] I present a list of further references with short excerpts in Appendix A.3.

the analysis.[2] This has generated new insights into long-standing economic questions such as a taste for policy diversification (Manski, 2009), robust social decisions (Danan, Gajdos, Hill, & Tallon, 2016), business cycle dynamics (Ilut & Schneider, 2014), no-trade results (Dow & da Costa Werlang, 1992), and the equity premium puzzle (Hansen, Sargent, & Tallarini Jr., 1999).

I structure the rest of the paper as follows. I first present the conceptual framework, clearly delineating the economic, mathematical, and computational model. I discuss alternative approaches to individual decision-making under risk and ambiguity, outline the model of a RMDP, and describe the prototypical model of human capital investment and its extension to accommodate model ambiguity. Then I explore the usefulness of my approach in a Monte Carlo exercise, where I focus on the modeling trade-off between model ambiguity and the psychic cost of schooling.

2 Conceptual Framework

I now present the economic, mathematical, and computational model for robust human capital investment under risk and ambiguity. I start with the discussion of the basic economic environment and alternative approaches to individual decision-making. I then turn to the corresponding mathematical model of a RMDP, outline its setup, contrast it to the standard Markov decision process (MDP), and describe the solution approach. Finally, I turn to the computational model and discuss assumptions about functional forms and the distributions of unobservables.

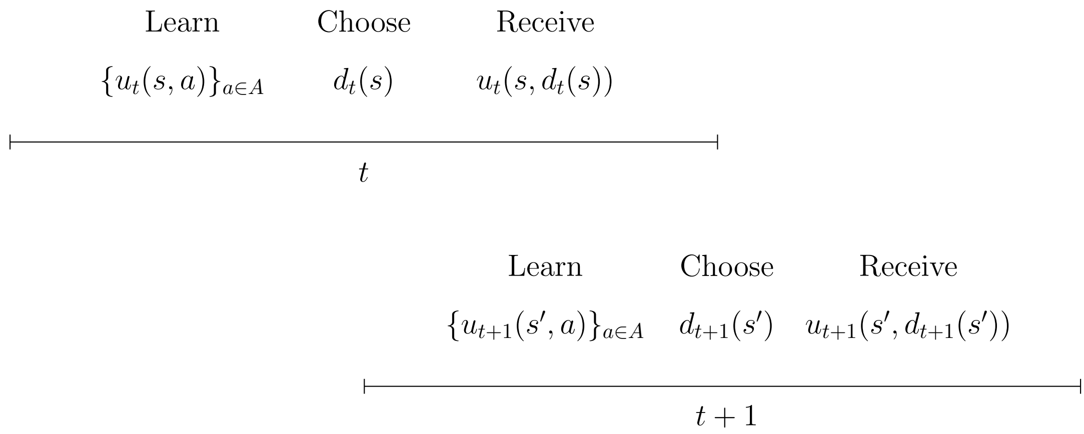
I will introduce acronyms and symbols as needed, but a full list of both is provided in Appendix B. Optimal decisions in the MDP and RMDP are a deterministic function of the current economic environment, i.e. an optimal decision rule is always deterministic and Markovian. I restrict my notation to this special case right from the beginning.

2.1 Economic Model

At time $t = 1, \ldots, T$ each individual observes the state of the economic environment $s \in S$ and chooses an action a from the set of admissible actions $\mathcal{A}$. The decision has two consequences: an individual receives an immediate utility $u_t(s, a)$ and the economy evolves to a new state s'. The transition from s to s' is affected by the action. Individuals are forward-looking, thus they do not simply choose the alternative with the highest immediate utility. Instead, they take the

[2] See Marinacci (2015) for a recent review and numerous additional references. Hansen & Sargent (2016) provide a textbook treatment in the context of macroeconomics.

Figure 2: Timing of events



future consequences of their current action into account.

Figure 2 depicts the timing of events in the model for two generic time periods. At the beginning of time t an individual fully learns about the immediate utility of each alternative, chooses one of them, and receives its immediate utility. Then the state evolves from s to s' and the process is repeated in $t+1$.

A decision rule d_t specifies the action at a particular time t for any possible state. A policy $\pi \equiv (d_1, \ldots, d_T)$ provides the individual with a prescription for choosing an action in any possible future state. It is a sequence of decision rules and its implementation generates a sequence of utilities. The evolution of states over time is at least partly unknown as future utilities depend on, for example, shocks to preferences. Let X_t denote the random variable for the state at time t.

Individuals use models about their economic environment to inform their beliefs about the future. For a given model, individuals thus face risk as each induces a unique objective transition probability distribution $p_t(s,a)$ for the evolution of state s to s' that depends on the action a. If, in addition, there is ambiguity about the model then individuals are faced with a whole set of conditional distributions $\mathcal{P}_t(s,a)$ and s' is determined by some $p_t(s,a) \in \mathcal{P}_t(s,a)$. In this setup, $\mathcal{P}_t(s,a)$ is often referred to as the set of priors and decision-making under risk corresponds to the special case where $\mathcal{P}_t(s,a)$ is a singleton.

Economic decision theory offers guidance on desirable decision principles in settings with risk as well as risk and ambiguity (Gilboa, 2009). There is broad agreement how reasonable individuals make decisions under risk. This is not the case, however, when individuals are faced with risk and ambiguity. Numerous competing approaches exist and any particular choice requires to assess the behavioral and computational trade-offs in their application.

I will discuss the main approaches for each setting in turn. I initially focus on static choices as this allows to study the basic ranking of actions with possibly ambiguous consequences in isolation. I then turn to the dynamic setting with sequential decisions, where new issues emerge such as the aggregation of the sequence of utilities, dynamic consistency of preferences, and updating of beliefs.

I will make several simplifications to explore a static model within the general setting of the dynamic model. I restrict attention to an individual in state s in the second-to-last period, set all immediate utilities to zero, and do not account for time discounting. This implies that individuals evaluate each action's total utility $v_{T-1}(s, a)$ entirely on the basis of its uncertain future utility $u_T(s', d_T(s'))$. Each action induces a different set of conditional distributions $\mathcal{P}_{T-1}(s, a)$ on the realization of future states s'.

2.1.1 Uncertainty as Risk

In the prototypical life cycle model, individuals make their decisions facing risk and have rational expectations (Muth, 1961; Lucas, 1972) as their model about the future also turns out to be true. In this case, there is a consensus that rational choices are expressed by expected utility preferences (Bernoulli, 1738; von Neumann & Morgenstern, 1944, 1947).

In a static setting, individuals in state s rank each action a according to the criterion formalized in equation (1):

$$v_{T-1}(s, a) = \int_S u_T(s', d_T(s'))\, dp_{T-1}(s, a). \tag{1}$$

Individuals simply evaluate the utility for all possible future states s' and weigh them by their objective probabilities. The von Neumann-Morgenstern utility function u captures an individual's attitude towards risk (Meyer, 2014).

The extension to the dynamic setting is straightforward. Individuals maximize their expected total discounted utility (Samuelson, 1937; Koopmans, 1960). A constant discount factor ensures dynamic consistency of preferences as the individual's future actions agree with the planned-for contingencies. Beliefs are updated according to Bayes's rule.[3]

[3]See Frederick, Loewenstein, & O'Donoghue (2002) for a critical review of the literature on time discounting and time preference. Fang & Silverman (2009), Fang & Yang (2015), and Chan (2017) are examples of hyperbolic discounting and thus potentially time-inconsistent preferences in settings similar to the one discussed here.

Equation (2) provides the formal representation of the individual's objective. Given an initial state s, individuals seek to implement the optimal policy π^* from the set of all possible policies Π that maximizes the expected total discounted utility over all T decision periods $v_1^{\pi^*}(s)$.

$$v_1^{\pi^*}(s) = \max_{\pi \in \Pi} \mathrm{E}_s^{\pi} \left[\sum_{t=1}^{T} \delta^{t-1} u_t(X_t, d_t(X_t)) \right] \tag{2}$$

The exponential discount factor $0 < \delta < 1$ captures a preference for immediate over future utility. The superscript of the expectation emphasizes that each policy π induces a different unique probability distribution over the sequences of utilities.

2.1.2 Uncertainty as Risk and Ambiguity

Numerous alternative approaches exist for rational decision-making in light of risk and ambiguity.[4] I will briefly outline the most common approaches to motivate and contrast my eventual modeling choice. Two groups can be broadly distinguished (Gilboa & Marinacci, 2013): Either all ambiguity can be probabilized by attaching a subjective belief to each possible model (Bayesian) or it cannot (non-Bayesian). All approaches are set up as a two-stage decision process (Anscombe & Aumann, 1963). Risk within a given model is first resolved by expected utility calculation and then model ambiguity is addressed in the second step.

Subjective expected utility preferences (Savage, 1954) are a natural extension of expected utility preferences that allows to capture model ambiguity. An individual simply considers the expected utility of each model and then averages them according to a subjective probability that captures his belief about the merit of each model. Decisions under ambiguity thus reduce to decision-making under risk. In equation (3), individuals treat the objective transition probabilities from a model and their subjective beliefs μ about each model equivalently:

$$v_{T-1}(s, a) = \int_{\mathcal{P}} \left(\int_S u_T(s', d_T(s'))\, dp_{T-1}(s, a) \right) d\mu(p). \tag{3}$$

However, a variety of experimental evidence such as the Ellsberg paradox (Ellsberg, 1961) indicates that individuals have a preference for risk over ambiguity.[5] This makes the distinction between risk for a given model and ambiguity about the model behaviorally meaningful (Hansen & Sargent, 2001).

To contrast different ambiguity attitudes across individuals and approaches, I adopt the char-

[4] See Al-Najjar & Weinstein (2009a,b) for a critical assessment of the whole endeavor as currently pursued.

[5] A recent survey of the experimental literature is provided by Trautmann & van de Kuilen (2015).

acterization from Ghirardato & Marinacci (2002). They build on the work by Yaari (1969) on comparative risk attitudes and apply his ideas to ambiguity. If an individual prefers an action with unambiguous consequences to an ambiguous one, a more ambiguity averse individual will do the same. Subjective expected utility preferences serve as the ambiguity neutral benchmark. Thus, preferences are ambiguity averse if they are more ambiguity averse than subjective expected utility preferences.[6]

Within the Bayesian paradigm, smooth ambiguity preferences are an attempt to accommodate differing attitudes towards risk and ambiguity (Klibanoff, Marinacci, & Mukerji, 2005, 2009). As with subjective expected utility preferences, individuals still need to specify a belief μ over the set of models. Equation (4) formalizes this idea, where $\phi : \mathbb{R} \mapsto \mathbb{R}$ is a strictly increasing function:

$$v_{T-1}(s,a) = \int_{\mathcal{P}} \phi \left(\int_{S} u_T(s', d_T(s'))\, dp_{T-1}(s,a) \right) d\mu(p). \tag{4}$$

Following a common interpretation, the curvature of ϕ captures an individual's attitude towards ambiguity. A negative attitude towards ambiguity is modeled by a concave ϕ, in the linear case the preferences are observationally equivalent to subjective expected utility. Subjective beliefs about the ambiguous consequences are exclusively represented by μ. Together, this creates a clear separation between ambiguity attitude and ambiguity itself. However, this feature of the model is contested as exemplified by the exchange between Epstein (2010) and Klibanoff, Marinacci, & Mukerji (2012).

Both approaches presented so far attach a subjective belief to each of the possible models. They do not permit to express pure ignorance and it remains unclear what informs subjective beliefs (Gilboa, Postlewaite, & Schmeidler, 2008, 2009, 2012).

Non-Bayesian models allow for pure ignorance about the true model. I will discuss the broad class of variational preferences (Maccheroni, Marinacci, & Rustichini, 2006a,b) in more detail. Maxmin expected utility (Gilboa & Schmeidler, 1989; Epstein & Schneider, 2003) and multiplier preferences (Hansen & Sargent, 2001), the two preferences most widely used in applications, are special cases of variational preferences. This allows me to contrast them within a unified framework and admits a comparison of ambiguity attitudes across these two types of preferences.

Under variational preferences, individuals value each alternative according to the criterion in

[6] Epstein (1999) takes an alternative approach and defines probabilistically sophisticated preferences (Machina & Schmeidler, 1992) as ambiguity neutral.

equation (5), where $c : \mathcal{P} \mapsto [0, \infty)$ assigns a cost to each model p under consideration:

$$v_{T-1}(s, a) = \min_{p \in \mathcal{P}} \left(\int_S u_T(s', d_T(s'))\, dp_{T-1}(s, a) + c(p) \right). \tag{5}$$

Maccheroni et al. (2006a) show that the cost function can be interpreted as an index of ambiguity aversion, allowing to rank individuals according to their ambiguity aversion. As the cost is always nonnegative, all variational preferences are at least weakly ambiguity averse. The representation of expected utility, maxmin expected utility, and multiplier preferences simply puts further restrictions on the cost function. For example, if c is zero for a reference model and infinity otherwise, then preferences are of the expected utility type. All deviations from the reference model are associated with a prohibitive penalty.

Multiplier preferences are a special case of variational preferences where $c(p) = \theta R(p \parallel q)$. The distance of each model under consideration p from a reference model q is assessed with respect to the Kullback-Leibler divergence R (Kullback & Leibler, 1951). The individual does not fully trust q and considers many other models to be plausible, with their plausibility diminishing proportionally to their distance from q. The proportionality parameter θ measures the degree of trust of the individual in the reference model q. Higher values of θ correspond to more trust. Strzalecki (2011) provides the behavioral conditions that characterize multiplier preferences within the class of variational preferences.

For maxmin expected utility preferences, $c(p)$ is an indicator function that takes value zero inside a set of revealed priors $\tilde{\mathcal{P}} \subseteq \mathcal{P}$ and infinity otherwise. Individuals rank each alternative according to the criterion in equation (6):

$$v_{T-1}(s, a) = \min_{p \in \tilde{\mathcal{P}}} \int_S u_T(s', d_T(s'))\, dp_{T-1}(s, a). \tag{6}$$

Each model within $\tilde{\mathcal{P}}$ is evaluated according to its expected utility and the worst-case realization is assigned. Maccheroni et al. (2006a) show how a strengthening of their choice axioms restricts variational preferences to maxmin expected utility.

There is a behavioral and a cognitive interpretation for the set of revealed priors $\tilde{\mathcal{P}}$. For the former, all that Gilboa & Schmeidler (1989) provide is a representation result. Under their axioms, individuals behave as if they base their decision on a set of revealed priors. It does not necessarily relate to the information available to the individual.[7] The size of $\tilde{\mathcal{P}}$ depends on a

[7] Gajdos, Hayashi, Tallon, & Vergnaud (2008) provide a link between the revealed set of priors and the full set through an individual's attitude towards imprecise information.

combination of beliefs and ambiguity attitudes of the individual and both cannot be separated (Ghirardato & Marinacci, 2002). In the latter case, however, its size corresponds directly to the set of all priors $\mathcal{P}$ that cannot be ruled out. Then maxmin expected utility preferences are an axiomatization of Wald (1950)'s maximin decision criterion and a special case of smooth ambiguity preferences under extreme ambiguity aversion (Klibanoff et al., 2005).[8]

The extension of non-Bayesian preferences with multiple priors to the dynamic setting is not straightforward. For example, several alternative updating schemes are proposed in the literature. Under full Bayesian updating (Pires, 2002), each prior is updated according to Bayes's rule but the set of priors itself is not revised in light of new information. As an alternative, Gilboa & Schmeidler (1993) propose maximum likelihood updating, where only the prior that assigned the highest probability to the newly arrived information is retained and updated according to Bayes's rule. Epstein & Schneider (2003) axiomatize a recursive approach for multiple prior preferences with dynamic consistency. This yields the requirement that the set of revealed priors is rectangular and revised according to full Bayesian updating. The assumption of rectangularity restricts the structure of information and is best interpreted in a game-theoretic perspective, where the individual views himself as playing a zero-sum game against a malevolent nature (Cerreia-Vioglio, Maccheroni, Marinacci, & Montrucchio, 2011). Rectangularity is a form of an independence assumption where the choice of any particular prior in a state-action pair does not limit nature's other choices, i.e. the choices are uncoupled. The individual thus gains nothing from having future actions depend explicitly on past realizations of uncertainty. I will provide a formal definition when I introduce the mathematical model in the next subsection.

Following maxmin expected utility preferences, the objective of the individual is to implement the policy π^* that maximizes the expected total discounted utility over the T decision periods under a worst-case scenario. The formal representation of the individual's objective is equation (7):

$$v_1^{\pi^*}(s) = \max_{\pi \in \Pi} \left\{ \min_{\mathbf{P} \in F_1^{\pi}} \mathrm{E}_s^{\mathbf{P}} \left[\sum_{t=1}^{T} \delta^{t-1} u_t(X_t, d_t(X_t)) \right] \right\}. \tag{7}$$

$\mathcal{F}_t^{\pi}$ denotes the set of transition probability distributions induced by adopting policy π from time t onwards and so F_1^{π} clarifies that there is now a whole set from which the worst-case element determines the ranking.

[8]See Stoye (2012) for a recent synthesis of the literature on statistical and axiomatic decision theory.

2.2 Mathematical Model

The life cycle model of human capital investment under risk is set up as a standard MDP where there is a unique transition probability distribution $p_t(s, a)$ associated with each state and action. However, when analyzing decisions under risk and ambiguity there is a whole set of conditional distributions $\mathcal{P}_t(s, a)$ to consider. Recent work in operations research extends the standard MDP and allows for ambiguous future transitions. It is motivated by the sensitivity of the optimal policy to perturbations in the transition probabilities that results in a serious degradation of performance (Ben-Tal, El Ghaoui, & Nemirowski, 2009). Hence I can rely on the mathematical model of a RMDP to capture the process of human capital investments under risk and ambiguity.

The MDP is the special case of the RMDP when $\mathcal{P}_t(s, a)$ is a singleton. So I will briefly review key elements of a MDP before turning to the new issues arising in the more general setting.[9]

Let me introduce some additional notation right at the beginning. The sequence of previous states and decisions up to time t is denoted by $h_t \equiv (s_1, a_1, \ldots, s_{t-1}, a_{t-1}, s_t)$ and $\mathcal{H}_T$ is the set of all possible histories over the T decision periods.

2.2.1 Uncertainty as Risk

When making sequential decisions under risk, the task is to determine the optimal policy π^* with the largest expected total discounted utility $v_1^{\pi^*}$ as formalized in equation (2). In principle, this requires to evaluate the performance of all policies based on all possible sequences of utilities and the probability that each occurs. Fortunately, however, the multistage problem can be solved by a sequence of simpler inductively defined single-stage problems.

Let $v_t^\pi(s)$ denote the expected total discounted utility under π from period t onwards:

$$v_t^\pi(s) = \mathrm{E}_s^\pi \left[\sum_{\tau=t}^{T} \delta^{\tau-t} u_\tau(X_\tau, d_\tau(X_\tau)) \right].$$

Then $v_1^\pi(s)$ can be determined for any policy by recursively evaluating equation (8):

$$v_t^\pi(s) = u_t(s, d_t(s)) + \delta \mathrm{E}_s^p \left[v_{t+1}^\pi(X_{t+1}) \right]. \tag{8}$$

Equation (8) expresses the utility $v_t^\pi(s)$ of adopting policy π going forward as the sum of its

[9] See Puterman (1994) and White (1993) for a textbook introduction to standard Markov decision processes and Rust (1994) for a review of its use in structural estimation.

immediate utility and all expected discounted future utilities.

The principle of optimality (Bellman, 1957; Puterman, 1994) allows to construct the optimal policy π^* by solving the optimality equations for all s and t in equation (9) recursively:

$$v_t^{\pi^*}(s) = \max_{a \in A} \left\{ u_t(s,a) + \delta \mathrm{E}_s^p \left[v_{t+1}^{\pi^*}(X_{t+1}) \right] \right\}. \tag{9}$$

The value function $v_t^{\pi^*}$ is the expected discounted utility in t over the remaining time horizon assuming the optimal policy is implemented going forward.

Algorithm (1) allows to solve the MDP by a simple backward induction procedure. In the final period T, there is no future to take into account and so the optimal decision is simply to choose the alternative with the highest immediate utility in each state. With the results for the final period at hand, the other optimal decisions can be determined recursively as the calculation of their expected future utility is straightforward given the relevant transition probability distribution.

Algorithm 1 Backward Induction Algorithm for MDP

for $t = T, \ldots, 1$ **do**

  if t == T **then**

    $v_T^{\pi^*}(s) = \max_{a \in A} \left\{ u_T(s,a) \right\} \qquad \forall\, s \in S$

  else

    Compute $v_t^{\pi^*}(s)$ for each $s \in S$ by

      $v_t^{\pi^*}(s) = \max_{a \in A} \left\{ u_t(s,a) + \delta \mathrm{E}_s^p \left[v_{t+1}^{\pi^*}(X_{t+1}) \right] \right\}$

    and set

      $d_t^{\pi^*}(s) = \arg\max_{a \in A} \left\{ u_t(s,a) + \delta \mathrm{E}_s^p \left[v_{t+1}^{\pi^*}(X_{t+1}) \right] \right\}.$

  end if

end for

2.2.2 Uncertainty as Risk and Ambiguity

Iyengar (2005) and Nilim & El Ghaoui (2005) establish that, given the assumption of rectangularity of the set of transition probability distributions, key results for the standard MDP such as the Bellman recursion and optimality of deterministic Markovian decision rules carry over to the RMDP. I will provide their formal definition of rectangularity and discuss the required modifications to the evaluation of policies and construction of an optimal policy in this general setting.

The set of all transition probability distributions associated with a deterministic Markovian decision rule $d_t(s)$ is given by:

$$\mathcal{F}^{d_t} = \{\mathbf{p} : \forall s \in S, p_t(s) \in P_t(s, d_t(s))\}.$$

For every state $s \in S$, the next state can be determined by any $p_t(s) \in P_t(s, d_t(s))$.

A policy π induces a set of probability distributions $\mathcal{F}_1^\pi$ on the set of all histories $\mathcal{H}_T$. There are many possible combinations of transition probabilities that result in history h_t. The key assumption is the rectangularity of $\mathcal{F}_1^\pi$ which is defined below.

Definition 1 ***Rectangularity*** *The set $\mathcal{F}_1^\pi$ of probability distributions associated with a policy π is given by*

$$\begin{aligned}
\mathcal{F}_1^\pi &= \left\{\mathbf{P} : \forall\, h_T \in \mathcal{H}_T,\, \mathbf{P}(h_T) = \prod_{t=1}^{T} p_t(s), p_t(s) \in \mathcal{F}^{d_t}, t = 1, \ldots, T\right\} \\
&= \mathcal{F}^{d_1} \times \mathcal{F}^{d_2} \times \ldots \times \mathcal{F}^{d_T},
\end{aligned}$$

where the notation simply denotes that each element in $\mathcal{F}_1^\pi$ is a product of $p_t(s) \in \mathcal{F}^{d_t}$, and vice versa (Iyengar, 2005).

If rectangularity holds, then the principle of optimality and the optimality of deterministic Markovian decision rules apply to the RMDP as well.

The decision maker's preferences are captured by the adapted optimality criterion in equation (7). The objective is to implement a policy π^* that maximizes the expected total discounted utility under a worst-case scenario. Equation (11) presents the adapted policy evaluation equations:

$$v_t^\pi(s) = \min_{\mathbf{P} \in F_t^\pi} \mathrm{E}_s^{\mathbf{P}} \left[\sum_{\tau=t}^{T} \delta^{\tau-t} u_\tau(X_\tau, d_\tau(X_\tau))\right]. \tag{10}$$

The inductive approach to construct $v_1^\pi(s)$ requires a similar modification as the future utility is evaluated under a worst-case scenario as well.

$$v_t^\pi(s) = u_t(s, d_t(s)) + \delta \min_{p \in \mathcal{P}_t(s, d_t(s))} \mathrm{E}_s^p \left[v_{t+1}^\pi(X_{t+1})\right] \tag{11}$$

The modified optimality equations are then straightforward:

$$v_t^{\pi^*}(s) = \max_{a \in A} \left\{ u_t(s,a) + \delta \min_{p \in \mathcal{P}_t(s,a)} \mathrm{E}_s^p \left[v_{t+1}^{\pi^*}(X_{t+1}) \right] \right\}.$$

Algorithm (2) allows to solve the RMDP using an only slightly modified backward induction procedure. The key modification to the standard backward induction procedure described earlier is the minimization step to determine the future utility of a decision under a worst-case scenario. This increases the computational burden considerably.

Algorithm 2 Backward Induction Algorithm for RMDP

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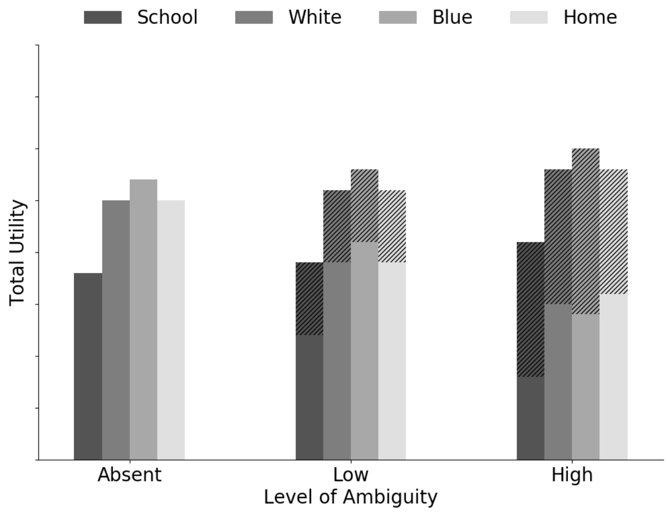
for $t = T, \ldots, 1$ **do**
  if t == T **then**
    $v_T^{\pi^*}(s) = \max_{a \in A} \left\{ u_T(s,a) \right\} \qquad \forall\, s \in S$
  else
    Compute $v_t^{\pi^*}(s)$ for each $s \in S$ by
      $v_t^{\pi^*}(s) = \max_{a \in A} \left\{ u_t(s,a) + \delta \min_{p \in \mathcal{P}_t(s,a)} \mathrm{E}_s^p \left[v_{t+1}^{\pi^*}(X_{t+1}) \right] \right\}$
    and set
      $d_t^{\pi^*}(s) = \arg\max_{a \in A} \left\{ u_t(s,a) + \delta \min_{p \in \mathcal{P}_t(s,a)} \mathrm{E}_s^p \left[v_{t+1}^{\pi^*}(X_{t+1}) \right] \right\}$
  end if
end for

Figure 3 illustrates the individual's decision problem at a state s for three different levels of ambiguity. To fix ideas and preview the Monte Carlo analysis, an individual is considering whether to work in a white- or blue-collar occupation, enroll in school, or remain at home. I plot each action's total utility for all elements of $\mathcal{P}_t(s,a)$ and each of the four possible choices under three different ambiguity scenarios. As the level of ambiguity increases, so does the number of potential conditional transition probability distributions.

In the absence of ambiguity, the individual is facing a standard decision problem under risk. Since there is only a single transition probability distribution, his decision is straightforward and he enrolls in school. However, as ambiguity is introduced there is a whole set of total utilities associated with each action as there are multiple transition probability distributions for future states. This is indicated by the dashed block which increases with the level of ambiguity. In the case of maxmin expected utility preferences, the decision is based on a comparison of worst-case evaluations. This corresponds to the bottom of each dashed block. For a low level of ambiguity the individual will still decide to enroll in school, as ambiguity increases, however,

Figure 3: Total utilities under different levels of ambiguity



he will stay at home.

Note that the choice of the preference model is crucial for the effect of ambiguity on choices. For example, in the case of subjective expected utility preferences a prior is attached to all models under consideration. If the level of ambiguity increases but only a very low belief is attached to the newly considered models, then choices might not be affected at all.

2.3 Computational Model

I now present the computational model and extend the setup of Keane & Wolpin (1994) to allow for model ambiguity. I discuss the assumptions about functional forms, the distribution of unobservables, and the ambiguity set in turn.

Individuals live for a total of T periods, are risk neutral, and make a decision about their human capital investment each period. They choose to either work in one of two occupations (a = 1, 2), attend school (a = 3), or stay at home (a = 4). The immediate utility from each alternative is the following:

$$u_t(s,a) = \begin{cases} w_{1t} = \exp\{\alpha_{10} + \alpha_{11} g_t + \alpha_{12} x_{1t} + \alpha_{13} x_{1t}^2 + \alpha_{14} x_{2t} + \alpha_{15} x_{2t}^2 + \epsilon_{1t}\} & \text{if } a = 1 \\ w_{2t} = \exp\{\alpha_{20} + \alpha_{21} g_t + \alpha_{22} x_{1t} + \alpha_{23} x_{1t}^2 + \alpha_{24} x_{2t} + \alpha_{25} x_{2t}^2 + \epsilon_{2t}\} & \text{if } a = 2 \\ \beta_0 - \beta_1 \mathbb{I}\,[\, g_t \geq 12\,] - \beta_2 (1 - \mathbb{I}\,[\, a_{t-1} = 3\,]) + \epsilon_{3t} & \text{if } a = 3 \\ \gamma_0 + \epsilon_{4t} & \text{if } a = 4. \end{cases}$$

g_t is the number of periods of schooling obtained by the beginning of period t, x_{1t} and x_{2t} are the number of periods that the individual worked in the two occupations respectively. The utility for each labor market alternative corresponds to its wage (w_{1t}, w_{2t}) and α_1 and α_2 are thus parameters associated with the wage functions. They capture the returns to schooling and occupation-specific human capital. β_0 is the consumption utility of schooling, β_1 is the post-secondary cost of schooling, and β_2 is an adjustment cost associated with returning to school. The mean utility of the home alternative is denoted γ_0. The ϵ_{at}'s are alternative-specific shocks to occupational productivity, the consumption utility of schooling, and the utility of home time. They are serially uncorrelated.

Given the structure of the utility functions and the lack of serial correlation, the state at time t is:

$$s_t = \{g_t, x_{1t}, x_{2t}, a_{t-1}, \epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}\}.$$

The observable components of s_t evolve according to the following rules:

$$\begin{aligned} x_{1,t+1} &= x_{1t} + \mathbb{I}\,[\,a_t = 1\,] \\ x_{2,t+1} &= x_{2t} + \mathbb{I}\,[\,a_t = 2\,] \\ g_{t+1} &= g_t \ \ + \mathbb{I}\,[\,a_t = 3\,]. \end{aligned}$$

The transitions of all observable components of s_t are deterministic. However, there is uncertainty about the realization of its unobservable components. I assume that all unobservable components are jointly normally distributed with mean zero and covariance matrix $\mathbf{\Sigma}$.

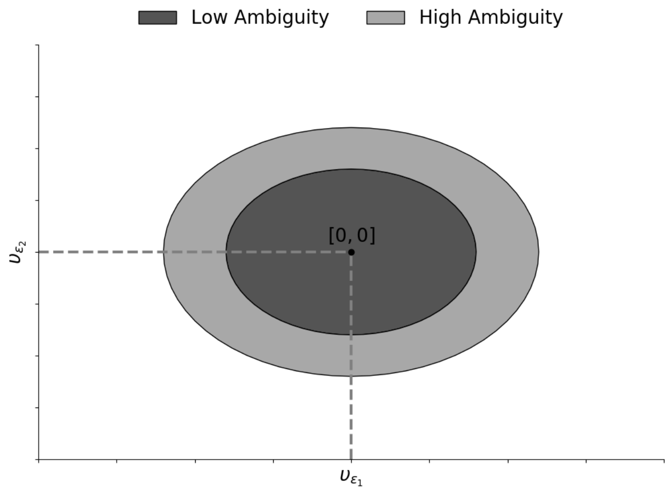
$$[\epsilon_{1t}, \epsilon_{2t}, \epsilon_{3t}, \epsilon_{4t}]^T \sim \mathcal{N}_0(\mathbf{0}, \mathbf{\Sigma})$$

Individuals face ambiguity about the true model and consider a set of normal distributions with unknown mean for the shocks to labor market productivity $\upsilon = (\upsilon_{\epsilon_1}, \upsilon_{\epsilon_2})$ when making their decisions. The set of admissible distributions is centered around the true model $\mathcal{N}_0$ and constrained by the Kullback-Leibler divergence.

$$\mathcal{E} = \{\mathcal{N} : R(\mathcal{N} \parallel \mathcal{N}_0) \leq \eta\}$$

The size of the ambiguity set $\mathcal{E}$ is governed by η and individuals ignore all distributions outside of it. This only weakens the assumption of rational expectations as the true model still serves as the reference. If η is equal to zero, then the set is a singleton and there is no model ambiguity. This is the original version of the model in Keane & Wolpin (1994) for human capital investments under risk. Here, however, ambiguity about the distribution of future shocks translates into ambiguous future transition probability distributions $p_t(s, a)$. Even though I only introduce ambiguity about the shocks to labor market productivity, there is a set of transition

Figure 4: Set of admissible means for shock distribution



probability distributions associated with all alternatives. The total utility of each alternative accounts for the possibility of working in the labor market in the future. The induced set of transition probability distributions associated with each policy is rectangular as the ambiguity set is unchanged over time, making the RMDP a suitable mathematical model for the analysis.

Figure 4 shows the set of admissible υ for alternative sizes of the ambiguity set. The black dot in the middle indicates the situation under risk, where there is just a single point $[0, 0]$ to consider. More generally, individuals consider all points within the set and base their decision on the worst possible realization υ^*. While the ambiguity set is always the same for all combinations of states and actions, the relevant worst-case realization will in general be different. For example, consider an individual that has worked in the first occupation in most periods, thus making continued employment in that occupation very likely due to positive returns to occupation-specific experience. The worst-case scenario for such an individual involves low future wages in the first occupation, thus υ^* will tend to involve a low value for future υ_{ϵ_1}. The opposite is true for an individual with long tenure in the second occupation.

When entering the model, individuals have no labor market experience ($x_{11} = x_{21} = 0$) but ten years of schooling ($g_1 = 10$). The idea is that individuals are about age 16 when entering the model in the first period and start out identically, different choices over the life cycle are then simply the cumulative effects of different shocks.

2.4 Estimation

The estimation of the model is standard and based on maximum simulated likelihood (Fisher, 1922; Manski & Lerman, 1977). Under all probability models the individual considers, the reference model $\mathcal{N}_0$ best describes the variability in the data. Thus, $\mathcal{N}_0$ is the relevant distribution for the shocks that are integrated out to construct the choice and wage probabilities.[10]

3 Monte Carlo Exploration

I now showcase the new economic insights generated by a life cycle model of robust human capital investment. I simulate a parameterized version of the computational model, provide a brief description of its key economic features, and present some basic descriptive statistics of the simulated sample. Then I investigate the comparative statics of the model by varying the size of the ambiguity set and study the modeling trade-off between model ambiguity and psychic cost. I conclude by quantifying the effect of model misspecification on psychic cost estimates and the ex ante evaluation of tuition policies.

3.1 Baseline

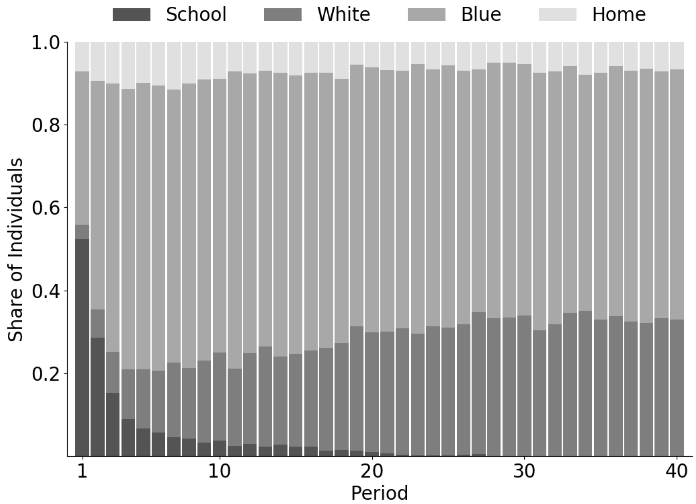
Keane & Wolpin (1994) outline a life cycle model of human capital investment under risk, so I rely on their parameterization as a baseline but instill individuals with an additional fear of model misspecification. The returns to schooling differ considerably between the two occupations. Schooling increases wages by only 4% in the first occupation compared to 8% in the second. I will thus refer to the former as blue-collar and the latter as white-collar going forward. Starting wages are considerably lower in the white-collar sector but wages increase more rapidly with occupation-specific experience compared to blue-collar wages. Own-work experience is highly valuable in both occupations. However, while white-collar wages increase with blue-collar experience as well, the opposite is not true. There is a consumption value of schooling of \$5,000 but the total cost of pursuing post-secondary education is considerable and amounts to \$5,000. Once leaving school, individuals incur a nearly prohibitive cost of \$15,000 for re-enrolling. Individuals are forward-looking but the future is discounted by 5%. The random shocks are not correlated across alternatives. Further details about the parameterization are available in Appendix 1.

I simulate the life cycle histories of 1,000 individuals for 40 periods. Figure 5 shows the share of individuals choosing each of the four alternatives by period. Initially, roughly 52% of individuals are enrolled in school but this share declines rapidly and only 19% attain any post-secondary education. Right away, about 35% of individuals are working in the blue-collar occupation. Blue-collar employment initially increases even further to peak at 67% as individuals are leaving school and entering the labor market. White-collar employment rises constantly over the life

[10]See Appendix C for further details about the computational implementation.

cycle but never reaches more than 35%. About 5% of individuals stay at home each period.

Figure 5: Choice patterns for baseline scenario



3.2 Comparative Statics

I now study the comparative statics of the model for three ambiguity scenarios. The size of the ambiguity set itself is hard to interpret and so I first compute for each of the scenarios the loss in expected total discounted utility for an individual entering the model compared to a world without any model ambiguity. Table 1 reports the results.

As a baseline, η is set to 0.01 which corresponds to an expected loss of about 5%. For a scenario with high ambiguity, I double η which results in an increase in the expected loss by only 1%. So increases in the size of the ambiguity set do not necessarily translate into a proportional loss in utility as individuals are able to adjust their decisions to the new situation. In addition, I also set up a scenario where any concern for model misspecification is removed from the individuals' decision problem by setting η to zero. This is the model of human capital investment under risk. I will label the three ambiguity scenarios as baseline, high, and absent going forward.

Table 1: Expected Utility Loss

Scenario	η	Loss
Absent	0.00	—
Baseline	0.01	5 %
High	0.02	6 %

Table 2 presents the average years an individual spends in each of the four alternatives. In the baseline scenario, individuals attend school for an additional 1.5 years and thus end up with roughly 11.5 years of schooling in total. They work in white-collar jobs for about 10 years. In the remaining 28 years, individuals take up blue-collar employment for 25 years and spend another 3 years at home. As ambiguity increases, schooling decreases as the incentive to invest is reduced by the lower expected worst-case wages in the labor market. Employment also shifts towards blue-collar jobs as these are less schooling intensive. The opposite is true for a decrease in ambiguity. Here, average education increases by roughly eight months.

Table 2: Effect of Ambiguity Set

Scenario	School	White	Blue	Home
Absent	12.29	11.77	23.44	2.50
Baseline	11.55	10.26	25.19	3.00
High	11.39	9.94	25.52	3.15

As mentioned before, the implication of this exercise depends on whether one assigns a cognitive or behavioral interpretation to the set of revealed priors $\tilde{\mathcal{P}}$. Following the cognitive interpretation, its increase corresponds directly to an increase in model ambiguity. In the behavioral interpretation, $\tilde{\mathcal{P}}$ represents a combination of ambiguity and the individual's attitude towards it and thus an increase can be the result of changes in both.

I now turn to the trade-off between model ambiguity and psychic cost of schooling. I show that by taking model ambiguity into account when studying the mechanisms driving schooling decisions, I can provide a more appealing economic interpretation for low enrollment in light of high returns than just large psychic cost.

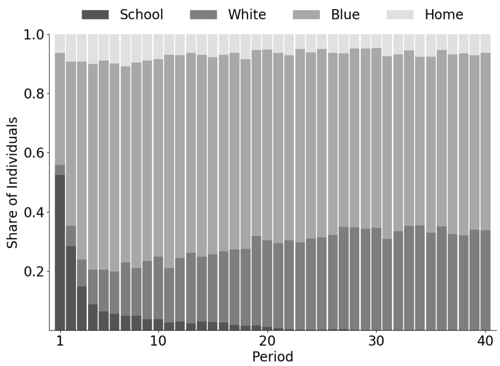
The parameter β_1 captures the total cost of pursuing any post-secondary education which includes the direct cost (e.g. tuition, room and board) as well as psychic cost (e.g. effort). Large psychic cost and model ambiguity both decrease the incentive to invest in schooling. But they do so for very different reasons. Large psychic cost decreases the immediate utility from attending school, while model ambiguity reduces the expected future utility due to the depressed labor market prospects.

I first look at combinations of η and β_1 that result in very similar observed life cycle histories. For each of the ambiguity scenarios, I run a simple grid search for β_1 that minimizes the root-mean-square error (RMSE) compared to the baseline scenario across all choice probabilities.

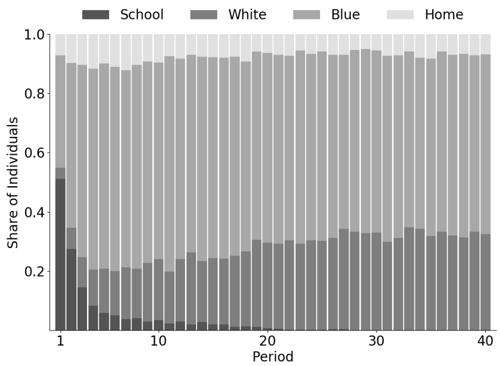
The RMSE is at most 0.01 in all cases and so the choice probabilities differ by less than 1% on average across the four alternatives in all periods.

Figure 6 shows the resulting choice patterns for a simulated sample of 1, 000 individuals. From the perspective of the econometrician they all appear very similar to the baseline sample, however, the underlying economics are quite different. In the left figure, individuals do not face any ambiguity about the model, but the psychic cost of post-secondary education are much larger compared to the baseline. The opposite is true for the right figure, here concerns about model misspecification are exacerbated but the psychic cost is lower.

Figure 6: Choice patterns for alternative scenarios



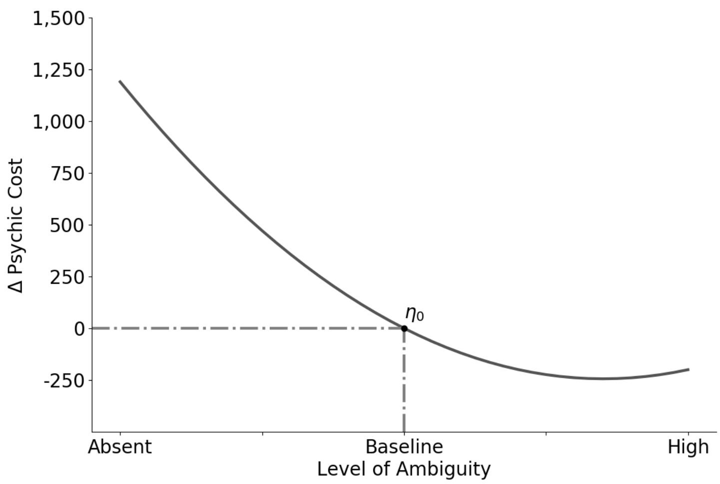
(a) Absent



(b) High

Figure 7 plots the different parameterizations (η, β_1) for the three scenarios. In the absence of any model ambiguity, the psychic cost needs to increase by roughly \$1,200. Faced with high model ambiguity, a decrease of about \$200 is required.

Figure 7: Modeling trade-off between psychic cost and model ambiguity



3.3 Model Misspecification

I now turn to the consequences of actually estimating a life cycle model of human capital investment under risk on a sample of individuals that make their decisions in light of risk and ambiguity.

As motivated in the beginning, the estimated total cost $\hat{\beta}_1$ of post-secondary education is usually considerably larger than the relevant direct cost. Thus a major role is attributed to psychic cost in driving schooling decisions. However, this might be due to model misspecification. If individuals make the schooling investment facing risk and ambiguity but econometricians analyze their decisions with a model of decision-making under risk, then this model misspecification shows up as a large psychic cost estimate.

I estimate a static and a (dynamic) risk model on the baseline sample of 1,000 individuals. There is no individual concern about model misspecification in both estimated models and so η is zero but the discount factor δ is specified as 0.00 for the static model and 0.95 for the risk model.

I start the estimations with all other parameter values from the baseline parameterization and allow for 40,000 evaluations of the criterion function. Based on the results, I simulate a new sample of 1,000 individuals to assess the in-sample fit. The observed choices from the estimated models are very similar to the baseline sample. The RMSE is small with 0.02 for the static model and 0.01 for the risk model.

I initially focus on the resulting psychic cost estimate and then turn to the consequences for the ex ante evaluation of tuition policies. I only discuss selected aspects of the estimation in the text. Additional material is available in Appendix A.2.

3.3.1 Psychic Cost

Table 3 shows the estimation results for the parameters of the immediate utility from attending school. In a static setting, the consumption value of education is much larger as there is no investment motive and so a much higher immediate utility is needed to result in the initially high enrollment in school. Overall, results from the risk model are much more in line with the true underlying parameters. However, the psychic cost estimate is increased by about \$160. The cost of re-enrollment remains prohibitive in either case as individuals in the baseline sample simply do not return to school once they left.

Table 3: Schooling Parameters

Parameter	Static	Risk	Correct
β_0	17,823.01	5,495.60	5,000.00
β_1	-644.16	5,159.35	5,000.00
β_2	37,439.73	18,464.96	15,000.00

3.3.2 Policy Evaluation

I now turn to the effect of model misspecification for the ex ante evaluation of alternative tuition policies. Following Keane & Wolpin (1994), I evaluate the effect of a tuition subsidy of \$500, \$1,000, and \$2,000. I simulate a sample of 1,000 individuals based on the estimation results for both misspecified models but decrease $\hat{\beta}_1$ according to the tuition subsidy.

In the case of decision-making under risk, tuition subsidies simply increase the immediate utility from schooling by the amount of the subsidy and decisions are revised based on expected utility calculations. When making decisions under risk and ambiguity based on maxmin expected utility preferences, then there is a second effect as individuals are adjusting their worst-case evaluation as well.

Table 4 compares the result from the misspecified models against the correct one. I report the average increase in years an individual spends in each of the four alternatives under the different policy regimes. As expected, average schooling increases with the size of the subsidy, employment shifts into the more schooling-intensive white-collar occupation, and home time decreases.

For the static model, there is only a very limited effect of any tuition subsidy. Even a subsidy of \$2,000 only increases average schooling by 0.11 years. This results from the myopic decision-making by individuals who only account for the subsidy on a year-to-year basis and ignore its cumulative effect over time. If individuals take the future into account, then the response is much more pronounced and even a \$500 subsidy has a larger effect than \$2,000 in a static setting. In a model of decision-making under risk, a subsidy of \$1,000 increases average schooling by about half a year. When doubling the size of the subsidy to \$2,000, the effect is nearly tripled. However, all estimated policy effects are lower compared to the correct model of decision-making under risk and ambiguity.

Table 4: Effect of Tuition Subsidy

Static				
	School	White	Blue	Home
Baseline	11.58	25.13	10.17	3.12
\$500	+0.02	+0.01	-0.02	-0.00
\$1,000	+0.05	+0.06	-0.10	-0.01
\$2,000	+0.11	+0.11	-0.19	-0.03
Risk				
	School	White	Blue	Home
Baseline	11.71	24.27	11.13	2.89
\$500	+0.19	+0.46	-0.53	-0.12
\$1,000	+0.52	+1.00	-1.25	-0.28
\$2,000	+1.43	+2.56	-3.31	-0.68
Correct				
	School	White	Blue	Home
Baseline	11.55	25.19	10.26	3.00
\$500	+0.26	+0.55	-0.65	-0.16
\$1,000	+0.63	+1.30	-1.57	-0.36
\$2,000	+1.87	+3.50	-4.45	-0.92

Thus, estimating a misspecified model of human capital investment does result in flawed policy conclusions. Here, the responsiveness of average schooling to tuition subsidies is underestimated.

4 Conclusion

The uncertainties involved in human capital investments are pervasive. I extended the standard life cycle model of human capital investment to capture a richer notion of uncertainty that includes ambiguity about the true model in addition to risk within a given model. I reviewed the main approaches to decision-making under risk and ambiguity. I identified maxmin expected utility preferences (Gilboa & Schmeidler, 1989; Epstein & Schneider, 2003) that provide an axiomatization for Wald (1950)'s maximin model as the intersection between the literature on decision theory in economics and robust optimization in operations research. I thus studied human capital investment through the lens of a robust Markov decision process and explored the economics of a simulated sample based on the canonical model of Keane & Wolpin (1994). I initially focused on the modeling trade-off between psychic cost and model ambiguity in isolation and then analyzed the consequences of estimating a misspecified model of decision-making under risk on a sample of individuals that make their investment decisions facing ambiguity as well. I found that psychic cost are overestimated and I am thus able offer an explanation for the large psychic cost estimates reported in the existing empirical literature. In addition, this also affects the ex ante evaluation of tuition policies as their impact on average schooling is understated.

The next step is to tackle the computational challenges of solving larger robust Markov decision problems and draw on the research summarized in Bertsimas, Brown, & Caramanis (2011) who review different specifications of the ambiguity set in light of their computational tractability. This will pave the way to analyze more elaborate models of human capital investment as presented in Keane & Wolpin (1997). Estimation of such a model on a suitable panel dataset will then allow to, for example, quantify the level ambiguity faced by individuals when making their human capital investments.

References

Abbott, B., Gallipoli, G., Meghir, C., & Violante, G. L. (2018). Education policy and intergenerational transfers in equilibrium. *Journal of Political Economy*, forthcoming.

Al-Najjar, N. I., & Weinstein, J. (2009a). The ambiguity aversion literature: A critical assessment. *Economics and Philosophy*, *25*(3), 249-284.

Al-Najjar, N. I., & Weinstein, J. (2009b). Rejoinder: "The ambiguity aversion literature: A critical assessment". *Economics and Philosophy*, *25*(3), 357-369.

Anscombe, F. J., & Aumann, R. J. (1963). A definition of subjective probability. *The Annals of Mathematical Statistics*, *34*(1), 199–205.

Arrow, K. J. (1951). Alternative approaches to the theory of choice in risk-taking situations. *Econometrica*, *19*(4), 404–437.

Becker, G. S. (1964). *Human capital* (1st ed.). New York City, NY: Columbia University Press.

Bellman, R. E. (1957). *Dynamic programming* (1st ed.). Princeton, NJ: Princeton University Press.

Ben-Tal, A., El Ghaoui, L., & Nemirowski, A. (2009). *Robust optimization* (1st ed.). Princeton, NJ: Princeton University Press.

Bernoulli, D. (1738). Specimen theoriae novae de mensura sortis. *Commentarii academiae scientiarum imperialis Petropolitanae*, *5*, 175–192.

Bertsimas, D., Brown, D. B., & Caramanis, C. (2011). Theory and applications of robust optimization. *SIAM Review*, *53*(3), 464–501.

Carneiro, P., Hansen, K. T., & Heckman, J. J. (2003). 2001 Lawrence R. Klein lecture: Estimating distributions of treatment effects with an application to the returns to schooling and measurement of the effects of uncertainty on college choice. *International Economic Review*, *44*(2), 361–422.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., & Montrucchio, L. (2011). Uncertainty averse preferences. *Journal of Economic Theory*, *146*(4), 1275–1330.

Chan, M. K. (2017). Welfare dependence and self-control: An empirical analysis. *Review of Economic Studies*, *84*(4), 1379–1423.

Cunha, F., Heckman, J. J., & Navarro, S. (2005). Separating uncertainty from heterogeneity in life cycle earnings. *Oxford Economic Papers*, *57*(2), 191–261.

Danan, E., Gajdos, T., Hill, B., & Tallon, J.-M. (2016). Robust social decisions. *American Economic Review*, *106*(9), 2407–2425.

Dow, J., & da Costa Werlang, S. R. (1992). Uncertainty aversion, risk aversion, and the optimal choice of portfolio. *Econometrica*, *60*(1), 197–204.

Eisenhauer, P., Heckman, J. J., & Mosso, S. (2015). Estimation of dynamic discrete choice models by maximum likelihood and the simulated method of moments. *International Economic Review*, *56*(2), 331–357.

Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms. *The Quarterly Journal of Economics*, *75*(4), 643–669.

Epstein, L. G. (1999). A definition of uncertainty aversion. *Review of Economic Studies*, *66*(3), 579-608.

Epstein, L. G. (2010). A paradox for the "smooth ambiguity" model of preference. *Econometrica*, *78*(6), 2085-2099.

Epstein, L. G., & Schneider, M. (2003). Recursive multiple priors. *Journal of Economic Theory*, *113*(1), 1–31.

Fang, H., & Silverman, D. (2009). Time-inconsistency and welfare program participation: Evidence from the NLSY. *International Economic Review*, *50*(4), 1043–1077.

Fang, H., & Yang, W. (2015). Estimating dynamic discrete choice models with hyperbolic discounting, with an application to mammography decisions. *International Economic Review*, *56*(2), 565–596.

Fisher, R. A. (1922). On the mathematical foundations of theoretical statistics. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, *222*(594-604), 309–368.

Frederick, S., Loewenstein, G., & O'Donoghue, T. (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature*, *40*(2), 351-401.

Gajdos, T., Hayashi, T., Tallon, J. M., & Vergnaud, J. C. (2008). Attitude toward imprecise information. *Journal of Economic Theory*, *140*(1), 27–65.

Ghirardato, P., & Marinacci, M. (2002). Ambiguity made precise: A comparative foundation. *Journal of Economic Theory*, *102*(2), 251–289.

Gilboa, I. (2009). *Theory of decision under uncertainty* (1st ed.). New York City, NY: Cambridge University Press.

Gilboa, I., & Marinacci, M. (2013). Ambiguity and the Bayesian paradigm. In D. Acemoglu, M. Arellano, & E. Dekel (Eds.), *Advances in economics and econometrics: Theory and applications* (1st ed., Vols. 1: Economic Theory, Tenth World Congress, pp. 179–242). New York City, NY: Cambridge University Press.

Gilboa, I., Postlewaite, A. W., & Schmeidler, D. (2008). Probability and uncertainty in economic modeling. *Journal of Economic Perspectives*, *22*(3), 173–188.

Gilboa, I., Postlewaite, A. W., & Schmeidler, D. (2009). Is it always rational to satisfy Savage's axioms? *Economics & Philosophy*, *25*(3), 285–296.

Gilboa, I., Postlewaite, A. W., & Schmeidler, D. (2012). Rationality of belief or: Why Savage's axioms are neither necessary nor sufficient for rationality. *Synthese*, *187*(1), 11–31.

Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, *18*(2), 141–153.

Gilboa, I., & Schmeidler, D. (1993). Updating ambiguous beliefs. *Journal of Economic Theory*, *59*(1), 33–49.

Hai, R., & Heckman, J. J. (2017). Inequality in human capital and endogenous credit constraints. *Review of Economic Dynamics*, *25*, 4–36.

Hansen, L. P., & Sargent, T. J. (2001). Robust control and model uncertainty. *American Economic Review*, *91*(2), 60–66.

Hansen, L. P., & Sargent, T. J. (2016). *Robustness* (Reprint ed.). Princeton, NJ: Princeton University Press.

Hansen, L. P., Sargent, T. J., & Tallarini Jr., T. D. (1999). Robust permanent income and pricing. *Review of Economic Studies*, *66*(4), 873–907.

Ilut, C. L., & Schneider, M. (2014). Ambiguous business cycles. *American Economic Review*, *104*(8), 2368–2399.

Iyengar, G. N. (2005). Robust dynamic programming. *Mathematics of Operations Research*, *30*(2), 257–280.

Keane, M. P., & Wolpin, K. I. (1994). The solution and estimation of discrete choice dynamic programming models by simulation and interpolation: Monte Carlo evidence. *Review of Economics and Statistics*, *76*(4), 648–672.

Keane, M. P., & Wolpin, K. I. (1997). The career decisions of young men. *Journal of Political Economy*, *105*(3), 473–522.

Klibanoff, P., Marinacci, M., & Mukerji, S. (2005). A smooth model of decision making under ambiguity. *Econometrica*, *73*(6), 1849–1892.

Klibanoff, P., Marinacci, M., & Mukerji, S. (2009). Recursive smooth ambiguity preferences. *Journal of Economic Theory*, *144*(3), 930–976.

Klibanoff, P., Marinacci, M., & Mukerji, S. (2012). On the smooth ambiguity model: A reply. *Econometrica*, *80*(3), 1303-1321.

Koopmans, T. C. (1960). Stationary ordinal utility and impatience. *Econometrica*, *28*(2), 287–309.

Kullback, S., & Leibler, R. A. (1951). On information and sufficiency. *The Annals of Mathematical Statistics*, *22*(1), 79–86.

Lucas, R. E. (1972). Expectations and the neutrality of money. *Journal of Economic Theory*, *4*(2), 103-124.

Maccheroni, F., Marinacci, M., & Rustichini, A. (2006a). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, *74*(6), 1447–1498.

Maccheroni, F., Marinacci, M., & Rustichini, A. (2006b). Dynamic variational preferences. *Journal of Economic Theory*, *128*(1), 4–44.

Machina, M. J., & Schmeidler, D. (1992). A more robust definition of subjective probability. *Econometrica*, *60*(4), 745–780.

Manski, C. F. (2009). The 2009 Lawrence R. Klein Lecture: Diversified treatment choice under ambiguity. *International Economic Review*, *50*(4), 1013-1041.

Manski, C. F., & Lerman, S. R. (1977). The estimation of choice probabilities from choice based samples. *Econometrica*, *45*(8), 1977–1988.

Marinacci, M. (2015). Model uncertainty. *Journal of the European Economic Association*, *13*(6), 1022–1100.

Marshall, A. (1890). *Principles of economics* (1st ed.). London, United Kingdom: Macmillan and Co., Ltd.

Meyer, J. (2014). The theory of risk and risk aversion. In M. J. Machina & K. Viscusi (Eds.), *Handbook of the economics of risk and uncertainty* (1st ed., Vol. 1, pp. 99–133). Amsterdam, Netherlands: North-Holland.

Muth, J. F. (1961). Rational expectations and the theory of price movements. *Econometrica*, *29*(3), 315–335.

Nilim, A., & El Ghaoui, L. (2005). Robust control of Markov decision processes with uncertain transition matrices. *Operations Research*, *53*(5), 780–798.

Pires, C. P. (2002). A rule for updating ambiguous beliefs. *Theory and Decision*, *53*(2), 137–152.

Puterman, M. L. (1994). *Markov decision processes: Discrete stochastic dynamic programming* (1st ed.). New York City, NY: John Wiley & Sons, Inc.

Rust, J. (1994). Structural estimation of Markov decision processes. In R. Engle & D. McFadden (Eds.), *Handbook of econometrics* (1st ed., Vol. 4, pp. 3081–3143). Amsterdam, Netherlands: North-Holland.

Samuelson, P. A. (1937). A note on measurement of utility. *Review of Economic Studies*, *4*(2), 155–161.

Savage, L. J. (1954). *The foundations of statistics* (1st ed.). New York City, NY: John Wiley & Sons, Inc.

Stoye, J. (2012). New perspective on statistical decisions under ambiguity. *Annual Review of Economics*, *4*, 257–282.

Strzalecki, T. (2011). Axiomatic foundations of multiplier preferences. *Econometrica*, *79*(1), 47–73.

Trautmann, S. T., & van de Kuilen, G. (2015). Ambiguity attitudes. In G. Keren & G. Wu (Eds.), *The Wiley Blackwell handbook of judgment and decision making* (1st ed., pp. 89–116). Oxford, United Kingdom: Wiley-Blackwell.

von Neumann, J., & Morgenstern, O. (1944). *Theory of games and economic behavior* (1st ed.). Princeton, NJ: Princeton University Press.

von Neumann, J., & Morgenstern, O. (1947). *Theory of games and economic behavior* (2nd ed.). Princeton, NJ: Princeton University Press.

Wald, A. (1950). *Statistical decision functions* (1st ed.). New York City, NY: John Wiley & Sons, Inc.

White, D. J. (1993). *Markov decision processes* (1st ed.). New York City, NY: John Wiley & Sons, Inc.

Yaari, M. E. (1969). Some remarks on measures of risk aversion and on their uses. *Journal of Economic Theory*, *1*(3), 315–329.

Appendix

Robust Human Capital Investment under Risk and Ambiguity

Philipp Eisenhauer

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Contents

A. Additional Information

This section presents additional background information.

A.1. Parameterization

Table 1 presents the full parameterization for the simulation of the baseline sample. Keane & Wolpin (1994) analyze three different parameterizations of the model. My analysis is based on their second parameterization as the cost of post-secondary education is set to zero in their first parameterization.

Table 1: Parameterization

Parameter	Value
η	0.01000
δ	0.95000
α_{10}	9.21000
α_{11}	0.04000
α_{12}	0.03300
α_{13}	-0.00050
α_{14}	0.00000
α_{15}	0.00000
α_{20}	8.20000
α_{21}	0.08000
α_{22}	0.02200
α_{23}	-0.00050
α_{24}	0.06700
α_{25}	-0.00100
β_0	5,000.00000
β_1	5,000.00000
β_2	15,000.00000
γ_0	14,500.00000
σ_{11}	0.16000
σ_{12}	0.00000
σ_{13}	0.00000
σ_{14}	0.00000
σ_2	0.25000
σ_{23}	0.00000
σ_{24}	0.00000
σ_{33}	36,000,000.00000
σ_{34}	0.00000
σ_{44}	36,000,000.00000

A.2. Estimations

Table 2 presents the estimation results for the two misspecified models. I use the baseline parameterization for the starting values of the free parameters. I allow for a total of 40,000 evaluations of the criterion function as further increases do not result in any meaningful improvements.

Table 2: Results

Parameter	Static	Risk
η	0.00000	0.00000
δ	0.00000	0.95000
α_{10}	9.42278	9.25884
α_{11}	0.02497	0.03602
α_{12}	0.02655	0.03056
α_{13}	-0.00031	-0.00043
α_{14}	-0.00682	0.00050
α_{15}	0.00050	-0.00008
α_{20}	8.26425	8.17113
α_{21}	0.08172	0.07879
α_{22}	0.02580	0.02379
α_{23}	-0.00059	-0.00054
α_{24}	0.06540	0.06727
α_{25}	-0.00118	-0.00101
β_0	17,823.00590	5,495.59528
β_1	-644.16468	5,159.35308
β_2	37,439.72955	18,464.95669
γ_0	6,720.32756	14,531.36764
σ_{11}	0.16426	0.16309
σ_{12}	0.01724	-0.01841
σ_{13}	0.01092	0.43065
σ_{14}	-0.00884	0.00559
σ_2	0.20005	0.26478
σ_{23}	0.00899	-0.04709
σ_{24}	0.00942	-0.00074
σ_{33}	332,622,917.52608	64,210,957.42185
σ_{34}	-2,260.00760	-71.40152
σ_{44}	93,961,068.27906	38,510,538.37437
Criterion	46.49548	45.40257
RMSE	0.02188	0.01303

Notes: RMSE = root-mean-square error; Criterion = value of criterion function at parameter values.

A.3. Psychic Cost

I present a list of papers and small excerpts that identify psychic cost as a driving force of schooling decisions.

Abbott, B., Gallipoli, G., Meghir, C., & Violante, G. L. (2018). Education policy and intergenerational transfers in equilibrium. *Journal of Political Economy,* forthcoming.

- *"Non-cognitive skills appear to be essential in reducing the costs of high school attendance, whereas cognitive skills are less important at that stage. Idiosyncratic preference shocks are also an important component of high school costs. However, conditional on gender, they only explain 8% of the variance of **psychic costs** of high school and 17% of that for college. Thus most of the variance in **psychic costs** of schooling is explained by cognitive and non-cognitive skills, the former being more important for college and the latter for high school."*
- *"The overall consumption value of **psychic costs** is substantial, especially when abilities are low. Among students with median abilities and education preference shocks, college and high-school **psychic costs** added together are worth about $135,000 on average, in year 2000 consumption terms. However, greater ability reduces these costs: a one standard deviation increase in cognitive skills reduced these costs by the equivalent of $36,000; the corresponding reduction for a change in non-cognitive skills is $30,000. Interestingly, for the very high skilled these costs can even be negative: for example, a student two standard deviations above the mean in cognitive and non-cognitive skills as well as education preferences would gain a utility equivalent to $41,000 of consumption by completing both high school and college. The average **psychic costs** that we compute are comparable in magnitude to those reported in Cunha, Heckman, & Navarro (2005) and Heckman, Lochner, & Todd (2006)."*

Carneiro, P., Hansen, K. T., & Heckman, J. J. (2003). Estimating distributions of treatment effects with an application to the returns to schooling and measurement of the effects of uncertainty on college choice. *International Economic Review, 44*(2), 361-422.

- *"As shown in Table 7 once we account for **psychic benefits or costs** of attending college (P) relative to attending high school, only 8% of college graduates regret going to college. This suggests a substantial part of the gain to college is due to nonpecuniary components. [⋯] Uncertainty in gains to schooling is substantial but knowledge of this uncertainty has a very small effect on the choice of schooling because the variance of gains is so much smaller than the variance of **psychic costs** or benefits, and it is the latter that drives most of the heterogeneity in schooling decisions. In addition, there is uncertainty about the level of both college and high school earnings."*

Cunha, F., & Heckman, J. J. (2008). A new framework for the analysis of inequality. *Macroeconomic Dynamics, 12*(S2), 315-354.

- *"The average contribution of ability to costs is positive for high school graduates (a true cost). It is negative for college graduates, so it is perceived as a benefit. This is the answer to our puzzle: people do not only (or even mainly) make their schooling decisions by looking at their monetary returns in terms of earnings. **Psychic costs** play a very important role. Differences in ability are one force behind this result."*

Cunha, F., Heckman, J. J. & Navarro, S. (2005). Separating uncertainty from heterogeneity in life cycle earnings. *Oxford Economic Papers, 57*(2), 191-261.

- *"In the human capital literature, a conventional maintained assumption used when computing rates of return from measured earnings data is that direct costs are only a small fraction of total earnings (see Heckman, Lochner, & Todd (2004)). Our evidence casts doubt on the validity of this assumption. **Psychic costs** (including expectational forecast errors) are a sizeable component of the net return, and they explain why agents who face high gross returns do not go to college. Ignoring direct costs overstates the rates of return. The existence of large ex post returns that could be realized by high school students who do not attend college are attributable in our model to **psychic costs** and expectational errors in some unknown proportion."*

Ehrenberg, R. G., & Smith, R. S. (2016). *Modern labor economics: Theory and public policy* (12th ed.). New York City, NY: Routledge.

- *"Beyond ability, however, economists have begun to recognize that "peer effects" can alter the **psychic costs** of attending school. If status with one's peers is enhanced by studying hard and getting good grades, the costs of studying are reduced - while the opposite occurs if status is reduced by academic achievement."*
- *"We saw in chapter 9 that the costs of learning a new skill are not the same for everyone; those who learn most quickly tend to have the lowest **psychic costs** of learning. The abrupt need to learn new skills, combined with differential learning costs across workers, generated two sources of greater inequality."*
- *"Second, a large part of the **costs** of migration is **psychic** - the losses associated with giving up friends, community ties, and the benefits of knowing one's way around. As we grow older, our ties to the community become stronger and the losses associated with leaving loom larger."*

Eisenhauer, P., Heckman, J. J. & Mosso, S. (2015). Estimation of dynamic discrete choice models by maximum likelihood and the simulated method of moments. *International*

Economic Review, 56(2), 331-357.

- *"**Psychic costs** are crucial determinants of net returns. For example, the median gross return for early and late college enrollment is positive, while the median net return is negative in both cases. As only agents with a positive net return choose to continue their education, this follows directly from our estimates (and the data) as more than half of the agents that are faced with the decision to enroll in college refrain from doing so."*
- *"At least 20% of agents have negative schooling costs in most states. They experience psychic benefits. For high school graduation, even the average cost is negative. **Psychic costs** play a dominant role in explaining schooling decisions. This is an unsatisfactory feature of the models in the literature."*

Hai, R., & Heckman, J. J. (2017). Inequality in human capital and endogenous credit constraints. *Review of Economic Dynamics, 25,* 4-36.

- *"Our paper extends the existing literature in other dimensions by analyzing how cognitive and noncognitive ability affects choices through* (*i*) ***psychic costs** of working and schooling;* (*ii*) *the technology of human capital production; and* (*iii*) *discount factors."*
- *"The psychic benefit of schooling is higher for individuals with higher cognitive and noncognitive abilities. Individuals whose parents have higher levels of education also have higher flow utility from schooling. The parameter estimates suggest that the **psychic cost** of working part-time while in school* ($\phi_{k,e} = -0.0773$) *is higher than the **psychic cost** of part-time working while not in school* ($\phi_{k,0} = -0.0332$)*. In our estimation, we do not impose any restrictions on the relative magnitudes of these two parameters. Forcing the **psychic cost** of working part-time while in school to be the same as the **psychic cost** of part-time working while not in school overpredicts the fraction of working students in school."*

Heckman, J. J., Lochner, L. J., & Todd, P. E. (2006). Earnings functions, rates of return and treatment effects: The Mincer equation and beyond. In E. A. Hanushek & F. Welch (Eds.), *Handbook of the economics of education* (Vol. 1, pp. 307-458). Amsterdam, Netherlands: North-Holland.

- *"**Psychic costs** play a very important role. More able people have lower **psychic costs** of attending college. The high estimated **psychic cost** is one reason why the rates of return that ignore **psychic costs** (and tuition) discussed in Section 4 are so high. This high **psychic cost** is one explanation why the college attendance rate is so low when the monetary returns are so high. One convention in the classical human capital literature - that tuition and **psychic costs** are negligible - is at odds with this* *evidence. The evidence against strict income maximization is overwhelming."*

- *"However, explanations based on **psychic costs** are intrinsically unsatisfactory. One can rationalize any economic choice data by an appeal to **psychic costs**. Heckman, Stixrud, & Urzua (2006) show the important role played by noncognitive skills as well as cognitive skills in explaining schooling (and other decisions). They show how, in principle, conventional risk aversion, time preference and leisure preference parameters can be related to psychometric measures of cognitive and noncognitive skills. Establishing this link will provide a better foundation for understanding what **psychic costs** actually represent."*
- *"**Psychic costs** may also be a stand in for credit constraints and risk aversion. However, the evidence on **psychic costs** in schooling choice equations is more sturdy than this discussion might suggest. Carneiro, Hansen, & Heckman (2003) obtain similar conclusions on the importance of **psychic costs** from a model where people are not allowed to borrow or lend and there is risk aversion. In Cunha et al. (2005) on the other hand, there are no constraints on borrowing or lending, and they also show sizeable components of **psychic costs**."*

Piatek, R., & Pinger, P. (2016). Maintaining (locus of) control? Data combination for the identification and inference of factor structure models. *Journal of Applied Econometrics, 31*(4), 734-755.

- *"Second, embedding our empirical results in a simple theoretical framework, we establish that locus of control only affects the **psychic cost** of education but is not directly rewarded on the labor market of young professionals."*

B. Acronyms and Symbols

Table 3: List of Acronyms

Acronym	**Meaning**
MDP	standard Markov decision process
RMDP	robust Markov decision process
RMSE	root-mean-square error

Table 4: List of Symbols

Symbol	**Meaning**
$\mathbb{R}$	set of real numbers
$\mathbb{I}\,[\,A\,]$	indicator function that takes value one if event A is true
Economic Model	
t	decision period
T	number of decision periods
$a \in \mathcal{A}$	set of admissible actions
$s \in \mathcal{S}$	set of possible states
$p_t(s,a) \in \mathcal{P}_t(s,a)$	set of conditional probability distributions for s_{t+1} when choosing action a in state s in period t
d_t	decision rule that specifies an action for all states s in period t

Table 4: List of Symbols

Symbol	**Meaning**
X_t	random variable for the state at t
$u_t(s, a)$	utility when choosing action a in state s in period t
δ	discount factor
v_t^{π}	expected total discounted utility of adopting policy π from period t going forward
$\pi \in \Pi$	set of all policies
$R(p \parallel q)$	Kullback-Leibler divergence from model p to q
θ	parameter for trust in reference model q
c	consideration function assigning a cost to each model p
μ	belief function assigning a subjective probability to each model p
ϕ	attitude function capturing individual's attitude towards ambiguity
Mathematical Model	
h_t	history of all realized states and actions up to time t
$h_T \in \mathcal{H}_T$	set of all possible histories up to period T
$\mathbf{p} \in \mathcal{F}^{d_t}$	set of all conditional distributions consistent with d_t

Table 4: List of Symbols

Symbol	Meaning
$\mathbf{P} \in \mathcal{F}_t^{\pi}$	set of all conditional distributions consistent with policy π from t onwards
	Computational Model
ϵ_{at}	random shock to utility of alternative a in period t
x_{jt}	number of periods worked in occupation j by the beginning of period t
g_t	number of periods enrolled in school by the beginning of period t
$\mathcal{N}_0$	true multivariate normal distribution for random shocks
Σ	covariance matrix of random shocks
$\mathcal{E}$	ambiguity set
η	size of ambiguity set
υ	admissible realization of means for future labor market shocks
α_j	parameters for utility function when working in occupation j
β	parameters for utility function when enrolling in school
γ	parameter for utility function when staying at home

C. Computational Details

The respy package (respy, 2018) provides the computational support for the project. Its online documentation is available at http://respy.readthedocs.io and thus I only outline the implementation details that are specific to this paper.

Optimization For the optimization of the criterion function, I rely on the Bound Optimization BY Quadratic Approximation (BOBYQA) algorithm (Powell, 2009). BOBYQA is a derivative-free trust region algorithm that forms quadratic models by interpolation and performs quite well during benchmarking exercises (Rios & Sahinidis, 2013). All tuning parameters are set to their default values.

Integration The solution and estimation of the model produces two types of integrals. I need to determine $E\max$ during the solution step and simulate the choice probabilities to construct the sample log-likelihood. I evaluate both using Monte Carlo integration. I use 200 random draws for the choice probabilities.

Function Smoothing I simulate the choice probabilities to evaluate the sample log-likelihood. With only a finite number of draws, there is always the risk of simulating zero probability for an individual's observed decision. So I use the logit-smoothed accept-reject simulator as suggested by McFadden (1989). The scale parameter is set to 500.

Miscellaneous Following Keane & Wolpin (1994), I impose that individuals cannot attain more than ten additional years of schooling.

I am indebted to several other open source tools among them matplotlib (Hunter, 2007) and Vagrant (Hashimoto, 2013).

Compute Machine

This study utilized the high-performance computational capabilities of the Acropolis Linux cluster hosted by Social Sciences Computing Services at the University of Chicago. Acropolis .uchicago.edu is a clustered Linux system maintained by Social Science Computing Services at the University of Chicago. The head node contains 80 CPUs, 2 TB of memory and 110 TB of storage. The compute nodes provide and additional 1792 CPUs, 8 TB of memory, and 4 NVIDIA Tesla GPUs.

References

Abbott, B., Gallipoli, G., Meghir, C., & Violante, G. L. (2018). Education policy and intergenerational transfers in equilibrium. *Journal of Political Economy*, forthcoming.

Carneiro, P., Hansen, K. T., & Heckman, J. J. (2003). 2001 Lawrence R. Klein lecture: Estimating distributions of treatment effects with an application to the returns to schooling and measurement of the effects of uncertainty on college choice. *International Economic Review*, *44*(2), 361–422.

Cunha, F., & Heckman, J. J. (2008). A new framework for the analysis of inequality. *Macroeconomic Dynamics*, *12*(S2), 315–354.

Cunha, F., Heckman, J. J., & Navarro, S. (2005). Separating uncertainty from heterogeneity in life cycle earnings. *Oxford Economic Papers*, *57*(2), 191–261.

Ehrenberg, R. G., & Smith, R. S. (2016). *Modern labor economics: Theory and public policy* (12th ed.). New York City, NY: Routledge.

Eisenhauer, P., Heckman, J. J., & Mosso, S. (2015). Estimation of dynamic discrete choice models by maximum likelihood and the simulated method of moments. *International Economic Review*, *56*(2), 331–357.

Hai, R., & Heckman, J. J. (2017). Inequality in human capital and endogenous credit constraints. *Review of Economic Dynamics*, *25*, 4–36.

Hashimoto, M. (2013). *Vagrant: Up and running* (1st ed.). Sebastopol, CA: O'Reilly Media.

Heckman, J. J., Lochner, L. J., & Todd, P. E. (2004). Earnings functions and rates of return: the Mincer equation and beyond. *University of Chicago*, Unpublished manuscript.

Heckman, J. J., Lochner, L. J., & Todd, P. E. (2006). Earnings functions, rates of return and treatment effects: The Mincer equation and beyond. In E. A. Hanushek & F. Welch (Eds.), *Handbook of the economics of education* (1st ed., Vol. 1, pp. 307–458). Amsterdam, Netherlands: North-Holland Publishing Company.

Heckman, J. J., Stixrud, J., & Urzua, S. (2006). The effects of cognitive and noncognitive abilities on labor market outcomes and social behavior. *Journal of Labor Economics*, *24*(3), 411–482.

Hunter, J. D. (2007). Matplotlib: A 2D graphics environment. *Computing in Science & Engineering*, *9*(3), 90–95.

Keane, M. P., & Wolpin, K. I. (1994). The solution and estimation of discrete choice dynamic programming models by simulation and interpolation: Monte Carlo evidence. *Review of Economics and Statistics*, *76*(4), 648–672.

McFadden, D. L. (1989). A method of simulated moments for estimation of discrete response models without numerical integration. *Econometrica*, *57*(5), 995–1026.

Piatek, R., & Pinger, P. (2016). Maintaining (locus of) control? Data combination for the identification and inference of factor structure models. *Journal of Applied Econometrics*, *31*(4), 734–755.

Powell, M. J. D. (2009). *The BOBYQA algorithm for bound constrained optimization without derivatives* (Tech. Rep.). Cambridge, United Kingdom: University of Cambridge.

respy. (2018). *respy: A Python package for the simulation and estimation of a prototypical discrete choice dynamic programming model.* Retrieved from http://doi.org/10.5281/zenodo.595547

Rios, L. M., & Sahinidis, N. V. (2013). Derivative-free optimization: A review of algorithms and comparison of software implementations. *Journal of Global Optimization*, *56*(3), 1247–1293.