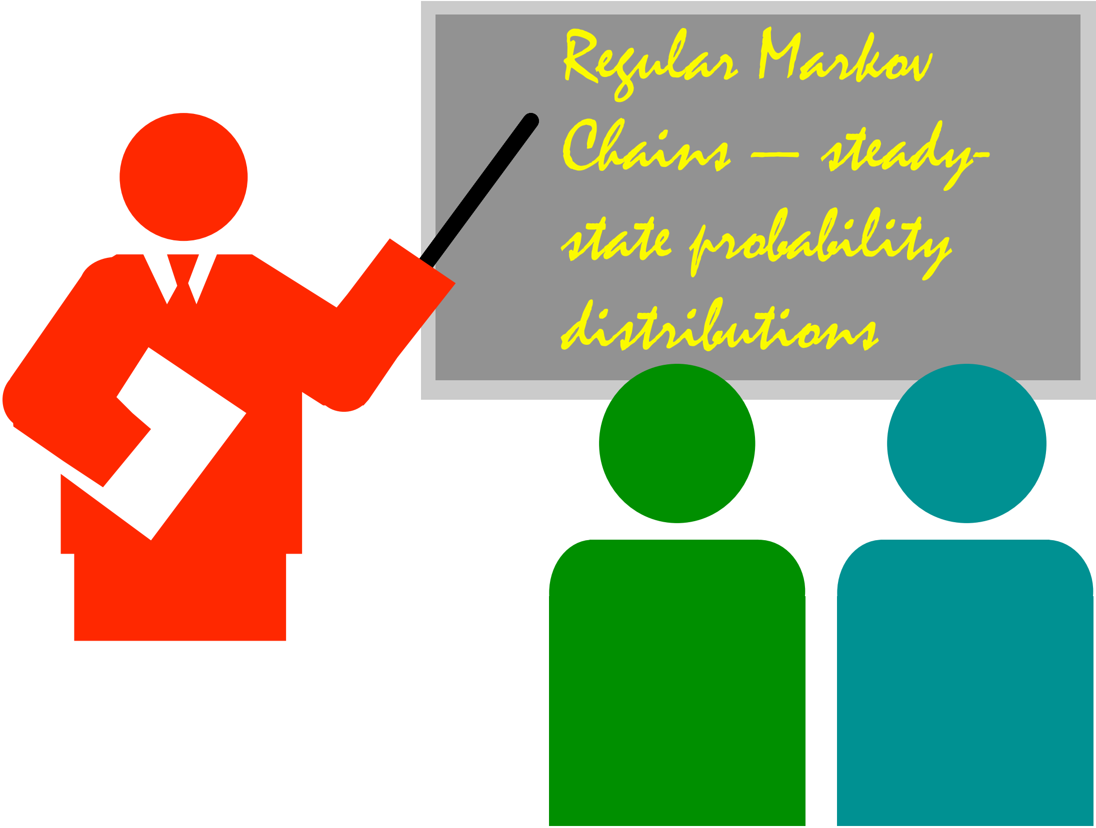
Regular Markov Chains – steady-state probability distributions

Example

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As n gets larger and larger, T^n gets closer and closer to the matrix

$$\begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

Notice

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$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

no matter what *a* and *b* are.

$$\text{Remember } T^n \approx \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

when n is large

Moral of the story: No matter what assumptions you make about the initial probability distribution, after a large number of steps have been taken the probability distribution is approximately

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Question: How could we determine this without computing large powers of T and estimating the limiting matrix?

Clue: Notice that the probability distribution (2/3 1/3) has the property that

$$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

Clue: Notice that the probability distribution (2/3 1/3) has the property that

$$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} .8 & .2 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$$

In other words, a solution of the matrix equation

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} .8 & .2 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

is $\boxed{x = 2/3,\ y = 1/3}$

Idea!!!

We can find the steady state probability distribution [(2/3 1/3) in this example] by solving for x and y below:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} .8 & .2 \\ .4 & .6 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

(Remember also that x + y =1)

We can find the steady state probability distribution [(2/3 1/3) in this example] by solving for *x* and *y* below:

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$$.8x + .4y = x$$

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$$x + y = 1$$

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$$\rightarrow x = 2y$$

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$\rightarrow x = 2y$

$x + y = 1 \rightarrow 2y + y = 1 \rightarrow$

$$\boxed{\begin{aligned} y &= 1/3 \\ x &= 2/3 \end{aligned}}$$

Example: Bob, Alice and Carol are playing Frisbee. Bob always throws to Alice and Alice always throws to Carol. Carol throws to Bob 2/3 of the time and to Alice 1/3 of the time. In the long run what percentage of the time do each of the players have the Frisbee?

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$$T = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} \end{array}$$

$$T = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} \end{array}$$

We must solve the matrix equation

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

and use the fact that $x + y + z = 1$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

We multiply out the left side and equate to the right side:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

We multiply out the left side and equate to the right side:

$$y + 1/3\, z = x$$

$$2/3\, z = y$$

$$x = z$$

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$$x = z$$

Remember also: $x + y + z = 1$

Equations to solve:

$$x + y + z = 1$$

$$y + 1/3\, z = x$$

$$2/3\, z = y$$

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Solution: $x + y + z = 1$ can be rewritten as

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$$x + y + z = 1$$

$$y + 1/3\, z = x$$

$$2/3\, z = y$$

$$x = z$$

Solution: $x + y + z = 1$ can be rewritten as

$$z + 2/3\, z + z = 1$$

so $z = 3/8$

Therefore $x = 3/8$ and $y = 1/4$

$x = 3/8,\ y = 1/4,\ z = 3/8$

Alice — 3/8 probability

Bob — 1/4 probability

Carol — 3/8 probability

In the truck rental problem we had the following transition matrix:

$$\begin{array}{c} \\ NC \\ SC \\ VA \end{array}\begin{array}{c} \begin{array}{ccc} NC & SC & VA \end{array} \\ \begin{bmatrix} .5 & .2 & .3 \\ .4 & .4 & .2 \\ .4 & .1 & .5 \end{bmatrix} \end{array}$$

What fraction of the time does a truck spend in each of the 3 states?

We have to solve for

$$[x \quad y \quad z]\begin{bmatrix} .5 & .2 & .3 \\ .4 & .4 & .2 \\ .4 & .1 & .5 \end{bmatrix} = [x \quad y \quad z]$$

remembering that $x + y + z = 1$.

We have to solve for

$$[x \quad y \quad z]\begin{bmatrix} .5 & .2 & .3 \\ .4 & .4 & .2 \\ .4 & .1 & .5 \end{bmatrix} = [x \quad y \quad z]$$

remembering that $x + y + z = 1$.
So the system of equations is

$$.5x + .4y + .4z = x$$

$$.2x + .4y + .1z = y$$

$$.3x + .2y + .5z = z$$

$$x + y + z = 1$$

$$.5x + .4y + .4z = x$$

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$$x + y + z = 1$$

Rewrite the equations in standard form:

$$x + y + z = 1$$

$$-.5x + .4y + .4z = 0$$

$$.2x + -.6y + .1z = 0$$

$$.3x + .2y + -.5z = 0$$

$$.5x + .4y + .4z = x$$
$$.2x + .4y + .1z = y$$
$$.3x + .2y + .5z = z$$
$$x + y + z = 1$$

Rewrite the equations in standard form:

$$x + y + z = 1$$
$$-.5x + .4y + .4z = 0$$
$$.2x + -.6y + .1z = 0$$
$$.3x + .2y + -.5z = 0$$

This system of equations could be solved using the augmented matrix method of Chapter 1.

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A regular Markov Chain is one where for some positive integer n, the matrix T^n has no 0 entries.

$$T = \begin{bmatrix} 0 & 0 & 1 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad T^2 = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{6} & \frac{3}{4} & \frac{1}{12} \\ \frac{7}{24} & \frac{9}{16} & \frac{7}{48} \end{bmatrix}$$

$$T^3 = \begin{bmatrix} \frac{1}{6} & \frac{3}{4} & \frac{1}{12} \\ \frac{13}{24} & \frac{3}{16} & \frac{13}{48} \\ \frac{43}{96} & \frac{21}{64} & \frac{43}{192} \end{bmatrix}$$

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T is regular because T^3 contains no 0 entries.