Topic 7 solutions

1. Suppose we have three samples d_1, d_2, and d_3 from three statistically independent, unbiased cues that have standard deviations σ_1, σ_2, and σ_3, respectively. The likelihood of depth d is

$$\begin{aligned} L(d) &= P(d_1, d_2, d_3|d) \\ &= P(d_1|d)P(d_2|d)P(d_3|d) \\ &= \phi(d_1, d, \sigma_1)\phi(d_2, d, \sigma_2)\phi(d_3, d, \sigma_3) \\ &= \phi(-d, -d_1, \sigma_1)\phi(-d, -d_2, \sigma_2)\phi(-d, -d_3, \sigma_3) \end{aligned}$$

We can combine the first two normal pdf's using the rule we covered in the lecture.

$$\propto \quad \phi(-d, \mu_{12}, \sigma_{12})\phi(-d, -d_3, \sigma_3)$$

where

$$\mu_{12} = \frac{\frac{1}{\sigma_1^2}(-d_1) + \frac{1}{\sigma_2^2}(-d_2)}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} \quad \text{and} \quad \frac{1}{\sigma_{12}^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

We can combine these two normal pdf's using the same rule.

$$\propto \quad \phi(-d, \mu_{123}, \sigma_{123})$$

where

$$\mu_{123} = \frac{\frac{1}{\sigma_{12}^2}\mu_{12} + \frac{1}{\sigma_3^2}(-d_3)}{\frac{1}{\sigma_{12}^2} + \frac{1}{\sigma_3^2}} = \frac{\frac{1}{\sigma_1^2}(-d_1) + \frac{1}{\sigma_2^2}(-d_2) + \frac{1}{\sigma_3^2}(-d_3)}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}}$$

and

$$\frac{1}{\sigma_{123}^2} = \frac{1}{\sigma_{12}^2} + \frac{1}{\sigma_3^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}$$

This normal pdf is at its maximum when $-d = \mu_{123}$, which means

$$d = \frac{\frac{1}{\sigma_1^2}d_1 + \frac{1}{\sigma_2^2}d_2 + \frac{1}{\sigma_3^2}d_3}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2}}$$

This is the maximum likelihood estimate of the depth d, given the three cue values d_1, d_2, and d_3.

2. The likelihood of depth d given cue samples d_1 and d_2 is

$$\begin{aligned} L(d) &= P(d_1, d_2|d) \\ &= P(d_1|d)P(d_2|d) \\ &= \phi(d_1, d, \sigma_1)\phi(d_2, d, \sigma_2) \\ &= \frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{(d_1-d)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-\frac{(d_2-d)^2}{2\sigma_2^2}} \end{aligned}$$

We are looking for the value of d that maximizes this function. The value of d doesn't affect the normalizing constants, so we can drop the constants and simplify this to

$$e^{-\frac{(d_1-d)^2}{2\sigma_1^2}} e^{-\frac{(d_2-d)^2}{2\sigma_2^2}} = e^{-\frac{(d_1-d)^2}{2\sigma_1^2}-\frac{(d_2-d)^2}{2\sigma_2^2}}$$

Similarly, we don't change the location of the maximum by taking the logarithm, so instead we can maximize

$$\log\left(e^{-\frac{(d_1-d)^2}{2\sigma_1^2}-\frac{(d_2-d)^2}{2\sigma_2^2}}\right) = -\frac{(d_1-d)^2}{2\sigma_1^2} - \frac{(d_2-d)^2}{2\sigma_2^2}$$

The derivative of this function with respect to d is

$$\frac{(d_1-d)}{\sigma_1^2} + \frac{(d_2-d)}{\sigma_2^2}$$

To find the maximum or minimum of $L(d)$ we set this derivative to zero and solve for d.

$$\begin{aligned}
&\frac{(d_1-d)}{\sigma_1^2} + \frac{(d_2-d)}{\sigma_2^2} = 0 \\
&d\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right) = \frac{d_1}{\sigma_1^2} + \frac{d_2}{\sigma_2^2} \\
&d = w_1 d_1 + w_2 d_2
\end{aligned}$$

where

$$w_1 = \frac{\frac{1}{\sigma_1^2}}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}} \quad \text{and} \quad w_2 = \frac{\frac{1}{\sigma_2^2}}{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}$$