Problem set 1 solutions

1. Let F be a random variable that indicates which door C chooses in step (2). Let G be a random variable that indicates the winning door. Both F and G take values 1, 2, or 3. Let W be a random variable that takes values 0 or 1 to indicate whether C wins. We can use the law of total probability to break up the probability of winning into nine simpler cases.

$$\begin{aligned}
P(W=1) &= P(W=1|F=1,G=1)P(F=1,G=1) \\
&+ P(W=1|F=1,G=2)P(F=1,G=2) \\
&+ P(W=1|F=1,G=3)P(F=1,G=3) \\
&+ P(W=1|F=2,G=1)P(F=2,G=1) \\
&+ P(W=1|F=2,G=2)P(F=2,G=2) \\
&+ P(W=1|F=2,G=3)P(F=2,G=3) \\
&+ P(W=1|F=3,G=1)P(F=3,G=1) \\
&+ P(W=1|F=3,G=2)P(F=3,G=2) \\
&+ P(W=1|F=3,G=3)P(F=3,G=3)
\end{aligned}$$

First we will find the the probability that C wins if he or she stays with the chosen door in step (4). With this strategy, $P(W=1|F=G)=1$ and $P(W=1|F\neq G)=0$. Using these facts to simplify the expansion above gives us:

$$P(W=1) = P(F=1,G=1) + P(F=2,G=2) + P(F=3,G=3)$$

F and G are independent, so for any i and j, $P(F=i,G=j) = P(F=i)P(G=j)$. We will assume that the winning door is chosen from the uniform distribution, so $P(G=j)=1/3$. (Notice that we do not need to assume anything about the distribution of F.) Then,

$$\begin{aligned}
P(W=1) &= P(F=1)P(G=1) + P(F=2)P(G=2) + P(F=3)P(G=3) \\
&= (P(F=1)+P(F=2)+P(F=3))/3 \\
&= 1/3
\end{aligned}$$

Next we will find the the probability that C wins if he or she switches doors in step (4). With this strategy, $P(W=1|F=G)=0$ and $P(W=1|F\neq G)=1$. Now simplifying

the expansion above gives us:

$$
\begin{aligned}
P(W=1) &= P(F=1, G=2) + P(F=1, G=3) \\
&\quad + P(F=2, G=1) + P(F=2, G=3) \\
&\quad + P(F=3, G=1) + P(F=3, G=2) \\
\\
&= P(F=1)P(G=2) + P(F=1)P(G=3) \\
&\quad + P(F=2)P(G=1) + P(F=2)P(G=3) \\
&\quad + P(F=3)P(G=1) + P(F=3)P(G=2) \\
\\
&= (2/3)(P(F=1) + P(F=2) + P(F=3)) \\
\\
&= 2/3
\end{aligned}
$$

The switch strategy has twice the probability of winning as the stay strategy.

Note that almost all the work here is done by using the law of total probability to break up $P(W = 1)$ into several cases, and then using facts about the problem to find the probabilities of the individual cases. This is often a good strategy for solving probability problems.

2. See `randnc.R`.

3. See `keeling.R`.