METIS

Lesson 10: Probability

Introduction

METIS

Lecture Overview:

Goals of the lecture:

1. Develop your probabilistic intuition to solve real-world problems
2. Understand the Sample Space and key rules to solve problems
3. Differentiate between Discrete and Continuous Random Variables
4. Common Discrete and Continuous Distributions

Probabilistic Intuition

METIS

Probability

Definition:

Probability is the study of theoretical possibilities and their likelihood of occurring

Testing your Probabilistic Intuition

Before we begin, let us test your intuition by discussing your perceived outcomes for a series of phenomena:

Example 1:

1. What is the probability of flipping heads with a fair coin?
2. What is the probability of getting a 1 or 2 with a fair die?
3. What is the probability of getting a Heart card from a 52 deck of cards?

Testing your Probabilistic Intuition

Before we begin, let us test your intuition by discussing your perceived outcomes for a series of phenomena:

Example 1:

1. What is the probability of flipping heads with a fair coin?

   0.5

2. What is the probability of getting a 1 or 2 with a fair die?

   2/6 = 1/3 = 0.33

3. What is the probability of getting a Heart card from a 52 deck of cards?

   13/52 = 1/4 = 0.25

Definitions

METIS

Random Experiment

Definition:

Random Experiment is the process of observing a phenomenon that has an uncertain outcome. Random Experiments are unpredictable in the short term and predictable in the long term

Example 2:

1. **Flipping a coin** (the outcome is uncertain – we may get heads or tails). In the short term, the experiment is unpredictable (i.e. probability of heads is unknown), but in the long term, the experiment becomes predictable (i.e. probability of heads becomes roughly 0.5 if fair coin)
2. **Rolling a 6-sided die** (the outcome is uncertain – we may get 1,2,3,4,5,6). In the short term, we can't predict if we get a 1, but in the long term, it converges to 1/6.

Sample Space

Definition:

Sample Space describes a list of all the possible outcomes of an experiment**.**

The Sample Space is typically denoted by Omega or S, enclosed in curly brackets and separated by commas: Ω = {Outcome_1, Outcome_2, Outcome_3, …, Outcome_n}

Example 3:

1. A fair coin has a sample space of {*H, T*}
2. A 6-sided die has a sample space of {1, 2, 3, 4, 5, 6}
3. Sentiment Analysis has a sample space of {“Positive”, “Negative”, “Neutral”}
4. A pregnancy test has a sample space of {“Pregnant”, “Not Pregnant”}

Random Experiment Outcomes

Definition:

The **Outcome** of a Random Experiment is simply the result that is obtained after running the experiment

Example 4:

1. We can flip a coin and observe the outcome as **“Heads”**
2. We can roll a die and observe the outcome as **“5”**
3. We can choose a card from a 52-card deck and observe the outcome as **“Ace of Spades”**
4. We can interpret the sentiment of a corpus of documents and observe the outcome as **“Positive”**
5. We can run a pregnancy test and observe the outcome as **“Not Pregnant”**

Events of a Random Experiment

Definition:

The **Events** of a Random Experiment are subsets of the outcomes in the Sample Space that we are interested in observing.

Example 5:

1. While a Sample Space of rolling a 6-sided die may contain all the numbers between 1 – 6, we may actually only be interested in the even numbers. Therefore, S = {1,2,3,4,5,6}, but E = {2,4,6}
2. While a Sample Space of a 52-card deck may contain all the combinations of 2-10,J,Q,K,A with the 4 suits, we may actually only be interested in the 4 aces. Therefore E = {A_spades, A_hearts, A_clubs, A_diamonds}

Problem 1

Problem 1:

If a Random Experiment consists of flipping a coin twice:

1. What is the Sample Space of this experiment?

2. What are the possible events where the first toss results in a Heads?

*Assume that the coin consist of two sides – “Heads” and “Tails”

Problem 1

Solution 1:

1. The Sample Space of the Random Experiment described above is:
   $S = \{TT, TH, HT, HH\}$
2. The events of the experiment where the first toss results in a Heads is:
   $E = \{HT, HH\}$

Probability Measure

Definition (1):

Probability Measure describes the chances that a specific event will occur and is denoted by $\mathbb{P}(E)$.

- If we assumed that each event had the same chance of occurring, then we can easily calculate the probability measure for any event in the experiment as:

$$\mathbb{P}(E) = \frac{\mathcal{N}(E)}{\mathcal{N}(S)} = \frac{\#\, of\ favorable\ outcomes}{\#\, total\ possible\ outcomes}$$

- OR, we can repeat the experiment for a large number of times and observe the frequency in which the event occurred:

$$\mathbb{P}(E) = \lim_{n \to \infty} \frac{f_n(E)}{n}$$

Probability Measure

Definition (2):

- The Probability Measure assigns a real number to each Event
- The probability of an event will have a value between 0 and 1
- The sum of the probabilities of the sample space is equal to 1

Example 6:

1. In the Sample Space associated with the Random Experiment of flipping a coin, the Probability Measure assigns a real number to each event: $\mathbb{P}(Heads) = 0.5$ and $\mathbb{P}(Tails) = 0.5$. Here, I am assigning numbers based on a probabilistic assumption of a fair coin
2. In the Sample Space associated with the Random Experiment of rolling a 6-sided die, the Probability Measure assigns a real number to each event: $\mathbb{P}(1) = \mathbb{P}(2) = \mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6) = \frac{1}{6}$.

Problem 2

Problem 2:

A roulette wheel has 37 numbers. 18 numbers are red, 18 numbers are black and 1 number is green.

1. What is the probability of getting a red number?
2. What is the probability of getting a black number?
3. What is the probability of getting a green number?

Problem 2

Solution 2:

$$\mathbb{P}(E) = \frac{\mathcal{N}(E)}{\mathcal{N}(S)} = \frac{\#\, of\ favorable\ outcomes}{\#\, total\ possible\ outcomes}$$

1. $\mathbb{P}(Red) = 18/37 = 0.486$
2. $\mathbb{P}(Black) = 18/37 = 0.486$
3. $\mathbb{P}(Green) = 1/37 = 0.027$

Probability Rules

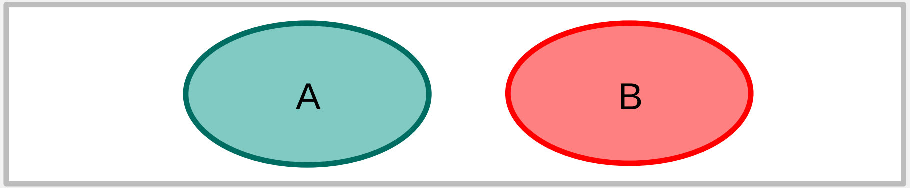
METIS

Disjoint Events

Definition:

If A and B are disjoint events

$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$



Example 7:

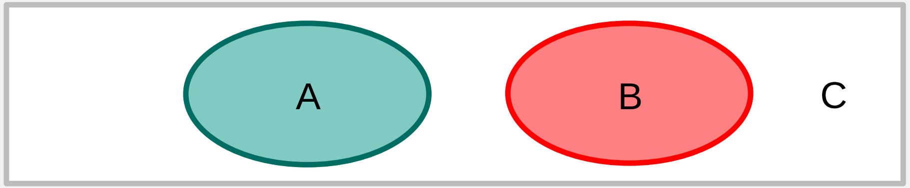
1. Probability of throwing a die and getting a 2 or 3

$$\mathbb{P}(2) + \mathbb{P}(3) = 1/6 + 1/6 = 2/6 = 1/3$$

Normalization of Total Probability

Definition:

If A, B and C are disjoint events
$\mathbb{P}(\Omega) = \mathbb{P}(A \cup B \cup C) =$
$\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) = \sum_i \mathbb{P}(x_i) = 1$



Example 8:

1. Probability of throwing a die and getting a 1, 2, 3, 4, 5 or 6

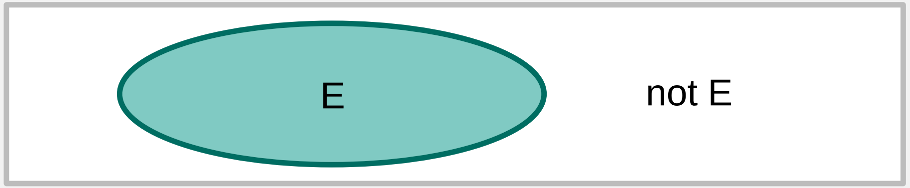
$$\mathbb{P}(1) + \mathbb{P}(2) + \mathbb{P}(3) + \mathbb{P}(4) + \mathbb{P}(5) + \mathbb{P}(6) = 1/6 + 1/6 + 1/6 + 1/6 + 1/6 + 1/6 = 1$$

Complementary

Definition:

For any event E

$\mathbb{P}(\text{not } E) = 1 - \mathbb{P}(E)$



Example 9:

1. Probability of throwing a die and not getting a 2

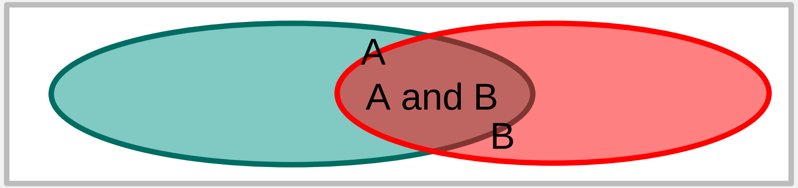
$\mathbb{P}(\text{not } 2) = 1 - \mathbb{P}(2) = 1-1/6 = 5/6$

Joint Events

Definition:

If A and B are joint events

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$



Example 10:

1. Probability of throwing a die and getting <4 OR even (e.g. {1,2,3,4,6})

$$\mathbb{P}(<4) + \mathbb{P}(\text{even}) - \mathbb{P}(<4 \cap \text{even}) = 3/6 + 3/6 - 1/6 = 5/6$$

Joint Events

Example 11:

Out of the students in this class,

- 40% live in Chicago,
- 60% have never programmed before, and
- 30% live in Chicago AND have never programmed before.

What is the probability that a randomly selected student does not live in Chicago and has programmed before?

Draw a diagram representing the Sample Space, and Events and label each part of the diagram.

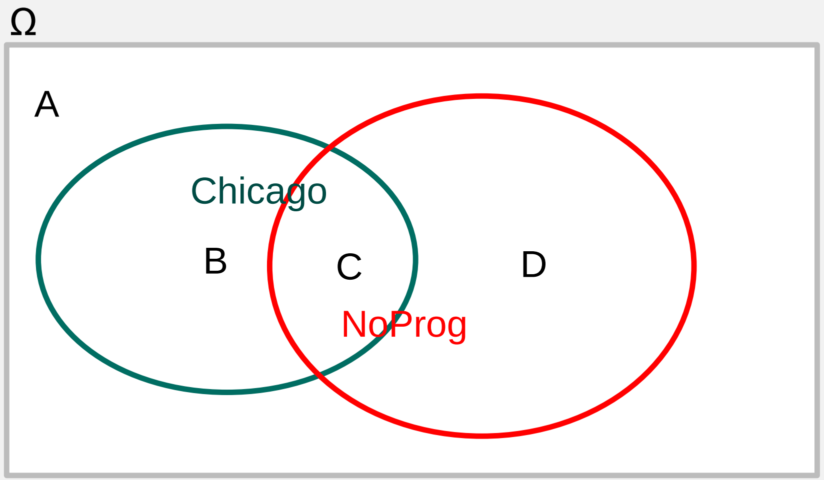
Joint Events

Example 11 (Solution – Part 1):

$\mathbb{P}(Chicago) = \mathbb{P}(B \cup C) = 0.4$
$\mathbb{P}(NoProg) = \mathbb{P}(C \cup D) = 0.6$
$\mathbb{P}(Chicago\ \cap\ NoProg) = \mathbb{P}(C) = 0.3$

A = NoChicago AND YesProg
B = Chicago AND YesProg
C = Chicago AND NoProg
D = NoProg AND NoChicago

Ω = A + B + C + D

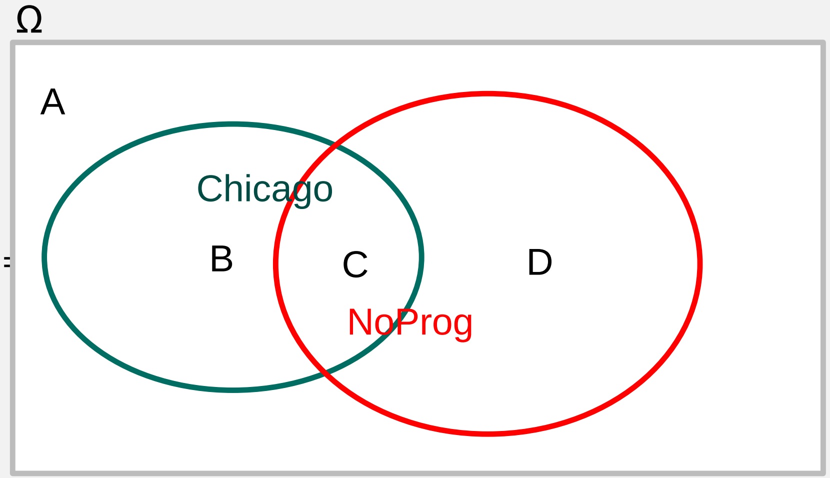


Joint Events

Example 11 (Solution – Part 1):

$\mathbb{P}(Chicago) = \mathbb{P}(B \cup C) = 0.4$
$\mathbb{P}(NoProg) = \mathbb{P}(C \cup D) = 0.6$
$\mathbb{P}(Chicago \cap NoProg) = \mathbb{P}(C) = 0.3$

$\mathbb{P}(A) = 1 - \left(\mathbb{P}(Chicago) + \mathbb{P}(NoProg) - \mathbb{P}(C)\right) =$
1-0.4-0.6+0.3 = 0.3



Problem 3

Problem 3:

1. What is the probability of throwing two dies and getting the same number on both (e.g. (1,1), (2,2), etc)?

Problem 3

Solution 3:

6 outcomes out of 36 meet the criteria:

$$\mathbb{P}(same) = \frac{6}{36} = \frac{1}{6}$$

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	4,5	5,5	5,6
6,1	6,2	6,3	4,6	6,5	6,6

Problem 4

Problem 4:

1. What is the probability of throwing two dies and the sum is an even number AND is greater than 7?

Problem 4

Solution 4:

9 outcomes out of 36 meet the criteria even AND >7:

$$\mathbb{P}(same) = \frac{9}{36} = \frac{1}{4}$$

1,1=2	1,2=3	1,3=4	1,4=5	1,5=6	1,6=7
2,1=3	2,2=4	2,3=5	2,4=6	2,5=7	2,6=8
3,1=4	3,2=5	3,3=6	3,4=7	3,5=8	3,6=9
4,1=5	4,2=6	4,3=7	4,4=8	4,5=9	4,6=10
5,1=6	5,2=7	5,3=8	4,5=9	5,5=10	5,6=11
6,1=7	6,2=8	6,3=9	4,6=10	6,5=11	6,6=12

Problem 5

Problem 5:

1. What is the probability of throwing two dies and the sum is an even number OR is greater than 7?

Problem 5

Solution 5:

$$\mathbb{P}(even) = \frac{18}{36} = \frac{1}{2}$$

1,1=2	1,2=3	1,3=4	1,4=5	1,5=6	1,6=7
2,1=3	2,2=4	2,3=5	2,4=6	2,5=7	2,6=8
3,1=4	3,2=5	3,3=6	3,4=7	3,5=8	3,6=9
4,1=5	4,2=6	4,3=7	4,4=8	4,5=9	4,6=10
5,1=6	5,2=7	5,3=8	4,5=9	5,5=10	5,6=11
6,1=7	6,2=8	6,3=9	4,6=10	6,5=11	6,6=12

Problem 5

Solution 5:

$$\mathbb{P}(even) = \frac{18}{36} = \frac{1}{2}$$

$$\mathbb{P}(> 7) = \frac{15}{36} = \frac{5}{12}$$

1,1=2	1,2=3	1,3=4	1,4=5	1,5=6	1,6=7
2,1=3	2,2=4	2,3=5	2,4=6	2,5=7	2,6=8
3,1=4	3,2=5	3,3=6	3,4=7	3,5=8	3,6=9
4,1=5	4,2=6	4,3=7	4,4=8	4,5=9	4,6=10
5,1=6	5,2=7	5,3=8	4,5=9	5,5=10	5,6=11
6,1=7	6,2=8	6,3=9	4,6=10	6,5=11	6,6=12

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Problem 5

Solution 5:

$$\mathbb{P}(even) = \frac{18}{36} = \frac{1}{2}$$

$$\mathbb{P}(> 7) = \frac{15}{36} = \frac{5}{12}$$

$$\mathbb{P}(> 7AND\ even) = \frac{9}{36} = \frac{1}{4}$$

1,1=2	1,2=3	1,3=4	1,4=5	1,5=6	1,6=7
2,1=3	2,2=4	2,3=5	2,4=6	2,5=7	2,6=8
3,1=4	3,2=5	3,3=6	3,4=7	3,5=8	3,6=9
4,1=5	4,2=6	4,3=7	4,4=8	4,5=9	4,6=10
5,1=6	5,2=7	5,3=8	4,5=9	5,5=10	5,6=11
6,1=7	6,2=8	6,3=9	4,6=10	6,5=11	6,6=12

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Problem 5

Solution 5:

$$\mathbb{P}(even) = \frac{18}{36} = \frac{1}{2}$$

$$\mathbb{P}(> 7) = \frac{15}{36} = \frac{5}{12}$$

$$\mathbb{P}(> 7 AND\ even) = \frac{9}{36} = \frac{1}{4}$$

1,1=2	1,2=3	1,3=4	1,4=5	1,5=6	1,6=7
2,1=3	2,2=4	2,3=5	2,4=6	2,5=7	2,6=8
3,1=4	3,2=5	3,3=6	3,4=7	3,5=8	3,6=9
4,1=5	4,2=6	4,3=7	4,4=8	4,5=9	4,6=10
5,1=6	5,2=7	5,3=8	4,5=9	5,5=10	5,6=11
6,1=7	6,2=8	6,3=9	4,6=10	6,5=11	6,6=12

$\mathbb{P}$(even U >7) = 1/2 + 5/12 – 1/4 = 8/12 = 2/3

$\mathbb{P}$(A U B) = $\mathbb{P}$(A) + $\mathbb{P}$(B) – $\mathbb{P}$(A ∩ B)

Independence

METIS

Independence

Definition:

When events are independent (e.g. the occurrence of one event does not influence the occurrence of another event), the probabilities can be multiplied.

$$\mathbb{P}(A,B) = \mathbb{P}(A) \times \mathbb{P}(B)$$

Example 12:

1. Probability of throwing a coin three times and getting exactly HTH

$$\mathbb{P}(HTH) = \mathbb{P}(H) \times \mathbb{P}(T) \times \mathbb{P}(H) = 1/2 \times 1/2 \times 1/2 = 1/8$$

Problem 6

Problem 6:

What is the probability of throwing a die 6 times and getting 1, 2, 3, 4, 5, and 6 (in that order)?

Problem 6

Solution 6:

$\mathbb{P}(1,2,3,4,5,6) = \mathbb{P}(1) \times \mathbb{P}(2) \times \mathbb{P}(3) \times \mathbb{P}(4) \times \mathbb{P}(5) \times \mathbb{P}(6) =$

$$\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6} = \frac{1}{6^6} = \frac{1}{46656}$$

Problem 7

Problem 7:

The probability of someone blinking on a photograph is 0.1. A picture of five people is being taken.

1. What is the probability that no one is blinking?
2. What is the probability that at least one person is blinking?
3. What is the probability that all are blinking?

Problem 7

Solution 7:

For one individual:
$\mathbb{P}(blink) = 0.1$
$\mathbb{P}(no\_blink) = 1 - 0.1 = 0.9$

For group:
1) $\mathbb{P}(no\_blink) = 0.9^5 = 0.59$
2) $\mathbb{P}(1\ or\ more\ blink) = 1 - 0.59 = 0.41$
3) $\mathbb{P}(all\_blink) = 0.1^5 = 0.00001$

Conditional Probability

Definition:

Two events are said to be dependent when the probability of one event occurring influences the likelihood of the other event occurring.

$$\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\ \mathbb{P}(B)$$

Example 13:

1. If you draw two cards without replacement,

What is the probability that the second card is an Ace if the first card is an Ace?

$$\mathbb{P}(A_2|A_1) = 3/51$$

What is the probability that the second card is an Ace if the first card is not an Ace?

$$\mathbb{P}(A_2|\text{not } A_1) = 4/51$$

Problem 8

Problem 8:

A bag contains 5 red balls, 3 blue balls and 4 green balls. If we chose two balls without replacement, what is the probability that the first ball will be green and the second blue?

Problem 8

Solution 8:

$\mathrm{R} = 5, \mathrm{B} = 3, \mathrm{G} = 4$

$\mathbb{P}(\mathrm{B} \cap \mathrm{G}) = \mathbb{P}(\mathrm{B}|\mathrm{G})\ \mathbb{P}(\mathrm{G}) = \frac{3}{11}\frac{4}{12} = \frac{1}{11}$

Problem 9

Problem 9:

I like dogs, they make me happy. Sometimes dogs greet me. What is the probability that I am happy?

$$\mathbb{P}(greet) = \mathbb{P}(G) = \frac{1}{4}$$

$$\mathbb{P}(happy|G) = \mathbb{P}(H|G) = \frac{9}{10}$$

$$\mathbb{P}(happy|NG) = \mathbb{P}(H|NG) = \frac{2}{10}$$

Problem 9

Solution 9:

$$\mathbb{P}(greet) = \mathbb{P}(G) = \frac{1}{4} \qquad \mathbb{P}(no\ greet) = \mathbb{P}(NG) = \frac{3}{4}$$

$$\mathbb{P}(happy|G) = \mathbb{P}(H|G) = \frac{9}{10} \qquad \mathbb{P}(not\ happy|G) = \mathbb{P}(NH|G) = \frac{1}{10}$$

$$\mathbb{P}(happy|NG) = \mathbb{P}(H|NG) = \frac{2}{10} \qquad \mathbb{P}(not\ happy|NG) = \mathbb{P}(NH|NG) = \frac{8}{10}$$

Problem 9

Solution 9:

$$\mathbb{P}(greet) = \mathbb{P}(G) = \frac{1}{4} \qquad \mathbb{P}(no\ greet) = \mathbb{P}(NG) = \frac{3}{4}$$

$$\mathbb{P}(happy|G) = \mathbb{P}(H|G) = \frac{9}{10} \qquad \mathbb{P}(not\ happy|G) = \mathbb{P}(NH|G) = \frac{1}{10}$$

$$\mathbb{P}(happy|NG) = \mathbb{P}(H|NG) = \frac{2}{10} \qquad \mathbb{P}(not\ happy|NG) = \mathbb{P}(NH|NG) = \frac{8}{10}$$

Problem 9

Solution 9:

$$\mathbb{P}(greet) = \mathbb{P}(G) = \frac{1}{4} \qquad \mathbb{P}(no\ greet) = \mathbb{P}(NG) = \frac{3}{4}$$

$$\mathbb{P}(happy|G) = \mathbb{P}(H|G) = \frac{9}{10} \qquad \mathbb{P}(not\ happy|G) = \mathbb{P}(NH|G) = \frac{1}{10}$$

$$\mathbb{P}(happy|NG) = \mathbb{P}(H|NG) = \frac{2}{10} \qquad \mathbb{P}(not\ happy|NG) = \mathbb{P}(NH|NG) = \frac{8}{10}$$

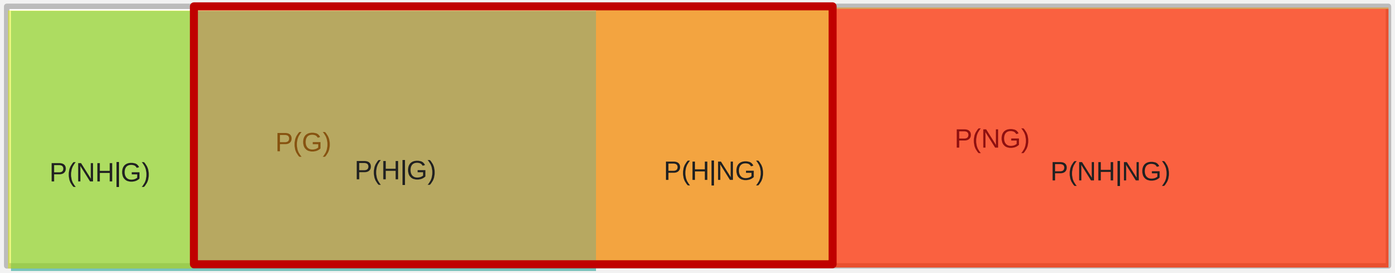
Problem 9

Solution 9:

$$\mathbb{P}(greet) = \mathbb{P}(G) = \frac{1}{4} \qquad \mathbb{P}(no\ greet) = \mathbb{P}(NG) = \frac{3}{4}$$

$$\mathbb{P}(happy|G) = \mathbb{P}(H|G) = \frac{9}{10} \qquad \mathbb{P}(not\ happy|G) = \mathbb{P}(NH|G) = \frac{1}{10}$$

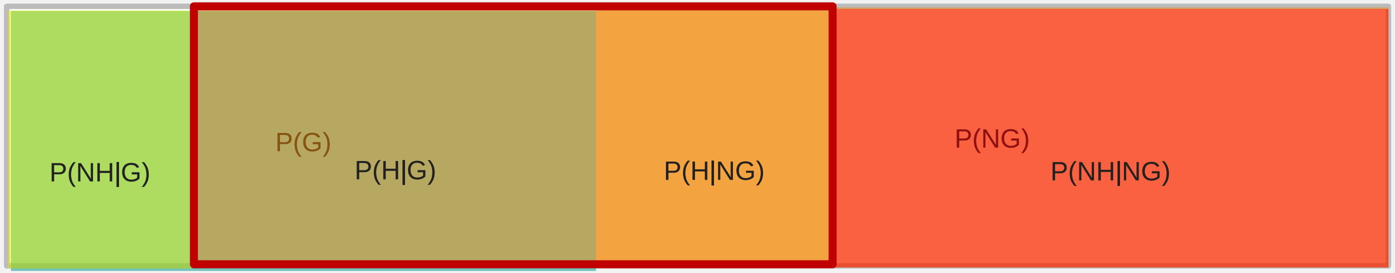
$$\mathbb{P}(happy|NG) = \mathbb{P}(H|NG) = \frac{2}{10} \qquad \mathbb{P}(not\ happy|NG) = \mathbb{P}(NH|NG) = \frac{8}{10}$$



Problem 9

Solution 9:

$$\mathbb{P}(H) = \mathbb{P}(H|G)\mathbb{P}(G) + \mathbb{P}(H|NG)\mathbb{P}(NG) = \frac{9}{10}\frac{1}{4} + \frac{2}{10}\frac{3}{4} = \frac{15}{40} = \frac{3}{8} = 0.375$$



Random Variables

METIS

Discrete Random Variables

Definition:

Discrete Random Variables are a specific type of Random Variable that can only take on finite number of values. All of the examples we have seen so far are cases of Discrete Random Variables

If we define the set of possible values for a Random Variable and find them to be a set of discrete points (i.e. $x_0, x_1, \ldots, x_n$), this is an example of a Discrete Random Variable

Example 14:

1. Die
2. Deck of cards
3. Age

Continuous Random Variables

Definition:

Continuous Random Variables are a specific type of Random Variable that can take on infinitely many values over a continuous range**.** Unlike Discrete Random Variables, the probability of any given value for a Continuous Random Variables is 0 $\mathbb{P}(X = a) = 0$

Example 15:

1. Height
2. Temperature
3. Distance

Probability Mass Functions

Definition:

PMF is a function for discrete random variables that indicates that the probability of an event is exactly equal to some value

$$\mathbb{P}_X(x_i) = \mathbb{P}\{X = x_i\}$$

Cumulative Distribution Function

Definition

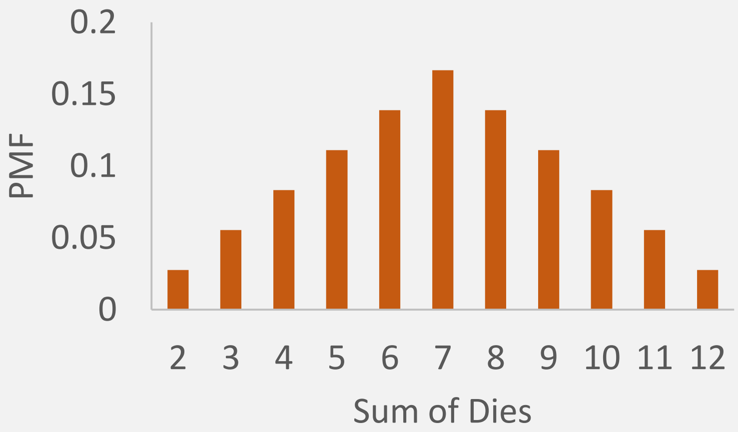
We can contrast the Probability Mass Function with a Cumulative Distribution Function (or just "Distribution Function"). The Cumulative Distribution Function is developed by summing probabilities in the Probability Mass Function in a cumulative fashion. In general, the formula for the CDF of a Random Variable X is:

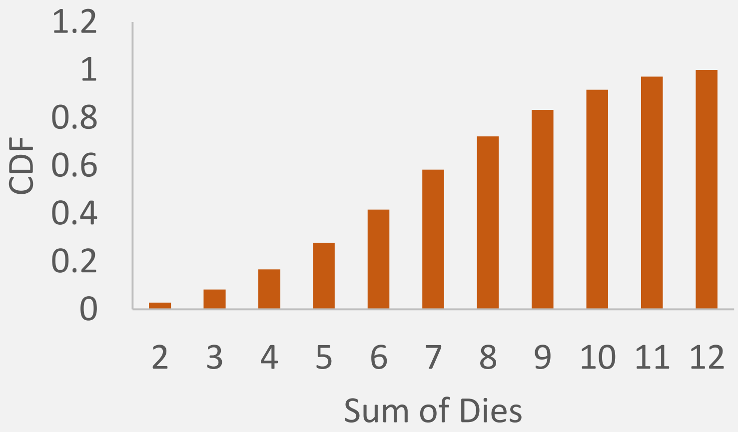
$$F_x(x) = \mathbb{P}(X \leq x) = \sum_{i=0}^{j} \mathbb{P}(X = x_i)$$

PMF and CMF

Example 16:

PMF and CDF of the sum of two dies





Problem 10

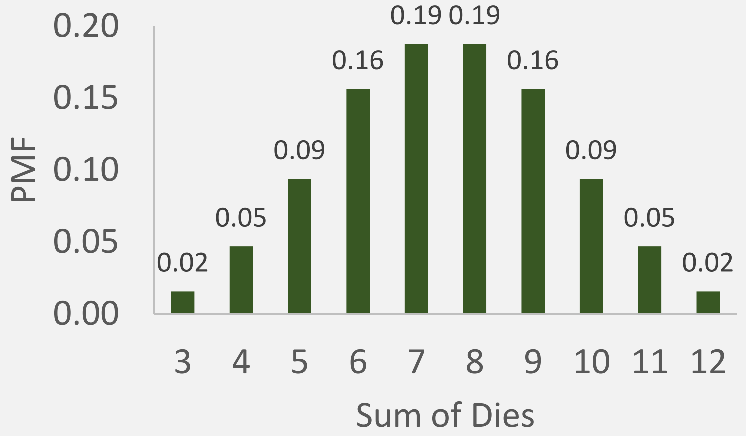
Problem 10:

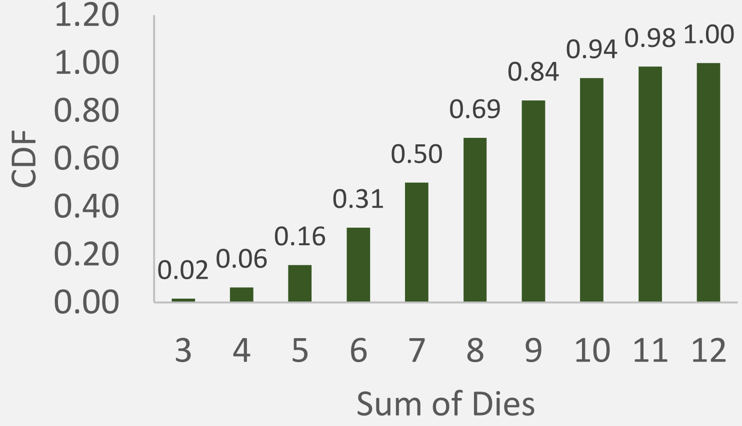
Assume you have a 4-sided die (S={1,2,3,4}), that is thrown three times.

1. Draw the PMF and CDF for the sum of the 3 dies. What is the most likely outcome?
2. What is the probability for the sum of the three dies to be a 10 or higher?

Problem 10

Solution 10:





1. Most likely outcome is a 7 or 8
2. $\mathbb{P}$(10,11,12) = 1-CFD(9) = 1-0.84 = 0.16

Probability Density Functions

Definition (1):

PDF is a function for continuous random variables, whose value at any given sample range can be interpreted as a relative likelihood that the value of the random value would equal that sample range. The (informal) formulation of the **pdf** is:

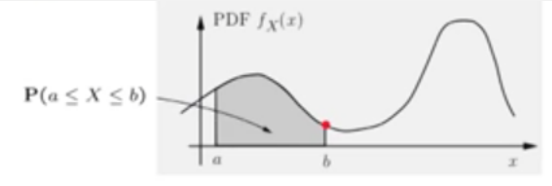
$$f_X(x)dx = \mathbb{P}(x \le X \le x + dx)$$

In the continuous case we measure **Density** (probability/unit of length) and NOT probability directly

Definition (2):

To calculate the probability of any range of x values, the area under the curve must be calculated as follows:

$$P(a \le X \le b) = \int_a^b f_X(x)dx$$



Cumulative Distribution Functions

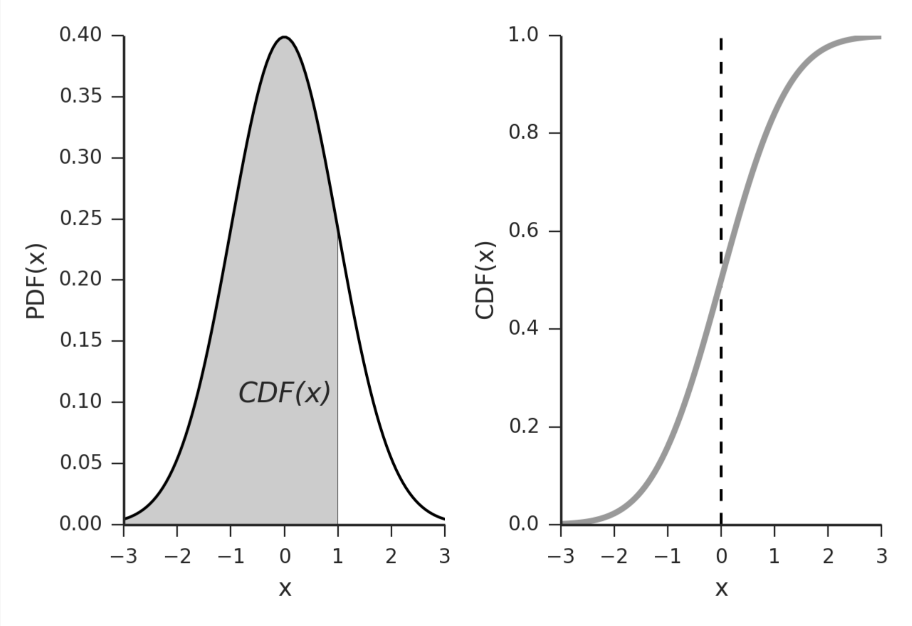
Definition

For Continuous Random Variables, we can no longer simply add probabilities in a cumulative fashion to develop the Cumulative Distribution Functions. Instead, we have to calculate the area under the curve from negative infinity to our desired point. Thus, for a Continuous Random Variable X with, we have:

$$F_x\left(x_j\right) = \int_{-\infty}^{x} f(x)dx$$

Continuous RV and Distribution Functions

Example 17:



Problem 11

Problem 11:

If we have a Continuous Random Variable X with the following Probability Density Function:

$$f_X(x) = \begin{cases} 0 \text{ if } x \ \le 0, \\ 0.2 \ if \ 0 < x \ \le 5, \\ 0 \ if \ x > 5 \end{cases}$$

Then what is the probability of randomly selecting a value less than 4?

Problem 11

Solution 11:

$$F_x(4) = \int_{-\infty}^{4} f_X(x)dx = \int_{0}^{4} 0.2dx = [0.2x]_0^4 = 0.2(4) = 0.8$$

Common Discrete Distributions

METIS

Discrete Distributions

Common Discrete Distributions:

1. Bernoulli Distribution
2. Binomial Distribution
3. Poisson Distribution

Common Continuous Distributions

METIS

Continuous Distributions

Common Continuous Distributions:

1. Exponential Distribution
2. Gamma Distribution
3. Chi-Squared Distribution
4. Normal (Gaussian) Distribution
5. Log-Normal Distribution

More Problems

METIS

Problem 12

Problem 12:

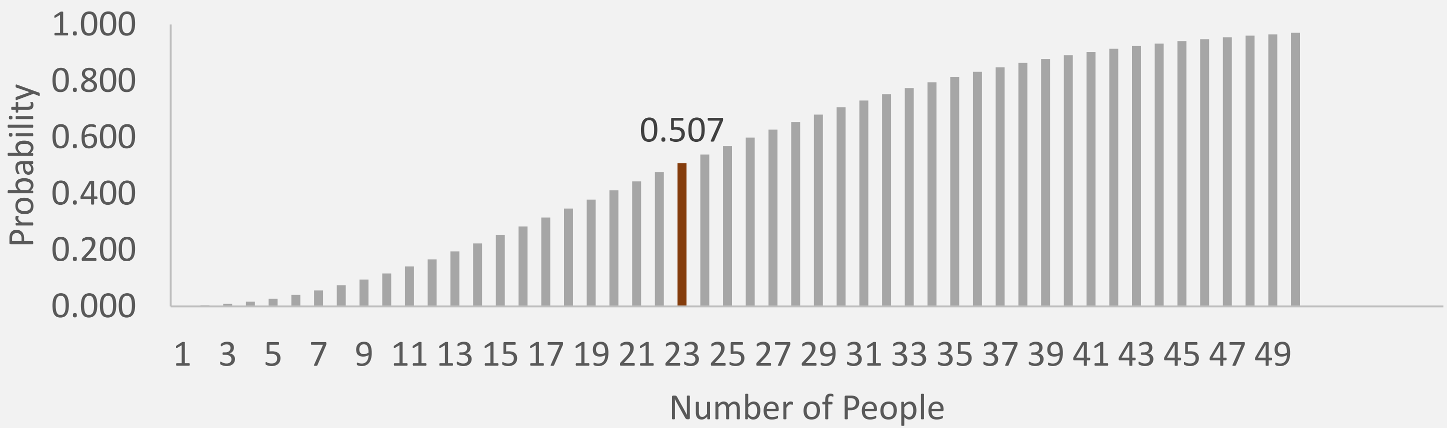
In a classroom of 23 people, what is the probability that at least two people have the same birthday? Assume 365 days in the calendar.

Problem 12

Solution 12:

$$\mathbb{P}(\text{Different Birthdays}) = \frac{365}{365}\frac{364}{365}\frac{363}{365}\cdots\frac{343}{365} = 0.4927$$

$$\mathbb{P}(\text{at least 2 Birthday}) = 1 - \mathbb{P}(\text{Different Birthdays}) = 1 - 0.4927 = 0.5072$$



Problem 13

Problem 13:

A survey determined the following probabilities for a group of students:
$\mathbb{P}$(at least play piano) = 0.3
$\mathbb{P}$(at least play guitar) = 0.2
$\mathbb{P}$(play piano and guitar) = 0.1

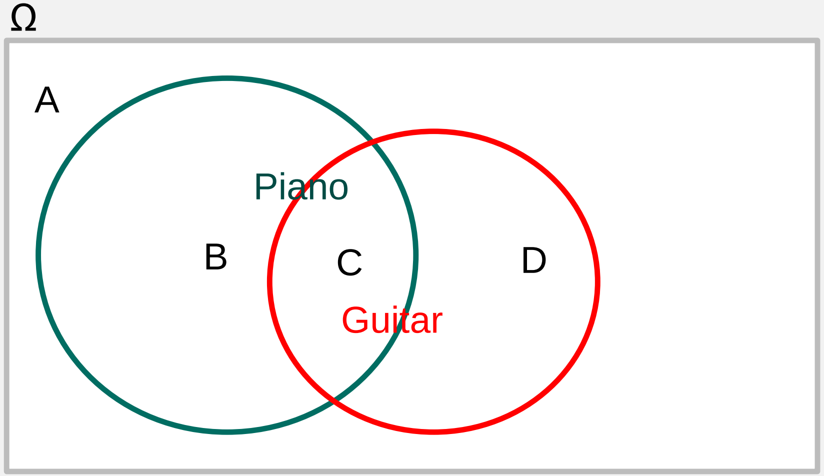
Find the probability that a student plays guitar, given that they play piano.

Problem 13

Solution 13:

$\mathbb{P}$(at least play piano) = 0.3
$\mathbb{P}$(at least play guitar) = 0.2
$\mathbb{P}$(play piano and guitar) = 0.1

$$\mathbb{P}(\mathrm{G}|\mathrm{P}) = \frac{\mathbb{P}(\mathrm{P}+\mathrm{G})}{\mathbb{P}(\mathrm{P})} = \frac{0.1}{0.3} = 0.33$$



Problem 14

Problem 14:

An ice cream shop noticed the following:
$\mathbb{P}(\text{only chocolate}) = 0.3$
$\mathbb{P}(\text{only vanilla}) = 0.1$
$\mathbb{P}(\text{chocolate and vanilla}) = 0.4$

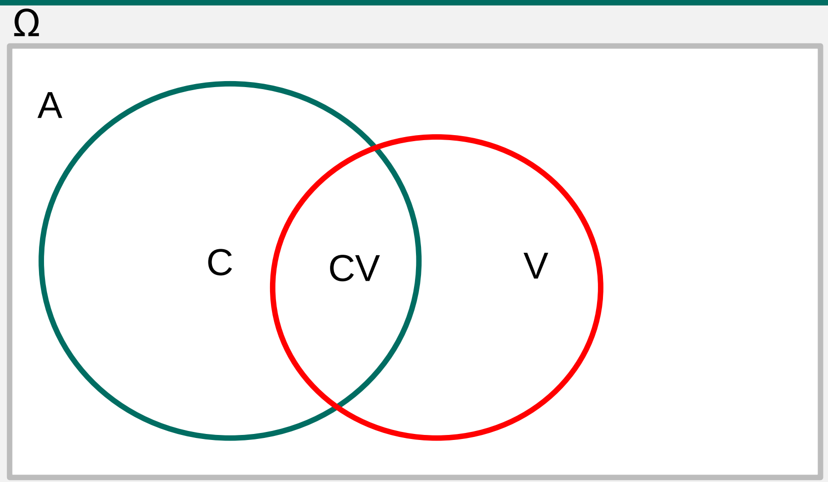
What is the probability that a random person ordered chocolate given that they ordered vanilla?

Problem 14

Problem 14:

An ice cream shop noticed the following:
$\mathbb{P}(\mathrm{C}) =$ $\mathbb{P}$(only chocolate) = 0.3
$\mathbb{P}(\mathrm{V}) =$ $\mathbb{P}$(only vanilla) = 0.1
$\mathbb{P}(\mathrm{CV}) =$ $\mathbb{P}$(chocolate and vanilla) = 0.4

$$\mathbb{P}(\mathrm{C}V|\mathrm{CV} \cup \mathrm{V}) = \frac{\mathrm{P(CV)}}{\mathrm{P(CV \cup V)}} = \frac{0.4}{0.5} = 0.8$$



Moments of Random Variables

METIS

Moments of Random Variables

Definition:

We described Random Variables in terms of their distributions. Well, if we use distributions to characterize Random Variables, we should also be able to characterize distributions and this is what **moments** are – characteristics of a distribution.

There are many moments but the two most important are:

1. **Expected Value**
2. **Variance (or Standard Deviation)**

- Depending on if the Random Variable is Discrete or Continuous, the moments will be formulated differently
- There exist many important properties of these two moments, some of which we will cover in the subsequent slides

Expected Value

Definition:

Expected Value of a Random Variable refers to the mean of the Random Variable. In other words, what value does the Random Variable take on average throughout many iterations of the Random Experiment.

Example 14:

Motivating Example: Let’s say I toss 2 coins in my Random Experiment and define the Random Variable to be the sum of the result (H – 1, T – 0) and this sum will be equal to my payout (i.e. get 2 Heads, make $2). I can then define my Probability Mass Function associated with the possible values that the Random Variable can take (0,1,2) and each possible value will be assigned a probability.

I might want to ask myself the following question: If I toss the coin 1000 times, on average, what is my payout after each toss?

Expected Value – Discrete Random Variables

Definition:

The **Expected Value** of a Discrete Random Variable is intuitive and what you would expect. It is simply the sum of the products of the possible values of the Random Variable and the probability that these values are obtained. Formally, we define the Expected Value as the following:

$$\mathbb{E}(X) = \sum_i x_i \mathbb{P}(X = x_i) = \mu_X$$

Example 15:

Following the motivating example from Example 14, the Probability Mass Function would be:

$$X = \begin{cases} 0, TT \text{ } with \text{ } probability \text{ } 0.25 \\ 1, HT, TH \text{ } with \text{ } probability \text{ } 0.5 \\ 2, HH \text{ } with \text{ } probability \text{ } 0.25 \end{cases}$$

Therefore, if I tossed the coins 1000 times, I would expect to get 250 TT, 500 HT/TH, and 250 HH

Expected Value – Discrete Random Variables

Example 15 (Continued):

If I were to calculate my expected payoff for any given toss after running the experiment 1000 times, I may intuitively calculate the formula as a sum of the recorded events of each possible values, weighted by their payoff, divided by the total number of events in the experiment:

$$\frac{0(250) + 1(500) + 2(250)}{1000} = \$1$$

Or, using the formula: $\mathbb{E}(X) = \sum_i x_i \mathbb{P}(X = x_i) = (0 * 0.25) + (1 * 0.50) + (2 * 0.25) = \$1$

Expected Value – Continuous Random Variables

Definition:

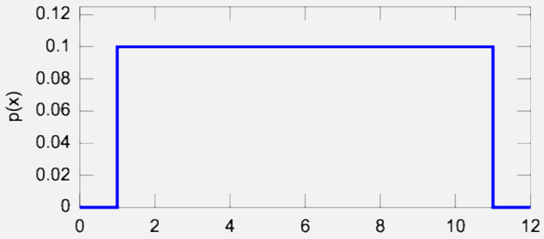
The **Expected Value** of a Continuous Random Variable is formulated differently than that of a Discrete Random Variable since we are now working with Probability Density Functions. The formula now involves an integral since we are interested in the area under the curve between two points:

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x) dx = \mu_X$$

Example 16:

If the Probability Density Function of a Random Variable X is represented as $\frac{1}{b-a}$ and shown in the picture below, then what is the Expected Value of X?

NOTE: We will name this specific Random Variable later on in the lecture as a Uniform Random Variable



Expected Value – Continuous Random Variables

Example 16 (Continued):

We can use the formula to calculate the Expected Value of the Random Variable X

$$\mathbb{E}(X) = \int_{1}^{11} x \frac{1}{11 - 1} dx$$

$$\mathbb{E}(X) = \left[\frac{x^2}{2} \times \frac{1}{10}\right] = \left[\frac{11^2}{2} \times \frac{1}{10}\right] - \left[\frac{1^2}{2} \times \frac{1}{10}\right] = 6$$

NOTE: Since this is actually a “special” Random Variable, we simply memorize the formula for its Expected Value as: $\frac{1}{2}(a + b)$

Variance

Definition:

Variance of a Random Variable refers to the spread of the distribution. In other words, on average, how far is each value of the Random Variable away from the Expected Value.

Example 17:

Motivating Example: Following from the motivating example in Example 14:

I might also want to ask myself the following question: If I toss the coin 1000 times, on average, how far away is the payout of each toss from the average payout?

Variance – Discrete Random Variables

Definition:

The **Variance** of a Discrete Random Variable is the sum of the products of the squared variance of the possible values of the Random Variable and the probabilities these values are obtained. Formally, we define the Variance as the following:

$$V(X) = \mathbb{E}\{[X - \mathbb{E}(X)]^2\} = \sum_i (x_i - \mu)^2 \mathbb{P}(X = x_i) = \sigma^2$$

Example 18:

Following the motivating example from Example 14, the Variance of the Random Variable X is:

$$V(X) = (0 - 1)^2(0.25) + (1 - 1)^2(0.5) + (2 - 1)^2(0.25) = 0.5$$

Variance – Continuous Random Variables

Definition:

The **Variance** of a Continuous Random Variable is again formulated differently than that of a Discrete Random Variable since we are now working with Probability Density Functions which requires the use of integrals:

$$V(X) = \mathbb{E}\{[X - \mathbb{E}(X)]^2\} = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \sigma^2$$

Example 19:

Following the example from Example 16, the Variance of the Random Variable X is:

$$V(X) = \int_1^{11} (x - 6)^2 \times \frac{1}{11-1} dx = [\frac{(x-6)^3}{3} \times \frac{1}{10}]_1^{11} = \left[\frac{(11-6)^3}{3} \times \frac{1}{10}\right] - \left[\frac{(1-6)^3}{3} \times \frac{1}{10}\right] = \frac{25}{3}$$

NOTE: Since this is a “special” Random Variable, we simply memorize the formula as $\frac{1}{12}(b - a)^2$

QUESTIONS?