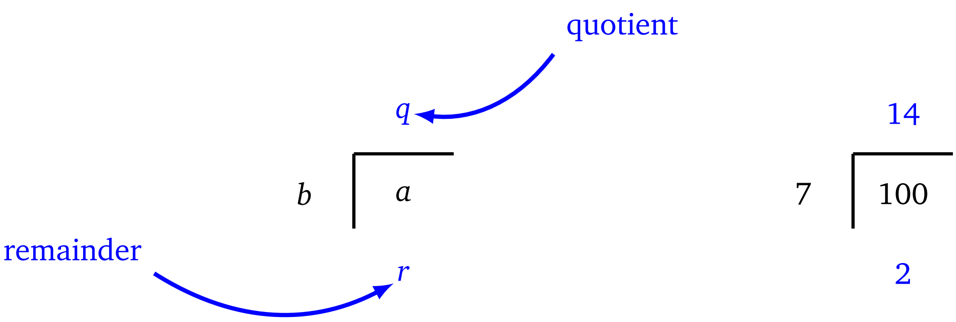
Arithmetic – While loop – I

The activities in this sheet focus on arithmetic: Euclidean division, prime numbers ... This is an opportunity to use the loop "while" intensively.

Lesson 1 (Arithmetic)**.**
We recall what Euclidean division is. Here is the division of a by b, a is a positive integer, b is a strictly positive integer (with an example of 100 divided by 7) :



We have the two fundamental properties that define q and r:

$$a = b \times q + r \qquad \text{and} \qquad 0 \leqslant r < b$$

For example, for the division of $a = 100$ by $b = 7$: we have the quotient $q = 14$ and the remainder $r = 2$ that verify $a = b \times q + r$ because $100 = 7 \times 14 + 2$ and also $r < b$ because $2 < 7$.

With `Python` :

- `a // b` returns the quotient,
- `a % b` returns the remainder.

It is easy to check that:

b is a divisor of a if and only if $r = 0$.

Activity 1 (Quotient, remainder, divisibility)**.**

Goal: use the remainder to find out if an integer divides another.

1. Program a function `quotient_remainder(a,b)` that does the following tasks for two integers $a \geqslant 0$ and $b > 0$:
   - It displays the quotient q of the Euclidean division of a per b,
   - it displays the remainder r of this division,

- it displays `True` if the remainder r is positive or zero and strictly less than b, and `False` otherwise,
- it displays `True` if you have equality $a = bq + r$, and `False` if not.

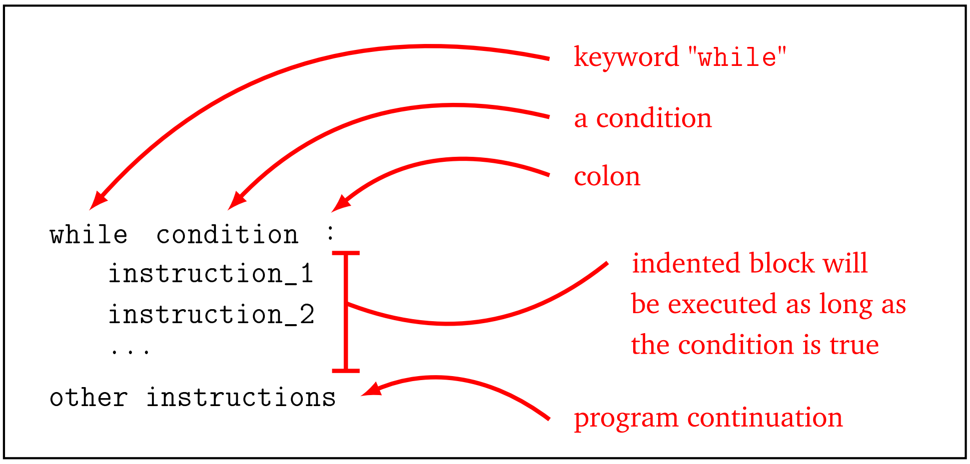
Here is for example what the call should display `quotient_remainder(100,7)`:

```
Division of a = 100 per b = 7
The quotient is q = 14
The remainder is r = 2
Check remainder: 0 <= r < b? True
Check equality: a = bq + r? True
```

Note. You have to check without cheating that we have $0 \leqslant r < b$ and $a = bq + r$, but of course it must always be true!

2. Program a function `is_even(n)` that tests if the integer n is even or not. The function returns `True` or `False`.
   Hints.
   - First possibility: calculate `n % 2` and discuss as appropriate.
   - Second possibility: calculate `n % 10` (which returns the digit of units) and discuss.
   - The smartest people will be able to write the function with only two lines (one for `def...` and the other for `return...`).
3. Program a function `is_divisible(a,b)` that tests if b divides a. The function returns `True` or `False`.

Lesson 2 (Loop "while").
The loop "while" executes instructions as long as a condition is true. As soon as the condition becomes false, it proceeds to the next instructions.



Example.

Here is a program that displays the countdown $10, 9, 8, \dots 3, 2, 1, 0$. As long as the condition $n \geqslant 0$ is true, we reduce n by 1. The last value displayed is $n = 0$, because then $n = -1$ and the condition "$n \geqslant 0$" becomes false so the loop stops.

```
n = 10
while n >= 0:
    print(n)
    n = n - 1
```

This is summarized in the form of a table:

Input: $n = 10$

n	"$n \geqslant 0$" ?	new value of n
10	yes	9
9	yes	8
...	...	...
1	yes	0
0	yes	-1
-1	no	

Display: $10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0$

Example.

This piece of code looks for the first power of 2 greater than a given integer n. The loop prints the values 2, 4, 8, 16,... It stops as soon as the power of 2 is higher or equal to n, so here this program displays 128.

```
n = 100
p = 1
while p < n:
    p = 2 * p
print(p)
```

Inputs: $n = 100$, $p = 1$

p	"$p < n$" ?	new value of p
1	yes	2
2	yes	4
4	yes	8
8	yes	16
16	yes	32
32	yes	64
64	yes	128
128	no	

Display: 128

Example.
For this last loop we have already prepared a function `is_even(n)` which returns `True` if the integer n is even and `False` otherwise. The loop does this: as long as the integer n is even, n becomes $n/2$. This amounts to removing all factors 2 from the integer n. As here $n = 56 = 2 \times 2 \times 2 \times 7$, this program displays 7.

```
n = 56
while is_even(n) == True:
    n = n // 2
print(n)
```

Input: $n = 56$

n	"is n even" ?	new value of n
56	yes	28
28	yes	14
14	yes	7
7	no	

Display: 7

For the latter example, it is much more natural to start the loop with

`while is_even(n):`

Indeed `is_even(n)` is already a value "True" or "False". We're getting closer to the English sentence "while n is even..."

Operation "+=". To increment a number you can use these two methods:

`nb = nb + 1` or `nb += 1`

The second writing is shorter but makes the program less readable.

Activity 2 (Prime numbers).

Goal: test if an integer is (or not) a prime number.

1. **Smallest divisor.**
   Program a function `smallest_divisor(n)` that returns, the smallest divisor $d \geqslant 2$ of the integer $n \geqslant 2$.
   For example `smallest_divisor(91)` returns 7, because $91 = 7 \times 13$.

   Method.
   - We remind you that d divides n if and only if `n % d` is equal to 0.
   - The bad idea is to use a loop "for d ranging from 2 to n". Indeed, if for example we know that 7 is a divisor of 91 it is useless to test if $8, 9, 10\ldots$ are also divisors because we have already found a smaller one.
   - The good idea is to use a loop "while"! The principle is: "as long as I haven't got my divisor, I keep looking for". (And so, as soon as I find it, I stop looking.)
   - In practice here are the main lines:
     – Begin with $d = 2$.
     – As long as d does not divide n then pass to the next candidate (d becomes $d + 1$).
     – At the end d is the smallest divisor of n (in the worst case $d = n$).
2. **Prime numbers (1).**
   Slightly modify your function `smallest_divisor(n)` to write a first function `is_prime_1(n)` which returns "True" if n is a prime number and "False" otherwise.

For example `is_prime_1(13)` returns `True`, `is_prime_1(14)` returns `False`.

3. **Fermat numbers.**
   Pierre de Fermat (∼1605–1665) thought that all integers of the form $F_n = 2^{(2^n)} + 1$ were prime numbers. Indeed $F_0 = 3$, $F_1 = 5$ and $F_2 = 17$ are prime numbers. If he had known Python he would probably have changed his mind! Find the smallest integer F_n which is not prime.
   Hint. With Python b^c is written `b ** c` and therefore $a^{(b^c)}$ is written `a ** (b ** c)`.

   We will improve our function which tests if a number is prime or not, it will allow us to test more quickly lots of numbers or very large numbers.

4. **Prime numbers (2).**
   Enhance your previous function to a `is_prime_2(n)` that does not test all divisors d from 2 to n, but only up to $\sqrt{n}$.
   Explanations.
   - For example, to test if 101 is a prime number, just see if it admits divisors among $2, 3, \ldots, 10$. The gain is significant!
   - This improvement is due to the following proposal: if an integer is not prime then it admits a divisor d that verifies $2 \leqslant d \leqslant \sqrt{n}$.
   - Instead of testing if $d \leqslant \sqrt{n}$, it is easier to test if $d^2 \leqslant n$.

5. **Prime numbers (3).**
   Improve your function into a function `is_prime_3(n)` using the following idea. We test if $d = 2$ divides n, but from $d = 3$, we just test the odd divisors (we test d, then $d + 2 \ldots$).
   - For example to test if $n = 419$ is a prime number, we first test if $d = 2$ divides n, then $d = 3$ and then $d = 5$, $d = 7 \ldots$
   - This allows you to do about half less tests!
   - Explanations: if an even number d divides n, then we already know that 2 divides n.

6. **Calculation time.**
   Compare the calculation times of your different functions `is_prime()` by repeating for example a million times the call `is_prime(97)`. See the course below for more information on how to do this.

Lesson 3 (Calculation time).
There are two ways to run programs faster: a good and a bad one. The bad way is to buy a more powerful computer. The good method is to find a more efficient algorithm!
With Python, it is easy to measure the execution time of a function in order to compare it with the execution time of another. Just use the module `timeit`.
Here is an example: we measure the computation time of two functions that have the same purpose, test if an integer n is divisible by 7.

```
# First function (not very clever)
def my_function_1(n):
    divis = False
    for k in range(n):
        if k*7 == n:
            divis = True
    return divis
```

```
# Second function (faster)
def my_function_2(n):
    if n % 7 == 0:
        return True
    else:
        return False

# Measurement of execution times
import timeit

print(timeit.timeit("my_function_1(1000)",
    setup="from __main__ import my_function_1",
    number=100000))
print(timeit.timeit("my_function_2(1000)",
    setup="from __main__ import my_function_2",
    number=100000))
```

Results.

The result depends on the computer, but allows the comparison of the execution times of the two functions.

- The measurement for the first function (called 100 000 times) returns 5 seconds. The algorithm is not very clever. We're testing if $7 \times 1 = n$, then test $7 \times 2 = n$, $7 \times 3 = n \ldots$
- The measurement for the second function returns 0.01 second! We test if the remainder of n divided by 7 is 0. The second method is therefore 500 times faster than the first.

Explanations.

- The module is named `timeit`.
- The function `timeit.timeit()` returns the execution time in seconds. It takes as parameters:
  - a string for the call of the function to be tested (here we ask if 1000 is divisible by 7),
  - an argument `setup="..."` which indicates where to find this function,
  - the number of times you have to repeat the call to the function (here `number=100000`).
- The number of repetitions must be large enough to avoid uncertainties.

Activity 3 (More prime numbers).

Goal: program more loops "while" and study different kinds of prime numbers using your function `is_prime()`.

1. Write a function `prime_after(n)` that returns the first prime number p greater than or equal to n.
   For example, the first prime number after $n = 60$ is $p = 61$. What is the first prime number after $n = 100\,000$?
2. Two prime numbers p and $p + 2$ are called ***twin prime numbers***. Write a function `twin_prime_after(n)` that returns the first pair p, $p + 2$ of twin prime numbers, with $p \geqslant n$.
   For example, the first pair of twin primes after $n = 60$ is $p = 71$ and $p + 2 = 73$. What is the first pair of twin primes after $n = 100\,000$?
3. An integer p is a ***Germain prime number*** if p and $2p + 1$ are prime numbers. Write a function `germain_after(n)` that returns the pair p, $2p + 1$ where p is the first Germain prime number $p \geqslant n$.

For example, the first Germain prime number after $n = 60$ is $p = 83$, with $2p + 1 = 167$. What is the first Germain prime number after $n = 100\,000$?