

Probabilities – Par- rondo's paradox

You will program two simple games. When you play these games, you are more likely to lose than to win. However, when you play both games at the same time, you have a better chance of winning than losing! It's a paradoxical situation.

Lesson 1 (Random – Winning – Expected value).

A player plays a game of chance: he throws a coin; depending on the result he wins or loses money. We cannot predict how much the player will win or lose for sure, but we will introduce a value that estimates how much the player can expect to win on average if he plays many times.

- At the start the player's total winnings are zero: $g = 0$.
- Each time he draws, he throws a coin. If he wins, he gets one dollar, if he loses he owes one dollar.
- In the games we study, the coin is not balanced (it is a little rigged). The player does not have as many chances to win as to lose.
- Draws are repeated a number of times N . After N draws, the player's winnings (which can be positive or negative) are totaled.
- The **expected value**, is the amount that the player can expect to win with each throw. The expected value is estimated by averaging the earnings of a large number of draws. In other words, we have the formula:

$$\text{expected value} \simeq \frac{\text{winnings after } N \text{ throws}}{N} \quad \text{with large } N.$$

For our games, the expected value will be a real number between -1 and $+1$.

- Examples:
 - If the coin is well balanced (50 chances out of 100 to win), then for a large number of N draws, the player will win about as many times as he loses; his winnings will be close to 0 and so the expected value will be close to $\frac{0}{N} = 0$. On average he earns 0 dollar per draw.
 - If the coin is faked and the player wins all the time, then after N draws, he has pocketed N dollars. The expected value is therefore $\frac{N}{N} = 1$.
 - If the coin is faked so that the player loses all the time, then after N draws, his win is $-N$ dollars. The expected value is therefore $\frac{-N}{N} = -1$.
 - An expected value of -0.5 means that on average the player loses 0.5 dollar per draw. This is possible with an unbalanced coin that wins in only one out of four cases. (Check the calculation!) If the player plays 1000 times, we can estimate that he will lose 500 dollars ($-0.5 \times 1000 = -500$).

Activity 1 (Game A: first losing game).

Goal: model a first simple game, which on average makes the player loses.

Game A. In this first game, we throw a slightly unbalanced coin: the player wins one dollar in 49 cases on 100; he loses one dollar in 51 cases on 100.

1. **One draw.** Write a function `throw_game_A()` that does not depend on any argument and that models a draw of the game A. For this purpose:
 - Pick a random number $0 \leq x < 1$ using the `random()` function of the `random` module.
 - Returns $+1$ if x is smaller than 0.49 ; and -1 otherwise.
2. **Gain.** Write a function `gain_game_A(N)` that models N draws from game A and returns the total winnings of these draws. Of course, the result depends on the actual runs, it can vary from one time to another.
3. **Expected value.** Write a function `expected_value_game_A(N)` that returns an estimate of the expected value of the game A according to the formula:

$$\text{expected value} \simeq \frac{\text{winnings after } N \text{ throws}}{N} \quad \text{with large } N.$$

4. **Conclusion.**
 - (a) Estimate the expected value by making at least one million draws.
 - (b) What does it mean that the expected value is negative?
 - (c) Deduct from the value of the expected value, the win (or loss) that I can expect by playing 1000 times in game A.

Activity 2 (Game B: second losing game).

Goal: model a second game that is a little more complicated which on average still makes the player loses.

Game B. The second game is a little more complicated. At the beginning, the player starts with a zero win: $g = 0$. Then, depending on this gain, he plays one of the following two subgames:

- **Subgame B1.** If the gain g is a multiple of 3, then he throws a very disadvantageous coin: the player wins a dollar in only 9 cases on 100 (so he loses a dollar in 91 cases on 100).
 - **Subgame B2.** If the gain g is not a multiple of 3, then he throws an advantageous coin: the player wins a dollar in 74 cases on 100 (so he loses a dollar in 26 cases on 100).
1. **One draw.** Write a `throw_game_B(g)` function that depends on the win already acquired and models a draw from game B. You can use the `g%3 == 0` test to find out if g is a multiple of 3.
 2. **Gain.** Write a function `gain_game_B(N)` that models N draws playing game B (starting from a zero initial gain) and returns the total gain of these draws.
 3. **Expected value.** Write a function `expected_value_game_B(N)` that returns an estimate of the expected value of the game B.
 4. **Conclusion.**
 - (a) Estimates the expected value by making at least one million draws.
 - (b) How much can I expect to win or lose by playing 1000 times in game B?

Activity 3 (Games A and B: a winning game!).

Goal: invent a new game by playing at each round game A or game B; strangely enough this game makes you win on average! That's Parrondo's paradox.

Games AB. In this third game, we play each round either game A or game B (the choice is made at

random). At the beginning the player starts with a zero win: $g = 0$. At each step, he chooses at random (50% of luck each):

- to play game A once,
 - or to play game B once; more precisely with subgame B1 or subgame B2 depending on the win already acquired g .
1. **One draw.** Write a `throw_game_AB(g)` function that depends on the win already acquired and models a draw of the AB game.
 2. **Gain.** Write a function `gain_game_AB(N)` that models N draws from the AB game (starting from a zero initial gain) and returns the total gain of these draws.
 3. **Expected value.** Write a function `expected_value_game_AB(N)` that returns an estimate of the expected value of the AB game.
 4. **Conclusion.**
 - (a) Estimate the expected value by performing at least one million game turns.
 - (b) What can we say this time about the sign of expected value?
 - (c) How much can I expect to win or lose by playing AB game 1000 times? Surprising, isn't it?

Reference: “Parrondo’s paradox”, Hélène Davaux, La gazette des mathématiciens, July 2017.