Binomial and the Exponential Family

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We need to show that the binomial is a member of the exponential family and to prove the sufficiency for the $\tau(x)$ that we get. The Exponential family: $f(x) = a(\theta)b(x)e^{\eta(\theta)\tau(x)}$

Let's play with the binomial distribution:

$$\begin{aligned} P(X = x) &= \binom{n}{x} p^x (1-p)^{n-x} \\ &= \binom{n}{x} e^{xlnp} e^{(n-x)xln(1-p)} \\ &= \binom{n}{x} \frac{e^{xlnp} e^{nln(1-p)}}{e^{xln(1-p)}} \\ &= \binom{n}{x} e^{xln\frac{p}{1-p}} e^{nln(1-p)} \end{aligned}$$

$b(x) = \binom{n}{x}$, $\tau(x) = x$ and $\eta(\theta) = ln\frac{p}{1-p}$

Let's decompose X:
since $x_1, .., x_n \overset{iid}{\sim} Ber(p)$

$$\begin{aligned} P(x_1, .., x_n) &= \prod_{i=1}^{n} p_i^x (1-p)^{1-x_i} \\ &= \prod_{i=1}^{n} e^{x_i lnp} e^{(n-x_i)x_i ln(1-p)} \\ &= \prod_{i=1}^{n} e^{x_i ln\frac{p}{1-p}} e^{ln(1-p)} \\ &= (1-p)^n e^{\sum_i^n x_i ln\frac{p}{1-p}} \end{aligned}$$

Thus, $\tau(x) = \sum x_i$

Sufficiency means $P(X = x|\tau(x) = t, \theta) = P(X = x|\tau(x) = t)$

$$
\begin{aligned}
P(X = x|\tau(x) = t, \theta) &= \frac{P(X = x, \tau(x) = t|\theta)}{P(\tau(x) = t|\theta)} \\
&= \frac{(1-p)^n e^{\sum x_i ln \frac{p}{1-p}}}{\binom{n}{\sum x_i} p^{\sum x_i} (1-p)^{n - \sum x_i}} \\
&= \frac{(1-p)^{\sum x_i} e^{\sum x_i ln \frac{p}{1-p}}}{\binom{n}{\sum x_i} p^{\sum x_i}} \\
&= \frac{(1-p)^{\sum x_i} \frac{p}{1-p}^{\sum x_i}}{\binom{n}{\sum x_i} p^{\sum x_i}} \\
&= \frac{1}{\binom{n}{\sum x_i}} \square
\end{aligned}
$$