Ordinary Least Squares (Linear) Regression

Department of Political Science and Government
Aarhus University

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Uses of Regression

1. Description
2. Prediction
3. Causal Inference

Descriptive Inference

1. We want to understand a *population* of cases
2. We cannot observe them all, so:
    1. Draw a *representative* sample
    2. Perform mathematical procedures on sample data
    3. Use assumptions to make inferences about population
    4. Express uncertainty about those inferences based on assumptions

Parameter Estimation

- We want to observe population *parameter* θ
- If we obtain a representative sample of population units:
  - Our sample statistic $\hat{\theta}$ is an unbiased estimate of θ
  - Our sampling procedure dictates how uncertain we are about the value of θ

An Example

- We want to know $\bar{Y}$ (population mean)
- Our *estimator* is the sample mean formula which produces the sample *estimate* $\bar{y}$:

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{1}$$

- The *sampling variance* is our uncertainty:

$$Var(\bar{y}) = \frac{s^2}{n} \tag{2}$$

where s^2 = sample element variance

Uncertainty

- We never know θ
- Our $\hat{\theta}$ is an estimate that may not equal θ
  - Unbiased due to **Law of Large Numbers**
  - For $\bar{y}$: $N(Y, \sigma^2)$
- The size of sampling variance depends on:
  - Element variance
  - Sample size!
- Note: $SE(\bar{y}) = \sqrt{Var(\bar{y})}$
- We may want to know $\hat{\theta}$ per se, but we are mostly interested in it as an estimate of θ

Causal Inference

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1. Everything that goes into descriptive inference

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2. Plus, philosophical assumptions

Causal Inference

1. Everything that goes into descriptive inference
2. Plus, philosophical assumptions
3. Plus, randomization *or* perfectly specified model

Questions about philosophical assumptions?

Ways of Thinking About OLS

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1. Estimating Unit-level Causal Effect
2. Ratio of $Cov(X, Y)$ and $Var(X)$
3. Minimizing residual sum of squares (SSR)
4. Line (or surface) of best fit

Bivariate Regression I

- Y is continuous
- X is a randomized treatment indicator/dummy $(0, 1)$
- How do we know if the treatment X had an effect on Y?

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- Y is continuous
- X is a randomized treatment indicator/dummy $(0, 1)$
- How do we know if the treatment X had an effect on Y?
- Look at mean-difference: $E[Y_i|X_i = 1] - E[Y_i|X_i = 0]$

Three Equations

1. Population: $Y = \beta_0 + \beta_1 X \ \ (+\epsilon)$

2. Sample estimate: $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$

3. Unit:

$$\begin{aligned} y_i &= \hat{\beta}_0 + \hat{\beta}_1 x_i + e_i \\ &= \bar{y}_{0i} + (y_{1i} - y_{0i}) x_i + (y_{0i} - \bar{y}_{0i}) \end{aligned}$$

Bivariate Regression I

- Mean difference ($E[Y_i|X_i = 1] - E[Y_i|X_i = 0]$) is the regression line slope
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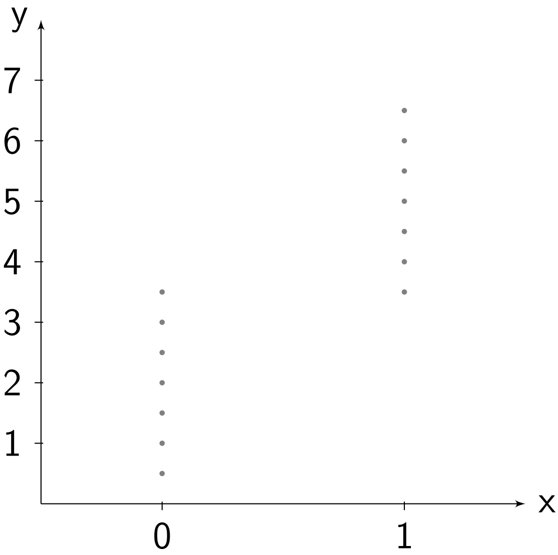
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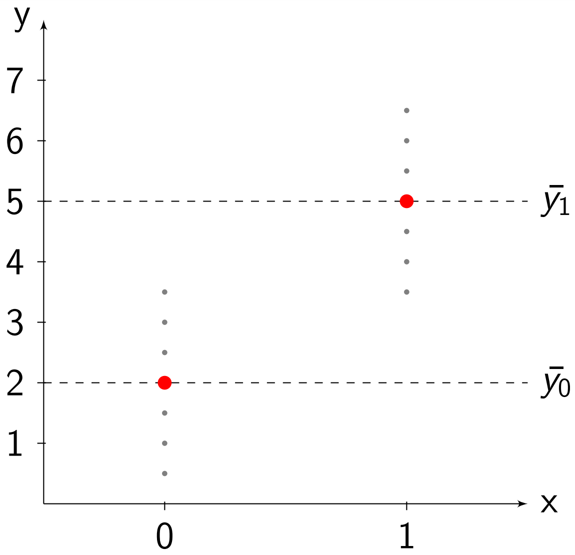
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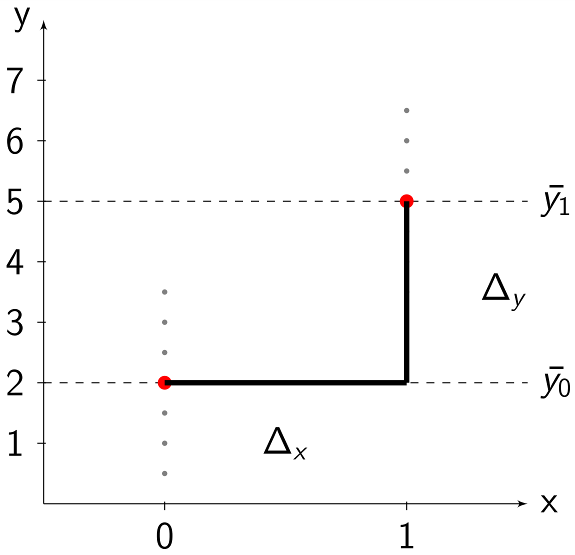
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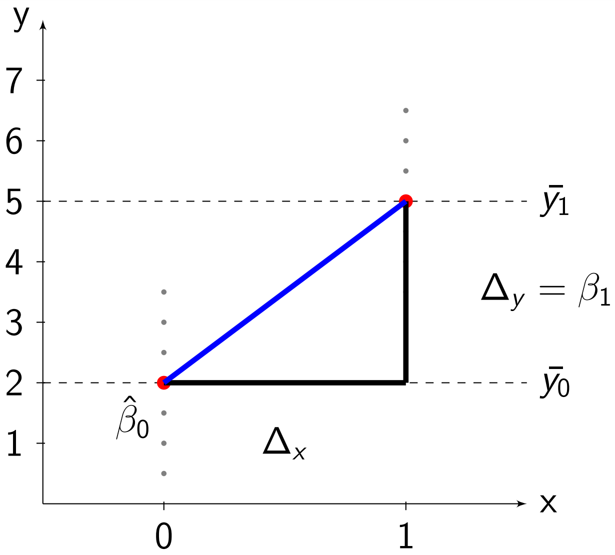
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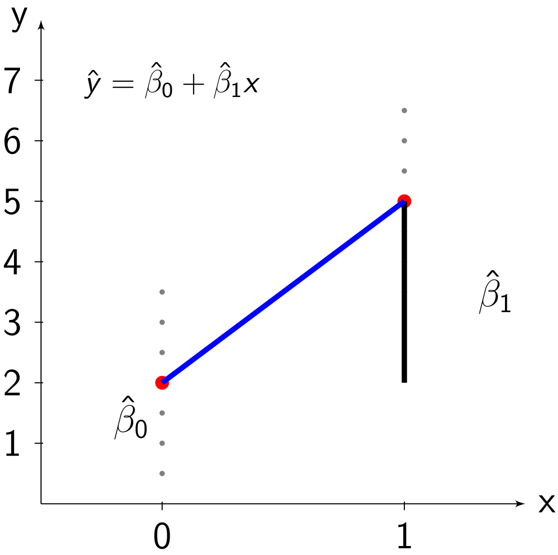
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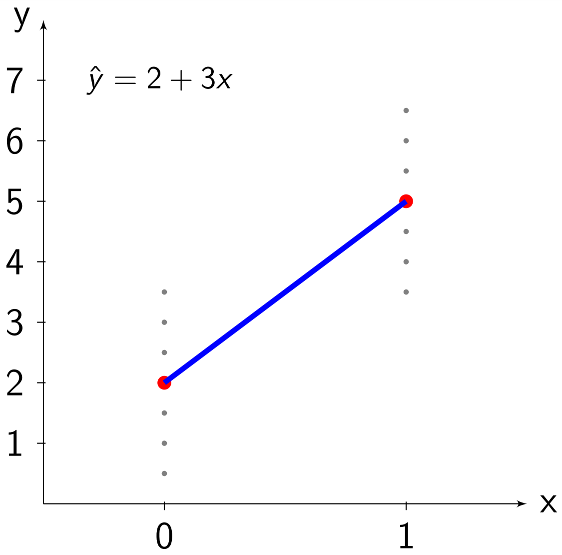
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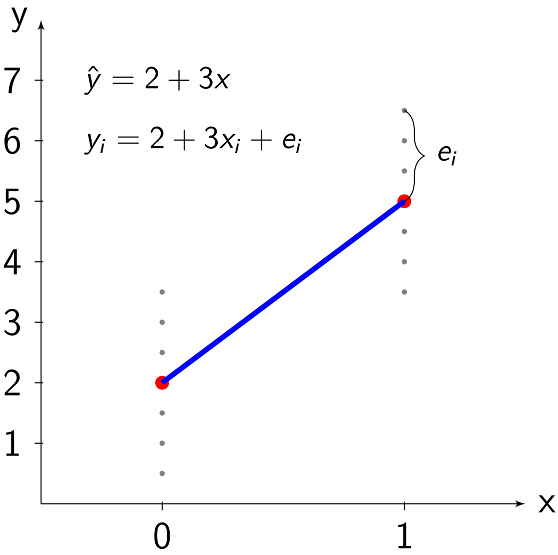
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$\hat{y} = 2 + 3x$
$y_i = 2 + 3x_i + e_i$
e_i
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x

Systematic versus unsystematic component of the data

- Systematic: Regression line (slope)
    - Linear regression estimates the conditional means of the population data (i.e., $E[Y|X]$)

- Unsystematic: Error term is the deviation of observations from the line
    - The difference between each value y_i and $\hat{y}_i$ is the *residual*: e_i
    - OLS produces an estimate of the relationship between X and Y that minimizes the *residual sum of squares*

Why are there residuals?

Why are there residuals?

- Omitted variables
- Measurement error
- Fundamental randomness

Bivariate Regression I

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- How do we know if this is a *significant* difference?
    - We'll come back to that

Ways of Thinking About OLS

1. Estimating Unit-level Causal Effect

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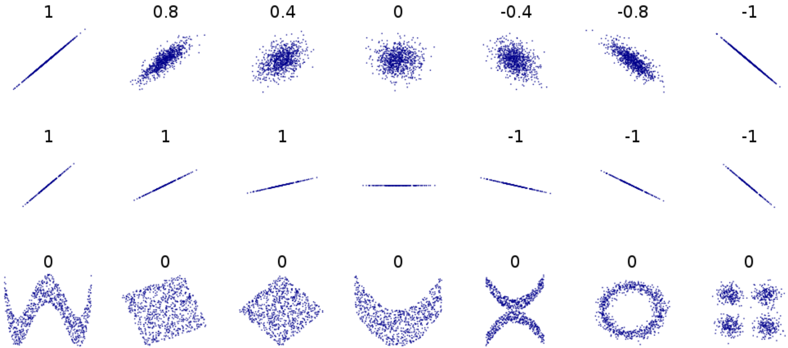
Bivariate Regression II

- Y is continuous
- X is continuous (and randomized)
- How do we know if the treatment X had an effect on Y?
  - Correlation coefficient (ρ)
  - Regression coefficient (slope; β_1)

Correlation Coefficient (ρ)

- Measures how well a scatterplot is represented by a straight (non-horizontal) line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Correlation Coefficient (ρ)

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- Formal definition: $\frac{Cov(X,Y)}{\sigma_X \sigma_y}$
- As a reminder:
  - $Cov(x, y) = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$
  - $s_x = \sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}$

OLS Coefficient (β_1)[1]

- Measures ΔY given ΔX

[1] Multivariate formula involves matrices; Week 20

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- $\hat{\rho}$ and $\hat{\beta}_1$ are just scaled versions of $\widehat{Cov}(x,y)$

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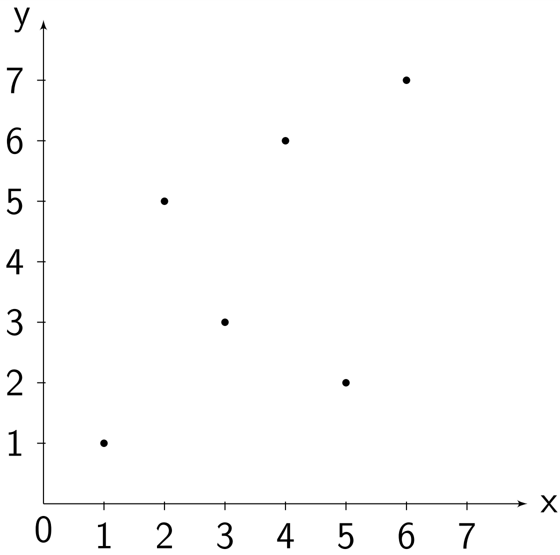
1. Do we need variation in X?
   - Yes, otherwise dividing by zero
2. Do we need variation in Y?
   - No, $\hat{\beta}_1$ can equal zero
3. How many observations do we need?
   - $n \geq k$, where k is number of parameters to be estimated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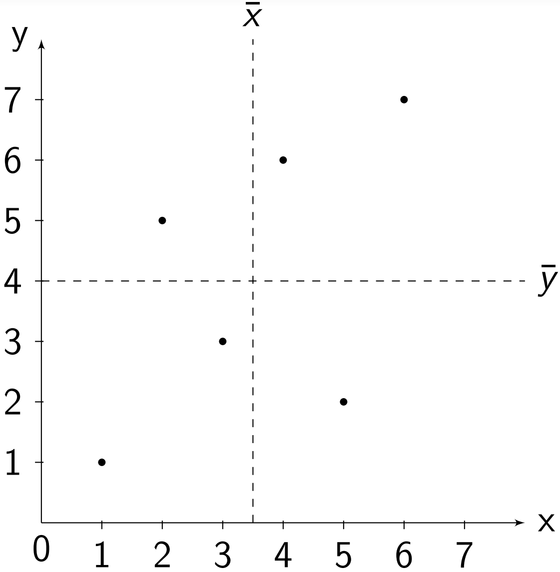
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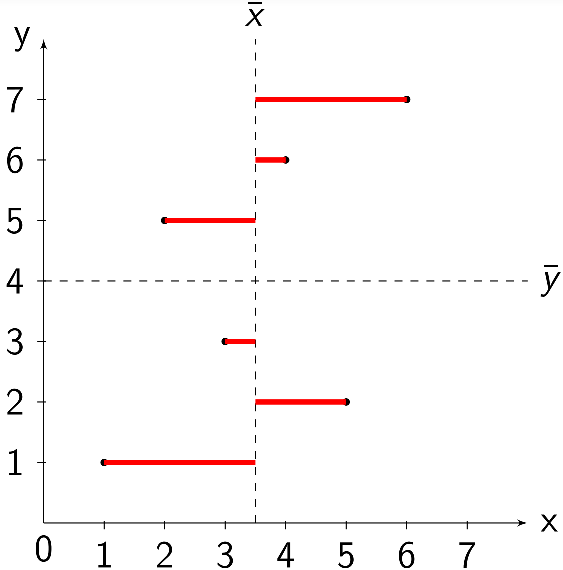
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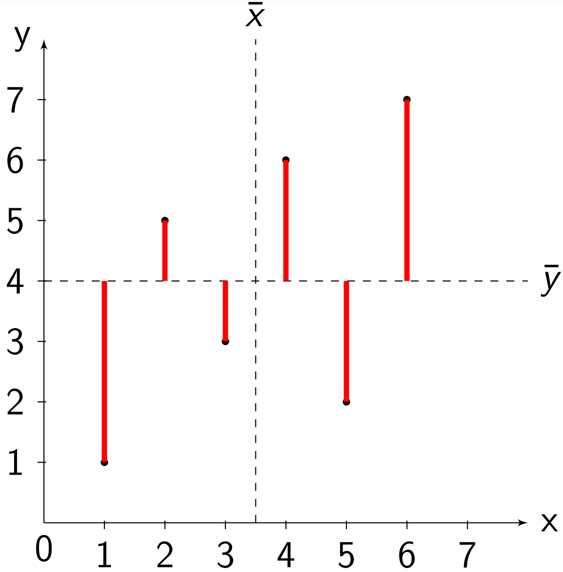
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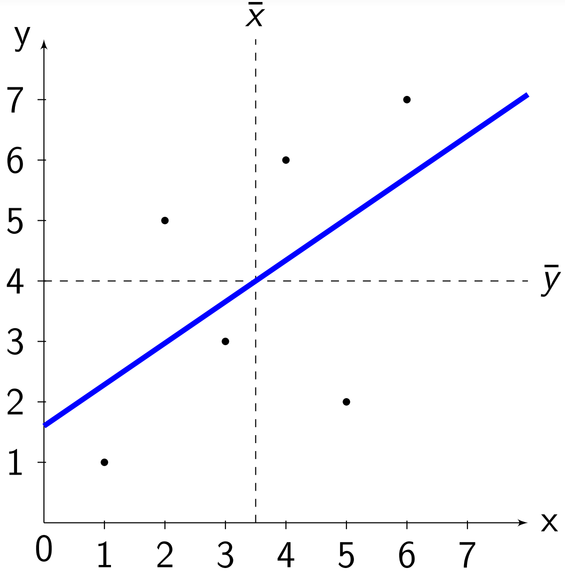
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Calculations

x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})(y_i - \bar{y})$	$(x_i - \bar{x})^2$
1	1	?	?	?	?
2	5	?	?	?	?
3	3	?	?	?	?
4	6	?	?	?	?
5	2	?	?	?	?
6	7	?	?	?	?

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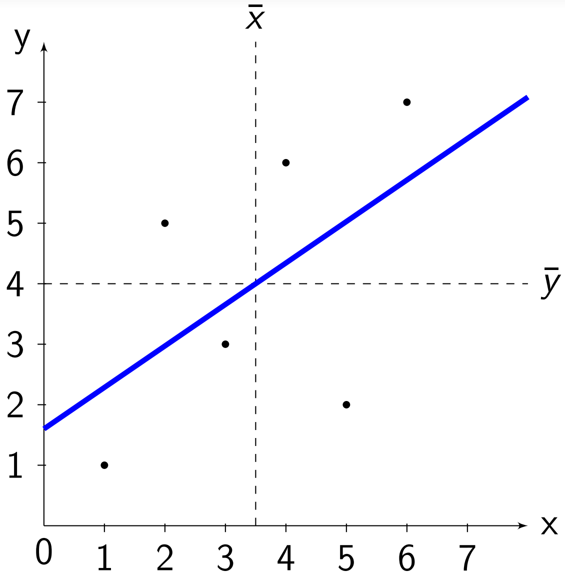
- Ex.: $\hat{\beta}_0 = 4 - 0.6857 * 3.5 = 1.6$
- $\hat{y} = 1.6 + 0.6857\hat{x}$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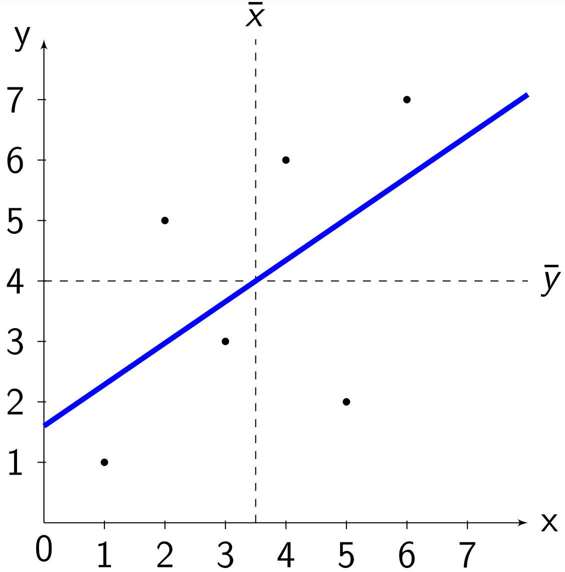
OLS Minimizes SSR

- Total Sum of Squares (SST): $\sum_{i=1}^{n}(y_i - \bar{y})^2$
- We can partition SST into two parts (ANOVA):
  - Explained Sum of Squares (SSE)
  - Residual Sum of Squares (SSR)
- $SST = SSE + SSR$
- OLS is the line with the lowest SSR



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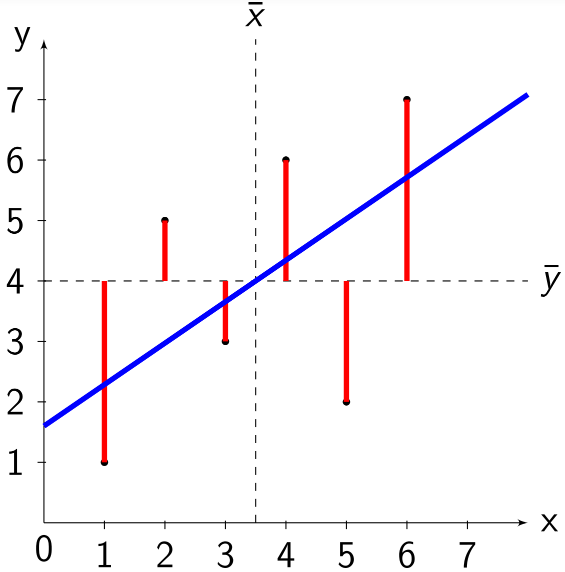
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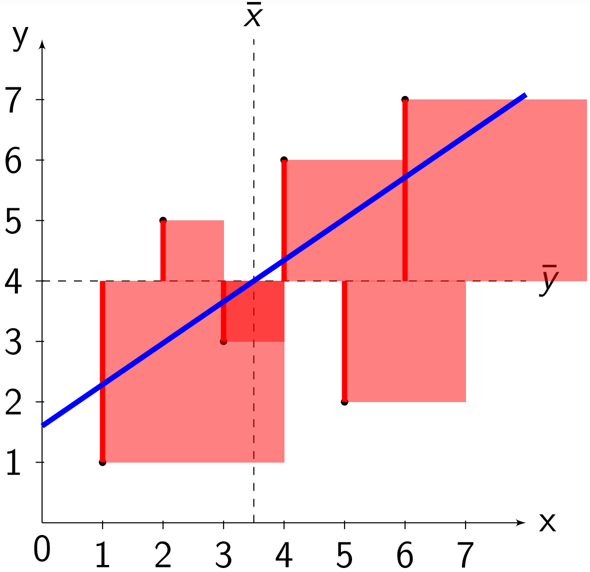
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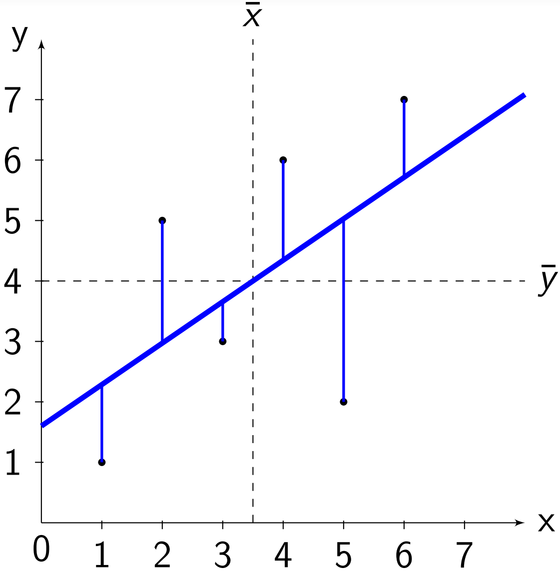

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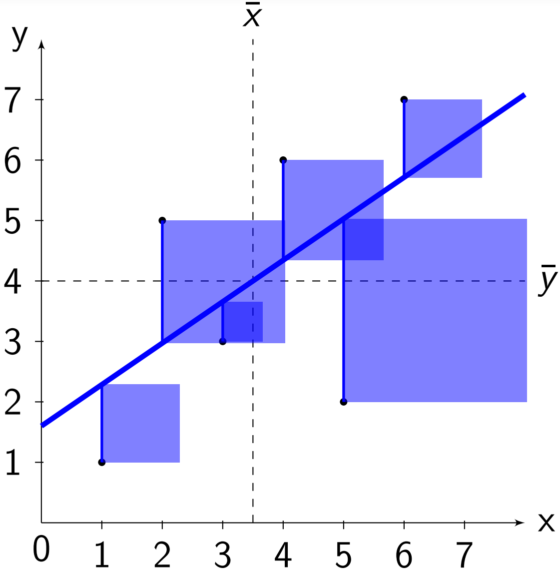

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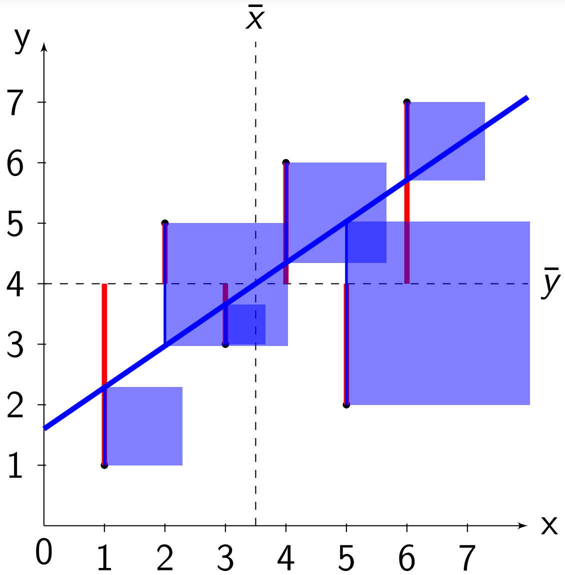
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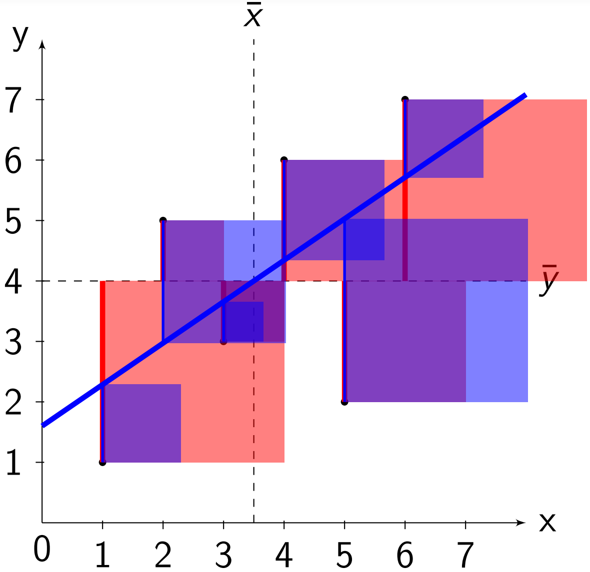

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Questions about OLS calculations?

Are Our Estimates Any Good?

Yes, if:

1. Works mathematically
2. Causally valid theory
3. Linear relationship between X and Y
4. X is measured without error
5. No missing data (or MCAR; see Lecture 5)
6. No confounding

Linear Relationship

- If linear, no problems
- If non-linear, we need to transform
  - Power terms (e.g., x^2, x^3)
  - log (e.g., $log(x)$)
  - Other transformations
  - If categorical: convert to set of indicators
  - Multivariate interactions (next week)

Coefficient Interpretation Activity

- Four types of variables:
  1. Indicator (0,1)
  2. Categorical
  3. Ordinal
  4. Interval
- How do we interpret a coefficient on each of these types of variables?

Notes on Interpretation

- Effect β_1 is constant across values of x

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- That is not true when there are:
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  - Nonlinear regression models (e.g., logit/probit)
- Interpretations are sample-level
  - Sample representativeness determines generalizability
- Remember uncertainty
  - These are *estimates*, not population parameters

Measurement Error in Regressor(s)

- We want effect of x, but we observe x^*, where $x = x^* + w$:

$$\begin{aligned} y &= \beta_0 + \beta_1 x^* + \epsilon \\ &= \beta_0 + \beta_1 (x - w) + \epsilon \\ &= \beta_0 + \beta_1 x + (\epsilon - \beta_1 w) \\ &= \beta_0 + \beta_1 x + v \end{aligned}$$

Measurement Error in Regressor(s)

- Produces *attenuation*: as measurement error increases, $\beta_1 \to 0$
- Our coefficients fit the observed data
- But they are *biased* estimates of our population equation
    - This applies to all $\hat{\beta}$ in a multivariate regression
    - Direction of bias is unknown

Measurement Error in Y

- Not necessarily a problem
- If *random* (i.e., uncorrelated with x), it costs us precision
- If *systematic*, who knows?!
- If *censored*, see Lectures 11 and/or 12

Missing Data

- Missing data can be a big problem
- We will discuss it in Lecture 5

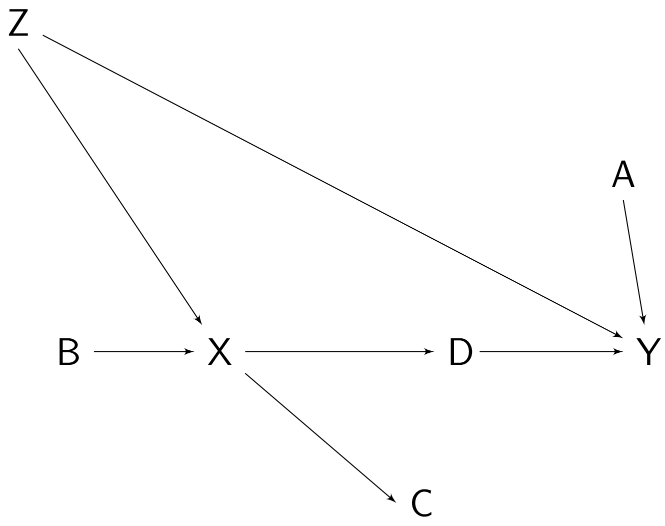
Confounding (Selection Bias)

- If x is not randomly assigned, potential outcomes are not independent of x
- Other factors explain why a unit i received their particular value x_i
- In matching, we obtain this *conditional independence* by comparing units that are identical on all confounding variables

Omitted Variables

$$\underbrace{E[Y_i|X_i=1]-E[Y_i|X_i=0]=}_{\text{Naive Effect}}$$

$$\underbrace{E[Y_{1i}|X_i=1]-E[Y_{0i}|X_i=1]}_{\text{Treatment Effect on Treated (ATT)}}+\underbrace{E[Y_{0i}|X_i=1]-E[Y_{0i}|X_i=0]}_{\text{Selection Bias}}$$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Omitted Variable Bias

- We want to estimate:

$$Y = \beta_0 + \beta_1 X + \beta_2 Z + \epsilon$$

- We actually estimate:

$$\begin{aligned}\tilde{y} &= \tilde{\beta}_0 + \tilde{\beta}_1 x + \epsilon \\ &= \tilde{\beta}_0 + \tilde{\beta}_1 x + (0 * z) + \epsilon \\ &= \tilde{\beta}_0 + \tilde{\beta}_1 x + \nu\end{aligned}$$

- Bias: $\tilde{\beta}_1 = \hat{\beta}_1 + \hat{\beta}_2 \tilde{\delta}_1$, where $\tilde{z} = \tilde{\delta}_0 + \tilde{\delta}_1 x$

Size and Direction of Bias

- Bias: $\tilde{\beta}_1 = \hat{\beta}_1 + \hat{\beta}_2 \tilde{\delta}_1$, where $\tilde{z} = \tilde{\delta}_0 + \tilde{\delta}_1 x$

	$Corr(x,z) < 0$	$Corr(x,z) > 0$
$\beta_2 < 0$	Positive	Negative
$\beta_2 > 0$	Negative	Positive

Aside: Three Meanings of "Endogeneity"

Formally endogeneity is when $Cov(X, \epsilon) \neq 0$

1. Measurement error in regressors
2. Omitted variables associated with included regressors
    - "Specification error"
    - Confounding
3. Lack of temporal precedence

Example: Englebert

- What is his research question?
- What is his theory? What does the graph look like?
- What is his analysis?

Common Conditioning Strategies

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Which of these are good strategies?

What goes in our regression?

- Use theory to build causal models
  - Often, a causal graph helps
- Some guidance:

What goes in our regression?

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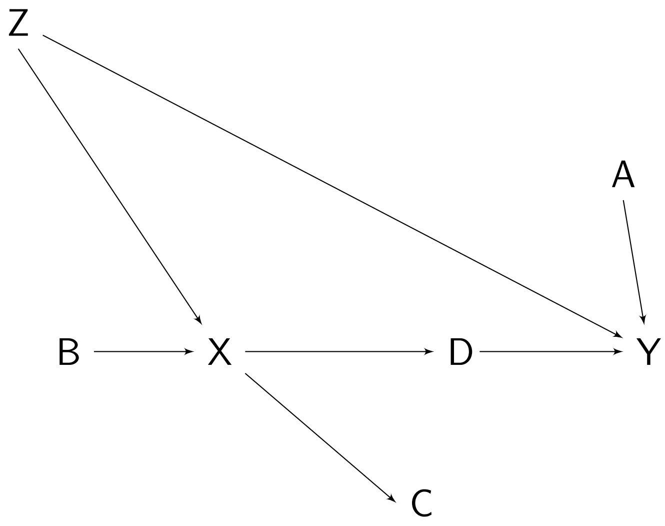
Z
A
B
X
D
Y
C

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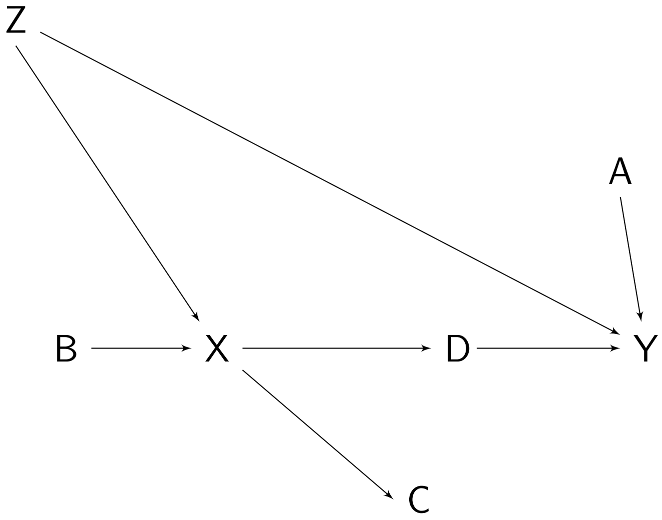
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Post-treatment Bias

- We usually want to know the **total effect** of a cause
- If we include a mediator, D, of the $X \rightarrow Y$ relationship, the coefficient on X:
    - Only reflects the **direct** effect
    - Excludes the **indirect** effect of X through M
- So don't control for mediators!

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Minimum Mathematical Requirements

1. Do we need variation in X?
   - Yes, otherwise dividing by zero
2. Do we need variation in Y?
   - No, $\hat{\beta}_1$ can equal zero
3. How many observations do we need?
   - $n \geq k$, where k is number of parameters to be estimated

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4. Can we have highly correlated regressors?
   - Generally no (due to multicollinearity)

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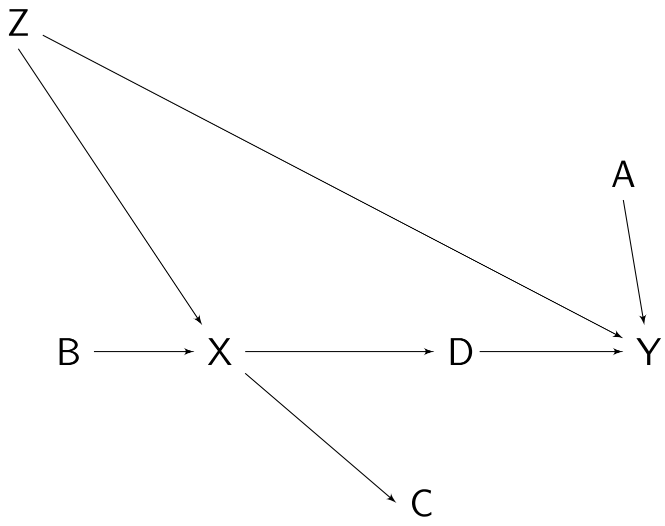
  - Including variables that affect Y alone increases certainty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Questions about specification?

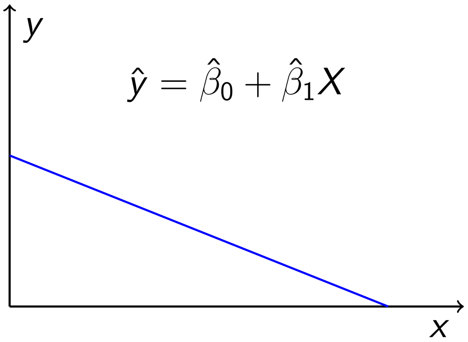
Multivariate Regression Interpretation

- All our interpretation rules from earlier still apply in a multivariate regression
- Now we interpret a coefficient as an effect "all else constant"
- Generally, not good to give all coefficients a causal interpretation
  - Think "forward causal inference"
  - We're interested in the $X \to Y$ effect
  - All other coefficients are there as "controls"

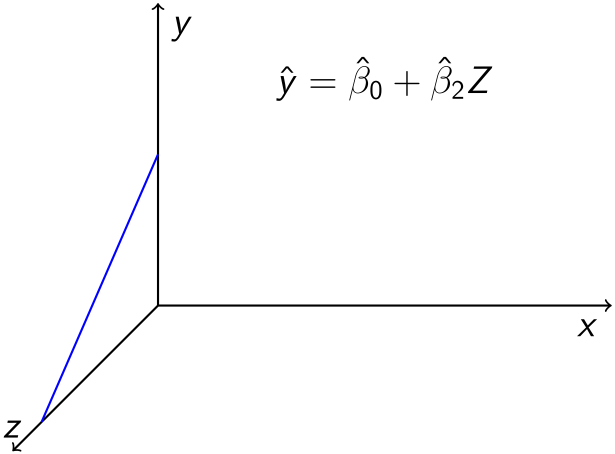
From Line to Surface I

- In simple regression, we estimate a **line**
- In multiple regression, we estimate a **surface**
- Each coefficient is the *marginal effect*, all else constant (at mean)
- This can be hard to picture in your mind

From Line to Surface II



From Line to Surface II



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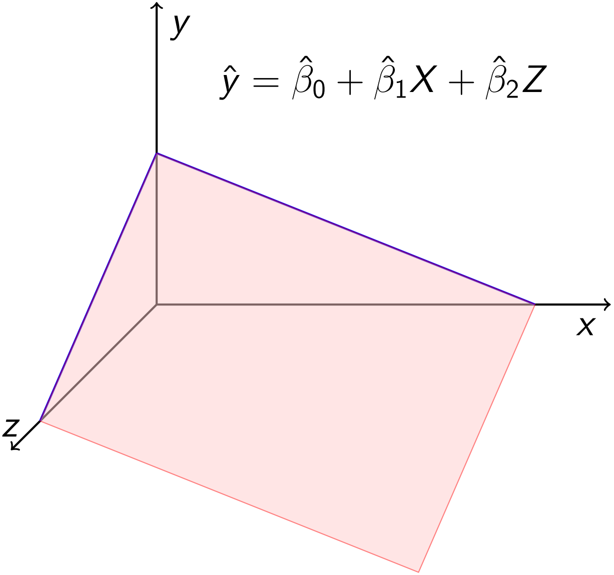
From Line to Surface II



$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 Z$$

Are Our Estimates Any Good?

Yes, if:

1. Works mathematically
2. Causally valid theory
3. Linear relationship between X and Y
4. X is measured without error
5. No missing data (or MCAR; see Lecture 5)
6. No confounding

OLS is BLUE

- BLUE: Best Linear Unbiased Estimator
- Gauss Markov Assumptions:
  1. Linearity in parameters
  2. Random sampling
  3. No multicollinearity
  4. Exogeneity ($E[\epsilon|\mathbf{X}] = 0$)
  5. Homoskedasticity ($Var(\epsilon|\mathbf{X}) = \sigma^2$)
- Assumptions 1–4 prove OLS is unbiased
- Assumption 5 proves OLS is the *best* estimator

Squared vs. Absolute Errors

- Conventionally use Sum of Squared Errors
- Using absolute errors is also unbiased
- Sum of Squared Errors:
  - more heavily weights outliers
  - has a smaller variance
- Thus OLS is **B**est**LUE**

Goodness-of-Fit

- We want to know: "How good is our model?"

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- We can answer:
  "How well does our model fit the observed data?"

Goodness-of-Fit

- We want to know: “How good is our model?”
- We can answer:
  “How well does our model fit the observed data?”
- Is this what we want to know?

Correlation

- Definition: $Corr(x, y) = \hat{r}_{x,y} = \frac{Cov(x,y)}{(n-1)s_x s_y}$
- Slope $\hat{\beta}_1$ and correlation $\hat{r}_{x,y}$ are simply different scalings of $Cov(x, y)$
- Interpretation: How well the bivariate relationship is summarized by a cloud of points?
- Units: none (range -1 to 1)

Coefficient of Determination (R^2)

- Definition: $R^2 = \hat{r}^2_{x,y} = \frac{SSE}{SST} = 1 - \frac{SSR}{SST}$
- Interpretation: How much of the total variation in y is explained by the model?
- But, R^2 increases simply by adding more variables
- So, Adjusted-$R^2 = R^2 - (1 - R^2)\frac{k}{n-k-1}$, where k is number of regressors
- Units: none (range 0 to 1)

Standard Error of the Regression (SER)

- "Root mean squared error" or just σ
- Definition: $\hat{\sigma} = \sqrt{\frac{SSR}{n-p}}$, where p is number of parameters estimated
- Interpretation: How far, on average, are the observed y values from their corresponding fitted values $\hat{y}$
  - $sd(y)$ is how far, on average, a given y_i is from $\bar{y}$
  - σ is how far, on average, a given y_i is from $\hat{y}_i$
- Units: same as y (range 0 to $sd(y)$)

The F-test

- Definition: Test of whether any of our coefficients differ from zero
  - In a bivariate regression, $F = t^2$
- Interpretation: Do any of the coefficients differ from zero?
  - Not a very interesting measure
- Units: none (range 0 to ∞)

```
. reg growth lcon

      Source |       SS       df       MS              Number of obs =      44
-------------+------------------------------           F(  1,    42) =    0.09
       Model |  .000038348     1  .000038348           Prob > F      =  0.7615
    Residual |  .017255198    42  .000410838           R-squared     =  0.0022
-------------+------------------------------           Adj R-squared = -0.0215
       Total |  .017293546    43  .000402175           Root MSE      =  .02027

------------------------------------------------------------------------------
      growth |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
        lcon |  -.0017819   .0058325    -0.31   0.761    -.0135524    .0099886
       _cons |   .0158988   .0390155     0.41   0.686    -.0628376    .0946353
------------------------------------------------------------------------------
```

The F-test for nested models

- Can use an F-test to compare fit of two nested models?
  - $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1$
  - $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \ldots$
- *Reduced* model is nested within *expanded* model
- Interpretation: Does adding additional variables significantly reduce SSR?

```
. nestreg: reg growth lcon (lconsq)

+-------------------------------------------------------------+
|       |          Block  Residual                     Change |
| Block |       F     df        df   Pr > F       R2    in R2 |
|-------+-----------------------------------------------------|
|     1 |    0.09      1        42   0.7615   0.0022          |
|     2 |    7.98      1        41   0.0073   0.1649   0.1626 |
+-------------------------------------------------------------+
```

Questions about model fit?

Inference from Sample to Population

- We want to know population parameter θ
- We only observe sample estimate $\hat{\theta}$
- We have a guess but are also uncertain

Inference from Sample to Population

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- What range of values for θ does our $\hat{\theta}$ imply?
- Are values in that range large or meaningful?

How Uncertain Are We?

- Our uncertainty depends on sampling procedures
- Most importantly, *sample size*
  - As $n \to \infty$, uncertainty $\to 0$
- We typically summarize our uncertainty as the *standard error*

Standard Errors (SEs)

- Definition: "The standard error of a sample estimate is the average distance that a sample estimate ($\hat{\theta}$) would be from the population parameter (θ) if we drew many separate random samples and applied our estimator to each."
- In bivariate regression: $Var(\hat{\beta}_1) = \frac{\frac{1}{n-2}SSR}{SST_x}$
- Thus, SE is a ratio of unexplained variance in y (weighted by sample size) and variance in x
- Units: same as coefficient ($\frac{y}{x}$)

What affects size of SEs?

- Larger variance in x means smaller SEs
- More unexplained variance in y means bigger SEs
- More observations reduces the numerator, thus smaller SEs
- Other factors:
  - Homoskedasticity
  - Clustering
- Interpretation:
  - Large SE: Uncertain about population effect size
  - Small SE: Certain about population effect size

Ways to Express Our Uncertainty

1. Standard Error
2. Confidence interval
3. t-statistic
4. p-value

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------------------------------------------------------------------------------
```

Confidence Interval (CI)

- Definition: Were we to repeat our procedure of sampling, applying our estimator, and calculating a confidence interval *repeatedly* from the population, a fixed percentage of the resulting intervals would include the true population-level slope.
- Interpretation: If the confidence interval overlaps zero, we are uncertain if β differs from zero

Confidence Interval (CI)

- A CI is simply a range, centered on the slope
- Units: Same scale as the coefficient ($\frac{y}{x}$)
- We can calculate different CIs of varying *confidence*
  - Conventionally, $\alpha = 0.05$, so 95% of the CIs will include the β

t-statistic

- A measure of how large a coefficient is relative to our uncertainty about its size

- Typically used to test a formal null hypothesis:
  - No effect null: $t_{\hat{\beta}_1} = \frac{\hat{\beta}_1}{SE_{\hat{\beta}_1}}$

  - Any other null: $\frac{\hat{\beta}_1 - \alpha}{SE_{\hat{\beta}_1}}$, where α is our null hypothesis effect size

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- Note: The t-statistic from a t-test of mean-difference is the same as the t-statistic from a t-test on an OLS slope for a dummy covariate

p-value

- A summary measure in a hypothesis test
- General definition: “the probability of a statistic as extreme as the one we observed, if the null hypothesis was true, the statistic is distributed as we assume, and the data are as variable as observed”
- Definition in a regression context: “the probability of a slope as large as the one we observed . . . ”

The p-value is not:

- The probability that a hypothesis is true or false
- A reflection of our confidence or certainty about the result
- The probability that the true slope is in any particular range of values
- A statement about the importance or substantive size of the effect

Significance

1. Substantive significance

2. Statistical significance

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   - Is the effect size (or range of possible effect sizes) *important* in the real world?

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1. Substantive significance
   - Is the effect size (or range of possible effect sizes) *important* in the real world?
2. Statistical significance
   - Is the effect size (or range of possible effect sizes) larger than a predetermined threshold?
   - Conventionally, $p \leq 0.05$

Questions about inference?