The Multinomial Distribution

or...Why You Shouldn't Play Roulette

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April 5, 2016

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- Describes the number of each possible outcome.
- Classic example: rolling a weighted die

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Common to bet simply on black or red.

Suppose you bet on black three times in a row. What's the probability that you win exactly twice?

Review: The Binomial Distribution

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$$P(2B) = 3 \cdot \frac{18}{38} \cdot \frac{18}{38} \cdot \frac{20}{38} \approx 35.4\%$$

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Let $X_i \sim Bern(p)$ for $i = 1 : n$, and $Y \sim Bin(n, p)$.

$$E[Y] = E[X_1 + \dots X_n]$$

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Example: the numbers of reds, blacks, and greens after 100 spins.
$n = 100$, $p = (\frac{18}{38}, \frac{18}{38}, \frac{2}{38})$.
One possible result is (43, 49, 8).

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Component i is binomial with parameters n and p_i.
(The number of blacks was the binomial example!)

Probability Mass Functions

Binomial pmf:

$$P(Y = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \tag{1}$$

Multinomial pmf:

$$P(Z = (k_1, k_2, ..., k_m)) = \frac{n!}{k_1! \cdots k_m!} p_1^{k_1} \cdots p_m^{k_m} \tag{2}$$

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Let Z be multinomial random vector with parameters n and probability vector p. Each component is binomial, so

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For our roulette example, if Z counts the number of RBG in 190 spins, $E[Z] = (90, 90, 10)$.

10 Spins

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```
n = 10 spins

RBG =

     3     6     1

Bet RED: -4
Bet BLACK: +2
```

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```
n = 100 spins

RBG =

    52    43     5

Bet RED: +4
Bet BLACK: -14
```

1000 Spins

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1000 Spins

Bet \$1 on either `RED` or `BLACK`, for 1000 spins.

```
n = 1000 spins

RBG =

   460   480    60

Bet RED: -80
Bet BLACK: -40
```

End

Questions?