The Binomial Distribution

An Introduction

Matt Johnson

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- Flip a fair coin 10 times. How many come up heads? $n = 10$, $p = 0.5$.
- If you roll two dice 5 times, how many of the rolls will add up to 7?

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  $n = 10$, $p = 0.5$.
- If you roll two dice 5 times, how many of the rolls will add up to 7?
  $n = 5$, $p = 1/6$

You should already know...

- Random Variables
- Probability Mass Functions
- Expected Value
- Variance
- Combinations and Permutations

Step 1: The Bernoulli Distribution

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Parameterized by the probability p that $X = 1$.

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$$X = 1 \Rightarrow \text{Success!}$$

$$X = 0 \Rightarrow \text{Failure...}$$

Bernoulli Expected Value and Variance

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Example: Biased Coin

We'll use this example throughout. Biased coin, with $P(H) = 0.6$. Define H as a success.

$$P(X = 1) = P(H) = 0.6.$$

X is a Bernoulli random variable with $p = 0.6$.

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$$P(2H) = 3 \cdot 0.6^2 \cdot 0.4$$

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$$P(k \text{ heads}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Step 2: The Binomial Distribution

The Binomial Distribution

X has a binomial distribution with number of trials n and probability of success p if its pmf is given by

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A binomial random variable with parameters n and p is equivalent to the sum of n independent Bernoulli random variables with parameter p.

Binomial Expected Value

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Binomial Variance

Let $X_i \sim Bern(p)$ for $i = 1 : n$, and $Y \sim Bin(n, p)$.

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Questions?