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STATS 244 HOMEWORK 1

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A deck of 52 cards has been shuffled thoroughly. What is the probability that the four aces are next to each other?

Let A be the event that 4 aces are next to each other. Consider 4 aces as being one card, then there are $(48!4!)$ ways to shuffle the deck with the four aces together. Since the 4 aces can be anywhere in the deck, there are 49 locations they can appear in the deck. Thus there are $\frac{(49)48!4!}{52!}$

$$\frac{49!4!}{52!} = \frac{4!}{(52)(51)(50)} = \frac{1}{5525}$$

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A factory runs three shifts. In a given day, 1% of items produced by the first shift are defective, 2% of the second shit's items are defective, and 5% of the third shift's items are defective. If the shifts all have the same productivity, what percentage of items produced in a day are defective? If an item is defective, what is the probability that it was produced by the third shift?

SUPPOSE WLOG, the factory produces 300 items a day. Since all shifts have the same productivity, each shift manufacturs 100 items. Then the percentage of defective item produced per day is $\frac{8}{300} \approx$ 2.7%. Conditional on a defective item being chose, the probability that it was produced by the third shift is $P(B \mid D) = \frac{\frac{.05}{3}}{\frac{.08}{3}} = \frac{5}{7}$ where D is a defective item.

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Suppose that n components are connected in series. For each unit, there is a backup unit, and the system fails if and only if both a unit and its backup fail. Assuming the all the units are independent and fail with probability p, what is the probability that the system works? For $n = 10$ and $p = .05$.

FOR ANY GIVEN component the probability of failure is

$$p^2 = .05^2 = .0025$$

Let F be probability of failure. Following example F, we'll find the probability of the compliment of this event, $P(F^c)$ or probability that all the components work.

$$P(F) = 1 - P(F^c) \tag{1}$$
$$= 1 - (1-p)^n \tag{2}$$
$$= 1 - (1 - .0025)^{10} \tag{3}$$
$$= 1 - (.975) \tag{4}$$
$$= .025 \tag{5}$$

The probability of failure in example F was .40 so this is a considerable improvement.

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This problem introduces some aspects of a simple genetic model. Assume that genes in an organism occur in pairs and that each member of the pair can be either of the types a or A. The possible genotypes of an organism are then AA, Aa and aa (Aa and aA are equivalent). When two organisms mate, each independently contributes one of its two genese; either one of the pair is transmitted with probability .5.

1. Suppose that the genotypes of the parents are AA and Aa. Find the possible genotypes of their offspring and the corresponding probabilities.

   CONSIDER THE TABLE

	A	A
A	AA	AA
a	Aa	Aa

   The possible genotypes are $\{AA, Aa\}$ with both with .5 probability.

2. Suppose that the probabilies of the genotypes AA, Aa and aa are p, $2q$, and r, respectively, in the first generation. Find the probabilities in the second and third generations and show that these are the same. This result is called the Hardy-Weinberg Law.

   THE PROBABILITIES TABLES for each genome type given parent types are as follows.

   P(AA)

	AA	Aa	aa
AA	1	.5	0
Aa	.5	.25	0
aa	0	0	0

P(Aa)

	AA	Aa	aa
AA	0	.5	1
Aa	.5	.5	.5
aa	1	.25	0

P(aa)

	AA	Aa	aa
AA	0	0	0
Aa	0	.25	.5
aa	0	.5	1

$P_g(x)$ is the probability of a genotype for generation g.

$$\begin{aligned}
P_2(AA) &= P_2(AA \mid AA, AA)P_1(AA, AA) \\
&+ 2(P_2(AA \mid AA, Aa)P_1(AA, Aa)) \\
&+ P_2(AA \mid Aa, Aa)P_1(Aa, Aa)) \\
&= (1)(p^2) + 2(.5)(2pq) + (.25)(4q^2) \\
&= p^2 + 2pq + q^2 \\
&= (p+q)^2
\end{aligned}$$

$$\begin{aligned}
P_2(Aa) &= P_2(Aa \mid Aa, Aa)P_1(Aa, Aa) \\
&+ 2(P_2(Aa \mid AA, Aa)P_1(AA, Aa)) \\
&+ 2(P_2(Aa \mid Aa, aa)P_1(Aa, aa)) \\
&+ 2(P_2(Aa \mid AA, aa)P_1(AA, aa)) \\
&= (.5)(4q^2) + 2(.5)(2pq) \\
&+ 2(.25)(2qr) + 2(1)(pr) \\
&= 2q^2 + 2pq + 2pqr_2pr \\
&= 2(q+p)(q+r)
\end{aligned}$$

$$\begin{aligned}
P_2(aa) &= P_2(aa \mid Aa, Aa)P(Aa, Aa) \\
&+ 2(P_2(aa \mid Aa, aa)P_1(Aa, aa) \\
&+ P_2(aa \mid aa, aa)P_1(aa, aa) \\
&= (.25)(4q^2) + 2(.5)(2pr) + (1)(r^2) \\
&= q^2 + 2qr + r^2 \\
&= (q+r)^2
\end{aligned}$$

Proof. It has been shown above that $P_2(AA) = (p+q)^2$, $P_2(Aa) = 2(q+p)(q+r)$ and $P_2(aa) = (q+r)^2$. To find the Hardy-Weinberg

Law result,we must show that $P_2(AA) = P_3(AA)$, $P_2(Aa) = P_3(Aa)$ and $P_2(aa) = P_3(aa)$.

First, it will be important to note that the three probabilities for a given generation's offspring will sum to 1. Ie...

$$(q+p)^2 + 2(q+p)(q+r) + (q+r)^2 = 1$$

Using the method to determine probabilities for generation 2 we can determine generation 3.

$$\begin{aligned}
P_3(AA) &= 1(p+q)^4 + 2(.5)(p+q)^2 \\
&\quad + 2(q+p)(q+r) + (.25)4(p+q)(q+r)^2 \\
&= (p+q)^4 + 2(p+q)^3(q+r) + (p+q)^2(q+r)^2 \\
&= (p+q)^2[(p+q)^2 + 2(p+q)(q+r) + (q+r)^2] \\
&= (p+q)^2
\end{aligned}$$

$$\begin{aligned}
P_3(Aa) &= (.25)(4)(q+p)^2(q+r)^2 + 2(.5)2(p+q)(q+r)(p+q)^2 \\
&= 2(.5)2(p+q)(q+r)(q+r)^2 + 2(p+q)^2(q+r)^2 \\
&= 2(p+q)^2(q+r)^2 + 2(p+q)^3(q+r) \\
&+ 2(p+q)(q+r)^3 + 2(p+q)^2(q+r)^2 \\
&= 2(q+p)(q+r)[(q+p)(q+r) + (p+q)^2 \\
&+ (q+r)^2(q+r)(q+r)] \\
&= 2(q+p)(q+r)[(p+q)^2 + 2(q+p)(q+r) + (q+r)^2] \\
&= 2(q+p)(q+r)
\end{aligned}$$

$$\begin{aligned}
P_3(aa) &= .25(4)(q+p)^2(q+r)^2 \\
&\quad + 2(.5)(q+r)^2 2(q+p)(q+r) + 1(q+r)^2 \\
&= (q+p)^2(q+r)^2 = 2(q+r)^3(q+p) + (q+r)^4 \\
&= (q+r)^2[(q+p)^2 + 2(q+r)(q+p) + (q+r)^2] \\
&= (q+r)^2
\end{aligned}$$

$\square$

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Show that the binomial probabilies sum to 1.

Proof. The binomial distribution (n, θ) is given as

$$b(x; n, \theta) = \binom{n}{x}\theta(1-\theta)^{n-x} \text{ for } x = 1, 2, ..., n$$

Since $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$

$$\begin{aligned}\sum b(x; n, \theta) &= \sum_{x=0}^{n}\binom{n}{x}\theta(1-\theta)^{n-x} \\ &= (\theta + 1 - \theta)^n \\ &= 1\end{aligned}$$

□

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Let $p_0, p_1, ..., p_n$ denote the probability mass function of the binomial distribution with parameters n and p. Let $q = 1 - p$. Show that the binomial probabilies can be compitered recursively by $p_0 = q^n$ and

$$p_k = \frac{(n-k+1)p}{kq}p_{k-1}, k = 1, 2, ..., n$$

Use this relation to find $P(X \leq 4)$ for $n = 9000$ and $p = .0005$.

Proof. The binomial mass function for the parameters n and p and $q = 1 - p$ for $P(X = 0)$ is

$$\begin{aligned}p_0 &= \binom{n}{0}p^0(1-p)^{n-0} \\ &= \frac{n!}{0!n!}p^0(q)^n \\ &= q^n\end{aligned}$$

Now consider

Binomial identity: $\binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$

$$\begin{aligned}p_1 &= \binom{n}{1}p^1 q^{n-1} \\ &= \frac{n+1-1}{1}\binom{n}{1-1}p^1 q^{n-1} \\ &= n\binom{n}{0}pq^{n-1} \\ &= npq^{n-1}\end{aligned}$$

Taking the ratio of p_1 and p_0

$$\begin{aligned}\frac{p_1}{p_0} &= \frac{npq^{n-1}}{q^n} \\ &= \frac{np}{q}\end{aligned}$$

Then

$$p_1 = \frac{np}{q} p_0$$

More generally, since

$$\begin{aligned} p_k &= \binom{n}{k} p^k q^{n-k} \\ &= \frac{n+1-k}{k} \binom{n}{k-1} p^k q^{n-k} \end{aligned}$$

and

$$p_{k-1} = \binom{n}{k-1} p^{k-1} q^{n-(k-1)}$$

then

$$\begin{aligned} \frac{p_k}{p_{k-1}} &= \frac{\frac{n+1-k}{k}\binom{n}{k-1} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-(k-1)}} \\ &= \frac{(n+1-k)}{k} pq^{-1} \\ &= \frac{(n+1-k)p}{kq} \end{aligned}$$

Therefore

$$p_k = \frac{(n+1-k)p}{kq} p_{k-1}$$

□

USING THIS RELATION we can find $P(X \leq 4)$ for $n = 9000$ and $p = .0005$.

$$\begin{aligned} P(X \leq 4) &= \sum_{k=0}^{4} p_k \\ &= (.011) + (.05) + (.11) + (.16) + (.18) \\ &= .511 \end{aligned}$$

Question 7

In heads up Texas hold 'em (two players, each dealt two cards), find the probability that neither is dealt a pair (two cards of the same rank). If there are three players, what is the probability that none have a pair?

LET A be the event that the first player gets a pair and let B be the event that the second player gets a pair. The probability that no player gets a pair is

$$1 - (P(A) + P(B) - P(A \cap B))$$
$$1 - 2(.0588) - (.0048)) = .8822$$

NOW LET C be the probability that the third player gets a pair. We can calcute the probability that none of the players get a pair using

$$\begin{aligned} 1 - P(A \cup B \cup C) = 1 - (&P(A) + P(B) + P(C) \\ &- P(A \cap B) - P(C \cap A) - P(B \cap C) \\ &+ P(A \cap B \cap C)) \end{aligned}$$

Where $P(A) = P(B) = P(C) = .0588$ and $P(A \cap C) \approx P(C \cap A) \approx P(B \cap C) \approx .0048$.

$$\begin{aligned} P(A \cap B \cap C) &= P(A)P(B \mid A)P(C \mid A, B) \\ &= (.0588)(.0816)(.0178) \end{aligned}$$

The probability that no player has a pair is .838.

Question 8

For a Poisson process $N(,)$ with parameter λ, find the probabilities of the following events:

1. $N((1,5]) > 1$ For a Poisson process the probability of k success over an interval of length t is

$$p(k) = \frac{\lambda t^k e^{-\lambda t}}{k!}$$

Since

$$P(N((1,5]) > 1) = 1 - P(N(1,5] = 0)) - P(N(1,5] = 1))$$

Compute and take the sum of the probabilities of each number of outcomes.

$$\begin{aligned} P(N((1,5]) > 1) &= 1 - \frac{\lambda(4)^0 e^{-\lambda(4)}}{0!} - \frac{\lambda(4)^1 e^{-\lambda(4)}}{1!} \\ &= 1 - e^{-4\lambda} - 4\lambda e^{-4\lambda} \\ &= 1 - e^{-4\lambda}(1 - 4\lambda) \end{aligned}$$

2. N((0,1]) = N((0,2])

   Take $P(N(0,1] = k) \cap P(N(0,2] = k))$

$$\frac{\lambda^k e^{-\lambda}}{k!} * \frac{\lambda 2^k e^{-\lambda 2}}{k!} = \frac{2\lambda^{2k} e^{-3\lambda}}{(k!)^2}$$

3. N((1,2]) + N((3,4]) = 6

   Let I be the index set of all pairs x, y for which $x + y = 6$ where $x, y \geq 0$. Denote $A = N(1,2])$ and $B = N(2,3])$ Take

$$\sum_{i \in I} P(A = x_i) P(B = y_i) \text{ for } x, y \in I.$$

   Which computes to $\frac{26.5}{180}(\lambda^{2k} e^{-2\lambda})$.

4. N((0,1]) = N((1,2])+m for m a nonnegative integer. Express you answer as an infinite series and then write your result in terms of a Bessel function (look it up). This result in terms of Bessel functions can be used to, for example, give a simple accurate approximation to this probability when m is large.

PROVED SOME integer m. Find $P(N(0,1]) = k + m \cap N(1,2] = k)$, denote these $P(A)$ and $P(B)$, respectively.

$$p(A) = \frac{\lambda t^{(k+m)} e^{-\lambda t}}{(k+m)!}$$

$$p(B) = p(k) = \frac{\lambda t^k e^{-\lambda t}}{k!}$$

Multiplying the two yields

$$\frac{1}{k!(k+m)!} \lambda^{(2k+m)} e^{-2\lambda}.$$

As m gets larger the probability of m outcomes will get closer and closer to 0.