STAT 244, HW1 Key

October 7, 2016

Total Score: 85 pts.

1 Problem 1: Rice, Ch1 Prob 20 - 10 pts

52 cards can be arranged/shuffled among themselves in $52!$ ways. There are 48 non aces and 4 ace cards. Treat 4 ace cards as 1 card. We then have 49 cards which can be shuffled among each other in $49!$ ways. On the other hand, the 4 aces representing 1 card so far (which are distinct in the sense that they are club, diamond, heart and spade), these can be shuffled among each other in $4!$ ways.

The number of ways the four aces can end up next to each other is $49! \times 4!$. This is out of total $52!$ ways. Take the ratio of the two quantities to get the probability.

Grading Scheme: 9 pts for the set up, 1 pt for the correct numerical result.

2 Problem 2: Rice, Ch 1 Prob 60- 10 pts

Consider the events

- **A**: first shift
- **B**: second shift
- **C**: third shift
- **D**: selecting a defective item

Percentage of defective items chosen

$$\begin{aligned}
Pr(D) &= Pr(A)Pr(D|A) + Pr(B)Pr(D|B) + Pr(C)Pr(D|C) && (1)\\
&= \frac{1}{3}\frac{1}{100} + \frac{1}{3}\frac{2}{100} + \frac{1}{3}\frac{5}{100} && (2)\\
&= \frac{8}{300} && (3)
\end{aligned}$$

Prob of a defective item from third shift

$$Pr(C|D) = \frac{Pr(D|C)Pr(C)}{Pr(D)} = \frac{\frac{1}{3}\frac{5}{100}}{\frac{8}{300}} = \frac{5}{8} \tag{4}$$

Grading Scheme: 4 pts each for the set ups for the two parts, 2 pt each for the correct numerical results.

3 Problem 3: Rice, Ch 1 Prob 72- 10 pts

The system works if all components are active, which has probability $(1-p^2)^n$ (p^2 because both the part and the back up fail). So, the probability that the system fails is $1-(1-p^2)^n$. If $n = 10$ and $p = 0.05$, then this probability is $1-(1-0.0025)^{10} = 0.02$.

This is better than the naive serial architecture and worse than the parallel architecture.

Grading Scheme: 7 pts for the derivation, 2 pts for the correct numerical result, and 1 pt for the comparison with the example from the text.

4 Problem 4: Rice, Ch 1 Prob 78 - 10 pts

Table 1: Genetic probability table

Genotype offspring — genotype parent	AA-AA	AA-Aa	AA-aa	Aa-Aa	Aa-aa	aa-aa
AA	1	0.5	0	0.25	0	0
Aa	0	0.5	1	0.5	0.5	0
aa	0	0	0	0.25	0.5	1

4.1 (i)

The offspring will get one A from a AA parent, and with probability 0.5 a A or a from the Aa parent. If he receives A from latter parent, he has genotype AA, else has genotype Aa, and the two happened with probability 0.5 each.

4.2 (ii)

Mating type	Random Mating prob	Offspring genotype
$AA \times AA$	p^2	$1 \times AA$
$AA \times Aa$	$2 \times p \times 2q$	$\frac{1}{2} \times AA + \frac{1}{2} \times Aa$
$AA \times aa$	$2 \times p \times r$	$1 \times Aa$
$Aa \times Aa$	$2q \times 2q$	$\frac{1}{4} \times AA + \frac{1}{2} \times Aa + \frac{1}{4} \times aa$
$Aa \times aa$	$2 \times 2q \times r$	$\frac{1}{2} \times Aa + \frac{1}{2} \times aa$
$aa \times aa$	$r \times r$	$1 \times aa$

Let the probabilities of AA, Aa and aa in second generation be p_2, q_2 and r_2.

$$p_2 = [p \times p \times 1] + [2 \times p \times 2q \times 0.5] + [2q \times 2q \times 0.25] = (p+q)^2$$

$$2q_2 = [2 \times p \times 2q \times 1] + [2 \times p \times r \times 1] + [2q \times 2q \times 0.5] + [2 \times 2q \times r \times 0.5] = 2(p+q)(q+r)$$

$$r_2 = [r \times r \times 1] + [2 \times r \times 2q \times 0.5] + [2q \times 2q \times 0.25] = (r+q)^2$$

So the new probabilities of the genotypes after 2nd gen are p_2, $2q_2$ and r_2 and then if we repeat for 3rd generation, we get

$$p_3 = (p_2 + q_2)^2$$
$$2q_3 = 2(p_2 + q_2)(q_2 + r_2)$$
$$r_3 = (q_2 + r_2)^2$$

Since we have $p + 2q + r = 1$

$$p_2 + 2q_2 + r_2 = (p+q)^2 + 2(p+q)(q+r) + (r+q)^2 = (p+2q+r)^2 = 1$$

$$p_2 r_2 = q_2^2$$

Also taking p_2, $2q_2$ and r_2 as start, we can create p_3, $2q_3$ and r_3 for third generation, and from set of equations above

$$p_3 + 2q_3 + r_3 = (p_2 + q_2)^2 + 2(p_2 + q_2)(q_2 + r_2) + (r_2 + q_2)^2 = (p_2 + 2q_2 + r_2)^2 = 1$$

$$p_3 r_3 = q_3^2$$

So $p_3, 2q_3, r_3$ satisfy same constraints as $p_2, 2q_2, r_2$. So they are the same probabilities.

$$p_3 = (p_2 + q_2)^2 = ((p+q)^2 + (p+q)(q+r))^2 = (p+q)^2(p+2q+r)^2 = (p+q)^2 = p_2$$

$$r_3 = (r_2 + q_2)^2 = ((r+q)^2 + (p+q)(q+r))^2 = (r+q)^2(p+2q+r)^2 = (r+q)^2 = r_2$$

From the relation $pr = q^2$, satisfied, we get if $q_3^2 = p_3 r_3$ and $q_2^2 = p_2 r_2 = p_3 r_3 = q_3^2$

So

$$p_2 = p_3 \qquad r_2 = r_3 \qquad q_2 = q_3$$

Grading Scheme: 2 pt for (a), 8 pts for (b). If someone does (c), ignore it.

5 Problem 5: Rice, Ch 2 Prob 8 - 5 pts

$$Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\sum_{x=0}^{n} Pr(X = x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \quad = (1-p)^n + \binom{n}{1} p(1-p)^{n-1} + ... = (p+1-p)^n \quad = 1 \tag{5}$$

6 Problem 6: Rice, Ch 2 Prob 28 - 10 pts

$p_0 = q^n$ is obvious. For $k \in \{1, \ldots, n\}$

$$\begin{aligned}
\frac{(n-k+1)p}{kq} p_{k-1} &= \frac{(n-k+1)p}{kq} \binom{n}{k-1} p^{k-1} q^{n-k+1} \\
&= \frac{(n-k+1)p}{kq} \cdot \frac{n!}{(k-1)!(n-k+1)!} p^{k-1} q^{n-k+1} \\
&= \frac{n!}{k!(n-k)!} p^k q^{n-k} \\
&= p_k.
\end{aligned}$$

$\Pr(X \leq 4) = 0.532$.

Grading Scheme: 8 pts for the derivation, 2 pts for the correct numerical result.

7 Problem 7 - 10 pts

Notice that there is ordering of player.
Let A denote the event that player 1 has a pair; let B denote the event that player 2 has a pair and; C denote the event that player 3 has a pair.
Then $\Pr(\text{at least a pair}) = \Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(B \cap C) - \Pr(C \cap A) + \Pr(A \cap B \cap C) = 3 \cdot \Pr(A) - 3 \cdot \Pr(A \cap B) + \Pr(A \cap B \cap C)$, by Inclusion-Exclusion Principle and symmetry of the probabilities.

$$\Pr(A) = \frac{(\text{no. of ways to choose the rank})(\text{no. of possible suits combination})}{\binom{52}{2}} = \frac{(13)\binom{4}{2}}{\binom{52}{2}}$$

$$\Pr(A \cap B) = \frac{\text{no. of poss. of form } \{1,1\} \text{ and } \{1,1\} + \text{no. of poss. of form } \{1,1\} \text{ and } \{2,2\}}{\binom{52}{2}\binom{50}{2}}$$

$$= \frac{\binom{13}{1}\binom{4}{2} + 13\binom{4}{2}12\binom{4}{2}}{\binom{52}{2}\binom{50}{2}}$$

For 2 players,

$$\Pr(\text{no pair in Players 1 and 2}) = 1 - [\Pr(A) + \Pr(B) - \Pr(A \cap B)] = 0.885$$

where $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cap B)$ are determined from above.

For three players, $\Pr(A \cap B \cap C) = (\text{no. of form } \{1,1\} \text{ and } \{1,1\} \text{ and } \{2,2\}$
$+\text{no. of form } \{1,1\} \text{ and } \{2,2\} \text{ and } \{1,1\}$
$+\text{no. of form } \{2,2\} \text{ and } \{1,1\} \text{ and } \{1,1\}$
$+\text{no. of form } \{1,1\} \text{ and } \{2,2\} \text{ and } \{3,3\})$
$/(\binom{52}{2}\binom{50}{2}\binom{48}{2}) = \frac{(3)(13)\binom{4}{2}\binom{2}{2}(12)\binom{4}{2}+(13)(12)(11)\binom{4}{2}^3}{\binom{52}{2}\binom{50}{2}\binom{48}{2}}$
In total, $\Pr(\text{no pair}) = 1 - \Pr(\text{at least a pair}) = 0.83$.

Grading Scheme: 10 pts for each part. Marks are assigned based on the number of correct counting made in each part.

8 Problem 8 - 20 pts

1. $N(1,5] \sim Poisson(4\lambda)$.

$$\begin{aligned}
\Pr\{N(1,5] > 1\} &= 1 - \Pr\{N(1,5] \leq 1\} && (6)\\
&= 1 - \Pr\{N(1,5] = 1\} - \Pr\{N(1,5] = 0\} && (7)\\
&= 1 - \frac{(4\lambda)^0}{0!}e^{-4\lambda} - \frac{(4\lambda)^1}{1!}e^{-4\lambda} && (8)\\
&= 1 - (1+4\lambda)e^{-4\lambda}. && (9)
\end{aligned}$$

2. $N(1,2] \sim Poisson(\lambda)$.

$$\Pr\{N(0,1] = N(0,2]\} = \Pr\{N(1,2] = 0\} = e^{-\lambda} \tag{10}$$

3. There are two ways to see this. The first of the two is to note that $N(1,2]$ and $N(3,4]$ are independent $Poisson(\lambda)$. Then,

$$\begin{aligned}
\Pr\{N(1,2] + N(3,4] = 6\} &= \sum_{k=0}^{6} Pr\{N(1,2] = k\} \Pr\{N(3,4] = 6-k\}\\
&= \sum_{k=0}^{6} \frac{\lambda^k e^{-\lambda}}{k!} \cdot \frac{\lambda^{6-k} e^{-\lambda}}{(6-k)!}\\
&= \lambda^6 e^{-2\lambda} \sum_{k=0}^{6} \frac{1}{k!(6-k)!}\\
&= \frac{\lambda^6 e^{-2\lambda}}{6!} \sum_{k=0}^{6} \frac{6!}{k!(6-k)!}\\
&= \frac{(2\lambda)^6 e^{-2\lambda}}{6!},
\end{aligned}$$

where in the last step, we have used the identity

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n.$$

Alternatively, one could observe that getting 6 on two disjoint intervals of length 1 is "like" getting 6 on a single interval of length 2. (This is an illustration of the more general property that if $X \sim Poisson(\mu_X)$ and $Y \sim Poisson(\mu_Y)$ are independent, then $X + Y \sim Poisson(\mu_X + \mu_Y)$.) Thus, $N(1,2] + N(3,4] \sim Poisson(2\lambda)$, from which the conclusion follows.

4. It has been pointed that there is an ambiguity in the wording of this problem, specifically whether we mean "a particular value of m", or "any m." We give credit for either solution, which differ by the presence or absence of a second infinite series, For a "particular m:" We note that $N(0,1]$

and $N(1,2]$ are independent $Poisson(\lambda)$.

$$
\begin{aligned}
Pr\{N(0,1] = N(1,2] + m\} &= \sum_{k=0}^{\infty} Pr\{N(1,2] = k\} \Pr\{N(0,1] = k+m\} \\
&= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{\lambda^{k+m} e^{-\lambda}}{(k+m)!} \\
&= e^{-2\lambda} \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left(\frac{2\lambda}{2}\right)^{2k+m} \qquad (*)
\end{aligned}
$$

A modified Bessel function of the first kind $I_m(x)$ is a solution to the second-order differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + m^2)y = 0.$$

For our purposes, it is enough to know that when m is a non-negative integer, $I_m(x)$ has the power series expansion

$$I_m(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m}.$$

Substituting the above into $(*)$, we obtain

$$\Pr\{N(0,1] = N(1,2] + m\} = e^{-2\lambda} I_m(2\lambda).$$

5. For "any m:": The probability is essentially $\Pr(N((0,1]) \geq N((1,2]))$, which is in turn

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \sum_{n=k}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} = e^{-2\lambda} \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{\lambda^{n+k}}{k!n!} = e^{-2\lambda} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\lambda^{(m+k)+k}}{k!(m+k)!}$$

$$= e^{-2\lambda} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+m+1)} \left(\frac{2\lambda}{2}\right)^{2k+m} = e^{-2\lambda} \sum_{m=0}^{\infty} I_m(2\lambda),$$

where I_m is a modified Bessel function of the first kind.

Grading Scheme: 2 pts for (a), 4 pts for (b), 6 pts for (c), and 8 pts for (d).