STAT 244, HW3 Key

October 14, 2016

Total Score: 120 pts.

1 Problem 1: 10 pts

- **[10pts] Rice 4.78**

Suppose $f(x)$ is a probability density function and symmetric about zero, then it is an even function ($f(x) = f(x)$). Let k be an odd number then $x^k f(x)$ is an odd function and hence

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x)dx = 0$$

So all odd moments of X is 0. As a result,

$$skewness = E((X - E(X))^3 = E(X^3) = 0$$

Grading Scheme: 5 pts for proving $E(X) = 0$, and 5 pts for the rest. Marks are assigned based on the progress made through.

2 Problem 2: 30 pts

(a)

$$\int_0^1 \int_0^1 f(x,y)dxdy = \frac{4}{5}\int_0^1 \int_0^1 (x+y+xy)dxdy = \frac{4}{5}\int_0^1 \frac{1}{2} + \frac{3}{2}ydy = 1$$

(b)

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 (x+y+xy)dx = \frac{2}{5}(3y+1)$$

for $0 < y < 1$.

(c)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{2(x+y+xy)}{3y+1}$$

Then

$$f_{X|Y=0.5}(x|y=0.5) = \frac{2(x+0.5+x/2)}{1.5+1} = \frac{2}{5}(3x+1)$$

for $0 < x < 1$.

(d) Note that $f(x,y)$ is symmetric in x and y, hence X and Y have the same distribution (they are not independent though).

$$E(X) = E(Y) = \frac{2}{5}\int_0^1 (3y+1)y dy = \frac{3}{5}$$

$$E(X^2) = E(Y^2) = \frac{2}{5}\int_0^1 (3y+1)y^2 dy = \frac{13}{30}$$

$$Var(X) == E(X^2) - (E(X))^2 = \frac{11}{150}$$

$$E(XY) = \frac{4}{5}\int_0^1\int_0^1 xy(x+y+xy)dxdy = \frac{4}{5}\int_0^1 \frac{1}{3}y + \frac{5}{6}y^2 dy = \frac{16}{45}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = -\frac{1}{225}$$

(e)

$$P(0.2 \le X \le 0.5, 0.4 \le Y \le 0.8) = \frac{4}{5}\int_{0.2}^{0.5}\int_{0.4}^{0.8}(x+y+xy)dydx$$
$$= \frac{4}{5}\int_{0.2}^{0.5}(0.64x+0.24)dx = 0.11136$$

(f)

$$P(X+Y \le 1) = \frac{4}{5}\int_0^1\int_0^{1-x}(x+y+xy)dydx = \frac{2}{5}\int_0^1 (x^3-3x^2+x+1)dx = \frac{3}{10}$$

Grading Scheme: 5 pts for each sub-problem. For each sub-problem, deduct 2 pts if answer is wrong but steps are valid. For (b) and (c), deduct 2 pts is domain is not given.

3 Problem 3: 20 pts

Let $X \sim Ber(p)$, then

$$M_X(t) = E(e^{tX}) = e^t p + (1-p)$$

For $k > 1$, the k-th derivatives are identical in this case, i.e.

$$M_X^{(k)}(t) = e^t p$$

As a result,

$$E(X) = M_X^{(1)}(0) = p$$

$$Var(X) = E(X^2) - (E(X))^2 = M_X^{(2)}(0) - (M_X^{(1)}(0))^2 = p(1-p)$$

$$E(X^3) = M_X^{(3)}(0) = p$$

Now let $Y \sim B(n,p)$, then $Y = \sum_{i=1}^n X_i$, where X_i are i.i.d. $Bernoulli(p)$ r.v.

$$M_Y(t) = E(e^{tY}) = E(e^{\sum_{i=1}^n tX_i}) = \prod_{i=1}^n E(e^{tX_i}) = M_X^n(t) = (e^t p + 1 - p)^n$$

where the 3rd equality follows from the independence of Xis and positiveness of the exponential function. You could still evaluate the derivative of $M_Y(t)$ at 0 to calculate the moments. Alternatively,

$$E(Y) = \sum_{i=1}^n E(X_i) = np$$

$$Var(Y) = \sum_{i=1}^n Var(X_i) = np(1-p)$$

Again, the covariance terms vanishes because X_is are independent.

Grading Scheme: 5 pts Bernoulli MGF, 5 pts for Bernoulli moments, 5 pts for Binomial MGF, 5 pts for Binomial moments.

4 Problem 4: 20 pts

Assume $X \sim Gamma(\alpha, \lambda)$, then its mgf is

$$M_X(t) = (1 - \frac{t}{\lambda})^{-\alpha}$$

Since $E(X) = \frac{\alpha}{\lambda}$,$Var(X) = \frac{\alpha}{\lambda^2}$, then the standardized r.v. is

$$Y = \frac{X - \frac{\alpha}{\lambda}}{\sqrt{\frac{\alpha}{\lambda^2}}} = \frac{\lambda}{\sqrt{\alpha}} X - \sqrt{\alpha}$$

Therefore,

$$M_Y(t) = e^{-\sqrt{\alpha}t} M_X(\frac{\lambda}{\sqrt{\alpha}}t) = e^{-\sqrt{\alpha}t}(1 - \frac{t}{\sqrt{\alpha}})^{-\alpha}$$

To see the behavior when $\alpha \to \infty$, we take the log and then use Taylors expansion up to the 2nd order.

$$log(M_Y(t)) = -\sqrt{\alpha}t - \alpha log(1 - \frac{t}{\sqrt{\alpha}}) = -\sqrt{\alpha}t - \alpha(-\frac{t}{\sqrt{\alpha}} + \frac{t^2}{2\alpha}) + o(\frac{1}{\alpha^3}) = \frac{t^2}{2} + o(1)$$

which is exactly $log(e^{t^2/2}) = log(M_Z(t))$, where $Z \sim N(0,1)$.

Grading Scheme: 5 pts for $M_X(t)$, 5 pts for $M_Y(t)$, 10 pts for the remaining deduction.

5 Problem 5: 10 pts

The prior of cure rate θ is $Beta(2,1)$, hence

$$f(\theta) \propto \theta$$

The probability model is

$$X \sim B(3, \theta)$$

So posterior of θ can be calculated as

$$f(\theta|X=k) \propto f(X=k|\theta)f(\theta) \propto \theta^k(1-\theta)^{3-k}\theta = \theta^{k+1}(1-\theta)^{3-k}$$

As a result,

$$\theta|X=k \sim Beta(k+2, 4-k)$$

Then

$$P(\theta \leq 0.2|X=k) = P(Beta(k+1, 4-k) \leq 0.2)$$

$$E(\theta|X=k) = \frac{k+2}{6}$$

The numerical values for each k are tabulated as follows: A simple R function that enables the

k	$P(\theta \leq 0.2 \vert X=k)$	$E(\theta \vert X=k)$
0	0.26	0.33
1	0.058	0.5
2	0.0067	0.67
3	0.00032	0.83

calculation is

```
Q4=function(k){
pr=pbeta(0.2,k+2,4-k)
ex=(k+2)/6
return(list(pr,ex))
}
```

Grading Scheme: 8 pts for posterior distribution, 2 pts for numerical calculation.

6 Problem 6: 10 pts

Denote the probability that the sun rises on a given day as θ. The prior of θ is uniform, so

$$f(\theta) \propto 1$$

Denote $X_i, i = 1, \cdots, n$ as the random variable that takes on value 1 if sun rises and 0 otherwise. So $X_i \sim Ber(\theta)$.
The posterior distribution of θ after observing n days of sunrise is therefore

$$f(\theta|X_1 = 1, \cdots, X_n = 1) \propto f(\theta)f(X_1 = 1, \cdots, X_n = 1|\theta) \propto \prod_{i=1}^{n} f(X_i|\theta) = \theta^n$$

So

$$\theta|X_1 = 1, \cdots, X_n = 1 \sim Beta(n + 1, 1)$$

So the posterior expectation of θ is

$$E(\theta|X_1 = 1, \cdots, X_n = 1) = \frac{n + 1}{n + 2}$$

Grading Scheme: 6 pts for posterior distribution, 4 pts for expectation.

7 Problem 7: 20 pts

(a) The likelihood

$$L(Data|\theta) = \binom{100}{3}\theta^3(1-\theta)^{97}$$

The prior

$$p(\theta) = \theta^{1-1}(1-\theta)^{1-1} = 1$$

The posterior

$$p(\theta|data) \propto L(Data|\theta)p(\theta)$$

$$p(\theta|data) \propto \theta^3(1-\theta)^{97} = Beta(4, 98)$$

So with this prior, the posterior distribution is $Beta(4, 98)$
The posterior mean is $\frac{4}{4+98} = \frac{4}{102} = 0.039$.
When the prior

$$p(\theta) = \theta^{0.5-1}(1-\theta)^{5-1} = \theta^{-0.5}(1-\theta)^4$$

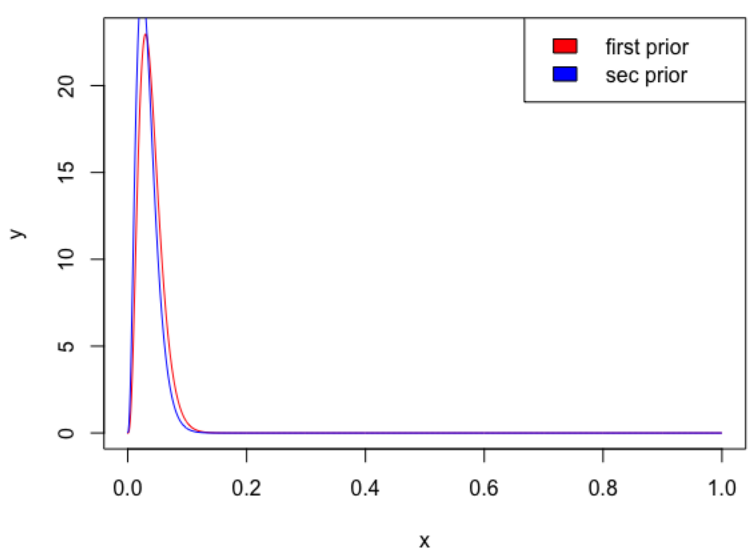
The posterior

$$p(\theta|data) \propto \theta^3(1-\theta)^{97} \times \theta^{-0.5}(1-\theta)^4 = \theta^{3.5-1}(1-\theta)^{102-1}$$

So with this prior, the posterior distribution is $Beta(3.5, 102)$
The posterior mean is $\frac{3.5}{3.5+102} = \frac{3.5}{105.5} = 0.033$.
The plots of the posterior distribution for the two cases can be given by

```
y <- dbeta(seq(0,1,length.out=1000),4, 98)
x <- seq(0,1,length.out=1000)
plot(x,y, type="l", col="red")

z <- dbeta(seq(0,1,length.out=1000),3.5, 102)
x <- seq(0,1,length.out=1000)
lines(x,z, type="l", col="blue")
legend("topright", c("first prior", "sec prior"), fill=c("red", "blue"))
```



Why is there the difference in these posterior means?
Note that the sample mean is $\frac{3}{100} = 0.03$ which is the sample proportion. The prior means on the other hand were $\frac{1}{1+1} = 0.5$ and $\frac{0.5}{0.5+5} = 0.09$. Note that the prior mean in second case was

lesser and the sample mean being 0.3, the posterior mean shifts more towards 0.3 due to prior mean being closer to it in the second case compared to the first.

(b) For part 8.64,

$$p(\theta) = 1$$

$$Pr(X = x) = \int_0^1 \theta^x (1-\theta)^{1-x} p(\theta) d\theta$$

$$Pr(X = x) = \int_0^1 \theta^x (1-\theta)^{1-x} d\theta$$

$$Pr(X = 1) = \int_0^1 \theta d\theta = 0.5$$

Also $Pr(X = 0) = 1 - Pr(X = 1) = 0.5$
For the second prior,

$$Pr(X = x) = \int_0^1 \theta^x (1-\theta)^{1-x} \theta^{0.5-1} (1-\theta)^{5-1} d\theta$$

$$Pr(X = x) \propto \int_0^1 \theta^{x+0.5-1} (1-\theta)^{1-x+5-1} d\theta \propto Beta(x + 0.5, 6 - x)$$

$Pr(X = 1) = Beta(1.5, 5)/(Beta(1.5, 5)+Beta(0.5, 6)) = 0.09$ and $Pr(X = 0) = 1-Pr(X = 1) = 0.91$
For the first prior the posterior was (as evaluated above)

$$p(\theta|samples) \propto \theta^3 (1-\theta)^{97}$$

$$Pr(X = x|samples) \propto \int_0^1 \theta^x (1-\theta)^{1-x} \theta^{4-1} (1-\theta)^{98-1} d\theta$$

$$Pr(X = x|samples) \propto \int_0^1 \theta^{x+4-1} (1-\theta)^{1-x+98-1} d\theta \propto Beta(x + 4, 99 - x)$$

$Pr(X = 1) = Beta(5, 98)/(Beta(5, 98)+Beta(4, 99)) = 0.039$ and $Pr(X = 0) = 1-Pr(X = 1) = 0.96$
For the second prior, the posterior was (as evaluated above)

$$p(\theta|samples) \propto \theta^{3.5-1} (1-\theta)^{102-1}$$

$$Pr(X = x|samples) \propto \int_0^1 \theta^{x+3.5-1} (1-\theta)^{1-x+102-1} d\theta \propto Beta(x + 3.5, 103 - x)$$

$Pr(X = 1) = Beta(4.5, 102)/(Beta(4.5, 102) + Beta(3.5, 103)) = 0.033$ and $Pr(X = 0) = 1 - Pr(X = 1) = 0.96$

Grading Scheme: 8.63, 12 pts (2+2 for two posterior, 2+2 for two posterior means, 2 for the plot, 2 for the explanation of difference in posterior means, 8.64, 8 pts for second part (2+2+2+2).