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STATS 244 HOMEWORK 3

Rice Chapter 4, Question 78

Show that if a density is symmetric about zero, its skewness is zero.

SKEWNESS can be determined using $E(X^3)$. For example if $E(X^3)$ is zero, then skewness is zero.

Let f_y be a density symmetric around zero. Then, because of this symmetry $-f_Y = f_y$. Which implies $E(Y^3) = E(-Y^3)$ Which means that $E(Y^3) = 0$.

Question 2

Consider the bivariate density of X and Y

$$f(x,y) = 4(x+y+xy)/5 \text{ for } 0 < x, y < 1, \text{ 0 otherwise}$$

1. Verify that this is a bivariate density
   (the total volume of $\int\int f(x,y)dxdy = 1$).

$$\begin{aligned}
\int\int f(x,y)dxdy &= \int_0^1\int_0^1 \frac{4}{5}(x+y+xy)dxdy \\
&= \int_0^1 [\frac{4}{5}(\int_0^1 x+y+xydx)]dy \\
&= \int_0^1 [\frac{4}{5}(\int_0^1 xdx + \int_0^1 ydx + \int_0^1 xydx)]dy \\
&= \int_0^1 [\frac{4}{5}(\frac{x^2}{2}|_0^1 + y + y(\frac{x^2}{2}|_0^1))]dy \\
&= \int_0^1 [\frac{4}{5}(\frac{1}{2} + 1\frac{y}{2})]dy \\
&= \int_0^1 \frac{4}{5}(\frac{3y}{2} + \frac{1}{2})dy \\
&= \int_0^1 \frac{4}{10}(3y+1)dy \\
&= \frac{4}{10}[\int_0^1 3ydy + \int_0^1 1dy] \\
&= \frac{4}{10}[3(\frac{1}{2}) + 1] \\
&= \frac{4}{10}[\frac{3}{2} + 1] \\
&= 1
\end{aligned}$$

2. Find the marginal density of Y

We basically found this in a step from part 1

$$
\begin{aligned}
f_Y(y) &= \int_0^1 f_{XY}(x,y)dx \\
&= \int_0^1 \frac{4}{5}(x+y+xy)dx \\
&= \frac{4}{5}\int_0^1 (x+y+xy)dx \\
&= \frac{4}{10}(3y+1)
\end{aligned}
$$

3. Find the conditional density of X given $Y = 0.5$.

$$
\begin{aligned}
f_{X|Y}(x \mid Y = .5) &= \frac{f(x,.5)}{f_y(.5)} \\
&= \frac{\frac{4}{5}(x+.5+.5x)}{\frac{4}{10}(2.5)} \\
&= \frac{4}{5}(1.5x+.5)
\end{aligned}
$$

4. Find $E(X)$, $E(X^2)$, $\mathrm{Var}(X)$, $E(XY)$, $\mathrm{Cov}(X,Y)$.

$$
\begin{aligned}
E(X) &= \int_0^1 x f_X(x)dx \\
&= \int_0^1 x\frac{4}{10}(3x+1)dx \\
&= \frac{4}{10}[\int_0^1 3x^2 + xdx] \\
&= \frac{4}{10}[\int_0^1 3x^2dx + \int_0^1 xdx] \\
&= \frac{4}{10}[1+\frac{1}{2}] \\
&= \frac{3}{5}
\end{aligned}
$$

$$\begin{aligned}
E(X^2) &= \int_0^1 x^2 f_X(x)dx \\
&= \int_0^1 x^2 \frac{4}{10}(3x+1)dx \\
&= \frac{4}{10}\int_0^1 3x^3 + x^2 dx \\
&= \frac{4}{10}[\frac{3}{4}x^3\Big|_0^1 + \frac{x^3}{3}\Big|_0^1] \\
&= \frac{4}{10}(\frac{3}{4} + \frac{1}{3}) \\
&= \frac{13}{30}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= E(X^2) - E(X)^2 \\
&= \frac{13}{30} - \frac{3}{5}^2 \\
&= \frac{11}{150}
\end{aligned}$$

$$\begin{aligned}
E(XY) &= \int_0^1 \int_0^1 xyf(x,y)dxdy \\
&= \int_0^1 \int_0^1 xy\frac{4}{5}(x+y+xy)dxdy \\
&= \int_0^1 [\frac{4}{5}\int_0^1 xy(x+y+xy)dx]dy \\
&= \int_0^1 [\frac{4}{5}y(\int_0^1 (x^2+xy+x^2y)dx]dy \\
&= \int_0^1 [\frac{4}{5}y(\frac{1}{3} + y(\frac{1}{2}) + y(\frac{1}{3})]dy \\
&= \int_0^1 \frac{4}{15}y(\frac{5y}{2}+1)dy \\
&= \frac{4}{15}\int_0^1 \frac{5y^2}{2} + ydy \\
&= \frac{4}{15}[\frac{5}{6} + \frac{1}{2}] \\
&= \frac{16}{45}
\end{aligned}$$

We should note that because the marginal density of X and Y are symmetric(?) $E(X) = E(Y) = \frac{3}{5}$. In anycase, we don't need to compute $E(Y)$ since the marginal densities look the same.

$$\begin{aligned}\text{Cov}(X,Y) &= E(XY) - E(X)E(Y)\\ &= \frac{16}{45} - \frac{3}{5}^2\\ &= -\frac{1}{225}\end{aligned}$$

5. Find $P(0.2 \leq X \leq .5 \text{ and } .4 \leq Y \leq .8)$

I asked for a clarification on this question on the discussion board and no one answered! The CPR rule takes effect, which says if there is no response, the affirmative is implied. It's a gamble, but whatever. There are many cases where I would NOT employ this rule.

$$\begin{aligned}P(0.2 \leq X \leq .5 \text{ and } .4 \leq Y \leq .8) &= \int_{.4}^{.8}\int_{.2}^{.5} f(x,y)dxdy\\ &= \int_{.4}^{.8}\int_{.2}^{.5} \frac{4}{5}(x+y+xy)dxdy\\ &= \int_{.4}^{.8}[\frac{4}{5}\int_{.2}^{.5} x+y+xydx]dy\\ &= \int_{.4}^{.8}[\frac{4}{5}(.405y+.105)dy\\ &= \frac{4}{5}(.0972+.042)\\ &= .11136\end{aligned}$$

6. Find $P(X+Y \leq 1)$ Set $y = v - x$

$$P(X+Y \leq 1) = \int_0^y\int_0^x f(x,v)dxdv$$

$$\int_0^1\int_0^{1-x} f(x,y)dydx$$

$$\begin{aligned}&= \int_0^1\int_0^{1-x}\frac{4}{5}(x+y+xy)dydx\\ &= \int_0^1[\frac{4}{5}(\int_0^{1-x} xdy + \int_0^{1-x} ydy + x\int_0^{1-x} ydy)]dx\\ &= \int_0^1 \frac{4}{5}[-(1-x)x + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^2x+]dx\\ &= \int_0^1 \frac{4}{10}(x^3-3x^2+x+1)dx\\ &= \frac{4}{10}[\frac{1}{4} - 1 + \frac{1}{2} + 1]\\ &= \frac{3}{10}\end{aligned}$$

Question 3, Rice 4.81 and 4.82

1. Find the moment-generating function of a Bernoulli random variable, and use it to find the mean, variance, and third moment.

A BERNOULLI random variable is one such that $f(x) = 1 - p$ where $f(0) = 1 - p$ and $f(1) = p$.

$$\begin{aligned} M(t) &= \sum e^{tx} f(x) \\ &= e^{t(0)} f(0) + e^{t(1)} f(1) \\ &= e^{t(0)}(1-p) + e^{t(1)}(p) \\ &= 1 - p + e^t p \end{aligned}$$

To find the first, second and third moments, take the following derivivative of $M(t)$.

$$\begin{aligned} M'(t) &= pe^t \\ M''(t) &= pe^t \\ M'''(t) &= pe^t \end{aligned}$$

Evauluating each of these at 0 gives us our moments, respectively.

$$\begin{aligned} E(X) &= p \\ E(X^2) &= p \\ E(X^3) &= p \end{aligned}$$

We also need find the variance.

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = p - p^2$$

If we let $q = 1 - p$ then $\mathrm{Var}(X) = p(1-p) = pq$

2. Use the result of Problem 81 to find the mgf of a binomial random variable and its mean and variance.

LET $X_1, X_2, X_3, ..., X_n$ be independently and identically distributed Bernoulli random variables with paremeter p.

Let $Y = X_1 + X_2 + X_3, ..., X_n$, so $Y = \sum_{i=1}^{n} X_i$. Then

$$\begin{aligned} M_Y(t) &= E(e^{ty}) \\ &= E(e^{(tx_1 + tx_2 + ... + tx_n)}) \\ &= E(e^{tx_1}) E(e^{tx_2}) ... E(e^{tx_n}) \\ &= M_{x_1}(t) M_{x_2}(t) ... M_{x_n}(t) \\ &= (1 - p + pe^t)^n \end{aligned}$$

Now we take the first and second derivates of $M_Y(t)$ in order to find the first and second moments.

$$
\begin{aligned}
M'_Y(t) &= \frac{d}{dt}(1-p+e^t p)^n \\
&= n(1-p+e^t p)^{n-1}\frac{d}{dt}(1-p+e^t p) \\
&= n(1-p+e^t p)^{n-1}e^t p \\
&= npe^t(1-p+pe^t)^{n-1}
\end{aligned}
$$