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# STATS 244

## HOMEWORK 2

*Rice Chapter 2, 44*

Let  $T$  be an exponential random variable with parameter  $\lambda$ . Let  $X$  be a discrete random variable defined as  $X = k$  if  $k \leq T < k + 1$ ,  $k = 0, 1, \dots$ . Find the frequency function of  $X$ .

THE EXPONENTIAL density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

The random variable  $X$  is essentially the space between two consecutive integers on  $f(x)$ ,  $k$  and  $k + 1$ . Therefore, using the CDF of  $f(x)$ , we can find the frequency function of  $X$ .

$$F_T(x) = \int_0^x f(u) du = 1 - e^{-\lambda x}$$

$$\begin{aligned} P(k \leq T < k + 1) &= P(T \leq k + 1) - P(T < k) \\ &= F_T(k + 1) - F_T(k) \\ &= (1 - e^{-\lambda(k+1)}) - (1 - e^{-\lambda k}) \\ &= -e^{-\lambda k - \lambda} + e^{-\lambda k} \\ &= -e^{-\lambda k}(e^{-\lambda} - 1) \end{aligned}$$

So the frequency function of  $X$  is  $f_X(x) = -e^{-\lambda x}(e^{-\lambda} - 1)$

*Rice Chapter 2, 46*

$T$  is an exponential random variable, and  $P(T < 1) = .05$ . What is  $\lambda$ ?

USING THE CDF of an exponential random variable, we find...

$$\begin{aligned} p(T < 1) &= .05 \\ 1 - e^{(-\lambda)} &= .05 \\ -e^{(-\lambda)} &= -.95 \\ e^{(-\lambda)} &= \frac{19}{20} \\ e^{\lambda} &= \frac{20}{19} \\ \lambda &= \ln\left(\frac{20}{19}\right) \\ \lambda &\approx .051 \end{aligned}$$

*Rice Chapter 3, 8*

Let  $X$  and  $Y$  have the joint density

$$f(x, y) = \frac{6}{7}(x + y)^2, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

1. By integrating over the appropriate regions, find (i)  $P(X > Y)$ , (ii)  $P(X + Y \leq 1)$ , (iii)  $P(X \leq \frac{1}{2})$ .]

(a)  $P(X > Y)$

$$P(X > Y) = \int_{-\infty}^{\infty} \int_{-\infty}^x f_{XY} dy dx$$

Since the joint density function applies to  $0 \leq x \leq 1$  set the limit of integration for  $X$  to be 0, 1 and 0,  $x$  for  $Y$ .

$$\begin{aligned} P(X > Y) &= \int_0^1 \int_0^x \frac{6}{7} (x+y)^2 dy dx \\ &= \int_0^1 \frac{6}{7} \left[ \frac{1}{3} (x+y)^3 \right]_0^x dx \\ &= \int_0^1 \frac{6}{7} \left[ \frac{1}{3} (2x)^3 - \frac{1}{3} (x)^3 \right] dx \\ &= \int_0^1 \frac{6}{7} \left[ \frac{1}{3} 7x^3 \right] dx \\ &= \int_0^1 2x^3 dx \\ &= \frac{1}{2} x^4 \Big|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

(b)  $P(X + Y \leq 1)$

$$\begin{aligned} P(X + Y \leq 1) &= \int_0^1 \int_0^{1-x} \frac{6}{7} (x+y)^2 dy dx \\ &= \int_0^1 \frac{6}{7} \left[ \frac{1}{3} (x+y)^3 \right]_0^{1-x} dx \\ &= \int_0^1 \frac{6}{7} \left[ \frac{1}{3} (x+1-x)^3 - \frac{1}{3} (x+0)^3 \right] dx \\ &= \int_0^1 \frac{6}{7} \left[ \frac{1}{3} (1-x^3) \right] dx \\ &= \frac{2}{7} \left[ \left( x - \frac{1}{4} x^4 \right) \right]_0^1 \\ &= \frac{3}{14} \end{aligned}$$

(c)  $P(X \leq \frac{1}{2})$ 

$$\begin{aligned}
P(X \leq \frac{1}{2}) &= \int_0^{\frac{1}{2}} \int_0^1 \frac{6}{7}(x+y)^2 dy dx \\
&= \int_0^{\frac{1}{2}} \frac{6}{7} \left[ \frac{1}{3}(x+y)^3 \right]_0^1 dx \\
&= \int_0^{\frac{1}{2}} \frac{6}{7} \left[ \frac{1}{3}((x+1)^3 - x^3) \right] dx \\
&= \int_0^{\frac{1}{2}} \frac{2}{7} [(x+1)^3 - x^3] dx \\
&= \frac{2}{7} \left[ \frac{1}{4}(x+1)^4 - \frac{1}{4}x^4 \right]_0^{\frac{1}{2}} \\
&= \frac{2}{28} \left[ \left( \frac{3^4}{2} - \frac{1^4}{2} \right) - 1 \right] \\
&= \frac{2}{14}
\end{aligned}$$

2. Find the marginal densities of  $X$  and  $Y$ .

$$\begin{aligned}
f(y) &= \int_{-\infty}^{\infty} f(x, y) dx \\
&= \int_0^1 \frac{6}{7}(x+y)^2 dx \\
&= \frac{6}{7} \left[ \frac{1}{3}(x+y)^3 \right]_0^1 \\
&= \frac{2}{7}(1+y)^3 + y^3 \\
&= \frac{2}{7}(3y^2 + 3y + 1), \quad 0 \leq y \leq 1
\end{aligned}$$

$$\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\
&= \int_0^1 \frac{6}{7}(x+y)^2 dy \\
&= \frac{6}{7} \left[ \frac{1}{3}(x+y)^3 \right]_0^1 \\
&= \frac{2}{7}(1+x)^3 + x^3 \\
&= \frac{2}{7}(3x^2 + 3x + 1), \quad 0 \leq x \leq 1
\end{aligned}$$

3. Find the two conditional densities.

$$\begin{aligned}
 f(x | y) &= \frac{f(x, y)}{f(y)} \\
 &= \frac{\frac{6}{7}(x+y)^2}{\frac{2}{7}(3y^2 + 3y + 1)} \\
 &= \frac{3(x+y)^2}{3y^2 + 3y + 1}
 \end{aligned}$$

$$\begin{aligned}
 f(y | x) &= \frac{f(x, y)}{f(x)} \\
 &= \frac{\frac{6}{7}(x+y)^2}{\frac{2}{7}(3x^2 + 3x + 1)} \\
 &= \frac{3(x+y)^2}{3x^2 + 3x + 1}
 \end{aligned}$$

*Rice Chapter 3, 14*

Suppose that

$$f(x, y) = xe^{-x(y+1)}, \quad 0 \leq x < \infty, \quad 0 \leq y < \infty$$

1. Find the marginal density of  $X$  and  $Y$ . Are  $X$  and  $Y$  independent?

Switch terms  $u = x(-y) - x$  so  
 $du = -xdy$

$$\begin{aligned}
 f_X(x) &= \int_0^\infty f_{XY} dy \\
 &= \int_0^\infty xe^{-x(y+1)} dy \\
 &= x \int_0^\infty e^{x(-y)-x} dy \\
 &= - \int_0^\infty e^u du \\
 &= -e^u \Big|_0^\infty \\
 &= -e^{-x(y+1)} \Big|_0^\infty \\
 &= 0 - (-e^{-x})
 \end{aligned}$$

So  $f_X(x) = e^{-x}$ .

To find

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)}$$

We'll integrate by parts,  $\int f dg = fg - \int g df$  where

$$\begin{aligned}f &= x \\df &= dx \\dg &= e^{x(-y-1)} dx \\g &= \frac{e^{x(-y-1)}}{-y-1}\end{aligned}$$

$$\begin{aligned}f_Y(y) &= \int_0^\infty x e^{-x(y+1)} dx \\&= \left. \frac{x e^{x(-y-1)}}{-y-1} \right|_0^\infty - \frac{1}{-y-1} \int_0^\infty e^{x(-y-1)} dx\end{aligned}$$

Substitute terms  $u = x(-y-1)$  and  $du = (-y-1)dx$ .

$$\begin{aligned}f_Y(y) &= 0 - \frac{1}{(-y-1)^2} \int_0^\infty e^u du \\&= -\frac{1}{(-y-1)^2} [e^u]_0^\infty \\&= \frac{1}{(y+1)^2}\end{aligned}$$

Therefore,  $f_Y(y) = \frac{1}{(y+1)^2}$ .

We say  $X, Y$  are independent if  $f(x, y) = f(x)f(y)$  but

$$x e^{-x(y+1)} \neq \frac{e^{-x}}{(y+1)^2}.$$

So  $X$  and  $Y$  are not independent.

2. Find the conditional densities of  $X$  and  $Y$ .

$$\begin{aligned}f(x | y) &= \frac{f(x, y)}{f(y)} \\&= \frac{x e^{-x(y+1)}}{\frac{1}{(y+1)^2}} \\&= (y+1)^2 x e^{-x(y+1)}\end{aligned}$$

$$\begin{aligned}f(x | y) &= \frac{f(x, y)}{f(x)} \\&= \frac{x e^{-x(y+1)}}{e^{-x}} \\&= x e^{-xy}\end{aligned}$$

*Question 5*

The Pareto distributions are a family of distributions of a continuous random variable  $X$  with probability density function given by

$$f(x; \alpha, \theta) = \begin{cases} \frac{\alpha\theta^\alpha}{x^{\alpha+1}} & \text{for } x \geq \theta \\ 0 & \text{for } x < \theta \end{cases}$$

where  $\alpha > 0$  and  $\theta > 0$  are the parameters of the family. The Pareto distribution arises as a model for the distribution of sizes of some measured quantity, such as personal or corporate income, or city population, or size of firm, given that it exceeds the threshold  $\theta$ .

1. Verify that this the formula for a density.

WHEN  $x > \theta$  we have  $f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}$ . To show that this is a density, the integral must sum to 1.

$$\begin{aligned} \int_{-\infty}^{\infty} f_X dx &= \int_{-\infty}^{\infty} \frac{\alpha\theta^\alpha}{x^{\alpha+1}} dx \\ &= \int_{\theta}^{\infty} \alpha\theta^\alpha x^{-(\alpha+1)} dx \\ &= \alpha\theta^\alpha \left[ \frac{-x^{-\alpha}}{\alpha} \right]_{\theta}^{\infty} \\ &= \alpha\theta^\alpha \left[ 0 - \left( \frac{-1}{\alpha\theta^\alpha} \right) \right] \\ &= 1 \end{aligned}$$

2. Find the formula for the cumulative distribution function.

$$\begin{aligned} F(x) &= \int_{-\infty}^{\infty} f(x) dx \\ &= \int_{\theta}^x \frac{\alpha\theta^\alpha}{x^{(\alpha+1)}} dx \\ &= \alpha\theta^\alpha \int_{\theta}^x x^{-(\alpha+1)} dx \\ &= \alpha\theta^\alpha \left[ \frac{-1}{\alpha x^\alpha} \right]_{\theta}^x \\ &= 1 - \left( \frac{\theta}{x} \right)^\alpha \end{aligned}$$

3. Assuming  $\alpha > 1$ , find  $E(X)$ . What is  $E(X)$  if  $0 < \alpha \leq 1$ ?

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_{\theta}^{\infty} x \frac{\alpha \theta^{\alpha}}{x^{\alpha+1}} dx \\
&= \alpha \theta^{\alpha} \int_{\theta}^{\infty} x^{-\alpha} dx \\
&= \alpha \theta^{\alpha} \left[ \frac{x^{1-\alpha}}{1-\alpha} \right]_{\theta}^{\infty} \\
&= \alpha \theta^{\alpha} \left[ \frac{-1}{(\alpha-1)(x^{\alpha-1})} \right]_{\theta}^{\infty} \\
&= \alpha \theta^{\alpha} \left[ 0 - \frac{-1}{(\alpha-1)(\theta^{\alpha-1})} \right] \\
&= \frac{\alpha \theta}{\alpha-1}, \quad \alpha > 1
\end{aligned}$$

When  $0 < \alpha \leq 1$  the integral does not converge. Therefore

$$E(X) = \begin{cases} \frac{\alpha \theta}{\alpha-1} & \text{for } \alpha > 1 \\ \text{DNE} & \text{for } 0 < \alpha \leq 1 \end{cases}$$

4. Show that if  $\alpha > 2$ ,  $\text{Var}(X) = \frac{\alpha \theta^2}{(\alpha-1)^2(\alpha-2)}$

$$= E(X^2) - \mu_x^2$$

Since  $\alpha > 1$  we already know  $E(X)$ . To find  $E(X^2)$ .

$$\begin{aligned}
E(X^2) &= \int_{\theta}^{\infty} x^2 \frac{\alpha \theta^{\alpha}}{x^{\alpha+1}} dx \\
&= \int_{\theta}^{\infty} \frac{\alpha \theta^{\alpha}}{x^{\alpha-1}} dx \\
&= \alpha \theta^{\alpha} \left[ \frac{-1}{(\alpha-2)x^{\alpha-2}} \right]_{\theta}^{\infty}, \quad \alpha > 2 \\
&= \alpha \theta^{\alpha} \left[ \frac{1}{(\alpha-2)(\theta^{\alpha-2})} \right] \\
&= \frac{\alpha \theta^2}{\alpha-2}, \quad \alpha > 2
\end{aligned}$$

Plugging what we just found back into the equation for variance, we have...

$$\text{Var}(X) = \frac{\alpha \theta^2}{\alpha-2} - \frac{\alpha \theta}{\alpha-1}$$

So  $\text{Var}(X) = \frac{\alpha \theta^2}{(\alpha-2)(\alpha-1)^2}$  when  $\alpha > 2$ .

5. Find (in terms of  $\alpha$  and  $\theta$ ) the median of the distribution of  $X$ .



THE MEDIAN can be found using the CDF.

$$\begin{aligned}\frac{1}{2} &= 1 - \left(\frac{\theta}{x}\right)^\alpha \\ \frac{1}{2} &= \left(\frac{\theta}{x}\right)^\alpha \\ x^\alpha &= \theta^\alpha\end{aligned}$$

$$\text{So } \tilde{\mu} = \theta^\alpha \sqrt{2}$$

6. Suppose  $X$  has a Pareto distribution with  $\alpha = 3$  and  $\theta = 1$ . Find  $P(1 < X < 4)$  and  $P(4 < X < 5)$ .

SINCE  $P(1 < X < 4) = P(X < 4) - P(X < 1)$  we consider the CDF.

$$\begin{aligned}P(X < 4) - P(X < 1) &= 1 - \left(\frac{1}{4}\right)^3 - \left(1 - \left(\frac{1}{1}\right)^3\right) \\ &= \frac{63}{64}\end{aligned}$$

$$\text{Similarly, } P(4 < X < 5) = \frac{61}{8000}$$

### Question 6

Suppose  $X$  and  $Y$  are independent random variables with  $X$  following a uniform distribution on  $(0, 1)$  and  $Y$  exponentially distributed with parameter  $\lambda = 1$ .

- Find the density for  $Z = X + Y$  (be careful with your limits of integration). Sketch the density and verify directly that it integrates to 1.
- Find the median (the value of  $Z$  for which  $P(Z \leq z) = \frac{1}{2}$ ).

THE FUNCTIONS PDFs can be written...

$$f_X = 1$$

and

$$f_Y = e^{-y}.$$

Since  $X$  and  $Y$  are independent we can find the density of using

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - x)f_Y(y)dy$$

However, there are two cases to consider since.

$$f_X(z - y) = \begin{cases} 1 & \text{for } 0 \leq z - y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

When  $0 \leq z \leq 1$  and  $z > 1$ . For the first we have.

$$\begin{aligned} f_Z(z) &= \int_0^z e^{-y}(1)dy \\ &= 1 - e^{-z}, \quad 0 \leq z \leq 1 \end{aligned}$$

When  $z > 1$ , we can say that  $z = y$  and we have

$$\begin{aligned} f_Z(z) &= \int_{z-1}^z e^{-y}(1)dy \\ &= e^{-(z-1)} - e^{-z} \end{aligned}$$

Therefore the PDF can be written

$$f_Z(z) = \begin{cases} 1 - e^{-z} & 0 \leq z \leq 1 \\ e^{-(z-1)} - e^{-z} & z > 1 \end{cases}$$

TO VERIFY this is a formula for a density we take (let  $u = -z$  and  $s = 1 - z$ ).

$$\begin{aligned} &\int_0^1 1 - e^{-z} + \int_1^\infty e^{-(z-1)} - e^{-z} \\ &- \int_0^1 e^{-z} + \int_0^1 1dz + - \int_1^\infty e^{-z}dz + \int_1^\infty e^{1-z} \\ &\int_0^1 e^u du + z + \int_1^\infty e^u du - \int_1^\infty e^s ds \\ &[z + e^{-z}]_0^1 + [-(e-1)e^{-z}]_1^\infty \end{aligned}$$

Which equals 1, verifying an equation for density!

NEXT we need to find the median. The familiar way to do this, starts with the CDF.

$$F_z(z) = \int_0^1 \int_0^{z-x} f(x, y) dy dx$$

let  $y = v - x$

$$\begin{aligned} &= \int_0^1 \int_0^z f(x, v - x) dv dx \\ &= \int_0^1 \int_0^z (1) e^{-(v-x)} dv dx \end{aligned}$$

let  $u = x - v$  and  $du = -dv$

$$\begin{aligned} &= \int_0^1 - \int_x^{z-x} e^u du dx \\ &= \int_0^1 [-e^u]_x^{z-x} dx \\ &= \int_0^1 (e^z - 1) e^{x-z} dx \\ &= (e^z - 1) \int_0^1 e^{x-z} dx \end{aligned}$$

now let  $u = x - z$  and  $du = dx$

$$\begin{aligned} &= (e^z - 1) \int_{-z}^1 1 - z^1 e^u du \\ &= (e^z - 1) (e^u) \Big|_{-z}^{1-z} \\ &= (e - 1) e^{-z} (e - 1) \end{aligned}$$

Now we solve for  $F_z(\tilde{\mu}) = \frac{1}{2}$

$$\begin{aligned} (e - 1) e^{-z} (e - 1) &= \frac{1}{2} \\ \frac{(e^z - 1)}{e^z} &= \frac{1}{2(e - 1)} \\ e^{-z} &= 1 - \frac{1}{2(e - 1)} \\ e^z &= \frac{1}{1 - \frac{1}{2(e-1)}} \\ z &= -\ln\left(1 - \frac{1}{2(e - 1)}\right) \end{aligned}$$

Hence,  $\tilde{\mu} \approx 0.34388$ .

2. Find  $E(X - Y)$  and  $\text{Var}(X - Y)$ . Find  $E(ZX)$ .

SINCE  $X$  and  $Y$  are independent, we can find  $E(X - Y) = E(X) - E(Y)$ .

$$\begin{aligned} E(X) &= \int_0^1 xf(x)dx \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} E(Y) &= \int_0^\infty ye^{-y}dy \\ &= -e^{-y}y \Big|_0^\infty + \int_0^\infty e^{-y}dy \\ &= e^{-y}y \Big|_0^\infty - e^{-y} \Big|_0^\infty \\ &= 1 \end{aligned}$$

$$\text{So } E(X - Y) = E(X) - E(Y) = \frac{-1}{2}$$

THE VARIANCE of X-Y can be find in a similiar way.

$$\text{Var}(X) = E(X^2) - \mu_x^2$$

So we need find  $E(X^2)$  and  $E(Y^2)$ .

$$\begin{aligned} E(X^2) &= \int_0^1 x^2 f_x(x) \\ &= \frac{x^3}{3} \Big|_0^1 \\ &= \frac{1}{3} \end{aligned}$$

$$\text{We have } \text{Var}(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Next, find  $E(Y^2)$

$$\begin{aligned} E(Y^2) &= \int_0^\infty y^2 e^{-y} dy \\ &= -e^{-y}y^2 \Big|_0^\infty + 2 \int_0^\infty e^{-y} y dy \\ &= -2e^{-x}x \Big|_0^\infty + 2e^{-x} \Big|_0^\infty \\ &= 2 \end{aligned}$$

$$\text{Now we have } \text{Var}(Y) = 2 - 1 = 1. \text{ Therefore } \text{Var}(X - Y) = \frac{-11}{12}$$

SINCE  $Z = X + Y$ ,

$$\begin{aligned} E(ZX) &= E(X^2 + XY) \\ &= E(X^2) + E(XY) \\ &= E(X^2) + E(X)E(Y) \\ &= \frac{1}{3} + \frac{1}{2} \\ &= \frac{5}{6} \end{aligned}$$