JOE SEIDEL

STAT 244 HOMEWORK 6

Question 1

Suppose X follows a geometric distribution with parameter p.

1. Derive the likelihood ratio for testing the hypothesis $p = p_0$ versus the alternative $p \neq p_0$.

   THE variable X has PMF given

   $$p(X = x) = p(1-p)^{x-1} \text{ , } x = 1, 2, ...$$

   We have a composite hypothesis a generalized likelihood ratio test is in order.

   $$\Lambda = \frac{\max_{p \in w_0}[L(p)]}{\max_{p \in \Omega}[L(p)]}$$

   Where the rejection region consists of small values for Λ. In this case $w_0 = \{p_0\}$ and $\Omega = \{0 < p < 1\}$

   $$\max_{p \in w_0}[L(p)] = p_0(1-p_0)^{x-1}.$$

   For the denomitor we have to maximize the likelihood for $p \in \Omega$. Which is the mle of $f(x \mid p) = p(1-p)^x$, $\hat{p} = \frac{1}{\overline{X}}$. Which makes

   $$\max_{p \in \Omega}[L(p)] = \frac{1}{\overline{X}}\left(1 - \frac{1}{\overline{X}}\right)^{x-1}.$$

   Therefore

   $$\Lambda = \frac{\overline{X} p_0 (1-p_0)^{x-1} \overline{X}^{x-1}}{(\overline{X}-1)^{x-1}}$$

   It's important to note that I've set this up observing only 1 X, otherwise this would look slightly different, in fact I will end up replacing x for $\overline{X}$ soon.

   Which is the generalized likelihood ratio that will test the hypothesis.

2. For $p_0 = 0.01$, by some combination of numerical experimentation and mathematical analysis, find the set of possible valyes of x for X for which the likelihood ratio is less than 0.1.

   EVALUATING Λ for a few values of X yields.

x	1	2	3	4	5
Λ	.01	.0396	.0066	.09	.117

   Table 1: GLR

   Hence for $x \leq 4$ we have likelihood ratio less than 0.1.

3. Find the probability of Type 1 error for the test the rejects $p_0 = 0.01$ when the likelihood ratio is less than 0.1. Find the power of this test when $p = 0.5$. Find the power of ths test when $p = 0.001$.

Question 2 Rice 9.12

Let $X_1, ..., X_n$ be a random sample from an exponential distribution with the density function $f(x \mid \theta) = \theta e^{-\theta x}$. Derive a likelihood ratio test of H_0: $\theta = \theta_0$ versus H_A: $\theta \neq \theta_0$, and show that the rejection region is of the form $\{\overline{X} e^{-\theta_0 \overline{X}} \leq c\}$.

FIRST, recall that the mle of $L(\theta)$ is $\hat{\theta} = \overline{X}$. Set up the likelihood ratio test

I've found this in previous homeworks.

$$\Lambda = \frac{\max_{p \in w_0}[L(p)]}{\max_{p \in \Omega}[L(p)]}.$$

The numerater will be

$$L(\theta_0) = \prod_{i=1}^{n} \theta e^{-\theta_0 x_i}$$

and the denomitaor

$$L(\hat{\theta}) = \prod_{i=1}^{n} \frac{1}{\overline{X}} e^{-\frac{x_i}{\overline{X}}}.$$

Then

$$\begin{aligned}
\Lambda &= \frac{\prod_{i=1}^{n} \theta e^{-\theta_0 x_i}}{\prod_{i=1}^{n} \frac{1}{\overline{X}} e^{-\frac{x_i}{\overline{X}}}} \\
&= \frac{\theta_0^n e^{-\theta_0 n \overline{X}}}{\frac{1}{\overline{X}^n} e^{-\frac{\overline{X} n}{\overline{X}}}} \\
&= \frac{\theta_0^n \overline{X}^n e^{-\theta_0 n \overline{X}}}{e^{-n}} \\
&= \left(e \theta_0 \overline{X} e^{(-\theta_0 \overline{X})}\right)^n
\end{aligned}$$

Where H_0 is rejected when Λ is small. Since e, n, θ are positive, Λ is small when $\overline{X} e^{(-\theta_0 \overline{X})}$ is small.

$$\begin{aligned}
\left(e \theta_0 \overline{X} e^{(-\theta_0 \overline{X})}\right)^n &< c_1 \\
e \theta_0 \overline{X} e^{(-\theta_0 \overline{X})} &< c_1^{\frac{1}{n}} \\
\overline{X} e^{(-\theta_0 \overline{X})} &< \frac{c_1^{\frac{1}{n}}}{\theta_0 e}
\end{aligned}$$

Therefore, we see the rejection region takes the form $\overline{X}e^{(-\theta_0\overline{X})} \leq c = \frac{c_1^{\frac{1}{n}}}{\theta_0 e}$.

Question 3 Rice 9.13

Suppose, to be specific, that in problem 12, $\theta_0 = 1$, $n = 10$, and that $\alpha = .05$. In order to use the test, we must find the appropriate value of c.

1. Show that rejection region is of the form $\{\overline{X} \leq x_0\} \cup \{\overline{X} \geq x_1\}$, where x_0 and x_1 are deterimined by c.

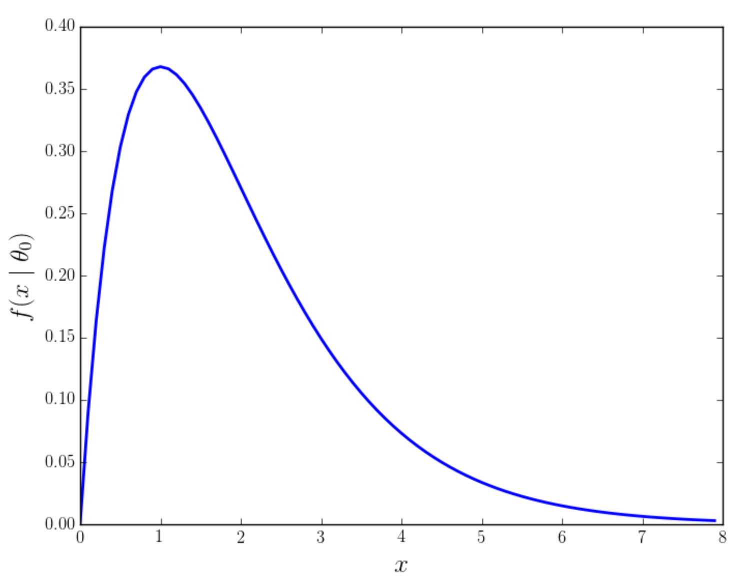
   Now that we are given a value for θ_0 we have

$$f(x \mid \theta_0) = xe^{-x}.$$

   But we also found in the previous question that our test rejects when

$$f(\overline{X} \mid \theta_0) = \overline{X}e^{(-\overline{X})}$$

   is small, specifically when less than c. To see why it takes the form $\{\overline{X} \leq x_0\} \cup \{\overline{X} \geq x_1\}$, consider the graph of the function.



   There would be a c chosen that corresponds with the Y axis of the graph and a horizontal line would interect the function at two values of x.

2. Explain why c should be chosen so that $P(\overline{X}e^{(-\overline{X})} \leq c) = .05$.

THIS is just a restatement of Type I error under Neyman-Pearson.

$$Pr(\text{Reject } H_0 \mid H_0) = Pr(x \in [0, c] \mid \theta_0)$$

Which can be worded to say, the probability that x falls in the rejection zone. Furthermore, we found that $P(\overline{X}e^{(-\overline{X})})$ provides a rejection zone for Λ. What remains is to determine how willing we are to make Type 1 error.

If we want $\alpha = .05$ then we should set

$$Pr(x \in [0, c] \mid \theta_0) = Pr(f(\overline{X} \mid \theta)) < c) = .05$$

to determine what c should be.

3. Explain why $\sum_{i=1}^{1} 0X_i$ and hence $\overline{X}$ follow gamma distributions when $\theta_0 = 1$. How could this knowledge be used to choose c?

Under H_0: $\theta_0 = 1$ $X_i \sim \textit{Exponetial}(1)$ which is a special ase of $\Gamma(1, \lambda)$ or in this particular case $\Gamma(1, 1)$. On the last homework, I found that $\sum_{i=1}^{n} X_i \sim \Gamma(n, 1)$ and $\overline{X} \sim \Gamma(n, n)$. Knowing the exact distribution, means I could compute the exact value of c for which 95 coverage for acceptable values of H_0.

In this day and age, a computer would be the easiest way to do this. The gist would be first solve $f(\overline{X}) = c$ to get $x_0(c)$ and $x_1(c)$. Then solve

$$\alpha(c) = F(x_0(c)) + 1 - F(x_1(c))$$

Where $F(x)$ is cdf of $\Gamma(10, 10)$.

4. Suppose you hadn't thought of the preceding fact. Explain how you could determine a good approximation to c by generating random numbers on a computer (simulation).

GENERATE a bunch of samples from $\textit{Exponential}(1)$ of size $n = 10$. Then compute $\overline{X}e^{\overline{X}}$ for each sample. Sort the result and set the cuttoff as the value indexed at the 5% of the number of samples generated. For example, if the results are stored in some list of size 10000 then you'd take value indexed at list[500] as the cuttoff value for a close approximation.

Question 4

Figure 1: James Mill, the economist

Call "This, it thus, and" Class I workds; Class II is "everything else". For each of 215 groups of 5 of James Mill's sentences, the number of Class I words was counted.

Test whether a Binomial Distribution $(n = 5, \theta)$ fits these data.

FIRST, knowing the MLE of a binomial will be useful.

$$L(\theta) = \prod_{i=1}^{n} \binom{n}{x_i} \theta^{x_i} (1-\theta)^{n-x_i}$$
$$= \prod_{i=1}^{n} \binom{n}{x_i} \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

Taking the log and the derivative we have

$$\frac{d}{d\theta} \log L(\theta) = \frac{\overline{X}n}{\theta} + \frac{n - \overline{X}n}{\theta - 1}$$

Setting the above equal to zero and solving for θ give the mle.

$$\hat{\theta} = \overline{X}$$

Using the table we can compute the mle, $\hat{\theta} = 339/215(5) = .31$. With this a few rows can be added to the table.

Expected is calculated

$$E_i = \binom{n}{x_i} \hat{\theta}_i^x (1-\hat{\theta})^{n-x_i} \cdot 225 \text{ for } x_i = 0, 1, 2, .., n = 5$$

Then calculate each component of

$$X^2 = \sum_{i=1}^{m} \frac{[x_i - np_i(\hat{\theta})]^2}{np_i(\hat{\theta})}$$

Where $m = 6$ cells.

No. Class I words	0	1	2	3	4	5
No. groups(observed)	87	11	51	42	20	4
No. groups(expected)	34	76	68	30	7	1
Component of Chi-Squared	82.62	55.59	4.25	4.8	24.14	9.0

Table 2: Class I words

Summing up the last row of the table, the chi-square statistic $X^2 = 180.4$ with 4 degrees of freedom (6 cells and one parameter was estimated from the data). Finally, our rejection region is $X^2 > \chi_4^2(\alpha)$. If we specify $\alpha = .005$ (giving our null hypothesis the best chance) our test is

$$180.4 = X^2 > \chi^2_4(.005) = 14.86$$

Which rejects the null hypothesis that $X \sim Bin(n = 5, \theta)$.

Question 5

The members of a community are classified by Blood type:

O	A	B	BA	Total
121	120	79	33	353

Table 3: Community blood types

Theory has is that the probabilities of those types depends on gene frequency parameters r, p, q, where $r + p + q = 1$ and $P(\text{"O"}) = r^2$, $P(\text{"A"}) = p^2 + 2pr$, $P(\text{"B"}) = q^2 + 2qr$, and $P(\text{"AB"}) = 2pq$. Using numerical methods (that is, a method such as that described in Chapter 5 of Stigler's notes) we can find the MLEs of r, p, q; they are

$$\hat{r} = 0.580$$
$$\hat{p} = 0.246$$
$$\hat{q} = 0.173$$

Test if the community fits the theory.

Can't afford to estimate three parameters with only 4 cells!

SINCE there are only 4 cells, I cannot afford to estimate all 3 parameters. Instead, I'll just estimate 2 and the substute the third $r' = 1 - .246 - .173 = .581$

	O	A	B	BA	Total
observed	121	120	79	33	353
expected	119.15	122.26	81.5	30.5	353.41
Comp of Chi-Square	0.02	0.04	0.08	0.2	.342

Table 4: Blood Types with Chi-Square

The table above was completed using similiar methods described in the previous question. The chi-squared statistic is $X^2 = .342$. We have 1 degree of freedom ($4 - 1 - 2 = 1$). Choose $\alpha = .1$, then

$$.342 = X^2 < \chi^2_1(.1) = 2.71$$

Which supports the null that the community fits the theory.

Question 6

Are finger print patterns genetic, or are the developmental? In 1892, Francis Galton compiled the following table on the relationship between the patterns on the same finger of 105 sibling pairs. Test the

Columbus sailed the ocean blue...or was it 1492?

hypothesis that the patterns are independent for example, that knowing one sibiling (A) has Whorl on the figunre does not help in predicting the pattern of the other (B).