

JOE SEIDEL

STAT 244

HOMEWORK 5

Question 1 Rice 8.16

Consider an i.i.d. sample of random variables with density function

$$f(x | \sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right)$$

1. Find the maximum likelihood estimate of σ .

SUPPOSE we have i.i.d. random variables with result x_1, x_2, \dots, x_n .
Then

$$\begin{aligned} f(x_1, x_2, \dots, x_n | \sigma) &= f(x_1 | \sigma) \cdot f(x_2 | \sigma) \cdots f(x_n | \sigma) \\ &= \frac{1}{2\sigma} e^{-\frac{|x_1|}{\sigma}} \cdot \frac{1}{2\sigma} e^{-\frac{|x_2|}{\sigma}} \cdots \frac{1}{2\sigma} e^{-\frac{|x_n|}{\sigma}} \\ &= \prod_{i=1}^n \frac{1}{2\sigma} e^{-\frac{|x_i|}{\sigma}} \end{aligned}$$

So the likelihood function of σ is given

$$L(\sigma) = \prod_{i=1}^n \frac{1}{2\sigma} e^{-\frac{|x_i|}{\sigma}}.$$

Estimate $\hat{\sigma}$ by taking the log and differentiate then solve by setting to 0.

$$\begin{aligned} \frac{d}{d\sigma} \log(L(\sigma)) &= \frac{d}{d\sigma} - n \log(2) - n \log(\sigma) - \sum_{i=1}^n \frac{|x_i|}{\sigma} \\ &= -\frac{n}{\sigma} + \frac{\sum_{i=1}^n |x_i|}{\sigma^2} \end{aligned}$$

Setting the above equal to 0 gives the result

$$\hat{\sigma} = \frac{1}{n} \sum_{i=1}^n |x_i|$$

2. Find the asymptotic variance of the mle.

In part 2 of this question, we'll verify that that this maximizes

When to use Likelihood function vs $f(x | \theta)$

$$\frac{1}{\tau^2} = -E\left[\frac{d^2}{d\sigma^2} \log L(\sigma)\right]$$

$$\begin{aligned}
\frac{d^2}{d\sigma^2} \log(L(\sigma)) &= \frac{n}{\sigma^2} - \frac{2 \sum_{i=1}^n |x_i|}{\sigma^3} \\
&= \frac{n}{\hat{\sigma}^2} - \frac{2n}{\hat{\sigma}^2} \\
&= -\frac{n}{\hat{\sigma}^2}
\end{aligned}$$

We sub σ with the estimator to eliminate the summation term.

Plugging the result into the equation that began this section

$$\begin{aligned}
\frac{1}{\tau^2} &= -E\left[-\frac{n}{\hat{\sigma}^2}\right] \\
&= \frac{n}{\hat{\sigma}^2}
\end{aligned}$$

Finally, $\tau^2 = \frac{\hat{\sigma}^2}{n}$.

Question 2

Suppose that X_1, X_2, \dots, X_n are i.i.d. random variables on the interval $[0, 1]$ with the density function

$$f(x | \alpha) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1} (1-x)^{2\alpha-1}$$

where $\alpha > 0$ is a parameter to be estimated from the sample. It can be shown that

$$\begin{aligned}
E(X) &= \frac{1}{3} \\
\text{Var}(X) &= \frac{2}{9(3\alpha + 1)}
\end{aligned}$$

1. What equation does the mle of α satisfy.

$$\begin{aligned}
L(\alpha) &= f(x_1 | \alpha) \cdot f(x_2 | \alpha) \cdots f(x_n | \alpha) \\
&= \prod_{i=1}^n \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x_i^{\alpha-1} (1-x_i)^{2\alpha-1}
\end{aligned}$$

$$\begin{aligned}
\log(L(\alpha)) &= n \log(\Gamma(3\alpha)) - n \log(\Gamma(\alpha)) - n \log(\Gamma(2\alpha)) \\
&\quad + \sum_{i=1}^n [(\alpha-1) \log(x_i) + (2\alpha-1) \log(1-x_i)]
\end{aligned}$$

$$\begin{aligned} \frac{d}{d\alpha} \log(L(\alpha)) &= \frac{n3\Gamma'(3\alpha)}{\Gamma(3\alpha)} - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{n2\Gamma'(2\alpha)}{\Gamma(2\alpha)} \\ &\quad + \sum_{i=1}^n [\log(x_i) + 2\log(1-x_i)] \end{aligned}$$

Setting the above equal 0.

$$\frac{3\Gamma'(3\alpha)}{\Gamma(3\alpha)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{2\Gamma'(2\alpha)}{\Gamma(2\alpha)} = -\frac{1}{n} \sum_{i=1}^n [\log(x_i) + 2\log(1-x_i)]$$

2. What is the asymptotic variance of the mle.

Use the property

$$\frac{1}{\tau^2} = -E\left[\frac{d^2}{d\alpha^2} \log(L(\alpha))\right]$$

$$\begin{aligned} \frac{d^2}{d\alpha^2} &= \frac{n9\Gamma''(3\alpha)}{\Gamma(3\alpha)} - \frac{n9\Gamma'(3\alpha)^2}{\Gamma(3\alpha)^2} - \frac{n\Gamma''(\alpha)}{\Gamma(\alpha)} + \frac{n\Gamma'(\alpha)^2}{\Gamma(\alpha)^2} \\ &\quad - \frac{n4\Gamma''(2\alpha)}{\Gamma(2\alpha)} + \frac{n4\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2} \end{aligned}$$

$$\begin{aligned} \frac{1}{\tau^2} &= -E\left[n\left(\frac{9\Gamma''(3\alpha)}{\Gamma(3\alpha)} - \frac{9\Gamma'(3\alpha)^2}{\Gamma(3\alpha)^2} - \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} + \frac{\Gamma'(\alpha)^2}{\Gamma(\alpha)^2} \right. \right. \\ &\quad \left. \left. - \frac{4\Gamma''(2\alpha)}{\Gamma(2\alpha)} + \frac{4\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2}\right)\right] \end{aligned}$$

Hence

$$\begin{aligned} \tau^2 &= -\frac{1}{n} \left[\frac{9\Gamma''(3\alpha)}{\Gamma(3\alpha)} - \frac{9\Gamma'(3\alpha)^2}{\Gamma(3\alpha)^2} - \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} + \frac{\Gamma'(\alpha)^2}{\Gamma(\alpha)^2} \right. \\ &\quad \left. - \frac{4\Gamma''(2\alpha)}{\Gamma(2\alpha)} + \frac{4\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2} \right]^{-1} \quad (1) \end{aligned}$$

Question 4

Suppose that X is the number of successes in a Binomial experiment with n trials and probability of success $\theta/(1+\theta)$, where $0 \leq \theta < \infty$.

1. Find the MLE of θ .

CONSIDER

$$p(x | \theta) = \binom{n}{x} \left(\frac{\theta}{1+\theta}\right)^x \left(1 - \frac{\theta}{1+\theta}\right)^{n-x}.$$

Then

$$L(\theta) = \binom{n}{x} \left(\frac{\theta}{1+\theta} \right)^x \left(1 - \frac{\theta}{1+\theta} \right)^{n-x}.$$

The log likelihood is

$$\log L(\theta) = \log \binom{n}{x} + x \log \left(\frac{\theta}{1+\theta} \right) + (n-x) \log \left(1 - \frac{\theta}{1+\theta} \right)$$

$$\begin{aligned} \frac{d}{d\theta} \log L(\theta) &= 0 + \frac{x(1+\theta)}{\theta} - (n-x)(\theta+1) \\ &= \frac{x(1+\theta)}{\theta} - (n-x)(\theta+1) \end{aligned}$$

Setting the above to 0 results in

$$\hat{\theta} = \frac{x}{n-x}$$

furthermore

$$\frac{d^2}{d\theta^2} = -n - \frac{x}{\theta^2} + x$$

verifying that this maximizes $\hat{\theta}$.

2. Use Fisher's Theorem to find the approximate distribution of the MLE when n is large.

Fisher's approximation allows the result for large n in the given case, $\hat{\theta}$ will have approximately a $N(\theta, \frac{\tau^2(\theta)}{n})$ distribution.

It remains to find $\tau^2(\theta)$.

$$\frac{1}{\tau^2(\theta)} = -E \left[\frac{d^2}{d\theta^2} \log f(X_1 | \theta) \right].$$

$$\begin{aligned} E \left[-n - \frac{x}{\theta^2} + x \right] &= -n - \frac{E(X)}{\theta^2} + E(X) \\ &= -n - \frac{n\theta}{1+\theta} * \frac{1}{\theta^2} + \frac{n\theta}{1+\theta} \\ &= n \left(-1 - \frac{1+\theta^2}{(1+\theta)\theta} \right) \\ &= -n \left(1 + \frac{1-\theta}{\theta} \right) \\ &= -\frac{n}{\theta} \end{aligned}$$

Hence $\tau^2(\theta) = \frac{\theta}{n}$

Applying Fisher, $\hat{\theta}$ will have an approximately normal $N(\theta, \theta)$ distribution.

Question 4

If X and Y are independent each with a Poisson distribution, show that $Z = X + Y$ is Poisson distributed.

SINCE X and Y are independent, use the discrete convolution formula. I'll assume X and Y have parameters λ_1 and λ_2 respectively. However, the result will be obvious if the parameter λ is the same.

$$P(X + Y = z) = p_Z(z) = \sum_{x=0}^z p_X(x) p_Y(z-x)$$

$$\begin{aligned} p_Z(z) &= \sum_{x=0}^z e^{-\lambda_1} \frac{\lambda_1^x}{x!} e^{-\lambda_2} \frac{\lambda_2^{z-x}}{(z-x)!} \\ &= e^{-(\lambda_1+\lambda_2)} \sum_{x=0}^z \frac{\lambda_1^x}{x!} \frac{\lambda_2^{z-x}}{(z-x)!} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{1}{z!} \sum_{x=0}^z \frac{z!}{x!(z-x)!} \lambda_1^x \lambda_2^{z-x} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{1}{z!} (\lambda_1 + \lambda_2)^z \\ &= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^z}{z!} \end{aligned}$$

Which is a Poisson probability distribution.

Question 5

In a famous example, Bortkiewicz tabulated the number of Cavalry men kicked to death by horses in the Prussian Cavalry, for 14 Corps over 20 years (1875-1894), giving $n = 280$ observations in all.

Number of deaths	Frequency count
0	144
1	91
2	32
3	11
4	2
More	0
Total	280

Table 1: Frequency tabulation

1. For this model, find the MLE of θ , assume the X_i 's are independent.

The likelihood function of θ is

$$L(\theta) = e^{-\theta n} \frac{\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n \frac{1}{x_i!}}.$$

Differentiating the log of this function gives

$$\frac{d}{d\theta} \log L(\theta) = \frac{\sum_{i=1}^n x_i}{\theta} - n.$$

Setting the above equal to zero we have

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{X}$$

2. Find the MSE of the MLE

SINCE the MLE is the sample mean the MLE is unbiased.

$$\text{MSE} = \text{Var}(\hat{\theta}) = \text{Var}(\bar{X}) = \frac{\text{Var}(X_i)}{n}$$

Since $P(X_i = k \mid \theta)$ is Poisson distributed, $\text{Var}(X_i) = \theta$.

So $\text{MSE} = \frac{\theta}{n}$.

3. From the Central Limit Theorem find the approximate distribution of the MLE when n is large.

Since the MLE is the sample mean for i.i.d X_i where $E(X_i) = \theta$ and $\text{Var}(X_i) = \theta$, $\hat{\theta}$ has an approximately $N(\theta, \frac{\theta}{n})$ distribution, which becomes a better approximation as n gets larger.

4. From Fisher's Approximation, find the approximate distribution of the MLE when n is large.

FISHER's approximation states that if data consist of independent random variables each with distribution $f(x \mid \theta)$, and $\hat{\theta}$ can be found by solving log likelihood equal to 0, then for large n , $\hat{\theta}$ has approximately a $N(\theta, \frac{\tau^2(\theta)}{n})$.

$$\frac{1}{\tau^2(\theta)} = -E \left[\frac{d^2}{d\theta^2} \log p(X_1 \mid \theta) \right]$$

Where

$$p(X_1 \mid \theta) = e^{-\theta} \frac{\theta^x}{x_1!}.$$

Taking the log of the above yields

$$\log p(X_1 \mid \theta) = -\theta + x \log(\theta) - \log(x!).$$

Differentiating,

$$\begin{aligned}\frac{d}{d\theta} \log p(X_1 | \theta) &= \frac{x_1}{\theta} - 1 \\ \frac{d^2}{d\theta^2} \log p(X_1 | \theta) &= -\frac{x_1}{\theta^2}\end{aligned}$$

Plugging this into the formula for asymptotic variance,

$$\frac{1}{\tau^2} = -E\left[-\frac{x_1}{\theta^2}\right] = \frac{E[X_1]}{\theta^2}$$

Recall that $E(X_1) = \theta$ so $\tau^2(\theta) = \theta$. Therefore, from Fisher the distribution of $\hat{\theta}$ can be approximated as $N(\theta, \frac{\theta}{n})$.

5. Evaluate the MLE for the given data.

We found MLE to be the sample mean. So take the total number of deaths by horse kicks and divide by $n = 280$.

$$\text{MLE} = \hat{\theta} = \frac{196}{280} = .7$$

6. If θ , the mean number of deaths per Corps in a year, is really 1.0, what (approximately) is the probability that the MLE would turn out to be below .85.

SINCE $\hat{\theta}$ has approximately $N(\theta, \frac{\theta}{n})$ distribution calculate this using

$$P(\hat{\theta} < .85) = \Phi\left(\frac{.85 - 1}{\sqrt{\frac{1}{280}}}\right) = .0062$$

Question 6

Let X_1, \dots, X_n be an i.i.d. sample from a Rayleigh distributions with parameter $\theta > 0$:

$$f(x | \theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, x \geq 0$$

1. Find the mle of θ

THE likelihood function of θ is

$$\begin{aligned}L(\theta) &= f(x_1, \dots, x_n | \theta) \\ &= f(x_1 | \theta) \cdots f(x_n | \theta) \\ &= \prod_{i=1}^n \frac{x_i}{\theta^2} e^{-x_i^2/(2\theta^2)}\end{aligned}$$

Taking the log gives

$$\log L(\theta) = \sum_{i=1}^n \left[\log(x_i) - 2 \log(\theta) - \frac{x_i^2}{2\theta^2} \right].$$

Differentiating

$$\frac{d}{d\theta} \log L(\theta) = -\frac{2n}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^3}.$$

Setting the derivative of the log likelihood to zero yields

$$\hat{\theta} = \sqrt{\frac{\sum_{i=1}^n x_i}{2n}}$$

2. Find the asymptotic variance of the mle.

Again, does asymptotic variance come from likelihood function or single $f(x | \theta)$.

$$\begin{aligned} \frac{d^2}{d\theta^2} \log L(\theta) &= \frac{d^2}{d\theta^2} \frac{2n}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta^3} \\ &= \frac{2n}{\theta^2} - \frac{3 \sum_{i=1}^n x_i}{\theta^4} \\ &= \frac{2n}{\hat{\theta}^2} - \frac{6n}{\hat{\theta}^2} \\ &= -\frac{4n}{\hat{\theta}^2} \end{aligned}$$

$$\frac{1}{\tau^2(\theta)} = -E \left[-\frac{4n}{\hat{\theta}^2} \right]$$

So the asymptotic variance is $\frac{\hat{\theta}^2}{4n}$