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STATS 244 HOMEWORK 4

Question 1

Let $Z \sim N(0,1)$, a stand normal distribution and let $X \sim N(\mu\sigma^2)$. Let $\Phi(z)$ be the cdf of Z. Suppose $X \sim N(-4,16)$; find.

It helps to know that

$$P(X < x) = \Phi\Big(\frac{x-\mu}{\sigma}\Big).$$

1. $P(X > 2)$.

$$P(X > 2) = 1 - P(X < 2) \qquad = 1 - \Phi\Big(\frac{2-(-4)}{\sqrt{16}}\Big)$$
$$= 1 - .9332$$
$$= .0688$$

2. $P(0 < X < 4)$

$$\begin{aligned} P(0 < X < 4) &= P(X < 4) - P(X < 0) \\ &= \Phi\Big(\frac{4-(-4)}{4}\Big) - \Phi\Big(\frac{0-(-4)}{4}\Big) \\ &= .1539 \end{aligned}$$

3. $P(|X+3| \geq 3)$

$$\begin{aligned} P(|X+3| \geq 3) &= P(X \geq 0) + P(X \leq -6) \\ &= 1 - \Phi\Big(\frac{4}{4}\Big) + \Phi\Big(\frac{-2}{4}\Big) \\ &= .1587 + .3085 \\ &= .4672 \end{aligned}$$

4. $P(X \leq 0 \text{ or } X \geq 3)$

$$\begin{aligned} P(X \leq 0 \text{ or } X \geq 3) &= P(X \leq 0) + P(X \geq 3) \\ &= \Phi\Big(\frac{4}{4}\Big) + 1 - \Phi\Big(\frac{7}{4}\Big) \\ &= .8413 + .0003 \\ &= .8416 \end{aligned}$$

Question 2

Based on student A's performace during the first two weeks of a course, the professor has approximately a normal $N(70, 8^2)$ prior distribution about the student's true ability, on a scale of 0 to 100.

Consider the midterm examination as an error-prone measure of the student's true ability, where if the true ability is x, the examination score can be modeled as approximately normally distributed, $N(x, 6^2)$. The student scores 90 on the midterm.

1. What are the posterior expectaion and the probability that the student's true ability is above 85?

   WE HAVE the following information.

   $$f(\theta) \sim N(70, 8^2)$$

   and

   $$f(x \mid \theta) \sim N(x, 6^2)$$

   Using Bayes, $f(\theta \mid X) \propto f(x \mid \theta) f(\theta)$ the posterior distribution is

   $$(\theta \mid X) = \frac{1}{\sqrt{2\pi}B} e^{\frac{(\theta - A)^2}{2B^2}}$$

   Where

   $$A = \frac{6^2(70) + 8^2(90)}{6^2 + 8^2}$$

   and

   $$B^2 = \frac{6^2 * 8^2}{6^2 + 8^2}$$

   Which give us a $N(82.8, 23.4)$ posterier distribution and means posterior of expectation of the student's true ability it 82.8. Using the the method from question 1, there is a .4625 probability that the students true ability is above 85.

2. Above 90?

   .3792

Question 3

A "psychic" uses a five-card deck of cards to demenonstrate ESP, and claims to be able to guess a card correctly with probability .5. A single experiment consists of making five guesses, reshuffling the deck after each guess. The experiment is treid and the "pyschic" guesses correctly 3 times out of give. Assuming the only two possibilities are "ESP" and "ordinary guessing", how how must the a priori ability be that "psychic" has ESP is at alteast .7?

LET $P(\theta)$ be the prior probability that an individual has ESP that we wish to find.

Bayes thereom is

$$P(\theta \mid X) = \frac{P(X \mid \theta)P(\theta)}{\sum_{i=1}^{2} P(X \mid \theta_i)P(\theta_i)}.$$

Where we can think of $P(\theta_1) = P(\theta)$ and $P(\theta_2) = 1 - P(\theta)$

Plugging in the observed data along with guessing probabilities provided.

$$P(X = 3 \mid \theta) = \binom{5}{3}(.5)^3(.5)^2 = .3125$$

and

$$P(X = 3 \mid \theta_2) = \binom{5}{3}(.2)^3(.8)^2 = .0512.$$

We can plug these values into Bayes and solve for $P(\theta \mid X) \leq .7$. Ie

$$.7 \leq \frac{.3125P(\theta)}{.3125P(\theta) + .0512(1 - P(\theta))}$$

Which gives $P(\theta) \geq .276$

Question 4

Suppose $\hat{\theta}_1$ and $\hat{\theta}_2$ are uncorrelated and both are unbiased estimators of θ, and that $\text{Var}(\hat{\theta}_1) = 2\text{Var}(\hat{\theta}_2)$.

1. Show that for any constant c, the weighted average $\hat{\theta}_3 = c\hat{\theta}_1 + (1 - c)\hat{\theta}_2$ is an unbiased estimator of θ.

$$\begin{aligned} E(\hat{\theta}_3) &= cE(\hat{\theta}_1) + (1 - c)E(\hat{\theta}_2) \\ &= c\theta + (1 - c)\theta \\ &= \theta \end{aligned}$$

   Hence $B(\hat{\theta}_3) = 0$.

2. Find c for which $\hat{\theta}_3$ has the smallest MSE.

   The MSE of $\hat{\theta}_3$ is

$$c^2\text{Var}(\hat{\theta}_1) + (1 - c)^2\text{Var}(\hat{\theta}_2)$$

   We can find the value, c, that minimizes MSE by differentation.

$$\begin{aligned} \frac{d}{dc} MSE(\hat{\theta}_2) &= 2cVar(\hat{\theta_1}) - 2Var(\hat{\theta_2})(1-c) \\ &= 2cVar(\hat{\theta_1}) - Var(\hat{\theta_1})(1-c) \\ &= 3cVar(\hat{\theta_1}) - Var(\hat{\theta_1}) \end{aligned}$$

Setting the above equal to 0 results in $c = \frac{1}{3}$. Furthermore, the second derivative of the MSE is positive so we can confirm that c minimizes.

3. Are there any values of c, $0 \leq c \leq 1$ for which $\hat{\theta}_3$ is better (in the sense of MSE) than both $\hat{\theta}_1$ and $\hat{\theta}_2$

We are given that $\text{Var}(\hat{\theta}_2)$ is less than $\text{Var}(\hat{\theta}_1)$. Since the bias as zero, the MSE is just the variance of each estimator.

$$\begin{aligned} \text{MSE } \hat{\theta}_3 &= c^2 \text{Var}(\hat{\theta}_1) + (1-c)^2 \text{Var}(\hat{\theta}_2) \\ &= c^2 2\text{Var}(\hat{\theta}_2) + (1-c)^2 \text{Var}(\hat{\theta}_2) \\ &= \end{aligned}$$