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STAT 244 HOMEWORK 5

Question 1 Rice 8.16

Consider an i.i.d. sample of random variables with density function

$$f(x \mid \sigma) = \frac{1}{2\sigma} \exp\Big(- \frac{|x|}{\sigma}\Big)$$

1. Find the maximum likelihood estimate of σ.

   SUPPOSE we have i.i.d. random variables with result $x_1, x_2, .., x_n$. Then

$$\begin{aligned} f(x_1, x_2, ..., x_n \mid \sigma) &= f(x_1 \mid \sigma) \cdot f(x_2 \mid \sigma) \cdots f(x_n \mid \sigma) \\ &= \frac{1}{2\sigma} e^{-\frac{|x_1|}{\sigma}} \cdot \frac{1}{2\sigma} e^{-\frac{|x_2|}{\sigma}} \cdots \frac{1}{2\sigma} e^{-\frac{|x_n|}{\sigma}} \\ &= \prod_{i=1}^{n} \frac{1}{2\sigma} e^{-\frac{|x_i|}{\sigma}} \end{aligned}$$

   So the likelihood function of σ is given

$$L(\sigma) = \prod_{i=1}^{n} \frac{1}{2\sigma} e^{-\frac{|x_i|}{\sigma}}.$$

   Estimate $\hat{\sigma}$ by taking the log and differentiate then solve by setting to 0.

$$\begin{aligned} \frac{d}{d\sigma} \log(L(\sigma)) &= \frac{d}{d\sigma} - n\log(2) - n\log(\sigma) - \sum_{i=1}^{n} \frac{|x_i|}{\sigma} \\ &= -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} |x_i|}{\sigma^2} \end{aligned}$$

   Setting the above equal to 0 gives the result

   In part 2 of this question, we'll verify that that this maximizes

$$\hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$$

2. Find the asymptotic variance of the mle.

$$\frac{1}{\tau^2} = -E\Big[\frac{d^2}{d\sigma^2} \log L(\sigma) \Big]$$

$$\begin{aligned}
\frac{d^2}{d\sigma^2}\log(L(\sigma)) &= \frac{n}{\sigma^2} - \frac{2\sum_{i=1}^{n}|x_i|}{\sigma^3} \\
&= \frac{n}{\hat{\sigma}^2} - \frac{2n}{\hat{\sigma}^2} \\
&= -\frac{n}{\hat{\sigma}^2}
\end{aligned}$$

We sub σ with the estimator to elimate the summation term.

Plugging the result into the equation that began this section

$$\begin{aligned}
\frac{1}{\tau^2} &= -E[-\frac{n}{\hat{\sigma}^2}] \\
&= \frac{n}{\hat{\sigma}^2}
\end{aligned}$$

Finally, $\tau^2 = \frac{\hat{\sigma}^2}{n}$.

Question 2

Suppose that $X_1, X_2, ..., X_n$ are i.i.d. random variables on the interval $[0, 1]$ with the density function

$$f(x \mid \alpha) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1}(1-x)^{2\alpha-1}$$

where $\alpha > 0$ is a parameter to be estimated from the sample. It can be shown that

$$\begin{aligned}
E(X) &= \frac{1}{3} \\
\mathrm{Var}(X) &= \frac{2}{9(3\alpha+1)}
\end{aligned}$$

1. What equation does the mle of α satisfy.