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# STATS 244 HOMEWORK 4

## *Question 1*

Let $Z \sim N(0,1)$, a stand normal distribution and let $X \sim N(\mu\sigma^2)$. Let $\Phi(z)$ be the cdf of $Z$. Suppose $X \sim N(-4,16)$; find.

It helps to know that

$$P(X < x) = \Phi\Big(\frac{x-\mu}{\sigma}\Big).$$

1. $P(X > 2)$.

$$P(X > 2) = 1 - P(X < 2) \qquad = 1 - \Phi\Big(\frac{2-(-4)}{\sqrt{16}}\Big)$$
$$= 1 - .9332$$
$$= .0688$$

2. $P(0 < X < 4)$

$$\begin{aligned} P(0 < X < 4) &= P(X < 4) - P(X < 0) \\ &= \Phi\Big(\frac{4-(-4)}{4}\Big) - \Phi\Big(\frac{0-(-4)}{4}\Big) \\ &= .1539 \end{aligned}$$

3. $P(|X+3| \geq 3)$

$$\begin{aligned} P(|X+3| \geq 3) &= P(X \geq 0) + P(X \leq -6) \\ &= 1 - \Phi\Big(\frac{4}{4}\Big) + \Phi\Big(\frac{-2}{4}\Big) \\ &= .1587 + .3085 \\ &= .4672 \end{aligned}$$

4. $P(X \leq 0 \text{ or } X \geq 3)$

$$\begin{aligned} P(X \leq 0 \text{ or } X \geq 3) &= P(X \leq 0) + P(X \geq 3) \\ &= \Phi\Big(\frac{4}{4}\Big) + 1 - \Phi\Big(\frac{7}{4}\Big) \\ &= .8413 + .0003 \\ &= .8416 \end{aligned}$$

## *Question 2*

Based on student A's performace during the first two weeks of a course, the professor has approximately a normal $N(70, 8^2)$ prior distribution about the student's true ability, on a scale of 0 to 100.

Consider the midterm examination as an error-prone measure of the student's true ability, where if the true ability is $x$, the examination score can be modeled as approximately normally distributed, $N(x, 6^2)$. The student scores 90 on the midterm.

1. What are the posterior expectaion and the probability that the student's true ability is above 85?

   WE HAVE the following information.

   $$f(\theta) \sim N(70, 8^2)$$

   and

   $$f(x \mid \theta) \sim N(x, 6^2)$$

   Using Bayes, $f(\theta \mid X) \propto f(x \mid \theta) f(\theta)$ the posterior distribution is

   $$(\theta \mid X) = \frac{1}{\sqrt{2\pi}B} e^{\frac{(\theta - A)^2}{2B^2}}$$

   Where

   $$A = \frac{6^2(70) + 8^2(90)}{6^2 + 8^2}$$

   and

   $$B^2 = \frac{6^2 * 8^2}{6^2 + 8^2}$$

   Which give us a $N(82.8, 23.4)$ posterier distribution and means posterior of expectation of the student's true ability it 82.8. Using the the method from question 1, there is a .4625 probability that the students true ability is above 85.

2. Above 90?

   .3792

## *Question 3*

A "psychic" uses a five-card deck of cards to demenonstrate ESP, and claims to be able to guess a card correctly with probability .5. A single experiment consists of making five guesses, reshuffling the deck after each guess. The experiment is treid and the "pyschic" guesses correctly 3 times out of give. Assuming the only two possibilities are "ESP" and "ordinary guessing", how how must the a priori ability be that "psychic" has ESP is at alteast .7?

LET $P(\theta)$ be the prior probability that an individual has ESP that we wish to find.

Bayes thereom is

$$P(\theta \mid X) = \frac{P(X \mid \theta)P(\theta)}{\sum_{i=1}^{2} P(X \mid \theta_i)P(\theta_i)}.$$

Where we can think of $P(\theta_1) = P(\theta)$ and $P(\theta_2) = 1 - P(\theta)$

Plugging in the observed data along with guessing probabilities provided.

$$P(X = 3 \mid \theta) = \binom{5}{3}(.5)^3(.5)^2 = .3125$$

and

$$P(X = 3 \mid \theta_2) = \binom{5}{3}(.2)^3(.8)^2 = .0512.$$

We can plug these values into Bayes and solve for $P(\theta \mid X) \leq .7$. Ie

$$.7 \leq \frac{.3125P(\theta)}{.3125P(\theta) + .0512(1 - P(\theta))}$$

Which gives $P(\theta) \geq .276$

## *Question 4*

Suppose $\hat{\theta}_1$ and $\hat{\theta}_2$ are uncorrelated and both are unbiased estimators of $\theta$, and that $\text{Var}(\hat{\theta}_1) = 2\text{Var}(\hat{\theta}_2)$.

1. Show that for any constant $c$, the weighted average $\hat{\theta}_3 = c\hat{\theta}_1 + (1 - c)\hat{\theta}_2$ is an unbiased estimator of $\theta$.

$$\begin{aligned} E(\hat{\theta}_3) &= cE(\hat{\theta}_1) + (1 - c)E(\hat{\theta}_2) \\ &= c\theta + (1 - c)\theta \\ &= \theta \end{aligned}$$

   Hence $B(\hat{\theta}_3) = 0$.

2. Find $c$ for which $\hat{\theta}_3$ has the smallest MSE.

   The MSE of $\hat{\theta}_3$ is

$$c^2\text{Var}(\hat{\theta}_1) + (1 - c)^2\text{Var}(\hat{\theta}_2)$$

   We can find the value, $c$, that minimizes MSE by evaulating.

$$
\begin{aligned}
MSE(\hat{\theta}_3) &= c^2\text{Var}(\hat{\theta}_1) + (1-c)^2\text{Var}(\hat{\theta}_2) \\
&= c^2 2\text{Var}(\hat{\theta}_2) + (1-c)^2\text{Var}(\hat{\theta}_2) \\
&= \text{Var}(\hat{\theta}_2)(3c^2 - 2c + 1)
\end{aligned}
$$

Setting the above equal to 0 results in $c = \frac{1}{3}$. Furthermore, the second derivative of the MSE is positive so we can confirm that $c$ minimizes.

3. Are there any values of $c$, $0 \leq c \leq 1$ for which $\hat{\theta}_3$ is better (in the sense of MSE) than both $\hat{\theta}_1$ and $\hat{\theta}_2$

We are given that $\text{Var}(\hat{\theta}_2)$ is less than $\text{Var}(\hat{\theta}_1)$. Since the bias as zero, the MSE is just the variance of each estimator.

Considering again the result from the question above...

$$\text{MSE } \hat{\theta}_3 = \text{Var}(\hat{\theta}_2)(3c^2 - 2c + 1)$$

implies that $0 < c < \frac{2}{3}$ will do better in the sene of MSE than the other two estimators.

*Question 5*

Consider observing $X$ from the density $f_\theta(x) = \frac{2}{\theta^2}(\theta - x)$, where $0 < x < \theta$ and $\theta > 0$ is unkown.

1. Verify that $f_\theta$ is indeed a valid density for all $\theta > 0$.

$$
\begin{aligned}
\int_0^\theta f_\theta(x)dx &= \int_0^\theta \frac{2}{\theta^2}(\theta - x)dx \\
&= \frac{2}{\theta^2}\int_0^\theta (\theta - x)dx \\
&= \frac{2}{\theta^2}[\theta^2 - \frac{\theta^2}{2}] \\
&= 1
\end{aligned}
$$

2. Find the MLE of $\theta$ based on $X$. Find the bias, the variance and the mean squared error of the MLE.

The MLE of $\theta$ can be found by solving $\frac{d}{d\theta}L(\theta) = 0$.

$$\frac{d}{d\theta}L(\theta) = \frac{d}{d\theta}\frac{2}{\theta^2}(\theta - x)$$
$$= 2(\frac{d}{d\theta}\theta^{-1} - x\theta^{-2}$$
$$= 4x\theta^{-3} - 2\theta^{-2}$$

Setting the above to 0 yields

$$4x\theta^{-3} = 2\theta^{-2}.$$

Hence $\hat{\theta} = 2X$. Furthermore the second derivative of $L(\theta)$ is negative when $\theta = 2x$ so we can be sure that this is a maximum when.

To find the bias, variance and MSE, tt helps to know the variance and expectation of $X$ which can be calculated from the given density. Hence,

$$E(X) = \frac{\theta}{3}$$
$$\text{Var}(X) = \frac{\theta^2}{18}$$

$$B(\hat{\theta}) = E(\hat{\theta}) - \theta$$
$$= E(2X) - \theta$$
$$= 2E(X) - \theta$$
$$= 2(\frac{\theta}{3}) - \theta$$
$$= -\frac{\theta}{3}$$

$$\text{Var}(\hat{\theta}) = \text{Var}(2X)$$
$$= 2^2\text{Var}(X)$$
$$= \frac{4\theta^2}{9}$$

$$\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta}) + B(\hat{\theta})^2$$
$$= \frac{5\theta^2}{9}$$

3. For the estimator $cX$ for $\theta$ (assume $c > 0$), find the bias, the variance and the mean squeated error of the estimator as a function of $\theta$. For what value of $c$ is $cX$ unbiased? For what value of $c$ is the mean squared error minimized? Plot the mean square error as a function of c.

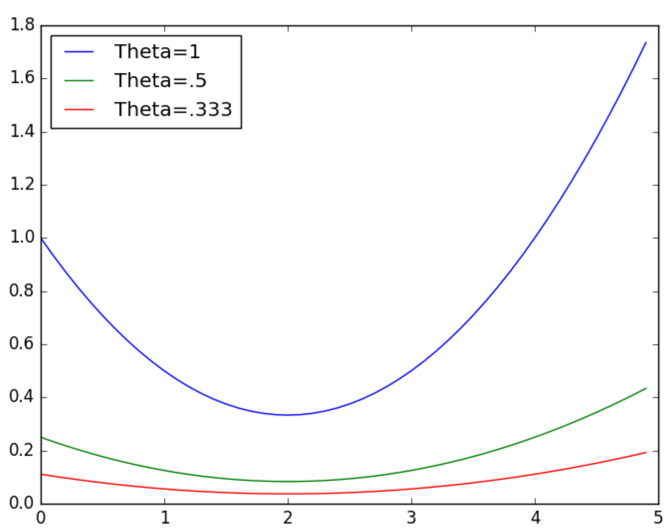
$$
\begin{aligned}
B(cX) &= E(cX) - \theta \\
&= cE(X) - \theta \\
&= c(\frac{\theta}{3}) - \theta \\
&= \frac{\theta(c-3)}{3}
\end{aligned}
$$

$$
\begin{aligned}
\text{Var}(cX) &= c^2\text{Var}(X) \\
&= \frac{c^2\theta^2}{18}
\end{aligned}
$$

$$
\begin{aligned}
\text{MSE}(cX) &= \text{Var}(cX) + B(cX)^2 \\
&= \frac{\theta^2(c^2 - 4c + 6)}{6}
\end{aligned}
$$

From the above, $cX$ is unbiased when $c = 3$. When $c = 2$ the MSE is minized.

Plotting the MSE as a function of $c$ keeping fixing $\theta$ at a view values yields.



4. Find the value of $c \neq \frac{1}{2}$ such that $cX$ has the same mean squared error as $\frac{1}{2}X$. Call this value $c'$. Discus which estimator of $\theta$ your prefer, $\frac{1}{2}X$ or $c'X$.

USING what we know about the MSE$(cX)$ as a function of $c$, $c' = \frac{7}{2}$.

Compairing the two, $c'$ has a larger variance but smaller bias than $\frac{1}{2}X$. So there is a tradeoff between the two, but I'd prefer the estimator with the smaller variance.

*Question 6*

Suppose the data consist of a signle number $X$, and the model is that $X$ has the following probabilty density:

$$f(x \mid \theta) = (1 + x\theta)/2 \text{ for } -1 \leq x \leq 1; = 0 \text{ otherwise.}$$

Supposing the possible values of $\theta$ are $0 \leq \theta \leq 1$; find the maximum likelihood estimate of $\theta$, $\hat{\theta}$, and its (exact) probability distibution. Is the MLE unbiased? Find its bias and MSE.

FIND the MLE for sample variables of X. For $X = 0.5$

$$\frac{d}{d\theta}L(\theta) = .25$$

For $X = -0.5$

$$\frac{d}{d\theta}L(\theta) = -.25$$

Which suggests that MLE$(\theta) = 0$.
The bias is $-\theta$. The MSE is $\theta^2$

*Question 7*

Suppose we observe $X \sim \text{Unif}(0, \theta)$, $\theta > 0$.

1. Find the mle of $\theta$.

$$p(x) = \frac{1}{\theta} \text{ for } \theta > 0$$

$$p(x \mid \theta) = \frac{1}{\theta} \text{ for } x = 1, 2, 3..., \theta$$

$$L(\theta) = \frac{1}{\theta} \text{ for } \theta \geq x$$

2. Find the mean squared error of the mle $\hat{\theta}$; that is $E_\theta\{(\hat{\theta}-\theta)^2\}$

First, find the bias and variance of $X$.

$$\begin{aligned} B(\hat{\theta}) &= E(X) - \theta \\ &= \frac{1+\theta}{2} - \theta \\ &= \frac{1-\theta}{2} \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(X) &= E(X^2) - E(X)^2 \\ &= \frac{(\theta+1)(2\theta+1)}{6} - \left(\frac{\theta+1}{2}\right)^2 \end{aligned}$$

$$\begin{aligned} \mathrm{MSE}(\hat{\theta}) &= \frac{(\theta+1)(2\theta+1)}{6} - \left(\frac{\theta+1}{2}\right)^2 + \left(\frac{1-\theta}{2}\right)^2 \\ &= \frac{2\theta^2 - 3\theta + 1}{6} \end{aligned}$$

3. Find the constant $c$ that makes $cX$ an unbiased extimate of $\theta$. Find the mean squared error of this estimate.

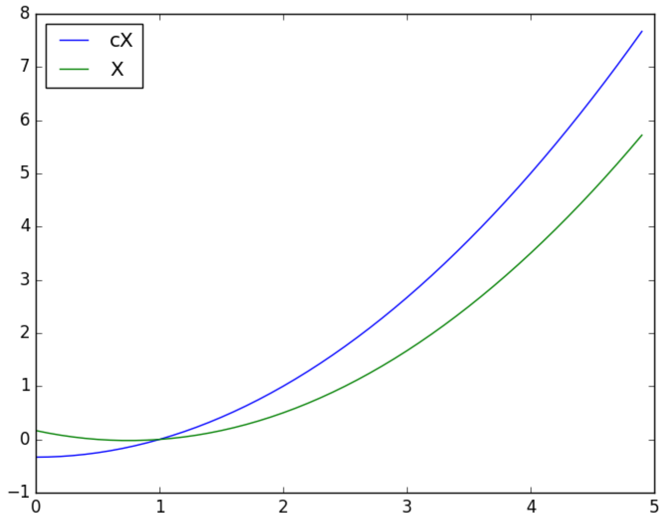
$c = \frac{2\theta}{1+\theta}$

Just a reminder, bias is 0 so just find variance.

$$\begin{aligned} \mathrm{MSE}(cX) &= c^2 \mathrm{Var}(X) \\ &= \frac{\theta^2 - 1}{3} \end{aligned}$$

4. Among all the estimates of $\theta$ of the form $cX$, is there a choice of $c$ that minimizes the resulting MSE for all $\theta$. If yes, find the value of $c$. If not, explain why not.

There is not. Since, $c$ is a function of $\theta$ and we observe that our baised estimate of $\theta$ has a smaller mse than $cX$, there is no way to minimize mse for all $\theta$. The graph below shows that when $\theta$ is less than 1 $cX$ does better, however as $\theta$ increases, $X$ starts to get better.




*Question 8, Rice 8.51*

The double exponential distribution is

$$f(x \mid \theta) = \frac{1}{2}e^{-|x-\theta|}, \ -\infty < x < \infty$$

For an i.i.d. sample of size $n = 2m + 1$, show that the mle of $\theta$ is the median of the sample.

Is the point of $n = 2m + 1$ to imply that the sample is atleast 3?

Suppose we have a bunch of data $X_1, X_2, ..., X_n$ which are iid variables with outcomes $x_1, x_2, ..., x_n$. Then

$$\begin{aligned} f(x_1, x_2, ..., x_n \mid \theta) &= f(x_1 \mid \theta) \cdot f(x_2 \mid \theta) \cdots f(x_n \mid \theta) \\ &= \frac{1}{2}e^{-|x_1-\theta|} \cdot \frac{1}{2}e^{-|x_2-\theta|} \cdots \frac{1}{2}e^{-|x_n-\theta|} \\ &= \frac{1}{2^n}e^{-\sum_{i=1}^{n}|x_i-\theta|} \end{aligned}$$

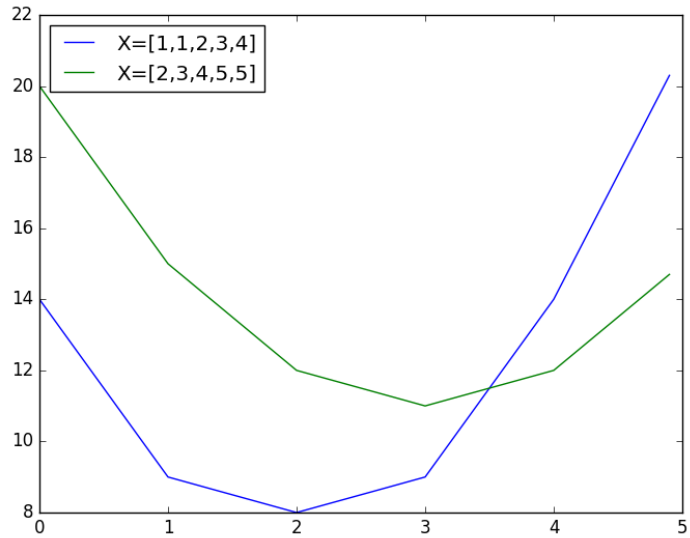
Hence

$$L(\theta) = \frac{1}{2^n}e^{-\sum_{i=1}^{n}|x_i-\theta|}$$

Which can be maximized by minimizing the exponent term

$$g(\theta) = \sum_{n=1}^{n} |x_i - \theta|.$$

However, as warned this function is not differetiable as is. A plot for small N reveals reveals that the slope changes at each instance where $x_i = \theta$.




It also appears that the function achieves a minimum when $\theta$ is equal to the median.

Let

$$h(x,\theta) = \begin{cases} 1 & \text{if } x_i < \theta \\ -1 & \text{if } x_i > \theta \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$\frac{dg(\theta)}{d\theta} = \sum_{i=1}^{n}(h(x_i,\theta))$$

Which will be positive if $\theta$ is greater than median of the data and negative if $\theta$ is less than the median of the data. Which implies that the $L(\theta)$ function is maximized at the median, i.e. when $\hat{\theta} = \text{median}(x_i)$.