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STATS 244 HOMEWORK 2

Rice Chapter 2, 44

Let T be an exponentional random variable with parameter λ. Let X be a discrete random variable defined as $X = k$ if $k \leq T < k+1$, $k = 0, 1, ...$ Find the frequency function of X.

THE EXPONENTIAL density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

The random variable X is essentially the space between two consecutive integers on $f(x)$, k and $k+1$. Therefore, using the CDF of $f(x)$, we can find the frequency function of X.

$$F_T(x) = \int_0^x f(u)du = 1 - e^{-\lambda x}$$

$$\begin{aligned} P(k \leq T < k+1) &= P(T \leq k+1) - P(T < k) \\ &= F_T(k+1) - F_T(k) \\ &= (1 - e^{-\lambda(k+1)}) - (1 - e^{-\lambda(k)}) \\ &= -e^{-\lambda k - \lambda} + e^{-\lambda k} \\ &= -e^{-\lambda k}(e^{-\lambda} - 1) \end{aligned}$$

So the frequency function of X is $f_X(x) = -e^{-\lambda x}(e^{-\lambda} - 1)$

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T is an exponential random variable, and $P(T < 1) = .05$. What is λ?

USING THE CDF of an exponential random variable, we find...

$$\begin{aligned} p(T < 1) &= .05 \\ 1 - e^(- \lambda) &= .05 \\ -e^(- \lambda) &= -.95 \\ e^(- \lambda) &= \frac{19}{20} \\ e^{\lambda} &= \frac{20}{19} \\ \lambda &= \ln(\frac{20}{19}) \\ \lambda &\approx .051 \end{aligned}$$

Rice Chapter 3, 8

Let X and Y have the joint density

$$f(x, y) = \frac{6}{7}(x+y)^2,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1$$

1. By integrating over the appropriate regions, find (i) $P(X > Y)$, (ii) $P(X + Y \leq 1)$, (iii) $P(X \leq \frac{1}{2})$.]

(a) $P(X > Y)$

$$P(X > Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY} dydx$$

Since the joint density function applies to $0 \leq x \leq 1$ set the limit of integration for X to be $0, 1$ and $0, x$ for Y.

$$\begin{aligned}
P(X > Y) &= \int_0^1 \int_0^x \frac{6}{7}(x+y)^2 dydx \\
&= \int_0^1 \frac{6}{7}[\frac{1}{3}(x+y)^3\Big|_0^x]dx \\
&= \int_0^1 \frac{6}{7}[\frac{1}{3}(2x)^3 - \frac{1}{3}(x)^3]dx \\
&= \int_0^1 \frac{6}{7}[\frac{1}{3}7x^3]dx \\
&= \int_0^1 2x^3 dx \\
&= \frac{1}{2}x^4\Big|_0^1 \\
&= \frac{1}{2}
\end{aligned}$$

(b) $P(X + Y \leq 1)$

$$\begin{aligned}
P(X + Y \leq 1) &= \int_0^1 \int_0^{1-x} \frac{6}{7}(x+y)^2 dydx \\
&= \int_0^1 \frac{6}{7}[\frac{1}{3}(x+y)^3\Big|_0^{1-x}]dx \\
&= \int_0^1 \frac{6}{7}[\frac{1}{3}(x+1-x)^3 - \frac{1}{3}(x+0)^3]dx \\
&= \int_0^1 \frac{6}{7}[\frac{1}{3}(1-x^3)]dx \\
&= \frac{2}{7}[(x - \frac{1}{4}x^4)\Big|_0^1] \\
&= \frac{3}{14}
\end{aligned}$$

(c) $P(X \leq \frac{1}{2})$

$$
\begin{aligned}
P(X \leq \frac{1}{2}) &= \int_0^{\frac{1}{2}} \int_0^1 \frac{6}{7}(x+y)^2 dy dx \\
&= \int_0^{\frac{1}{2}} \frac{6}{7}[\frac{1}{3}(x+y)^3\Big|_0^1] dx \\
&= \int_0^{\frac{1}{2}} \frac{6}{7}[\frac{1}{3}((x+1)^3 - x^3)] dx \\
&= \int_0^{\frac{1}{2}} \frac{2}{7}[((x+1)^3 - x^3)] dx \\
&= \frac{2}{7}[\frac{1}{4}(x+1)^4 - \frac{1}{4}x^4\Big|_0^{\frac{1}{2}}] \\
&= \frac{2}{28}[(\frac{3}{2}^4 - \frac{1}{2}^4) - 1] \\
&= \frac{2}{14}
\end{aligned}
$$

2. Find the marginal densities of X and Y.

$$
\begin{aligned}
f(y) &= \int_{-\infty}^{\infty} f(x,y) dx \\
&= \int_0^1 \frac{6}{7}(x+y)^2 dx \\
&= \frac{6}{7}[\frac{1}{3}(x+y)^3\Big|_0^1] \\
&= \frac{2}{7}(1+y)^3 + y^3 \\
&= \frac{2}{7}(3y^2 + 3y + 1),\ 0 \leq y \leq 1
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \int_{-\infty}^{\infty} f(x,y) dy \\
&= \int_0^1 \frac{6}{7}(x+y)^2 dy \\
&= \frac{6}{7}[\frac{1}{3}(x+y)^3\Big|_0^1] \\
&= \frac{2}{7}(1+x)^3 + x^3 \\
&= \frac{2}{7}(3x^2 + 3x + 1),\ 0 \leq x \leq 1
\end{aligned}
$$

3. Find the two conditional densities.

$$
\begin{aligned}
f(x \mid y) &= \frac{f(x,y)}{f(y)} \\
&= \frac{\frac{6}{7}(x+y)^2}{\frac{2}{7}(3y^2+3y+1)} \\
&= \frac{3(x+y)^2}{3y^2+3y+1}
\end{aligned}
$$

$$
\begin{aligned}
f(y \mid x) &= \frac{f(x,y)}{f(x)} \\
&= \frac{\frac{6}{7}(x+y)^2}{\frac{2}{7}(3x^2+3x+1)} \\
&= \frac{3(x+y)^2}{3x^2+3x+1}
\end{aligned}
$$

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Suppose that

$$f(x,y) = xe^{-x(y+1)},,\ 0 \le x < \infty,\ 0 \le y < \infty$$

1. Find the marginal density of X and Y. Are X and Y independent?

Switch terms $u = x(-y) - x$ so $du = -xdy$

$$
\begin{aligned}
f_X(x) &= \int_0^\infty f_{XY} dy \\
&= \int_0^\infty xe^{-x(y+1)} dy \\
&= x \int_0^\infty e^{x(-y)-x} dy \\
&= -\int_0^\infty e^u du \\
&= -e^u \Big|_0^\infty \\
&= -e^{-x(y+1)} \Big|_0^\infty \\
&= 0 - (-e^{-x})
\end{aligned}
$$

So $f_X(x) = e^{-x}$.

To find

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)}$$

We'll integrate by parts, $\int f dg = fg - \int g df$ where

$$\begin{aligned} f &= x \\ df &= dx \\ dg &= e^{x(-y-1)}dx \\ g &= \frac{e^{x(-y-1)}}{-y-1} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_0^\infty xe^{-x(y+1)}dx \\ &= \frac{xe^{x(-y-1)}}{-y-1}\Big|_0^\infty - \frac{1}{-y-1}\int_0^\infty e^{x(-y-1)}dx \end{aligned}$$

Substitute terms $u = x(-y-1)$ and $du = (-y-1)dx$.

$$\begin{aligned} f_Y(y) &= 0 - \frac{1}{(-y-1)^2}\int_0^\infty e^u du \\ &= -\frac{1}{(-y-1)^2}[e^u\Big|_0^\infty] \\ &= \frac{1}{(y+1)^2} \end{aligned}$$

Therefore, $f_Y(y) = \frac{1}{(y+1)^2}$.

We say X, Y are independent if $f(x,y) = f(x)f(y)$ but

$$xe^{-x(y+1)} \neq \frac{e^{-x}}{(y+1)^2}.$$

So X and Y are not independent.

2. Find the conditional densities of X and Y.

$$\begin{aligned} f(x \mid y) &= \frac{f(x,y)}{f(y)} \\ &= \frac{xe^{-x(y+1)}}{\frac{1}{(y+1)^2}} \\ &= (y+1)^2 xe^{-x(y+1)} \end{aligned}$$

$$\begin{aligned} f(x \mid y) &= \frac{f(x,y)}{f(x)} \\ &= \frac{xe^{-x(y+1)}}{e^{-x}} \\ &= xe^{-xy} \end{aligned}$$

Question 5

The Pareto distributions are a family of distributions of a continuous random variable X with probability density function given by

$$f(x;\alpha,\theta) = \begin{cases} \frac{\alpha\theta^\alpha}{x^{\alpha+1}} & \text{for } x \geq \theta \\ 0 & \text{for } x < \theta \end{cases}$$

where $\alpha > 0$ and $\theta > 0$ are the paremeters of the family. The Pareto distribution arises as a model for the distribution of sizes of some measured quantity, such as personal or corperate income, or city population, or size of firm, given that it exceeds the threshold θ.

1. Verify that this the formula for a density.

WHEN $x > \theta$ we have $f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}$. To show that this is a density, the integral must sum to 1.

$$\begin{aligned} \int_{-\infty}^{\infty} f_X dx &= \int_{-\infty}^{\infty} \frac{\alpha\theta^\alpha}{x^{\alpha+1}} dx \\ &= \int_{\theta}^{\infty} \alpha\theta^\alpha x^{-(\alpha+1)} dx \\ &= \alpha\theta^\alpha [\frac{-x^{-\alpha}}{\alpha}\Big|_\theta^\infty] \\ &= \alpha\theta^\alpha [0 - (\frac{-1}{\alpha\theta^\alpha}] \\ &= 1 \end{aligned}$$

2. Find the formula for the cumulative distribution function.

$$\begin{aligned} F(x) &= \int_{-\infty}^{\infty} f(x) dx \\ &= \int_{\theta}^{x} \frac{\alpha\theta^\alpha}{x^{(\alpha+1)}} dx \\ &= \alpha\theta^\alpha \int_{\theta}^{x} x^{-(\alpha+1)} dx \\ &= \alpha\theta^\alpha [\frac{-1}{\alpha x^\alpha}\Big|_\theta^x] \\ &= 1 - (\frac{\theta}{x})^\alpha \end{aligned}$$

3. Assuming $\alpha > 1$, find $E(X)$. What is $E(X)$ if $0 < \alpha \leq 1$?

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_{\theta}^{\infty} x \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}} dx \\
&= \alpha\theta^{\alpha} \int_{\theta}^{x} x^{-\alpha} dx \\
&= \alpha\theta^{\alpha} [\frac{x^{1-\alpha}}{1-\alpha}\Big|_{\theta}^{\infty}] \\
&= \alpha\theta^{\alpha} [\frac{-1}{(\alpha-1)(x^{\alpha-1})}\Big|_{\theta}^{\infty}] \\
&= \alpha\theta^{\alpha} [0 - \frac{-1}{(\alpha-1)(\theta^{\alpha-1})}] \\
&= \frac{\alpha\theta}{\alpha-1},\ \alpha > 1
\end{aligned}$$

When $0 < \alpha \leq 1$ the integral does not converge. Therefore

$$E(X) = \begin{cases} \frac{\alpha\theta}{\alpha-1} & \text{for } \alpha > 1 \\ \text{DNE} & \text{for } 0 < \alpha \leq 1 \end{cases}$$

4. Show that if $\alpha > 2$, $\text{Var}(X) = \frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}$

$$= E(X^2) - \mu_x^2$$

Since $\alpha > 1$ we already know $E(X)$. To find $E(X^2)$.

$$\begin{aligned}
E(X^2) &= \int_{\theta}^{\infty} x^2 \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}} dx \\
&= \int_{\theta}^{\infty} \frac{\alpha\theta^{\alpha}}{x^{\alpha-1}} dx \\
&= \alpha\theta^{\alpha} [\frac{-1}{(\alpha-2)x^{\alpha-2}}\Big|_{\theta}^{\infty}],\ \alpha > 2 \\
&= \alpha\theta^{\alpha} [\frac{1}{(\alpha-2)(\theta^{\alpha-2})}] \\
&= \frac{\alpha\theta^2}{\alpha-2},\ \alpha > 2
\end{aligned}$$

Plugging what we just found back into the equation for variance, we have...

$$\text{Var}(X) = \frac{\alpha\theta^2}{\alpha-2} - \frac{\alpha\theta}{\alpha-1}$$

So $\text{Var}(X) = \frac{\alpha\theta^2}{(\alpha-2)(\alpha-1)^2}$ when $\alpha > 2$.

5. Find (in terms of α and θ) the median of the distribution of X.

THE MEDIAN can be found using the CDF.

$$\begin{aligned}\frac{1}{2} &= 1 - (\frac{\theta}{x})^{\alpha}\\ \frac{1}{2} &= (\frac{\theta}{x})^{\alpha}\\ x^{\alpha} &= \theta^{\alpha}\end{aligned}$$

So $\tilde{\mu} = \theta^{\alpha}\sqrt{2}$

6. Suppose X has a Pareto distribution with $\alpha = 3$ and $\theta = 1$. Find $P(1 < X < 4)$ and $P(4 < X < 5)$.

SINCE $P(1 < X < 4) = P(X < 4) - P(X < 1)$ we consider the CDF.

$$\begin{aligned}P(X < 4) - P(X < 1) &= 1 - (\frac{1}{4})^3 - (1 - (\frac{1}{1})^3)\\ &= \frac{63}{64}\end{aligned}$$

Similarly, $(P(4 < X < 5) = \frac{61}{8000}$

Question 6

Suppose X and Y are independent random variables with X following a uniform distribution on $(0,1)$ and Y exponentially distributed with parameter $\lambda = 1$.

1. Find the density for $Z = X + Y$ (be careful with your limits of integration). Sketch the density and verify directly that it integrates to 1. Find the medain (the value of Z for which $P(Z \leq z) = \frac{1}{2}$.

THE FUNCTIONS PDFs can be writtin...

$$f_X = 1$$

and

$$f_Y = e^{-y}.$$

Since X and Y are independent we can find the density of using

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-x)f_Y(y)dy$$

However, there are two cases to consider since.

$$f_x(z-y) = \begin{cases} 1 & \text{for } 0 \leq z-y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

When $0 \leq z \leq 1$ and $z > 1$. For the first we have.

$$\begin{aligned} f_Z(z) &= \int_0^z e^{-y}(1)dy \\ &= 1 - e^{-z},\ 0 \leq z \leq 1 \end{aligned}$$

When $z > 1$, we can say that $z = y$ and we have

$$\begin{aligned} f_Z(z) &= \int_{z-1}^z e^{-y}(1)dy \\ &= e^{-(z-1)} - e^{-z} \end{aligned}$$

Therefore the PDF can be written

$$f_Z(z) = \begin{cases} 1 - e^{-z} & 0 \leq z \leq 1 \\ e^{-(z-1)} - e^{-z} & z > 1 \end{cases}$$

To VERIFY this is a formula for a density we take (let $u = -z$ and $s = 1 - z$.

$$\begin{gathered} \int_0^1 1 - e^{-z} + \int_1^\infty e^{-(z-1)} - e^{-z} \\ - \int_0^1 e^{-z} + \int_0^1 1dz + - \int_1^\infty e^{-z}dz + \int_1^\infty e^{1-z} \\ \int_0^1 e^u du + z + \int_1^\infty e^u du - \int_1^\infty e^s ds \\ [z + e^{-z}\Big|_0^1] + [-(e-1)e^{-z}\Big|_1^\infty] \end{gathered}$$

Which equals 1, verifying an equation for density!

NEXT we need to find the median. The familiar way to do this, starts with the CDF.

$$
\begin{aligned}
F_Z(z) &= \int_0^1 \int_0^{z-x} f(x,y)dydx \\
\text{let } y = v - x & \\
&= \int_0^1 \int_0^z f(x, v-x)dvdx \\
&= \int_0^1 \int_0^z (1)e^{-(v-x)}dvdx \\
\text{let } u = x - v \text{ and } du = -dv & \\
&= \int_0^1 - \int_x^{z-x} e^u dudx \\
&= \int_0^1 [-e^u \Big|_x^{x-z}]dx \\
&= \int_0^1 (e^z - 1)e^{x-z}dx \\
&= (e^z - 1)\int_0^1 e^{x-z}dx \\
\text{now let } u = x - z \text{ and } du = dx & \\
&= (e^z - 1)\int_{-z} 1 - z^1 e^u du \\
&= (e^z - 1)(e^u)\Big|_{-z}^{1-z} \\
&= (e-1)e^{-z}(e-1)
\end{aligned}
$$

Now we solve for $F_Z(\tilde{\mu}) = \frac{1}{2}$

$$
\begin{aligned}
(e-1)e^{-z}(e-1) &= \frac{1}{2} \\
\frac{(e^z - 1)}{e^z} &= \frac{1}{2(e-1)} \\
e^{-z} &= 1 - \frac{1}{2(e-1)} \\
e^z &= \frac{1}{1 - \frac{1}{2(e-1)}} \\
z &= -\ln(1 - \frac{1}{2(e-1)})
\end{aligned}
$$

Hence, $\tilde{\mu} \approx 0.34388$.

2. Find $E(X - Y)$ and $\text{Var}(X - Y)$. Find $E(ZX)$.

SINCE X and Y are independent, we can find $E(X - Y) = E(X) - E(Y)$.

$$
\begin{aligned}
E(X) &= \int_0^1 xf(x)dx \\
&= \frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
E(Y) &= \int_0^\infty ye^{-y}dy \\
&= -e^{-y}y\Big|_0^\infty + \int_0^\infty e^{-y}dy \\
&= e^{-y}y\Big|_0^\infty - e^{-y}\Big|_0^\infty \\
&= 1
\end{aligned}
$$

So $E(X-Y) = E(X) - E(Y) = \frac{-1}{2}$

The variance of X-Y can be find in a similiar way.

$$
\mathrm{Var}(X) = E(X^2) - \mu_x^2
$$

So we need find $E(X^2)4$ and $E(Y^2)$.

$$
\begin{aligned}
E(X^2) &= \int_0^1 x^2 f_x(x) \\
&= \frac{x^3}{3}\Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}
$$

We have $\mathrm{Var}(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$
Next, find $E(Y^2)$

$$
\begin{aligned}
E(Y^2) &= \int_0^\infty y^2 e^{-y}dy \\
&= -e^{y}y^2\Big|_0^\infty + 2\int_0^\infty e^{-y}ydy \\
&= -2e^{-x}x|\Big|_0^1 + 2e^{-x}\Big|_0^\infty \\
&= 2
\end{aligned}
$$

Now we have $\mathrm{Var}(Y) = 2 - 1 = 2$. Therefore $\mathrm{Var}(X-Y) = \frac{-11}{12}$

SINCE $Z = X + Y$,

$$\begin{aligned} E(ZX) &= E(X^2 + XY) \\ &= E(X^2) + E(XY) \\ &= E(X^2) + E(X)E(Y) \\ &= \frac{1}{3} + \frac{1}{2} \\ &= \frac{5}{6} \end{aligned}$$

Question 7

For X following a standard normal distribution, find $E(X^3)$ and $E(X^4)$. For $X \sim N(\mu, \sigma^2)$, find $E(X^3)$ and $E(X^4)$ using your answers for the standard normal and no futher calculus.

The PDF of the standard normal is

$$\Phi = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}.$$

Also, note that $E(X) = \mu = 0$ and $E(X^2) = 1$, since $1 = E(X^2) - 0$.

$$\begin{aligned} E(X^3) &= \int_{-\infty}^{\infty} x^3 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \\ &= \frac{1}{\sqrt(2\pi} \int_{-\infty}^{\infty} x^3 e^{\frac{-x^2}{2}} dx \\ \text{let } u = x^2 &\text{ and } du = 2xdx \\ &= \frac{1}{2\sqrt(2\pi} \int_{-\infty}^{\infty} u e^{\frac{-u}{2}} du \end{aligned}$$

Integrate by parts using $f = u$, $df = du$, and $g = \frac{-1}{e^{\frac{u}{2}}}$

$$\begin{aligned} E(X^3) &= \frac{1}{2\sqrt{2\pi}} ([ue^{\frac{-u}{2}} \Big|_{-\infty}^{\infty}] + \int_{-\infty}^{\infty} 2e^{\frac{-u}{2}} du) \\ &= 0 + \int_{-\infty}^{\infty} \frac{2e^{-\frac{u}{2}}}{2\sqrt{2\pi}} du \\ &= \int_{-\infty}^{\infty} \frac{e^{-\frac{u}{2}}}{\sqrt{2\pi}} du \\ &= 2\int_{-\infty}^{\infty} x \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \\ &= 2 * E(X) \\ &= 0 \end{aligned}$$

Now, we'll find $E(X^4)$. Let $u = \frac{x^2}{2}$ and $du = xdx$.

$$
\begin{aligned}
E(X^4) &= \int_{-\infty}^{\infty} x^4 \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (2u)^{\frac{3}{2}} e^{-u} du \\
&\text{*since standard normal distribution is symmetric} \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} (2u)^{\frac{3}{2}} e^{-u} du \\
&= \frac{2^{\frac{5}{2}}}{\sqrt{2\pi}} \int_{0}^{\infty} (u)^{\frac{5}{2}-1} e^{-u} du \\
&= \frac{2^{\frac{5}{2}}}{\sqrt{2\pi}} \Gamma(\frac{5}{2}) \\
&= \frac{4\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \frac{3\sqrt{\pi}}{4} \\
&= 3
\end{aligned}
$$

For $X \sim (\mu, \sigma^2)$, $E(X^3) = 0$ and $E(X^4) = 3\sigma^4$. Admittedly, I'm not sure how or why from my answers above. I imagine that since standard deviation will be a constant it will just fall out of the intergrals...

Question 8

Consider a Poisson process with paremter λ. Let X be the number of events in $(0, t_2]$ and Y the number of events in $(t_1, t_3]$ for $0 < t_1 < t_2 < t_3$ so that the intervals are guaranteed to overlap.

1. Find the mean and variance of $X - Y$.

Consider first the expected values of X and Y, independently.

$$
\begin{aligned}
E(X) &= \sum_{k=0}^{\infty} \frac{k\lambda(t_2)^k}{k!} e^{-\lambda(t_2)} \\
&= [\lambda(t_2)] e^{-\lambda(t_2)} \sum_{k=1}^{\infty} \frac{[\lambda(t_2)]^{k-1}}{(k-1)!} \\
&= [\lambda(t_2)] e^{-\lambda(t_2)} \sum_{j=1}^{\infty} \frac{[\lambda(t_2)]^{j}}{(k-1)!} \\
&= [\lambda(t_2)] e^{-\lambda(t_2)} e^{\lambda t_2} \\
&= \lambda(t_2)
\end{aligned}
$$

For $E(Y)$ we can following the same process, resuling in

$$E(Y) = \lambda(t_3 - t_1).$$

So

$$E(Y) - E(X) = \lambda(t_3 - t_1) - \lambda(t_2)$$

The variance is defined.

$$\text{Var}(Y - X) = \text{Var}(Y) - \text{Var}(X) - 2\text{Cov}(X, Y)$$

We can use the the moment generating function for the Poisson distribution.

$$\begin{aligned} M(t) &= \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} e^{-\lambda} \\ &= e^{-\lambda} e^{\lambda(e^t - 1)} \\ &= e^{\lambda(e^t - 1)} \end{aligned}$$

Now differentiate

$$\begin{aligned} M'(t) &= \lambda e^t e^{\lambda(e^t-1)} \\ M''(t) &= \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)} \end{aligned}$$

We already know the value at the first moment $E(X)$ so just evualate the second derivative at $t = 0$.

$$E(X^2) = (\lambda t_2)^2 + \lambda t_2$$

Therefore we have

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \lambda t_2.$$

Similarily

$$\text{Var}(Y) = E(Y^2) - [E(X)]^2 = \lambda(t_3 - t_1).$$

It remains to find $\text{Cov}(XY)$.

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

The last item to find is $E(XY)$. The joint PMF

$$p_k = \frac{[\lambda(t_2 - t_1)]^k e^{-[\lambda(t_2 - t_1)]}}{k!}$$

Using the same methods for find the $E(X)$ and $E(Y)$

$$E(XY) = \lambda(t_2 - t_1)$$

Plugging values in

$$\text{Cov}(X, Y) = \lambda(t_2 - t_1) - \lambda(t_3 - t_1)\lambda(t_2)$$

So

$$\begin{aligned}\text{Var}(Y - X) &= \text{Var}(Y) - \text{Var}(X) - 2\text{Cov}(X, Y) \\ &= \lambda(t_3 - t_1) - \lambda t_2 - 2[\lambda(t_2 - t_1) - \lambda^2(t_3 - t_1)(t_2)]\end{aligned}$$

2. Find $E(Y \mid X)$. Verify that $E[E(Y \mid X)]$ equals $E(Y)$.

In general $P(Y \mid X) = \frac{P(X,Y)}{P(Y)}$. In this case, if p is the PMF of the Poisson distribution.

$$p_{Y|X} = \frac{p_{XY}}{p_Y}$$

$$p_{XY} = \frac{(t_2\lambda)^x e^{-(t_2)\lambda}}{x!} \frac{[(t_3 - t_1)\lambda]^{y-x} e^{-(t_3 - t_1)\lambda}}{(y - x)!}$$

The conditional expection of Y given X is

$$E(Y \mid X) = \sum_y y p_{Y|}(y \mid x)$$

$$\begin{aligned}p_{Y|X}(y \mid n) &= \frac{(t_2\lambda)^x e^{-(t_2)\lambda}}{x!} \frac{[(t_3 - t_1)\lambda]^{y-x} e^{-(t_3 - t_1)\lambda}}{(y - x)!} \\ &* \frac{1}{\frac{[(t_3 - t_1)\lambda]^{y-x} e^{-(t_3 - t_1)\lambda}}{(y!)}} \\ &= \frac{x!}{x!(y - x!)} t_2^x (t_3 - t_1)^{y-x}\end{aligned}$$

THE LAST thing to do is verifty $E[Y \mid X = E(Y)$.

$$\begin{aligned}
E[E(Y \mid X)] &= \sum_y y P_{Y|X}(y \mid x) \\
&= \sum_y y \frac{x!}{x!(y-x!)} t_2^x (t_3 - t_1)^{y-x} \\
E[E(Y \mid X)] \sum_x p_X(x) &= \sum_y y \sum_x \frac{x!}{x!(y-x!)} t_2^x (t_3 - t_1)^{y-x} p_X(x)
\end{aligned}$$

The law of total probability states

$$p_Y(y) = \sum_x p_{Y|X}(y \mid x) p_X(x)$$

So

$$\begin{aligned}
\sum_y y \sum_x \frac{x!}{x!(y-x!)} t_2^x (t_3 - t_1)^{y-x} p_X(x) &= \sum_y p_y(Y) \\
&= E(Y)
\end{aligned}$$