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STAT 244 HOMEWORK 5

Question 1 Rice 8.16

Consider an i.i.d. sample of random variables with density function

$$f(x \mid \sigma) = \frac{1}{2\sigma} \exp\Big(-\frac{|x|}{\sigma}\Big)$$

1. Find the maximum likelihood estimate of σ.

SUPPOSE we have i.i.d. random variables with result $x_1, x_2, .., x_n$. Then

$$\begin{aligned} f(x_1, x_2, ..., x_n \mid \sigma) &= f(x_1 \mid \sigma) \cdot f(x_2 \mid \sigma) \cdots f(x_n \mid \sigma) \\ &= \frac{1}{2\sigma} e^{-\frac{|x_1|}{\sigma}} \cdot \frac{1}{2\sigma} e^{-\frac{|x_2|}{\sigma}} \cdots \frac{1}{2\sigma} e^{-\frac{|x_n|}{\sigma}} \\ &= \prod_{i=1}^{n} \frac{1}{2\sigma} e^{-\frac{|x_i|}{\sigma}} \end{aligned}$$

So the likelihood function of σ is given

$$L(\sigma) = \prod_{i=1}^{n} \frac{1}{2\sigma} e^{-\frac{|x_i|}{\sigma}}.$$

Estimate $\hat{\sigma}$ by taking the log and differentiate then solve by setting to 0.

$$\begin{aligned} \frac{d}{d\sigma} \log(L(\sigma)) &= \frac{d}{d\sigma} - n\log(2) - n\log(\sigma) - \sum_{i=1}^{n} \frac{|x_i|}{\sigma} \\ &= -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} |x_i|}{\sigma^2} \end{aligned}$$

Setting the above equal to 0 gives the result

In part 2 of this question, we'll verify that that this maximizes

$$\hat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$$

2. Find the asymptotic variance of the mle.

$$\frac{1}{\tau^2} = -E\Big[\frac{d^2}{d\sigma^2} \log L(\sigma)\Big]$$

$$\frac{d^2}{d\sigma^2}\log(L(\sigma)) = \frac{n}{\sigma^2} - \frac{2\sum_{i=1}^n |x_i|}{\sigma^3}$$
$$= \frac{n}{\hat{\sigma}^2} - \frac{2n}{\hat{\sigma}^2}$$
$$= -\frac{n}{\hat{\sigma}^2}$$

We sub σ with the estimator to elimate the summation term.

Plugging the result into the equation that began this section

$$\frac{1}{\tau^2} = -E[-\frac{n}{\hat{\sigma}^2}]$$
$$= \frac{n}{\hat{\sigma}^2}$$

Finally, $\tau^2 = \frac{\hat{\sigma}^2}{n}$.

Question 2

Suppose that $X_1, X_2, ..., X_n$ are i.i.d. random variables on the interval $[0, 1]$ with the density function

$$f(x \mid \alpha) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1}(1-x)^{2\alpha-1}$$

where $\alpha > 0$ is a parameter to be estimated from the sample. It can be shown that

$$E(X) = \frac{1}{3}$$
$$\text{Var}(X) = \frac{2}{9(3\alpha+1)}$$

1. What equation does the mle of α satisfy.

$$L(\alpha) = f(x_1 \mid \alpha) \cdot f(x_2 \mid \alpha) \cdots f(x_n \mid \alpha)$$
$$= \prod_{i=1}^n \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x_i^{\alpha-1}(1-x_i)^{2\alpha-1}$$

$$\log(L(\alpha)) = n\log(\Gamma(3\alpha)) - n\log(\Gamma(\alpha)) - n\log(\Gamma(2\alpha))$$
$$+ \sum_{i=1}^n [(\alpha-1)\log(x_i) + (2\alpha-1)\log(1-x_i)]$$

$$\frac{d}{d\alpha}\log(L(\alpha)) = \frac{n3\Gamma'(3\alpha)}{\Gamma(3\alpha)} - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{n2\Gamma'(2\alpha)}{\Gamma(2\alpha)} + \sum_{i=1}^{n}[\log(x_i) + 2\log(1-x_i)]$$

Setting the above equal 0.

$$\frac{3\Gamma'(3\alpha)}{\Gamma(3\alpha)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{2\Gamma'(2\alpha)}{\Gamma(2\alpha)} = -\frac{1}{n}\sum_{i=1}^{n}[\log(x_i) + 2\log(1-x_i)]$$

2. What is the asymptotic variance of the mle.

   Use the property

$$\frac{1}{\tau^2} = -E\Big[\frac{d^2}{d\alpha^2}\log(L(\alpha))\Big]$$

$$\frac{d^2}{d\alpha^2} = \frac{n9\Gamma''(3\alpha)}{\Gamma(3\alpha)} - \frac{n9\Gamma'(3\alpha)^2}{\Gamma(3\alpha)^2} - \frac{n\Gamma''(\alpha)}{\Gamma(\alpha)} + \frac{n\Gamma'(\alpha)^2}{\Gamma(\alpha)^2} - \frac{n4\Gamma''(2\alpha)}{\Gamma(2\alpha)} + \frac{n4\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2}$$

$$\frac{1}{\tau^2} = -E\Big[n(\frac{9\Gamma''(3\alpha)}{\Gamma(3\alpha)} - \frac{9\Gamma'(3\alpha)^2}{\Gamma(3\alpha)^2} - \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} + \frac{\Gamma'(\alpha)^2}{\Gamma(\alpha)^2} - \frac{4\Gamma''(2\alpha)}{\Gamma(2\alpha)} + \frac{4\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2}\Big]$$

Hence

$$\tau^2 = -\frac{1}{n}\Big[\frac{9\Gamma''(3\alpha)}{\Gamma(3\alpha)} - \frac{9\Gamma'(3\alpha)^2}{\Gamma(3\alpha)^2} - \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} + \frac{\Gamma'(\alpha)^2}{\Gamma(\alpha)^2} - \frac{4\Gamma''(2\alpha)}{\Gamma(2\alpha)} + \frac{4\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2}\Big]^{-1} \quad (1)$$

Question 3

Suppose that X is the number of successes in a Binomial experiment with n trials and probability of success $\theta/(1+\theta)$, where $0 \le \theta < \infty$.

1. Find the MLE of θ.

   CONSIDER

$$p(x \mid \theta) = \binom{n}{X}\Big(\frac{\theta}{1+\theta}\Big)^X\Big(1 - \frac{\theta}{1+\theta}\Big)^{n-X}.$$

Then
$$L(\theta) = \binom{n}{X}\Big(\frac{\theta}{1+\theta}\Big)^X \Big(1 - \frac{\theta}{1+\theta}\Big)^{n-X}.$$
The log likelihood is
$$\begin{aligned}\log L(\theta) &= \log\binom{n}{X} + X\log(\theta) - X\log(1+\theta) - (n-X)\log(1+\theta)\\ &= \log\binom{n}{X} + X\log(\theta) - n\log(1+\theta)\end{aligned}$$

Differentiating,
$$\frac{d}{d\theta}\log L(\theta) = \frac{X}{\theta} - \frac{n}{\theta+1}$$

Setting the above to 0 results in
$$\hat{\theta} = \frac{X}{n-X}$$
furthermore evaluating the second derivative using $\hat{\theta}$
$$\begin{aligned}\frac{d^2\log L(\theta)}{d\theta^2} &= -\frac{X}{\theta^2} + \frac{n}{(\theta+1)^2}\\ &= -\frac{X}{\hat{\theta}^2} + \frac{n}{(\hat{\theta}+1)^2}\\ &= -\frac{(n-X)^2}{X} + \frac{(n-X)^2}{n} < 0\end{aligned}$$

verifying that this maximizes $\hat{\theta}$.

2. Use Fisher's Theorem to find the approximate distribution of the MLE when n is large.

Fisher's approximation allows the result for large n in the given case, $\hat{\theta}$ will have approximately a $N(\theta, \frac{\tau^2(\theta)}{n})$ distribution.
It remains to find $\tau^2(\theta)$.
$$\frac{1}{\tau^2(\theta)} = -E\Big[\frac{d^2}{d\theta^2}\log f(X \mid \theta)\Big].$$
$$\begin{aligned}E[\frac{X}{\theta^2} + \frac{n}{(\theta+1)^2}] &= -\frac{E(X)}{\theta^2} + \frac{n}{(\theta+1)^2}\\ &= -\frac{n}{(1+\theta)\theta} + \frac{n}{(1+\theta)^2}\\ &= -\frac{n}{(1+\theta)^2\theta}\end{aligned}$$

Hence $\tau^2(\theta) = \frac{\theta(1+\theta)^2}{n}$

Applying Fisher, $\hat{\theta}$ will have an approximately normal $N(\theta, \frac{\theta(1+\theta)^2}{n})$ distribution.

It is important to recognize that we evaulated the entire likelihood function, so we take $\tau^2(\theta)$ as is.

Question 4

If X and Y are independent each with a Poisson distribtuion, show that $Z = X + Y$ is Poisson distributed.

Since X and Y are independent, use the discrete convolution formula. I'll assume X and Y have paramters λ_1 and λ_2 respectively. However, the result will be obvious if the parameter λ is the same.

$$P(X+Y=z) = p_Z(z) = \sum_{x=0}^{z} p_X(x) p_Y(z-x)$$

$$\begin{aligned}
p_Z(z) &= \sum_{x=0}^{z} e^{-\lambda_1} \frac{\lambda_1^x}{x!} e^{-\lambda_2} \frac{\lambda_2^{z-x}}{(z-x)!} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{x=1}^{z} \frac{\lambda_1^x}{x!} \frac{\lambda_2^{z-x}}{(z-x)!} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{1}{z!} \sum_{x=1}^{z} \frac{z!}{x!(z-x)!} \lambda_1^x \lambda_2^{z-x} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{1}{z!} (\lambda_1 + \lambda_2)^z \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^z}{z!}
\end{aligned}$$

Which is a Poisson probability distribution.

Question 5

In a famous example, Bortkiewicz tabuluated the number of Cavalry men kicked to death by horses in the Prussian Cavalry, for 14 Corps over 20 years (1875-1894), giving $n = 280$ observations in all.

Number of deaths	Frequency count
0	144
1	91
2	32
3	11
4	2
More	0
Total	280

Table 1: Frequency tabulation

1. For this model, find the MLE of θ, assume the X_i's are independent.

   The likelihood function of θ is

$$L(\theta) = e^{-\theta n} \frac{\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} \frac{1}{x_i!}}.$$

   Differentiating the log of this function gives

$$\frac{d}{d\theta} \log L(\theta) = \frac{\sum_{i=1}^{n} x_i}{\theta} - n.$$

   Setting the above equal to zero we have

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} x_i = \overline{X}$$

2. Find the MSE of the MLE

   SINCE the MLE is the sample mean the MLE is unbiased.

$$\text{MSE} = \text{Var}(\hat{\theta}) = \text{Var}(\overline{X}) = \frac{\text{Var}(X_i)}{n}$$

   Since $P(X_i = k \mid \theta)$ is Poisson distributed, $\text{Var}(X_i) = \theta$.

   So $\text{MSE} = \frac{\theta}{n}$.

3. From the Central Limit Theorem find the approximate distribution of the MLE when n is large.

   Since the MLE is the sample mean for i.i.d X_i where $E(X_i) = \theta$ and $\text{Var}(X_i) = \theta$, $\hat{\theta}$ has an approximately $N(\theta, \frac{\theta}{n})$ distrubution, which becomes a better aproximiation as n gets larger.

4. From Fisher's Approximation, find the approximate distribution of the MLE when n is large.

   FISHER'S approximation states that if data consit of independent random variables each with distribution $f(x \mid \theta)$, and $\hat{\theta}$ can be found by solving log likelihood equal to 0, then for large n, $\hat{\theta}$ has approximately a $N(\theta, \frac{\tau^2(\theta)}{n})$.

$$\frac{1}{\tau^2(\theta)} = -E\Big[\frac{d^2}{d\theta^2} \log p(X_1 \mid \theta)\Big]$$

   Where

$$p(X_1 \mid \theta) = e^{-\theta} \frac{\theta_1^x}{x_1!}.$$

Taking the log of the above yields

$$\log p(X_1 \mid \theta) = -\theta + x\log(\theta) - \log(x!).$$

Differientiating,

$$\frac{d}{d\theta}\log p(X_1 \mid \theta) = \frac{x_1}{t} - 1$$

$$\frac{d^2}{d\theta^2}\log p(X_1 \mid \theta) = -\frac{x_1}{\theta^2}$$

Plugging this into the formula for asymptotic variance,

$$\frac{1}{\tau^2} = -E\Big[-\frac{x_1}{\theta^2}\Big] = \frac{E[X_1]}{\theta^2}$$

Recall that $E(X_1) = \theta$ so $\tau^2(\theta) = \theta$. Therefore, from Fisher the distrubition of $\hat{\theta}$ can be approximated as $N(\theta, \frac{\theta}{n})$.

5. Evaluate the MLE for the given data.

   We found MLE to be the sample mean. So take the total number of deaths by horse kicks and divide by $n = 280$.

$$\text{MLE} = \hat{\theta} = \frac{196}{280} = .7$$

6. If θ, the mean number of deaths per Corps in a year, is really 1.0, what (approximately) is the probability that the MLE would turn out to be below .85.

   Since $\hat{\theta}$ has approximately $N(\theta, \frac{\theta}{n})$ distribution calculate this using

$$P(\hat{\theta} < .85) = \Phi\Big(\frac{.85 - 1}{\sqrt{\frac{1}{280}}}\Big) = .0062$$

Question 6

Let $X_1, ..., X_n$ be an i.i.d. sample from a Rayleigh distributions with parameter $\theta > 0$:

$$f(x \mid \theta) = \frac{x}{\theta^2}e^{-x^2/(2\theta^2)}\ ,\ x \geq 0$$

1. Find the mle of θ

THE likelihood function of of θ is

$$\begin{aligned} L(\theta) &= f(x_1, ..., x_n \mid \theta) \\ &= f(x_1 \mid \theta) \cdots f(x_n \mid \theta) \\ &= \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-x_i^2/2\theta^2} \end{aligned}$$

Taking the log gives

$$\log L(\theta) = \sum_{i=1}^{n} \Big[\log(x_i) - 2\log(\theta) - \frac{x_i^2}{2\theta^2} \Big].$$

Differentiating

$$\frac{d}{d\theta} \log L(\theta) = -\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} x_i^2}{\theta^3}.$$

Setting the derivative of the log likelihood to zero yields

$$\hat{\theta} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}}$$

2. Find the asymptotic variance of the mle.

$$\begin{aligned} \frac{d^2}{d\theta^2} \log L(\theta) &= \frac{d^2}{d\theta^2} - \frac{2n}{\theta} + \frac{\sum_{i=1}^{n} x_i^2}{\theta^3} \\ &= \frac{2n}{\theta^2} - \frac{3\sum_{i=1}^{n} x_i^2}{\theta^4} \\ &= \frac{2n}{\hat{\theta}^2} - \frac{6n}{\hat{\theta}^2} \\ &= -\frac{4n}{\hat{\theta}^2} \end{aligned}$$

$$\frac{1}{\tau^2(\theta)} = -E\Big[-\frac{4n}{\theta^2} \Big]$$

So the asymptotic variance is $\frac{\theta^2}{4n}$