Worksheet 07 (Solutions)

1. Let X be a random variable with the following probability mass function:

$$p_X(x) = \begin{cases} 0.6, & \text{if } x = 1 \\ 0.2, & \text{if } x = 2 \\ 0.2, & \text{if } x = 3 \end{cases}$$

Find $\mathbb{E}X$.

The expected value is:

$$\begin{aligned} \mathbb{E}X &= 0.6 \cdot 1 + 0.2 \cdot 2 + 0.2 \cdot 3 \\ &= 1.6 \end{aligned}$$

2. Let Y be a random variable with the following probability mass function:

$$p_Y(y) = \begin{cases} (1-p), & \text{if } y = 0 \\ p, & \text{if } y = 1 \end{cases}$$

For some $p \in [0,1]$. Find (a) $\mathbb{E}Y$ and (b) $Var(Y)$. (c) Sketch a plot of $Var(Y)$ in terms of p. (d) What value of p maximizes the variance?

The expected value (a) is simply:

$$\begin{aligned} \mathbb{E}Y &= p(0) \cdot 0 + p(1) \cdot 1 \\ &= p \end{aligned}$$

And the variance is given by:

$$\begin{aligned} \mathbb{E}(Y-p)^2 &= p(0) \cdot p^2 + p(1) \cdot (1-p)^2 \\ &= (1-p)p^2 + p \cdot (1-p)^2 \\ &= p^2 - p^3 + p + p^3 - 2p^2 \\ &= p - p^2 \\ &= p(1-p) \end{aligned}$$

(c) Is a parabola with a maximum at 0.5, and intercepts 0 and 1. Therefore (d) is 0.5.

3. Let X be a random variable defined as follows:

$$p_X(x) = \frac{1}{n}, \quad x \in \{1, 2, \ldots, n\}$$

Calculate $\mathbb{E}X$ and simplify the result.

The expected value is equal to:

$$\begin{aligned}\mathbb{E}X &= \sum_i p(i) \cdot i \\ &= \sum_{i=1}^{n} \frac{1}{n} \cdot i \\ &= \frac{1}{n} \cdot \sum_{i=1}^{n} i \\ &= \frac{1}{n} \cdot \frac{n \cdot (n+1)}{2} \\ &= \frac{n+1}{2}\end{aligned}$$

Using the formula for sum of the first n integers.

4. Let Z be a random variable uniformly distributed over the integers $\{-n, -(n-1), \ldots - 1, 0, 1, \ldots, n\}$. Calculate $\mathbb{E}Z$ using the transformation of variance theorem and your solution to the previous question.

For $m = 2n+1$, we can define $Z_n = X_m - n - 1$. Then:

$$\begin{aligned}\mathbb{E}Z_n &= \mathbb{E}\left(X_m - n - 1\right) \\ &= \mathbb{E}X_m - n - 1 \\ &= \frac{m+1}{2} - n - 1 \\ &= \frac{2n+2}{2} - n - 1 \\ &= 0\end{aligned}$$

Which we should have expected via the symmetry of the distribution.

5. Consider flipping a fair coin until it comes up heads. Let X be a random variable equal to the number of flips that are made. Calculate (a) $p_X(1)$, (b) $p_X(2)$, and (c) $p_X(3)$. (d) Write a general formula for $p_X(n)$. (e) Write down the quantity $\mathbb{E}X$. Notice that the summation is very difficult to simplify (you may leave it as is). (f) Write down the quantity $Var(X)$, for X defined as above. It is also very difficult to simplify.

For (a-d) have the following formulas:

$$\begin{aligned} p_X(1) &= 1/2 \\ p_X(2) &= (1/2)^2 \\ p_X(3) &= (1/2)^3 \\ &\vdots \\ p_X(n) &= (1/2)^n \end{aligned}$$

The (d) expected value is then given by:

$$\mathbb{E}X = \sum_{i=1}^{\infty} (1/2)^i \cdot i$$

And the (f)[1] variance is given by:

[1] yes, I forgot a part (e)!

$$Var(X) = \sum_{i=1}^{\infty} (1/2)^i \cdot (i - \mathbb{E}X)^2$$

We will see clever tricks for simplifying these in the next handout.