Worksheet 08 (Solutions)

1. For a random variable X, let $Y = a \cdot X$ for some constant a. Find a formula for $Var(Y)$ in terms of the variance of X.

Using the definitions, we have:

$$\begin{aligned}
Var(Y) &= \mathbb{E}\left((Y - \mathbb{E}Y)^2\right) \\
&= \mathbb{E}\left((aX - \mathbb{E}aX)^2\right) \\
&= \mathbb{E}\left((aX - a\mathbb{E}X)^2\right) \\
&= a^2 \cdot \mathbb{E}\left((X - \mathbb{E}X)^2\right) \\
&= a^2 \cdot Var(X).
\end{aligned}$$

2. For a random variable X, let $Y = X + b$ for some constant b. Find a formula for $Var(Y)$ in terms of the variance of X.

Using the definitions, we have:

$$\begin{aligned}
Var(Y) &= \mathbb{E}\left((Y - \mathbb{E}Y)^2\right) \\
&= \mathbb{E}\left((X + b - \mathbb{E}(X + b))^2\right) \\
&= \mathbb{E}\left((X - \mathbb{E}X)^2\right) \\
&= Var(X).
\end{aligned}$$

3. Let $X \sim Bin(n, p)$ and $Y = X/n$. Find the expected value and variance of Y. What is the limit of both quantities as $n \to \infty$? What is the intuition for these results?

The expected value is just:

$$\begin{aligned}
\mathbb{E}Y &= \mathbb{E}(X/n) \\
&= \frac{1}{n} \cdot \mathbb{E}X \\
&= \frac{np}{n} \\
&= p
\end{aligned}$$

And, using the previous result from (1), the variance is just:

$$\begin{aligned}
Var(Y) &= Var(X/n) \\
&= \frac{1}{n^2} Var(X) \\
&= \frac{np \cdot (p - 1)}{n^2} \\
&= \frac{p \cdot (p - 1)}{n}
\end{aligned}$$

In the limit, the expected value goes to p and the variance goes to zero. So, with a large sample size, the number of successes limits to p, which makes quite a lot of sense.

4. For any n, define $p_n = \lambda/n$ for some fixed $\lambda > 0$. If $X \sim Bin(n, p_n)$, find the following:

$$\lim_{n\to\infty} \mathbb{E}X = ?$$
$$\lim_{n\to\infty} Var(X) = ?.$$

The first quantity is just:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}X &= \lim_{n\to\infty} np_n \\ &= \lambda\end{aligned}$$

Because the mean is actually a constant. The variance is given by:

$$\begin{aligned}\lim_{n\to\infty} Var(X) &= \lim_{n\to\infty} (np(1-p)) \\ &= \lim_{n\to\infty} (\lambda - \lambda \cdot p_n) \\ &= \lambda - \lambda \cdot lim_{n\to\infty}(p_n) \\ &= \lambda.\end{aligned}$$

Because p_n limits to zero as n goes to infinity.