Worksheet 12 (Solutions)

1. Let X and Y be random variables defined as:

$$f(x,y) = x + y, \quad x, y \in (0,1)$$

(a) Compute $f(x)$, (b) $f(y|x)$.

For (a) $f(X)$, this is just:

$$\begin{aligned} f(x) &= \int f(x,y)dy && = \int_{y=0}^{1} x + ydy \\ &= \left[xy + y^2/2\right]_{y=0}^{1} \\ &= x + \frac{1}{2} \end{aligned}$$

For (b), we have:

$$\begin{aligned} f(y|x) &= \frac{f(x,y)}{f(x)} \\ &= \frac{x+y}{x+1/2} \end{aligned}$$

2. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a $N(\mu, \sigma^2)$ distribution.

From the independence we can just multiply the densities together. Therefore:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= \prod_i \frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\frac{1}{2\sigma^2}\cdot(x_i-\mu)^2} \\ &= (2\pi\sigma^2)^{-n/2} \times \exp\left\{-\frac{1}{2\sigma^2}\sum_i (x_i - \mu)^2\right\} \end{aligned}$$

3. The negative binomial $NB(k,p)$ is the sum of k independent geometric random variables with parameter p. Find the MGF, expected value, and variance of the negative binomial.

Let $Y_1, \ldots, Y_k$ be i.i.d. random variables with $Geom(p)$ distributions and $X = \sum_i Y_i$. We already know that the MFG is just the MFG of a single geometric raised to the n'th power:

$$m_X(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^n$$

The expected value and variance come almost immediately from the theorems on today's handout:

$$\begin{aligned}\mathbb{E}X &= \sum_i \mathbb{E}Y_i \\ &= k \cdot \frac{1}{p} \\ &= k/p.\end{aligned}$$

And the variance is:

$$\begin{aligned}Var(X) &= \sum_i Var(Y_i) \\ &= k \cdot \frac{1-p}{p^2} \\ &= \frac{k(1-p)}{p^2}.\end{aligned}$$

4. Consider random variables X and Y such that:

$$\begin{aligned}p &\sim Beta(\alpha, \beta) \\ X|p &\sim Bin(n, p)\end{aligned}$$

For part (a), we have (throwing away constants):

$$\begin{aligned}f(p|x) &\propto f(X|p)f(p) \\ &\propto p^x(1-p)^{n-x} \times p^{\alpha-1}(1-p)^{\beta-1} \\ &\propto p^{x+\alpha-1} \cdot (1-p)^{n-x+\beta-1}\end{aligned}$$

If we set $\alpha' = x + \alpha$ and $\beta' = n - x + \beta$, then this is proportion to a $Beta(k+\alpha, n-x+\beta)$ distribution. Therefore, the constant must be the constant for this distribution and:

$$f(p|x) = \left(\frac{\Gamma(n+\alpha+\beta)}{\Gamma(x+\alpha)\Gamma(n-x+\beta)}\right) \times p^{x+\alpha-1} \cdot (1-p)^{n-x+\beta-1}$$

More importantly, the (c) expected value is just:

$$\frac{x+\alpha}{n+\alpha+\beta}.$$