Handout 10: Continuous RVs

Continuous Random Variables

A random variable is continuous if it may take on any value in some interval of real numbers. While discrete random variables are typically associated with the process of counting (e.g., the number of times a particular event occurs in repeated trials of an experiment), continuous variables are generally the result of measurements rather than counts. Examples include the measurement of physical characteristics like height, weight, specific gravity, length of life, or intensity of sound. A variable is considered to be continuous if, in theory, it can take any value in an interval, even though its value may be "discretized" in practical applications. For example, we consider a person's height as a continuous variable even though we would often round that height off to the nearest inch.

The most common way to model a continuous random variable is by means of a curve called a *probability density function* (pdf) or simply a *density*. A probability density function $f\ (x)$ is a real valued function having two basic properties:

- (i) $f\ (x) \geq 0$ for all x $\in (-\infty\ ,\ \infty\ )$,
- (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$.

As a practical matter, we will wish to have $f\ (x)$ be an integrable function, as it is through the integration of f that we will be able to compute probabilities from the model. As defined above, a density has a natural interpretation as a probability model. If the random variable X has density $f\ (x)$, then for any real numbers $a < b$, the probability that X $\in (a,\ b)$ may be computed as

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

Our conditions above guarantee that the integral is between 0 and 1, and the probability of the universal set is equal to 1. Therefore this definition satisfies the Axioms of Probability.

Let X be a continuous random variable with density $f(x)$. Then the *expected value* of X is given by:

$$\mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx$$

As with discrete variables, we refer to EX here as the mean of X and sometimes denote it by μ_X or simply by μ.[1]

[1] We will not formally re-state and prove each theorem we have so far regarding expected values and variances, but all hold for continuous random variables and this definition of expectation.

For any random variable X, the cumulative distribution function (cdf) of X is the function $F(x) = P(X \leq x)$. This function may be evaluated for any $x \in (-\infty, \infty)$ and is well defined for both discrete and continuous random variables. If X is discrete, then $F(x)$ is simply the sum of the probabilities of all values less than or equal to x that the variable X may take on. If X is continuous, then $F(x)$ is evaluated as the area under the density of X from $-\infty$ to x, that is,

$$F(z) = \int_{-\infty}^{z} f_X(t)dt.$$

It should be clear from the discussion above that the cdf F of any random variable will have the following properties:

- (1) For any $x \in (-\infty, \infty)$, $0 \leq F(x) \leq 1$. This is because, for any x, $F(x)$ represents the probability of a particular event, and thus it must take values in the interval [0, 1].
- (2) *F(x) is a non-decreasing function of x.* This is because if $x < y$, the occurrence of the event $\{X \leq x\}$ implies the occurrence of the event $\{X \leq y\}$, so the probability of the former event cannot be larger than that of the latter event.
- (3) $F(x)$ has the following limits: $\lim_{x\to -\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. This is so because, if it were not, the distribution of X would give probability less than 1 to the entire set of real numbers and thus, it would give positive probability to either ∞ or $-\infty$ or both. But random variables map sample spaces onto the real line so that the set of real numbers receives total probability 1. One final property of note is best demonstrated graphically. A distribution function need not be a continuous function, but
- (4) $F(x)$ is right-continuous, that is, $\lim_{x\to A+} F(x) = F(A)$. This means that if a sequence of x values are approaching the number A from the right, then $F(x) \to F(A)$. It need not be true that $F(x) \to F(A)$ when a sequence of x value approach the number A from the left.