Handout 09: MFGs and Poisson

Moment Generating Functions

We have seen how the variance is related the expected value of X^2. Expected values of other powers of X provide meaningful information about the character or shape of a distribution.

Definition 1 (Moments) *For positive integers $k = 1, 2, 3$...the k^{th} moment of a random variable X is the expected value:*

$$EX^k = \sum_k x^k p(x).$$

These moments lead us to a power probability tool. Because of its unexpected form, you'd be unlikely to think of it yourself (at least in your first semester of probability).

Definition 2 (Moment Generating Function (MGF)) *Let X be a discrete random variable with pmf $p(x)$. The moment-generating function (mgf) of X, denoted by $m_X(t)$, is given by:*

$$m_X(t) = \mathbb{E}\left[e^{tX}\right].$$

Among the important properties of moment-generating functions (mgfs) is the fact that, if they exist (i.e., if the expectation is finite), there is only one distribution with that particular mgf and there is, in this sense, a one-to-one correspondence between random variables and their mgfs. An immediate consequence is the fact that one may be able to recognize what the distribution of X is (in certain complex situations in which that distribution hasn't been explicitly stated) by recognizing the form of the mgf. But the utility of mgfs goes well beyond this fact. Its very name suggests a connection between the mgf and the moments of X.

Theorem 1 *Let X be a discrete random variable with moment generating function m_X (t), and suppose that $Y = aX + b$ where a and b are arbitrary constants. Then the mgf of Y is given by m_Y $(t) = e^{bt}m_X(at)$, for all t for which m_X (at) exists.*

Theorem 2 *Let X be a discrete random variable with moments $EX^k < \infty$ for $k = 1, 2, 3, \ldots$. Let $m_X(t)$ be the mgf of X, and assume that m_X (t) exists and is finite in an open interval of real numbers containing 0. Then*

$$\frac{\partial^k}{\partial t^k} m_X(t)|_{t=0} = \mathbb{E}X^k.$$

This is where the moment generating functions gets its name.

Theorem 3 (Uniqueness Property) *If two distributions have the same moment-generating function, then they are identical at almost all points. That is, if for all values of t,*

$$M_X(t) = M_Y(t)$$

Then,

$$F_X(t) = F_Y(t).$$

Theorem 4 (Linear Combination of Random Variables) *If $S_n = \sum_i a_i X_i$ where X_i are independent random variables*[1] *and the a_i are constants, then:*

$$M_{S_n}(t) = M_{X_1}(a_1 t) \cdot M_{X_2}(a_2 t) \cdots M_{X_n}(a_n t).$$

[1] We will formalize the notion of independent random variables next week.

Theorem 5 (Continuity Property) *Let X_1, X_2, X_3,... be a sequence of random variables with respective moment-generating functions $m_1(t)$, $m_2(t)$, $m_3(t)$,.... Suppose that the sequence $\{m_n(t)\}$ converges to a function $m(t)$, that is, suppose that as $n \to \infty$, $m_n(t) \to m(t)$ for all values of t in an open interval of real numbers containing 0. If $m(t)$ if the mgf of a random variable X, then as $n \to \infty$ we have $X_n \to X$.*[2] *That is, the probability distribution of X_n converges to the probability distribution of X.*

[2] We will also formalize the notion of random variable converging to another at a later point.

Poisson Distribution

The Poisson distribution describes a random variable that models the number of events occuring in a given interval of time or space. For example, the number of patients arriving at an emergency room or the number of students arriving at DHall. If $X \sim Poisson(\lambda)$, then:

$$\mathbb{P}[X = k] = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$$

On today's worksheet you will prove the following theorem:

Theorem 6 (Law of Rare Events) *Let $X_n \sim Bin(n, p)$. If $n \to \infty$ and $p_n = \lambda/n$ so that $\mathbb{E}X = \lambda$, then*

$$X_n \to X, \quad X \sim Poisson(\lambda).$$