Worksheet 16 (Solutions)

1. Let $X \sim Bin(1, p)$. Find the maximum likelihood estimator for p.

The MLE can be calculated by maximizing the log-likelihood

$$\begin{aligned}\frac{d}{dp} l(X; p) &\propto \frac{d}{dp} \left[x \log(p) + (n - x) \log(1 - p)\right] \\ &= \frac{x}{p} - \frac{n - x}{1 - p}\end{aligned}$$

Which is zero at:

$$\begin{aligned}\frac{x}{\widehat{p}} &= \frac{1 - x}{1 - \widehat{p}} \\ \widehat{p}(1 - x) &= x \cdot (1 - \widehat{p}) \\ \widehat{p} &= x.\end{aligned}$$

2. Let $X \sim N(\mu, 1)$. Find the maximum likelihood estimator for p.

The MLE can be calculated by maximizing the log-likelihood

$$\begin{aligned}\frac{d}{d\mu} l(X; p) &\propto \frac{d}{d\mu} \left[\frac{-1}{2}(x - \mu)^2\right] \\ &= (x - \mu)\end{aligned}$$

And therefore the MLE is at μ.

3. Let $X_1, \ldots, X_n \sim U(a, b)$. Find the maximum likelihood estimator for a and b.

The likelihood is equal to $(b-a)^{-n}$ as long as all of the data is between a and b, and equal to zero otherwise. So we want to set:

$$\begin{aligned}\widehat{a} &= min_i(X_i) \\ \widehat{b} &= max_i(X_i)\end{aligned}$$

4. Let $X \sim Gamma(1, \beta)$. Find the maximum likelihood estimator for β.

The MLE can be calculated by maximizing the log-likelihood

$$\begin{aligned}\frac{d}{d\beta} l(X; p) &\propto \frac{d}{d\beta} \left[x/\beta - \log(\beta)\right] \\ &= -x/\beta^2 - \frac{1}{\beta}\end{aligned}$$

Which yields:

$$\widehat{\beta} = x$$

5. Let $X_1, \dots, X_n \sim_{i.i.d.} N(\mu, \sigma^2)$. Find the maximum likelihood estimator for μ and σ^2.

The MLE can be calculated by maximizing the log-likelihood for both μ and σ^2:

$$\begin{aligned}\frac{d}{d\mu} l(X; p) &\propto \frac{d}{d\mu}\left[\frac{-1}{2\sigma^2}\sum_i (X_i - \mu)^2\right] \\ &= \frac{-1}{\sigma^2} \cdot \sum_i (X_i - \mu)\end{aligned}$$

And therefore the MLE for the mean in terms of the variance is:

$$\widehat{\mu} = \bar{X}$$

As before. The variance is:

$$\begin{aligned}\frac{d}{d\sigma^2} l(X; p) &\propto \frac{d}{d\sigma^2}\left[\frac{1}{2}\log(\sigma^2)\frac{-1}{2\sigma^2}\sum_i (X_i - \mu)^2\right] \\ &= \frac{1}{\sigma^2} - \frac{1}{(\sigma^2)^2} \cdot \sum_i (X_i - \mu)^2\end{aligned}$$

Which is zero at:

$$\widehat{\sigma}^2 = \frac{1}{n} \cdot \sum_i (X_i - \mu)^2$$

And plugging in $\mu = \bar{X}$:

$$\begin{aligned}\widehat{\mu} &= \bar{X} \\ \widehat{\sigma}^2 &= \frac{1}{n} \cdot (X_i - \bar{X})^2\end{aligned}$$