Handout 18: Z-Scores

We have mentioned that there is no closed form solution for computing the following integral for arbitrary values a and b:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \int_a^b e^{-\frac{1}{2}(x-\mu)^2}$$

This integral come from taking $X \sim N(\mu, \sigma^2)$, and finding the probability:

$$\mathbb{P}\left[a \leq X \leq b\right].$$

Given the important of the normal distribution, it seems that we should have a way of approximating such probabilities. The solution is to calculate a Z score, which is a centered and standardized version of random variable:

$$Z = \frac{X - \mu}{\sigma}$$

Notice that then we have $Z \sim N(0, 1)$. If we want to find the probability that X is between a and b, we can write this is terms of Z:

$$\begin{aligned}
\mathbb{P}\left[a \leq X \leq b\right] &= \mathbb{P}\left[a - \mu \leq X - \mu \leq b - \mu\right] \\
&= \mathbb{P}\left[\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right] \\
&= \mathbb{P}\left[\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right] \\
&= \mathbb{P}\left[Z \leq \frac{b-\mu}{\sigma}\right] - \mathbb{P}\left[Z \leq \frac{a-\mu}{\sigma}\right]
\end{aligned}$$

So, we could calculate the probability that a normally distribution variable X is between two extremes as long as we had a table giving the probability that Z is less than any particular α. A table of such values is called a **Z-table**. These usually only include values for a positive α, as negative values can be inferred by symmetry. Such a table is included in this handout and will given on the final two exams.

Two particularly important scores are:

$$\begin{aligned}
\mathbb{P}\left[Z \leq 1.96\right] &\approx 0.9750 = 1 - 0.05/2 \\
\mathbb{P}\left[Z \leq 2.56\right] &\approx 0.9948 = 1 - 0.01/2
\end{aligned}$$

Which yield what we refer to as tail bounds:

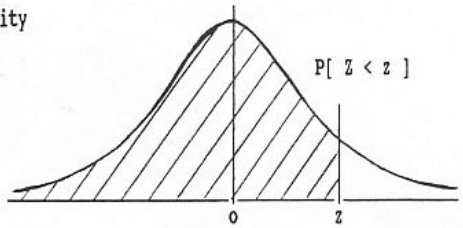
$$\begin{aligned}
\mathbb{P}\left[|Z| \geq 1.96\right] &\approx 0.05 \\
\mathbb{P}\left[|Z| \leq 2.56\right] &\approx 0.01
\end{aligned}$$

STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z

i.e. $$P[\, Z < z \,] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}Z^2)\, dZ$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000