Worksheet 22 (Solutions)

1. Let $X \sim N(5, 4^2)$. What is the probability that X is between than 4 and 10?

We want to re-write this in terms of a standard normal:

$$\begin{aligned}
\mathbb{P}[4 \le X \le 10] &= \mathbb{P}[(4-5) \le (X-5)/4 \le (10-5)] \\
&= \mathbb{P}[(4-5)/4 \le (X-5)/4 \le (10-5)/4] \\
&= \mathbb{P}[(4-5)/4 \le Z \le (10-5)/4] \\
&= 1 - \mathbb{P}[(4-5)/4 \ge Z \quad \text{or} \quad Z \ge (10-5)/4] \\
&= 1 - \mathbb{P}[(4-5)/4 \ge Z] - \mathbb{P}[Z \ge (10-5)/4]
\end{aligned}$$

Where Z is a standard normal $N(0,1)$ random variable and we used the mutually exclusivity of the two events in the or statement to separate them. From the tables we have:

$$\begin{aligned}
\mathbb{P}[(4-5)/4 \ge Z] &= \mathbb{P}[(5-4)/4 \le Z] \\
&= 1 - \mathbb{P}[Z \le (5-4)/4] \\
&\approx 1 - 0.5987 \\
&= 0.4013
\end{aligned}$$

And

$$\begin{aligned}
\mathbb{P}[Z \ge (10-5)/4] &= 1 - \mathbb{P}[Z \le (10-5)/4] \\
&= 1 - \mathbb{P}[Z \le 1.25] \\
&\approx 1 - 0.8944 \\
&= 0.1056
\end{aligned}$$

Which gives:

$$\begin{aligned}
\mathbb{P}[4 \ge X \ge 10] &\approx 1 - 0.4013 - 0.1056 \\
&= 0.4931.
\end{aligned}$$

One could skip a lot of these steps by drawing good pictures, but I wanted to show a full completely worked out example.

2. Let $X \sim Bin(100, 0.3)$. Using a normal approximation, estimate the probability that X is between 33 and 40.

The normal approximation gives $X \sim N(\cdot, \cdot)$ with the mean and variance computed by:

$$\begin{aligned}
\mathbb{E}X &= np = 100 \cdot 0.3 = 30, \\
V(X) &= np(1-p) = 100 \cdot 0.3 \cdot (1 - 0.3) = 21 \approx (4.58)^2
\end{aligned}$$

We then proceed exactly as before: We want to re-write this in terms of a standard normal:

$$\begin{aligned}
\mathbb{P}[33 \le X \le 40] &= \mathbb{P}[(33-30) \le (X-30)/4 \le (40-30)] \\
&= \mathbb{P}[(33-30)/4.58 \le (X-30)/4.58 \le (40-30)/4.58] \\
&= \mathbb{P}[(33-30)/4.58 \le Z \le (40-30)/4.58] \\
&= 1 - \mathbb{P}[(33-30)/4.58 \ge Z \quad \text{or} \quad Z \ge (40-30)/4.58] \\
&= 1 - \mathbb{P}[(33-30)/4.58 \ge Z] - \mathbb{P}[Z \ge (40-30)/4.58]
\end{aligned}$$

From the tables we have:

$$\begin{aligned}
\mathbb{P}[Z \le (33-30)/4.58] &= \mathbb{P}[Z \le 0.655] \\
&\approx 0.7422
\end{aligned}$$

And

$$\begin{aligned}
\mathbb{P}[Z \ge (40-30)/4.58] &= 1 - \mathbb{P}[2.183406 \ge Z] \\
&\approx 1 - 0.9854 \\
&= 0.0146
\end{aligned}$$

Which gives:

$$\begin{aligned}
\mathbb{P}[33 \ge X \ge 40] &\approx 1 - 0.7422 - 0.0146 \\
&= 0.2432.
\end{aligned}$$

And again, this could be done with less fuss using a diagram.

3. Assume that $X_1, \ldots, X_4 \sim_{i.i.d.} N(\mu, \sigma^2)$ and you observe the values: 1, 4, 6, and 9. Find a confidence interval at the 95% confidence level for μ.

We simply plug into the formulas for this one:

$$\begin{aligned}
\bar{X} &= \frac{1}{4} \cdot (1+4+6+9) = 5 \\
s^2 &= \frac{1}{3} \cdot \left[(1-5)^2 + (4-5)^2 + (6-5)^2 + (9-5)^2\right] = 11.33.
\end{aligned}$$

And then the confidence interval is just:

$$\begin{aligned}
\mu &\in \bar{X} \pm \frac{s}{\sqrt{n}} \cdot z_{\alpha/2} \\
\mu &\in 5 \pm 12.8211 \\
\mu &\in (-7.82, 17.82)
\end{aligned}$$

Where I used 1.96 as $z_{\alpha/2}$ because we wanted a 95%-confidence interval.

4. Sometimes we want a one-sided confidence interval of the form:

$$\mathbb{P}[b < \mu] = \alpha$$

Using the same data (1, 4, 6, and 9) find the one-sided confidence interval at the 95% confidence level.

This is exactly the same as above, but we have only an upper bound and change the z-score to z_α, which is 1.65 in this case:

$$\begin{aligned}\mu \in \left(-\infty, \bar{X} + \frac{s}{\sqrt{n}} \cdot z_{\alpha/2}\right) \\ \mu \in (-\infty, 5 \pm 10.8) \\ \mu \in (-\infty, 15.8)\,.\end{aligned}$$