Worksheet 09

1. Let $X \sim Bin(1, p)$. Compute $m_X(t)$. Using the moment generating function, what is $\mathbb{E}X^5$?

2. Describe $Y \sim Bin(n, p)$ in terms of independent random variables[1] $X_1, \ldots X_n \sim Bin(1, p)$. What is $m_Y(t)$?

[1] Again, we'll formalize this soon. Just use your intuition here.

3. We want to compute the moment generating function for the Poisson distribution. To start, show that if $X \sim Poisson(\lambda)$, then:

$$m_X(t) = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k e^{tk}}{k!}$$

Now, show to re-write this as:

$$m_X(t) = e^{-\lambda} e^{\lambda e^t} \cdot \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k e^{-(e^t \lambda)}}{k!}$$

Set $\delta = e^t \lambda$ and notice that the value under the sum is a known quantity. Simply the result.

4. For any n, define $p_n = \lambda/n$ for some fixed $\lambda > 0$, and let $X_n \sim Bin(n, p_n)$. Show that $m_{X_t}(t) \to m_Y(t)$ for $Y \sim Poisson(\lambda)$ in the limit of $n \to \infty$. By the uniqueness property, this shows that X_n converges to Y. You may need the following limit, which can be used without proof:

$$\lim_{\to \infty} \left[1 + \frac{a}{n}\right]^n = e^a, \quad \text{for fixed a.}$$

5. Let $X \sim Bin(n, p)$. Find the quantity:

$$\mathbb{E}\left[X(X-1)(X-2)\right]$$

Hint: Use the technique of Theorem 5 on Handout 8.

6. Let $m_X(t) = (1 - 2t)^{-3}$. What are $\mathbb{E}X$ and $Var(X)$?