Worksheet 03 (Selected Solutions)

1. Let A, B, and C be three events in a random experiment with sample space S. Write expressions for each of the following sets in terms of the set operations "union," "intersection", "complement," and "difference":

- (a) **only A occurs**: $A - B - C$
- (b) **A and B occur but C does not occur**: $(A \cup B) - C$
- (c) **exactly one of the events occurs**: $(A - B - C) \cup (B - A - C) \cup (C - A - B)$
- (d) **at least one of the events occurs**: $A \cup B \cup C$
- (e) **at most one of the events occurs**: $S - (A \cap B) - (A \cap C) - (A \cap B)$
- (f) **exactly two of the events occur**: $(A \cap B - C) \cup (A \cap C - B) \cup (B \cap C - A)$
- (g) **at least two of the events occur**: $(A \cap B) \cup (A \cap C) \cup (B \cap C)$
- (h) **at most two of the events occur**: $S - (A \cap B \cap C)$
- (i) **all three events occur**: $A \cap B \cap C$
- (j) **none of the events occur**: $S - A - B - C$

3. Suppose $\boldsymbol{S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}}$ and the sets $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are given by $\boldsymbol{A = \{2, 4, 6, 8, 10\}}$, $\boldsymbol{B = \{2, 5, 6, 7, 10\}}$, and $\boldsymbol{C = \{1, 6, 9\}}$. Identify each of the following sets:

- (a) $\boldsymbol{A \cup B}$: $\{2, 4, 5, 6, 7, 8, 10\}$
- (b) $\boldsymbol{A \cap B}$: $\{2, 6, 10\}$
- (c) $\boldsymbol{A - B}$: $\{4, 8\}$
- (d) $\boldsymbol{A \cup B^c}$: $\{1, 2, 3, 4, 6, 8, 9, 10\}$
- (e) $\boldsymbol{A \cap B \cap C}$: $\{6\}$
- (f) $\boldsymbol{B \cap (A \cup C)^c}$: $\{2, 6, 10\}$
- (g) $\boldsymbol{(A \cap C) \cup (B \cap C)}$: $\{6\}$
- (h) $\boldsymbol{(A - C) \cup (C - A)}$: $\{1, 2, 4, 8, 9, 10\}$
- (i) $\boldsymbol{A^c \cap B \cap C^c}$: $\{5, 7\}$

5. For arbitrary sets A and B, give a set theoretic proof that $\boldsymbol{A \cap B^c}$ is equal to $\boldsymbol{A - B}$.

Let $x \in A \cap B^c$, then by definition $x \in A$ and $x \in B^c$. From the second statement, we know that $x \notin B$. This fits the definition of $A - B$. Let $x \in A - B$; then by definition $x \in A$ and $x \notin B$. The second statement implies that $x \in B^c$, which finishes the result.

6. For arbitrary sets A and B, prove that $\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{A} \cup (\boldsymbol{B} - \boldsymbol{A})$.

If $x \in A \cup (B - A)$ we know that $x \in A$ or $x \in B - A$. From the second statement we know that $x \in A$ or $x \in B$, and from this $x \in A \cup B$. The other direction is the trickier one. Let $x \in A \cup B$, then either $x \in A$ or $x \in B$. If $x \in A$, then clearly $x \in A \cup (B - A)$; if $x \notin A$, then x must be in B, and is therefore in $B - A$, which completes the result.

7. Specify the sample space for the experiment consisting of three consecutive tosses of a fair coin. Using that model, compute the probability that you (a) obtain exactly one head, (b) obtain more heads than tails, (c) obtain the same outcome each time.

The sample space is:

$$U = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

The probabilities, based on simple counting, are (a) 3/8, (b) 4/8, and (c) 2/8.

8. (Review) Certain members of the eight-member city council are feuding at present and absolutely refuse to work together on council projects. Specifically, Mr. T refuses to work with Ms. B, and Mr. U refuses to work with Dr. P. How many three-person committees can be formed (to serve as the city's Public Relations Task Force) that involve only council members willing to work together amicably?

If the council contains none of the feuding members, there are $\binom{4}{3}$ options. If it picks members from both feuding pairs there are $\binom{2}{1} \cdot \binom{2}{1} \cdot \binom{4}{1}$ options. If one of T and B are choosen but neither of U and P, there are $\binom{2}{1} \cdot \binom{4}{2}$; there are an equal number of choices for the number of councils that pick one of U and P but neither of T and B. So, totaling these up (there is no double counting because each set is uniquely defined), the answer is:

$$\binom{4}{3} + \binom{2}{1} \cdot \binom{2}{1} \cdot \binom{4}{1} + 2 \cdot \binom{2}{1} \cdot \binom{4}{2}$$

9. (Review) Morse code is made up of dots and dashes. A given sequence of dots and dashes stands for a letter. For example, $- \cdot -$ might be one letter, and $\cdot \cdot - \cdot$ might be another. Suppose we are not interested in our own alphabet, but in a more general alphabet with more letters, and suppose we

use a Morse code with at least one, and at most *n*, dots and dashes. How many different letters could be represented by such a code?

There are exactly 2^n letters that can be represented by words that are exactly length n. A perfectly valid solution is given by:

$$\text{number words} = 2^1 + 2^2 + \cdots 2^n = \sum_{i=1}^{n} 2^i$$

It is a fairly easy proof by induction to show that this is equal to:

$$\text{number words} = 2^{n+1} - 2$$

While the proof is easy, it is by no means simply to get the result if you have not seen this trick before.