Worksheet 11 (Solutions)

1. Let $X \sim Gamma(\alpha, \beta)$. Find $\mathbb{E}X^{1/2}$.

We can calculate this as follows:

$$\begin{aligned}
\mathbb{E}X^{1/2} &= \int_0^\infty \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} \times x^{1/2} dx \\
&= \int_0^\infty \frac{\beta^\alpha x^{(\alpha+0.5)-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} dx \\
&= \left(\frac{1}{\Gamma(\alpha)\cdot\beta^\alpha}\right)\cdot\left(\frac{\beta^{\alpha+1/2}\Gamma(\alpha+1/2)}{\beta^{\alpha+1/2}\Gamma(\alpha+1/2)}\right) \times \int_0^\infty x^{(\alpha+0.5)-1}e^{-x/\beta}dx \\
&= \left(\frac{\beta^{\alpha+1/2}\Gamma(\alpha+1/2)}{\Gamma(\alpha)\cdot\beta^\alpha}\right) \times \int_0^\infty \frac{x^{(\alpha+0.5)-1}e^{-x/\beta}}{\Gamma(\alpha+1/2)\beta^{\alpha+1/2}} dx \\
&= \left(\frac{\beta^{\alpha+1/2}\Gamma(\alpha+1/2)}{\Gamma(\alpha)\cdot\beta^\alpha}\right) \\
&= \beta^{1/2} \times \frac{\Gamma(\alpha+1/2)}{\Gamma(\alpha)}
\end{aligned}$$

2. Let $X \sim N(\mu, \sigma^2)$. Show that $\mathbb{E}(X)$ is equal to μ using the moment generating function.

3. Let $X \sim N(\mu, \sigma^2)$. Show that $Var(X)$ is equal to σ^2 using the moment generating function.

The solution for both 2 and 3 are given by taking the derivatives of the moment generating function for $Z \sim N(0, 1)$:

$$\begin{aligned}
\frac{\partial}{\partial t} m_Z(t) &= t \cdot e^{t^2/2} \\
\frac{\partial^2}{\partial^2 t} m_Z(t) &= 1 \cdot e^{t^2/2} + t^2 \cdot e^{t^2/2}
\end{aligned}$$

Which gives:

$$\begin{aligned}
\mathbb{E}X &= 0 \\
\mathbb{E}\left(X - \mathbb{E}X\right)^2 &= \mathbb{E}X^2 \\
&= 1
\end{aligned}$$

Now the result now follows immediately from the two theorems on transformations of normal variables.

4. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ be independent random variables. Let $W = X + Y$. Show that W, as defined above, is a normally distributed random variable. Find its mean and variance.

We know that the moment generating function of W is the product of the moment generating functions of X and Y:

$$\begin{aligned} m_W(t) &= m_X(t) \cdot m_Y(t) \\ &= e^{\mu_1 t + \frac{1}{2}\cdot\sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2}\cdot\sigma_2^2 t^2} \\ &= e^{(\mu_1+\mu_2)t + \frac{1}{2}\cdot(\sigma_1^2+\sigma_2^2)t^2} \end{aligned}$$

This is the mgf of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. By the uniqueness theorem we have $W \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.