Worksheet 06 (Solutions)

1. At Betty Boop's birthday party, each of her ten guests received a party favor. The nicest two were wrapped in white paper, the next three nicest party favors were wrapped in blue, and the remaining five were wrapped in red. As it happens, the Bobsy twins (Bobo and Mimi) attended the party, and they tend to have huge fights when they don't get exactly the same thing. Mrs. Boop plans to distribute the party favors completely at random. (a) What's the probability that neither of the Bobsy twins got favors wrapped in white paper? (b) Given that neither of the Bobsy twins got a white favor, what's the probability that their party favors were the same color?

This is just a standard counting problem. For (a) there are $\binom{10}{2}$ total outcomes, but only $\binom{8}{2}$ ways that neither twin can get a non-white favors. This gives a probability of:

$$\begin{aligned}\mathbb{P}(\text{both non-white favors}) &= \frac{\binom{8}{2}}{\binom{10}{2}} \\ &\approx 0.622.\end{aligned}$$

For (b) there are $\binom{8}{2}$ total outcomes, but only $\binom{3}{2}$ plus $\binom{5}{2}$ where they get the same color favor. Therefore the probability is:

$$\begin{aligned}\mathbb{P}(\text{both same color}) &= \frac{\binom{3}{2}+\binom{5}{2}}{\binom{8}{2}} \\ &\approx 0.464.\end{aligned}$$

2. Four male members of the Davis Opera Society went to see La Boheme at the Mondavi Center last night. They turned in their top hats at the hat check stand and they were placed by themselves in a small box. When they left, the hat check clerk was nowhere to be seen. Since all four hats looked the same, they each took one of the hats at random. What's the probability that none of them got their own hat? Hint: You may need to brute force this problem.

By brute force, we can see that 9 permutations of 4 numbers yield none of the numbers i in position i. This is known as a *derangement*. So the probability is just $9/4!$ or $9/24 = 0.375$.

It is not possible to set this up as a multi-stage experiment using our theorem because (as far as I can see, anyway) there is no way to set this up where each stage has a fixed number of valid choices.

3. Mr. Phelps plays poker on Wednesday nights. About 70% of the time, he will return home inebriated. Mrs. Phelps will retire before he comes home, but she leaves the porch light on for him. When Mr. Phelps comes home inebriated, he forgets to turn off the porch light with probability 0.6**; otherwise, he forgets with probability** 0.2**. On those Thursday mornings on which Mrs. Phelps finds the porch light on, what probability should she ascribe to her husband having been inebriated the night before?**

One more, just to get the hang of it. Let D be the even that Mr. Phelps is inebriated (drunk) and T be the event that he turns off the light.

$$\begin{aligned}\mathbb{P}(D|T^c) &= \frac{\mathbb{P}(D \cap T^c)}{\mathbb{P}(T^c)} \\ &= \frac{0.7 * 0.6}{0.7 * 0.6 + 0.3 * 0.2} \\ &= 0.875\end{aligned}$$

This is a classical example of Bayesian statistics. Without further information Mrs. Phelps assumes Mr. Phelps was inebriated with probability 0.7. After noticing that the lights were not turned off she increases this probability to 0.875. Likewise, if we computed $\mathbb{P}(D|T^c)$, she would decrease this probability if the light were turned off. Updating prior knowledge with data is what statistics is all about!

4. Sally will go to the homecoming dance with probability .5, while John will go with probability .2. If the probability that at least one of them goes is .6, can Sally's decision on whether or not to go be considered independent of John's?

Let S be the event that Sally goes and J be the event that John goes. We see that:

$$\begin{aligned}\mathbb{P}(S \cap J) &= 0.2 \\ &\neq \mathbb{P}S \cdot \mathbb{P}J \\ &= 0.5 \cdot 0.2 \\ &= 0.1\end{aligned}$$

So by definition S and J are not independent.

5. Just before going on stage, a magician crams 3 bunnies and 2 squirrels into his top hat. At the beginning of his act,

he takes off his hat, nervously wipes his brow, and shows the audience what appears to be an empty hat. Then he sets the top hat on a table and proceeds to draw animals from the hat. (a) What is the probability that one of the first two draws is a squirrel and the other is a bunny? (b) Given that the same kind of animal was drawn in the first two draws, what is the probability that they were both bunnies?

These can all be solved as simple counting problems. For (a) this derived from:

$$\begin{aligned}\mathbb{P}(\text{first two draws unique}) &= \frac{\binom{3}{1}\cdot\binom{2}{1}}{\binom{5}{2}}\\ &= 0.6.\end{aligned}$$

Because there are three ways of picking one bunny and 2 ways of choosing the squirrel. For (b) there are $\binom{3}{2}$ ways of picking two bunnies and $\binom{3}{2}$ plus $\binom{2}{2}$ of picking the same animal:

$$\begin{aligned}\mathbb{P}(\text{both bunnies}|\text{both same}) &= \frac{\binom{3}{2}}{\binom{3}{2}+\binom{2}{2}}\\ &= 0.75.\end{aligned}$$

6. The Pennsylvania School of Psychic Arts and Sciences has 1000 students. The admission criteria at PSPAS include the requirement that students have vivid dreams every night. It is known that the chance that a given dream will be predictive (that is, entirely come true within 30 days) is 1 in 10,000. What is the probability that at least one student at the school will have a predictive dream in the next seven days?

It is much easier to solve for the probability that nobody has a dream and then take 1 minus this answer. The probability that a given student does not have a dream on a given night is $\frac{9999}{1000}$. All of the trials are independent, and there are 7000 of them, so this yields:

$$\begin{aligned}\mathbb{P}(\text{at least one dream}) &= 1 - \mathbb{P}(\text{no dreams})\\ &= 1 - \left(\frac{9999}{10000}\right)^{7000}\\ &\approx 0.503.\end{aligned}$$

So nearly half of the time one student will have a predictive dream.