Let $X_i \sim_{i.i.d.} Poisson(\lambda)$. Find the MLE for λ.

We first need to write down the joint density of the samples X_i, given as the product of their respective densities (because they are independent)

$$L(X_1, \ldots, X_n) = \prod_i \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \tag{1}$$

$$= \frac{\lambda^{\sum_i x_i} e^{-n\lambda}}{\prod_i x_i!} \tag{2}$$

This is a case, as most are, where taking the logarithm of the density simplifies things greatly. We see that:

$$\log(L) = \sum_i x_i \cdot \log(\lambda) - \lambda n - \log(\prod_i x_i!). \tag{3}$$

The final step is to differentiate with respect to λ:

$$\frac{\partial}{\partial \lambda} \log(L) = \sum_i x_i \cdot \frac{1}{\lambda} - n \tag{4}$$

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Which we then set to zero:

$$\sum_i x_i \cdot \frac{1}{\widehat{\lambda}} = n \tag{5}$$

$$\widehat{\lambda} = \frac{1}{n} \sum_i x_i. \tag{6}$$

And so this is just the average of the samples.

So, if we observe the values 2, 4, 5, and 7 and believe that they come from a Poisson distribution, a good estimator of the rate λ is:

$$\widehat{\lambda} = \frac{1}{4}(2 + 4 + 5 + 7) \tag{7}$$

$$= 4.5 \tag{8}$$

Let $X_i \sim_{i.i.d.} Geometric(p)$. Find the MLE for p.

We again need to write down the joint density of the samples X_i:

$$L(X_1, \ldots, X_n) = \prod_i (1-p)^{x_i - 1} \cdot p \tag{9}$$

$$= (1-p)^{\sum_i x_i - n} \cdot p^n \tag{10}$$

And we again take the logarithm of the density to simplify:

$$\log(L) = (\sum_i x_i - n)\log(1-p) + n\log(p) \tag{11}$$

And then differentiate with respect to p:

$$\frac{\partial}{\partial \lambda} \log(L) = \frac{n}{p} - \frac{\sum_i x_i - n}{1-p} \tag{12}$$

And then differentiate with respect to p:

$$\frac{\partial}{\partial \lambda} \log(L) = \frac{n}{p} - \frac{\sum_i x_i - n}{1-p} \tag{12}$$

Which we then set to zero:

$$\frac{n}{\widehat{p}} = \frac{\sum_i x_i - n}{1-\widehat{p}} \tag{13}$$

$$n(1-\widehat{p}) = \widehat{p}(\sum_i x_i - n) \tag{14}$$

$$n - n\widehat{p} = \widehat{p}\sum_i x_i - \widehat{p}n \tag{15}$$

$$n = \widehat{p}\sum_i x_i \tag{16}$$

And so this gives the estimator:

$$\widehat{p} = \frac{n}{\sum_i x_i} \tag{17}$$

$$\tag{18}$$

Does this make sense? (it should!)

If we observe the number of free throws a basketball player makes before missing one, and get the data points 10, 8, 12, and 2, a good estimator for how often they miss a free throw would be:

$$\begin{aligned} \widehat{p} &= \frac{4}{10+8+12+2} && (19) \\ &= 0.125 && (20) \end{aligned}$$

Or, in other words, they make aroun 87.5% of their throws.