Handout 03: Set Theory and Randomness

Set theory basics

In the mathematical treatment of set theory,[1] the term "set" is undefined. It is assumed that we all have an intuitive appreciation for the word and what it might mean. Informally, we will have in the back of our minds the idea that there are some objects we are interested in, and that we may group these objects together in various ways to form sets. We will use capital letters A, B, C to represent sets and lower case letters a, b, c to represent the objects, or "elements," they may contain. The various ways in which the notion of "containment" arises in set theory is the first issue we face. We have the following possibilities to consider:

[1] Today we are talking about *elementary*, not formal axiomatic set theory.

Definition 1 (Set Containment) *The following definitions hold for arbitrary sets A and B.*

- *(i) $a \in A$: a is an element in the set A; $a \notin A$ is taken to mean that a is not a member of the set A.*
- *(ii) $A \subseteq B$: The set A is contained in the set B, that is, every element of A is also an element of B. A is then said to be a "subset" of B.*
- *(iii) $A \subset B$: A is a "proper subset" of B, that is, A is contained in B ($A \subseteq B$) but B contains at least one element that is not in A.*
- *(iv) $A = B$: $A \subseteq B$ and $B \subseteq A$, that is, A and B contain precisely the same elements.*

In typical applications of set theory, one encounters the need to define the boundaries of the problem of interest. It is customary to specify a "universal set" U, a set which represents the collection of all elements that are relevant to the problem at hand. It is then understood that any set that subsequently comes under discussion is a subset of U. For logical reasons, it is also necessary to acknowledge the existence of the "empty set" $\emptyset$, the set with no elements. This set constitutes the logical complement of the notion of "everything" embodied in U, but also plays the important role of zero in the set arithmetic we are about to discuss. We use arithmetic operations like addition and multiplication to create new numbers from numbers we have in hand. Similarly, the arithmetic of sets centers on some natural ways of creating new sets. The set operations we will use appear below:

Definition 2 (Set Operations) *The following definitions hold for arbitrary sets A and B defined over a universal set U.*

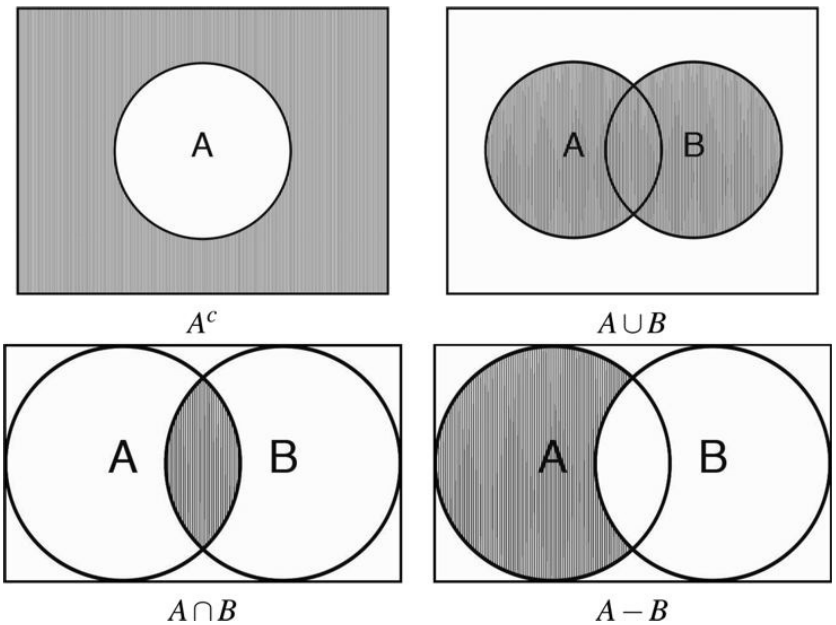
- *(i) A^c: "the complement of A," that is, the set of elements of U that are not in A.*

- *(ii) $A \cup B$: "A union B," the set of all elements of U that are in A or in B or in both A and B.*
- *(iii) $A \cap B$: "A intersect B," the set of all elements of U that are in both A and B.*
- *(iv) $A - B$: "A minus B," the set of all elements of U that are in A but are not in B.*

We often use pictorial displays called Venn diagrams to get an idea of what a set looks like. The Venn diagrams for common set operations are shown below:[2]

[2] I often use diagrams from various textbooks. Like this one, they often come with some numbering scheme (such as Figure 1.1.1); you should just blissfully ignore these.

Figure 1.1.1. Venn diagrams of A^c, $A \cup B$, $A \cap B$, and $A - B$.



Venn diagrams are often helpful in determining whether or not two sets are equal. You might think "I know how to check set equality; just see if the sets contain the same elements. So what's the big deal?" This reaction is entirely valid in a particular application where the elements of the two sets can actually be listed and compared. On the other hand, we often need to know whether two different methods of creating a new set amount to the same thing. To derive a general truth that holds in all potential applications, we need tools geared toward handling abstract representations of the sets involved. Consider, for example, the two sets: $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$. The equality of these two sets (a fact which is called the distributive property of intersection with respect to union) can be easily gleaned from a comparison of the Venn diagrams of each.

Theorem 1 (De Morgan's Law for Two Sets) *For two arbitrary sets*

A and B,

$$(A \cup B)^c = A^c \cap B^c.$$

Proof. The two sets A and B can be shown to be equal by showing that each contains the other. Let's first show that $(A \cup B)^c \subseteq A^c \cap B^c$. Assume that x is an arbitrary element of the set $(A \cup B)^c$ that is, assume that:

$$x \in (A \cup B)^c.$$

This implies that

$$x \notin (A \cup B)$$

which implies that

$$x \notin A \quad \text{and} \quad x \notin B$$

which implies that

$$x \in A^c \quad \text{and} \quad x \in B$$

which implies that

$$x \in A^c \cap B^c.$$

The same sequence of steps, in reverse, proves that $A^c \cap B^c \subseteq (A \cup B)^c$ ■.

Although harder to rationalize via Venn Diagrams, the same result holds for an arbitrarily large collection of sets.

Theorem 2 (General De Morgan's Law) *For a collection of n arbitrary sets $A_1, A_2, \ldots A_n$,*

$$\left(\bigcup_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} (A_i^c).$$

The proof is nearly exactly the same as the single set version by replacing a single union or interesection with one over all available sets.

Approaches to Modeling Randomness

Randomness is a slippery idea. Even the randomness involved in a simple phenomenon like flipping a coin is difficult to pin down. What is important about the models we will employ to describe what might happen in particular experiments is that they constitute reasonable approximations of reality. Under normal circumstances, we'll find that the coin tossing we do is quite well described by the assumption that heads and tails are equally likely. Our criterion for the validity of a model will be the level of closeness achieved between real experimental outcomes and the array of outcomes that our model would have predicted. Our discussion of randomness will require a bit of jargon. We collect some key phrases in the following:

Definition 3 (Random Experiment) *A random experiment is an experiment whose outcome cannot be predicted with certainty. All other experiments are said to be deterministic.*

Definition 4 (Sample Space) *The sample space plays the role of the universal set in problems involving the corresponding experiment, and it will be denoted by S.*

Definition 5 (Simple and Compound Events) *A single outcome of the experiment, that is, a single element of S, is called a simple event. A compound event is simply a subset of S. While simple events can be viewed as compound events of size one, we will typically reserve the phrase "compound event" for subsets of S with more than one element.*

Developing a precise description of the sample space of a random experiment is always the first step in formulating a probability model for that experiment. Over the first month and half of this course we will deal exclusively with "discrete problems," that is, with problems in which the sample space is finite or, at most, countably infinite. (A countably infinite set is one that is infinite, but can be put into one-to-one correspondence with the set of positive integers. For example, the set $\{2^n : n = 1, 2,3,...\}$ is countably infinite.)

Simple examples of random experiments often involve sampling coins, dice, and cards. For example, suppose that you toss a single coin once. Since you catch it in the palm of one hand, and quickly turn it over onto the back of your other hand, we can discount the possibility of the coin landing and remaining on its edge. The sample space thus consists of the two events "heads" and "tails" and may be represented as:

$$\{H, T\}$$

If we instead toss a coin twice, the sample space becomes:

$$\{HH, HT, TH, TT\}$$

We can see that sets are useful for describing the outcomes of a random experiment, but where will all of the machinery of set theory actually be used in our study of probability? Our next handout will do this by more rigorously define the axioms of probability.