# Worksheet 18 (Solutions)

**1. Let $X \sim N(4, 10^2)$. What is the probability that $X$ is greater than 20?**

We have:

$$\begin{aligned}
\mathbb{P}[X > 20] &= \mathbb{P}[(X-4)/10 > (20-4)/10] \\
&= \mathbb{P}[Z > 1.6] \\
&= 1 - 0.9452 \\
&= 0.0548
\end{aligned}$$

**2. Let $X \sim N(1, 3^2)$. What is the probability that $X$ is between than 0 and 5?**

We have:

$$\begin{aligned}
\mathbb{P}[X < 5] &= \mathbb{P}[(X-1)/3 < (5-1)/3] \\
&= \mathbb{P}[Z < 1.333333] \\
&= 0.9082
\end{aligned}$$

The solution comes from taking $0.9082 - 0.5$, which is $0.4082$.

**3. Let $X \sim N(0, \sigma^2)$. What is the probability that $X$ is greater than $3\sigma$?**

We have:

$$\begin{aligned}
\mathbb{P}[X > 3\sigma] &= \mathbb{P}[X/\sigma > 3] \\
&= \mathbb{P}[Z > 3] \\
&= 1 - 0.9986 \\
&= 0.0014
\end{aligned}$$

**4. Let $X \sim Bin(100, 0.2)$. Using a normal approximation, estimate the probability that $X$ is between 10 and 25.**

The mean is equal to 20 and the variance is $np(1-p)$, or 16, so we have:

$$\begin{aligned}\mathbb{P}\left[X < 10\right] &= \mathbb{P}\left[(X-20)/4 < (10-20)/4\right] \\ &= \mathbb{P}\left[Z < -2.5\right] \\ &= \mathbb{P}\left[Z > 2.5\right] \\ &= 1 - 0.9938 \\ &= 0.0062\end{aligned}$$

And the probability that $X$ is greater than 25 is:

$$\begin{aligned}\mathbb{P}\left[X > 25\right] &= \mathbb{P}\left[(X-20)/4 > (25-20)/4\right] \\ &= \mathbb{P}\left[Z > 1.25\right] \\ &= 1 - 0.8944 \\ &= 0.1056\end{aligned}$$

And, adding the regions, we get:

$$\begin{aligned}\mathbb{P}\left[10 < X < 25\right] &= 1 - 0.1056 - 0.0062 \\ &= 0.8882\end{aligned}$$

**5. Let $X_1, \ldots, X_n \sim_{i.i.d.} N(\mu, 1)$. If $n$ is equal to 16 what is the probability that $\bar{X}$ is more than 0.1 away from $\mu$? What if $n$ is equal to 100?**

We know that $\bar{X} \sim N(\mu, n^{-1})$, so:

$$\begin{aligned}\mathbb{P}\left[|\bar{X} - \mu| > 1\right] &= 2 \cdot \mathbb{P}\left[\bar{X} - \mu > 0.1\right] \\ &= 2 \cdot \mathbb{P}\left[(\bar{X} - \mu) \cdot \sqrt{n} > 0.1 \cdot \sqrt{n}\right] \\ &= 2 \cdot \mathbb{P}\left[Z > 0.1\sqrt{n}\right]\end{aligned}$$

For $n = 16$, we then have

$$\begin{aligned}2 \cdot \mathbb{P}\left[Z > 0.4\right] &= 2 \cdot (1 - 0.6554) \\ &= 0.6892\end{aligned}$$

So the estimator is more than 0.1 of the actual value about 69% of the time. Now with $n = 100$ we instead have:

$$\begin{aligned}2 \cdot \mathbb{P}\left[Z > 1\right] &= 2 \cdot (1 - 0.8413) \\ &= 0.3174\end{aligned}$$

So, in this case, the estimator is not within 0.1 with a probability of 31.7%.