Worksheet 14 (Solutions)

1. Let $X \sim Bin(n,p)$ and consider the estimator $\widehat{p} = X/n$. Find the variance, bias, and MSE of $\widehat{p}$.

We have the following:

$$\begin{aligned} Var(\widehat{p}) &= \frac{1}{n^2} Var(X) \\ &= \frac{np(1-p)}{n^2} \\ &= \frac{p(1-p)}{n} \end{aligned}$$

And the Bias is:

$$\begin{aligned} \mathbb{E}(\widehat{p}) - p &= \frac{1}{n}\mathbb{E}X - p \\ &= \frac{np}{p} - p \\ &= 0 \end{aligned}$$

So the MSE is just equal to the variance.

2. Let $X \sim Bin(n,p)$ and consider the estimator $\widehat{p} = 1/2$. Find the variance, bias, and MSE of $\widehat{p}$.

We have the following:

$$Var(\widehat{p}) = 0$$

Because the estimator is a constant. The Bias is:

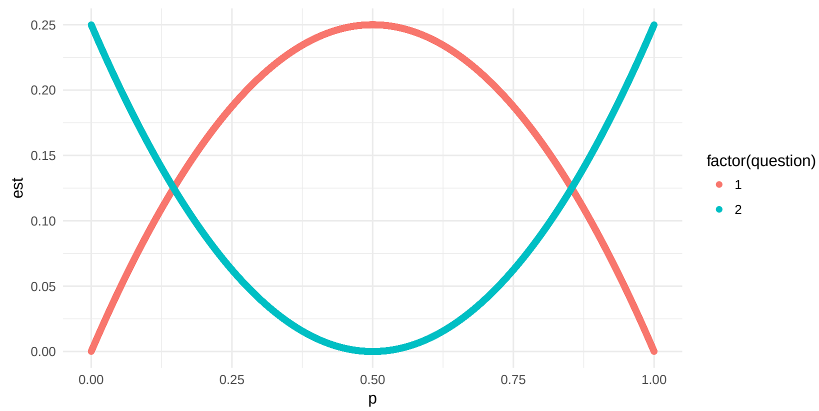
$$\mathbb{E}(\widehat{p}) - p = \frac{1}{2} - p$$

And so the MSE is:

$$MSE(\widehat{p}) = \left(\frac{1}{2} - p\right)^2$$

3. Sketch a graph showing for which p the estimator from question 2 is better than the one from question 1 (in terms of MSE)? Note: You do not need to formally work out what the bounds are; it is doably but messy algebra.

Here is a graph with $n = 1$:



4. Let $X \sim Bin(n, p)$. Consider:

$$\widehat{p} = \frac{x + a}{n + 2a}$$

Notice that this is like assuming that you have observered a 1's and a 0's before seeing any data. Find the Bias, Variance, and MSE of $\widehat{p}$. How does this compare to the result from question 1?

We have the following:

$$\begin{aligned} Var(\widehat{p}) &= \left(\frac{1}{n + 2a}\right)^2 \cdot Var(X) \\ &= \frac{np(1-p)}{(n + 2a)^2} \end{aligned}$$

And the Bias is:

$$\mathbb{E}(\widehat{p}) - p = \frac{p + a}{n + 2a}$$

And so the MSE is:

$$\begin{aligned} MSE(\widehat{p}) &= \frac{np(1-p) + (p + a)^2}{(n + 2a)^2} \\ &= \frac{np - np^2 + p^2 + a^2 + 2ap}{(n + 2a)^2} \end{aligned}$$

5. Let $X \sim Bin(2, p)$ and define $\theta = p^2$. Let $\widehat{\theta}$ be equal to $(X/n)^2$. Is $\widehat{\theta}$ unbiased?

We have:

$$\begin{aligned}\mathbb{E} = \frac{1}{2^2} \cdot \mathbb{E}X^2 \\ &= \frac{2p(1-p+2p)}{2^2} \\ &= \frac{p+p^2}{2}\end{aligned}$$

We want this to be equal to p^2, which is only true if p is equal to 0 or 1. And so, no, it is not (generally) unbiased.

6. Let $X \sim Gamma(\alpha, 1)$. Let $\widehat{\alpha}$ be equal to $n^{-1}\sum_i X_i$. Find the variance, bias, and MSE of $\widehat{\alpha}$.

We have the following:

$$\begin{aligned}Var(\widehat{p}) &= \frac{1}{n^2}\sum_i Var(X_i) \\ &= \frac{\alpha}{n}\end{aligned}$$

And the Bias is:

$$\begin{aligned}\mathbb{E}(\widehat{p}) - \alpha &= \frac{1}{n}\sum_i \mathbb{E}X_i - \alpha \\ &= \alpha - \alpha \\ &= 0\end{aligned}$$

And so the MSE is just the variance.