Worksheet 15 (Solutions)

1. Find the MME for $\widehat{p}$ when $X \sim Bin(1, p)$.

There is only one parameter, so we need only to match the mean. We have:

$$\mathbb{E}X = \frac{1}{n}\sum_i X_i$$
$$\widehat{p} = \frac{1}{n}\sum_i X_i$$

So it is just the sample mean.

2. Find the MME for $\widehat{\mu}$ when $X \sim N(\mu, 1)$.

There is only one parameter, so we again only need to match the mean. We have:

$$\mathbb{E}X = \frac{1}{n}\sum_i X_i$$
$$\widehat{\mu} = \frac{1}{n}\sum_i X_i$$

3. Let $X \sim U(a, b)$, the continuous uniform distribution between a and b ($a < b$). Find the mean and variance of X.

The mean is equal to $\frac{1}{2}(a+b)$ and the variance is $\frac{1}{12}(b-a)^2$.

4. Find the MMEs for $\widehat{a}$ and $\widehat{b}$ when $X \sim U(a, b)$.

Now we need two parameters, and therefore need both the mean and variance:

$$\mathbb{E}X = \frac{1}{n}\sum_i X_i$$
$$Var(X) = \frac{1}{n-1}\sum_i (X_i - \bar{X})^2$$

Which yields:

$$\frac{1}{2}(\widehat{a} + \widehat{b}) = \bar{X}$$
$$\frac{1}{12}(\widehat{b} - \widehat{a})^2 = \frac{1}{n-1}\sum_i (X_i - \bar{X})^2$$

Solving for $\widehat{a}$ we have:

$$\widehat{a} = 2\bar{X} - \widehat{b}$$

And plugging in to the second line:

$$\begin{aligned}\frac{1}{12}(\widehat{b} - 2\bar{X} + \widehat{b})^2 &= s^2 \\ \frac{4}{12}(\widehat{b} - \bar{X})^2 &= s^2 \\ \widehat{b} &= \sqrt{3}s - \bar{X}\end{aligned}$$

Which yields:

$$\begin{aligned}\widehat{a} &= \bar{X} - \sqrt{3}s \\ \widehat{b} &= \bar{X} + \sqrt{3}s\end{aligned}$$

So, basically, go out three standard deviations from the mean.

5. Find the MMEs for $\widehat{\mu}$ and $\widehat{\sigma^2}$ when $X \sim N(\mu, \sigma^2)$.

Now we need two parameters, and therefore need both the mean and variance:

$$\begin{aligned}\mathbb{E}X &= \bar{X} \\ Var(X) &= s^2\end{aligned}$$

Which yields:

$$\begin{aligned}\mu &= \bar{X} \\ \sigma^2 &= s^2\end{aligned}$$

So, it's quite straightforward.

6. Find the MMEs for $\widehat{\alpha}$ and $\widehat{\beta}$ when $X \sim Gamma(\alpha, \beta)$.

Now we need two parameters, and therefore need both the mean and variance:

$$\begin{aligned}\mathbb{E}X &= \bar{X} \\ Var(X) &= s^2\end{aligned}$$

Which yields:

$$\begin{aligned}\alpha\beta &= \bar{X} \\ \alpha\beta^2 &= s^2\end{aligned}$$

And gives:

$$\begin{aligned}\bar{X}\widehat{\beta} &= s^2 \\ \widehat{\beta} &= \frac{s^2}{\bar{X}}\end{aligned}$$

Which gives:

$$\widehat{\alpha} = \frac{\bar{X}^2}{s^2}.$$