Worksheet 04 (Solutions)

1. Draw a Venn-Diagram with one circle representing Females (F) and the other representing Smokers (S). Using the sets Males (M) and Non-smokers (N), label the four areas as the intersections of two sets.

I won't draw the sets (a pain in LaTeX), but the decomposition should just include all four permutations: $F \cap S$, $F \cap N$, $M \cap S$, and $M \cap N$. Where to place them should be straightforward.

2. The workers in a particular factory are 65% male, 70% married, and 45% married male. If a worker is selected at random from this factory, find the probability that the worker is (a) a married female, (b) a single female, (c) married or male or both.

Build a table!

	Male	Female	Total
Married	0.45	0.25	0.70
Single	0.20	0.10	0.30
Total	0.65	0.35	1.00

The answers just roll right off the table: (a) 25%, (b) 20%, and (c) 90%. The last one is just 1 minus the number of single females.

3. Let A and B be two events. Identify the conditions on A and B that must hold in order to guarantee the validity of the identity: $P(B - A) = P(B) - P(A)$. Prove this identity.

The result holds is when $\mathbb{P}(A - B)$ is equal to zero. A slightly less general result that you may have arrived at is that $A \subseteq B$, from which one can quickly confirm the more general statement.

To prove this, notice that $B = (B - A) \cup (B \cap A)$ and that this is a disjoint union. Therefore:

$$\begin{aligned} \mathbb{P}B &= \mathbb{P}(B - A) + \mathbb{P}(B \cap A) \\ \mathbb{P}(B - A) &= \mathbb{P}B - \mathbb{P}(B \cap A) \end{aligned}$$

Similarly, we have that $A = (A - B) \cup (A \cap B)$ and that this is also a disjoint union. This then gives:

$$\begin{aligned}\mathbb{P}A &= \mathbb{P}(A-B) + \mathbb{P}(A\cap B)\\ &= \mathbb{P}(A\cap B)\end{aligned}$$

With the second line coming from the fact that we assume $\mathbb{P}(A-B)$ is equal to zero. The result comes from substituting this last line into the first derivation.

4. Formally prove that the third axiom implies the two set case (it explicitly applies only a countable infinite collection of sets). That is, show that if $A_1 \cap A_2 = \emptyset$ then $\mathbb{P}(A_1 \cup A_2)$ must be equal to $\mathbb{P}A_1 + \mathbb{P}A_2$.

To get this from the axioms, simply set $A_i = \emptyset$ for $i \geq 2$. All of the A_i are still mutually disjoint (yes, the empty set is disjoint from itself). We then get:

$$\begin{aligned}\mathbb{P}\bigcup_{i=1}^{\infty} A_i &= \sum_{i=1}^{\infty} \mathbb{P}A_i\\ &= \mathbb{P}A_1 + \mathbb{P}A_2\end{aligned}$$

The result hinges on the result that $\mathbb{P}\emptyset$ is always equal to zero. The same argument can be applied to any finite collection of disjoint sets.

5. Psychology majors are required to take two particular courses: Psychology 100 and Psychology 200. It is a rare student indeed who does outstanding work in both courses. It is known that the chances of getting an A in PSY 100 is .4 and the chances of getting an A in PSY 200 is .3, while the chances of getting an A in both courses are .05. What are the chances that a randomly selected student will get at least one A in the two courses?

Another table!

	Psych 100: A	Psych 100: Non-A	Total
Psych 200: A	0.05	0.25	0.30
Psych 200: Non-A	0.35	0.35	0.70
Total	0.40	0.60	1.00

So the answer is $0.05 + 0.25 + 0.35$, or 65%.

6. Let A and B be two events in a random experiment. Prove that the probability that exactly one of the events occurs (that

is, the event $(A \cap B^c) \cup (A^c \cap B)$ occurs) in a given trial of this experiment is equal to $\mathbb{P}(A) + \mathbb{P}(B) - 2 \cdot \mathbb{P}(A \cap B)$**.**

We can quickly see that $(A \cap B^c)$ and $(A^c \cap B)$ are disjoint. Therefore:

$$\mathbb{P}\left((A \cap B^c) \cup (A \cap B)\right) = \mathbb{P}(A \cap B^c) + \mathbb{P}(A^c \cap B)$$

Now, we can also show that:

$$\begin{aligned} \mathbb{P}A &= \mathbb{P}(A \cup B) + \mathbb{P}(A \cup B^c) \\ \mathbb{P}B &= \mathbb{P}(B \cup A) + \mathbb{P}(B \cup A^c) \end{aligned}$$

Re-arranging these two equations to be in terms of the first result and plugging in gives the desired result.