Worksheet 09 (Solutions)

1.Let $X \sim Bin(1, p)$. Compute $m_X(t)$. Using the moment generating function, what is $\mathbb{E}X^5$?

This is a rare case where we can compute the mfg directly:

$$\begin{aligned} m_X(t) &= \mathbb{E}e^{tX} \\ &= \mathbb{P}(X = 0) \cdot e^{t \cdot 0} + \mathbb{P}(X = 1) \cdot e^{t \cdot 1} \\ &= (1 - p) \cdot e^{t \cdot 0} + p \cdot e^{t \cdot 1} \\ &= (1 - p) + pe^t. \end{aligned}$$

2. Describe $Y \sim Bin(n, p)$ in terms of independent random variables[1] $X_1, \ldots X_n \sim Bin(1, p)$. What is $m_Y(t)$?

[1] Again, we'll formalize this soon. Just use your intuition here.

If $Y = \sum_i X_i$, then Y should have the desired distribution. Then, we see that:

$$\begin{aligned} m_Y(t) &= \sum_{i=1}^{n} m_{X_i}(t) \\ &= \sum_{i=1}^{n} \big((1 - p) + pe^t\big) \\ &= \big((1 - p) + pe^t\big)^n \end{aligned}$$

3. Compute the moment generating function for the Poisson distribution.

The steps are fairly straightforward given the hints, but the solution is ultimately this:

$$m_X(t) = e^{\lambda(e^t - 1)}.$$

4. For any n, define $p_n = \lambda/n$ for some fixed $\lambda > 0$, and let $X_n \sim Bin(n, p_n)$. Show that $m_{X_t}(t) \to m_Y(t)$ for $Y \sim Poisson(\lambda)$ in the limit of $n \to \infty$.

We can rewrite the moment generating function of X_n as:

$$\begin{aligned} m_{X_n}(t) &= \big[(1 - p_n) + p_n \cdot e^t\big]^n \\ &= \big[1 + p_n(e^t - 1)\big]^n \\ &= \left[1 + \frac{\lambda(e^t - 1)}{n}\right]^n \end{aligned}$$

Using the given limit theorem and setting $a = \lambda(e^t - 1)$ yields:

$$m_{X_n}(t) \to e^{\lambda(e^t-1)}.$$

This is exactly equal to Poisson distribution.

5. Let $X \sim Bin(n,p)$. Find the quantity:

$$\mathbb{E}\left[X(X-1)(X-2)\right]$$

Just as on Handout 8, we re-write the density in terms of a known probability mass function:

$$\begin{aligned}
\mathbb{E}(X(X-1)(X-2)) &= \sum_{x=0}^{n} x \cdot (x-1) \cdot (x-2) \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=3}^{n} x \cdot (x-1) \cdot (x-2) \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}, \text{ as above is zero when } x < 3 \\
&= \sum_{x=3}^{n} \frac{n!}{(x-3)!(n-x)!} p^x (1-p)^{n-x}, \text{ canceling first 3 terms of factorial} \\
&= n(n-1)(n-2)p^3 \cdot \sum_{x=3}^{n} \frac{(n-3)!}{(x-3)!(n-x)!} p^{x-3} (1-p)^{n-x}
\end{aligned}$$

Making the substitution $y = x - 2$ cancels the summation term, as it is a $Bin(n-3,p)$ probability mass function. The result is then:

$$\mathbb{E}(X(X-1)(X-2)) = n(n-1)(n-2)p^3.$$

6. Let $m_X(t) = (1-2t)^{-3}$. What are $\mathbb{E}X$ and $Var(X)$?

The derivatives are

$$\begin{aligned}
\frac{\partial}{\partial t} m_X(t) &= \left((1-2t)^{-3}\right) \\
&= (-3)(1-2t)^{-4}(-2) \\
&= 6(1-2t)^{-4}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial^2}{\partial^2 t} m_X(t) &= \left(6(1-2t)^{-4}\right) \\
&= 6(-4)(1-2t)^{-5}(-2) \\
&= 48(1-2t)^{-5}
\end{aligned}$$

Giving:

$$\begin{aligned}
\mathbb{E}X &= 6 \\
\mathbb{E}X^2 &= 48
\end{aligned}$$

And the variance:

$$\begin{aligned} Var(X) &= 48 - 6^2 \\ &= 12. \end{aligned}$$