Worksheet 13 (Solutions)

1. Let $X \sim Gamma(\alpha, \beta)$ and $Y \sim Bin(n, p)$ be independent random variables. Find the following two quantities:

$$\begin{aligned} \mathbb{E}\left(2X + 5Y\right) &= ? \\ Var\left(2X + 5Y\right) &= ? \end{aligned}$$

We can do this quickly using the rules about expectation, variance, and independence and looking up the means and variance of each known distribution:

$$\begin{aligned} \mathbb{E}\left(2X + 5Y\right) &= 2\mathbb{E}X + 5\mathbb{E}Y \\ &= 2np + 5\alpha\beta. \end{aligned}$$

And:

$$\begin{aligned} Var\left(2X + 5Y\right) &= 2^2 Var(X) + 5^2 Var(Y) \\ &= 4np(1-p) + 25\alpha\beta^2 \end{aligned}$$

2. Let $X_s \sim Geom(\lambda/s)$ and $Y_s = X_s/s$. Find the moment generating function of Y_s.

The moment generating function of Y_s is, given the rules about additive constants, is just:

$$M_Y(t) = \frac{\lambda/s \cdot e^{t/s}}{1 - (1 - \lambda/s)e^{t/s}}$$

3. Show that if $Y_s \to Y$, then $Y \sim Exp(\lambda)$ by evaluating the limit the moment generating function.

We can re-write the moment generating function by multiplying the top and bottom by $s \cdot e^{-t/s}$.

$$\begin{aligned} M_Y(t) &= \frac{\lambda/s \cdot e^{t/s}}{1 - (1 - \lambda/s)e^{t/s}} \times \frac{s \cdot e^{-t/s}}{s \cdot e^{-t/s}} \\ &= \frac{\lambda}{s \cdot e^{-t/s} + \lambda - s} \\ &= \frac{\lambda}{\lambda - s[e^{-t/s} - 1]} \end{aligned}$$

In the limit of $s \to \infty$, using the result on the worksheet, we then have:

$$M_Y(t) \to \frac{\lambda}{\lambda - t}$$

This is the moment generating function of the exponential distribution with rate λ, and completes the result.

4. Let the arrival of students to my office hours follow a Poisson distribution with an average of 2 students arriving each hour. What is the probability that nobody comes during the first hour?

The probability that nobody arrives is just the density of the Poisson distribution with rate $\lambda = 2$ at zero:

$$\mathbb{P}(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$
$$\mathbb{P}(X = 0) = e^{-\lambda}$$

So the answer is e^{-2}, or about 13.5%.