Worksheet 05 (Solutions)

1. The workers in a particular factory are 65% male, 70% married, and 45% married male. If a worker is selected at random from this factory, find the probability that (a) a female worker is married, (b) a married worker is female.

We first build the same table as in the previous worksheet:

Build a table!

	Male	Female	Total
Married	0.45	0.25	0.70
Single	0.20	0.10	0.30
Total	0.65	0.35	1.00

Now, we can calculate the conditional probabilities directly:

$$\begin{aligned}\mathbb{P}(\text{married}|\text{female}) &= \frac{\mathbb{P}(\text{married} \cap \text{female})}{\mathbb{P}(\text{female})} \\ &= \frac{0.25}{0.35} \\ &\approx 0.714.\end{aligned}$$

And:

$$\begin{aligned}\mathbb{P}(\text{female}|\text{married}) &= \frac{\mathbb{P}(\text{married} \cap \text{female})}{\mathbb{P}(\text{married})} \\ &= \frac{0.25}{0.7} \\ &\approx 0.357.\end{aligned}$$

Notice that these are not asking the same question!

2. Psychology majors are required to take two particular courses: Psychology 100 and Psychology 200. It is a rare student indeed who does outstanding work in both courses. It is known that the chances of getting an A in PSY 100 is .4 and the chances of getting an A in PSY 200 is .3, while the chances of getting an A in both courses are .05. What are the chances (a) that a student who get's an A on the first exam will get an A on the second exam and (b) that a student who doesn't get an A on the first exam will get an A on the second exam?

We again grab the table from last class:

	Psych 100: A	Psych 100: Non-A	Total
Psych 200: A	0.05	0.25	0.30
Psych 200: Non-A	0.35	0.35	0.70
Total	0.40	0.60	1.00

And compute the conditional probabilities:

$$\begin{aligned}\mathbb{P}(\text{Psych 200: A}|\text{Psych 100: A}) &= \frac{\mathbb{P}(\text{Psych 200: A} \cap \text{Psych 100: A})}{\mathbb{P}(\text{Psych 100: A})}\\ &= \frac{0.05}{0.40}\\ &= 0.125.\end{aligned}$$

And

$$\begin{aligned}\mathbb{P}(\text{Psych 200: A}|\text{Psych 100: Non-A}) &= \frac{\mathbb{P}(\text{Psych 200: A} \cap \text{Psych 100: Non-A})}{\mathbb{P}(\text{Psych 100: Non-A})}\\ &= \frac{0.25}{0.60}\\ &\approx 0.417.\end{aligned}$$

Paradoxically, doing well on the first exam signals that you will likely do worse in the second exam.

3. You give a friend a letter to mail. He forgets to mail it with probability .2. Given that he mails it, the Post Office delivers it with probability .9. Given that the letter was not delivered, what's the probability that it was not mailed?

Let M be the event that the letter is mailed and D be the event that it is delivered. Notice that $M^c \cap D^c$ is not being deliver and not being mailed, and is therefore equal to M^c (since the post office cannot deliver mail it has not received). Using conditional probabilities we then have:

$$\begin{aligned}\mathbb{P}(M^c|D^c) &= \frac{\mathbb{P}(M^c \cap D^c)}{\mathbb{P}(D^c)}\\ &= \frac{\mathbb{P}(M^c)}{1-\mathbb{P}(D)}\\ &= \frac{\mathbb{P}(M^c)}{1-\mathbb{P}(D|M)\cdot\mathbb{P}M}\\ &= \frac{0.2}{1-0.9*0.8}\\ &\approx 0.714.\end{aligned}$$

In the second to last step I used $\mathbb{P}M = 1 - \mathbb{P}M^c = 0.8$.

4. The Rhetoric Department offers the popular course "Statistically Speaking" two times a year. The final (and only)

exam in the course is a fifty-question multiple-choice test. If rhetoric major Rhett Butler knew the answer to forty of the questions, and selected an answer at random (from among five possible answers) for the remaining ten questions, what is the probability that he actually knew the answer to a particular question that he got right?

This is just another conditional probability question. For a given question, let K be the student knows the answer and C be the event that it is correct. Once again, we see that $K \subseteq C$, which simplifies the calculation:

$$\begin{aligned}
\mathbb{P}(K|C) &= \frac{\mathbb{P}(K \cap C)}{\mathbb{P}(C)} \\
&= \frac{\mathbb{P}(K \cap C)}{\mathbb{P}(C \cap K) + \mathbb{P}(C \cap K^c)} \\
&= \frac{\mathbb{P}(K \cap C)}{\mathbb{P}(C \cap K) + \mathbb{P}(K^c) \cdot \mathbb{P}(C|K^c)} \\
&= \frac{40/50}{40/50 + 10/50 * 0.2} \\
&\approx 0.952.
\end{aligned}$$

5. Two people work independently on deciphering a coded message. Their probabilities of success are 1/2 and 1/4, respectively. What is the probability that the message will be decoded? Try to apply the addition rule and the definition of independence to get to the result.

Let D_1 be the event that the first person is successful and D_2 the event that the third is successful. We want $\mathbb{P}(D_1 \cup D_2)$. Using the addition rule and independence of these events we see:

$$\begin{aligned}
\mathbb{P}(D_1 \cup D_2) &= \mathbb{P}D_1 + \mathbb{P}D_2 - \mathbb{P}(D_1 \cap D_2) \\
&= \mathbb{P}D_1 + \mathbb{P}D_2 - \mathbb{P}(D_1) \cdot \mathbb{P}(D_2) \\
&= 0.5 + 0.25 - 0.25 * 0.5 \\
&= 0.625.
\end{aligned}$$

6. At an upcoming holiday gathering, cousins William and Brenda will play a game, repeatedly and independently, until one of them wins. A given game ends in a tie with probability p_1. The probability that William wins an individual game is p_2, while the probability that Brenda wins an individual game is p_3. Find the probability that William is the eventual winner.

Notice that we don't care how many rounds of this game are played, we only care about what happens in the round where one of the two players win. Let W be the result that William wins and B the result that Brenda wins. We see that:

$$\begin{aligned}\mathbb{P}(W|W \cup B) &= \frac{\mathbb{P}((W \cup B) \cap W)}{\mathbb{P}(W \cup B)} \\ &= \frac{\mathbb{P}(W)}{\mathbb{P}(W \cup B)} \\ &= \frac{p_2}{p_2 + p_3}\end{aligned}$$

You can do the exact same calculation by defining the result T as a tie and conditioning on T^c, which is exactly the same as $W \cup B$.