Worksheet 20 (Solutions)

For each question, compute: (a) joint distribution of the data x_i, (b) the posterior distribution, and (c) a formula for the Bayes estimator of the unknown parameter.

1. Consider the following for the parameter p:

$$\begin{aligned} p &\sim Beta(\alpha, \beta) \\ X|p &\sim Geometric(p) \end{aligned}$$

The (a) joint distribution of X's is given by:

$$\begin{aligned} f(x_1, \ldots, x_n) &= \prod_i (1-p)^{x_i - 1} p \\ &= p^n (1-p)^{\sum_i (x_i) - n} \end{aligned}$$

The (b) posterior distribution is proportional to:

$$\begin{aligned} f(p|X) &\propto f(p) \cdot f(X|p) \\ &=\propto p^{\alpha-1}(1-p)^{\beta-1} \cdot p^n (1-p)^{\sum_i (x_i) - n} \\ &= p^{(\alpha+n)-1} \cdot (1-p)^{\beta + \sum_i (x_i) - n - 1} \qquad =\sim Beta(\alpha + n, \beta + \sum_i (x_i) - n) \end{aligned}$$

And so the (c) Bayes estimator is given by:

$$\widehat{p} = \frac{\alpha + n}{\alpha + \beta + \sum_i (x_i)}$$

2. Consider the following for the parameter μ:

$$\begin{aligned} \mu &\sim N(0, 1) \\ X|\mu &\sim N(\mu, 1) \end{aligned}$$

The (a) joint distribution of X's is given by:

$$\begin{aligned} f(x_1, \ldots, X_n) &= \prod_i \frac{1}{\sqrt{2\pi}} e^{-1/2(X_i - \mu)^2} \\ &= (2\pi)^{-n/2} e^{-1/2 \sum_i (X_i - \mu)^2} \end{aligned}$$

The (b) posterior distribution is proportional to:

$$
\begin{aligned}
f(\mu|X) &\propto f(\mu)\cdot f(X|\mu)\\
&=\propto e^{-1/2(\mu)^2}\cdot e^{-1/2\sum_i (X_i-\mu)^2}\\
&= \exp\left\{-1/2\sum_i (X_i^2 - X_i\mu+\mu^2)+\mu^2\right\}\\
&\propto \exp\left\{-1/2\left((n+1)\mu^2-2\bar{X}\mu n\right)\right\}
\end{aligned}
$$

To finish this, we need to complete the square in the exponent:

$$
\begin{aligned}
f(\mu|X) &\propto f(\mu)\cdot f(X|\mu)\\
&\propto \exp\left\{-1/2(n+1)\left(\mu-\frac{n}{n+1}\cdot\bar{X}\right)^2\right\}
\end{aligned}
$$

And so, we have the following distribution for the posterior:

$$\mu|X \sim N(\frac{n}{n+1}\cdot\bar{X},(n+1)^{-1})$$

For part (c) the mean of the posterior is just:

$$\widehat{\mu}=\frac{n}{n+1}\cdot\bar{X}$$

This is a slightly shrunken version of the MLE.

3. Consider the following for the parameter θ:

$$
\begin{aligned}
\theta &\sim F(\alpha,\beta)\\
X|\theta &\sim U(0,\theta)
\end{aligned}
$$

Where F is the distribution given by:

$$p(\theta)=\frac{\alpha m^\alpha}{\theta^{\alpha+1}},\quad \theta>m.$$

This is a generalization of the Pareto.

The (a) joint distribution of X's is given by:

$$
\begin{aligned}
f(x_1,\dots,x_n) &= \prod_i \frac{1}{\theta}\cdot 1_{x_i\in(0,\theta)}\\
&= \theta^{-n}\cdot 1_{x_i\in(0,\theta)\forall i}
\end{aligned}
$$

The (b) posterior distribution is proportional to:

$$
\begin{aligned}
f(\theta|X) &\propto f(\theta)\cdot f(X|\theta)\\
&\propto \theta^{-n}\cdot\theta^{-(\alpha+1)}\cdot 1_{\theta>m}\cdot 1_{x_i\in(0,\theta)\forall i}\\
&= \theta^{-(\alpha+1+n)}\cdot 1_{\theta>m}\cdot 1_{x_i\in(0,\theta)\forall i} \qquad \sim F(\alpha+n,max(m,x_i))
\end{aligned}
$$

We need to actually calculate the mean of this distribution as we do not already have it. Let $Y \sim F(\alpha, m)$; then:

$$\begin{aligned}
\mathbb{E}Y &= \int_m^\infty y \cdot \frac{\alpha m^\alpha}{y^{\alpha+1}} dy \\
&= \alpha m^\alpha \cdot \int_m^\infty y^{-\alpha} dy \\
&= \alpha m^\alpha \cdot \left[\frac{-y^{-\alpha+1}}{-\alpha+1} \right]_{y=m}^\infty \\
&= \alpha m^\alpha \cdot \frac{m^{-\alpha+1}}{1-\alpha} \\
&= \frac{\alpha m}{\alpha - 1}
\end{aligned}$$

So, letting m^* be the maximum of m and all of the x_i's, the (c) Bayesian estimator in this case is just:

$$\widehat{\theta} = \frac{(\alpha + n)}{\alpha + n - 1)} \cdot m^*$$