Handout 07: Random Variables

Definitions

While the concept of a sample space provides a comprehensive description of the possible outcomes of a random experiment, it turns out that, in a host of applications, the sample space provides more information than we need or want.

Definition 1 (Random Variables) *A random variable is a function whose domain is the sample space of a random experiment and whose range is a subset of the real line.*

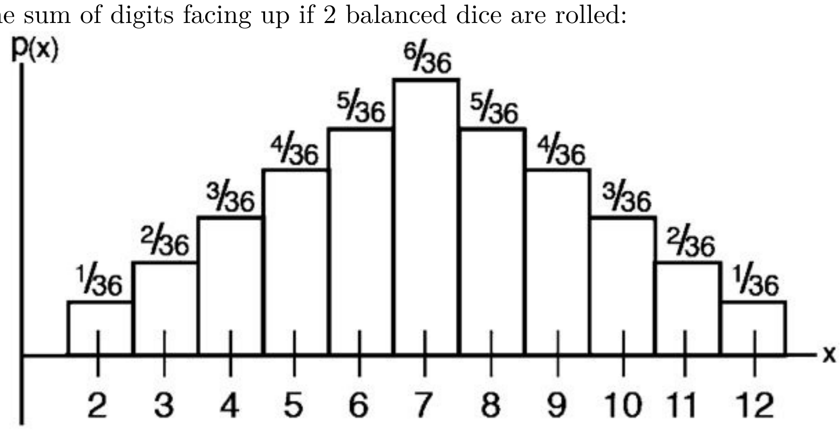
This definition simply states that a random variable is a "mapping" that associates a real number with each simple event in a given sample space. Since different simple events can map onto the same real number, each value of the random variable actually corresponds to a compound event, namely, the set of all simple events which map on to this same number.

It is helpful to have a quantity that defines the analogue of probabilites to the values of a random variables.

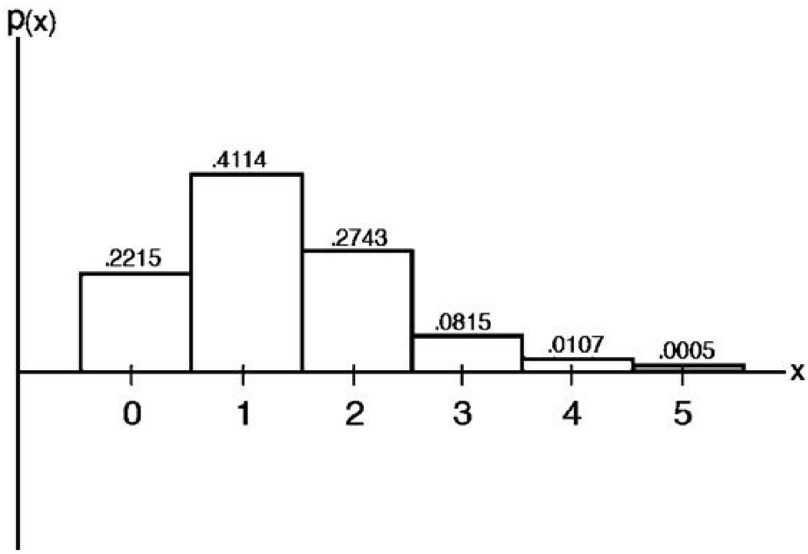
Definition 2 (Probability Mass Function) *A probability mass function (pmf) is a function that gives the probability that a discrete random variable is exactly equal to some value.*

We will use this object when describing the probabilities associated with any discrete random variable. The word "mass" refers to the weight (or probability) assigned to given values of X.

As an example, here is the probability mass function for X equal to the sum of digits facing up if 2 balanced dice are rolled:



And here is the probability mass function for X equal to the number of spades in a random 5-card poker hand.



Mathematical Expectation

One numerical summary of the distribution of a random variable X that is quite meaningful and very widely used is the *mathematical expectation* of X. Often, this numerical summary is referred to as the *expected value* of X and is denoted by $\mathbb{E}(X)$ or simply $\mathbb{E}X$. As the name suggests, this number represents the value we would "expect" X to be if the random experiment from which X is derived were to be carried out. We recognize, of course, that the actual value of the X that we observe might be larger or smaller than $\mathbb{E}X$, but $\mathbb{E}$ is meant to be interpreted as what we would expect X to be, "on the average," in a typical trial of the experiment. The expected value of any discrete random variable X may be computed by the following formula.

Definition 3 (Expected Value) *If X is a discrete random variable with probability mass function $p(x)$, then the expected value of X is given by:*

$$\mathbb{E}X = \sum_{all\ x} x \cdot p(x)$$

The expected value of a random variable X is often called its mean and denoted by μ_X or simply by μ.

Theorem 1 (Transformation of Expected Value) *Let X be a discrete random variable, and let p(x) be its pmf. Let $Y = g(X)$. If $\mathbb{E}g(X) < \infty$, then this expectation may be calculated as*

$$\mathbb{E}g(X) = \sum_{all\ x} g(x) \cdot p(x)$$

We now define a number of other expected values of interest. The first involves the expectation of a useful class of random variables associated with a given random variable X – the class of linear functions of X. The following simple result will often prove useful.

Theorem 2 (Linearity of Expectation) *Let X be a discrete random variable with finite mean $\mathbb{E}X$. Let a and b be real numbers, and let $Y = aX + b$. Then $\mathbb{E}Y = a \cdot \mathbb{E}X + b$.*

Proof. Using the Tranformation of Expected Value theorem, we have that:

$$\begin{aligned}
\mathbb{E}g(X) &= \sum_{\text{all x}} g(x) \cdot p(x) \\
&= \sum_{\text{all x}} (a \cdot x + b) \cdot p(x) \\
&= a \cdot \sum_{\text{all x}} x \cdot p(x) + b \cdot \sum_{\text{all x}} p(x) \\
&= a\mathbb{E}X + b \cdot 1 \\
&= a\mathbb{E}X + b
\end{aligned}$$

Which finished the result $\blacksquare$.

Definition 4 (Variance) *Let X be a discrete random variable with mean μ. The variance of X, denoted interchangeably by V(X), σ_X^2, or when there can be no confusion, σ^2, is defined as the expected value:*

$$\sigma_X^2 = \mathbb{E}(X - \mu)^2.$$

The variance of X (or of the distribution of X) is a measure of how concentrated X is about its mean. Since the variance measures the expected value of the squared distance of a variable X from its mean, large distances from the mean are magnified and will tend to result in a large value for the variance.