Worksheet 02 (Solutions)

1. Assume for the sake of simplicity that the probability that someone is born in a particular month is $1/12$. What is the probability that 6 randomly selected people do not share a birth month? In this case, try to simplify the result to a decimal version.

We can use the formula in the notes to derive this result quickly:

$$\begin{aligned}\mathbb{P}(\text{no match}) &= \frac{P(n,k)}{P^*(n,k)} \\ &= \frac{P(12,6)}{P^*(12,6)} \\ &= \frac{12!}{6!12^6} \\ &\approx 0.222\end{aligned}$$

2. Three students get on a bus to downtown at the same time. The bus makes three stops once it arrives in downtown Richmond. If each student randomly decides which stop to disembark, what is the probability that everyone gets off at the same stop?

Using our naïve counting definition we get:

$$\begin{aligned}\mathbb{P}(event) &= \frac{\#\{\text{ways get off same stop}\}}{\#\{\text{ways of getting off bus}\}} \\ &= \frac{3}{3^3} \\ &\approx 0.111.\end{aligned}$$

3. Assume for the sake of simplicity that the probability that someone has a particular birthday is $1/365$. What is the probability that 28 randomly chosen people share a birthday?

We can use the counting formula once again:

$$\begin{aligned}\mathbb{P}(\text{all same birthday}) &= \frac{\#\{\text{ways same birthday}\}}{\#\{\text{ways of not all same birthday}\}} \\ &= \frac{365}{365^2 8} \\ &= 365^{-27}\end{aligned}$$

4. Write the formula for the log-probability that k randomly

selected elements from a set of n items (with replacement) have no 'matches'. In writing your result, use the *log-factorial* function $lf(n) = \log(n!)$. Why might taking logs be useful?

We can use the matching formula for generic n and k to get:

$$\begin{aligned}
\log\left(\mathbb{P}(\text{no match})\right) &= \frac{P(n,k)}{P^*(n,k)} \\
&= \log\left(\frac{n!}{(n-k)!n^k}\right) \\
&= \log(n!) - \log((n-k)!) - k \cdot \log(n) \\
&= lf(n) - lf(n-k) - k * \log(n)
\end{aligned}$$

Logs are useful because they do not blow up like factorials and powers. Most programming languages have a function for computer the $lf(\cdot)$ function. This is how I got the answer for the previous question, in fact.

5. Queen Elizabeth I of England (1533– 1603) was fond of picnics. It is said that William Shakespeare attended several and that Sir Walter Raleigh attended most of them. An Elizabethan picnic consisted of the Queen and seven guests. The eight of them would sit on the ground in a circle, each buoyed by an overstuffed goose down pillow. The picnics were legendary for the sparkling conversation, but they were also famous as a culinary experience. Because the Queen always served herself first and then passed food to the right, sitting at the Queen's right was considered the greatest honor, and sitting as close to her right as possible was considered desirable. The phrases "right-hand man" and "left-handed compliment" might have originated at these picnics. How many different seating orders were possible at these picnics? Should Ben Jonson have been miffed because he never got closer than fourth place at the three picnics he attended?

The first question is easily answered since, after the queen has been seated, one merely needs to count the number of permutations possible for the seven places to her right. Thus, there are $7! = 5040$ possible seating orders. If Ben Jonson is placed in any one of the last four places, there are $6! = 720$ possible orderings of the other guests; there are thus $4 \cdot 720 = 2880$ such seatings. If seating order had been determined at random, the chances that Jonson would get an unfavorable seating on a given occasion is $2880/5040 = 4/7$ (which will agree with your intuition), and the chances it would happen on three independent seatings is $(4/7)^3 = .187$. That likelihood is not so small that Jonson could justifiably conclude that his seating was discriminatory.

6. Kindergarten teacher Monica Speller gave each of her students a box of twenty crayons on the first day of class. Miss Speller (an unfortunate moniker for a teacher) has told her students that she would like them to draw a new picture every day using exactly three different colors. She also wants the three colors chosen to be a different combination each day. Given that her kindergarten class will meet 150 times during the course of the year, is she asking the impossible? Anyone who knows a five-year-old will know that she is! But suppose she is willing to replace crayons as needed, and will personally keep track of the color combinations used so far. Are there at least 150 combinations possible?

There are plenty of options, $\binom{20}{3} = 1140$ to be exact.

7. Let's take a look at the lottery (in an imaginary principality referred to as "the State"). After the public has selected its numbers, the State, using a time-honored procedure involving air-blown ping pong balls, will identify the six numbers against which a player's chosen numbers must be matched. The State's numbers may be thought of as having been chosen at random from the integers 1, 2,...,52,53. The number of possible six-somes that can be chosen is

$$\binom{53}{6} = \frac{53!}{6!47!} = 22,957,480.$$

You win five dollars when exactly three of your choices match up with the State's. What is the probability that you get exactly 3 numbers to match?

The probability of exactly 3 matches is given by:

$$\frac{\binom{6}{3} \cdot \binom{47}{3}}{\binom{53}{6}} \approx 0.01413.$$

8. Back to poker, what is the probability that a randomly dealt hand of 5 cards yields a full house? That is, three cards of the same type and two cards of a different type.

The basic rule of counting will come into play in this example because we will take the view that putting together a full house is a four-stage experiment: pick a value you will be taking three cards from, take three cards of this value, pick a value you will be taking two cards of, take two cards of this value. Notice that the two values that are picked play different roles; if you pick a king in stage one and a four in stage three, you get a different full house than if the order was reversed. Since we are assuming that cards are dealt randomly, so that all the possible poker hands are equally likely, we can represent the probability of a full house as the ratio of the number of ways of getting a full house to the total number of poker hands possible. Visualizing the situation as a four-stage experiment, we thus obtain.

$$\begin{aligned}\mathbb{P}(\text{full house}) &= \frac{13 \cdot \binom{4}{3} \cdot 12\binom{4}{2}}{\binom{52}{5}} \\ &\approx 0.00144.\end{aligned}$$

To emphasize the fact that we are keeping track of the order of the two values we pick, let's do the computation again, but in a different way. Think of the process of forming a full house as a three-stage experiment: pick two ordered values, take three cards of the first value, take two cards of the second value. Visualizing the process this way leads to

$$\mathbb{P}(\text{full house}) = \frac{P(13,2) \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{52}{5}}$$

which, of course, is the same answer as obtained above. The most common mistake made when computing probabilities in this type of problem is the mistake of ignoring order. If you were to replace P(13,2) by $\binom{13}{2}$, you would be off by a factor of two. In not 2 accounting for order, you would be implicitly equating full houses having three kings and two fours with full houses having three fours and two kings.

9. Now, what is the probability of getting a three of a kind in poker, but not a full house or a four of a kind? That is, you have three cards of equal types by the other two are of different types.

It is helpful to think of assembling the hand as a five-stage experiment. We will need to choose a value (among 2, 3,..., J, Q, K, A) to get three cards from, then choose three cards of that value, then choose two additional values from which a single card will be drawn and then choose one of the four cards from each of these latter two values. I will need to keep track of order, at least partially. Simply choosing the three numbers that will be in my hand will not do, as I also need to know precisely which of them corresponds to my threesome. So I will proceed as follows. I will first pick a number that I will be drawing three cards from. I can choose that number in 13 ways. I will then choose those three cards, which I can do in $\binom{4}{3}$ ways. Then I will choose the two numbers (from the remaining twelve) that I will get 3 singletons from. I don't need to keep track of the order of the latter two numbers because these numbers will not be treated differently. I can choose these two numbers in $\binom{12}{2}$ ways. Then I will choose one card with each of these numbers. Taking these actions in sequence, I get

$$\begin{aligned}\mathbb{P}(\text{three of a kind}) &= \frac{13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot \binom{4}{1} \cdot \binom{4}{1}}{\binom{52}{5}} \\ &\approx 0.02113.\end{aligned}$$

10. Suppose that two evenly matched teams (say team A and team B) make it to the baseball World Series. The series ends as soon as one of the teams has won four games. Thus, it can end as early as the 4th game (a "sweep") or as late as the 7th game, with one team winning its fourth game compared to the other team's three wins. What's the probability that the series ends in 4 games, 5 games, 6 games, 7 games?

This is just an application of the naïve definition of probability again. For a sweep, we have just two ways that sweeps can occur (4 straight wins by A and four straight by B):

$$\begin{aligned}\mathbb{P}(\text{sweep}) &= \frac{\#\{\text{ways get getting sweep}\}}{\#\{\text{total outcomes}\}} \\ &= \frac{2}{2^4} = 0.125.\end{aligned}$$

How many ways can a game end in the fifth game? Well, this is equal to the number of ways that either team can win 3 of the first 4 games. If we think of selecting the wins for team A, this is $\binom{4}{3}$ (if team A is doing better) and $\binom{4}{1}$ in the other direction: $\binom{4}{1}$:

$$\begin{aligned}\mathbb{P}(\text{five games}) &= \frac{\binom{4}{3} + \binom{4}{1}}{2^5} \\ &= \frac{8}{2^5} = 0.25.\end{aligned}$$

And for six it's the same logic with $\binom{5}{3}$ and $\binom{5}{2}$:

$$\begin{aligned}\mathbb{P}(\text{six games}) &= \frac{\binom{5}{3} + \binom{5}{2}}{2^6} \\ &= 0.3125.\end{aligned}$$

And finally for seven it is just $\binom{6}{3}$ and $\binom{6}{3}$:

$$\begin{aligned}\mathbb{P}(\text{seven games}) &= \frac{\binom{6}{3} + \binom{6}{3}}{2^7} \\ &= 0.3125.\end{aligned}$$