Handout 02: More Counting

Today we continue our study of counting. The two types of counting problems we will treat here in detail differ from each other in one essential respect: in one type of problem, you are concerned with the order in which items occur, while in the other type of problem, you are not. Two license plates are not the same just because they contain the same numbers and letters. The order in which the numbers and letters are arranged is important. On the other hand, two ham and cheese omelets are certainly the same whether the ham was added before the cheese or after it.

In many of the situations we will face, we will speak of forming groups of objects selected from a population of n "distinguishable" objects. The group of balls, numbered 1 to 15, used in the game of pool is an example of a population of distinguishable objects. A collection of fifteen red marbles is not. When the items in your population can be separately identified, each subgroup of a given size can be distinguished from every other. We use the word *permutation* in reference to an ordered collection of objects, and the word *combination* for collections in which order is irrelevant.

Definition 1 *A permutation of a collection of n objects is a fixed ordering of these objects.*

The six permutations of the numbers 1, 2, and 3 are:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), \text{ and } (3, 2, 1).$$

Why would you want to consider these six orderings to be different from one another?

Consider the process of drawing k objects from a population of n distinguishable objects, keeping track of order. The number of different ways this can be done, first when the draws are made with replacement and secondly when the draws are made without replacement, is identified in the two fundamental counting formulas below:[1]

[1] The first you should have identified yourself from the first worksheet.

Theorem 1 *The number of ways of obtaining a permutation of k objects drawn with replacement from a population of n objects is denoted by P^* (n, k), and is given by:*

$$P^*(n, k) = n^k.$$

Theorem 2 *The number of ways of obtaining a permutation of k objects drawn without replacement from a population of n objects is denoted by $P(n, k)$, and is given by:*

$$P(n, k) = n \cdot (n-1) \cdots (n-k-1) = \frac{n!}{(n-k)!}$$

There is one important class of problems that involves both of these counting formulas, that is, both P and P^*. Determining the probability of k randomly selected people not sharing the same birth month is one example of such as problem. In "matching problems" like this, we are generally interested in the probability that all of the items drawn from a given population are different. We can calculate this by:

$$P(\text{no match}) = \frac{P(n,k)}{P^*(n,k)}.$$

We can thus identify the desired probabilities fairly quickly.

Suppose you have a population of n items, and you wish to assemble a sub-population of k distinct items from that population. In sampling from the population, you obviously would choose k items *without replacement*, as you do not wish to allow for possible duplications in separate draws. We will therefore restrict our attention to sampling without replacement, which is the only case of any practical interest. We use the term *combination* when referring to a set of objects for which order is irrelevant. How many possible combinations are there of n things taken k at a time? The answer is contained in the following:

Theorem 3 *The number of ways of drawing a combination of k objects without replacement from a population of n objects is denoted by C(n, k), and is given by:*

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Proof. If the order of the draw was important, the number of possibilities would be $P(n,k)$. In the case of combinations, we are overcounting by a factor of how many permutations exist for k items. This is given by $P(k,k)$. So then:

$$\begin{aligned} C(n,k) &= \frac{P(n,k)}{P(k,k)} \\ &= \frac{n!}{(n-k)!} \cdot \frac{(k-k)!}{k!} \\ &= \frac{n!}{k!(n-k)!} \end{aligned}$$

By using the fact that 0! is equal to 1; this finishes the result ■.

A commonly used alternative to the function $C(n,k)$ is a notation called a binomial coefficent:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This new symbol is pronounced "n choose k," and this will be our standard notation for combinations as we proceed. You will notice some inherent symmetry in the combinations formula; it is clear, for example, that

$$\binom{n}{k} = \binom{n}{n-k}.$$

This identity is obvious algebraically when one replaces the two symbols by what they stand for, but it also follows from the fact that choosing k items among n to be in a set is equivalent to choosing the $(n-k)$ items to be left out.

These results will get us a long way in our study of discrete probability. There are, of course, many other counting theorems. If you are interested, I suggest you find a good combinatorics book (or better yet, course), which extends on the basic ideas explored here.[2]

[2] A good place to start is the *Twelve-fold way* page on Wikipedia, which explains 12 different counting theorems, of which we have only shown 3.