Worksheet 10 (Solutions)

1. Let Let X be a random variable defined over the set $[0, b]$ for some $b > 0$ with a density function:

$$f(x) = C \cdot (b - x)$$

For some constant $C > 0$. Find the constant C that makes this a valid density function.

We need the integral of the density to sum to 1, so we get:

$$\begin{aligned}
1 &= \int_{x=0}^{b} C \cdot (b - x)dx \\
&= C \cdot \left[bx - \frac{x^2}{2}\right]_{x=0}^{b} \\
&= C \cdot \left[b^2 - \frac{b^2}{2}\right] \\
&= \frac{b^2 \cdot C}{2}
\end{aligned}$$

So $C = \frac{2}{b^2}$.

2. What are $\mathbb{E}X$ and $Var(X)$ for X as defined in question 1?

The expected value is given by:

$$\begin{aligned}
\mathbb{E}X &= \frac{2}{b^2} \cdot \int_{x=0}^{b} x \cdot (b - x)dx \\
&= \frac{2}{b^2} \cdot \left[\frac{bx^2}{2} - \frac{x^3}{3}\right]_{x=0}^{b} \\
&= \frac{2}{b^2} \cdot \left[\frac{b^3}{2} - \frac{b^3}{3}\right] \\
&= \frac{b}{3}.
\end{aligned}$$

The expected square is given by:

$$\begin{aligned}
\mathbb{E}X^2 &= \frac{2}{b^2} \cdot \int_{x=0}^{b} x^2 \cdot (b - x)dx \\
&= \frac{2}{b^2} \cdot \left[\frac{bx^3}{3} - \frac{x^4}{4}\right]_{x=0}^{b} \\
&= \frac{2}{b^2} \cdot \left[\frac{b^4}{3} - \frac{b^4}{4}\right] \\
&= \frac{b^2}{6}.
\end{aligned}$$

And so the variance is:

$$\begin{aligned} Var(X) &= \mathbb{E}X^2 - (\mathbb{E}X)^2 \\ &= \frac{b^2}{6} - \frac{b^2}{9} \\ &= \frac{b^2}{18}. \end{aligned}$$

3. Let X be a continuous random variable with density $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and some fixed $\lambda > 0$. This is called the exponential distribution, which we can write $X \sim Exp(\lambda)$. What is the cumulative distribution $F(x)$? Find $\mathbb{P}[x \geq 1]$.

The cumulative distribution is given by:

$$\begin{aligned} F(z) &= \int_0^z f(x)dx \\ &= \int_0^z \lambda \cdot e^{-\lambda \cdot x} dx \\ &= \lambda \cdot \int_0^z e^{-\lambda \cdot x} dx \\ &= \lambda \cdot \left[-\lambda^{-1} e^{-\lambda x}\right]_{x=0}^{z} \\ &= 1 - e^{-\lambda z} \end{aligned}$$

The desired probability comes from the CDF:[1]

$$\begin{aligned} \mathbb{P}[x \geq 1] &= 1 - \mathbb{P}[x < 1] \\ &= 1 - [1 - e^{-\lambda}] \\ &= e^{-\lambda}. \end{aligned}$$

[1] With continuous random variables, we do not need to be careful about the different between $\geq$ and $>$ because the probability that the variable will take on an exact value at the endpoint is zero.

4. Find the MGF of the exponential distribution for $t < \lambda$.

The MFG is given by:

$$\begin{aligned} m_X(t) &= \mathbb{E}e^{tX} \\ &= \lambda \cdot \int_0^\infty e^{tx} \cdot e^{-\lambda x} dx \\ &= \lambda \cdot \int_0^\infty e^{(t-\lambda)x} dx \\ &= \lambda \cdot \left[\frac{1}{t-\lambda} \cdot e^{(t-\lambda)x}\right]_{x=0}^{\infty} \\ &= \frac{\lambda}{t-\lambda} \cdot \left[1 - e^{(t-\lambda)\cdot\infty}\right] \\ &= \frac{\lambda}{\lambda - t}. \end{aligned}$$

Where in the last step we used the fact that $t < \lambda$ to argue that $e^{(t-\lambda)\cdot\infty}$ should be zero.

5. If $X \sim Exp(\lambda)$, find $\mathbb{E}X$ and $Var(X)$.

These come from the moment generating function. The first two derivatives are:

$$\begin{aligned}\frac{\partial}{\partial t} m_X(t) &= \frac{\partial}{\partial t}\left(\frac{\lambda}{\lambda - t}\right)\\ &= \lambda \cdot (-1) \cdot (\lambda - t)^{-2} \cdot (-1)\\ &= \lambda \cdot (\lambda - t)^{-2}\end{aligned}$$

And,

$$\begin{aligned}\frac{\partial^2}{\partial^2 t} m_X(t) &= \frac{\partial}{\partial t}\left(\lambda \cdot (\lambda - t)^{-2}\right)\\ &= \lambda \cdot (-2) \cdot (\lambda - t)^{-3} \cdot (-1)\\ &= 2\lambda \cdot (\lambda - t)^{-3}\end{aligned}$$

Evaluating at $t = 0$ we get:

$$\begin{aligned}\mathbb{E}X &= \lambda^{-1}\\ \mathbb{E}X^2 &= 2\lambda^{-2}\end{aligned}$$

And, finally, the variance formula yields,

$$\begin{aligned}Var(X) &= 2\lambda^{-2} - \left(\lambda^{-1}\right)^2\\ &= \lambda^{-2}.\end{aligned}$$