Worksheet 19 (Solutions)

1. Assume that $X_1, \ldots, X_5 \sim_{i.i.d.} N(\mu, 3^2)$ and you observe the values: 1, 3, 4, 6, 10. Find a confidence interval at the 95% confidence level for μ. Hint: You can let s be the real standard deviation here.

Plugging into our equation we have:

$$\begin{aligned}\bar{X} \pm \frac{s}{\sqrt{n}} z_{\alpha/2} &= 4.8 \pm \frac{3}{\sqrt{5}} \cdot 1.96 \\ &= 4.8 \pm 2.63 \\ &= 2.17 \text{ to } 7.43\end{aligned}$$

2. Assume that $X_1, \ldots, X_5 \sim_{i.i.d.} N(\mu, \sigma^2)$ and you observe the values: 1, 3, 4, 6, 10. Find a confidence interval at the 95% confidence level for μ. How does it compare to the one from question 1?

Here, the estimator s^2 will be:

$$\begin{aligned}s^2 &= \frac{1}{n-1} \sum_i (X_i - \bar{X}) \\ &= \frac{1}{4} \sum (X_i - 4.8) \\ &= \frac{46.8}{4} = 11.7\end{aligned}$$

Plugging into our equation we have:

$$\begin{aligned}\bar{X} \pm \frac{s}{\sqrt{n}} z_{\alpha/2} &= 4.8 \pm \sqrt{\frac{11.7}{5}} \cdot 1.96 \\ &= 4.8 \pm 3.00 \\ &= 1.80 \text{ to } 7.80\end{aligned}$$

So the confidence interval has the same center but is wider when the standard deviation is unknown.

3. Assume that $X_1, \ldots, X_5 \sim_{i.i.d.} N(\mu, \sigma^2)$ and you observe the values: 1, 3, 4, 6, 10. Find a confidence interval at the 80% confidence level for μ. How does it compare to the one from question 2?

Looking at the z-table, we see that the cut-off is 1.282. Plugging into our equation once again we have:

$$\begin{aligned}\bar{X} \pm \frac{s}{\sqrt{n}} z_{\alpha/2} &= 4.8 \pm \sqrt{\frac{11.7}{5}} \cdot 1.282 \\ &= 4.8 \pm 1.96 \\ &= 2.84 \text{ to } 6.76\end{aligned}$$

So the confidence interval has the same center but is narrower with a lower confidence level.

4. Sometimes we want a one-sided confidence interval of the form:

$$\mathbb{P}\left[b < \mu\right] = \alpha$$

Using the same data as from question 1 (with the known variance), find the value of b to get a one-sided confidence interval at the 95% confidence level.

We see that this comes from manipulating the probability statements just as we did on the worksheet:

$$\mathbb{P}\left[\frac{\bar{X} - \mu}{s/\sqrt{n}} \leq z_\alpha\right] \approx 1 - \alpha$$

Looking at the table then, we need a value of 1.65. This gives:

$$\begin{aligned}\text{less than } \bar{X} + \frac{s}{\sqrt{n}} z_{\alpha/2} &= \text{ less than } 4.8 + \sqrt{\frac{11.7}{5}} \cdot 1.65 \\ &= \text{ less than } 5.68\end{aligned}$$

5. Consider estimating the probability that a coin comes up heads when tossed. How many tosses are needed to guarantee that a 95% confidence interval will be smaller than 0.1? Hint: We know that the variance is maximized when p is equal to 0.5.

My original version of this was overly complicated by using the $n-1$ form of the s^2 estimator. Instead, let us assume that $X \sim Bin(n,p)$ and that $Y = X/n$. Then, for a large enough n we know that $Y \sim N(p, p(1-p)/n)$. We don't know the exact value of p, but the worst case scenario (i.e., the widest interval) has $p = 0.5$ so we can just plug that into the formula. So, let $W \sim N(0, 0.25/n)$. How large does n have to be so that:

$$\begin{aligned}\mathbb{P}\left[|W| \geq 0.05\right] &= 0.05 \\ \mathbb{P}\left[W \geq 0.05\right] &= 0.025 \\ \mathbb{P}\left[W \leq 0.05\right] &= 0.975 \\ \mathbb{P}\left[Z \leq 0.1 \cdot \sqrt{n}\right] &= 0.975\end{aligned}$$

And we know then that this is true when $0.2 \cdot \sqrt{n}$ is equal to 1.96:

$$\begin{aligned} 1.96 &= 0.1\sqrt{n} \\ n &= 384.16 \end{aligned}$$