Spring 2016 **Statistics 330** **Exam II**
(100 points)

Please write your first and last name and Section here:

Name ______________________________

Instructions:

- Partial credit will be given only if you show your work.
- Reason out your answers. In many cases, a line or two of justification is enough.
- The questions are roughly in the order in which the material is presented in class, so they are not necessarily ordered easiest to hardest.
- If you get stuck on one, it may be a good idea to move on and come back to that question at the end.
- You may use your prepared notes (1 page, both sides) and a calculator only.

1. The number of computer shutdowns during any month has a Poisson distribution, averaging 0.25 shutdowns per month.

   (a) What is the probability of exactly 1 computer shutdown in a month? (6 points)

   Answer:

   $$\frac{e^{-\frac{1}{4}}\left(\frac{1}{4}\right)^1}{1!} = 0.195$$

   (b) What is the probability of at least 2 computer shutdowns during the next year? (6 points)

   Answer: # Shutdowns in the next year $\sim$ Pois$(12 \cdot 0.25)$ = Pois(3) because this is the sum of 12 i.i.d. Pois(0.25) random variables.

   $$1 - \left(\frac{e^{-3}\,(3)^0}{0!} + \frac{e^{-3}\,(3)^1}{1!}\right) = 0.801$$

   (c) During the next year, what is the probability of at least 3 months (out of 12) with exactly 1 computer shutdown in each? Assume the number of shut downs per month are independent. (6 points)

   Answer: First, note that this is a binomial-type problem, where the "coin flips" are months and the "probability of success" is $\frac{e^{-\frac{1}{4}}\left(\frac{1}{4}\right)^1}{1!} = 0.195$.
   So the probability is:

   $$1-\left[\binom{12}{0}(0.195)^0(0.805)^{12} + \binom{12}{1}(0.195)^1(0.805)^{11} + \binom{12}{2}(0.195)^2(0.805)^{10}\right] = 0.424$$

2. Suppose X and Y are two random variables on the same sample space Ω and their joint pmf is given by this table:

		X		
		1	2	3
	2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
Y	3	$\frac{1}{6}$	0	$\frac{1}{6}$
	4	0	$\frac{1}{3}$	0

(a) Show that X and Y are dependent. (8 points)

Answer: $P(X=2)\cdot P(Y=3) = \frac{1}{2}\cdot\frac{1}{3} \neq 0 = P(X=2, Y=3)$

(b) Give a probability table (like we have above for X and Y) for random variables U and V that have the same marginal distributions as X and Y but are independent. (8 points)

Answer: Using the table above, note that $X = 1$, 2, and 3 with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively. $Y = 2$, 3, and 4 all with probability $\frac{1}{3}$. Now construct U and V having the same marginal distributions as X and Y, and multiply the respective elements together to get the joint distribution table below.

V $\backslash U$	1	2	3
2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
3	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
4	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

3. The random number generator on a certain calculator is not well chosen in that values it generates are not adequately described by a distribution uniform on the interval (0,1). Let X = the next value generated by the calculator's random number generator. Suppose the following probability density function (pdf) is a more appropriate model for X:

$$f_X(x) = \begin{cases} c(5-x) & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $c = \frac{2}{9}$ makes $f_X(x)$ a valid pdf. (7 points)

Answer:

$$1 = \int_0^1 c(5-x)dx = c[5x - \frac{x^2}{2}]\bigg|_0^1 = \frac{9c}{2}$$

so $c = \frac{2}{9}$.

(b) What is $E(X)$? (8 points)

Answer:

$$E[X] = \int_0^1 \frac{2x}{9}(5-x)dx = \frac{10x^2}{18} - \frac{2x^3}{27}\bigg|_0^1 = \frac{13}{27} = 0.4814$$

(c) Find $P(X < .85)$. Compare this value to what you would get if the calculator truly generated uniform(0,1) values, i.e. give the difference in probabilities. (10 points)

Answer: Integrating $f_X(x)$ from 0 to .85 gives .864. For a uniform(0,1), the probability is .85. So the difference is .014.

4. Cows are sent to the slaughterhouse at an average rate of 5 per hour starting at some time $t = 0$. Assume we can model the time between cows being sent using the exponential distribution.

   (a) What is the probability that it takes more than 20 minutes for the first cow to be sent? (8 points)

   Answer: Use the cdf of the exponential distribution, and the fact that 20 mins is one third of an hour:

$$1 - [1 - e^{-5 \cdot \frac{1}{3}}] = 0.189$$

   (b) You arrive at time $t = 2$ hours and are told that 11 cows have already been sent. How long should you expect to wait until the next cow enters the slaughterhouse? (8 points)

   Answer: The expected value of an Expo(5) distribution is $\frac{1}{5}$, so we should expect to wait one fifth of an hour, or 12 minutes.

   (c) Starting from some time $t = 0$, how long do you have to wait to observe the first cow being sent with a probability of 95%? (8 points)

   Answer:

$$\begin{aligned} 1 - e^{-5t} &= 0.95 \\ e^{-5t} &= 0.05 \\ t &= \frac{\log(0.05)}{-5} \\ &= 0.599 \end{aligned}$$

   So we need to wait 0.599 hours, or about 35.94 minutes.

5. The resistance of an assembly of several resistors connected in series is the sum of the resistances of the individual resistors. Suppose that a large lot of resistors has mean resistance $\mu = 9.91$ ohms and standard deviation of resistances $\sigma = .08$ ohms. Suppose that 30 resistors are randomly selected from this lot and connected in series. Let $S =$ resistance of the assembly. Then $S = X_1 + \ldots + X_{30}$, where $X_i =$ resistance of the ith resistor.

   (a) Give the mean and standard deviation of S. (8 points)

   Answer:

   $E(S) = n\mu = 30(9.91) = 297.3$.
   $Var(S) = n\sigma^2 = 30(.08^2)$, so the standard deviation of S is $\sqrt{30(.08^2)} = \sqrt{30}(.08) = .438$

   (b) Approximate the probability that resistance of the assembly exceeds 298.2 ohms. State any results you use. (9 points)

   Answer: By the CLT, S is approximately normally distributed with mean and variance given above.
   Thus, $P(S > 298.2) \approx P(Z > \frac{298.2-297.3}{.438}) = P(Z > 2.05) = .0202$.