Fall 2014 **Statistics 330** **Exam III**
(100 points)

Please write your first and last name here:

Name ______________________________

Instructions:

- Partial credit will be given only if you show your work.
- Reason out your answers. In many cases, a line or two of justification is enough.
- The questions are roughly in the order in which the material is presented in class, so they are not necessarily ordered easiest to hardest.
- If you get stuck on one, it may be a good idea to move on and come back to that question at the end.
- You may use your prepared notes (1 page, both sides) and a calculator only.

1. For any given hour, a particular website has a probability of 0.99 of being up next hour, if it is currently up this hour, and probability of 0.9 of being down next hour, if it is currently down this hour.

   (a) What is the 1-step transition matrix? (5 points)

   Answer:

$$P = \begin{array}{c c} & \begin{array}{cc} U & D \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{pmatrix} 0.99 & 0.01 \\ 0.1 & 0.9 \end{pmatrix} \end{array}$$

   (b) Suppose the website is up now, what is the probability it will be up in 2 hours? (8 points)

   Answer:

$$P \cdot P = \begin{array}{c c} & \begin{array}{cc} U & D \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{pmatrix} 0.9811 & 0.0189 \\ 0.1890 & 0.8110 \end{pmatrix} \end{array}$$

   So the probability it will be up in two hours is 0.9811.

   (c) At some point in the distant future, what is the probability it will be up? (7 points)

   Answer: Since $\pi P = \pi$ defines the stationary distribution for π, we know

$$\pi_U = 0.99\pi_U + 0.1\pi_D$$
$$\pi_D = 0.01\pi_U + 0.9\pi_D$$

   Using one of the above equations and the equation $\pi_U + \pi_D = 1$, we can find that $\pi_U = 10/11 = 0.909$.

2. The mailing list at `jobs@bayesian.org` receives, on average, 50 spam messages per day. **For each part below, state the distribution you are using with the appropriate parameter values!!** (5 points each)

   (a) What is the expected number of spam messages after one week?

   Answer: Let $X(7)$ be the number of spam messages after one week (7 days). Then $X(7) \sim Po(50 \times 7)$ and $E[X(7)] = 350$.

   (b) What is the expected time between spam messages?

   Answer: Let I_j be the inter arrival time between the $j-1$ and j spam message. Then $I_j \overset{ind}{\sim} Exp(50)$ and $E[I_j] = 1/50 = 0.02$ days $= 0.48$ hours.

   (c) What is the probability that the next spam message occurs within one hour?

   Answer: We have the same distribution as in the previous question. $P(I_1 < 1/24) = 1 - e^{-50\times 1/24} = 88\%$.

   (d) What is the expected waiting time until 5 spam messages arrive?

   Answer: Let O_j be the occurrence time of the jth spam message. Then $O_j \sim Ga(5, 50)$ and $E[O_j] = 5/50 = 1/10$ days $= 2.4$ hours.

3. A library has only one librarian to help borrowers check out their books. Borrowers arrive according to a Poisson process at a rate of 5 per hour. The service time for each customer follows an Exponential distribution with expectation 0.1 hour. If the librarian is serving someone, arriving borrowers will stay in a line.

   (a) What is this queuing system? (5 points)

   Answer: It is a M/M/1 queuing system.

   (b) What is the traffic intensity? (5 points)

   Answer: $\lambda = 5$ per hour and $\mu = 1/0.1 = 10$ per hour. Thus the traffic intensity $a = \lambda/\mu = 0.5$.

   (c) What is the (steady state) probability that there is no one in the *queue*? (5 points)

   Answer: That means $P(X(t) = 0 \text{ or } 1) = p_0 + p_1$. $p_0 = 1 - a = 1 - 0.5 = 0.5$ and $p_1 = p_0 a = 0.25$.

   (d) What is the (steady state) expected waiting time to check out (time in the queue plus time in service)? (5 points)

   Answer:

$$L = \frac{a}{1-a} = \frac{0.5}{1-0.5} = 1$$

   Expected waiting time in the system $W = L/\lambda = 1/5 = 0.2$ hour.

4. A computing server operates as an M/M/1/3 queue, with jobs arriving as a Poisson process with rate of 3 per day. The service time of a job is an exponential random variable expectation 1/6 per day.

   (a) Find the (steady state) probability that the server is busy (i.e. the server is not idle). (7 points)

   Answer: $K = 3$, $\lambda = 3$, $\mu = 6$ $a = \lambda/\mu = 3/6 = 0.5$

$$S = \frac{1 - a^{K+1}}{1 - a} = \frac{1 - 0.5^4}{1 - 0.5} = 1.875$$

   $p_0 = 1/S = 0.5333$. $P(\text{server is busy}) = 1 - p_0 = 0.4667$.

   (b) What is the (steady state) average number of jobs in the server (including jobs in service and in the queue)? (7 points)

   Answer:

$$L = \frac{a}{1 - a} - \frac{(K+1)a^{K+1}}{1 - a^{K+1}} = 1 - \frac{(4)(0.5^4)}{1 - 0.5^4} = 0.7333$$

   (c) What is the (steady state) rate of arriving packets being discarded? (6 points)

   Answer:

$$p_3 = a^3 p_0 = 0.5^3(0.5333) = 0.06667$$

   The required rate is $p_3 \lambda = 0.2$.

5. A gas station has two car wash systems. Each system can clean one car at a time and its expected cleaning time is 1 minute. Customers arrive at a rate of 0.5 per minute. Assume both inter-arrival times and service times are independently exponetially distributed. If both car wash systems are being used, the arriving customer stays in one queue.

   (a) Find the (steady state) probability that there is no job in the system. (5 points)

   Answer: It is a M/M/2 queuing system with $\lambda = 0.5$, $\mu = 1$, $c = 2$, $a = \lambda/\mu = 0.5$ and $\rho = a/c = 0.25$.

$$p_0 = \left(\sum_{k=0}^{c-1} \frac{a^k}{k!} + \frac{a^c}{c!} \frac{1}{1-\rho} \right)^{-1} = \left(1 + 0.5 + \frac{0.5^2}{2!} \cdot \frac{1}{1-0.25} \right)^{-1} = 3/5 = 0.6.$$

   (b) Find the (steady state) average length of the queue. (4 points)

   Answer:

$$L_q = p_0 \cdot \frac{a^c}{c!} \cdot \frac{\rho}{(1-\rho)^2} = 0.03333.$$

   (c) Find the average service time. (4 points)

   Answer:

$$W_s = \frac{1}{\mu} = 1$$

   (d) Find the (steady state) average number of jobs in the system. (7 points)

   Answer: First,

$$W_q = L_q/\lambda = (0.03333)/(0.5) = 0.06667.$$

   Then,

$$W = W_s + W_q = 0.06667 + 1 = 1.067.$$

   Thus,

$$L = W\lambda = (1.067)(0.5) = 0.5335.$$