Spring 2016 **Statistics 330** **Exam III**
(100 points)

Please write your first and last name here:

Name ____

Instructions:

- Partial credit will be given only if you show your work.
- Reason out your answers. In many cases, a line or two of justification is enough.
- The questions are roughly in the order in which the material is presented in class, so they are not necessarily ordered easiest to hardest.
- If you get stuck on one, it may be a good idea to move on and come back to that question at the end.
- You may use your prepared notes (1 page, both sides) and a calculator only.

1. Suppose that for a particular breed of dog, an offspring of a black dog is black with probability .7 and brown with probability .3. An offspring of a brown dog is black with probability .1 and brown with probability .9.

   (a) Write down the 1-step transition matrix. (6 points)

   Answer:

$$P = \begin{array}{c c} & \begin{array}{cc} \mathit{Black} & \mathit{Brown} \end{array} \\ \begin{array}{c} \mathit{Black} \\ \mathit{Brown} \end{array} & \left(\begin{array}{cc} .7 & .3 \\ .1 & .9 \end{array} \right) \end{array}$$

   (b) Write down the 2-step transition matrix, $P^{(2)}$. (6 points)

   Answer:

$$P \cdot P = \begin{array}{c c} & \begin{array}{cc} \mathit{Black} & \mathit{Brown} \end{array} \\ \begin{array}{c} \mathit{Black} \\ \mathit{Brown} \end{array} & \left(\begin{array}{cc} .52 & .48 \\ .16 & .84 \end{array} \right) \end{array}$$

   (c) Suppose a dog is brown. What is the probability that its grandchild will be a brown dog? (6 points)

   Answer: We want $P_{22}^{(2)} = .84$

(d) Give the steady state distribution of the dog color by solving $\pi P = \pi$ (6 points)

Answer: Since $\pi P = \pi$ defines the stationary distribution for π, we know

$$
\begin{aligned}
.7\pi_{Black} + .1\pi_{Brown} &= \pi_{Black} \\
.3\pi_{Brown} + .9\pi_{Brown} &= \pi_{Brown}
\end{aligned}
$$

Using one of the above equations and the equation $\pi_{Black} + \pi_{Brown} = 1$, we can find that the steady state distribution is $\pi_{Black} = 1/4$ and $\pi_{Brown} = 3/4$

2. Suppose we have the following 3 by 3 transition matrix for states 1, 2, and 3.

$$
P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{pmatrix} .4 & .3 & a \\ .5 & b & .5 \\ c & .2 & .6 \end{pmatrix} \end{array}
$$

(a) Give the values for a, b and c. (6 points)

Answer: Since the rows have to add to 1, a = .3, b = 0, c = .2

(b) The 2-step transition matrix $P^{(2)}$ is:

$$
P^{(2)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}
\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{pmatrix} .37 & .18 & .45 \\ .30 & .25 & .45 \\ .30 & .18 & .52 \end{pmatrix} \end{array}
$$

Give the probability of going from state 2 to state 1 in two steps. (4 points)

Answer: We want $P^{(2)}_{23}$ which is .3

3. Suppose the mailing list at a website receives, on average, 50 spam messages per day. Assume the arrival of spam messages follows a homogeneous poisson process.

   (a) What is the expected number of spam messages after one week? (6 points)

   Answer: Let X_7 be the number of spam messages after one week (7 days). Then $X_7 \sim Po(50 \times 7)$ and $E[X_7] = 350$.

   (b) What is the expected time between spam messages? (6 points)

   Answer: Let I_j be the inter arrival time between the $j-1$ and j spam message. Then $I_j \overset{ind}{\sim} Exp(50)$ and $E[I_j] = 1/50 = 0.02$ days $= 0.48$ hours.

   (c) What is the probability that the next spam message occurs within one hour? (6 points)

   Answer: We have the same distribution as in the previous question. $P(I_1 < 1/24) = 1 - e^{-50\times 1/24} = 88\%$.

   (d) What is the expected waiting time until 5 spam messages arrive? (6 points)

   Answer: Let O_j be the occurrence time of the jth spam message. Then $O_j \sim Ga(5, 50)$ and $E[O_j] = 5/50 = 1/10$ days $= 2.4$ hours.

4. Customers come to a teller's window according to a Poisson process with a rate of 10 customers every 25 minutes. Service times are Exponential. The average service takes 2 minutes. Using what you know about M/M/1 queuing systems, compute:

   (a) the traffic intensity value, a. (5 points)

   Answer:

$$a = \frac{\lambda}{\mu} = \frac{\frac{10}{25}}{\frac{1}{2}} = 0.8$$

   (b) the average number of customers in the system **and** the average number of customers waiting in a line. (7 points)

   Answer:

$$L = \frac{a}{1-a} = 4 \text{ customers}$$

$$L_q = \frac{a^2}{1-a} = 3.2 \text{ customers}$$

   (c) the fraction of time when the teller is busy with a customer. (5 points)

   Answer:

$$1 - \pi_0 = 1 - (1-a) = a = 0.8$$

   (d) the fraction of time when the teller is busy and at least 2 other customers are waiting in a line. (7 points)

   Answer:

$$1 - \pi_0 - \pi_1 - \pi_2 = a - a(1-a) - a^2(1-a) = a^3 = 0.512$$

5. A computing server operates as an M/M/1/3 queue, with jobs arriving as a Poisson process at a rate of 3 per day. The service time of a job is exponentially distributed with a mean of 1/6 days.

   (a) Find the (steady state) probability that the server is busy (i.e. the server is not idle). (6 points)

   Answer: We have $K = 3$, $\lambda = 3$, $\mu = 6$, and $a = \frac{\lambda}{\mu} = \frac{1}{2}$.

$$S = \frac{1 - a^{K+1}}{1 - a} = \frac{1 - 0.5^4}{1 - 0.5} = 1.875$$

$$\pi_0 = \frac{1}{S} = 0.5333 \rightarrow P(\text{server is busy}) = 1 - \pi_0 = 0.4667$$

   (b) What is the (steady state) average number of jobs in the server (including jobs in service and in the queue)? (6 points)

   Answer:

$$L = \frac{a}{1 - a} - \frac{(K+1)a^{K+1}}{1 - a^{K+1}} = 0.7333$$

   (c) What is the (steady state) rate of arriving jobs being discarded? (6 points)

   Answer:

$$\pi_3 = a^3 \pi_0 = 0.0667$$

   Therefore, the rate is $\pi_3 \lambda = 0.2$