Please write your first and last name here:

Name ____

Instructions:

- Partial credit will be given only if you show your work.
- Reason out your answers. In many cases, a line or two of justification is enough.
- The questions are roughly in the order in which the material is presented in class, so they are not necessarily ordered easiest to hardest.
- If you get stuck on one, it may be a good idea to move on and come back to that question at the end.
- You may use your prepared notes (1 page, both sides) and a calculator only.

1. Consider two events $A, B \subset \Omega$. If $P(A) = \frac{1}{3}$ and $P(\bar{B}) = \frac{1}{4}$, can A and B be disjoint? If not, why not?

   Answer: They cannot be disjoint. If they were,

$$P(A \cup B) = P(A) + P(B) = \frac{1}{3} + \frac{3}{4} = \frac{13}{12} > 1$$

2. A small company has 12 employees. 7 are programmers and 5 are graphic designers. Suppose a team of 4 is going to be selected for a project.

   (a) How many possible teams can be selected? (5 points)

   Answer:

$$\binom{12}{4} = 495$$

   (b) If a team is selected randomly, what is the probability that there are exactly 2 programmers and 2 graphic designers on the team? (7 points)

   Answer:

$$\frac{\binom{7}{2} \times \binom{5}{2}}{\binom{12}{4}} = .4242$$

   (c) If a team is selected randomly, what is the probability that there is at least 1 graphic designer on the team? (7 points)

   Answer:

$$1 - \frac{\binom{7}{4} \times \binom{5}{0}}{\binom{12}{4}} = .9292$$

3. Suppose that 5% of men and .25% of women are color-blind, and that we are randomly drawing people from a group with equal numbers of males and females.

   (a) What is the probability that a randomly chosen person is both male and color-blind?

   Answer: Let C denote the event that the person chosen is color-blind, and M denote the event that the person is male.

   $$P(C, M) = P(C|M) \cdot P(M) = 0.05 \cdot 0.5 = 0.025$$

   (b) Suppose a person chosen at random is color-blind. What is the probability that this person is male?

   Answer:

   $$P(M|C) = \frac{P(C, M)}{P(C)} = \frac{0.025}{P(C, M) + P(C, F)} = \frac{0.025}{0.025 + (0.0025 \cdot 0.5)} \approx 0.952$$

4. Let $c_1, \ldots, c_k$ denote the k components in a serial system. Assume the k components operate independently, and denote $P(c_j \text{ works})$ as p_j. Derive the formula for the reliability of the system in terms of the probabilities p_j. Clearly state which axioms or results you are using at each stage of the development.
Note: Giving only the formula here will result in *no credit*; we want you to show that you understand where the formula comes from and how it is derived from basic facts about probabilities.

Answer:

$$\begin{aligned}
P(\text{system works}) &= P(\text{every component works}) && (1)\\
&= P(\{c_1 \text{ works}\} \cap \ldots \cap \{c_k \text{ works}\}) && (2)\\
&= \Pi_{j=1}^{k} P(\{c_j \text{ works}\}) && (3)\\
&= \Pi_{j=1}^{k} p_j && (4)
\end{aligned}$$

where equation (1) holds because this is a serial system, equation (3) holds by the independence of the components (and thus the events $\{c_j \text{ works}\}$), and equation (4) holds by the definition of p_j.

5. In a large manufacturing run of PCs, let X be the number of minor flaws in a randomly chosen PC. Suppose the probability mass function for X is given by:

x	0	1	2	3	4
$p(x)$	.45	.35	.10	.05	.05

(a) What are the expectation and variance of X? (10 points)

Answer:

$$\begin{aligned} E(X) &= \sum_x xp(x) \\ &= (0)(.45) + (1)(.35) + (2)(.1) + (3)(.05) + (4)(.05) = .9 \\ Var(X) &= \sum_x (x - E(X))^2 p(X) \\ &= (0 - .9)^2(.45) + (1 - .9)^2(.35) + (2 - .9)^2(.1) \\ &\quad + (3 - .9)^2(.05) + (4 - .9)^2(.05) \\ &= 1.19 \end{aligned}$$

or $Var(X) = E(X^2) - (EX)^2 = 2 - (.9)^2 = 1.19$.

Suppose a PC is labeled "defective" if there are at least two minor flaws in it. For a randomly chosen PC we can define a random variable Y_i as:

$$Y_i = \begin{cases} 1 & X \geq 2 \\ 0 & X < 2 \end{cases}$$

(b) For a randomly chosen PC, what is $P(Y_i = 1)$? (5 points)

Answer:

$$P(Y_i = 1) = P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) = .2$$

Suppose ten PCs are randomly selected. Let Y = the number of PCs that are "defective". Then $Y = \sum_{i=1}^{10} Y_i$, where Y_i is defined above. Assume PCs are independent of each other in terms of flaws.

(c) What distribution does Y follow? Give the name and relevant values. (5 points)

Answer: Y follows a binomial distribution with $n = 10$ and $p = .2$

(d) What is the probability that in ten randomly chosen PCs, there is more than 1 "defective" PC? (5 points)

Answer: We want $P(Y > 1) = 1 - P(Y \leq 1) = .624$