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Static spatio-temporal models

The etiology of the process is limited

Hierarchical structure

[data | process, parameter] **First** Observation model
[process | parameter] **Second** Process model
[parameter] **Third** Parameter model

Gelfand (2012)

[data | process, parameter] **First** Observation distribution

$$\{Z(s,t) : s \in D_s, t \in D_t\}$$ continuous continuous

$$\{Z_t(s) : s \in D_s, t \in \{0,\dots,T\}\}$$ continuous discrete

$$\{Z_{s,t} : s \in \{0,\dots,S\}, t \in \{0,\dots,T\}\}$$ discrete discrete

$$Z_t(s) = H(y_t(s)) + \varepsilon_t(s)$$

Mapping function

$\varepsilon_t(s)$ Gaussian, independent in space and time

$\varepsilon_{t,s}$ Poisson, independent in space and time

[data | process, parameter] **First** Observation distribution

spatio-temporal observation

spatio-temporal realization

$$Z_t(s) = H(y_t(s)) + \varepsilon_t(s)$$

Mapping function

$\varepsilon_t(s)$ Gaussian, independent in space and time

$\varepsilon_{t,s}$ Poisson, independent in space and time [process | parameter] **Second** Process distribution

spatio-temporal realization — mean field — residual field

$$g(y_t(s)) = \mu_t(s) + v_t(s)$$

link function

temporal basis

$$\mu_t(s) = \alpha + \sum_{l=1}^{L} \gamma_l x_{l,t}(s) \quad + \quad \sum_{k=1}^{m} \phi_k(s)\beta_k(t)$$

spatio-temporal covariates

spatial field

Spatio-temporal field

[process | parameter] **Second** Process distribution

spatio-temporal realization

residual field

$$g(y_t(s)) = \mu_t(s) + v_t(s)$$

link function

$\varepsilon_t(s)$ Gaussian $g = 1$ $v_t(s) \sim N$ Gaussian process

$\varepsilon_{t,s}$ Poisson $g = \log$ $v_t(s) \sim N$ log-Gaussian Cox process mean field

$$g(y_t(s)) = \mu_t(s) + v_t(s)$$

Several options

$\{\beta_k(t)\}$

temporal basis

$\{\phi_k(s)\}$

spatial field

Orthogonal polynomials
Harmonic functions
Piecewise linear bases
Wavelets
Splines
Empirical orthogonal functions
Moran's I bases
Fourier bases

residual
field

$$g(y_t(s)) = \mu_t(s) + v_t(s)$$

$v_t(s)$ Gives the spatio-temporal variance-covariance structure

$$v_t(s) \sim N(0, \Sigma_v(\theta_s))$$

residual
field

$$g(y_t(s)) = \mu_t(s) + v_t(s)$$

$v_t(s)$ Gives the spatio-temporal variance-covariance structure

$$v_t(s) \sim N(0, \Sigma_v(\theta_s))$$

$\Sigma_v(\theta_s)$ This has N^2 elements residual field

$$g(y_t(s)) = \mu_t(s) + v_t(s)$$

$v_t(s)$ Gives the spatio-temporal variance-covariance structure

$$v_t(s) \sim N(0, \Sigma_v(\theta_s))$$

$\Sigma_v(\theta_s)$ This is usually assumed stationary residual field

$$g(y_t(s)) = \mu_t(s) + v_t(s)$$

$v_t(s)$ Gives the spatio-temporal variance-covariance structure

In general

$v(s,t)$ **stationary** $\mathrm{cov}(s,t,s+h,t+l) = C(h,l)$

$y(s,t)$

stationary $\mu(s,t) = 0, g = 1$ Kriging

weakly stationary $\mu(s,t) = a, g = 1$

residual
field

$$g(y_t(s)) = \mu_t(s) + v_t(s)$$

$v_t(s)$ Gives the spatio-temporal variance-covariance structure

Also it can be assumed **separable**

$$\mathrm{cov}(s,t,s+h,t+l) = C(h,l) = C_S(h)C_T(l)$$

it is often difficult to see the difference between separability and non-separability in realizations from a process

R packages

`spacetime`	Visualization tools
`SpatioTemporal`	Gaussian linear models
`spTimer`	Gaussian linear models with time hierarchy
`lgcp`	Multivariate log-Gaussian Cox processes

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Dynamic spatio-temporal models

The current values of the process at a location evolve from past values of the process at various locations

$y(s,t)$ needs to be expressed in terms of its changes in *s* and *t*

Example PDE Reaction-diffusion equation in continuous 1-D *s* and *t*

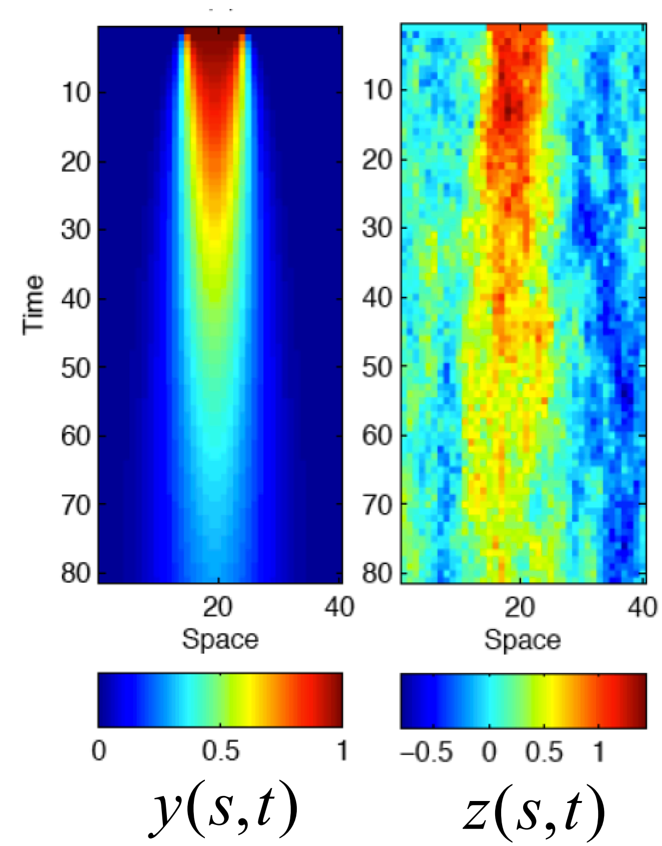
$$\frac{\partial y(s,t)}{\partial t} = \beta \frac{\partial^2 y(s,t)}{\partial s^2} - \alpha y(s,t)$$

The current values of the process at a location evolve from past values of the process at various locations

$y(s,t)$ needs to be expressed in terms of its changes in *s* and *t*

Example PDE

$$\frac{\partial y(s,t)}{\partial t} = \beta \frac{\partial^2 y(s,t)}{\partial s^2} - \alpha y(s,t)$$



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Example PDE

$$\frac{\partial y(s,t)}{\partial t} = \beta \frac{\partial^2 y(s,t)}{\partial s^2} - \alpha y(s,t)$$

Hierarchical structure

[data | process, parameter]	**First** Observation model
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[parameter]	**Third** Parameter model

Gelfand (2012)

spatio-temporal mean residual
realization field field

$$g(y_t(s)) = \mu_{t,t-1,\ldots}(s) + \nu_t(s)$$

mean field

residual field

$$g(y_t(s)) = \mu_{t,t-1,\ldots}(s) + v_t(s)$$

$$\mu_{t,t-1,t-2,\ldots}(s) = f(y_{t-1}, y_{t-2}, \ldots, X_t, X_{t-1}, \ldots, s)$$

$$\mu_{t,t-1}(s) = f(y_{t-1}, X_t(s))$$

mean
field

$$g(y_t(s)) = \mu_{t,t-1,\ldots}(s) + v_t(s)$$

$$\mu_{t,t-1,t-2,\ldots}(s) = f(y_{t-1}, y_{t-2}, \ldots, X_t, X_{t-1}, \ldots, s)$$

$$\mu_{t,t-1}(s) = f(y_{t-1}, X_t(s))$$

spatial
correlation

$$\mu_{t,t-1}(s) = \alpha + \sum_{l=1}^{L} \gamma_l x_{l,t}(s) + \gamma \int m_s(r) y_{t-1}\, dr$$

temporal
correlation

$$g(y_t(s)) = \mu_{t,t-1,\ldots}(s) + v_t(s)$$

$v_t(s)$ Usually assumed stationary but not separable

Space Kalman filter model

mean field

$$y_t(s) = \mu_{t,t-1}(s) + v_t(s)$$

$$\mu_{t,t-1}(s) = \sum_{l=1}^{L} \gamma_l x_{l,t}(s) + K(s) \cdot u_t$$

$$u_t = G u_{t-1} + \eta_t$$

p-dimensional spatial field

$$\eta_t \sim N(0, \Sigma_\eta)$$

$$\text{cov}(s,t,s+h,t+l) = C_S(h) C_T(l) = \sigma_v^2 C_S(h)$$

$$C_S(h) = \exp(-\theta h)$$

Spectral model used in Stochastic advection-diffusion processes

mean
field

$$g(y_t(s)) = \mu_{t,t-1,\ldots}(s) + v_t(s)$$

$$\mu_{t,t-1}(s) = \alpha + \sum_{l=1}^{L} \gamma_l x_{l,t+1}(s) + \phi(\beta_{t+1}(k^c, k^s))$$

$$\beta_{t+1}(k^c, k^s) = g(\beta_t(k^c, k^s)) + v_t(k^c, k^s) \; v_t(k^c, k^s) \sim N(0, Q)$$

ϕ Fourier transform

$\{\beta_{t+1}\}$ Fourier dynamic field

g propagator

Q innovation covariance matrix

R packages

`Stem`	Spatial Kalman filter model
`spate`	Stochastic advection-diffusion models
`crawl`	Correlated random walks
`ctmcmove`	Continuous-time discrete-space animal movement models
`moveHMM`	Hidden Markov models