1 Linear Kalman Filter

- System Model:
$$x_t = Fx_{t-1} + Gu_t + w_t$$
  - F is the transition matrix
  - x_{t-1} is the state vector
  - G is a transition matrix, u_t is a input (control) vector
  - w_t is process noise vector, $w_t \sim$ N(0, Q)
- Observation Model:
$$z_t = Hx_t + v_t$$
  - H is a transition matrix, x_t is the state vector
  - v_t is the observation noise vector, $v_t \sim$ N(0, R)
- Assumptions
  - w and v are uncorrelated
  - Initial system state, x_0 is uncorrelated to w and v
  - The initial conditions, $\hat{x}_{0|0} = E(x_0), P_{0|0} = E((\hat{x}_{0|0} - x_0)(\hat{x}_{0|0} - x_0)^T)$ are known
- Notation Note: $\hat{x}_{t|i}$ is estimation of x at time t based on all the observations up to and including time i
- First, we would like an estimate of our state vector x_t, using $z_1, z_2, ...z_{t-1}$, our observations up to and including time $t-1$:
  - Predicted state estimate:
$$\hat{x}_{t|t-1} = F\hat{x}_{t-1|t-1} + Gu_{t-1}$$
  - Predicted error covariance matrix:
$$P_{t|t-1} = FP_{t-1|t-1}F^T + Q$$
- Then at time t we observe a new observation, z_t. We would like to use this observation to update our original state estimate $\hat{x}_{t|t-1}$:
  - Update the state estimate:
$$\hat{x}_{t|t} = (I - K_tH)\hat{x}_{t|t-1} + K_tz_t$$
  - Update the error covariance:
$$P_{t|t} = (I - K_tH)P_{t|t-1}$$
  - Where the Kalman Gain, K_t, is given by
$$K_t = P_{t|t-1}H^T[HP_{t|t-1}H^T + R]^{-1}$$
- Other Notation
  - Innovation Gain at time k:
$$z_t = H\hat{x}_{t|t-1}$$
  - Innovation Covariance matrix at time k:
$$S = HP_{t|t-1}H^T + R$$

2 Extended Kalman Filter

- System Model:
$$x_t = f(x_{t-1}, u_t) + w_t$$
  - f is an arbitrary nonlinear function
  - u_t is a input (control) vector
  - w_t is process noise vector, $w_t \sim \text{N}(0,\, Q_t)$
- Observation Model:
$$z_t = h(x_t) + v_t$$
  - h is an arbitrary nonlinear function, x_t is the state vector
  - v_t is the observation noise vector, $v_t \sim \text{N}(0,\, R_t)$
  - for covariance estimation, the nonlinear terms in system and observation model are linearized using first order Taylor series approximation
- Prediction Equations (Priori):
  - State equation
$$\hat{x}_{t|t-1} = f(\hat{x}_{t-1|t-1}, u_t)$$
  - Covariance equation
$$P_{t|t-1} = F_{t-1} P_{t-1|t-1} F_{t-1}^T + Q_t$$
  - where F_{t-1} is
$$F_{t-1} = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{t-1|t-1}, u_t}$$
- Update Equations (Posteriori):
  - Kalman Gain
$$K_t = P_{t|t-1} H_t^T \left[H_t P_{t|t-1} H^T + R_t\right]^{-1}$$
  - State equation
$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t \left(z_t - h(X_{t|t-1}\right)$$
  - Covariance
$$P_{t|t} = (1 - K_t H_t)\, P_{t|t-1}$$
  - where H_t is
$$H_t = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{t|t-1}}$$
- EKF is not optimal because of the error due to linear approximation of nonlinearity

2.1 Non-Additive Noise

- Model Equation
$$x_t = f(x_{t-1}, u_t, w_t)$$
$$z_t = h(x_t, v_t)$$
  - Where $w_t \sim \text{N}(0,\, \tilde{Q}_t)$ and $v_t \sim \text{N}(0,\, \tilde{R}_t)$

- Noise terms can also be linearized using first order taylor series approximation and the additive noise covariance terms in EKF equations become

$$Q_t = L_t \tilde{Q}_t L_t^T$$

$$R_t = M_t \tilde{R}_t M_t^T$$

  - where L and M are

$$L_t = \left.\frac{\partial f}{\partial w}\right|_{w_t}$$

$$M_t = \left.\frac{\partial f}{\partial v}\right|_{v_t}$$

3 Parameter Estimation

- Model:

$$y = G(x, w)$$

  - where x is the input of the process, y is the output of the process and G is the nonlinear map parametrized by the vector w.
  - let $\{x_k,\ \tilde{y}_k\ \}$ k=1,2..,n are the pairs consisting of inputs and measured outputs.
  - The error is $e_k = \tilde{y}_k - G(x_k, w)$
- State space model can be rewritten to estimate the parameter w

$$w_k = w_{k-1} + u_k$$

$$y_k = G(x_k, w_k) + e_k$$

  - In the context of EKF w is treated as stationary process driven by noise u and y is the nonlinear observation
  - $\hat{w} = E[w|y]$ is the optimal value of the parameter w, which can be recursively estimated using EKF in section 2.
  - Convergence depends on the choice of u_k
- A similar framework can be used for dual estimation of state and parameter (see reference).

4 Kalman Filtering in R

	`FKF`	`dlm`	`dse`	`KFAS`
System Model	$\alpha_t = T\alpha_{t-1} + d + H\eta_t$ $\eta_t \sim N(0, I)$ $H\eta_t \sim N(0, HH^T)$	$\alpha_t = G\alpha_{t-1} + w_t$ $w_t \sim N(0, W)$	$\alpha_t = F\alpha_{t-1} + Gu_t + Q\eta_t$ $\eta_t \sim N(0, I)$ $Q\eta_t \sim N(0, QQ^T)$	$\alpha_t = T\alpha_{t-1} + R\eta_t$ $\eta_t \sim N(0, Q)$ $R\eta_t \sim N(0, RQR^T)$
Observation Model	$y_t = Z\alpha_t + c + G\epsilon_t$ $\epsilon_t \sim N(0, I)$ $G\epsilon_t \sim N(0, GG^T)$	$y_t = F\alpha_t + v_t$ $v_t \sim N(0, V)$	$z_t = H\alpha_t + R\epsilon_t$ $\epsilon_t \sim N(0, I)$ $R\epsilon_t \sim N(0, RR^T)$	$y_t = Z\alpha_t + \epsilon_t$ $\epsilon_t \sim N(0, H)$
Function	`fkf(a0, P0, dt, ct, Tt, Zt, HHt, GGt, yt)`	`dlm(m0, C0, FF, V, GG, W)`	`SS (F, G, H, Q, R, z0, P0)`	`KFAS(a1, P1, Z, T, H, Q, R, y, u)`
Function Arguments	`a0` = initial mean `P0` = initial covariance `dt, ct` = d, c `Tt, Zt` = Z, T `HHt, GGt` = HH^T, GG^T	`m0` = initial mean `C0` = initial covariance `FF, GG` = F, G `V, W` = V, W	`z0` = initial mean `P0` = initial covariance `F, G, H` = F, G, H `Q, R` = Q, R	`a1` = initial mean `P1` = initial covariance `T, Z` = T, Z `H, Q, R` = H, Q, R `y` = data `u` = parameters in Non-Gaussian case
Non-Gaussian				✓
Time Variant	✓	✓		✓
Missing Values	✓	✓		✓

5 Exercises

5.1 `dlm`

Required File

`dlmlab.rmd`

Exercise

Go through the `dlmlab.r` file to understand how the `dlm` package works with a simple linear model. To familiarize yourself with the different `R` packages one can use for Kalman Filtering, try repeating this example with one of the following packages: `FKF, dse, KFAS`

5.2 EKF - Tracking

Required File

`trackinglab.r`

Background Information

A logistic growth model with rate r and carrying capacity k can be written as

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{k}\right)$$

with initial guess p_0, the logistic growth model can be analytically solved as.

$$p = \frac{kp_0 \exp(rt)}{k + p_0(\exp(rt) - 1)}$$

A population data is modeled based on the analytical solution of logistic growth model with additive random process error

$$p_t = \frac{kp_{t-1} \exp(r\Delta t)}{k + p_{t-1}(\exp(r\Delta t) - 1)} + v$$

An extended Kalman Filter is used to track the population given that the variance of process and observation error is known. state space is assumed as $x = [r\ p]^T$ and observation model is given as

$$z = [0\ 1][r\ p]^T + w$$

In the given code the sample data is synthetically generated and tracking algorithm is implemented.

Exercise 1

In the sample code only population in the state space is assumed to have additive process error. Modify the code such that rate of population growth also has additive process error.

Exercise 2

Implement Kalman filter tracking algorithm for the same logistic population growth by using Euler's explicit time stepping scheme.

$$p_t = p_{t-1} + \Delta t\left(rp_{t-1}\left(1 - \frac{p_{t-1}}{k}\right) + v\right)$$

5.3 EKF - Parameter Estimation

Required File

`paramestimationlab.r`

Background Information

A sample code for estimating parameter a in the equation

$$y = a^2 x^2 + x + 1$$

is given in the sample code. The data is synthetically generated to observe the error of estimated parameter form the true value of the parameter.

Exercise 1

Estimate the initial velocity (v_0) of the projectile for the given data in projectile.txt using initial angle as $\theta = pi/4$. Projectile motion equation is

$$x = v_0 t \cos(\theta)$$

$$y = v_0 t \sin(\theta) - \frac{gt^2}{2}$$

6 References

6.1 Linear Kalman Filter

- An Introduction to the Kalman Filter
- Derivation of Kalman Filter
- Wikipedia Article on Kalman Filter

6.2 Extended Kalman Filter

- Yaakov Bar-Shalom, X. Rong Li, and Thiagalingam Kirubarajan. *Estimation with Applications to Tracking and Navigation.* Wiley-Interscience, New York, 1st edition, June 2001
- Wikipedia Article on Extended Kalman Filter

6.3 Parameter Estimation

- Simon Haykin, editor. *Kalman Filtering and Neural Networks.* Wiley-Interscience, New York, 1st edition, October 2001
- Eric Wan, Ronell Van Der Merwe, and others. The unscented Kalman filter for nonlinear estimation. In *Adaptive Systems for Signal Processing, Communications, and Control Symposium 2000. AS-SPCC. The IEEE 2000*, pages 153–158. IEEE, 2000

6.4 `R` Resource

- Kalman Filtering in R
- `FKF`
- `dlm`
- `dse`
- `KFAS`

References

[1] Yaakov Bar-Shalom, X. Rong Li, and Thiagalingam Kirubarajan. *Estimation with Applications to Tracking and Navigation*. Wiley-Interscience, New York, 1st edition, June 2001.

[2] Simon Haykin, editor. *Kalman Filtering and Neural Networks*. Wiley-Interscience, New York, 1st edition, October 2001.

[3] Eric Wan, Ronell Van Der Merwe, and others. The unscented Kalman filter for nonlinear estimation. In *Adaptive Systems for Signal Processing, Communications, and Control Symposium 2000. AS-SPCC. The IEEE 2000*, pages 153–158. IEEE, 2000.