1 Linear Kalman Filter

- System Model:
$$x_t = Fx_{t-1} + Gu_t + w_t$$
  - F is the transition matrix
  - x_{t-1} is the state vector
  - G is a transition matrix, u_t is a input (control) vector
  - w_t is process noise vector, $w_t \sim$ N(0, Q)
- Observation Model:
$$z_t = Hx_t + v_t$$
  - H is a transition matrix, x_t is the state vector
  - v_t is the observation noise vector, $v_t \sim$ N(0, R)
- Assumptions
  - w and v are uncorrelated
  - Initial system state, x_0 is uncorrelated to w and v
  - The initial conditions, $\hat{x}_{0|0} = E(x_0), P_{0|0} = E((\hat{x}_{0|0} - x_0)(\hat{x}_{0|0} - x_0)^T)$ are known
- Notation Note: $\hat{x}_{t|i}$ is estimation of x at time t based on all the observations up to and including time i
- First, we would like an estimate of our state vector x_t, using $z_1, z_2, ...z_{t-1}$, our observations upto and including time $t-1$:
  - Predicted state estimate:
$$\hat{x}_{t|t-1} = F\hat{x}_{t-1|t-1} + Gu_{t-1}$$
  - Predicted error covariance matrix:
$$P_{t|t-1} = FP_{t-1|t-1}F^T + Q$$
- Then at time t we observe a new observation, z_t. We would like to use this observation to update our original state estimate $\hat{x}_{t|t-1}$:
  - Update the state estimate:
$$\hat{x}_{t|t} = (I - K_tH)\hat{x}_{t|t-1} + K_tz_t$$
  - Update the error covariance:
$$P_{t|t} = (I - K_tH)P_{t|t-1}$$
  - Where the Kalman Gain, K_t, is given by
$$K_t = P_{t|t-1}H^T[HP_{t|t-1}H^T + R]^{-1}$$
- Other Notation
  - Innovation Gain at time k:
$$z_t = H\hat{x}_{t|t-1}$$
  - Innovation Covariance matrix at time k:
$$S = HP_{t|t-1}H^T + R$$

2 Extended Kalman Filter

- System Model:
$$x_t = f(x_{t-1}, u_t) + w_t$$
  - f is an arbitrary nonlinear function
  - u_t is a input (control) vector
  - w_t is process noise vector, $w_t \sim \mathrm{N}(0,\, Q_t)$
- Observation Model:
$$z_t = h(x_t) + v_t$$
  - h is an arbitrary nonlinear function, x_t is the state vector
  - v_t is the observation noise vector, $v_t \sim \mathrm{N}(0,\, R_t)$
  - for covariance estimation, the nonlinear terms in system and observation model are linearized using first order Taylor series approximation
- Prediction Equations (Priori):
  - State equation
$$\hat{x}_{t|t-1} = f(\hat{x}_{t-1|t-1}, u_t)$$
  - Covariance equation
$$P_{t|t-1} = F_{t-1} P_{t-1|t-1} F_{t-1}^T + Q_t$$
  - where F_{t-1} is
$$F_{t-1} = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}_{t-1|t-1}, u_t}$$
- Update Equations (Posteriori):
  - Kalman Gain
$$K_t = P_{t|t-1} H_t^T \left[H_t P_{t|t-1} H^T + R_t \right]^{-1}$$
  - State equation
$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t \left(z_t - h(X_{t|t-1} \right)$$
  - Covariance
$$P_{t|t} = (1 - K_t H_t)\, P_{t|t-1}$$
  - where H_t is
$$H_t = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}_{t|t-1}}$$
- EKF is not optimal because of the error due to linear approximation of nonlinearity

2.1 Non-Additive Noise

- Model Equation
$$x_t = f(x_{t-1}, u_t, w_t)$$
$$z_t = h(x_t, v_t)$$
  - Where $w_t \sim \mathrm{N}(0,\, \tilde{Q}_t)$ and $v_t \sim \mathrm{N}(0,\, \tilde{R}_t)$

- Noise terms can also be linearized using first order taylor series approximation and the additive noise covariance terms in EKF equations become
$$Q_t = L_t \tilde{Q}_t L_t^T$$
$$R_t = M_t \tilde{R}_t M_t^T$$
  - where L and M are
$$L_t = \left.\frac{\partial f}{\partial w}\right|_{w_t}$$
$$M_t = \left.\frac{\partial f}{\partial v}\right|_{v_t}$$

3 Parameter Estimation

- Model:
$$y = G(x, w)$$
  - where x is the input of the process, y is the output of the process and G is the nonlinear map parametrized by the vector w.
  - let $\{x_k,\ \tilde{y}_k\ \}$ k=1,2..,n are the pairs consisting of inputs and measured outputs.
  - The error is $e_k = \tilde{y}_k - G(x_k, w)$
- State space model can be rewritten to estimate the parameter w
$$w_k = w_{k-1} + u_k$$
$$y_k = G(x_k, w_k) + e_k$$
  - In the context of EKF w is treated as stationary process driven by noise u and y is the nonlinear observation
  - $\hat{w} = E[w|y]$ is the optimal value of the parameter w, which can be recursively estimated using EKF in section 2.
  - Convergence depends on the choice of u_k
- A similar framework can be used for dual estimation of state and parameter (see reference).

4 References

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