Particle Filtering

STATS 744

Oct. 22nd, 2015

Outline

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Introduction

- Usually, prior knowledge about the phenomenon being modelled is available.
- Inferences on unknown quantities are based on the posterior.
- It is necessary to update the posterior distribution when new data is available.
- New observations come sequentially.
- For example, tracking an aircraft using radar, estimating the volatility of financial instruments using stock market data.

- If the data are modelled by a linear Gaussian state-space model, we can derive an exact analytical expression to compute the evolving sequence of posterior distribution by Kalman filter.
- If the data are modelled as partially observed, finite state-space Markov chain, we can obtain the analytical solution by hidden Markov model.
- Sequential Monte Carlo methods are a set of simulation-based methods which provide a convenient and attractive approach to computing the posterior distribution.

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Importance Sampling

- Suppose one is interested in evaluating the expected value
$$E_\pi(f(X)) = \int f(x)\pi(x)\,dx.$$
- Sampling from π is hard but we can easily sample from g.
- If g is an *importance density* having property that $g(x) = 0$ implies $\pi(x) = 0$, then one can write
$$E_\pi(f(X)) = \int f(x)\frac{\pi(x)}{g(x)}g(x)dx = E_g(f(X)\omega(X)).$$
where $\omega(x) = \frac{\pi(x)}{g(x)}$ is *importance function*.
- Approximate the expected value of interest by generating a random sample of size N from g and computing
$$\frac{1}{N}\sum_{i=1}^{N} f(x^{(i)})\omega(x^{(i)}) \approx E_\pi(f(X)).$$

Hidden Markov Model

- Consider an unobserved true discrete-state Markov process X_t that can only be inaccurately measured through a variable Y_t.
- Assume an initial distribution for X_0;
- The true process evolves to the next time step

$$X_{t+1} \sim f_1(X_t, \theta);$$

- The observed process for each time point

$$Y_t \sim f_2(X_t, \theta),$$

where f_i are some known distributions and θ is a parameter vector.

- From the observations $Y_{1:t}$, we can sample X_t from the target posterior distribution $p(X_t|Y_{1:t})$ by using

$$\begin{aligned} p(X_t|Y_{1:t}) &= \frac{p(X_t, Y_{1:t})}{p(Y_{1:t})} \\ &= \frac{p(Y_t|X_t)p(X_t|Y_{1:t-1})}{p(Y_{1:t})} \\ &\propto p(Y_t|X_t)p(X_t|Y_{1:t-1}) \end{aligned}$$

- We can sample from importance density $p(X_t|Y_{1:t-1})$ by f_1 and f_2.
- $p(Y_t|X_t)$ is importance function.

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Basic Particle Filtering

- Particle filtering is how sequential Monte Carlo is usually referred to in applications to state space models.
- This method is easier to understand when viewed as an extension of importance sampling.
- Generate starting particles based on the distribution assumption for X_0.
- Resample X_0 with replacement according to weights (also called a *bootstrap filter*)
- The resampled particles are used to predict X_1.
- Take X_1 as starting point for the next iteration.
- The algorithm can be described as next page.

Data: Observation time series $y_1, ..., y_T$

Input: number of particles N, initial sampling distribution p_0

```
/* Initialization                                                    */
Sample N particles from p0: x^1, ..., x^N
for (t = 1, 2, ..., T) do
    /* Weights from likelihood (measurement update)                  */
    for (i = 1, 2, ..., N) do
        w^i ← p(y_t|x^i) = f_2(x^i, θ)
    end
    /* Normalize weights                                             */
    for (i = 1, 2, ..., N) do
        w^i ← w^i / Σ_k w^k
    end
    /* Resample according to weights                                 */
    x̃^{1:N} ← sample(x^{1:N}, prob = w, replace = true)
    /* Update next time step X from the resampled particles:  predicts the new particles
       */
    for (i = 1, 2, ..., N) do
        x^i ← f_1(x̃^i, θ)
    end
end
```

Result: A set of N samples approximating the continuous posterior distribution $p(x_t|y_t)$ at every time step $t = 1, ..., T$.

Figure. Pseudocode by David Champredon

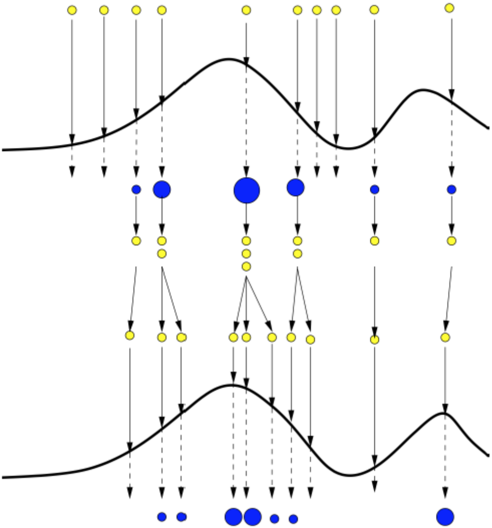

Figure. Design by David Champredon

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Numerical Example (Doucet *et al.* ch. 1)

- The data are generated from a local level model with system variance 1 and observation variance 2.
- The initial distribution is $N(10, 9)$.
- We record the observations Y.
- The number of particles is 1000.
- Setting the threshold step to 500 then we do resampling whenever the effective sample size drops below one half of the number of particles.
- We can compare the filtering state estimates and their deviations computed with the Kalman filter and particle filter.

Part 1: initialization

```
library("dlm")
### Generate data
mod <- dlmModPoly(1,dV=2,dW=1,m0=10,C0=9)
n <- 100
set.seed(23)
simData <- dlmForecast(mod=mod,nAhead=n,sampleNew=1)
y <- simData$newObs[[1]]

### Basic Particle Filter - optimal importance density
N <- 1000
N_0 <- N/2
pfOut <- matrix(NA_real_,n+1,N)
wt <- matrix(NA_real_,n+1,N)
importanceSd <- sqrt(drop(W(mod)-W(mod)^2/(W(mod)+V(mod))))
predSd <- sqrt(drop(W(mod)+V(mod)))
```

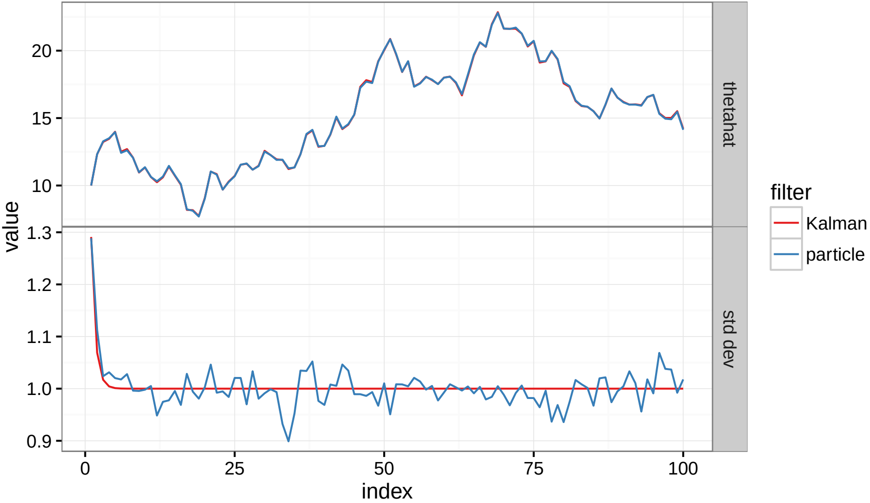
Part 2: sampling

```
pfOut[1,] <- rnorm(N,mean=m0(mod),sd=sqrt(C0(mod))); wt[1,] <- rep(1/N,N)
for (it in 2:(n+1)) {
    ## generate particles
    means <- pfOut[it-1,]+W(mod)*(y[it-1]-pfOut[it-1,])/(W(mod)+V(mod))
    pfOut[it,] <- rnorm(N,mean=means,sd=importanceSd)
    ## update the weights
    wt[it,] <- dnorm(y[it-1],mean=pfOut[it-1,],sd=predSd)*wt[it-1,]
    wt[it,] <- wt[it,]/sum(wt[it,])
    N.eff <- 1/crossprod(wt[it,])  ## need to resample?
    if (N.eff < N_0) {              ## multinomial resampling
        index <- sample(N,N,replace=TRUE,prob=wt[it,])
        pfOut[it,] <- pfOut[it,index]
        wt[it,] <- 1/N
    }
}
```

Part 3

```
### Compare exact filtering distribution with PF approximation
modFilt <- dlmFilter(y,mod)
thetaHatKF <- modFilt$m[-1]
sdKF <- with(modFilt,sqrt(unlist(dlmSvd2var(U.C,D.C))))[-1]
pfOut <- pfOut[-1,]
wt <- wt[-1,]
thetaHatPF <- sapply(1:n,function(i) weighted.mean(pfOut[i,],wt[i,]))
sdPF <- sapply(1:n,function(i)
    sqrt(weighted.mean((pfOut[i,]-thetaHatPF[i])^2,wt[i,])))
```

Particle vs Kalman filter results



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Iterative Filtering

- Instead of performing a single particle filter from the observed data $Y_{1:T}$, we estimate the posterior distribution of X_t at each time step.
- This method iterates several times through the process.
- The idea is changing the constant parameter θ into the dynamic $\theta(t)$ following a Gaussian process.
- It converges to maximum likelihood estimator since the variance of $\theta(t)$ tends to 0.
- The algorithm is listed on next page.

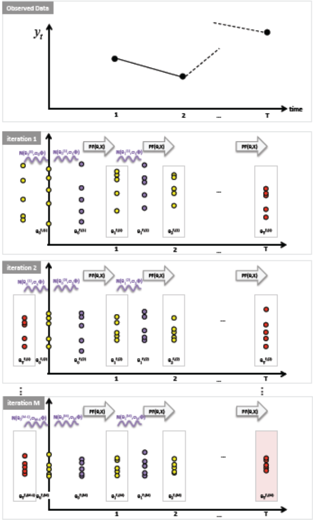

Figure. Design by David Champredon

Data: Observation time series $y_1, ..., y_T$

Input: number of main iterations M, number of particles J, J initial parameter particles $\Theta_j^{(0)}$, perturbation density $h(mean, var)$, perturbation sequence matrix $\sigma_{1:T}$, function f evaluating the process X at the next time step, observation process $g(y|X, \theta)$

for $m = 1, 2, ..., M$ **do**
  for $j = 1, ..., J$ **do**
    /* Process starting position */
    $\theta_F^{(m)}(0, j) \sim h(\Theta^{(m-1)_j}, \sigma_m)$
    $X_F(0, j) = f(X(0), \theta_F^{(m)}(0, j))$
  end
  /* Loop through all observation dates */
  for $t = 1, 2, ..., T$ **do**
    /* Particle filter from t-1 to t: */
    for $j = 1, ..., J$ **do**
      /* Perturbation of parameters: */
      $\theta_P^{(m)}(t, j) \sim h(\theta_F^{(m)}(t-1, j)), \sigma_m)$
      /* Prediction based on perturbation: */
      $X_P(t, j) \leftarrow f(X_F(t-1, j), \theta_P^{(m)}(t, j))$
      /* weights from perturbed likelihood */
      $w(t, j) \leftarrow g(y_t | X_P(t, j), \theta_P^{(m)}(t, j))$
    end
    $k_1, ..., k_J \leftarrow \texttt{sample}(1 : J, prob = w(t, \bullet), replace = true)$
    /* Update parameter and process */
    for $j = 1, ..., J$ **do**
      $\theta_F^{(m)}(t, j) \leftarrow \theta_P^{(m)}(t, k_j)$
      $X_F(t, j) \leftarrow X_P(t, k_j)$
    end
    /* ((End particle filter)) */
  end
  /* Set all particles for next main iteration */
  $\Theta^{(m)_j} \leftarrow \theta_F^{(m)}(t, j)$ for all $j = 1, ..., J$
end

Figure. Pseudocode by David Champredon

Thanks For Your Patience!