YOUNG-GEUN statistics

Statistical Computing

R Lab

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R Lab for Statistical Computing

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Contents

Welcome

Statistical computing mainly treats useful simulation methods.

Statistical Computing

We first look at *random generation* methods. Lots of simulation methods are built based on this random numbers.

Sampling from a fininte population

Generating random numbers is like sampling. From finite population, we can sample data with or without replacement. For example of sampling with replacement, we toss coins 10 times.

```
sample(0:1, size = 10, replace = TRUE)
```

```
 [1] 1 0 0 1 0 1 1 0 1 1
```

Sampling without replacement: Choose some lottery numbers which consist of 1 to 100.

```
sample(1:100, size = 6, replace = FALSE)
```

```
[1] 61 83 50 74 34 35
```

Random generators of common probability distributions

`R` provides some functions which generate random numbers following famous distributions. Although we will learn some skills generating these numbers in basis levels, these functions do the same thing more elegantly.

```
gg_curve(dbeta, from = 0, to = 1, args = list(shape1 = 3, shape2 = 2)) +
  geom_histogram(
    data = tibble(
      rand = rbeta(1000, 3, 2),
      idx = seq(0, 1, length.out = 1000)
    ),
    aes(x = rand, y = ..density..),
    position = "identity",
    bins = 30,
    alpha = .45,
    fill = gg_hcl(1)
  )
```

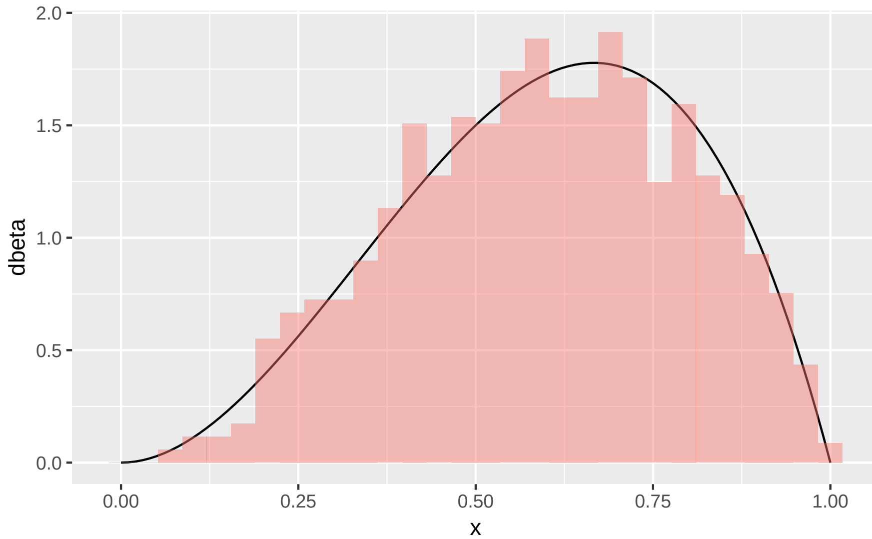


Figure 1: Beta(3,2) random numbers

Figure 1 shows that `rbeta()` function generate random numbers very well. Histogram is of the random number, and the curve is the true beta distribution.

Chapter 1

Methods for Generating Random Variables

1.1 Introduction

Most of the methods so-called *computational statistics* requires generation of random variables from specified probability distribution. In hand, we can spin wheels, roll a dice, or shuffle cards. The results are chosen randomly. However, we want the same things with computer. Here, `r`. As we know, computer cannot generate complete uniform random numbers. Instead, we generate **pseudo-random** numbers.

1.2 Pseudo-random Numbers

Definition 1.1 (Pseudo-random numbers)**.** Sequence of values generated deterministically which have all the appearances of being independent $unif(0,1)$ random variables, i.e.

$$x_1, x_2, \ldots, x_n \stackrel{iid}{\sim} unif(0,1)$$

- behave *as if* following $unif(0,1)$
- typically generated from an *initial seed*

1.2.1 Linear congruential generator

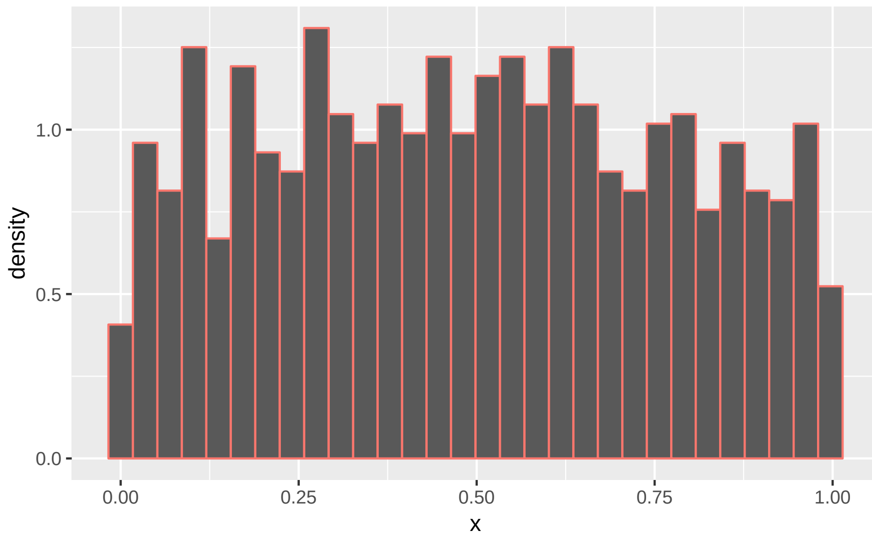
Let $x_0, x_1, \ldots \in \mathbb{Z}_+$.

1. Set x_0 as initial seed.
2. Generate $x_n, n = 1, 2, \ldots$ recursively:
    a. $x_n = (ax_{n-1} + c) \mod m$
    b. where $a, c \in \mathbb{Z}_+, m$: modulus
3. Compute $u_n = \frac{x_n}{m} \in (0,1)$

Then $u_1, u_2, \ldots \sim unif(0,1)$

```r
lcg <- function(n, seed, a, b, m) {
  x <- rep(seed, n + 1)
  for (i in 1:n) {
    x[i + 1] <- (a * x[i] + b) %% m
  }
  x[-1] / m
}
```

```r
tibble(
  x = lcg(1000, 0, 1664525, 1013904223, 2^32)
) %>%
  ggplot(aes(x = x)) +
  geom_histogram(aes(y = ..density..), bins = 30, col = gg_hcl(1))
```



1.2.2 Multiplicative congruential generator

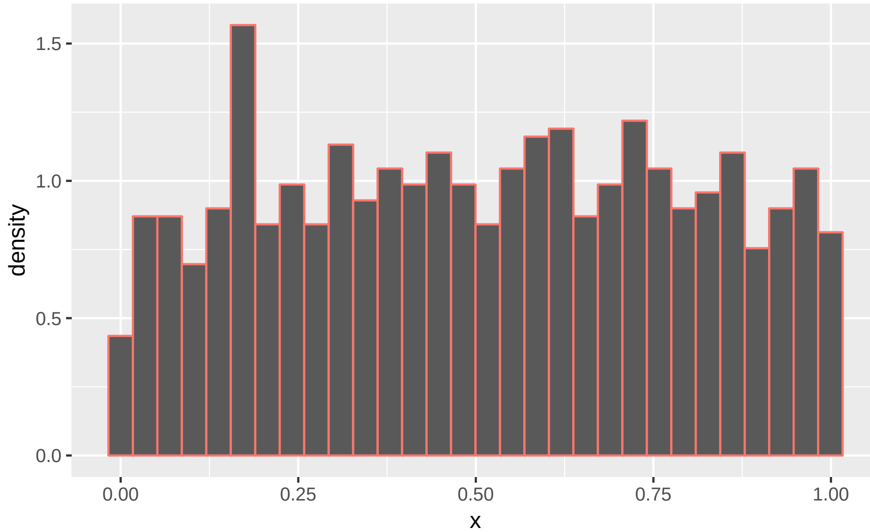
As we can expect from its name, this is congruential generator with $c = 0$.

1. Set x_0 as initial seed.
2. Generate $x_n, n = 1, 2, \ldots$ recursively:
   a. $x_n = a x_{n-1} \mod m$
   b. where $a \in \mathbb{Z}_+, m$: modulus
3. Compute $u_n = \frac{x_n}{m} \in (0, 1)$

Then $u_1, u_2, \ldots \sim unif(0, 1)$

We just set `b = 0` in our `lcg()` function. The **seed must not be zero**.

```r
tibble(
  x = lcg(1000, 5, 1664525, 0, 2^32)
) %>%
  ggplot(aes(x = x)) +
  geom_histogram(aes(y = ..density..), bins = 30, col = gg_hcl(1))
```



1.2.3 Cycle

Generate LCG $n = 32$ with $a = 1$, $c = 1$, and $m = 16$ from the seed $x_0 = 0$.

```
lcg(32, 0, 1, 1, 16)
```

```
 [1] 0.0625 0.1250 0.1875 0.2500 0.3125 0.3750 0.4375 0.5000 0.5625 0.6250
[11] 0.6875 0.7500 0.8125 0.8750 0.9375 0.0000 0.0625 0.1250 0.1875 0.2500
[21] 0.3125 0.3750 0.4375 0.5000 0.5625 0.6250 0.6875 0.7500 0.8125 0.8750
[31] 0.9375 0.0000
```

Observe that we have the cycle after m-th number. Against this problem, we give different seed from every $(im + 1)$th random number.

1.3 The Inverse Transform Method

Definition 1.2 (Inverse of CDF)**.** Since some cdf F_X is not strictly increasing, we difine $F_X^{-1}(y)$ for $0 < y < 1$ by

$$F_X^{-1}(y) := inf\{x : F_X(x) \geq y\}$$

Using this definition, we can get the following theorem.
Theorem 1.1 (Probability Integral Transformation)**.** *If X is a continuous random variable with cdf $F(x)$, then*

$$U \equiv F_X(X) \sim unif(0, 1)$$

Probability Integral Transformation. Let $U \sim unif(0, 1)$. Then

$$\begin{aligned} P(F_X^{-1}(U) \leq x) &= P(\inf\{t : F_X(t) = U\} \leq x) \\ &= P(U \leq F_X(x)) \\ &= F_U(F_X(x)) \\ &= F_X(x) \end{aligned}$$

□

Thus, to generate n random variables $\sim F_X$,

1. form of $F_X^{-1}(u)$
2. For each $i = 1, 2, \ldots, n$:
   a. Generate $u_i \sim unif(0, 1)$
   b. $x_i = F_X^{-1}(u_i)$

Collect $x_1, x_2, \ldots, x_n \stackrel{iid}{\sim} F_X$.

1.3.1 Continuous case

Denote that the *probability integral transformation* holds for a continuous variable. When generating continuous random variable, applying above algorithm might work.

Example 1.1 (Exponential distribution)**.** If $X \sim Exp(\lambda)$, then $F_X(x) = 1 - e^{-\lambda x}$. We can derive the inverse function of cdf

$$F_X^{-1}(u) = \frac{1}{\lambda} \ln(1 - u)$$

Note that

$$U \sim unif(0, 1) \Leftrightarrow 1 - U \sim unif(0, 1)$$

Then we just can use U instead of $1 - U$.

```
inv_exp <- function(n, lambda) {
  -log(runif(n)) / lambda
}
```

If we generate $x_1, \ldots, x_{500} \sim Exp(\lambda = 1)$,

```
gg_curve(dexp, from = 0, to = 10) +
  geom_histogram(
    data = tibble(x = inv_exp(500, lambda = 1)),
    aes(x = x, y = ..density..),
    bins = 30,
    fill = gg_hcl(1),
    alpha = .5
  )
```

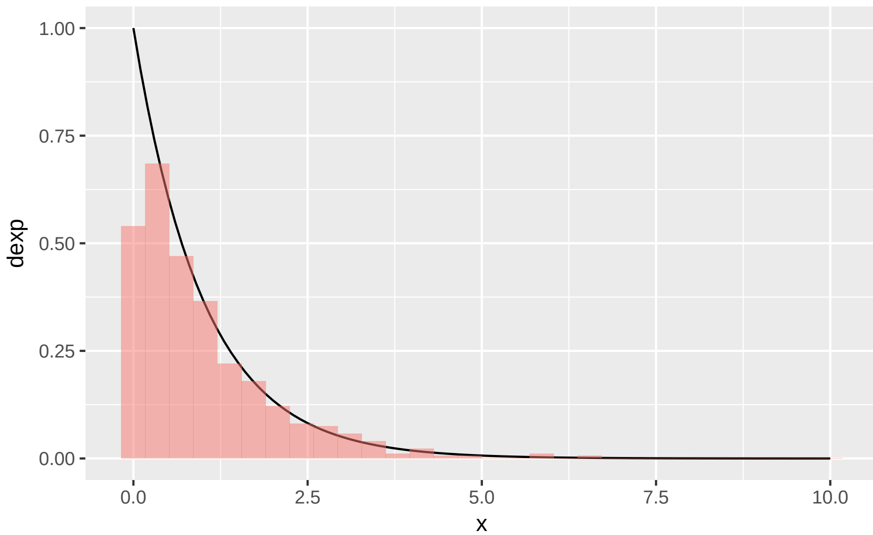


Figure 1.1: Inverse Transformation: Exp(1)

1.3.2 Discrete case

1. For each $i = 1, 2, \ldots, n$:
   a. Generate $u_i \sim unif(0, 1)$
   b. Take x_i s.t. $F_X(x_{i-1}) < U \le F_X(x_i)$

Collect $x_1, x_2, \ldots, x_n \sim F_X$.

```
pmf <-
  tibble(
    x = 0:4,
    p = c(.1, .2, .2, .2, .3)
  )
```

Example 1.2 (Discrete Random Variable)**.** Consider a discrete random variable X with a mass function as following.

x	p
0	0.1
1	0.2
2	0.2
3	0.2
4	0.3

i.e.

```
pmf %>%
  ggplot() +
  geom_segment(aes(x = x, y = 0, xend = x, yend = p)) +
  ylab(expression(p(x)))
```

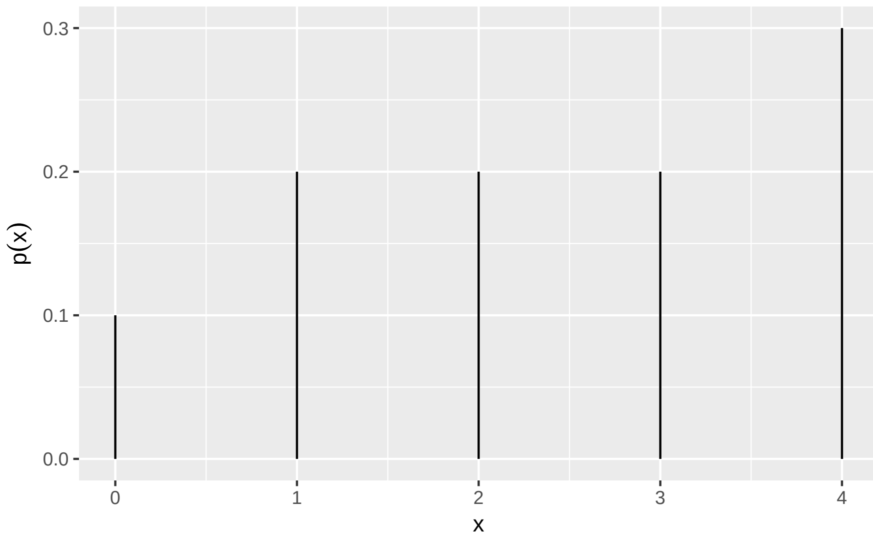


Figure 1.2: Probability Mass Function

Then we have the cdf

```
pmf %>%
  mutate(
    fx = cumsum(p),
    x_end = lead(x, default = 5),
    u = fx,
    u = ifelse(u == .5, .6, u),
    fx1 = lead(fx, default = 1),
    rand = u > fx & u <= fx1
  ) %>%
  ggplot() +
  geom_segment(aes(x = x, y = fx, xend = x_end, yend = fx)) +
  ylab(expression(F(x))) +
  geom_segment(
    aes(x = 0, y = u, xend = x_end, yend = u, colour = rand),
    linetype = "dashed",
    arrow = arrow(length = unit(.5, "cm")),
    show.legend = FALSE
  ) +
  geom_segment(
    aes(x = x_end, y = u, xend = x_end, yend = 0, colour = rand),
    linetype = "dashed",
    arrow = arrow(length = unit(.5, "cm")),
    show.legend = FALSE
  ) +
  scale_colour_manual(
    values = c("TRUE" = gg_hcl(1), "FALSE" = "#00000000")
  )
```

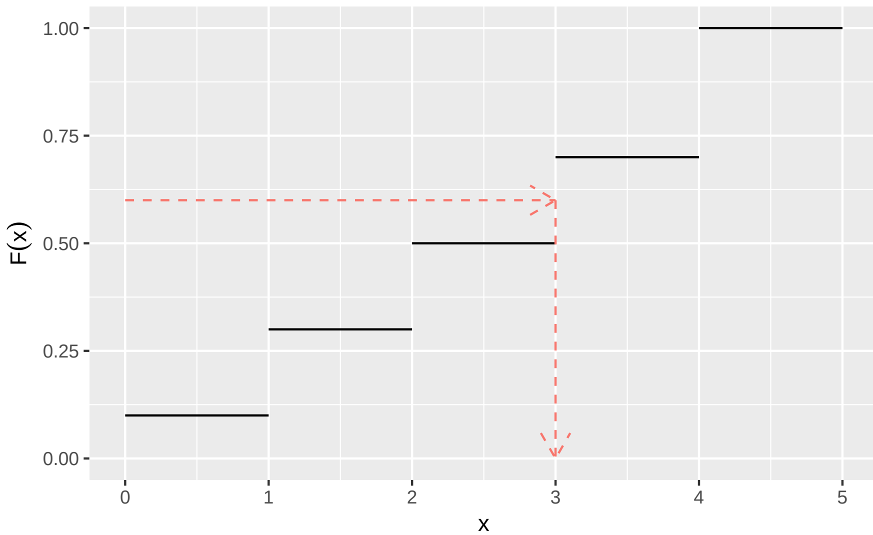


Figure 1.3: CDF of the Discrete Random Variable: Illustration for discrete case

Remembering the algorithm, we can implement `dplyr::case_when()` here.

```
rcustom <- function(n) {
  tibble(u = runif(n)) %>%
    mutate(
      x = case_when(
        u > 0 & u <= .1 ~ 0,
        u > .1 & u <= .3 ~ 1,
        u > .3 & u <= .5 ~ 2,
        u > .5 & u <= .7 ~ 3,
        TRUE ~ 4
      )
    ) %>%
    select(x) %>%
    pull()
}
```

```
tibble(
  x = rcustom(100)
) %>%
  ggplot(aes(x = x)) +
  geom_histogram(aes(y = ..ndensity..), binwidth = .1)
```

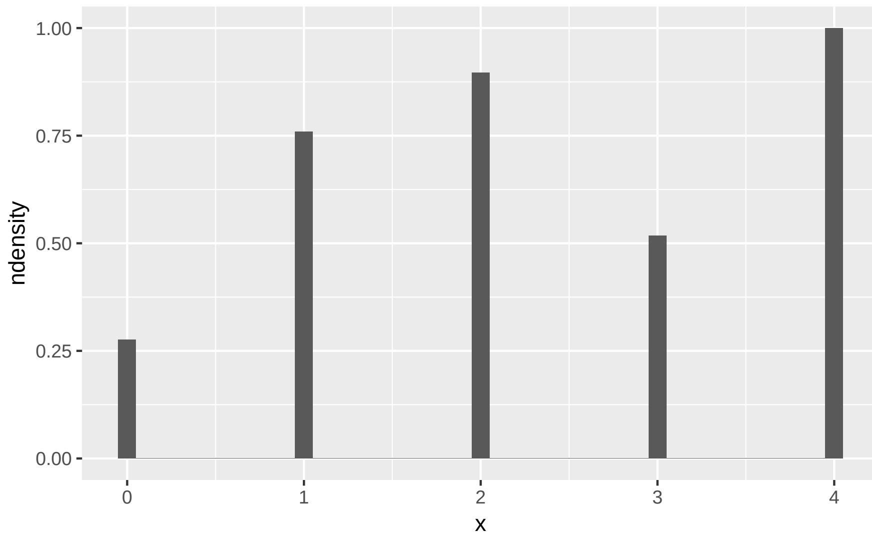


Figure 1.4: Generated discrete random numbers

See Figure 1.2 and 1.4. Comparing the two, the result can be said okay.

1.3.3 Problems with inverse transformation

Examples 1.1 and 1.2. We could generate these random numbers because we aware of

1. analytical F_X
2. F^{-1}

In practice, however, not all distribution have analytical F. Numerical computing might be possible, but it is not efficient. There are other approaches.

1.4 The Acceptance-Rejection Method

Chapter 2

Monte Carlo Integration and Variance Reduction