* Show $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ is a pivot for μ, σ known, using (9.2.11)

from C & B pg 428

Let $T=\bar{X}$. Then

$$f_T(t) = \frac{1}{\sqrt{2\pi}\,\sigma/\sqrt{n}} e^{-\frac{(\cancel{t}-\mu)^2}{2\sigma^2/n}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{t-\mu}{\sigma^2/n}\right]^2} \left|\frac{-1}{\sigma/\sqrt{n}}\right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[Q(t,\mu)]^2} \left|\frac{d}{dt} Q(t,\mu)\right|$$

where $Q(t,\mu) = \frac{t-\mu}{\sigma/\sqrt{n}} = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ is monotone in $t=\bar{x}$

Note that $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = g(z)$ is the distribution of a $N(0,1)$

which $z = Q(t,\mu)$ is...

* Show $\frac{n\hat{\sigma}^2}{\sigma^2}$ is a pivot for σ^2, when μ known.

we could just use the fact that

$$\frac{n\hat{\sigma}^2}{\sigma^2} = \Sigma\left(\frac{X_i-\mu}{\sigma}\right)^2 = \sum_i Z^2 \sim \chi^2_n \text{ is free of } \sigma$$

But for illustration, let $T = n\hat{\sigma}^2 = \Sigma(X_i-\mu)^2$

Then $T \sim \Gamma\left(\frac{n}{2}, 2\sigma^2\right)$ as a sum of $n\ [N(0,\sigma^2)]^2$ variables

So $$f_T(t) = \frac{1}{\Gamma(\frac{n}{2})[2\sigma^2]^{n/2}} t^{n/2-1} e^{-t/2\sigma^2}$$

$$= \underbrace{\frac{1}{\Gamma(\frac{n}{2})2^{n/2}} \left[\frac{t}{\sigma^2}\right]^{\frac{n}{2}-1} e^{-\frac{1}{2}\left[\frac{t}{\sigma^2}\right]}}_{g(Q(t,\sigma))} \cdot \underbrace{\left|\frac{1}{\sigma^2}\right|}_{\left|\frac{d}{dt}Q(t,\sigma)\right|}$$

where $g(Q(t,\sigma))$ is the χ^2_n distribution

and $Q(t,\sigma) = \frac{t}{\sigma^2}$

* when μ is unknown, we need Cochran's Theorem to show $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$