3.1 We consider three nested regression models for modelling the number of car accidents according to region (`region`). The variables risk class (`risk`) has 3 categories and the number of years of driving experience (`exp`) is broken down into 4 categories.

Model	variables	$p+1$	$\ell(\widehat{\boldsymbol{\beta}})$	AIC	BIC
M_1	`risk`	3	−244.566	495.132	510.362
M_2	`risk + region`	⋆	−151.620	⋆	⋆
M_3	`risk + region + exp`	10	−139.734	299.468	350.235

Table 1: Goodness-of-fit measures for three nested regression models with the number of parameters in each model $(p+1)$, the value of the log-likelihood function evaluated at the maximum likelihood estimate ($\ell(\widehat{\boldsymbol{\beta}})$) and information criteria.

What is the difference between AIC and BIC of Model M_2 (in absolute value)?

Solution

The difference is approximately 35.54. The only information required is the number of parameters of M_2 and the sample size. We can use the first line to compute the latter, since

$$\mathsf{BIC} = -2\ell(\mathrm{M}_1) + 3\ln(n)$$

and we find upon solving the equation $n = 1184$. It only remains to compute the number of levels for the region variable. This is $\mathrm{DF}(\mathrm{M}_3) - \mathrm{DF}(\mathrm{M}_1) - 3 = 7$; there are $K = 4$ categories for years of experience, but only three parameters. The difference $\mathsf{BIC} - \mathsf{AIC} = 7 \cdot \{\ln(1184) - 2\}$.

3.2 A random variable X follows a geometric distribution with parameter p if its probability mass function is

$$\mathsf{P}\,(X = x) = (1-p)^{x-1}p, \qquad x = 1, 2, \ldots$$

(a) Write the likelihood and the log-likelihood of the n sample.
(b) Derive the maximum likelihood estimator for the parameter p.
(c) Compute the observed information matrix.
(d) Suppose we have a sample of 15 observations, $\{5, 6, 3, 7, 1, 2, 11, 8, 7, 34, 1, 7, 10, 1, 0\}$, whose sum is 103. Compute the maximum likelihood estimate and its approximate standard error.
(e) Compute the likelihood ratio and the Wald test statistics. Perform a test at level 5% for $\mathscr{H}_0 : p_0 = 0.1$ against the two-sided alternative $\mathscr{H}_a : p_0 \neq 0.1$.

Solution

(a)

$$L(p;\boldsymbol{x}) = \prod_{i=1}^{n}(1-p)^{x_i-1}p = (1-p)^{\sum_{i=1}^{n}(x_i-1)}p^n$$
$$\ell(p;\boldsymbol{x}) = \ln(1-p)\sum_{i=1}^{n}(x_i-1) + n\ln(p)$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}p}\ell(p;\boldsymbol{x}) = -\frac{1}{(1-p)}\sum_{i=1}^{n}(x_i-1) + \frac{n}{p}$$

Setting the score to zero and re-arranging the expression, we find

$$\frac{1}{p} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Since the second derivative is negative, the maximum likelihood estimator is the reciprocal mean, $\widehat{p} = \overline{X}^{-1}$.

(c) The observed information function is the negative of the hessian matrix. We have

$$\frac{\mathrm{d}^2}{\mathrm{d}p^2}\ell(p;\boldsymbol{x}) = -\frac{n(\overline{x}-1)}{(1-p)^2} - \frac{n}{p^2}$$

and the observed information evaluated at the maximum likelihood estimate is $j(\widehat{p}) = n\overline{x}^3/(\overline{x}-1)$.

(d) The maximum likelihood estimate is 0.1456 and its standard error is 0.0347.

(e) The Wald statistic and the likelihood ratio statistic equal

$$\begin{aligned} W &= (\widehat{p}-p_0)^2/\mathsf{Va}(\widehat{p}) = (0.1456-0.1)^2/0.0347 = 1.72\\ R &= 2\{\ell(\widehat{p}) - \ell(p_0)\} = 2\{-45.11+45.39\} = 0.558. \end{aligned}$$

Both can be compared to χ^2_1: the respective p-values are 0.19 and 0.45, so we fail to reject the null $\mathscr{H}_0: p = 0.1$.

3.3 We consider the failure time of an engine based on its level of corrosion w. Failure time T is modelled with an exponential distribution with density $f(t) = \lambda\exp(-\lambda t)$, but whose rate parameter $\lambda = aw^b$; if $b = 0$, the failure time is a^{-1}. Assume we have an n sample of independent observations with w_i assumed known. [Coles (2001)]

(a) Write down the log-likelihood of the model.

(b) Derive the observed and the Fisher information matrices.

(c) Show that the profile log-likelihood for b is

$$\ell_{\mathsf{p}}(b) = n\ln(\widehat{a}_b) + b\sum_{i=1}^{n}\ln(w_i) - \widehat{a}_b\sum_{i=1}^{n} w_i^b t_i,$$

and give an explicit formula for the partial maximum likelihood estimator $\widehat{a}_b$.

Solution

(a) The log-likelihood is

$$\ell(a,b;\boldsymbol{w},\boldsymbol{t}) = n\ln(a) + b\sum_{i=1}^{n}\ln(w_i) - a\sum_{i=1}^{n} w_i^b t_i$$

(b) The Fisher information is the expected value of the negative of the hessian matrix. Setting

$$j(a,b) = -\begin{pmatrix} \partial^2\ell/\partial a^2 & \partial^2\ell/\partial a\partial b\\ \partial^2\ell/\partial b\partial a & \partial^2\ell/\partial b^2\end{pmatrix} = \begin{pmatrix} na^{-2} & \sum_{i=1}^n w_i^b t_i\ln(w_i)\\ \sum_{i=1}^n w_i^b t_i \ln(w_i) & a\sum_{i=1}^n w_i^b t_i\ln^2(w_i)\end{pmatrix}.$$

The Fisher information is obtained by computing the expectation of each entry of the matrix. Only the T_i terms are constant, and since the formulae only involve terms that are linear in T_i, the Fisher information is

$$I(a,b) = \begin{pmatrix} na^{-2} & a^{-1}\sum_{i=1}^n \ln(w_i)\\ a^{-1}\sum_{i=1}^n\ln(w_i) & \sum_{i=1}^n \ln^2(w_i)\end{pmatrix}$$

given that $\mathsf{E}(T_i) = a^{-1}w_i^{-b}$.

(c) The partial maximum $\widehat{a}_b$ is obtained by differentiating the log-likelihood as a function of a while treating b as fixed. We get

$$\frac{\partial \ell}{\partial a} = \frac{n}{a} - \sum_{i=1} w_i^b t_i = 0$$

and isolating a shows that $\widehat{a}_b = n / \sum_{i=1}^n w_i^b t_i$ is the maximum, since the second derivative calculated previously is negative.