Final concepts of SLR

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This material is part of the ***statsTeachR*** *project*

Today's lecture

- Simple Linear Regression Continued
- Multiple Regression Intro

Simple linear regression model

- Observe data (y_i, x_i) for subjects $1, \ldots, I$. Want to estimate β_0, β_1 in the model

E[y|x]

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i; \ \epsilon_i \overset{iid}{\sim} (0, \sigma^2)$$

Normality

- Note the assumptions on the variance:
  - $E(\epsilon \mid x) = E(\epsilon) = 0$
  - Constant variance
  - Independence
  - [Normally distributed is not needed for least squares, but is needed for inference]

Some definitions / SLR products

$:=$

- *Fitted values*: $\hat{y}_i := \hat{\beta}_0 + \hat{\beta}_1 x_i$ obs fitted
- *Residuals / estimated errors*: $\hat{\epsilon}_i := y_i - \hat{y}_i$
- *Residual sum of squares*: $\text{RSS} := \sum_{i=1}^{n} \hat{\epsilon}_i^{\,2}$
- *Residual variance*: $\hat{\sigma^2} := \frac{RSS}{n-2}$
- *Degrees of freedom*: $n - 2$

Notes: residual sample mean is zero; residuals are uncorrelated with fitted values.

R^2

Looking for a measure of goodness of fit.

- RSS by itself doesn't work so well:

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

- Coefficient of determination (R^2) works better:

$$R^2 = 1 - \frac{\sum(y_i - \hat{y}_i)^2}{\sum(y_i - \bar{y})^2}$$

prop of var. explained by model

prop of all variance described by errors

R^2

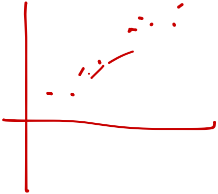
Some notes about R^2

- Interpreted as proportion of outcome variance explained by the model.
- Alternative form

R_2^2

$$R^2 = \frac{\sum(\hat{y}_i - \bar{y})^2}{\sum(y_i - \bar{y})^2}$$

- R^2 is bounded: $0 \leq R^2 \leq 1$
- For simple linear regression only, $R^2 = \rho^2$



$R_1^2 = R_2^2$

$\rho_1 \neq \rho_2$

ANOVA

Lots of sums of squares around.

- Regression sum of squares $SS_{reg} = \sum(\hat{y}_i - \bar{y})^2$
- Residual sum of squares $SS_{res} = \sum(y_i - \hat{y}_i)^2$
- Total sum of squares $SS_{tot} = \sum(y_i - \bar{y})^2$
- All are related to sample variances

Analysis of variance (ANOVA) seeks to address goodness-of-fit by looking at these sample variances.

ANOVA

ANOVA is based on the fact that $SS_{tot} = SS_{reg} + SS_{res}$

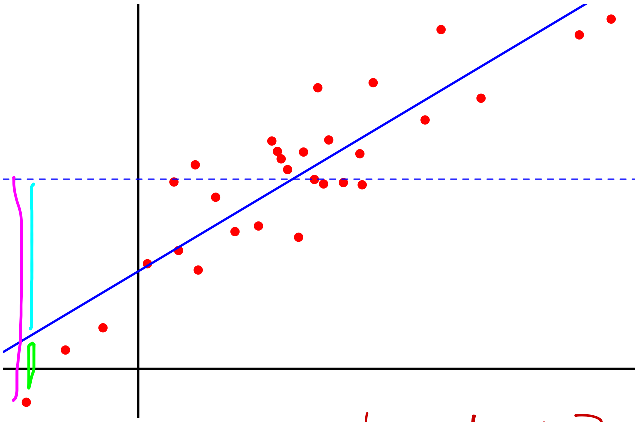
$$SS_{tot} = \sum (y_i - \bar{y})^2$$

$$= \sum \Big(\underbrace{(y_i - \hat{y})}_{A} + \underbrace{(\hat{y} - \bar{y})}_{B} \Big)^2$$

$\vdots$

ANOVA

ANOVA is based on the fact that $SS_{tot} = SS_{reg} + SS_{res}$



See textbook?

ANOVA and R^2

- Both take advantage of sums of squares
- Both are defined for more complex models
- ANOVA can be used to derive a "global hypothesis test" based on an F test (more on this later)

R^2 never used for hypothesis test

R example

```
require(alr3)
data(heights)
linmod <- lm(Dheight ~ Mheight, data = heights)
linmod

##
## Call:
## lm(formula = Dheight ~ Mheight, data = heights)
##
## Coefficients:
## (Intercept)      Mheight
##      29.917        0.542
```

R example

```
summary(linmod)

##
## Call:
## lm(formula = Dheight ~ Mheight, data = heights)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -7.397 -1.529  0.036  1.492  9.053
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   29.917      1.623    18.4   <2e-16 ***
## Mheight        0.542      0.026    20.9   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.27 on 1373 degrees of freedom
## Multiple R-squared:  0.241, Adjusted R-squared:  0.24
## F-statistic:  435 on 1 and 1373 DF,  p-value: <2e-16
```

R example

class(linmod)
"lm"

```
names(linmod)

##  [1] "coefficients"  "residuals"     "effects"       "rank"
##  [5] "fitted.values" "assign"        "qr"            "df.residual"
##  [9] "xlevels"       "call"          "terms"         "model"
```

linmod$coef
?lm

R example

```
head(linmod$residuals)

##      1      2      3      4      5      6
## -7.160 -4.947 -6.747 -6.001 -7.397 -2.084

head(resid(linmod))

##      1      2      3      4      5      6
## -7.160 -4.947 -6.747 -6.001 -7.397 -2.084

head(linmod$fitted.values)

##     1     2     3     4     5     6
## 62.26 61.45 62.75 62.80 63.40 59.98

head(fitted(linmod))

##     1     2     3     4     5     6
## 62.26 61.45 62.75 62.80 63.40 59.98
```

R example

```
names(summary(linmod))

##  [1] "call"          "terms"         "residuals"     "coefficients"
##  [5] "aliased"       "sigma"         "df"            "r.squared"
##  [9] "adj.r.squared" "fstatistic"    "cov.unscaled"

summary(linmod)$coef[1,4]

##              Estimate Std. Error t value  Pr(>|t|)
## (Intercept)   29.9174    1.62247   18.44 5.212e-68
## Mheight        0.5417    0.02596   20.87 3.217e-84

summary(linmod)$r.squared

## [1] 0.2408
```

R example

```
anova(linmod)

## Analysis of Variance Table
##
## Response: Dheight
##             Df Sum Sq Mean Sq F value Pr(>F)
## Mheight      1   2237    2237     435 <2e-16 ***
## Residuals 1373   7052       5
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

SSreg

SSres

$$R^2 = 1 - \frac{SS_{res}}{SS_{tot}} = 1 - \frac{SS_{res}}{SS_{res} + SS_{reg}}$$

R example

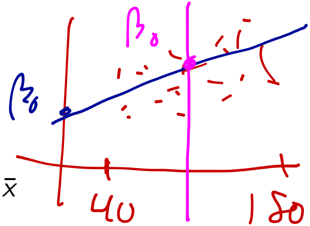
```
anova(linmod)

## Analysis of Variance Table
##
## Response: Dheight
##             Df Sum Sq Mean Sq F value Pr(>F)
## Mheight      1   2237    2237     435 <2e-16 ***
## Residuals 1373   7052       5
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(r2 <- 1 - 7052/(7052 + 2237))

## [1] 0.2408
```

Note on interpretation of β_0



Recall $\beta_0 = E(y|x = 0)$

- This often makes no sense in context
- "Centering" x can be useful: $x^* = x - \bar{x}$
- Center by mean, median, minimum, etc
- Effect of centering on slope:

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

$$\hat{\beta}_1^* = \frac{\sum \qquad (y_i - \bar{y})}{\sum (x_i - c - \bar{x}^*}$$

$$\vdots$$

$$= \hat{\beta}_1$$

$$\bar{x}^* = \frac{\sum x_i^*}{n} = \frac{\sum (x_i - c)}{n} = \vdots$$

Note on interpretation of β_0, β_1

- The interpretations are sensitive to the scale of the outcome and predictors (in reasonable ways)
- You can't get a better model fit by rescaling variables

$$x_i^* = c \cdot x_i$$

$$\hat{\beta}_1 \neq \hat{\beta}_1^*$$

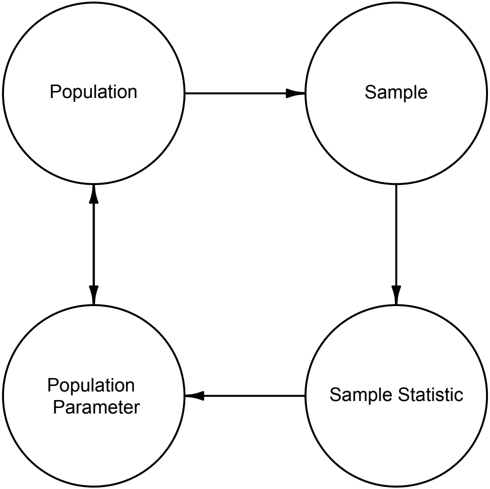
but inference won't change

R example

```
heights$centeredMheight <- heights$Mheight - mean(heights$Mheight)
centeredLinmod <- lm(Dheight ~ centeredMheight, data = heights)
summary(centeredLinmod)

##
## Call:
## lm(formula = Dheight ~ centeredMheight, data = heights)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -7.397 -1.529  0.036  1.492  9.053
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)      63.7511     0.0611  1043.1   <2e-16 ***
## centeredMheight   0.5417     0.0260    20.9   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.27 on 1373 degrees of freedom
## Multiple R-squared:  0.241, Adjusted R-squared:  0.24
## F-statistic:  435 on 1 and 1373 DF,  p-value: <2e-16
```

Properties of $\hat{\beta}_0, \hat{\beta}_1$



Properties of $\hat{\beta}_0, \hat{\beta}_1$

Estimates are unbiased:

$E(\hat{\beta}_0) = \beta_0$

$E(\hat{\beta}_1) = \beta_1$

Properties of $\hat{\beta}_0, \hat{\beta}_1$

Variances of estimates

$Var(\hat{\beta}_0) = \frac{\bar{x}\sigma^2}{\sum x^2}$

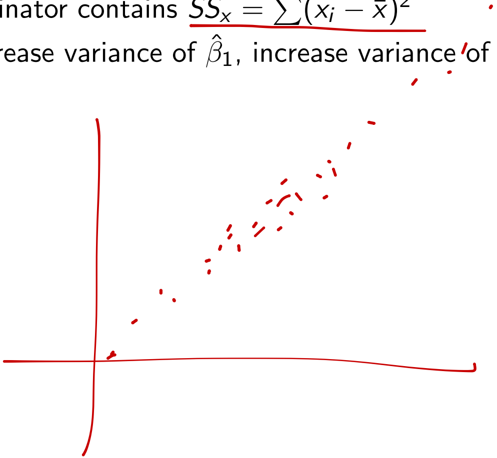
$Var(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{S_{xx}}$

where $S_{xx} = \sum(x - \bar{x})^2$

Properties of $\hat{\beta}_0, \hat{\beta}_1$

Note about the variance of β_1:

- Denominator contains $SS_x = \sum(x_i - \bar{x})^2$
- To decrease variance of $\hat{\beta}_1$, increase variance of x



One slide on multiple linear regression

- Observe data $(y_i, x_{i1}, \ldots, x_{ip})$ for subjects $1, \ldots, n$. Want to estimate $\beta_0, \beta_1, \ldots, \beta_p$ in the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_1 x_{ip} + \epsilon_i; \;\; \epsilon_i \overset{iid}{\sim} (0, \sigma^2)$$

- Assumptions (residuals have mean zero, constant variance, are independent) are as in SLR
- Notation is cumbersome. To fix this, let
  - $\mathbf{x}_i = [1, x_{i1}, \ldots, x_{ip}]$ 1xp
  - $\boldsymbol{\beta}^T = [\beta_0, \beta_1, \ldots, \boldsymbol{\beta}_p]$
  - Then $y_i = \mathbf{x}_i \boldsymbol{\beta} + \epsilon_i$

β = px1 vector

= [1xp] [px1]

Summary

Today's big ideas

- Simple linear regression definitions
- Properties of least squares estimates

Coming up soon

- More on MLR