SDS 383D Ex 02:
Bayes and the Gaussian Linear Model

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A Simple Gaussian Location Model

Part A

The marginal prior $p(\theta)$ is a gamma mixture of normals. Show it takes the form of a centered, scaled t distribution: $p(\theta) \propto \left(1 + \frac{1}{\nu} \cdot \frac{(x-m)^2}{s^2}\right)^{-\frac{\nu+1}{2}}$.

$$(\theta|\omega) \sim N(\mu, \omega^{-1}k^{-1}) and \omega \sim gamma(\frac{d}{2}, \frac{\eta}{2})$$

Joint prior for $p(\theta, \omega)$ has form

$$p(\theta, \omega) = p(\theta|\omega)p(\omega) \propto \omega^{\left(\frac{(d+1)}{2}-1\right)} \exp[-\omega \cdot \frac{k(\theta-\mu)^2}{2}] \exp[-\omega \frac{\eta}{2}]$$

Marginal prior for θ is calculated as follows:

$$\begin{aligned}
p(\theta) &= \int_0^\infty p(\theta, \omega) \delta\omega \\
&= \int_0^\infty \omega^{\left(\frac{(d+1)}{2}-1\right)} \exp[-\omega \cdot \frac{k(\theta-\mu)^2}{2}] \exp[-\omega \frac{\eta}{2}] \delta\omega \\
&= \int_0^\infty \omega^{\left(\frac{(d+1)}{2}-1\right)} e^{-\omega\left(\frac{k(\theta-\mu)^2}{2} + \frac{\eta}{2}\right)} \delta\omega
\end{aligned}$$

This is the kernel of a $gamma\left(\frac{(d+1)}{2}, \frac{k(\theta-\mu)^2}{2} + \frac{\eta}{2}\right)$ distribution.
The integral will therefore equal $\frac{1}{c}$, where c is the constant of proportionality for this gamma density, $\frac{1}{\beta^\alpha/\Gamma(\alpha)} = \frac{\Gamma(\alpha)}{\beta^\alpha}$.

$$\begin{aligned}
&= \Gamma(\frac{d+1}{2}) \left[\frac{k(\theta-\mu)^2}{2} + \frac{\eta}{2}\right]^{-\frac{d+1}{2}} \\
&\propto \left[\left(\frac{\eta}{2}\right)\left(1 + \frac{k(\theta-\mu)^2}{\eta}\right)\right]^{-\frac{d+1}{2}} \\
&= \left(\frac{\eta}{2}\right)^{-\frac{d+1}{2}} \left[\left(1 + \frac{k(\theta-\mu)^2}{\eta}\right)\right]^{-\frac{d+1}{2}}
\end{aligned}$$

The left term cancels, as it is constant wrt θ.

$$\propto \left[\left(1 + \frac{k(\theta-\mu)^2}{\eta}\right)\right]^{-\frac{d+1}{2}}$$

Multiply numerator and denominator by d.

$$\begin{aligned}
&= \left[\left(1 + \frac{dk(\theta-\mu)^2}{d\eta}\right)\right]^{-\frac{d+1}{2}} \\
&= \left[\left(1 + \left(\frac{1}{d}\right)\frac{(\theta-\mu)^2}{\frac{\eta}{dk}}\right)\right]^{-\frac{d+1}{2}}
\end{aligned}$$

This has the desired central-scaled t form, with mean $m = \mu$, df $\nu = d$, and scale $s^2 = \frac{\eta}{dk}$. ■

Part B

Assume the sampling model $(y_i|\theta,\sigma^2) \sim N(\theta,\sigma^2)$ for $i = 1...n$, so $y = (y_1, ..., y_n)^T$. Work with precision $\omega = 1/\sigma^2$.

Assume the same normal-gamma prior has form

$$p(\theta,\omega) \propto \omega^{\left(\frac{(d+1)}{2}-1\right)} \exp[-\omega \cdot \frac{k(\theta-\mu)^2}{2}] \exp[-\omega\frac{\eta}{2}]$$

Calculate the joint posterior up to a constant of proportionality (factors not dependent on θ or ω.) Show the joint posterior is also normal-gamma.

The posterior $\propto$ sampling model * prior, so

$$p(\theta,\omega|y) \propto p(y|\theta,\omega)p(\theta,\omega)$$

Sampling model (from the normal likelihood), using $S_y = \sum_{i=1}^n (y_i - \bar{y})^2$, is

$$p(y|\theta,\omega) \propto \omega^{\frac{n}{2}} exp\left[-\frac{\omega}{2}\left(S_y + n(\bar{y}-\theta)^2\right)\right]$$

Therefore, the joint posterior is

$$\begin{aligned}
p(\theta,\omega|y) &\propto \omega^{\left(\frac{(d+1)}{2}-1\right)} \exp[-\omega \cdot \frac{k(\theta-\mu)^2}{2}] \exp[-\omega\frac{\eta}{2}] \cdot \omega^{\frac{n}{2}} \exp\left[-\frac{\omega}{2}\left(S_y + n(\bar{y}-\theta)^2\right)\right] \\
&= \omega^{\left(\frac{n+d+1}{2}-1\right)} \cdot \exp\left[-\frac{\omega}{2}\left(S_y + n(\bar{y}-\theta)^2 + k(\theta-\mu)^2 + \eta\right)\right]
\end{aligned}$$

Simplify the exponential term multiplied by $-\frac{\omega}{2}$.

$$\begin{aligned}
&\left(S_y + n(\bar{y}-\theta)^2 + k(\theta-\mu)^2 + \eta\right) \\
&= S_y + n(\bar{y}-\theta)(\bar{y}-\theta) + k(\theta-\mu)(\theta-\mu) + \eta \\
&= S_y + n\bar{y}^2 - 2n\bar{y}\theta + n\theta^2 + k\theta^2 - 2k\theta\mu + k\mu^2 + \eta \\
&= (n+k)\theta^2 - 2(n\bar{y}+k\mu)\theta + \left(S_y + n\bar{y}^2 + k\mu^2 + \eta\right)
\end{aligned}$$

This has the form $ax^2 - 2bx + c$. We can complete the square as follows:

$$\begin{aligned}
&ax^2 - 2bx + c \\
&\quad = a\left[x^2 - 2\frac{b}{a}x + \frac{c}{a}\right] \\
&\quad = a\left[x^2 - 2\frac{b}{a}x + (\frac{b}{a})^2 - (\frac{b}{a})^2 + \frac{c}{a}\right] \\
&\quad = a\left[(x-\frac{b}{a})^2 - (\frac{b}{a})^2 + \frac{c}{a}\right] \\
&\quad = a(x-\frac{b}{a})^2 - \frac{b^2}{a} + c
\end{aligned}$$

Plugging in the appropriate a, b and c terms yields:

$$
\begin{aligned}
&= (n+k)\left[\theta - \frac{n\bar{y}+k\mu}{n+k}\right]^2 - \frac{(n\bar{y}+k\mu)^2}{n+k} + S_y + n\bar{y}^2 + k\mu^2 + \eta \\
&= (n+k)\left[\theta - \frac{n\bar{y}+k\mu}{n+k}\right]^2 + \frac{nk(\bar{y}-\mu)^2}{(n+k)} + S_y + \eta
\end{aligned}
$$

Plug this rearranged term back into the entire posterior:

$$
p(\theta,\omega|y) \propto \omega^{\left(\frac{n+d+1}{2}-1\right)} \cdot \exp\left[-\frac{\omega}{2}\left((n+k)\left[\theta - \frac{n\bar{y}+k\mu}{n+k}\right]^2 + \frac{nk(\bar{y}-\mu)^2}{(n+k)} + S_y + \eta\right)\right]
$$

$$
p(\theta,\omega|y) \propto \omega^{\left(\frac{n+d+1}{2}-1\right)} \cdot \exp\left[-\frac{\omega}{2}\left((n+k)\left[\theta - \frac{n\bar{y}+k\mu}{n+k}\right]^2\right)\right] \cdot \exp\left[-\frac{\omega}{2}\left(\frac{nk(\bar{y}-\mu)^2}{(n+k)} + S_y + \eta\right)\right]
$$

This is the desired form of the normal-gamma distribution, with posterior parameters as follows.

$$
\begin{aligned}
d^* &= (n+d) && (1)\\
k^* &= (n+k) && (2)\\
\mu^* &= \frac{n\bar{y}+k\mu}{(n+k)} && (3)\\
\eta^* &= \frac{nk(\bar{y}-\mu)^2}{(n+k)} + S_y + \eta && (4)
\end{aligned}
$$

Part C

The conditional posterior of $p(\theta|y,\omega)$ is

$$
p(\theta|y,\omega) \propto \exp\left[-\frac{\omega}{2}\cdot(n+k)\left(\theta - \frac{n\bar{y}+k\mu}{(n+k)}\right)^2\right]
$$

This is the Normal distribution form, parameterized with precision.

$$
p(\theta|y,\omega) \sim N\left(\frac{n\bar{y}+k\mu}{(n+k)}, -\omega(n+k)\right)
$$

Part D

From the joint posterior in Part A, what is the marginal posterior $p(\omega|y)$?

Note: Will indicate posterior parameters with a *, for notational simplicity, and will substitutde values of posterior parameters at end.

$$\begin{aligned}
p(\omega|y) &= \int_{-\infty}^{\infty} p(\theta,\omega|y)\delta\theta \\
&\propto \int_{-\infty}^{\infty} \omega^{\left(\frac{(d^*+1)}{2}-1\right)} \exp[-\frac{\omega}{2}k^*(\theta-\mu^*)^2]\exp[-\omega\frac{\eta^*}{2}]\delta\theta \\
&= \omega^{\left(\frac{(d^*+1)}{2}-1\right)} e^{\left(-\omega\frac{\eta^*}{2}\right)} \int_{-\infty}^{\infty} \exp[-\frac{\omega}{2}k^*(\theta-\mu^*)^2]\delta\theta
\end{aligned}$$

The integral is a normal kernel: $N(\mu^*, \omega k^*)$. It integrates to $\frac{1}{c}$, where c is the constant of proportionality for the normal density. The integral $= \frac{1}{\omega^{1/2}} = \omega^{-\frac{1}{2}}$.

$$\begin{aligned}
&= \omega^{\left(\frac{(d^*+1)}{2}-1\right)} e^{\left(-\omega\frac{\eta^*}{2}\right)} \omega^{-\frac{1}{2}} \\
&= \omega^{\left(\frac{d^*}{2}-1\right)} e^{\left(-\omega\frac{\eta^*}{2}\right)}
\end{aligned}$$

This is the kernel for *gamma* $\left(\frac{d^*}{2}, \frac{\eta^*}{2}\right)$, where

$$\begin{aligned}
d^* &= (n+d) \\
\eta^* &= \frac{nk(\bar{y}-\mu)^2}{(n+k)} + S_y + \eta
\end{aligned}$$

Part E

From C and D, we know the marginal posterior $p(\theta|y)$ is a gamma mixture of normals. Show this takes the form of a centered, scaled t distribution of form $p(\theta) \propto \left(1 + \frac{1}{\nu} \cdot \frac{(x-m)^2}{s^2}\right)^{-\frac{\nu+1}{2}}$ and state what m, s and v are.

Joint posterior $p(\theta, \omega|y)$:

$$p(\theta, \omega|y) \propto \omega^{\left(\frac{n+d+1}{2}-1\right)} \cdot \exp\left[-\frac{\omega}{2}\left((n+k)\left[\theta - \frac{n\bar{y}+k\mu}{n+k}\right]^2\right)\right] \cdot \exp\left[-\frac{\omega}{2}\left(\frac{nk(\bar{y}-\mu)^2}{(n+k)} + S_y + \eta\right)\right]$$

Marginal posterior $p(\theta|y)$:

$$\begin{aligned} p(\theta|y) &= \int_0^\infty p(\theta, \omega|y)\delta\omega \\ &= \int_0^\infty \omega^{\frac{d^*+1}{2}-1} \cdot \exp\left[-\frac{\omega}{2}\left(k^*(\theta-\mu^*)^2 + \frac{\eta^*}{2}\right)\right]\delta\omega \end{aligned}$$

This is the kernel of a *gamma* $\left(\frac{(d^*+1)}{2}, \frac{k^*(\theta-\mu^*)^2+\eta^*}{2}\right)$ distribution.

The integral will therefore equal $\frac{1}{c}$, where c is the constant of proportionality for this gamma density, $\frac{1}{\beta^\alpha/\Gamma(\alpha)} = \frac{\Gamma(\alpha)}{\beta^\alpha}$.

$$\begin{aligned} &= \Gamma\left(\frac{d^*+1}{2}\right)\left[\frac{k^*(\theta-\mu^*)^2}{2} + \frac{\eta^*}{2}\right]^{-\frac{d^*+1}{2}} \\ &\propto \left[\left(\frac{\eta^*}{2}\right)\left(1 + \frac{k^*(\theta-\mu^*)^2}{\eta^*}\right)\right]^{-\frac{d^*+1}{2}} \\ &= \left(\frac{\eta^*}{2}\right)^{-\frac{d^*+1}{2}}\left[\left(1 + \frac{k^*(\theta-\mu^*)^2}{\eta^*}\right)\right]^{-\frac{d^*+1}{2}} \end{aligned}$$

The left term cancels, as it is constant wrt θ.

$$\propto \left[\left(1 + \frac{k^*(\theta-\mu^*)^2}{\eta^*}\right)\right]^{-\frac{d^*+1}{2}}$$

Multiply numerator and denominator by d^*.

$$\begin{aligned} &= \left[\left(1 + \frac{d^*k^*(\theta-\mu^*)^2}{d\eta^*}\right)\right]^{-\frac{d^*+1}{2}} \\ &= \left[\left(1 + \left(\frac{1}{d^*}\right)\frac{(\theta-\mu^*)^2}{\frac{\eta^*}{d^*k^*}}\right)\right]^{-\frac{d^*+1}{2}} \end{aligned}$$

This has the desired central-scaled t form, with mean $m = \mu^*$, df $\nu = d^*$, and scale $s^2 = \frac{\eta^*}{d^*k^*}$. $\blacksquare$

Part G

True or False? In the limit, as prior parameters $k, d, \eta \to 0$, priors $p(\theta)$ and $p(\omega)$ are valid probability distributions. (Must integrate to 1, or something finite so can be normalized over their domains.)

$p(\theta)$:
The prior $p(\theta)$ is a gamma mixture of normals, which can arrange as a scaled, centered t.

$$\lim_{d,k,\eta \to 0} \Gamma\left(\frac{d+1}{2}\right)\left[\frac{k(\theta-\mu)^2}{2}+\frac{\eta}{2}\right]^{-\frac{d+1}{2}}$$

Goes to:

$$\Gamma\left(\frac{1}{2}\right)\left[\frac{0(\theta-\mu)^2}{2}+\frac{0}{2}\right]^{-\frac{1}{2}} \to 0$$

This is not a valid probability distribution. It is a point mass at 0. It is an improper prior when the prior parameters go to zero.

$p(\omega)$:
The prior $p(\omega)$ is $gamma(\frac{d}{2}, \frac{\eta}{2})$.

$$p(\omega) = \frac{(\frac{\eta}{2})^{(\frac{d}{2})}}{\Gamma(\frac{d}{2})}\omega^{\frac{d}{2}-1}e^{-\omega\frac{\eta}{2}}$$

$$\lim_{d,k,\eta \to 0} \frac{(\frac{\eta}{2})^{(\frac{d}{2})}}{\Gamma(\frac{d}{2})}\omega^{\frac{d}{2}-1}e^{-\omega\frac{\eta}{2}}$$

Goes to:

$$(1)(\omega^{-1})(1) = \omega^{-1} = \omega^{0-1}e^{-0\omega}$$

This is the form of $gamma(0,0)$, which is not a valid probability distribution, since gamma limits both parameters to values greater than zero. This is also an improper prior.

The answer is False for both prior distributions; neither are proper probability distributions.

Part H

True or False? In the limit, as prior parameters $k, d, \eta \to 0$, priors $p(\theta|y|)$ and $p(\omega|y|)$ are valid probability distributions. (Must integrate to 1, or something finite so can be normalized over their domains.)

As the hyperparameters go to zero, the posterior parameters are as follows.

$$d^* = (n+d) \to d^* = n \tag{5}$$
$$k^* = (n+k) \to k^* = n \tag{6}$$
$$\mu^* = \frac{n\bar{y} + k\mu}{(n+k)} \to \mu^* = \bar{y} \tag{7}$$
$$\eta^* = \frac{nk(\bar{y}-\mu)^2}{(n+k)} + S_y + \eta \to \eta^* = S_y \tag{8}$$

$\underline{p(\theta|y)}$:

The marginal posterior $p(\theta|y)$ is also a gamma mixture of normals, which can be written as a scaled, centered t. Plug the updated posterior parameters into the central t form.

$$\left[\left(1 + \left(\frac{1}{d^*}\right)\frac{(\theta-\mu^*)^2}{\frac{\eta^*}{d^*k^*}}\right)\right]^{-\frac{d^*+1}{2}} \qquad \to \qquad \left[\left(1 + \left(\frac{1}{n}\right)\frac{(\theta-\bar{y})^2}{\frac{S_y}{n^2}}\right)\right]^{-\frac{n+1}{2}}$$

This is a valid probability distribution. It has form of the centered, scaled t distribution, with parameters

$$m = \bar{y}$$
$$s^2 = \frac{S_y}{n} = \frac{\sum_{i=1}^{n}(y-\bar{y})^2}{n^2}$$
$$\nu = n$$

$\underline{p(\omega|y)}$:

The marginal posterior $p(\omega|y)$ is *gamma* $\left(\frac{d^*}{2}, \frac{\eta^*}{2}\right)$. Plug the updated posterior parameters into the gamma density.

$$p(\omega|y|) = \frac{(\frac{\eta^*}{2})^{(\frac{d^*}{2})}}{\Gamma(\frac{d^*}{2})}\omega^{\frac{d^*}{2}-1}e^{-\omega\frac{\eta^*}{2}} \qquad \to \qquad p(\omega|y|) = \frac{(\frac{S_y}{2})^{(\frac{n}{2})}}{\Gamma(\frac{n}{2})}\omega^{\frac{n}{2}-1}e^{-\omega\frac{S_y}{2}}$$

This is a valid probability distribution. It is *Gamma* $\left(\frac{n}{2}, \frac{S_y}{2}\right)$.

The answer is True for both posterior marginal distributions. Both are valid probability distributions. Morevoer, their parameters are intuitive based on priors assuming zero information.

Part I

The Bayesian Credible Interval has the form $\theta \in m \pm t^* \cdot s$ where m and s are the posterior mean and scale, and t^* is the appropriate t critical value using the posterior degrees of freedom: $t^* = t_{\nu^*, 1-\frac{\alpha}{2}}$.

True or False? In the limit, as prior parameters $k, d, \eta \to 0$, the Bayes Credible Interval equals the frequentist confidence interval for θ.

This is true. We can use the marginal posterior of $\theta|y$ from Part E, which is centered and scaled t.

$$p(\theta|y) \sim t\left(m = \mu^*, \nu = d^*, s^2 = \frac{\eta^*}{d^* k^*}\right)$$

where, in the limit, the posterior parameters go to

$$\begin{aligned}
d^* &= (n+d) \to d^* = n \\
k^* &= (n+k) \to k^* = n \\
\mu^* &= \frac{n\bar{y} + k\mu}{(n+k)} \to \mu^* = \bar{y} \\
\eta^* &= \frac{nk(\bar{y}-\mu)^2}{(n+k)} + S_y + \eta \to \eta^* = S_y
\end{aligned}$$

The Bayes Credible Interval goes to

$$m \pm t^* \cdot s \qquad \to \qquad \bar{y} \pm t^* \cdot \sqrt{\frac{S_y}{n^2}}$$

THIS IS CLOSE - but there is an extra n hanging out in denoninator. Look into this.

Appendix: R Code