Macroeconometrics
Problem Set 1 *

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Contents

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1 Introduction

This problem set covers topics such as simulation of stochastic processes, ARIMA modelling and order of integration analysis of time series.

The Stata do-file for creating the outputs shown and commented below is downloadable from the Github repository at bit.ly/2PshW3X

2 Questions

2.1 Data Generating Process

A stochastic progress x_t is given by the DGP

$$\begin{gathered} x_t = f(t) + \phi x_{t-1} + \varepsilon_t, \\ \text{with } f(t) = \mu \neq 0, \quad |\phi| < 1 \text{ and } \varepsilon_t \sim iid(0, \sigma_\varepsilon^2) \end{gathered} \tag{2.1}$$

x_t is the realization of an ergodic stochastic process if it is stationary and for any point in time t satisfies

$$\lim_{T \to \infty} \left(T^{-1} \sum_{s=1}^{T} Cov(x_t, x_{t+s}) \right) = 0 \tag{2.2}$$

That is, on average the memory of the process is bounded. Thus, covariances goes towards 0 for two different point in time so that two parts of the same stochastic process can be independent from each other.

By iterative substitution we see that equation 2.1 can be rewritten as

$$\begin{aligned} x_t &= \mu + \phi x_{t-1} + \varepsilon_t \\ &= \mu + \phi(\mu + \phi x_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \mu(1+\phi) + \phi^2 x_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t \\ &= \mu(1+\phi) + \phi^2(\mu + \phi x_{t-3} + \varepsilon_{t-2}) + \phi\varepsilon_{t-1} + \varepsilon_t \\ &= \mu(1+\phi+\phi^2) + \phi^3(x_{t-3}) + \phi^2\varepsilon_{t-2} + \phi\varepsilon_{t-1} + \varepsilon_t \\ &\vdots \\ &= \phi^T x_{t-T} + \sum_{s=1}^{T} \left(\mu\phi^{s-1} + \phi^{s-1}\varepsilon_{t-s+1} \right) \\ &\text{rewriting the sum of the geometric series:} \\ &= \phi^T x_{t-T} + \mu \left(\frac{1-\phi^T}{1-\phi} \right) + \sum_{s=1}^{T} \phi^{s-1}\varepsilon_{t-s+1}, \\ &\text{where } T \text{ is the number of periods } s \text{ prior to } t. \end{aligned} \tag{2.3}$$

If the process was a unit root the constant terms with μ would accumulate and produce a deterministic trend, but as $|\phi| < 1$ then $\phi^T \xrightarrow[T\to\infty]{} 0$ such that for a high number of periods T equation 2.3 converges to

$$x_t \xrightarrow[T\to\infty]{} \frac{\mu}{1-\phi} + \sum_{s=1}^{T} \phi^{s-1}\varepsilon_{t-s+1} \tag{2.4}$$

And as ε_t is a stochastic term the expexted realization of $_t$ will be

$$E(x_t) = \frac{\mu}{1-\phi} \tag{2.5}$$

As the expected mean is a constant then $\{x_t\}_{t=1}^T$ is an integrable *iid* stochastic process and for $\mu < 0$ the Law of large numbers implies that the mean converge in probability

$$\bar{x}_T \xrightarrow{p} \frac{\mu}{1-\phi} \tag{2.6}$$

Furthermore, we assume $T \to \infty$ then x_t can be rewritten to a sequence of random variables $\{\tilde{x}_T\}_{T=1}^{\infty}$ that are drawn from a population with a probability distribution with finite mean $\frac{\mu}{1-\phi}$ and finite variance σ_ε^2. The Central Limit Theorem then implies convergence to the normal distribution

$$\sqrt{T}\left(\bar{x} - \frac{\mu}{1-\phi}\right) \xrightarrow{d} N(0, \sigma_\varepsilon^2) \tag{2.7}$$

The derivations above shows that the memory of x_t is bounded as the stationarity condition $|\phi| < 1$ is fullfilled, thus the memory of the initial value x_{t-T} and chock ε_{t-T+1} decreases to zero the further away in time t is from the initial period $t - T$.

Furthermore, the deterministic term given by the constant $\mu \neq 0$ does not give rise to a linear trend, as $|\phi| < 1$.

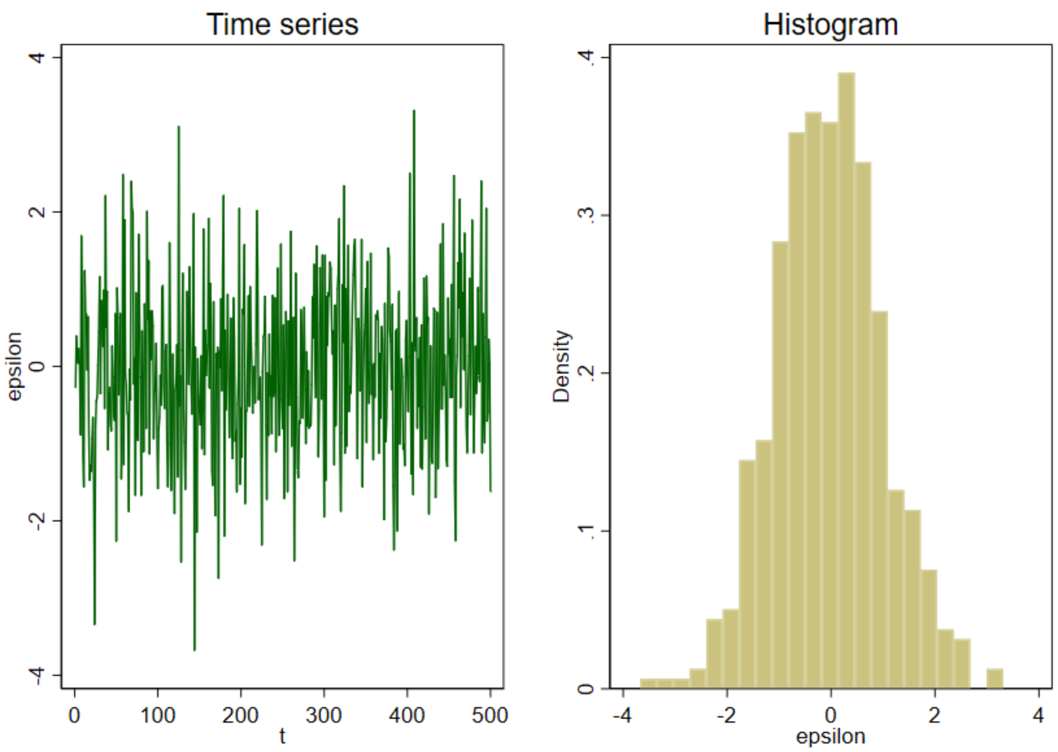
In conclusion, x_t in equation 2.1 describes an ergodic stochastic process.

2.2 DGP in Stata

a) Generate a realization of a Gaussian stochastic process of sample size T = 500, and plot the time series against time

As the data generating process (DGP) is stochastic without any autoregressive element, we simply perform 500 draws from a standard normal distribution. Panel A in figure 1 shows the sime series, whereas the histogram in panel B confirms that the realizations are approximately standard normal distributed.

Figure 1: Gaussian stochastic process



b) Generate AR(1) processes with three different autoregressive coefficients $\phi = \{0.5, 0.8, 0.95\}$

Keeping the sample size at 500 and including a burn-in period of 2.000 iterations, the time series for the autoregressive models are plotted in figure 2. The autocorrelation functions and partial autocorrelation functions are plottes in figure 3.

While all three processes are mean-reverting as the stationarity condition $|\phi| < 1$ is fulfilled, it is clear that persistence increases as $\phi \to 1$. This translates into longer lasting and more extreme cycles in figure 2 as the autoregressive element dominates the stochastic error term. Likewise, the autocorrelation functions (ACFs) show that temporal dependence is significant three periods back for $\phi = 0.5$ while the autocorrelation is significant for the past 12-13 periods for $\phi = 0.95$. While I have only shown 25 lags in the figures, extending it even shows that the the latter process is borderline-significantly but negatively autocorrelated with period 30-37 as well.

Figure 3 shows for the ACFs as well as for the partial autocorrelation functions (PACFs) that the coefficient of correlation with the first lag is close to the actual ϕ-values in the DGP. Except for a few outliers, the PACFs are inconsistent with AR(p) models of order $p > 1$.

As the number of lags lags that are significantly autocorrelated are much higher in the ACFs as opposed to the PACFs this indicate that the processes are AR(1) and not MA(q) or ARMA(1,q) processes.

Figure 2: AR(1) processes with different autocorrelation coefficients

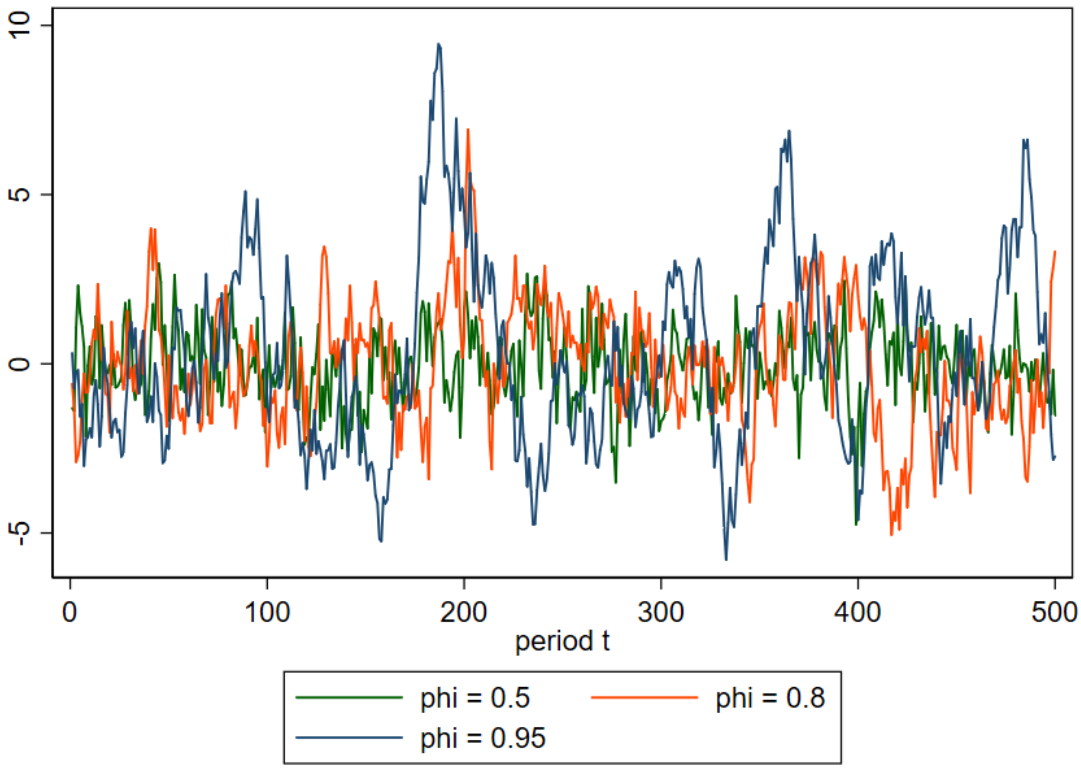
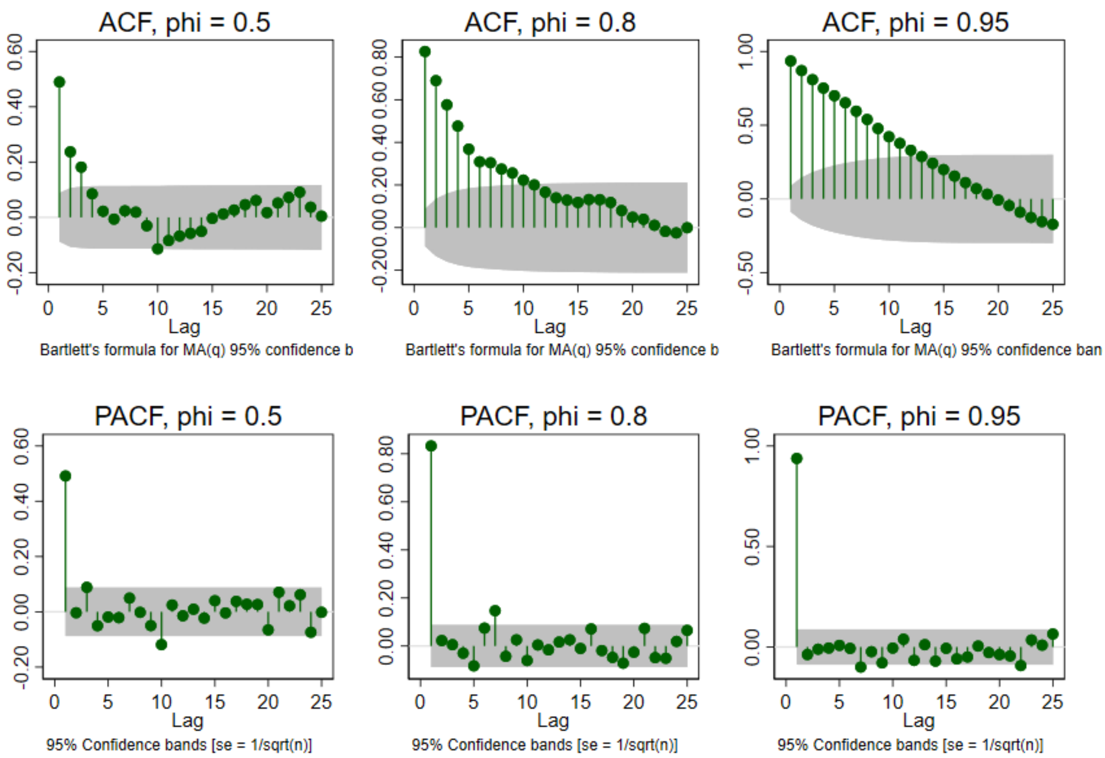


Figure 3: Autocorrelation and Partial Autocorrelation functions for AR(1) processes



b) Generate MA(1) processes with three different moving average coefficients $\theta = \{0.5, 0.8, 0.95\}$

The Moving Average processes are likewise generated with a sample size at 500m including a burn-in period of 2.000 iterations. The time series are plotted in figure 4 and the autocorrelation functions and partial autocorrelation functions are plottes in figure 5 on the next page.

As opposed to the AR(1) processes there are as good at no signs of persistence in the time series for the MA(1) processes. Though a bit difficult to tell from comparison of figure 1 and 4, the summary statistics show that the standard deviation is a little higher for the MA(1) processes than for the standard normal distribution, and increase slightly with $\theta -> 1$.

The ACFs in figure 5 confirms that there are not any persistence after the 1st lag, while the PACFs show significant lags besides the first but with alternating sign. For $\theta = 0.5$ x_t is linearly correlated with x_{t-1}, x_{t-2} and x_{t-3} and arguably with x_{t-3} (borderline significant). The MA(1) process with $\theta = 0.95$ on the other hand shows significant correlation with all lags 1-8.

The number of significant lags are much higher in the PACFs as opposed to the AACFs where only the first lag is significant which is consistent with an MA(q) model of order $q = 1$ while inconsistent with the processes being AR(p) or ARMA(p,q) processes.

Figure 4: MA(1) processes with different autocorrelation coefficients

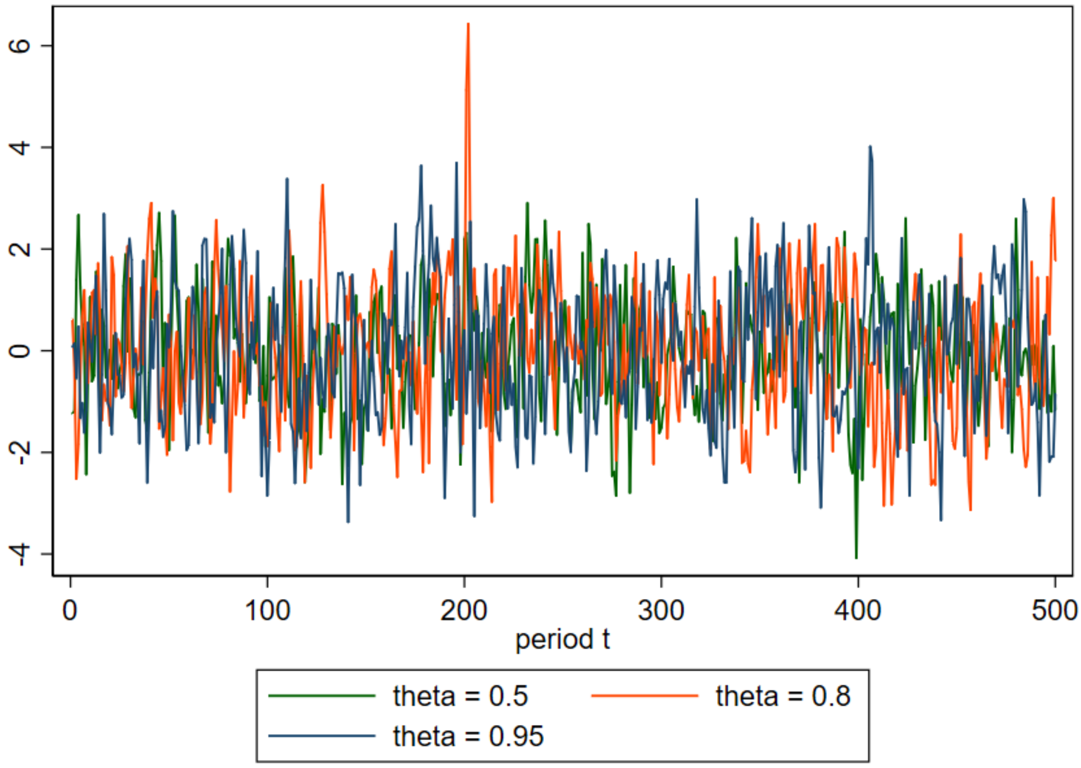
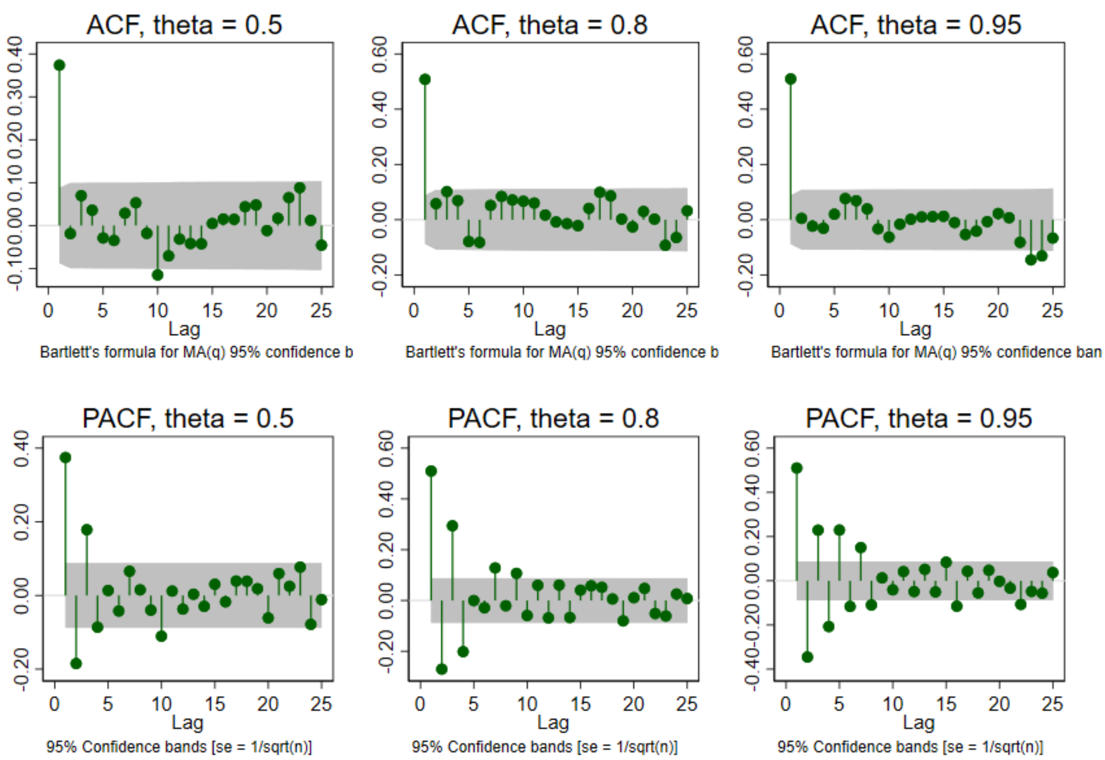


Figure 5: Autocorrelation and Partial Autocorrelation functions for MA(1) processes



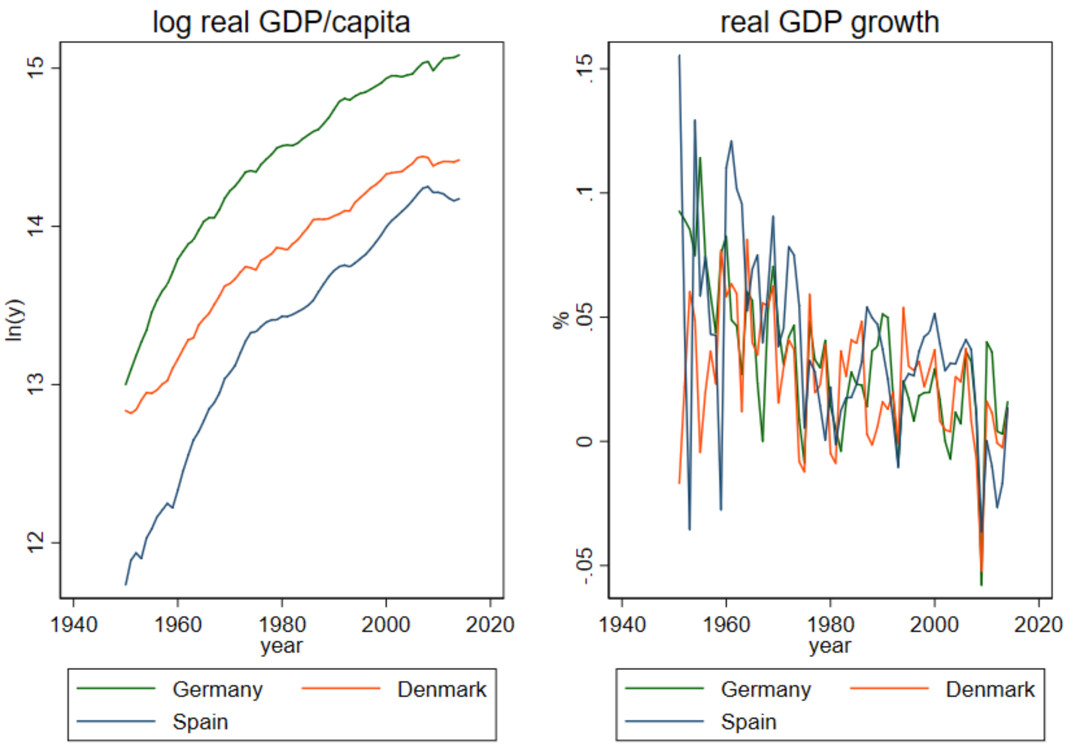
2.3 ARIMA modelling

Using the "Penn World Tables v. 9.0"[1] I analyze the development in the logarithm of real GDP per capita (at 2011 national prices) from 1950-2014 for Spain, (Western) Germany, and Denmark[2].

a) Plot the time series

From figure 6 below it is obvious that the series for actual values of log GDP are not stationary while the transformed series, namely the first differences might be. Thus we will work with theese series instead.

Figure 6: Time series and first differences of log real GDP/capita



However, the much higher growth rates in the 50s and 60s does raise the question of functional form even for the transformed processes of 1st order. Is there a negative trend or a level shift? Also business cycles and especially negative shocks are evident. All three countries experience negative growth due to the busts in 1993 and 2009. Furthermore Denmark is hit significantly by the oil crisis in 1974-75 and Germany in 1975. Denmark also starts the 80s with negative growth rates due to the recession, while Germany again only experience detraction for one year in 1982 and again in 2003. The negative shock

[1] Downloadable from `https://www.rug.nl/ggdc/productivity/pwt/`

[2] Before taking the log, the Danish Kroner (DKK) is transformed to Euros by multiplication with 7.46038 which is the central exchange rate at which the DKK is pegged to the Euro within a tiny 2.5% band, according to the European Exchange Rate Mechanism (ERM).

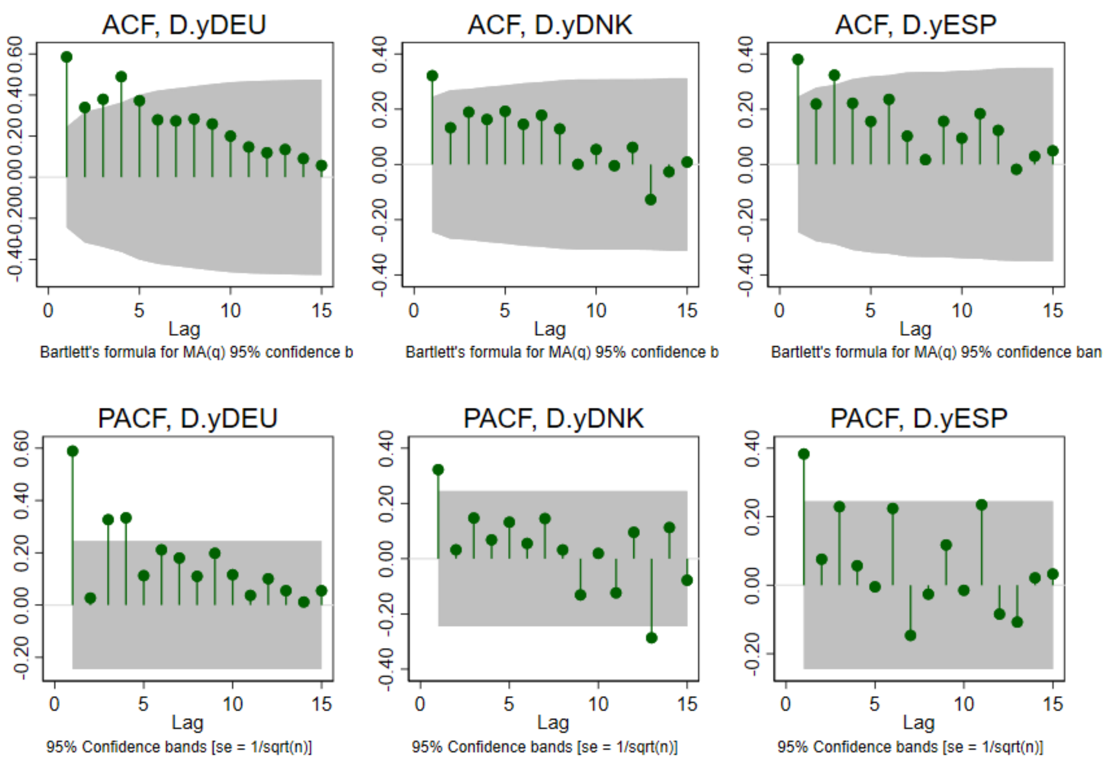
also shows persistence in Denmark in 2008-2009, even more so in Spain with detraction through 2011-13.

b) ***Identification: using autocorrelation and partial autocorrelation functions, identify the orders of the ARIMA model. Compute both the numerical and graphical autocorrelation functions (including confidence bands)***

The ACFs and PACFs in figure 7 below show first of all that we do not need to worry about unit roots as none of the lags have an autocorrelation coefficient above 0.6.

The GDP growth of Germany show signs of autocorrelation for up to 14 lags in the numerical correlogram and in the partial autocorrelation for lag 1, 3-4, 6-7, 9, as well as with several 16+ lags. The high number of lags in both the ACF and PACF suggests that German log real GDP can be approximated by an ARIMA(p,1,q) model. The graphical representation with the confidence bands however is more consistent with 4 ACF lags and 4 PACF lags, thus suggesting an ARIMA(4,1,4) model. However I choose to follow the Parsimonious principle of estimating the more simple model, thus, my more cautious suggestion would be an ARIMA(1,1,4) model as the coefficient of the second PACF lag is close to zero, well-knowing that we do loose information as lag 3 and 4 that both are clearly significant.

Figure 7: ACF and PACF of the first differences of log real GDP



While the graphical plots of ACF and PACF for real GDP growth in Denmark only show that the first lag is within the 95% confidence interval, the correlogram still show signs of autocorrelation and partial autocorrelation. In fact it is not untill 12^{th} lag is included, that the The Box-Pierce Qtest (Ljung-Box) rejects the null-hypothesis that there is no autocorrelation in the sample. This gives confidence that we can try an ARIMA(1,1,1) model. Especially if controlling for some of the no less than 12 out of the 64 periods with negative growth.

Like for Denmark, the stochastic process for Spain also show a little more evidence of autocorrelation than of partial autocorrelation. We will likewise test an ARIMA(1,1,1) model though a MA(1) or MA(3) model for growh in real GDP would be just as well-foundes.

c) Estimation: fit the identified ARIMA specification, using MLE estimation procedures

Table 1 on the next page show the baseline ARIMA(1,1,1) estimations for each country in column 1, 6, and 7 respectively. First of all it is worth noting that all models are poorly-specified. |MA-roots|<1 but While the point estimates barely fulfills AR-roots<1, the size of the standard errors show that none of the models are significantly different from a autoregressive unit root process.

For Germany the ARIMA(1,1,1) is tested against the ARIMA(1,1,4) (the ARIMA(1,1,2) does not even converge). As the Akaike information criterion (AIC) points towards ARIMA(1,1,4) and the Bayesian IC (BIC) points towards the ARIMA(1,1,1) specification we still have a subjective choice to make. In the spirit of Milton Friedman's positive economics, we go with the ARIMA(1,1,4) model in order to maximize predictive power at the cost of using a less realistic model.

d) Validation: check the estimated residuals for misspecification errors

The residuals are computed and investigaterd for the estimated models. For Germany none of the lags in the ACF for the residuals are significant. In the PACF only lag 15, 22, 25 are significant but we cannot model an AR(p) model where only lag 15, 22, 25 are significant. Thus, they must be so-called 'false signals' which could be explained by model shifting due to one of the five structural breaks that we found earlier. The more sizable negative residuals are for the 2009-crisis followed by the 1974-75 oil-crisis and the 1993-recession. 2009 is the only year where the residual exceeds the standard error of the autoregressive coefficient, but it is not able to produce 'false signals' 15+ lags behind, but including dummies for these 4 years should certainly improve the estimate. The estimation of the ARIMA(1,1,4) and ARIMA(1,1,1) for Germany controlling for these 4 outliers is shown in column 3-4 of table 1 below. As expected both IC improve. However the MA coefficinents become highly insignificant for both models and for the ARIMA(1,1,4) the standard errors explode to unacceptable dimensions. Therefore a AR(1) model for real GDP growth is tested (column 5 in the table). While both information criteria increase a little compared to the ARIMA(1,1,4) model with dummies it is an improvement compared to the ARIMA(1,1,1) model with dummies. Most notably, when controlling for structural breaks the AR-coefficient is now clearly different from a unit root.

Like for Germany the residuals are not autocorrelated for Denmark nor Spain with a few pointless partial autocorrelations with high order lags. For all countries the correlation of all lags are close to zero and for all lags the Q test clearly reject the H_0 that there is no-autocorrelation, thus, there is nothing left to be explained and our models are well-specified.

Table 1: Estimation of ARMA models for real GDP growth

	(1) Germany b/se	(2) Germany b/se	(3) Germany b/se	(4) Germany b/se	(5) Germany b/se	(6) Denmark b/se	(7) Spain b/se
D.y74			-0.014	-0.006	-0.013		
			(0.013)	(0.008)	(0.013)		
D.y75			-0.033	-0.037***	-0.033		
			(0.025)	(0.008)	(0.025)		
D.y93			-0.016	-0.023***	-0.017		
			(0.020)	(0.005)	(0.017)		
D.y09			-0.043***	-0.037***	-0.045***		
			(0.009)	(0.008)	(0.010)		
cons	0.042*	0.043*	0.034***	0.043*	0.035***	0.023***	0.038
	(0.024)	(0.023)	(0.009)	(0.025)	(0.010)	(0.007)	(0.024)
ARMA							
L.ar	0.980***	0.970***	0.698***	0.984***	0.762***	0.869***	0.923***
	(0.031)	(0.044)	(0.130)	(0.025)	(0.084)	(0.152)	(0.117)
L.ma	-0.731***	-0.535***	0.136	-0.915		-0.680***	-0.688***
	(0.096)	(0.148)	(0.195)	(.)		(0.206)	(0.164)
L2.ma		-0.384**		-0.702			
		(0.175)		(1163.033)			
L3.ma		0.021		-0.915			
		(0.175)		(3033.344)			
L4.ma		0.294*		1.000			
		(0.155)		(3313.741)			
sigma	0.022***	0.020***	0.018***	0.010	0.018***	0.023***	0.034***
	(0.002)	(0.002)	(0.002)	(16.083)	(0.002)	(0.002)	(0.002)
aic	-296.996	-301.882	-313.860	-332.106	-315.598	-291.739	-244.462
bic	-288	-287	-297	-311	-300	-283	-236

Standard errors are in parentheses. * p<0.10, ** p<0.05, *** p<0.01

e) Forecasting: compute the one-step ahead forecast of the log of real GDP per capita time series

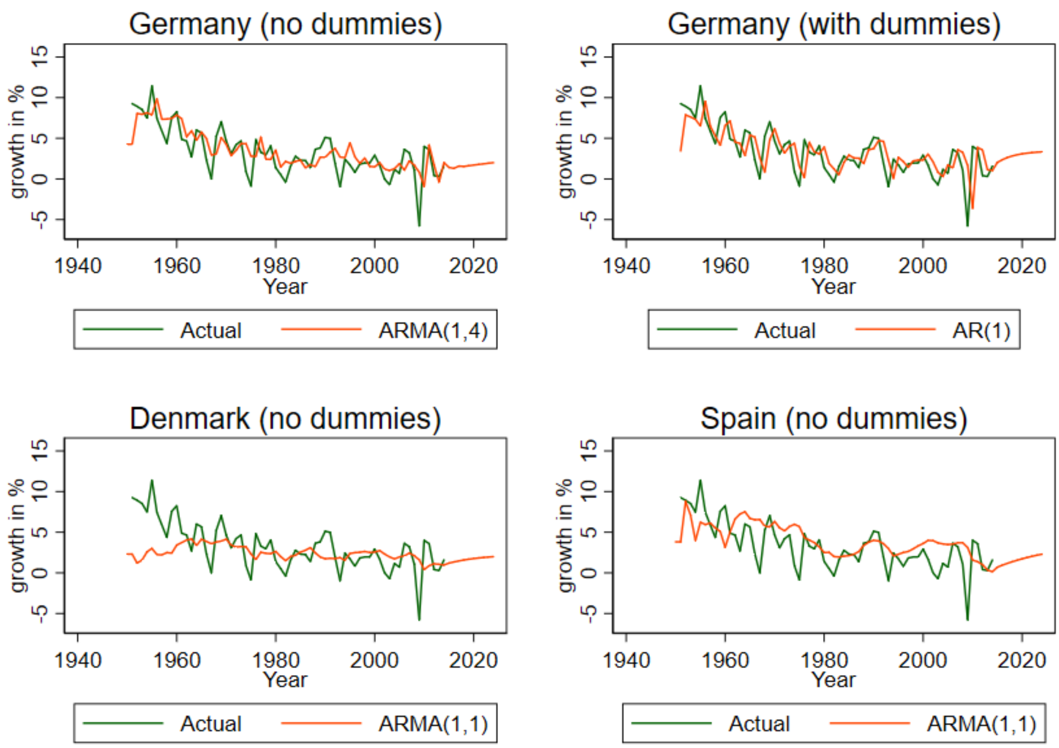
Forecasting one step ahead (2015) predicts a growth rate in real GDP per capita in Germany of 1.4 and 2.0 (ARMA(1,1) without dummies and AR(1) model with dummies respectively). For Denmark the growth forecast is 1.2, and for Spain 0.7 in 2015.

f) Plot both the actual and predicted time series for each country

Figure 8 evaluates the predictive power of each models by plotting them against the predicted values against the actual values as well as forecasting real GDP growth the coming 10 years.

While including time dummies for negative chocks clearly increases within-sample predictability, the downfall becomes very evident as well. The AR(1) model for the growth rate in real GDP predicts that the growth rate in Germany goes from 1.6 in 2014 to 2.0 in 2015. Thus, we see that omitting the negative chock of the 2009-crisis removes persistence of the chock to a degree where the model not only predicts a drastic jump, but also a high-pace 'recovery' towards the pre-2009 trend, quickly converging towards a highly unrealistic real growth rate of 3.3 % in 2024.

Figure 8: Actual vs predicted growth in real GDP



2.4 Order of integration analysis