

Position Auctions

One application of auctions which has become increasingly important over the last few years is their application to online advertizing. Hallerman¹ estimates that online advertizing revenues in the US in 2007 were at least \$21 billion. (?) estimates that revenues associated with the auction of search terms by industry leaders Google and Yahoo in 2005 were at least \$11 billion. (?) provide a less specific and more conservative estimate of \$10 billion. Either way, a very large chunk of total advertizing revenue on the internet.

Auctions are used to determine which urls are displayed in the “Sponsored Links” column of the google search page, and the “Sponsored Results” column on Yahoo. These auctions are typically referred to as *Position Auctions*. The technology behind these is remarkable. When a search word is entered, a page is returned with a set of search results (sometimes referred to as ‘organic links’) along with a series of sponsored links. The primary difference between the two is that the sponsored links almost always offer to sell you something. The organic links may or may not involve sales. The sponsored links are useful provided they contain items the person looking at the page is likely to want to buy.

The way the urls are made relevant is to associate with each search term, a set of *positions*.² The first position is the place at the top right of the page where the first sponsored link will be placed, the second position is the second highest link that will be displayed, and so on. Advertisers who are interested in a particular search term can then bid for that position in an auction. When you enter your search term, google checks for bids associated with that search term, then puts the link associated with the advertiser who submitted the highest bid in the

¹Hallerman, D (2008) “US Online Advertizing: Resilient in a Rough Economy”, *eMarketer*, March.

²It is even more sophisticated than this since Google has quite a bit of information about the viewer who has entered the search term. When the viewer sends his search request to Google, his browser passes a cookie to the google server. The cookie doesn’t identify the viewer directly, but it provides an id that can be used to view a database that contains information about past searches that have been made from the same browser. By viewing these, Google can make a good guess about whether the viewer is male or female, rich or poor, young or old, etc. The ip adress of the browser can be used to determine where the viewer is located geographically. Finally, Google knows how likely it is that the viewer using the current browser will actually follow a sponsored link. Bids that advertisers submit can be conditioned on all these things.

first position³, the second highest bidder in the second highest position, etc, before sending the page back to you. In other words, every Google page view results in a new auction being held. In this way, we are all involved in a large number of auctions.

The payment that a bidder makes depends on the number of viewers who actually click on the ad. To give some idea of what is involved, Google estimates that bidders who want to have their link displayed in the third position of the auction associated with the search phrase “luxury hotels in London” would have to pay about \$3.04 per click.

Given the enormous number of times Google and Yahoo are used to conduct internet searches each day, even a tiny probability of clicking on a sponsored ad will result in large revenues for the search sites. Tweaking the design of the auction by modifying reserve prices and limiting search slots could make a lot of difference to total revenue. Squeezing revenue is not the only consideration that is important of course. The search engines also want to attract bidders and viewers who will click on their ads. We deal with some of these issues later in the chapter on competition among auctions. This chapter focuses on the case where the search site faces an exogenous set of advertisers bidding for positions whose value to them is also exogenous.

1. COMPLETE INFORMATION

To begin, we will simply assume that there are K positions which have associated with them exogenously given values $x_1 > x_2 > \dots > x_K > 0$. The idea is that the slot in the first position is the one that consumers are most likely to click. There are two reasons for this. The first is simply that it is closest to the cursor which has just clicked the search box in the top right corner of the browser. A more compelling argument is a sort of self fulfilling prophecy that we will explain in more detail later. If consumers think that the most useful ads are in the first position, then firms will bid more aggressively for that position. However, the firms that are most likely to win the auction for that position are the ones who are most likely to make a sale. So at least when consumers want to buy, their expectation that the firms from whom they are most likely to buy will be in the first position are going to be correct. Interpret the x_j as the expected number of clicks at position j .

³This isn't quite correct. In fact what they do is to give each advertiser a score based on how much revenue they expect to earn from that advertiser given the bid they have submitted. The winner of the auction is the advertiser whose bid generates the highest expected revenue given his score. We use the simpler high bid auction to make things a bit simpler below.

There are $m > K$ firms bidding for the various positions on the web page. Firm i has characteristic v_i that describes its profit per click. For simplicity, assume that $v_1 \geq v_2 \geq \dots \geq v_m$. The expected revenue of a firm who successfully places their ad in position j is $v_i x_j$. Every firm wants the first position, but not all firms will be willing to pay the same amount for it. All firms, on the other hand, still earn profits by placing their ads in lower positions on the web page.

The position auction works as follows: each firm places a bid b_i . Lets order the bids so that b_1 is the highest bid, b_2 the second highest, and so on.

The first and most desirable position is awarded to the bidder who submits the highest bid b_1 . He or she then pays the second highest bid b_2 for each click. Similarly, the bidder who submits the k^{th} highest bid wins the k^{th} position, and then pays the $k + 1^{st}$ highest bid for each click he receives. Let the bids be ordered from highest to lowest. If firm i wins the k^{th} best position (with a bid b_k) and pays b_{k+1} , then his or her revenue per click is $(v_i - b_{k+1})$. Notice that the firm that wins the K^{th} position (i.e., the lowest or worst position) pays the highest bid of among bidders who did not win a slot. Given some array b_1, \dots, b_m of bids, bidder i 's total payoff when he wins the k^{th} slot is

$$(v_i - b_{k+1}) x_k,$$

while his payoff is zero if he doesn't win a slot.

As for any problem in which we want to use Nash equilibrium to predict the outcome, we need to specify the entire game. In other words, we have to describe the payoff a player earns for any array of bids, and for any bid that he might submit instead. This is a little complex here because we have to figure out what will happen when there are ties. The first complication we have to deal with is that we can't simply order the bids $b_1 > b_2$ etc, we need to tie a bid to a bidder. The reason for this complication is that we want to allow a bidder i to consider submitting bids that are higher than all the others, higher than all but one of the others, lower than all the others, etc. The rank of a bidder i 's bid is something we are going to determine as part of our equilibrium.

The strategies of the players are just their bids. So lets make the switch and imagine that the player whose name is i submits a bid b^i . The rank that this bid has depends on the bids of the other bidders. So lets write the bids submitted by all the players as $b = \{b^1, \dots, b^m\}$. So lets write $b_k(b)$ to mean the k^{th} highest bid in the array of bids b . For example, $b_1(b)$ is the highest bid submitted by any player, $b_2(b)$ is the second highest bid in the array b , and so on.

We could also write the bids b in a slightly different way. For example, we could write b^{-i} to mean the array of bids submitted by the players other than i , then $b_k(b^{-i})$ represents the k^{th} highest bid of the bidders *other* than bidder i . Also notice that the vector (b^i, b^{-i}) is the same vector as $b = (b^1, \dots, b^{i-1}, b^i, \dots, b^m)$ except that the elements have been re-ordered. So $b_k(b) = b_k(b^i, b^{-i})$.

Now we can write the payoff to each player for every array of bids in a position auction where the k^{th} position is awarded to the k^{th} highest bidder who is then charged the $k + 1^{st}$ highest bid for each click. I'll take a shortcut here to make the formula somewhat simpler.

(1.1)

$$\pi^i(b', b^{-i}) = \begin{cases} (v_i - b_1(b^{-i})) x_1 & b' > b_1(b^{-i}) \\ (v_i - b_{k+1}(b^{-i})) x_{k+1} & \exists k < K : b_k(b^{-i}) > b' > b_{k+1}(b^{-i}) \\ 0 & \text{otherwise.} \end{cases}$$

This is a complicated expression, so we'll go over it. When a bidder considers a bid b' , she first checks the bids of the other players (i.e., b^{-i}) and puts them in order $\{b_1(b^{-i}), \dots, b_m(b^{-i})\}$. She then looks to see if her bid b' is higher than one of the K highest bids that the others have submitted. For example, if there are four bidders (as there will be in the graphical illustration below), and they submit the bids $\{2, 4, 3, 6\}$ (where 2 is submitted by the bidder with identity 1, 4 by bidder with identity 2, etc, then bidder 3 first takes the other bids $\{2, 4, 6\}$ and puts them in descending order. This gives her the vector $b_1(\{2, 4, 6\}) = 6$, $b_2(\{2, 4, 6\}) = 4$ and $b_3(\{2, 4, 6\}) = 2$. Suppose there are $K=3$ positions to be won in the auction. She sees that there is a value for j , i.e., 2, which is less than the number of positions up for auction, i.e., 3, and such that her bid is still at least as large as $b_3(\{2, 4, 6\}) = 2$. That means she will win a slot with this bid, that is slot $j + 1$ or 3, and pay $b_3(\{2, 4, 6\}) = 2$ for it. Her profit is then the difference between her value and the price 2 multiplied by the average number of clicks associated with slot 3. The math expression above is certainly a more concise way of saying this. It even works when there are more bidders than slots once you think it through.

Now you might have noticed already what the shortcut is - I left out the possibility of equality. The reason I did this (apart from avoiding writing out a long formula) is that I already know that placing a bid exactly equal to the bid of another player is a weakly dominated strategy. It might not matter if the bid doesn't win a slot, but if it does, then the slot I get would be randomly decided in a lottery with the bidder with whom I had tied. If I raise the bid ever so slightly, I would be sure to get the highest of the two slots, which is better. So I

couldn't possibly have an equilibrium is which two bidders submit the same bid and one of them actually wins a slot.

Now that we have the profit function (1.1), we basically know the payoffs that each player will get for every array of bids. That is like knowing what payoffs we put in each cell of the matrix in a bimatrix game. A Nash equilibrium of this game is an array of m bids such that no player can improve his payoff given the bids of the other players.

We can't really do the gambit thing and check every profile of bids for profitable deviations. There is a continuum of profiles to worry about. Solving games like this usually means making a good guess about what the equilibrium looks like, then trying to verify your guess is an equilibrium. The educated guess here comes from the observation that if you win a slot at all, you won't pay what you bid. Instead, you'll pay what one of the other players bid. So your only real choice is to decide which slot, if any, you want. The others' bids completely determine what you pay when you win it.

The second part of the guess comes from the intuition that it is likely to be the case that the bidder with the highest value will want the best slot, the bidder with the second highest value will want the second best slot, and so on. Why? Well, at this stage in your reasoning you can't really articulate why, but it just seems right. Nash equilibrium is going to give us a conceptual tool that will make it crystal clear why that intuition will work.

With that start, let's just guess that each slot will have a price, say $p_1 \geq p_2 \geq \dots \geq p_K$, and that the bidder with the highest value will have to win the best slot, and so on as we described above. Since bidders are really only choosing which slot to win, this will only work if the set of prices satisfies

$$(1.2) \quad v_i - p_i \geq 0$$

for $i = 1, \dots, K$ and $v_i - p_K \leq 0$ for all $i > K$. If this weren't true, then some of the winners would be sorry to have won.

Also, since players can also win other players' slots by outbidding them (and they don't have to worry about their actual bid), then for $K \geq i \geq 1$, player i would prefer to win slot i at price p_i , than to win any other slot ,

$$(1.3) \quad (v_i - p_i) x_i \geq (v_i - p_j) x_j$$

for each i and j .

If we could find a solution to these inequalities, then we could just have each bidder submit the corresponding bid, i.e., the bidder with the highest value could submit the bid v_1 , etc. No bidder would want

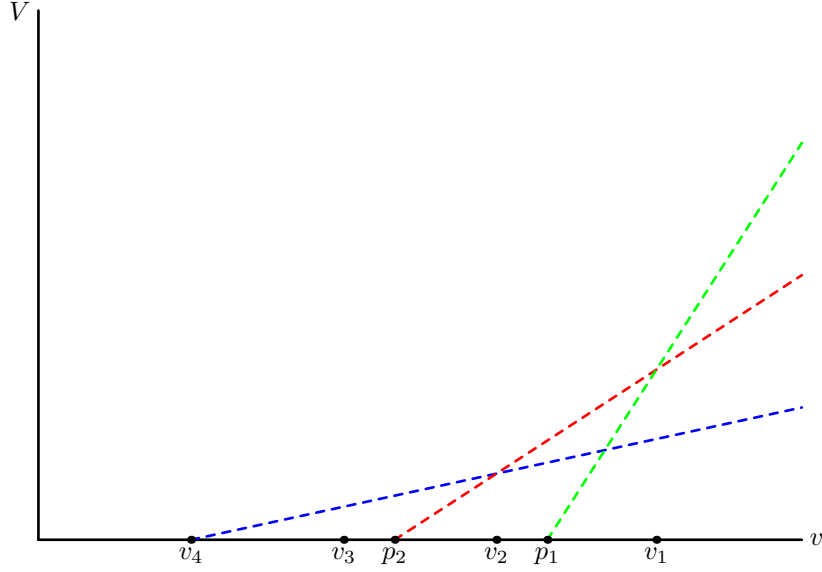


FIGURE 1.1. Recursive Construction of Slot Prices

to outbid any other bidder, because that would violate (1.3). This would constitute a Nash equilibrium. Finding a solution to a set of inequalities – now we are back in algorithm mode.

To begin, start with the firm with the $K + 1^{st}$ highest value, and suppose that this firm bids its value v_{K+1} . This determines the price of the K^{th} and worst slot to be v_{K+1} . If firm K buys this slot, it earns $(v_K - v_{K+1})x_K \geq 0$. Now for $i < K$, recursively define

$$(v_i - p_i)x_i = (v_i - p_{i+1})x_{i+1}.$$

A diagram will illustrate why this simple construction will generate a set of competitive prices. We illustrate for the case where there are three slots.

The horizontal axis in the picture measures the possible values that firms can have for slots. The values of the top four firms are marked. The dashed line with the lowest slope (the blue line in the picture) is the graph of the function

$$(v - v_4)x_3$$

which describes the payoff to different firm types when they buy slot 3 at price v_4 . It is flat because the slope of the line is x_3 , the click through rate of the worst available slot. The advantage of drawing this is that it makes it possible to see the profits the higher valued firms would make from this slot as well. Following our construction, we choose the price p_2 for slot 2 that makes the bidder with value v_2 just indifferent

between purchasing slot 2 at price p_2 , and buying slot 3 at price v_4 . The dashed red line in the picture represents the payoff function $(v - p_2)x_2$. This line is steeper than the line for the third slot because its slope is equal to the higher click through rate x_2 . Since the click through rate x_2 is higher than the click through rate for slot 3, p_2 is going to have to be higher than v_4 . Since the payoff function given by the red line is steeper than the one associated with the blue line, firm 3 is going to prefer slot 3 to slot 2 when they are priced this way. The green line (the one with the steepest slope) shows how to construct the price p_1 for the most valuable slot.

Now these prices v_4 , p_2 and p_1 satisfy the inequalities (1.2) and (1.3). To see why, look at the outcome for the bidder with value v_3 . He is supposed to win slot 3 at price v_4 , so the profit he gets from that can be read as the distance up to the blue dashed line. Similarly he could try the other slots by paying p_2 or p_1 , and you can read the consequences of these choices by measuring the distance up to the red and green lines - at value v_3 , up is down to those lines, which means that bidder 3 would lose money on these other slots.

On the other hand, bidder 2 is supposed to win slot 2 at price p_2 . His profit is the distance up to the red line above v_2 . He could win the lower slot at price v_4 , but if he did, he would get the same profit because of the way we chose p_2 . Again, if he tried to win the higher slot, his profit would be negative, which you can see by reading from the graph.

So the bids, v_4 , p_2 , p_1 and v_1 will support the slot prices that satisfy (1.2) and (1.3).

It isn't hard to understand why the firms do this in this example. They never expect to have to pay what they bid, so there is no reason for them to worry about their bid, beyond the fact that it secures the slot they want. Firm 3, for example, knows in this equilibrium that he will only have to pay v_4 for the slot.

Second, observe that we can support many Nash equilibria this way. All we need to do is to make sure that each of the top three firms makes a profit on the slot they win, and that each prefers their slot to any other slot. In the picture, just slide the red line to the left to any position in which firm 3 prefers slot 3 to slot 2 (which just means the blue line has to be higher than the red line at v_3). Similarly, slide the green line to the left lowering p_1 to any position at which the blue line is higher than the green line at v_2 . These changes will generate new Nash equilibria with new prices for the different slots. These prices will be lower for slots 1 and 2.

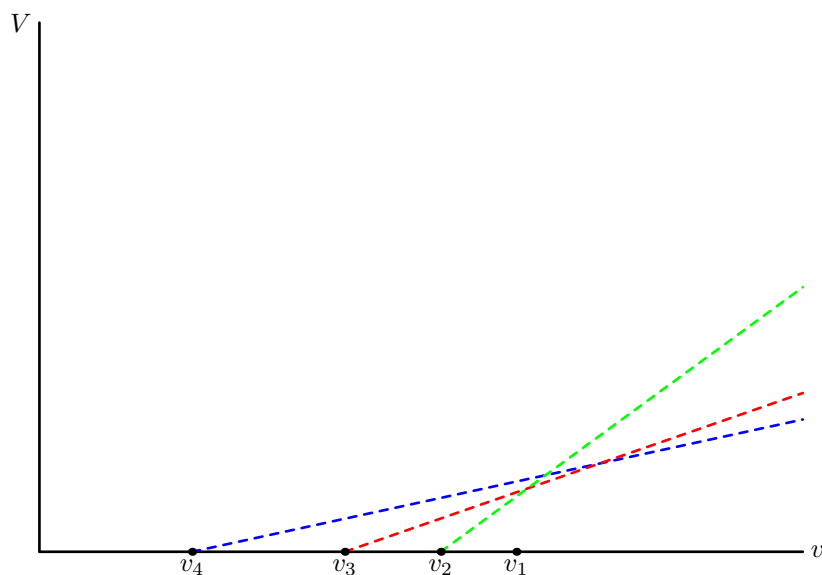


FIGURE 1.2. Bidding Values?

On the other hand, we could go the other way and raise revenues that are generated by the position auction. We can't just raise bidder 3's bid this time, because if we do, bidder 2 will switch and want to bid on the worst slot instead of on the second best slot. We could raise revenues by increasing the bid by bidder 4 however. This may be one reason that google doesn't explain exactly how they set their prices. The Nash equilibrium leaves some money on the table for them if they can set the price of the lowest slot to the value of the third highest bidder.

A second price auction of the kind we are studying here often has the feature that bidders will bid their 'valuations'. This won't necessarily support an equilibrium here as Figure 1.2 illustrates.

In this Figure, each bidder bids his own value, which would give prices v_4 for slot 3, v_3 for slot 2 and v_2 for slot 1. The it is easy to see from the figure, that at these prices, all three of the top bidders would prefer to win the third slot. The reason is that the higher prices that are supported by the bidders values, aren't warranted given the marginally higher click through rates associated with the better slots. We have no problem constructing an equilibrium for this problem, however, it won't support an outcome where bidders bid their true values.

Exercises.

- (1) Can you give an algebraic condition (i.e., an inequality) that will ensure that there is a Nash equilibrium for this game in which each bidder bids their value?
- (2) Using the diagrammatic method, show how to compute prices for the case where the bidder with the fourth highest value bids v_3 instead of v_4 .
- (3) Write down a constrained maximization problem that shows how google would maximize its ad revenues from this action by setting slot prices satisfying (1.2) and (1.3). Can you suggest an algorithmic method to solve this problem? Could you code it?

2. CONSUMER SEARCH

Apart from the fact that firms bid more than a slot is truly worth to them in the position auctions, there are a number of characteristics of this model that seem unrealistic. First, each firm knows exactly what the values of the other firms are. This is what allows them to confidently bid way more for a slot than it is worth to them. Perhaps more important, the click through rates that the firms are bidding on are unrelated to the characteristics of the firms that are actually making the bids. Most consumers will only click on sponsored ads if they feel that these ads will provide them useful information. Whether they do or not must surely depend on the characteristics they believe the bidding firms have. Secondly, there is a strong connection between click through rates and the position of the ad on the web page. Presumably this reflects the fact that consumers think that ads in a higher slot are more informative. It would be nice to know exactly why this might be.

In this section we try to incorporate some of these things into the analysis. Let's assume first that consumers bear costs when they click on a sponsored link. There are two reasons this is a reasonable assumption. First, consumers' interest in following a commercial link differ. Some consumers have a direct and immediate need, while others may be viewing search results for other reasons. A consumer whose primary interest is to locate a fact may find it very costly to follow a sponsored link that may or may not lead him to something that he can buy. Secondly, even consumers with immediate needs may find sponsored links costly. The link itself probably won't directly provide the price and product information they want, causing them to have to search a specific website while they may be able to find the information they want more effectively by searching organic links.

Suppose the search cost for consumer i is s_i drawn using a uniform distribution on $[0, 1]$. Lets make things simple and assume that there are only two consumers like this. Firms, on the other hand, are distinguished by the probability with which they will be able to supply the good the consumers are looking for. Let q_j be the probability with which firm j can supply a consumer. Again, we'll assume this is drawn using a uniform distribution on $[0, 1]$. We'll also assume that there are 3 firms competing for two slots.

Consumers have value 1 if the link that they click leads a product they want. The firm at that link also receives revenue 1 in this case. So the way the auction works is that consumers open a page with 2 sponsored links. A consumer has cost s_i of clicking a sponsored link and discovering whether or not the good that is advertized is one that they want. Depending on the cost of clicking on the link, the consumer decides whether or not to click. If the cost is low enough that the consumer does want to click, we assume that she clicks the top link first and explain why it makes sense for her to do this in a bit. If she decides it isn't worth it to follow the sponsored link, she leaves with zero payoff.

If the firm behind the link has quality q_j , then a transaction happens with probability q_j . In that event the consumers payoff is $1 - s_i$ and the firms payoff is $1 - b_i$ where b_i is the amount that the firm agreed to pay for its advertizing slot when it won it at auction. If the transaction doesn't happen, the firm gets nothing from the transaction, but still pays b_i for the click. The consumer then decides again based on her cost, whether or not to click on the second sponsored link. If she doesn't click, then she simply leaves the auction having born the search costs associated with her first click. If she does decide to continue searching, she meets a firm with whom she transacts with whatever probability q characterizes that firms quality. If no transaction occurs in this case, she leaves the auction site with payoff equal to minus twice her search cost.

This model (which is based on (?)) is relatively simple, but captures a number of interesting features. First, the value of the different slot positions to firms is endogenous since the click through rate will depend on how many consumers have search costs low enough to induce them to follow the link. The identity of these consumers is also endogenous since whether or not they want to follow a link will depend on how likely it is the link will provide them a transaction. If the firms who are most likely to be able to supply consumers (that is, firms with high q) think that consumers are clicking the top ad first, then they will be willing to bid more for the first slot. If they do, then consumers will be

quite rational in thinking that they are more likely to find a transaction if they click the first link than if they click the second.

This model also resolves one of the issues associated with the position auction we discussed in the first part of this chapter, since firms won't know exactly the values of their competitors. This helps make bidding strategies of the firms more reasonable.

As before, we assume the firms submit bids to the position auction. The highest bidder is awarded the top slot on the page, and required to pay the bid of the second highest bidder. Similarly, the second highest bidder is given the second slot and required to pay the bid of the third highest bidder.

2.1. Equilibrium. To see how the equilibrium unfolds, let's suppose for the moment, that each firm uses the same bidding strategy $b(q)$ that is monotonically increasing in q . If the realized qualities of the three firms are $q_1 > q_2 > q_3$, then $b_1 > b_2 > b_3$ and firm 1 will win the first slot on the page, while firm 3 won't win any slot at all. From consumers' perspective, the expected quality of the firm whose advertisement appears in the first slot is the expected value of the highest of the three qualities of firms who are bidding, while the expected quality of the firm in the second slot is the expectation of the second highest quality. As these qualities are uniformly distributed, it isn't so hard to figure out what these are. For example, there are three firms who qualify as candidates for the highest quality firm. Each quality in the interval $[0, 1]$ is equally likely. For any particular quality q and firm, the probability that the other two firms have lower qualities is q^2 . So the expected value of the highest quality firm is

$$\int_0^1 q (3q^2) dq = \frac{3}{4}.$$

Similarly, the expected quality of the firm who wins the second slot can be calculated from the following observation: if this firm has quality q , then the probability that one of the others has lower quality, while the other has higher quality is $q(1 - q)$. There other two can be ordered in two different ways, and there are three different firms who could have the second highest quality. So the expected quality of the firm in the second slot is

$$\int q 6q (1 - q) dq = \frac{1}{2}.$$

This begins to explain why it makes sense for consumers to click on the top ad, since the probability with which they will find what they are looking for if they click there is $\frac{3}{4}$, while if they click on the second

ad it is only $\frac{1}{2}$. So we will assume from now on that consumers whose search costs are low enough that they will want to click on some ad, will choose the top url first.

Their search problem doesn't end at this point, because there is still a ex ante probability of $\frac{1}{4}$ that they won't find what they want after clicking on the first ad. The first ad represents their best chance of finding a useful product. So if they don't find what they want, then it is bad news, and they should revise their estimate of the expected quality of the firm at the second position downward.

To do this, we need to calculate the probability that the consumer will trade if he clicks on the second link conditional on failing to trade after clicking on the first link. The easiest way to do this computation is directly using the distribution of the second order statistic of firm qualities. If the firm with the second highest quality has quality x , then the expected quality of the firm with the highest quality is $x + \frac{1-x}{2}$. Conditional on the second best firm's quality, the probability that the consumer trades with the second best firm and fails with the first is $x(1 - x - \frac{1-x}{2})$. Integrating this across the possible qualities of the second best firm using the density of the second best firm's quality that we derived above, then dividing by the probability $\frac{1}{4}$ with which the consumer fails to trade with the best firm (since we want the trading probability conditional on failure on the first try), we derive the probability with which the consumer trades with the second best firm after failing to find what she want with the first to be $\frac{2}{5}$. Notice that this is much lower than the expected quality of the second best firm at the beginning of the search process. Once again, this is because the failure to trade with the first firm suggests that the best firm has a relatively low quality. This makes the consumer more pessimistic about the quality of the firm available at the second link.

Now we know exactly how the consumer is going to conduct her search. She perceives a cost of s_i of clicking on any link, so she simply compares the expected quality available by clicking on the link with her cost. Consumers whose costs exceed $\frac{3}{4}$ won't bother to click on the sponsored links at all. If their costs are below $\frac{3}{4}$, they will click on the first sponsored link on the page. Some consumers who do this will find just what they want, and will leave the search process.

Those who fail to find what they want after the first link, become much more pessimistic as we explained. If their costs are between $\frac{2}{5}$ and $\frac{3}{4}$ they won't bother to click on another sponsored link. However, those whose costs are below $\frac{2}{5}$ will continue their search and click on the second link.

This is exactly the information needed to determine how valuable the different slots are to firms. There are two consumers. No matter what the firm bids, two consumers will click on the first slot with probability $\left(\frac{3}{4}\right)^2$ (which is the probability that both consumers have search costs below $\frac{3}{4}$). The probability that only one consumer clicks is $2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$, while the probability there are no clicks at all is $\left(\frac{1}{4}\right)^2$. The revenue that a firm of type q gets from slot 1 is then

$$R_1(q) = \left(\frac{3}{4}\right)^2 2q + 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)q = \frac{6}{4}q.$$

On the other hand, a sale through the second slot requires two things to occur. First, consumers must be unsatisfied at the first link. Then, in addition, their search costs have to be low enough to induce them to click on the second link.

The probability that consumers who want to search fail to transact after clicking on the first link depends on the quality of the firm who wins the first link. This characteristic of the position auction makes it unusual. What it takes to win the top slot depends on what the firm who wins the second slot bids. So the *revenue* earned by the firm in the second position depends on its bid. Exactly how it depends is determined by the equilibrium bidding rule.

To see how, let $b(q)$ be some monotonic bidding rule that firms are using to submit bids in the position auction. If a firm bids b' , then it wins the first auction if both of the other firms have qualities such that $b(q) < b'$. If we write $b^{-1}(b')$ to be the firm type who bids b' , then the probability that the firm wins the first slot with a bid b' is $(b^{-1}(b'))^2$. It earns $R_1(q)$ in this event as we explained above.

The probability that it wins the second slot is $2b^{-1}(b')(1 - b^{-1}(b'))$. In this case estimates the expected quality of the firm who won the best slot to be

$$b^{-1}(b') + \frac{1 - b^{-1}(b')}{2}.$$

Then the probability that a consumer both has search cost less than $\frac{2}{5}$ and fails to buy from the firm at the first slot is $\frac{2}{5} \frac{1 - b^{-1}(b')}{2}$. This gives the revenue to a firm when it wins the second slot as

$$R_2(q, b', b(\cdot)) = \left(\frac{1 - b^{-1}(b')}{5}\right)^2 2q + 2\left(\frac{1 - b^{-1}(b')}{5}\right)\left(1 - \frac{1 - b^{-1}(b')}{5}\right)q =$$

$$2q\left(\frac{1 - b^{-1}(b')}{5}\right).$$

We can use all this to work out the equilibrium bidding rule. Assuming the bidding rule is monotonic, the expected payoff of a firm of type q who bids as if her quality were q' is given by

$$(q')^2 \frac{6}{4} q + 2q'(1 - q') 2q \left[\left(\frac{1 - q'}{5} \right) \right] - b(q').$$

The equilibrium bidding rule should have the property that a firm of quality q actually wants to bid as if its quality were q . This requires that the derivative of the expression above with respect to q' evaluated at q should be uniformly equal to zero in q . Taking the first derivative in the expression above is a straightforward but tedious exercise. Since a firm of quality q obviously wants to bid 0, this yields a differential equation whose solution provides the equilibrium bidding rule.