1 Practise Question

In the public goods problem let both consumers have identical utility functions given by

$$u_1(x, y) = u_2(x, y) = x^\alpha y^{(1-\alpha)}$$

Suppose that the aggregate endowment of the private good is 1 and that consumer 1 owns the share ω_1 of this aggregate endowment. The production function is such that one additional unit of the private good always yields exactly one unit of the public good (i.e. $y = x$). Find the equilibrium for the voluntary contribution game. Find the Lindahl prices and allocations.

1.1 Answer

Each consumer takes the consumption of the private good chosen by the other to be fixed. With Cobb Douglas preferences, 0 consumption of either good is never optimal. However, eating all the endowment may be optimal if consumer 2 is contributing enough to the public good. If consumer 1 chooses consumption less than his endowment the choice must be a solution to the first order condtion

$$\alpha x^{\alpha-1}(1 - x_1 - x_2)^{1-\alpha} - (1-\alpha)x^\alpha(1 - x_1 - x_2)^{-a} = 0$$

which solves to

$$x_1 = (1-\alpha)x_2 \tag{1}$$

so the overall solution is

$$x_1 = \min[\omega_1, (1-\alpha)x_2]$$

This is the reaction function given in Figure 2 of the reading. Reproduce that diagram and fill in this function labelling all the axis for practise.

To keep things simple suppose that the endowments are such that both choose to contribute to the public good. Since their first order conditions are the same $x_1 = x_2$ in this case, so substituting this into (1) gives

$$x_1 = x_2 = \frac{1}{1+\alpha}$$

1.1.1 Lindahl Prices

Let x and y be the output choices of the firm. Profits are then $(p_1 + p_2)y + x$ so consumer 1 has budget

$$\omega_1((p_1 + p_2)y + x)$$

while consumer 2 has budget

$$\omega_2((p_1 + p_2)y + x)$$

Since consumer 1 has Cobb Douglas preferences, his demands are

$$x_1 = \alpha\omega_1 \left(\left(p_1 + p_2\right) y + x\right)$$

for the private good and

$$y_1 = (1-\alpha)\,\omega_1 \frac{\left(\left(p_1 + p_2\right) y + x\right)}{p_1}$$

The sum of the demand for the private goods from 1 and 2 must equal the total output of the private good by the firm, so

$$\alpha\omega_1 \left(\left(p_1 + p_2\right) y + x\right) + \alpha\left(1-\omega_1\right)\left(\left(p_1 + p_2\right) y + x\right) = x$$

or just

$$\alpha\left(\left(p_1 + p_2\right) y + x\right) = x \tag{2}$$

Each of the two consumers has to demand exactly the amount of the public good that the firm produces so

$$(1-\alpha)\,\omega_1 \frac{\left(\left(p_1 + p_2\right) y + x\right)}{p_1} = y \tag{3}$$

and

$$(1-\alpha)\left(1-\omega_1\right)\frac{\left(\left(p_1 + p_2\right) y + x\right)}{p_2} = y \tag{4}$$

Finally, the production technology says that $y = 1 - x$. That means as well that the slope of the PPF must be 1 for profit maximization to occur, so $p_1 + p_2 = 1$. Finally, notice from (3) and (4) that if we set

$$p_1 = p_2 \frac{\omega_1}{1-\omega_1}$$

then whenever the second condition holds, the final one will as well. Now substitute all this information into (2) to get

$$\alpha = x$$

From (3)

$$(1-\alpha)\,\omega_1 \frac{1}{p_1} = (1-\alpha)$$

or $p_1 = \frac{1}{\omega_1}$ and the rest of the question follows. Plot this information in Figure 4 in the reading for practise.