

Centipede Game

Here is the extensive game we discussed in class:

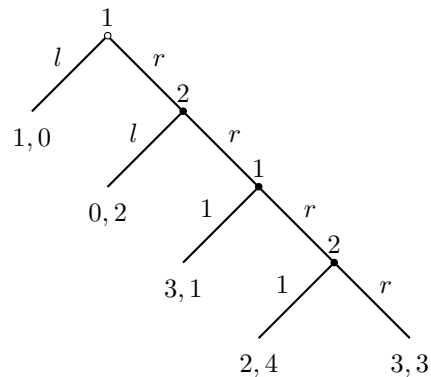


Figure 1: Centipede Game

Player 1 moves first, and can end the game by choosing l . Alternatively, he can pass the move on to 2 who has the same choice. We want to find all the subgame perfect equilibrium outcomes of this game. The branches of the tree represent actions in the game. The points where the branches split are referred to as decision nodes. Player 1 has a decision node at the start of the game. The label above a decision node identifies the player whose turn it is to move. A connected set of branches is called a history, because it specifies a sequence of actions for the players. Notice that the centipede game has the property that each player sees the whole history of the game in each of his decision nodes. This isn't always true. A game in which each player sees the full history at each of his decision nodes (i.e., at every instance where he has to make a decision) is called a game of perfect information. The game ends at points where no new branches emerge. These points are called terminal nodes. They are labeled with the payoffs, the payoff to player 1 being listed first.

To find the subgame perfect equilibria for this problem, you work backward - starting with each decision node that leads to a terminal node, and determining what action is most favorable to the player whose turn it is to move in that decision node. In this game, we begin with the information set labeled 2 on the bottom right. Player 2 has two choices, move left which gives her 4, and move right which gives her 3. Evidently she moves left as indicated by the pink line in the next diagram.

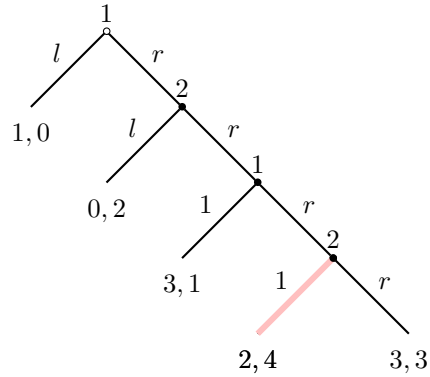


Figure 1: Centipede Game

What this says is that, if the game reaches the final decision node (it won't in equilibrium), then player 1 should expect player 2 to choose to go left.

We can now proceed to the decision node before that in which player 1 makes a decision. If he goes left, he ends the game and receives a payoff 3. If he goes right, then he passes the move to 2. As we just argued, 2 should choose to go left in that case, so 1's payoff will be 2. Since 3 is bigger than 2, player 1 should choose to go left at his last decision node - as in the following diagram.

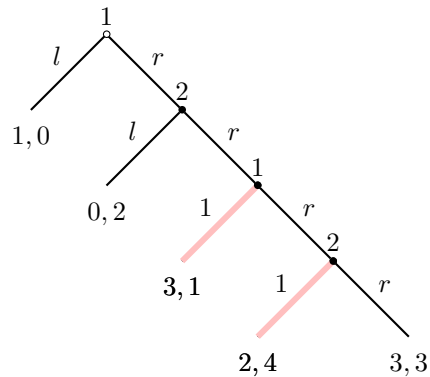


Figure 1: Centipede Game

We can go further. At her first decision node, player 2 can go left and end the game for a payoff of 2. She could choose to go right, in which case, she should expect 1 to go left, giving 2 a payoff of 1. So 2 has to choose left at her first decision node. Finally 1 must choose down.

A strategy for each player in this game is a specification of an action for each of their decision nodes. This makes it possible to describe the normal form of the game. The notation ij means that the corresponding player goes i (either left or right) at his or her first decision node, and j (either left or right) at her second. The normal form of this game looks like this

	ll	lr	rl	rr
ll	1, 0	1, 0	1, 0	1, 0
lr	1, 0	1, 0	1, 0	1, 0
rl	0, 2	0, 2	3, 1	3, 1
rr	0, 2	0, 2	2, 4	3, 3

Make sure you understand the payoffs. For example, the pair of strategies (rl, rl) say that 1 will choose to go right at his first decision node, and left at his second decision node, similarly for 2. The outcome is then that 1 goes right, 2 goes right, then 1 chooses left, leading to the payoff $(3, 1)$ as indicated in the box.

There are no strictly dominated strategies in this game, as you should verify for yourself. By exhaustively checking each cell of the payoff matrix for profitable deviations, you should be able to verify that there are 4 Nash equilibria coinciding with the four cells in the upper left hand corner of the table.

To see how subgame perfection works, let's focus on the Nash equilibrium (lr, lr) . We would explain that this is a *Nash* equilibrium because neither player can benefit by unilaterally deviating to another of their strategies. For example, player 1, who chooses the row, would lower his payoff from 1 to 0 if he deviates and uses one of his strategies which select r at his first decision node. This is because he expects player 2 to follow by playing l . Player 2 doesn't think there is anything he can do to affect his payoff because he expects player 1 to start the game by choosing l . His strategy to play l in his first decision node is the best he can do.

Subgame perfectionists would then argue that, though this is a Nash equilibrium, it isn't a very good one. In fact, they would say it is completely implausible. The reason is that if 2 ever found himself in his first decision node actually making a choice, and if he believed that player 1 were using the strategy lr , then at that point he would understand that he should switch his action to r in order to take advantage of the fact that 1 would then follow by playing r in his second decision node, giving him the opportunity to take l on his last move, and earn 4.

You might try to counter that 2 shouldn't expect 1 to choose r at his second decision node. She should understand that 1 will see that she only wants to get to that decision node in order to choose l . If you see this, then you are using subgame perfect reasoning already, and you will see that the way to make things work out is to work backwards through the extensive form, calculating what action each player will take at each decision node. This yields the unique subgame perfect equilibrium in which each player uses the strategy l, l . It is

much easier to do this in the extensive form than it is in the normal form of the game.

1 A Non-Generic Variant

The centipede game as it is usually presented is very simple. The subgame perfect equilibrium is unique and readily computed by backward induction as we did above. This is often true for games of perfect information - but not always. To see what can happen, we could modify the payoffs in the centipede game slightly as follows:

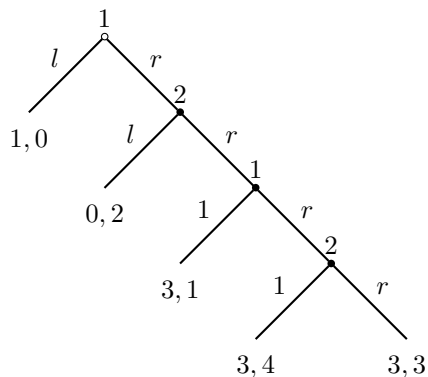


Figure 1: Centipede Game - Non-generic

The difference between this game and the original centipede game is that the payoff for the leftward terminal node has been changed from $(2, 4)$ to $(3, 4)$. Without comment, let's just start to do backward induction. At her second decision node, 2 has only 1 sensible choice, which is to go left. As we move back up the tree, we then see that in 1's second decision node, he can go left, for a payoff of 3, or go right, at which point he should expect 2 to go left. Now, 1 is indifferent. It is now reasonable for 1's behavior at this decision node to be a bit unpredictable. Let's use the mixed strategy approach and assume that 1 plays left in his second decision node with probability π .

We might depict this situation in the following way:

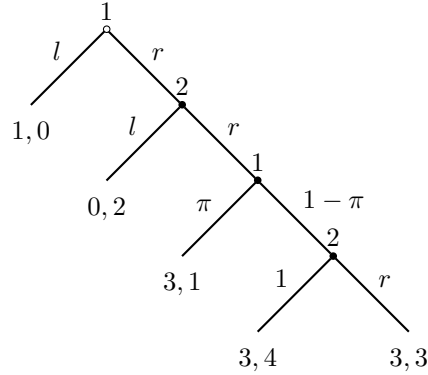


Figure 1: Centipede Game - Non-generic

Notice that we have labeled the branches at 1's second decision node with the probabilities with which 1 chooses each action. This makes it possible for 2 to evaluate the action r in her first decision node. The payoff to 2 of using r is evidently $\pi + (1 - \pi)4$. So 2 should use the action r at her first decision node whenever 1 chooses l in his second decision node with probability less than $\frac{2}{3}$.

Finally, this allows us to figure out what 1 will do in his first decision node. If $\pi < \frac{2}{3}$, then 2 should choose r in her first decision node. If that occurs, 1 can choose either of his actions in his last decision node, and get 3 - evidently much better than the payoff 1 he gets by choosing l . If $\pi > \frac{2}{3}$, then 2 will choose l in her first decision node, in which case 1 should choose l as well. What we have just described is a continuum of different subgame perfect equilibrium outcomes in which every player is using an action in each decision node that maximizes their payoff given continuation play.

What happens when $\pi = \frac{2}{3}$. Then 2 is indifferent between l and r at her first decision node. Let ρ be the probability with which 2 is supposed to choose left. If 2 chooses right at this first decision node (and 1 continues to choose l with probability $\frac{2}{3}$), 1's payoff is 3, if 2 chooses l it is zero. So the expected payoff for 1 when he chooses r is $(1 - \rho)3$. Evidently, there are a continuum of subgame perfect equilibrium in which 1 plays left at his first decision node and plays left at his second decision node with probability $\frac{2}{3}$, while two chooses left with probability between $\frac{2}{3}$ and 1.

Exercise: Write down the normal form of this non-generic centipede game, and find the mixed Nash equilibrium in the usual way.

2 Imperfect Information

You have already seen a whole bunch of games with imperfect information. Lets use the game we discussed in class as an example. If you recall the story, the

NHL wants to continue to have a hockey team in Phoenix. Quebec city would like to have the team moved to Quebec. The two cities then have the option of trying to bribe the NHL by subsidizing arenas and cutting taxes for the hockey team. In class, we discussed the possibility that the cities would simply build a new arena for the team. The payoff matrix looked like this (though I switched the payoffs around to make Quebec the row player):

	d	b
d	10, 30	10, 20
b	20, 10	0, 20

Quebec is the row player in this example. The actions d (don't build the arena) and b (build a new arena) generate payoffs that depend on what the other city does. If neither city builds a new arena, the NHL does what it wanted in the first place and leaves the team in Phoenix. Phoenix gets 30 in this case. Quebec get 10 because they like teams no matter where they are. If costs each city 10 to build an arena, so if they both build, the NHL again stays in Phoenix, and both cities lose the cost of the arena, Phoenix ends up with 20, Quebec 10. If Phoenix builds unilaterally, it just loses the cost of the arena, and its payoff falls to 20. If Quebec builds unilaterally, then the NHL decides to move the franchise to Quebec to make more money. Quebec likes this because gaining the hockey team more than makes up for the cost of the arena. If you remember, this game has a unique Nash equilibrium in which each city builds with probability $\frac{1}{2}$.

Lets look at this problem in a slightly different way. Each city makes a unilateral decision about whether or not to build the arena. Then they both announce their decisions at the same news conference. In this story, Quebec makes a decision about whether or not to build first, but Phoenix doesn't get to see this decision before they make their own. The following Figure shows how we represent this:

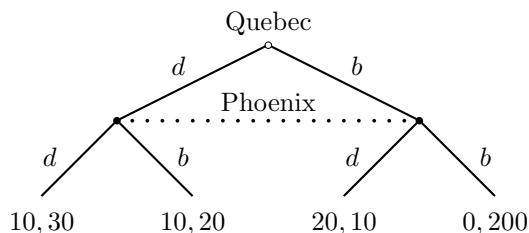


Figure 1: An extensive game

To capture the fact that Phoenix doesn't actually know which choice Quebec made, we draw a dashed line between the two decision nodes at which Phoenix makes a choice. We refer to this collection of decision nodes as an *information set*. Generally, we think of players as making decisions in information sets and note in passing that in a game of perfect information every information set consists of exactly one decision node.

Yet in this representation of the game, there are two decision nodes in Phoenix's information set. When some information sets have more than one decision nodes, we call the game a game of *imperfect information*. To specify a strategy for this game, we need to specify a choice for a player for each of his or her information sets. This is easy here because Phoenix only has one information set. So all we have to do to describe Phoenix behavior is to describe a mixed strategy for that information set. Quebec has just a single information set as well consisting of a single decision node - even simpler. A Nash equilibrium is a pair of strategies that are best replies to one another. Evidently both cities should announce a commitment with probability $\frac{1}{2}$.

Finally, let's give Quebec a second option, by allowing them to hold a press conference announcing their decision the week before Phoenix does. However, we want this to be an *option* for Quebec - they can choose to do it if they want, but they could also wait a week and make an announcement at the same time as Phoenix.

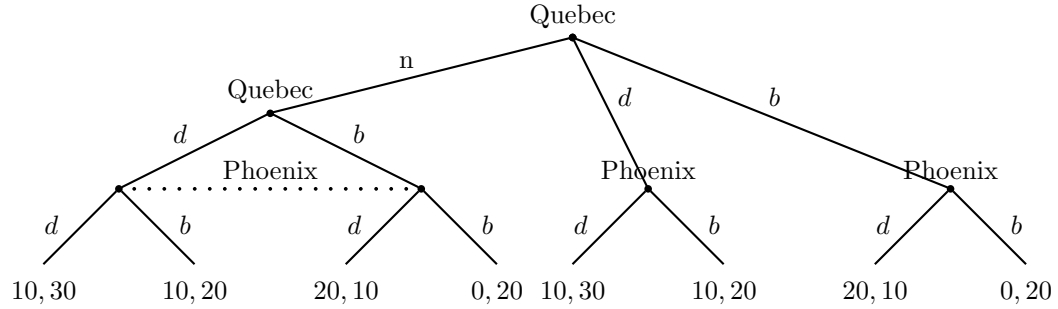


Figure 1: An extensive game

Lets go over this structure. Quebec begins the game by making a decision about whether or not to jump the gun and make their announcement the week before Phoenix does. We represent this by giving Quebec a decision node at the top of the game in which it has three possible actions, n which means cancel the press conference until next week, d , which means hold the press conference this week and announce you won't build, and b , which means hold the press conference this week and announce that you will build. If Quebec chooses either of these latter two decisions, then Phoenix simply decides at its final decision node whether to build or not after learning what Quebec is going to do.

The alternative for Quebec is to put the press conference off until next week. At that point Quebec again has a chance to make a decision, but its position is a little different in the sense that Phoenix will no longer hear what it is going to do before it makes its own decision.

Part of this looks just like the centipede game. For example if we follow the rightmost branch where Quebec chooses to build, then Phoenix has a decision node. By backward induction, Phoenix should choose to build at this decision

node, since it gets 20 if it builds and only 10 if it doesn't. Similarly if Quebec goes down the central branch and makes a pre-emptive announcement that it won't build, then Phoenix should also choose not to build.

How about the leftmost branch where Quebec chooses to postpone its press conference until next week. Then it forces itself to make a choice at the same time as Phoenix. Subgame perfection is going to force us to expect that when Quebec chooses this option, it *must* anticipate that the choices that Phoenix makes will be part of a Nash equilibrium of our original game - i.e., when Phoenix reaches its non-degenerate information set, it will choose to build and not to build with equal probability. Once we know that, we can determine the outcome of the whole game since should Quebec choose to postpone its new conference, it should expect a payoff of 10. Knowing that, we can compare this with what would have happened had Quebec chosen one of its other actions. If it announced a week early that it was going to build, it should expect Phoenix to build as well, giving it a payoff of zero. If it announces early that it won't build, then Phoenix won't build and Quebec's payoff would be 10, the same as if it had deferred its press conference to the next week.

Exercise: Find all the mixed subgame perfect equilibrium for this game.

For completeness, we can write down the normal form of this game. Quebec has two decision nodes in the game for which it must specify an action. Lets call this strategy a pair ij where i is what it chooses at the beginning of the game, and j is what it chooses when it defers the news conference. Phoenix must specify an action for each of its two decision nodes, and for its information set. We can list them in order left to right - so ijk means that Phoenix takes action i in its information set, action j when Quebec announces that it won't build at its first press conference, and action k if Quebec announces that it will build. Here is the normal form:

	ddd	ddb	dbd	dbb	bdd	bdb	bbd	bbb
nd	10, 30	10, 30						
nb								
dd								
db								
bd								
bb								0, 20

Exercise: Fill in the other payoffs in this normal form. Check to see if any strategies are strictly dominated, or whether you can solve the game by iterated deletion of strictly dominated strategies. Check to see if any strategies are weakly dominated. What do you get if you iteratively delete weakly dominated strategies.