The Price of Anarchy

This short piece is an attempt to explain a theorem from computer science which provides a lower bound on the performance of unregulated networks. You can read about this concept in wikipedia (just type in the title into google to get the link). The basic idea (at least as I'll describe it here) is to compare Nash equilibrium with regulation in a matrix game. This is a useful exercise because it forces you to work you way through all the properties of a matrix game. We will also come back to this idea when we study directed search a little later in the course.

The very simple version we study here describes a network in which websites are trying to send 'packets' through a network of routers to a final user. Routers are just computers, as are websites. We are interested in how many packets the websites are able to transmit to the final user.

The websites face two problems. First, the routers inevitably drop packets, so some information may not get through. Routers differ in their ability to process packets - some transmit more information than others under all traffic conditions. Secondly, if too many packets travel through the same router, congestion will slow things down. If you don't like computers and computer networks, then you can think of traffic networks. Cars try to get to destinations using different routes. Some routes are longer than others, but if all cars try to take the same route there will be congestion which will slow down even the faster route.

At this point we are studying simple matrix games, so we are going to model this is a very special way. Here is the game:

| | | Website B | |
|---|---|---|---|
| | | Router 1 | Router 2 |
| Website A | Router 1 | $\frac{1}{2}, \frac{1}{2}$ | $1, \beta$ |
| | Router 2 | $\beta, 1$ | $\frac{\beta}{2}, \frac{\beta}{2}$ |

The idea behind this game is that Router 1 is more efficient. We are just going to assume that if a website sends a packet through Router 1, and there is no congestion, then the packet gets through for sure. If a website sends a packet through router 2, then the packet might be dropped even if there is no congestion. In particular, we will just assume that if there is no congestion, then the packet gets through with probability $1 > \beta > 0$. The fraction $\beta$ represents the relative inefficiency of Router 2. This is just a simple way to approximate efficiency in networks. In a real computer network, Router 2 might just be slower, or further from the final destination in the sense that there are more 'hops' to other routers between Router 2 and the final destination.

If two packets end up at the same router at the same time, then there is a congestion problem. To capture this, we just assume that the router randomly drops one of the two packets. So if both packets go to router 1, each will get through to its final destination with probability $\frac{1}{2}$. If both websites send their packet to router 2, then the router will randomly select one of them and try to transmit it - except it will only succeed with probability $\beta$. So each website will get its packet to the final destination with probability $\frac{\beta}{2}$ in this case.

You will notice that in this simple matrix game, there are no dominated strategies if $\beta \geq \frac{1}{2}$. In that case, Website $A$ strictly prefers to use Router 1 if Website $B$ is using Router 2 and conversely. If $\beta$ is less than $\frac{1}{2}$, then both Websites will use Router 1, which can be deduced by using iterated elimination of dominated strategies. What that means is that when both Websites are using Router 1, each gets their packet through with probability $\frac{1}{2}$. This is better than unilaterally deviating and using Router 2, where a packet gets through with probability $\beta < \frac{1}{2}$.

Reasoning the same way, when $\beta > \frac{1}{2}$, there are a pair of pure strategy equilibrium in which Website $A$ uses Router 1 while Website $B$ uses Router 2, or the converse. We'll come back to the pure strategy equilibria later when we discuss the price of anarchy, but observe that there is a sense in which the pure strategy equilibrium are a little implausible. It seems unlikely in a large computer network than individual websites would be able to coordinate their packet sending strategies quite so precisely - there are millions of websites that would have to communicate to accomplish this kind of coordination. A more plausible story is that each website uses a 'mixed strategy' that sends some proportion of their packets to each of the different routers. We can capture this kind of logic by describing the mixed strategy equilibrium of this simple matrix game.

As we discussed in class, we can find this 'mixed' equilibrium by find a probability $\pi$ with which Website $B$ sends its packet through Router 1 which has the property that Website A will be indifferent between which of the two routers it uses. If we do this, then we might reasonably expect Website A to use a random strategy about where to send its packet and we could pick this random strategy so that Website B was indifferent. This would give us a mixed Nash equilibrium for the little matrix game we described.

To make Website A indifferent, Website B needs to send its packet to Router 1 with a probability $\pi$ that satisfies

$$\frac{\pi}{2} + (1-\pi) = \pi\beta + (1-\pi)\frac{\beta}{2}$$

The solution is $\pi = \frac{2-\beta}{\beta+1}$ (so $1-\pi = \frac{2\beta-1}{\beta+1}$).

There is something you should notice about this solution, which is that if $\beta < \frac{1}{2}$, then $\pi$ is larger than 1, which doesn't make any sense. This is the kind of signal you should check for when you are doing your calculations. In this case $\pi$ larger than 1 is telling you that using Router 2 is a dominated strategy.

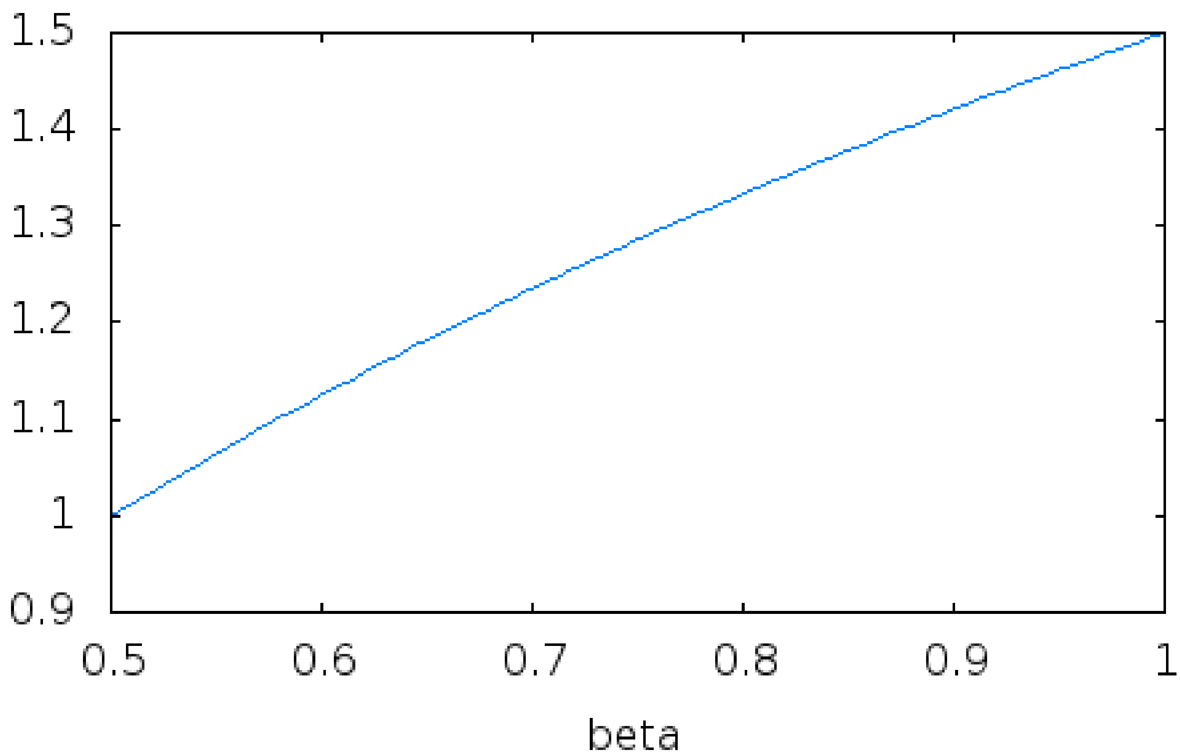
Now that we know what the Nash equilibrium is, we can compute how well the network functions. To compute the expected number of packets that get through the network we just evaluate

$$\pi^2 + 2(1-\pi)\pi(1+\beta) + (1-\pi)^2\beta$$

at the Nash equilibrium value of $\pi = \frac{2-\beta}{\beta+1}$. According to wxMaxima, this is

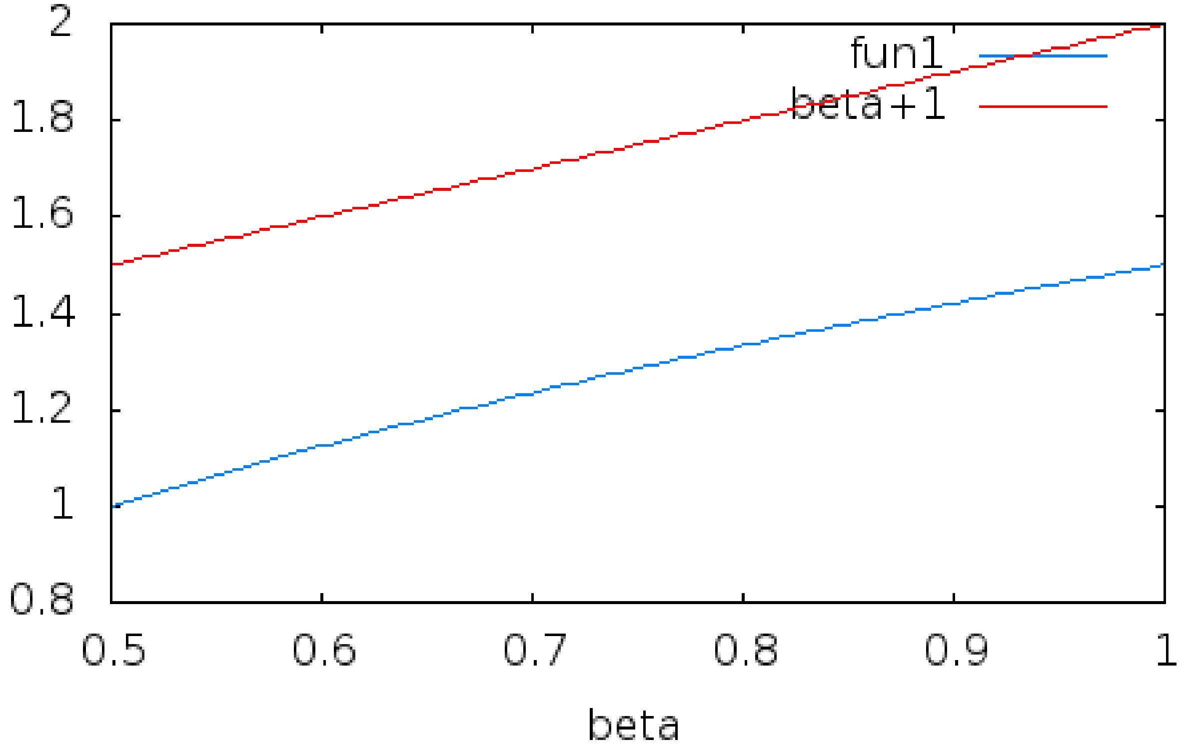
$$\frac{\beta}{\beta+1}\left\{\frac{(2\beta-1)^2}{\beta+1}+\frac{(\beta-2)^2}{\beta+1}-2\left(\beta-2\right)\left(2\beta-1\right)\right\}$$

Here is a picture of this function for values of $\beta$ between $\frac{1}{2}$ and 1 (also drawn with wxMaxima).




When $\beta$ is equal to $\frac{1}{2}$, remember that Router 2 becomes a dominated strategy for both Websites. In that case, they both use Router 1 in the unique Nash equilibrium - which randomly selects one of their packets and gets it through for sure - i.e., the expected number of packets that gets through is exactly 1 as it appears in the picture. As $\beta$ gets larger, Router 2 is becoming more efficient, so more packets get through on average. When $\beta$ reaches 1, then the routers are equally efficient and both Websites choose each of them with probability $\frac{1}{2}$ - the network achieves its best outcome with mixed strategies - 1.5 packets get through on average.

For the case in which $\beta > \frac{1}{2}$, this mixed equilibrium gives the worst possible performance for the network. If the network were regulated by someone whose objective is to maximize the number of packets that get through, then by instructing the Websites to use different routers (for sure), this regulator could ensure that $1+\beta$ packets get through. The picture that follows illustrates the worst that can happen then




The red function at the top gives the expected number of packets that are transmitted in a fully regulated network, the blue curve below it gives the expected number of packets that are transmitted in a Nash equilibrium, both as function of $\beta$, the efficiency of Router 2. Notice that the difference isn't too big - the fully regulated network never achieves more than $\frac{4}{3}$ what the unregulated network does. Anarchy simply can't be that costly.

If you want to read about how well anarchy does in more realistic (but obviously much more complex networks), try the paper "Selfish Routing and the Price of Anarchy" by Tim Roughgarden (Stanford).

As we mentioned above, this is a little generous toward the regulator, since we assume he can individually direct where the website should send their packets. A more realistic approach in a large network would be to allow the regulator to send a message to both websites telling them the probability with which they should use Router 1 - the constraint being that we require the regulator to give each website the same recommendation. Assuming you could force them to carry out that recommendation, the average number of packets that the regulator could get through the network is

$$\pi^2 + 2\left(1-\pi\right)\pi\left(1+\beta\right) + \left(1-\pi\right)^2\beta$$

where $\pi$ is the common recommendation. The best recommendation is the one that maximizes this expression, and we can find it by differentiating and setting the result to zero - i.e by solving

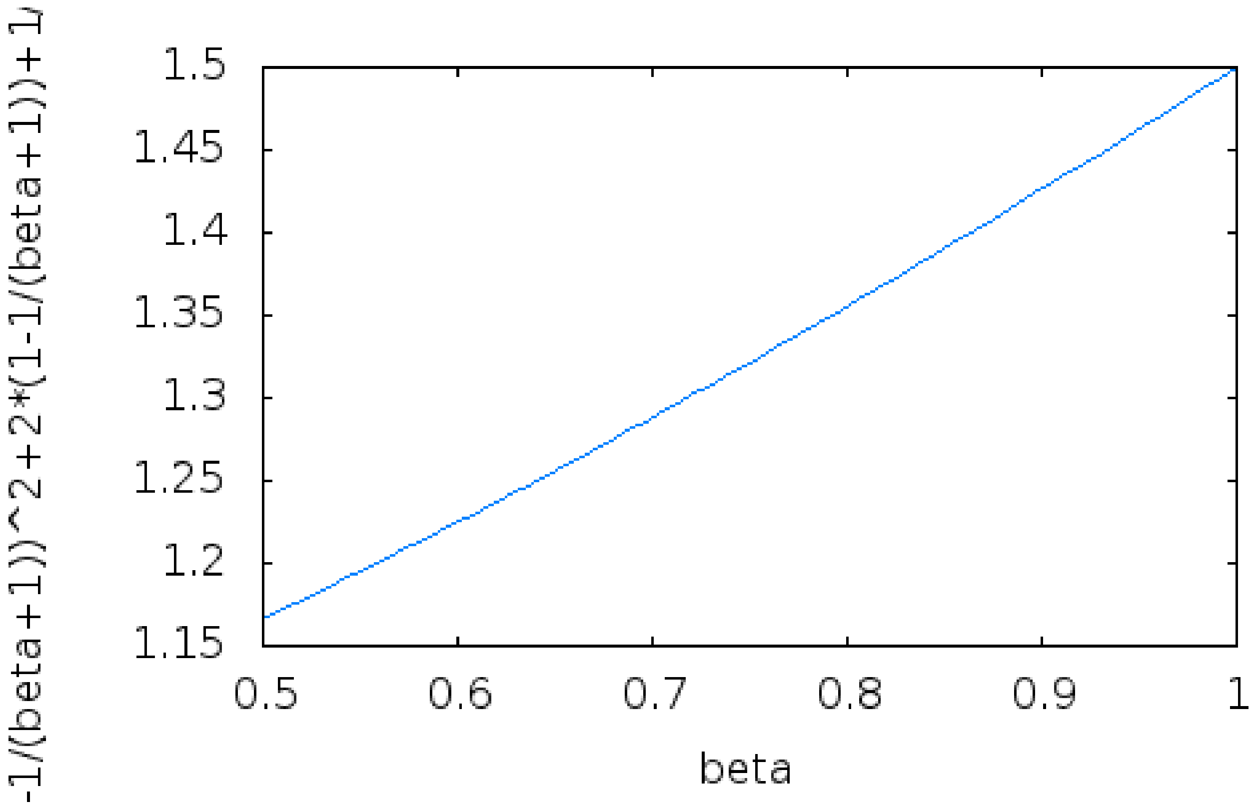
$$2\pi + 2\left(1-\pi\right)\left(1+\beta\right) - 2\pi\left(1+\beta\right) - 2\left(1-\pi\right)\beta = 0$$

The solution (wxMaxima) is

$$\pi = \frac{1}{\beta+1}.$$

Notice a couple of things. First, as $\beta$ goes to 1 (the routers are equally efficient), then the regulator who is constrained to have both websites use the same strategy will have them choose router 1 with probability 1/2. That is intuitive - the regulator would rather send the websites through different routers, but he can't do that because he is constrained to have them use the same strategy. If they are equally efficient then he is indifferent about which of the two they use.

Perhaps the more surprising thing is what happens when $\beta = \frac{1}{2}$. In that event router 2 is dominated and the websites start to focus on Router 1 in the Nash equilibrium. The regulator, however, wants them to use router 1 with probability $\frac{2}{3}$. The average number of packets that get through as a function of $\beta$ is shown in the following picture:




Now notice that when the regulator tells the websites to use router 1 with probability $\frac{2}{3}$, he actually manages to get $\frac{7}{6}$ packets through the network, instead of the single packet that gets through in the Nash equilibrium.

Exercise: Now do your own calculation for the case where $\beta < \frac{1}{2}$. Draw a graph showing the average number of packets that the regulator gets through the network for different values of $\beta < \frac{1}{2}$ and compare this with what happens in the symmetric Nash equilibrium. Might be a good time for you to try to figure out how to use a computer algebra package - this is the kind of computation you will often find yourself faced with.