How to Characterize Solutions to Constrained Optimization Problems

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1 Introduction

A common technique for characterizing maximum and minimum points in math is to use first order conditions. When a function reaches its maximum, its derivative must be zero. The zero derivative can often be interpreted. When the maximization is subject to an *equality* constraint, the problem isn't much harder (see the special case below). However, in economics, maximization typically occurs subject to various inequality constraints. For example, demand for a particular commodity can't be negative (though it could be zero). Labour supply can't exceed 24 hours, but will only exceptionally be equal to 24 hours.

To deal with this, use the following theorem: Let $f(x_1, \ldots x_n)$ be a real valued function with n real arguments. Let $c_1 - G_1(x_1, \ldots x_n) \leq 0$, $\ldots c_m - G_m(x_1, \ldots x_n)$ be a series of m constraints, where each of the $G_i(\cdot)$ functions is a real valued function. Let $x^* = \{x_1^*, \ldots x_n^*\}$ be a solution to the problem

$$\max f(x_1, \ldots x_n)$$

subject to

$$c_1 - G_1(x_1, \ldots x_n) \leq 0$$

$$\vdots$$

$$c_m - G_m(x_1, \ldots x_n) \leq 0$$

and suppose that the functions G_1 through G_m satisfy the *constraint qualification*[1]. Then there is a set of m constants $\lambda^* = \{\lambda_1^*, \ldots \lambda_m^*\}$ such that

$$L[x_1, \ldots x_n, \lambda_1, \ldots \lambda_m] \equiv f(x_1, \ldots x_n) + \sum_{i=1}^{m} \lambda_i [c_i - G_i(x_1, \ldots x_n)]$$

[1]The constraint qualification says that the vectors $[\partial G_i(x^*)/\partial x_1, \ldots \partial G_i(x^*)/\partial x_n]$ for each i for which the constraint holds with equality in the optimal solution are linearly independent.

and

$$\frac{\partial L\left[x_1^*, \dots x_n^*, \lambda_1^*, \dots \lambda_m^*\right]}{\partial x_j} = 0 \tag{1}$$

for $j = 1, \dots n$ and

$$\frac{\partial L\left[x_1^*, \dots x_n^*, \lambda_1^*, \dots \lambda_m^*\right]}{\partial \lambda_i} \leq 0; \lambda_i \leq 0 \tag{2}$$

for all i with complementary slackness.[2]

The idea is roughly that once you have found a solution, it might be that you are prevented from getting the outcome you would like by one of the constraints. The derivative of your objective function won't be zero. If that is so, then it should be possible to calculate the *marginal* gain you could attain by violating that constraint by a little bit and imposing a penalty for marginal violations (that is what the λ_j's are for). The *Lagrangian* function $L[\cdot, \cdot]$ is the sum of your main objective and all these penalties. At your optimal x^* the derivative of this Lagrangian function could be made to be exactly zero if you could impose penalties that exactly offset the marginal gains of violating the constraints.

Notice that since the constraints are of the form $c_i - G_i(x_1, \dots x_m) \leq 0$ we want the penalty to be positive whenever $c_i - G_i(x_1, \dots x_m) > 0$. That is why we want the 'fines' for violating the constraint to be negative in (2). This also explains complementary slackness. Since the Lagrangian function is linear in the λ_j, the derivative $\frac{\partial L}{\partial \lambda_i}$ will always be equal to the value on the left hand side of one of the constraints. If this value is strictly negative at the solution, we don't want to impose a penalty because that would force the decision maker away from the right solution. As a consequence, we need to have the penalty equal to zero. On the other hand, if the constraint were exactly satisfied at the optimal solution, then we would typically be able to do strictly better by violating the constraint and the penalty would need to be positive.

1.1 Some simple examples

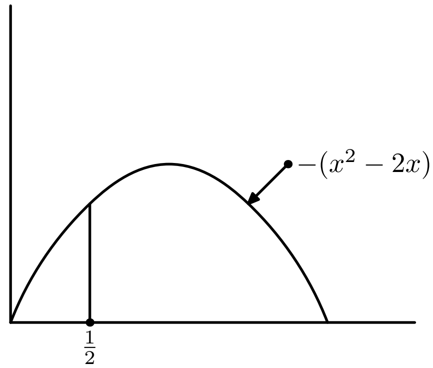
The function $-\left(x^2 - 2x\right)$ has a maximum point at $x = 1$ where the function takes on a value equal to 1. This is illustrated in Figure 1.

Suppose we want to maximize the function $-\left(x^2 - 2x\right)$ subject to the constraint that $0 \leq x \leq \frac{1}{2}$. The solution x^* is obviously equal to $\frac{1}{2}$ as shown in the picture. At this point the derivative of $-\left(x^2 - 2x\right)$ is equal to 1. So if we were to raise x and violate the constraint by a little bit, we would gain 1. To prevent this we would intuitively like to create a penalty that imposes a cost 1 for each unit by which the constraint is violated. Then on the margin, increasing x a little would cause the objective to rise by 1, but the penalty cost would rise

[2] Complementary slackness means that

$$\lambda_i \cdot \frac{\partial L\left[x_1^*, \dots \lambda_m^*\right]}{\partial \lambda_i} = 0$$

for all i.



by exactly the same amount - the derivative of the Lagrangian function would be exactly zero. Then we could think about solutions to constrained optimization problems by looking at first order conditions exactly as in unconstrained problems.

We could put this into the form needed by the theorem. We want to maximize

$$f(x) = -\left(x^2 - 2x\right)$$

There are two constraints, x needs to be at least zero and no more than $\frac{1}{2}$. We need to write them both as less than or equal to zero constraints, i.e.,

$$-x \leq 0$$

and

$$x - \frac{1}{2} \leq 0$$

so $c_1 = 0$ and $G_1(x) = x$, while $c_2 = -\frac{1}{2}$ and $G_2(x) = -x$.

The theorem now says that when $x = x^* = \frac{1}{2}$ we will be able to find a pair of penalties λ_1^* and λ_2^* such that if we define the function

$$L[x, \lambda_1, \lambda_2] = -\left(x^2 - 2x\right) + \lambda_1[-x] + \lambda_2\left[-\frac{1}{2} - (-x)\right]$$

then

$$\left.\frac{\partial L[x, \lambda_1, \lambda_2]}{\partial x}\right|_{x=\frac{1}{2}, \lambda_1=\lambda_1^*, \lambda_2=\lambda_2^*} = 0$$

$$\left.\frac{\partial L[x, \lambda_1, \lambda_2]}{\partial \lambda_1}\right|_{x=\frac{1}{2}, \lambda_1=\lambda_1^*, \lambda_2=\lambda_2^*} \leq 0; \lambda_1^* \leq 0$$

and

$$\left.\frac{\partial L[x, \lambda_1, \lambda_2]}{\partial \lambda_2}\right|_{x=\frac{1}{2}, \lambda_1=\lambda_1^*, \lambda_2=\lambda_2^*} \leq 0; \lambda_2^* \leq 0$$

where the last two conditions both hold with complementary slackness.

So could we characterize $x^* = \frac{1}{2}$ using a first order condition? The derivative of the objective is positive at $\frac{1}{2}$ so we need a penalty, and since the derivative of objective is 1, we should intuitively set this penalty to 1. We could accomplish this by setting $\lambda_2^* = -1$. At the optimal solutions we don't need to impose any penalties for the constraint at zero, because it is not *binding* at the optimal solution. So set $\lambda_1^* = 0$. Then we have

$$\left.\frac{\partial L[x, \lambda_1, \lambda_2]}{\partial x}\right|_{x=\frac{1}{2}, \lambda_1=\lambda_1^*, \lambda_2=\lambda_2^*} = \left. -(2x-2) - \lambda_1 + \lambda_2 \right|_{x=\frac{1}{2}, \lambda_1=\lambda_1^*, \lambda_2=\lambda_2^*} = 1-1 = 0$$

just as we want. You can check that the other two inequalities hold with complementary slackness as the theorem suggests.

So the idea is to find the optimal solution, then calculate how much could be gained by violating the constraints so that you can set appropriate penalties to prevent that. You may also see that if we pick the penalties properly, you won't want to violate the constraints, so penalties will never actually be paid and the value of the Lagrangian at the optimal solution using the λ^*'s will be equal to the value of the original objective evaluated at the optimum.

You might notice that this requires that we actually know the solution to the problem in order to be able to calculate the appropriate penalties. This seems to defeat the purpose of using this to find a solution in the first place. You don't actually use this to method to calculate solutions (though it will sometimes take you that far). You use it to describe the properties of solutions.

An obvious problem to try this with is the problem of maximizing a utility function subject to a budget constraint. In particular, suppose that the utility function is $u(x, y) = x^\alpha y^{(1-\alpha)}$ where $0 < \alpha < 1$ is some constant. We need to satisfy the budget constraint $M \geq px + qy$ where p and q are the prices for x and y respectively. We also need both consumption levels to be non-negative. So to put this in the form needed by the theorem let $c_1 = -M$ and $G_1(x, y) = -px - qy$, $c_2 = c_3 = 0$, with $G_2(x, y) = x$ and $G_3(x, y) = y$.

The Lagrangian is then

$$x^\alpha y^{(1-\alpha)} + \lambda_1 [-M - (-px - qy)] + \lambda_2[-x] + \lambda_3 [-y]$$

I don't know what the optimal solution to this problem is. Even if I find a solution to the first order conditions described above, this doesn't really help since the theorem *does not* say that things that satisfy the first order conditions are solutions, it is the other way around.

Nonetheless I can combine things that I do know about the solution to make some progress. For example if there is positive income $M > 0$, then I should never pick $x = 0$ or $y = 0$ since this will make the objective 0 and I can get strictly more than that if I make both x and y strictly positive, even if they are very small. Furthermore, if I increase x or y or both of them a little bit, I strictly increase the object. So the solution can never lie strictly inside the budget constraint. So I already know that I will need to impose a penalty to prevent myself from exceeding the budget, though I won't want to impose any penalties for the non-negativity constraints since I would never want to violate them anyway.

So at the optimal solution I will have constants λ_1, λ_2 and λ_3 such that

$$\frac{\partial L[x, y, \lambda_1, \lambda_2, \lambda_3]}{\partial x} = \alpha x^{\alpha-1} y^{(1-\alpha)} + \lambda_1 p - \lambda_2 = 0$$

$$\frac{\partial L[x, y, \lambda_1, \lambda_2, \lambda_3]}{\partial y} = (1-a)\, x^{\alpha} y^{-\alpha} + \lambda_1 q - \lambda_3 = 0$$

$$-M - (-px - qy) \leq 0; \lambda_1 \leq 0$$

$$-x \leq 0; \lambda_2 \leq 0$$

$$-y \leq 0; \lambda_3 \leq 0$$

where the last three conditions hold with complimentary slackness. Since I know that x and y both have to be strictly positive at the optimal solution, then I also know that $\lambda_2 = \lambda_3 = 0$ by complementary slackness. Also M should exactly equal $px + qy$ and the penalty for violating the constraint λ_1 should be strictly negative. So this leaves me with

$$\alpha x^{\alpha-1} y^{(1-\alpha)} + \lambda_1 p = 0$$

$$(1-a)\, x^{\alpha} y^{-\alpha} + \lambda_1 q = 0$$

$$M = px + qy$$

These three equations in the three unknowns must have a solution from the theorem above (at least provided you are sure that a solution exists). Divide the first equation by the second to get

$$\frac{\alpha}{(1-\alpha)} \frac{y}{x} = \frac{p}{q}$$

or $y = \frac{px}{q} \frac{(1-\alpha)}{\alpha}$. Substitute this into the final equation in the first order condition to get

$$M = px + q \frac{px}{q} \frac{(1-\alpha)}{\alpha} = px \left[\frac{\alpha + (1-\alpha)}{\alpha} \right]$$

or $x = \frac{\alpha M}{p}$. Similarly, $y = \frac{(1-\alpha) M}{q}$. So, you can give a complete characterization of the result by using the first order conditions, along with the properties that you know about solution.