Public Goods

Michael Peters

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1 Introduction

In this reading, a *public good* is one having the property that once it has been produced, it can be made freely available to everyone. An mp3 file, or a computer program would fit into this category. Contrast this with the *private goods* we have discussed so far. If one person consumes a private good, no other consumer can have that good. Some of the more important goods in the modern economics are actually public in this sense. For example, an idea or a bit of information is a public good. Whether or not you know the idea, or have the information does not impact on my ability to know the same thing or have the same information (though whether or not you have the same information as I do may determine whether or not I can make money off you). A music file shared on my computer is a public good – if you download it from my website, my ability to enjoy it is not affected at all. A theorem, like the first welfare theorem that we studied last week, is a public good: I do not forget it when you learn it. Contrast this with my car or my lunch which are both private goods: if you take them from me I cannot enjoy them at all.

This note describes an equilibrium for the *voluntary contribution game*, which is a common way to think about how public goods are provided. It explains why the amount of the public good in the voluntary contribution game is under produced (because the outcome is not Pareto optimal: there is another outcome that will make everyone better off).

1.1 The voluntary contribution game

Suppose there are two goods, one public (y) and one private (x). Let $f(x)$ denote the amount of the public good that can be produced from x units of the private good. Suppose there are two consumers with utility functions $u_1(x, y)$ and $u_2(x, y)$ respectively. Their endowments of the private good are ω_1 and ω_2. The set of points $\{(x, y) : y = f(\omega_1 + \omega_2 - x)\}$ is the *production possibilities frontier.*

If the first consumer decides to consume x_1 (and devote the rest of his endowment ω_1 to production of the public good) while consumer 2 decides to consume x_2 the utilities of each of the consumers are given by

$$u_1\left(x_1, f\left(\omega_1 + \omega_2 - x_1 - x_2\right)\right)$$

for consumer 1 and

$$u_2\left(x_2, f\left(\omega_1 + \omega_2 - x_1 - x_2\right)\right)$$

for consumer 2. The important point is that if consumer 1, say, decides to consume a bit less of the private good and produce a bit more of the public good, then consumer 2 will enjoy the additional public good too without any cost at all.

The voluntary contribution game equilibrium is one way to predict how much of the private good each consumer will choose. Each of the consumers simply picks the amount of the private good they want on their own. This is a bit hard to do because the amount that each consumer will choose to contribute depends on how much they expect the other consumer to contribute. A good way to do this is to use a *Nash equilibrium* in place of a Walrasian equilibrium. Instead of taking prices to be fixed, each consumer takes the contribution of the other consumer to be fixed and chooses the contribution that maximizes his utility given this expectation. In a Walrasian equilibrium, when a consumer acts as if he believes that prices are fixed, he has to be physically able to purchase the bundle that maximizes his utility at these prices. That is why the price expectations have to be such that markets clear. Analogously, when the consumer chooses his optimal contribution given a fixed expectation about the contribution of the other consumer, he has to end up with exactly the amount of the public good that he expected to get.

Formally, a Nash equilibrium for the voluntary contribution game is a pair of private consumptions x_1^* and x_2^* such that

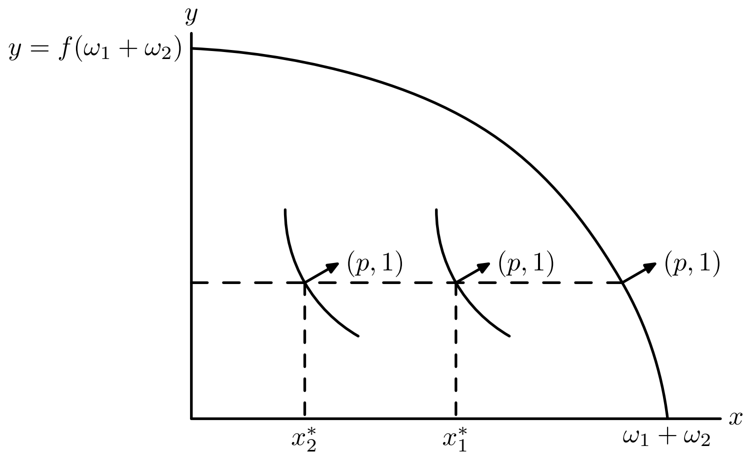
$$u_1\left(x_1^*, f\left(\omega_1 + \omega_2 - x_1^* - x_2^*\right)\right) \geq u_1\left(x', f\left(\omega_1 + \omega_2 - x' - x_2^*\right)\right) \tag{1}$$

for any alternative contribution $x' \in [0, \omega_1]$ and

$$u_2\left(x_2^*, f\left(\omega_1 + \omega_2 - x_1^* - x_2^*\right)\right) \geq u_2\left(x', f\left(\omega_1 + \omega_2 - x_1^* - x'\right)\right) \tag{2}$$

for any alternative contribution $x' \in [0, \omega_2]$.

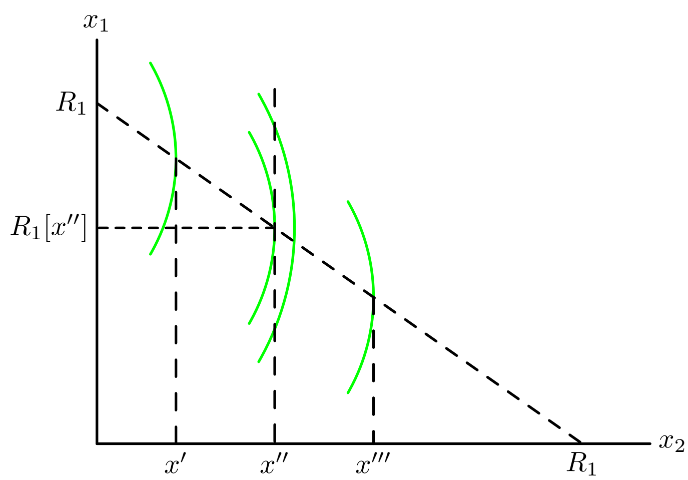
One way to view the outcome of this game is given in Figure 1 where the two consumers choose x_1^* and x_2^*. The 'budget line' that consumer 1 faces, for example, when consumer 2 chooses consumption x_2^* is the set of all pairs $\{(x_1, y) : y = f\left(\omega_1 + \omega_2 - x_1 - x_2^*\right)\}$. The slope of this is exactly the same as the slope of the production possibilities frontier at the point $(x_1^* + x_2^*, y^*)$. The same is true for consumer 2. So, in the equilibrium of the voluntary contribution game, each consumer has the same marginal rate of substitution and the same marginal rate of transformation in production.



With private goods, this is exactly what you want. Recall that, in the Edgeworth box, both consumers' indifference curves were tangent and the common slope of their indifference curves was equal to the slope of the production possibilities frontier. With public goods, this is not the outcome that you want.

An alternative approach – that helps to explain how contributions are determined and why these contributions are not Pareto optimal – is to try find the best choice for consumer 1 to make for all the different possible contribution levels that consumer 2 might choose. This approach is more common in game theory and involves the construction of something called the *best reply function.* The best reply functions are put together to understand the final outcome.

In Figure 2, the various *consumption choices* that player 2 can make are given along the bottom axis. Each such choice implies a contribution to the production of the public good - just take the difference between the consumption level and the endowment to find it.



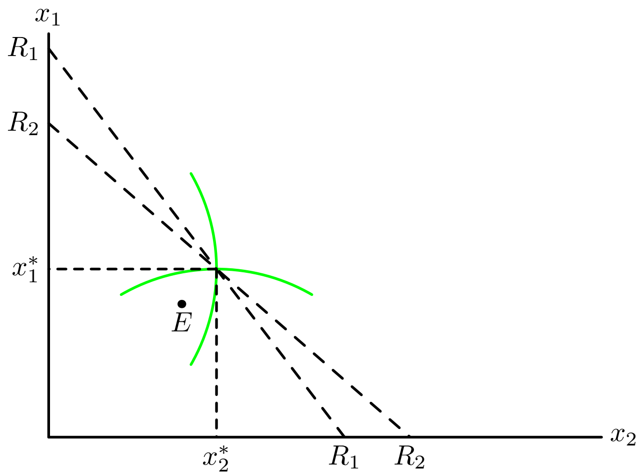
The green lines represent iso-utility curves for consumer 1. They are solutions to equations of the form

$$u_1 \left(x_1, f \left(\omega_1 + \omega_2 - x_1 - x_2 \right) \right) = K$$

where K is some constant. Consumer 1 achieves higher utility (holding his own consumption level constant) the *lower* is the consumption level of consumer 2. Lower consumption by consumer 2 means that consumer 2 is contributing more of her endowment to produce the public good.

In the Nash equilibrium, consumer 1 forms some belief about the consumption level that consumer 2 will pick. Suppose for the moment, that he believes that consumer 2 will choose consumption x''. Then he can attain any (x_1, x_2) combination that lies on the vertical line through x''. The best such point is the one that lies on the highest iso-utility curve. That is the one through the point $(R_1 [x''], x'')$ where an iso-utility curve is just tangent to this vertical line. (If he were to lower his planned consumption by moving down this vertical line, he would end up on a lower iso-utility curve like the one that lies just to the right of the point $(R_1 [x''], x'')$.

There would be a different best choice like this for every different choice that consumer 2 makes. The picture shows the corresponding tangencies at point x' and x'''. If you joined all these best choices together they would form a line (not necessarily straight as it is in the picture) called consumer 1's reaction function. This is the line $R_1 R_1$ in the picture. It explains what consumer 1 would choose to do for every possible different belief that he might have about the consumption choice of consumer 2.



Doing the same exercise for consumer 2 yields a similar curve, which is drawn in Figure 3 as $R_2 R_2$. The point where these two curves intersect is the Nash equilibrium. Each consumer chooses his best consumption given what he expects the other consumer to choose; and, as it turns out, the other consumer always does exactly what he expects him to do.

When you try to construct the iso-utility curves for person 2, he will choose one that is tangent to the flat line through x_1^* since he expects consumer 1's consumption choice to be x_1^* no matter what he does. This means that the iso-utility curves for the two consumers must cut through each other as shown in the diagram. There must be a point like E where both of the consumers would be better off if they could jointly agree to move there. That would involve each of them reducing their own consumption of the private good and using that to increase production of the public good.

2 What is wrong with the outcome in the voluntary contribution game.

Since it is so important to understand how externalities work in economics, lets make the argument in a different way. This exercise also helps to understand the methods that micro economists sometimes use to describe why institutions work badly, and the sorts of ideas they propose to fix them.

To begin, lets revert to the picture we have used so far to describe allocations, with the total consumption of the private good on the horizontal axis and the output of the putblic good on the vertical axis. For the purposes of this argument, lets just imagine that the private good is money. The public good is any good that can be reproduced at zero marginal cost. I'll call it software.

Typical software programs use existing libraries to carry out simple tasks that come about during the creation of some higher level process. For example, a web page that displays an ad may request the ad from some other computer on a network. The 'process' of requesting an ad or any bit of infomration over a network in the course of building a webpage is patentable. The actual code that carries out the request is a software program that is subject copyright - two different levels of protection. If there is no patent on the process, then the copyright holder can prevent someone else from using his or her program to make a network request unless they pay a fee. If you aren't willing to pay the fee, then you can write your own version of the program. If the *process* of making a network request is also patented, then you can't actually make a network request without paying a fee.

In Canada and the US, patents have nothing to do with invention, the first person to apply for a patent gets it no matter who 'invented', if that is the right word, the process. In many ways the reason that companies apply for patents is to protect themselves from other patent holders. If they have a patent *portfolio* then they can make cross licensing agreements with other large companies that also have portfolios. These agreements mean that the companies agree not to sue each other for patent infringement. The sort of companies that make these agreements – Microsoft, Apple, Dell, Samsung. Of course, this creates an even bigger pool of patents these companies can use to sue other entrants.

What has saved the tech sector from software patents is open source software – Google, Facebook and the other big social media giants have been developed using open source software. At least in my experience, they tend to give back by giving their own code open source licenses so that other firms can use it.

In the story that follows software means the bundles of library code and software methods that users develop to carry out their day to day activities over the computer. Each of the two consumers below writes software for this purpose. The voluntary contribution game approximates the open source method. We'll try to compare that with what happens in the patent sector.

As in the voluntary contribution story given above, each consumer has a payoff $u_i(x, y)$ that depends on x, the amount of their money income they keep for themselves, and y the total amount of software code that they use. The software is freely redistributable since an essentially unlimited number of copies of the software can be made at no extra cost, so it satisfies the definition of a public good as we described it above. Indifference curves for consumer 1 are bundles that give him or her the same level of utility. For consumer 1, this is really straightforward. The indifference curve is just the collection of (x, y) pairs that satisfy

$$u_1(x, y) = K.$$

The graphs of these curves look no different than they do in any other problem. Higher indifference curves (further away from the origin) represent higher values of K in the equation above, and correspondingly higher payoffs to consumers.

The leap we are going to make here is that when we choose a bundle (x, y) for consumer 1, this bundle induces a corresponding bundle for consumer 2 given by

$$\left(\omega_1 + \omega_2 - f^{-1}(y) - x, y\right).$$

In words, we figure out how much money is required to produce y, that is the $f^{-1}(y)$, then subtract that from the total endowment $\omega_1 + \omega_2$ to get the total amount of money left over after producing y units of software. Subtract the x we want to give to consumer 1, then give the rest to consumer 2. The function $f^{-1}(y)$ is sometimes referred to as the cost function for the public good.

Given this logic, we could define an indifference curve for *consumer 2* by finding all the consumption bundles for *consumer 1* that induce the same payoff for player 2. Formally, an indifference curve for player 2 is the collection of all bundles for player 1 that satisfy

$$u_2\left(\omega_1 + \omega_2 - f^{-1}(y) - x, y\right) = K.$$

To figure out what these indifference curves look like is a bit daunting. Holding y constant, the less money consumer 1 has, the more is left over for consumer 2. In this sense, player 2's indifference curves represent higher payoffs for 2 the closer they are to the origin.

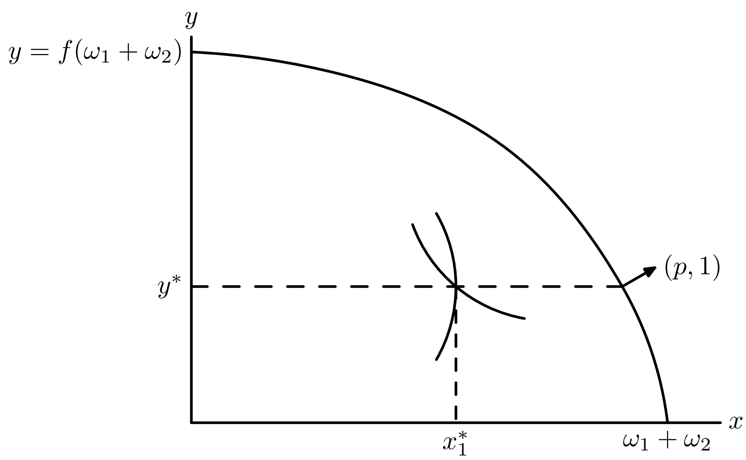
For the rest, we'll rely on the idea that the public good provides diminishing marginal utility to consumer 2. What that means is that the more of the public good that is being produced, the less valuable an additional unit of the public good will be. Fix the consumption of the private good of consumer 1 at some value x_0 and travel up the vertical line through x_0. Remember that changing the public good in this case means that all the cost of producing the public good is borne by consumer 2 since the amount of money consumer 1 has is being held constant at x_0. Initially as you increase the public good, this makes consumer 2 better off since the public good is very valuable to her when there isn't much of it. Eventually the thrill will wear off, and as the production of the public good gets higher and higher, consumer 2's payoff will begin to decline. What that means is that consumer 2's indifference curves will typically cross any vertical line twice.

Of course, there will be one indifference curve that is just tangent to the vertical line. This one is illustrated in the picture below.

The indifference curves should look like backward C's as in the following diagram, where I have superimposed consumer 2's indifference curve into the original diagram depicting the Nash equilibrium of the voluntary contribution game. Let me explain why. At the point where the indifference curve is just tangent to the vertical line, consumer 2 could increase her production of software y and travel further up the vertical line. If she travels directly upward, it means that she is holding consumer 1's holdings of money constant. In other words, if she increases production, she pays all the costs herself. At the tangency, she doesn't want to do this - the marginal benefit she gets from increasing production is exactly equal to the marginal cost. Formally

$$\frac{\partial u_2\left(\omega_1+\omega_2-f^{-1}\left(y^*\right)-x_1^*,y^*\right)}{\partial x}\frac{1}{f'\left(y^*\right)}=\frac{\partial u_2\left(\omega_1+\omega_2-f^{-1}\left(y^*\right)-x_1^*,y^*\right)}{\partial y}.$$

Then if you travel up the vertical line a bit above (x_1^*, y^*), then consumer 2 is strictly worse off. How to restore her payoff? You have to give her more money, which means reducing the amount of money you leave for consumer 1. That means that you have to travel left of the vertical line to restore 2's payoff. The indifference curve must lie to the left of the vertical line at every point except at the tangency.



Since the point (x_1^*, y^*) is supposed to coincide with the Nash equilibrium in the voluntary contribution game, consumer 1's indifference curve has the same slope as the production possibilities frontier at the point where y^* is produced. In other words, it is downward sloping, not vertical. The lens between the curves represents a situation in which both consumers could be made better off - same thing we showed above.

Theoretically, in an open source software world, too little software will be produced.

3 Patents

In real life, the fact that an outcome isn't pareto optimal doesn't necessarily make it bad. As I mentioned above, the software products you probably use the most, Google search, Facebook and Twitter all use opensource software and give back a bunch of the code they write. The programming languages which drive most web pages, php, python, and javascript, are all open source.

Thanks to good old fashioned economic theory, the fact that the equilibrium of the voluntary contribution game is not pareto optimal makes it sound like this is a problem and that SOMETHING MUST BE DONE! Big corporations have lots of talented people working for them and this gave them an idea. Patents are 'something' therefore WE MUST HAVE PATENTS.

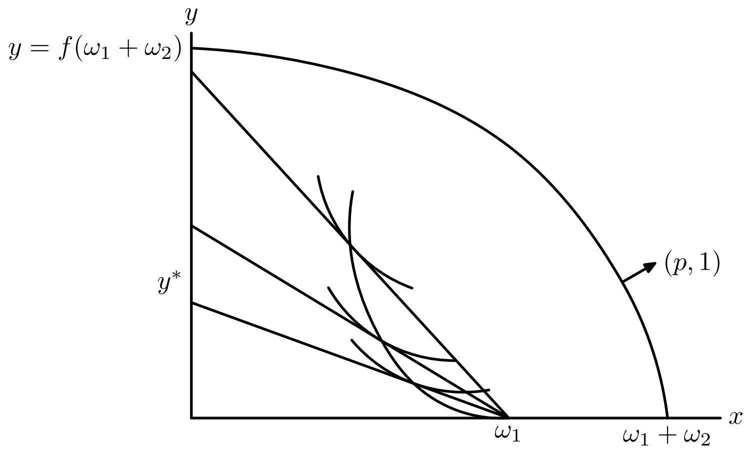
You can read about US copyright and patent laws in a lovely (free) book called "Against Intellectual Monopoly" by David Levine and Michele Boldrin (http://www.dklevine.com/general/intellectual/againstfinal.htm). The book is full of historical anecdotes and simple economic models to explain what is going on. The gist of their argument is that copyright and patent laws in the US are designed to give corporations levers that they can use to prevent entry and suppress competition. Evidence that these laws encourage creation of public goods is basically non-existent, what evidence there is suggests the opposite. This is important for Canada since the US almost always insists that countries adopt US copyright and patent law before they will engage in trade negotiation.

The way patents work is that one of the two consumers applies for a patent, and is then given a monopoly over production of the patented product, in this case our software. As I mentioned above, the US (and now Canada) has what is called a 'First to Patent' law, which means that whoever files a patent on something first gets the patent whether they developed the idea or not. This creates its own problems. For example, patents are sometimes granted after methods are being widely used - one click shopping, and dynamic pricing being examples. Lets ignore this, and just suppose consumer 2 wins this race. Consumer 2 now owns the public good, and can charge whatever price she likes for it. Suppose she sets the price p for the public good.

Normally we do all our stuff using the price of good x and letting the price of good 1 be normalized to 1. We'll switch that here, but this shouldn't create too much of a problem. The public good y has a price p , while the private good has price 1. Given the price set by consumer 2, consumer 1 now finds the best bundle of public and private goods he can afford. This is given by the point where his indifference curve is tangent to the budget line that passes through the point $(\omega_1, 0)$. Why care? The reason is that consumer 1 will no longer be able to produce the public good on his own (of if he does, he will have to pay consumer 2 a fee because she owns the public good). So the price that consumer 2 sets will change consumer 1's desire to have the public good. If consumer 2 sets a price too high, consumer 1 won't want any.

What this means is that the patent doesn't by itself give consumer 2 any incentive to produce more of the public good - production depends on price, which is chosen by consumer 2. What the patent does give her is a device that she can use to extract money from consumer 1.

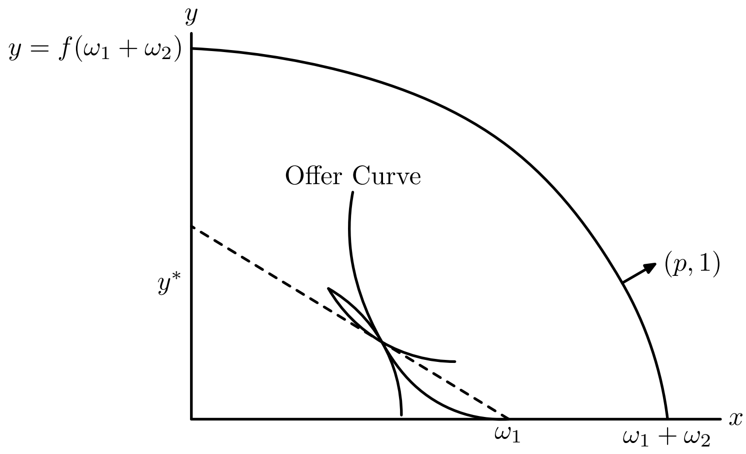
The figure below depicts what consumer 1 would want for three different prices.



The steepest budget line is the one that has the lowest price for the public good, the flattest budget line has the highest price. Since 2 now has the 'intellectual' monopoly, she can set any price that she likes. She will pick the price that maximizes her payoff.

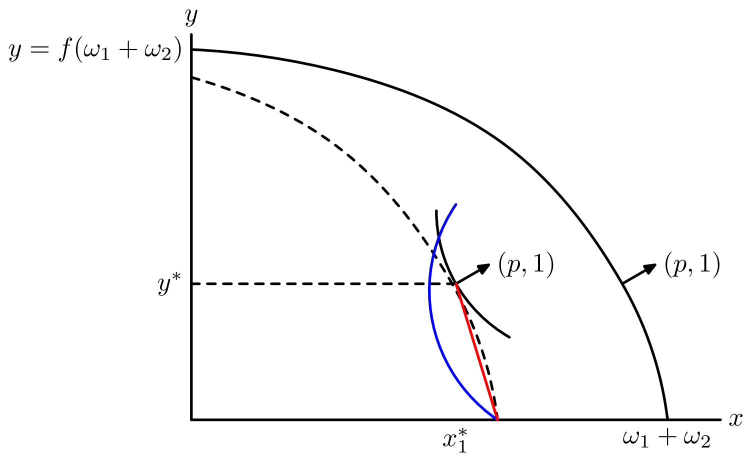
It would seem quite daunting to find this price, since the change in price will change consumer 1's consumption of the public good, which will in turn effect how much consumer 2 needs to produce.

Fortunately we have just the graphic device we need figure out what consumer 2 will do, since we just figured out what her indifference curves looked like in the space depicting consumer 1's consumption of the private and public good. She will pick the price that put her on the highest indifference curve consistent with consumer 1 maximizing subject to his budget constraint. You'll notice in the curve above that there is a smooth line that connects all the tangency points for different prices. This curve is sometimes referred to as the offer curve. The picture that follows shows what happens when consumer 2 chooses the point on this offer curve that is tangent to her own indifference curve.



The indifference curve for consumer 2 is now tangent to consumer 1's offer curve so consumer 2 is getting the best payoff she can get. Consumer 1 is getting the best payoff he can afford. Notice that because of the way the offer curve is drawn the two consumers indifference curves won't be tangent. Both consumers would be better off if more of the public good were produced. So the patent solution just produces another inefficient outcome. This is why proponents of patents will never talk about how well patents work. Instead they focus on how bad the outcome is likely to be without them - we went over that - not really so bad.

We can continue with graphic reasoning a bit more, but then we'll need to switch to algebra. The following figure reproduces the outcome for consumer 1 in the equilibrium of the voluntary contribution game.

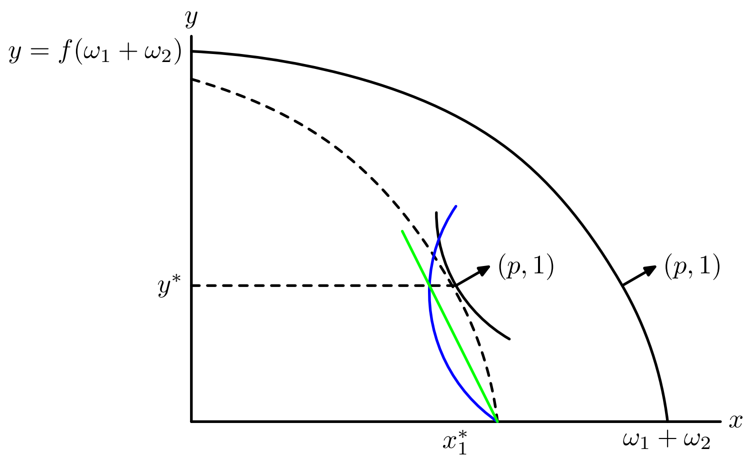


Recall that the budget line for consumer 1 with patents starts at his or her endowment point $(0, \omega_1)$. If consumer 2 sets a price for software which allows consumer 1 to buy the outcome he enjoyed in the volunary contribution game, then the budget line consumer 1 faces would be given by the red line in the picture. Since this budget line always lies below the production possibilities curve that consumer 1 faces in the volutary contribution game, it will be steeper than consumer 1's indifference curve at his allocation in the equilibrium of that game.

So far, this is completely working. If the price of the software were set so that consumer 1 could reproduce the outcome in the volunary contribution game, then consumer 1 would be able to purchase the additional software that he wants from consumer 2. Consumer 2 would in turn be willing to produce it because she is making more than enough money to compensate her for her additional cost.

What patents get wrong is what they do next - they give consumer 2 control over price. Most economics students understand at some level that markets don't work when some market participant can control the price. In this example, it is easy to see why this is the case. The patent holder now has a new device for earning money that has nothing at all to do with having to produce software. It would first occur to her than since she controls price, she can actually get the same level of the public good y^* as in the voluntary contribution game, while ending up with a lot more money for herself.

In our formalism, we can use consumer 1's offer curve to understand this. Since the budget line that consumer 1 faces when the price is set in such a way that he can choose the equilibrium outcome in the voluntary contribution game is steeper than his indifference curve at that point, it means that consumer 1's offer curve cuts the horizontal line through y^* at a point to the left of consumer 1's original equilibrium allocation. The offer curve should look like the solid blue line in the figure above. So consumer 2 can raise the price of software until the budget line faced by consumer 1 looks like the green line in the next figure.



If you recall that the definition of the offer curve is all the points at which consumer 1's indifference curve is tangent to a line that runs from that point back to the endowment point, consumer 1 will voluntarily choose y^* units of software when the price is set to the green line is the budget line. By allowing consumer to control of the price, the patent basically creates a subsidy to the patent holder.

Consumer 2 might not be satisfied with this subsidy. She has really embarked at this point on a new venture - surplus extraction. Software is somewhat secondary at this point and she receives her reward by manipulating price. It is theoretically possible that she might want to raise output of software, but as in the figures above, this isn't her main objective, it is to find the place where her indifference curve is tangent to consumer 1's indifference curve.

Whether she does or not can be determined by travelling along the dashed horizontal line to the left of the equilibrium point in the voluntary contribution game until you reach the offer curve, given by the solid blue line in the figure above. Since consumer 2 can achieve any point on the offer curve that she wants, what she does will depend on what her indifference curve looks like through that point. Generally, the tangency with the offer curve may involve either more or less software than in the voluntary contribution game. So patents are as likely from a theoretical perspective to lower output of software as they are to raise it.

If this graphical reason is getting too complicated, then we can also do the argument algebraically. Suppose that each consumer has money income ω and that one unit of software costs one dollar to build. Lets also suppose that the consumers both have Cobb Douglas utility functions with

$$u(x, y) = x^{\alpha} y^{1-\alpha}.$$

Consumer 1 maximizes

$$x^{\alpha} (2\omega - x - x_2)^{1-\alpha}$$

where x_2 is the amount of income consumer 1 expects consumer 2 to retain for herself (which makes her contribution to the production of software $\omega - x_2$). This is done subject to the constraint that $x \leq \omega$.

The first order condition gives

$$x = \alpha (2\omega - x_2)$$

which has a solution between 0 and ω provided 2α is less than 1 and $x_2 < \omega$. The symmetric equilibrium for the voluntary contribution game is then

$$x = \frac{2\alpha}{1+\alpha}\omega$$

for each of the two players. The total output of software would then be

$$2\omega - \frac{4\alpha}{1+\alpha}\omega = 2\omega \frac{(1-\alpha)}{1+\alpha}. \tag{3}$$

On the other hand, if 2 has the patent, then as you know, consumer 1 will keep the fraction α of his money income and choose to buy $\frac{(1-\alpha)\omega}{p}$ units of the public good. Since $\frac{2}{1+\alpha}\alpha\omega$ is obviously strictly larger than $\alpha\omega$ you can see that consumer 1 will have a lot less income with patents.

So when 2 charges p for the public good, her payoff is

$$\left(\omega\left(2-\alpha\right)-\frac{\left(1-\alpha\right)\omega}{p}\right)^{\alpha}\left(\frac{\left(1-\alpha\right)\omega}{p}\right)^{(1-\alpha)}.$$

To figure out whether 1 will end up with more of the public good, we have to figure out what price 2 will choose. The price that maximizes the expression above will also maximize

$$\alpha\ln\left(\omega\left(2-\alpha\right)-\frac{\left(1-\alpha\right)\omega}{p}\right)+\left(1-\alpha\right)\ln\left(\frac{\left(1-\alpha\right)\omega}{p}\right)$$

which is a little simpler. Differentiate it to get

$$\frac{\alpha}{\omega\left(2-\alpha\right)-\frac{\left(1-\alpha\right)\omega}{p}}\frac{\left(1-\alpha\right)\omega}{p^{2}}-\frac{1-\alpha}{p}.$$

This means that the price 2 will set satisfies

$$\alpha\omega=p\left(\omega\left(2-\alpha\right)-\frac{\left(1-\alpha\right)\omega}{p}\right).$$

This has solution.

$$p=\frac{1}{2-\alpha}.$$

Now lets use the diagram above to figure out whether 2 will want more of less software than is created in the voluntary contribution game. We can decompose her price adjustment into two parts. The first thing we could imagine is that she takes the subsidy and raises the price to the point where 1 will want the same level of the public good as he did in the equilibrium of the voluntary contribution game. With Cobb-Douglas preferences, the consumer will always want to keep $\alpha\omega$ of his income. So we want to find a price that make consumer 1 purchase the level of output in the voluntary contribution game. From (3) this is $2\omega\frac{(1-\alpha)}{1+\alpha}$. So we want a price such that

$$2\omega\frac{\left(1-\alpha\right)}{1+\alpha}=\frac{\left(1-\alpha\right)\omega}{p}$$

or

$$p=\frac{1+\alpha}{2}.$$

It isn't too hard to show that $\frac{1+\alpha}{2}>\frac{1}{2-\alpha}$ for all $0<\alpha<1$, so consumer 2 will set a lower price for software than the price that would have supported the same output of software as in the voluntary contribution game.

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(%i22) wxplot2d([((x*2)^x)*(2*2*(1-x)*(2-x))^(1-x),((x/(1-x))^x)*(4*(1-x)*(1+x))], [x,.1,.9], [y,.1,10])$
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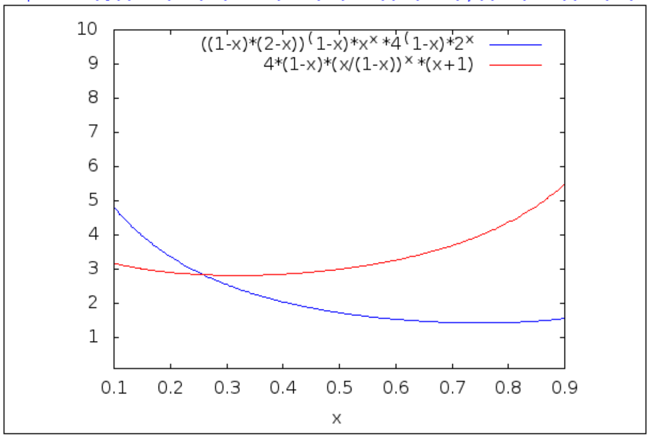
(%t22)



We could have figured this out more directly. Once consumer 2 gets a patent, she automatically receives a transfer of income from consumer 1. However, the size of the transfer doesn't change as 2 varies the price. So if consumer 1 has Cobb Douglas preferences, the only way consumer 2 can increase her payoff is by using this transfer to pay for more software for herself.

4 The Samuelson Condition for Efficient Production of the Public Good.

Lets rephrase the problem a bit. Instead of thinking that the two consumers in the problem above as choosing how much private good they want to consume, lets just address the equivalent question of how much they want to contribute to the production of the public good. Think of e_1 as the contribution (for example the effort) that consumer 1 makes to production of the public good. e_2 is consumer 2's contribution or effort. What ever is left over from the endowment, i.e., $\omega_1 - e_1$, is consumption of the private good for consumer 1. This is exactly the same problem as the one discussed above.

An equilibrium in the voluntary contribution game then ensues, and consumers jointly make some contribution e^* to production of the public good. In other words, in the equilibrium of the voluntary contribution game, consumers contribute at total of $e^* = \omega_1 - x_1^* + \omega_2 - x_2^*$ to produce $y^* = f(e^*)$ units of the public good in all.

If you look back at the conditions (1) and (2) that describe the equilibrium of the voluntary contribution game, each consumer is maximizing his payoff given the contributions of the other player. To rephrase consumer 1's condition (1), for example, we would write

$$u_1\left(\omega_1 - e_1^*, f\left(e_1^* + e_2^*\right)\right) \geq u_1\left(\omega_1 - e_1^*, f\left(e_1' + e_2^*\right)\right)$$

for all feasible contributions e_1' for consumer 1. The first order condition for this is

$$u_1^x(\omega_1 - e_1^*, f(e_1^* + e_2^*)) = u_1^y(\omega_1 - e_1^*, f(e_1^* + e_2^*)) f'(e_1^* + e_2^*).$$

The way this is often said in words is that the marginal rate of substitution of the public for the private good $\left(\frac{u_1^y(\omega_1 - e_1^*, f(e_1^*+e_2^*))}{u_1^x(\omega_1 - e_1^*, f(e_1^*+e_2^*))}\right)$ is equal to the marginal cost $\frac{1}{f'(e_1^*+e_2^*)}$ of producing the public good.

Why do we say this isn't pareto optimal? This is because we can actually do something that would make consumer 1 strictly better off without harming consumer 2 at all. To see how, take the payoff $u_2^* = u_2(\omega_2 - e_2^*, f(e_1^* + e_2^*))$ to be fixed, and try to maximize

$$u_1(\omega_1 - e_1, f(e_1 + e_2))$$

subject to the constraint that $\omega_1 \geq e_1$, $\omega_2 \geq e_2$ and

$$u_2(\omega_2 - e_2, f(e_1 + e_2)) = u^*.$$

In words, we want to find the best payoffs we could achieve for consumer 1 conditional on requiring that consumer 2 continues to receive the same payoff he or she gets in the equilibrium of the voluntary contribution game. You know how to solve this problem - it is just a Lagrangian problem

$$\mathcal{L}(e_1, e_2, \lambda_1, \lambda_2, \lambda_3) =$$

$$u_1(\omega_1 - e_1, f(e_1 + e_2)) +$$

$$\lambda_1(u_2(\omega_2 - e_2, f(e_1 + e_2)) - u^*) + \lambda_2(e_1 - \omega_1) + \lambda_2(e_2 - \omega_2).$$

To make life simple, suppose we happen to find a solution to this problem where both e_1 and e_2 are less than ω_1 and ω_2 respectively. Then by complementary slackness, the first order conditions are given by

$$-u_1^x(\omega_1 - e_1, f(e_1 + e_2)) + u_1^y(\omega_1 - e_1, f(e_1 + e_2)) f'(e_1 + e_2)$$

$$= -\lambda_1 u_2^y(\omega_2 - e_2, f(e_1 + e_2)) f'(e_1 + e_2).$$

The corresponding condition for consumer 2 is

$$u_1^y(\omega_1 - e_1, f(e_1 + e_2)) f'(e_1 + e_2) =$$

$$-\lambda_1(-u_2^x(\omega_2 - e_2, f(e_1 + e_2)) + u_2^y(\omega_2 - e_2, f(e_1 + e_2)) f'(e_1 + e_2)).$$

Divide the top equation by the bottom equation to get

$$\frac{-u_1^x + u_1^y f'}{u_1^y f'} = \frac{u_2^y f'}{-u_2^x + u_2^y f'}$$

where I have left out the arguments of the various functions to make things a bit simpler. Now cross multiply to get rid of the denominators to get

$$\left(u_2^y f' - u_2^x\right)\left(u_1^y f' - u_1^x\right) = u_1^y u_2^y f'^2$$

which gives

$$u_1^y f' u_2^x + u_2^y f' u_1^x = u_2^x u_1^x$$

which finally yields what is called the Samuelson condition for optimality of public good provision

$$\frac{u_1^y}{u_1^x} + \frac{u_2^y}{u_2^x} = \frac{1}{f'}.$$

The term $\frac{u_1^y}{u_1^x}$ is the ratio of the marginal utility of the public good to the marginal utility of the private good. In words, it it the amount of the private good that consumer 1 is willing to give up to get one additional unit of the public good. The term $\frac{1}{f'}$ is the amount of the private good that has to be given up to get one additional unit of the public good - that is, it is the marginal cost of providing the public good. The Samuelson condition then says that the sum of the marginal willingness to pay for both consumers should be equal to the marginal cost of providing the public good.

5 Lindahl Prices

Like the *First Welfare Theorem* that is associated with the private goods markets we studied before (Recall the theorem says that every Walrasian Equilibrium is pareto optimal), competitive arguments can be misused with public goods as well. Provided firms and consumers are all price takers, it is actually possible to turn public goods into private goods by treating each unit of the public good that the firm produces as if it were multiple copies of the same good, then charging each consumer independently for each of these copies. The idea is that when the firm produces additional units of the public good, each consumer will pay on the margin what the extra unit is worth to them. This trick overcomes the externality.

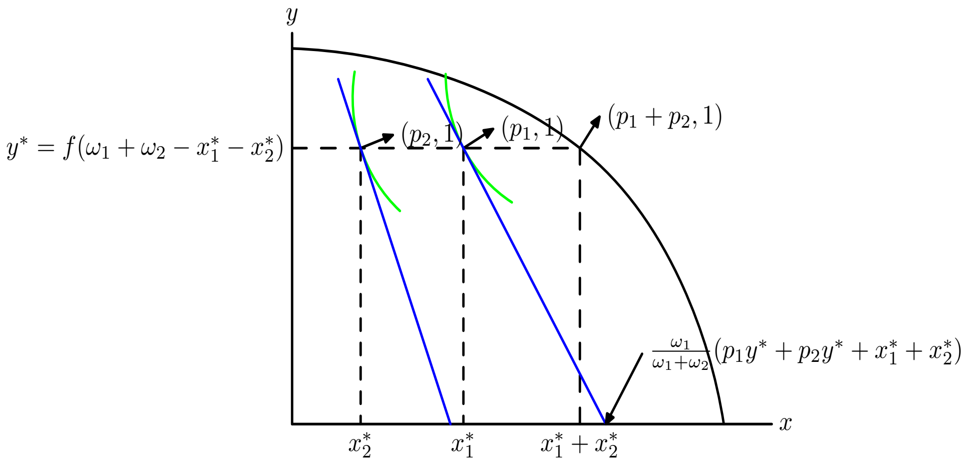
Since this theorem is at the heart of much of current copyright and patent legislation, I'll explain it first, then illustrate the ways it is misused.

First declare that there are actually 3 goods, y_1, y_2, and x. The first is public good for person 1, the second public good for person 2 and the third the private good. All production of these three goods will be undertaken by a single profit-maximizing firm whose production possibilities frontier is just

$$\{(y_1, y_2, x) : y_1 = y_2 = f(\omega_1 + \omega_2 - x)\}$$

All endowments are owned by the firm, but consumer 1 owns the share $\omega_1/(\omega_1 + \omega_2)$ of the firm and will receive that share of its profits. Consumer 2 will own the complementary share $\omega_2/(\omega_1 + \omega_2)$. We will make one big assumption, which is that this single firm is a price-taker.

Consumer 1 only cares about his consumption of the private good and his consumption of good 1; consumer 2 only cares about her consumption of the private good and good 2. Consumer 1 doesn't care how much y_2 consumer 2 consumes, so there are no externalities in consumption. The firm 'internalizes' all the externalities associated with the public good. The physical connection between y_1 and y_2 is simply a part of its production process, so there are no production externalities. So, all we need to do is to find the Walrasian equilibrium of this economy with production and that will give us a Pareto optimal allocation by the first welfare theorem that we studied last week. The solution is given in Figure 4.



Notice that, when the firm increases production of the public good, it receives revenue twice on each unit it produces. Consumer 1 pays p_1 for that unit, but consumer 2 also pays p_2 for it. So, the iso-profit curve for the profit maximizing firm that must be tangent to the production possibilities frontier has slope $\frac{1}{p_1+p_2}$. The firm earns its profits for its production decision then distributes these profits to its owners. Consumer 1 receives income

$$\frac{\omega_1}{\omega_1 + \omega_2}(p_1 y^* + p_2 y^* + x_1^* + x_2^*)$$

which is labeled on the horizontal axis in Figure 4 . Consumer 2 receives

$$\frac{\omega_2}{\omega_1 + \omega_2}(p_1 y^* + p_2 y^* + x_1^* + x_2^*)$$

which is the point where 2's budget line intersects the horizontal axis (that has not been labeled in the figure to keep things simpler).

Consumer 1 now faces a budget line (the blue line in the picture) along which he chooses his best consumption bundle. Notice that since consumer 1 only buys good y_1 at price p_1 – the slope of this budget line is $\frac{1}{p_1}$ *not* $\frac{1}{p_1+p_2}$. So in this equilibrium, consumers marginal rates of substitution will be different from the firm's marginal rate of substitution and generally different from each other.

The market clearing conditions are twofold. First, consumers must both choose to purchase the common level of output of the public good that has been offered by the firm. Second, the sum of the private good demand of each consumer must be equal to the total amount of the private good that the firm has chosen to produce.

The prices that support this outcome are often know as Lindahl prices. The reciprocal of the slope of the production possibilities frontier is the marginal cost of producing one extra unit of the public good (expressed in terms of units of the private good). Since the iso-profit line must be tangent to the production possibilities curve, this marginal cost is equal to $p_1 + p_2$. The reciprocal of the slopes of the consumers indifference curves are equal to their marginal willingness to pay for the public good (again expressed in terms of the private good). Since the indifference curves are tangent to the individual budget lines, these willingnesses to pay are p_1 and p_2 respectively. So, the Lindahl prices ensure that the marginal cost of producing the public good is exactly equal to the sum of the two consumers' willingness to pay.