Lecture 11: Regularization

STAT 598z: Introduction to computing for statistics

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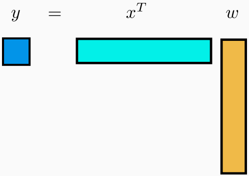
Consider linear regression:

$$y = \mathbf{x}^\top \mathbf{w} + \epsilon$$



Ordinary least squares

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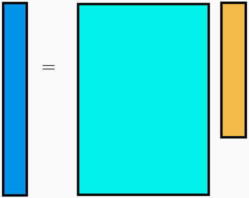
In vector notation:

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \epsilon, \quad \mathbf{y} \in \Re^n, \mathbf{X} \in \Re^{n \times p}$$



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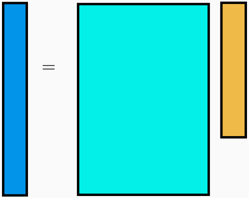

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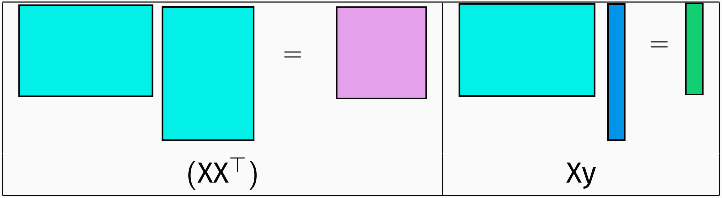

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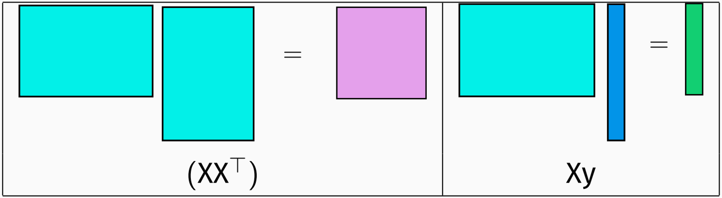

How to do this in `R` (without using `lm`)?

- Do not invert with `solve` and multiply!
- Directly solve $(\mathbf{X}\mathbf{X}^\top)\hat{\mathbf{w}} = \mathbf{X}\mathbf{y}$

Prediction error

$\hat{\mathbf{w}}$ is an unbiased estimate of the true $\mathbf{w}$

For a test vector $\mathbf{x}^{test}$ we predict $\mathbf{w}^\top \mathbf{x}^{test}$.

(Squared) prediction error: $PE = \frac{1}{k} \sum_{i=1}^{k} (y_i^{test} - \mathbf{w}^\top \mathbf{x}_i^{test})^2$

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What if $p > n$?

- $\mathbf{X}\mathbf{X}^\top$ is singular

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λ trades-off bias and variance

Maybe a nonzero λ is actually good?

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ℓ_2/ridge/Tikhonov regularization

Ridge regression (solution)

Simple modification of the least-squares solution:

$$\hat{\mathsf{w}}_\lambda = (\mathsf{X}^\top \mathsf{X} + \lambda \mathsf{I}_p)^{-1} \mathsf{X}^\top \mathsf{y}$$

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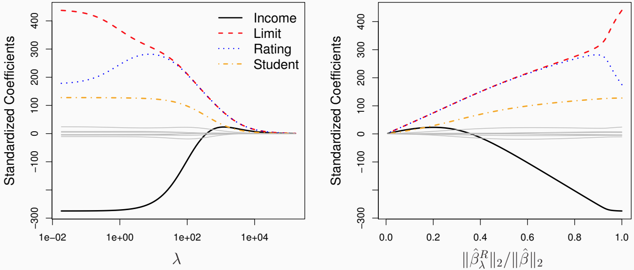
Shrinks least-squares solution.

Credit data set (average credit card debt)



James, Witten, Hastic and Tibshirani

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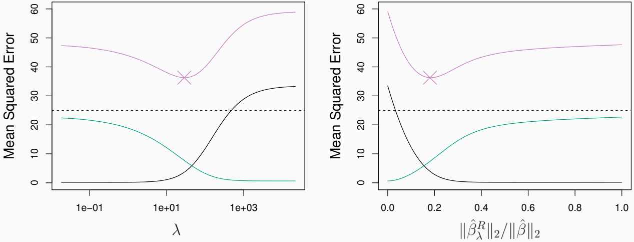
- Having chosen $\hat{\lambda}$ solve regularized least square on all data

Does this work?



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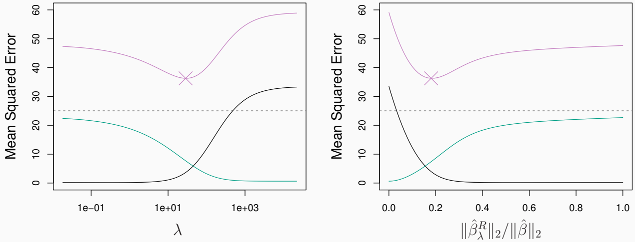

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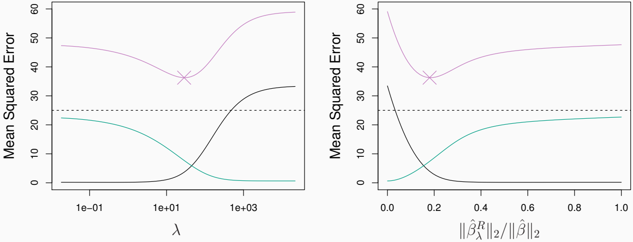

Ridge regression improves performance by reducing variance

- does not perform feature selection
- just shrinks components of $\mathbf{w}$ towards 0

For the former: Lasso

Regularization as constrained optimization

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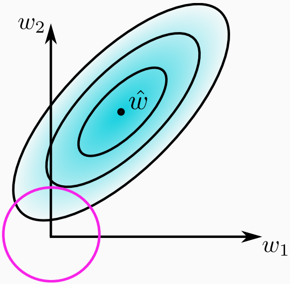
First problem: regularized optimization
Second problem: constrained optimization



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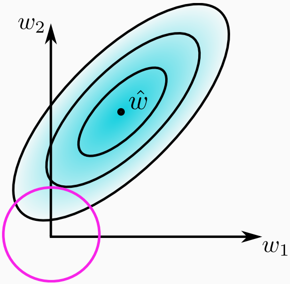

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Result:

- $\hat{\mathbf{w}}_{LASSO}$ has some components *exactly* equal to zero.
- Performs feature selection.

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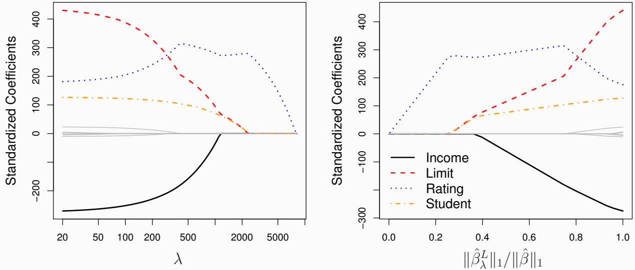

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