Solving Hamilton-Jacobi-Bellman equations in a search and matching model

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Model overview

- People moving from job to job transmit productive ideas between firms.
- Firms search for “managers” to improve the productive knowledge of their firm.
- Workers at productive firms look for manager jobs to get more pay.

Model overview

- Focus on the steady-state (drop ‘time’ from the model where possible) — Ignore the growth dimension of the model
- Complicated problem because:
  - Two-sided search market
  - Heterogeneous agents on both sides
  - Continuous time model (Hamilton-Jacobi-Bellman)

Firms

- Hire 1 manager and $l(z)$ number of workers
- Productivity determined by manager's knowledge
- Can try to improve productivity/profitability by posting vacancies for a "better" mananger

Firms

Value function:

$$
\begin{aligned}
r\Pi(z) &= p(z)y(z,l(z)) - wl(z) - m(z) + \frac{\partial \Pi(z)}{\partial t} \\
&\quad + \max_{v} \Bigg[- c_M(v) + q(\theta)v \int_{z}^{\infty} \int_{\bar{x}(z,y)}^{y} [\Pi(x) - \Pi(z)] g(x,z,y)
\end{aligned}
$$

Policy rule:

$$
\frac{\partial c_M(v(z))}{\partial v} = q(\theta) \int_{z}^{\infty} \int_{\bar{x}(z,y)}^{y} [\Pi(x) - \Pi(z)] g(x,z,y)\, dx\, \phi_{Hs}(y) dy
$$

Workers

- Supplys 1 unit of labor inelastically, earns wage w
- Passively absorb the productivity knowledge of their firm.
- Can search for a managerial job in another firm
- If the worker finds a job, he is able to impart some of his learned knowledge on the new firm

Workers

Value function:

$$
\begin{aligned}
rW(z) \quad = \quad & w + \frac{\partial W(z)}{\partial t} + \max_s \Bigg\{ - c_s(s) \\
& + s\tilde{\theta} q(\tilde{\theta}) \int_0^z \int_{\bar{x}(y,z)}^z \max\{M(x) - W(z)\ ,\ 0\} g(x,y,z)\, dx\, \phi_{Fv} \\
& + [\text{Prob. fired}] \int_0^\infty (W(y) - W(z)) \phi_{Fw}(y)\, dy \Bigg\} \\
& + q(\theta) v_M(z) \int_z^\infty \int_{\bar{x}(z,y)}^y (W(x) - W(z)) g(x,z,y)\, dx\, \phi_{Hs}(y)
\end{aligned}
$$

Policy rule:

$$
\begin{aligned}
c_s^{'}(s(z)) \quad = \quad & \theta q(\theta) \int_0^z \int_{\bar{x}(y,z)}^z \max\{M(x) - W(z)\ ,\ 0\} g(x,y,z)\, dx\, \phi_{Fv}(\\
& + \frac{\partial [\text{Prob. fired}]}{\partial s} \left[\int_0^\infty (W(y') - W(z)) \phi_{Fw}(y') dy' \right]
\end{aligned}
$$

Managers

- No incentive to move
- Stay at the firm earning $m(z)$ each period,
- When the firm finds a better manager, you become a worker for the new manager firm

Algorithm objective

For each grid point find:

- the value of being a Firm ($\Pi(z)$), Manager ($M(z)$), and Worker ($W(z)$).
- the policy rules ($v(z)$ and $s(z)$) that maximize the value functions.

Methods to Solve HJB equations

Unlike discrete-time Bellman equations, no convergence theorem exists!

But in general, we can usually find a solution:

- Candler – Iterate between policy rules and value functions.
- Kushner-Dupuis – Discrete time approximation.

Numerical approximation

- state space $z \in [0, \infty)$ bounded to $z \in [z_{min}, z_{max}]$
- Continuous State space approximated by finite grid

Algorithm overview

1. Given distribution of productivity and value functions, solve for *policy rules*
2. Given distribution and policy rules, solve for *value functions*

Iterate between (1) and (2) until the value functions converge.

Algorithm overview

1. Given distribution of productivity and value functions, solve for *policy rules*
   1.1 Solve for new $v(z_n)$
   1.2 Update the distribution of vacancies and labor market tightness
   1.3 Solve for new $s(z_n)$
   1.4 Update distribution of searchs and labor market tightness
   1.5 Repeat for all z_n in the state-space grid
2. Given distribution and policy rules, solve for *value functions*

Iterate between (1) and (2) until the value functions converge.

Algorithm overview

1. Given distribution of productivity and value functions, solve for *policy rules*
2. Given distribution and policy rules, solve for *value functions*
    2.1 Holding fixed the policy rules, rewrite $W(z)$ in matrix form
    2.2 Solve for new $W(z)$
    2.3 Using new $W(z)$, solve for $M(z)$
    2.4 Using $M(z)$, compute $m(z)$
    2.5 Using $m(z)$, solve for $\Pi(z)$

Iterate between (1) and (2) until the value functions converge.

Code

Lets look at some code . . .