Solving Hamilton-Jacobi-Bellman equations in a search and matching model

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Model overview

- People moving from job to job transmit productive ideas between firms.
- Firms search for “managers” to improve the productive knowledge of their firm.
- Workers at productive firms look for manager jobs to get more pay.

Model overview

- Focus on the steady-state (drop ‘time’ from the model where possible) — Ignore the growth dimension of the model
- Wont show the complete model here
- Complicated problem because:
  - Two-sided search market with heterogeneous agents on both sides
  - Continuous time model (Hamilton-Jacobi-Bellman)
  - *A lot* of `for` loops — Computationally slow

Firms

- Hire 1 manager and $l(z)$ number of workers
- Productivity determined by manager's knowledge
- Can try to improve productivity (and profitability) by posting vacancies for a "better" mananger

Firms

Value function:

$$
\begin{aligned}
r\Pi(z) &= p(z)y(z,l(z)) - wl(z) - m(z) + \frac{\partial \Pi(z)}{\partial t} \\
&\quad + \max_{v}\left[-c_M(v) + q(\theta)v \int_z^{\infty} \int_{\bar{x}(z,y)}^{y} [\Pi(x) - \Pi(z)] g(x,z,y)\, dx\, \phi_{Hs}(y)\, dy \right]
\end{aligned}
$$

Policy rule:

$$
\frac{\partial c_M(v(z))}{\partial v} = q(\theta) \int_z^{\infty} \int_{\bar{x}(z,y)}^{y} [\Pi(x) - \Pi(z)] g(x,z,y)\, dx\, \phi_{Hs}(y) dy
$$

Workers

- Supplys 1 unit of labor inelastically, earns wage w
- Passively absorb the productivity knowledge of their firm.
- Can search for a managerial job in another firm
- If the worker finds a job, he is able to impart some of his learned knowledge on the new firm

Workers

Value function:

$$\begin{aligned} rW(z) &= w + \frac{\partial W(z)}{\partial t} + \max_s \Bigg\{ -c_s(s) \\ &+s\tilde{\theta}q(\tilde{\theta}) \int_0^z \int_{\bar{x}(y,z)}^z [M(x) - W(z)] g(x,y,z)\, dx\, \phi_{Fv}(y)\, dy \\ &+[\text{Prob. fired}] \int_0^\infty (W(y) - W(z))\phi_{Fw}(y)\, dy \Bigg\} \\ &+q(\theta)v_M(z) \int_z^\infty \int_{\bar{x}(z,y)}^y (W(x) - W(z)) g(x,z,y)\, dx\, \phi_{Hs}(y)\, dy \end{aligned}$$

Policy rule:

$$\begin{aligned} c_s^{'}(s(z)) &= \theta q(\theta) \int_0^z \int_{\bar{x}(y,z)}^z [M(x) - W(z)] g(x,y,z)\, dx\, \phi_{Fv}(y) dy \\ &+ \frac{\partial[\text{Prob. fired}]}{\partial s} \left[\int_0^\infty (W(y') - W(z))\phi_{Fw}(y') dy' \right] \end{aligned}$$

Managers

- No incentive to move
- Stay at the firm earning $m(z)$ each period,
- When the firm finds a better manager, you become a worker for the new manager firm

Managers

Value function:

$$\begin{aligned} rM(z) &= \mu(z) + \frac{\partial M(z)}{\partial t} \\ &+q(\theta)v_M(z)\int_z^\infty \int_{\bar{x}(y)}^y [W(x) - M(z)]g(x,z,y)\,dx\,\phi_{Hs}(y)\,dy \end{aligned}$$

Algorithm objective

For each grid point find:

- the value of being a Firm ($\Pi(z)$), Manager ($M(z)$), and Worker ($W(z)$).
- the policy rules ($v(z)$ and $s(z)$) that maximize the value functions.

Methods to Solve HJB equations

Unlike discrete-time Bellman equations, no convergence theorem exists!

But in general, we can usually find a solution:

- Candler – Iterate between policy rules and value functions.
- Kushner-Dupuis – Discrete time approximation.

Numerical approximation

- state space $z \in [0, \infty)$ bounded to $z \in [z_{min}, z_{max}]$
- Continuous state space approximated by finite grid $z \in [z_1, z_2, \ldots, z_N]$

Algorithm overview

1. **Given distribution of productivity and value functions, solve for *policy rules***
2. **Given distribution and policy rules, solve for *value functions***

Iterate between (1) and (2) until the value functions converge.

Algorithm overview

1. **Given distribution of productivity and value functions, solve for *policy rules***
    1.1 Solve for new $v(z_n)$
    1.2 Update the distribution of vacancies and labor market tightness
    1.3 Solve for new $s(z_n)$
    1.4 Update distribution of searchers and labor market tightness
    1.5 Repeat for all z_n in the state-space grid, and until all policy rules converge
2. **Given distribution and policy rules, solve for *value functions***

Iterate between (1) and (2) until the value functions converge.

Algorithm overview

1. **Given distribution of productivity and value functions, solve for *policy rules***
2. **Given distribution and policy rules, solve for *value functions***
   2.1 Holding fixed the policy rules, rewrite $W(z)$ in matrix form
   2.2 Solve for new $W(z)$
   2.3 Using new $W(z)$, solve for $M(z)$
   2.4 Using $M(z)$, compute $m(z)$
   2.5 Using $m(z)$, solve for $\Pi(z)$

Iterate between (1) and (2) until the value functions converge.

Code

Lets look at some code . . .

Policy rule: Firms

$$\frac{\partial c_M(v)}{\partial v} = q(\theta) \int_z^{\infty} \left(\int_{\bar{x}(y)}^{y} [\Pi(x) - \Pi(z)]\, g(x, z, y)\, dx \right) \phi_{Hs}(y)\, dy$$

$$\psi_s v(z) = q(\theta) \sum_{y=z}^{z_{max}} \underbrace{\left(\sum_{x=z}^{y} \overbrace{\mathbb{I}_{M(x)>W(y)}}^{\texttt{moving\_matrix}} [\Pi(x) - \Pi(z)]\, \overbrace{g(x, z, y)}^{\texttt{TransProbVec}} \right)}_{\texttt{inner\_intgr[indx\_y]}} \phi_{Hs}(y)$$

Appendix: Model Equation

$$\phi_{Hw}(z) = \frac{z^{\frac{1/\sigma-1}{\alpha(1-1/\sigma)-1}}\phi_F(z)}{\int_0^\infty x^{\frac{1/\sigma-1}{\alpha(1-1/\sigma)-1}}\phi_F(x)\,dx}$$

$$Y(t) \quad = \quad N^\alpha \left[\int_0^\infty z^{\frac{1/\sigma-1}{\alpha(1-1/\sigma)-1}}\phi_F(z)\,dz\right]^{\sigma/(\sigma-1)-\alpha}$$

$$w \quad = \quad N^{\alpha-1}\alpha(1-1/\sigma)\left[\int_0^\infty z^{\frac{1-\sigma}{\alpha(\sigma-1)-\sigma}}\phi_F(z)\,dz\right]^{\frac{1}{\sigma-1}+1-\alpha}$$

$$v(z) = \frac{1}{\psi} q(\theta) \int_z^\infty \left(\int_{\bar{x}(y)}^y \left[\Pi(x) - \Pi(z)\right] g(x,z,y)\, dx \right) \phi_{Hs}(y)\, dy$$

$$\phi_{Fv}(z) = \frac{v(z)\phi_F(z)}{\int_0^\infty v(y)\phi_F(y)\, dy}$$

$$s(z) < \frac{\gamma z l(z)}{l(z)\theta q(\theta) \int_0^z [\int_y^z \mathbb{I}_{M(x)>W(z)} g(x,y,z) dx] \phi_{Fv}(y)\, dy}$$

Case 1: $s \leq s_{theshold}$ (people are getting fired)

$$\begin{aligned} c_s^{'}(s(z)) &= \theta q(\theta) \int_0^z \left(\int_y^z \mathbb{I}_{M(x)>W(y)}[M(x) - W(z)] g(x, y, z)\, dx \right) \phi_{Fv}(y)\, dy \\ & -\theta q(\theta) \left(\int_0^z [1 - G(\bar{x}(z)] \phi_{Fv})(y)\, dy \right) \left(\int_0^\infty (W(y) - W(z)) \phi_{Fw}(y)\, dy \right) \end{aligned}$$

Case 2: $s > s_{theshold}$ (people are not getting fired)

$$c_s^{'}(s(z)) = \theta q(\theta) \int_0^z \left(\int_y^z \mathbb{I}_{M(x)>W(z)}[M(x) - W(z)] g(x, y, z)\, dx \right) \phi_{Fv}(y)\, dy$$

$$\phi_{Hs}(z) = \frac{s(z)\phi_{Hw}(z)}{\int_0^\infty s(x)\phi_{Hw}(x)\,dx}$$

$$\theta = \frac{N\int_0^\infty v(z)\phi_F(z)\,dz}{\int_0^\infty s(z)\phi_{Hw}(z)\,dz}$$

$$
\begin{aligned}
rW(z) \quad = \quad & w + \gamma W(z) - \gamma z \frac{z_n - W(z_{-1})}{dz} - \tilde{c}_s(s(z)) \\
& + s(z)\theta q(\theta) \sum_{x=z_1}^{z} [M(x) - W(z)] \sum_{y=z_1}^{x} \mathbb{I}_{M(x)>W(z)} g(x,y,z_n)\, \phi_{Fv}(y)\, dy\, dx \\
& + \mathbb{I}_{\Delta l<0} \left[\frac{\gamma z l^{'}(z)}{l(z)} - s(z)\theta q(\theta) \sum_{y=0}^{z} \sum_{x=y}^{z} \mathbb{I}_{M(x)>W(z)} g(x,y,z) dx \phi_{Fv}(y) dy \right] \times \\
& \sum_{y'=0}^{\infty} (W(y') - W(z)) \phi_{Fw}(y')\, dy' \\
& + q(\theta) v_M(z) \sum_{x=z_n}^{\infty} (W(x) - W(z)) \sum_{y=x}^{\infty} \mathbb{I}_{M(x)>W(y)} g(x,z,y)\, \phi_{Hs}(y)\, dy\, dx
\end{aligned}
$$

$$
\begin{aligned}
rM(z) &= \mu(z) + \gamma M(z) - \gamma z \frac{M(z) - M(z_{-1})}{\Delta z} \\
&+ q(\theta) v_M(z) \sum_{y=z}^{\infty} \sum_{z}^{y} \mathbb{I}_{M(x) > W(y)} [W(x) - M(z)] g(x, z, y)\, dx\, \phi_{Hs}(y)\, dy
\end{aligned}
$$

$$
\begin{aligned}
\Pi(z) &= p(z) y(z, l(z)) - wl(z) - mu(z) - \tilde{c}_M(v_M(z)) + \gamma \Pi(z) - z\gamma \frac{\Pi(z) - \Pi(z_{-1})}{dz} \\
&+ q(\theta) \nu_M(z) \sum_{y=z}^{\infty} \left(\sum_{z}^{y} \mathbb{I}_{M(x) > W(y)} [\Pi(x) - \Pi(z)] g(x, z, y)\, dx \right) \phi_{Hs}(y)\, dy
\end{aligned}
$$