The Dynamics of Bertrand Price Competition with Cost-Reducing Investments†

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Abstract: We extend the classic Bertrand duopoly model of price competition to a dynamic setting with uncertain technological progress, where competing duopolists can undertake cost reducing investments in an attempt to "leapfrog" their rival to attain temporary low cost leadership. We analytically characterize the set of *all* Markov perfect equilibria of this game, which include *investment preemption* where only one of the firms does all of the investing as well as *leapfrogging* with piece-wise flat equilibrium price paths punctuated by occasional discontinuous price declines that occur when the high cost follower leapfrogs its low cost rival. Unlike the static Bertrand model, the equilibria of the dynamic Bertrand model are generally *inefficient* due to excessively frequent and duplicative investments.

Keywords: duopoly, Bertrand-Nash price competition, Bertrand investment paradox, leapfrogging, cost-reducing investments, technological improvement, dynamic models of competition, Markov-perfect equilibrium, tacit collusion, price wars, coordination and anti-coordination games, strategic preemption

JEL classification: D92, L11, L13

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1 Introduction

Given the large theoretical literature since the original work of Bertrand (1883), it is surprising that our understanding of price competition in the presence of production cost uncertainty is still rudimentary. For example, in the introduction to his paper on the *static* Bertrand model, Routlege (2010) states “However, there is a notable gap in the research. There are no equilibrium existence results for the classical Bertrand model when there is discrete cost uncertainty.” (p. 357). Less is known about Bertrand price competition in *dynamic* models where firms compete by undertaking cost-reducing investments. In these environments firms face uncertainty about their rivals’ investment decisions as well as uncertainty about the timing of technological innovations that can affect future prices and costs of production.

This paper analyses a simple dynamic extension of the textbook Bertrand-Nash duopoly model of price competition where the firms can make investment decisions as well as pricing decisions. At any time t a firm can decide whether to replace its current production plant with a new state of the art production facility that enables it to produce at a lower marginal cost than its rival. A key assumption of our model is that the state of the art technology improves stochastically and exogenously, whereas technology adoption decisions are endogenous.

The term *leapfrogging* describes the longer run investment competition where a higher cost firm purchases a state of the art production technology that reduces its marginal cost relative to its rival and allows it to attain, at least temporarily, a position of low cost leadership. The assumption that the state of the art technology evolves exogenously differentiates our model from earlier examples of leapfrogging in the literature by, for example, Fudenberg et. al. (1983) and Reinganum (1985).

This earlier work on *patent races* and models of *research and development* focused on firms' continuous choice of R&D expenditures with the goal of producing a patent or a drastic innovation that could not be easily duplicated by rivals.

However, in many industries firms do little or no R&D themselves but can obtain a production cost advantage by investing in a state of the art production technology that is developed and sold by third parties. We model this investment as a binary decision: each firm faces a decision of whether or not to incur the substantial fixed investment cost to replace their current legacy production technology with the latest technology in order to become the current low cost leader. Since all firms have equal opportunity to acquire the state of the art production technology the markets we study are different from those studied in the earlier literature on leapfrogging in the context of R&D and patent races. Though the market we study abstracts from entry and exit, it is *contestable due to ease of investment.* This may promote competition in ways that are similar to markets that are *contestable due to ease of entry* (Baumol, Panzar and Willig, 1982).

However the ease by which either firm can invest in our model leads to an issue we call the *"Bertrand investment paradox"*. If both firms invest at the same time, then Bertrand price competition drives *ex post* profits to zero. If the firms expect this, the *ex ante* return on their investments will be negative, so it is possible that no firm would have an incentive to undertake cost-reducing investments. But if no firm invests, it may make sense for at least one firm to invest. Thus, the investment problem has the structure of an *anti-coordination game.*

Riordan and Salant (RS,1994) showed how the Bertrand investment paradox can be resolved. They analyzed a model of Bertrand price competition between duopolists who can both invest to acquire a *deterministically improving* state of the art technology to try to gain a temporary cost advantage over their rival. RS proved that investment does incur in equilibrium, but *by only one of the firms.* In this *preemption equilibrium* consumers never benefit from technological improvements, because the price remains at the high marginal cost of the non-adopting firm. Further, they showed that the preemption equilibrium is completely inefficient: to discourage entry of its rival the preempting firm adopts new technologies so frequently that all of its profits (and thus all social surplus) is completely dissipated. Their result implies that *leapfrogging is incompatible with Bertrand price competition.*

Though RS stressed that their investment preemption result was "narrow in that it need not hold for other market structures" their analysis "suggests a broader research agenda exploring market structure dynamics" to answer questions such as "Under what conditions do other equilibrium patterns emerge such as action-reaction (Vickers, 1986) or waves of market dominance in which the identity of the market leader changes with some adoptions but not others?" (p. 258).

Giovannetti (2001) advanced the literature by showing that a particular type of leapfrogging — *alternating adoptions* — can be an equilibrium outcome in a discrete time duopoly model of Bertrand price competition under assumptions that are broadly similar to RS. Though Giovannetti did not cite or specifically address RS's work, he showed that both preemption and alternating adoptions can be equilibrium outcomes depending on the elasticity of demand.

Giovannetti's analysis was done in the context of a game where firms make *simultaneous* investment decisions, whereas RS modeled the investment choices as an *alternating move game.* The alternating move assumption seems to be a reasonable way to approximate decisions made in continuous time, where it is unlikely that two firms would be informed of a new technological innovation and make investment decisions at precisely the same instant. However, the change in timing assumptions could have significant consequences, since Theorem 1 of RS shows that *preemption is the only equilibrium* in the continuous time limit of a sequence of discrete-time alternating move investment games as the time between moves tends to zero. RS conjectured that whether firms move simultaneously or alternately makes no difference with respect to the conclusion that preemption is the unique equilibrium of the continuous time limiting game. "We believe the same limit holds if the firms move simultaneously in each stage of the discrete games in the definition. The alternating move structure obviates examining mixed strategy equilibria for some subgames of sequence of discrete games." (RS, p. 255).

Giovannetti's finding that an equilibrium with alternating investments investments is possible if firms move simultaneously suggests that Riordan and Salant's conjecture is incorrect. Giovannetti did not consider whether his results hold if firms move alternately rather than simultaneously, or whether leapfrogging is sustainable in the continuous time limit. Further neither Giovannetti nor RS considered the effect of uncertain technological progress on their conclusions: both assumed that the state of the art production cost declines deterministically over time. Stochastic technological change could create investment opportunities that could upset the preemption equilibrium and lead to more complex adoption dynamics. In particular, deterministic technological progress rules out the possibility of *drastic innovations* in the sense of Arrow (1962), where there is a there is sudden large improvement in technology. Riordan and Salant conjectured that the preemption result was a robust conclusion that would continue to hold in the presence of drastic innovations:

"We conjecture that there exists an equilibrium adoption pattern featuring increasing dominance and rent dissipation quite generally. The heuristic reason is the standard one (Gilbert and Newbery, 1982, Vickers, 1986) that the leading firm always has a weakly greater incentive to preempt to protect its incumbent profit flow." (RS, p. 257).

This paper is the first to characterize the set of *all* Markov perfect equilibria (MPE) — both pure and mixed (behavioral) strategies — of a dynamic duopoly model of Bertrand price competition with stochastic technological progress — under both simultaneous and alternating move assumptions (including stochastic alternating move versions of the game). We provide a unifying framework and reconcile the conflicting results of Giovannetti and RS, and by allowing for stochastic technological progress we also study a much wider range of environments than either of these analyses were able to consider. In particular, by allowing for stochastic technological progress we analyze firm behavior and industry dynamics when there is a possibility of drastic innovations that Arrow (1962) contemplated. Similar to the result of Routledge (2010) in the static context, we establish existence of equilibria in the dynamic Bertrand investment game.

We show that rent dissipating investment preemption will not be an equilibrium outcome if any of the three key assumptions (deterministic technological progress, alternating moves, continuous time) is removed, contrary to RS's conjectures. Instead, we show that very complex patterns of dynamic investment competition are supported, with leapfrogging occurring in many other forms than simple patterns of deterministically alternating investments of Giovannetti (2001). In fact, we show, via numerical calculation of *all* MPE of example games using the *recursive lexicographical search* (RLS) algorithm of Iskhakov, Rust and Schjerning (IRS2016), that various types of leapfrogging equilibria are the typical outcome of the Bertrand investment game.

In the simultaneous move version of the game there is a vast multiplicity of equilibria, and our characterization the set of all MPE of this game is reminiscent of the Folk Theorem: the convex hull of the set of initial node pay-offs of the game is a triangle, whose vertices include two monopoly pay-offs and the origin — a mixed strategy equilibrium with zero pay-offs to both firms. In the *monopoly* MPE of our game one duopolist never invests and the other does all of the investing and keeps the price equal to the higher marginal cost of production, earning full monopoly profit. However the monopoly equilibrium is very different than the investment preemption equilibrium studied by Riordan and Salant (1994). In particular, we prove that the monopoly equilibria are *fully efficient* whereas preemption equilibria are completely inefficient due to the excessively rapid rate of adoption of new technologies by the low cost leader to avoid being leapfrogged by the high cost follower.

When firms invest in an alternating fashion (under deterministic and stochastically alternating move variations), we show that the convex hull of the set of equilibrium pay-offs is a strict subset of the same triangle, and in particular neither the monopoly or the zero profit mixed strategy MPE is supportable in this case. We provide a sufficient condition for the uniqueness of equilibrium: in the alternating moves specification when technology improves in every time period with probability one, the Bertrand investment game has a unique MPE. This condition is satisfied in RS's and Giovannetti's frameworks where technological progress is deterministic. However, when the probability of no improvement in the state of the art in any single period is sufficiently large, the set of MPE is no longer a singleton and will in general include a large number of equilibria that exhibit various types of leapfrogging.

Besides analytic characterization of the set of equilibrium pay-offs, we utilize the RLS algorithm of IRS2016 to numerically compute *all* MPE in a finite state version of the Bertrand investment game. Using the RLS algorithm we can calculate the empirical distribution of the efficiency of all MPE in the Bertrand investment game. We find that the equilibria in our model are typically *inefficient* due to investments that occur *too frequently* relative to the social optimum and due to *duplicative investments* that are a reflection of coordination failures in this game. The most inefficient equilibria are those involving preemption and mixed strategies. However we show that there are also *fully efficient equilibria* that take the form of *asymmetric pure strategy equilibria* and include the monopoly equilibria mentioned above.

Though most of the leapfrogging equilibria display some degree of inefficiency due to duplicative investments, the overall efficiency is generally very high in the Bertrand investment games we have solved. Calculating an efficiency score (defined as the fraction of the total surplus achieved by the social planner that consumers and producers obtain under the duopoly equilibrium), we find that the median efficiency of all equilibria in examples we provide in section 4 is over 95%. Although investment competition in the non-monopoly equilibria of the model does benefit consumers by lowering costs and prices in the long run, it does generally come at the cost of some inefficiency due to coordination failures. However, we provide examples (and thus establish existence) of perfectly coordinated, fully efficient leapfrogging equilibria as well.

In the next section we present our model and summarize the solution method we used to compute all MPE of the game. Section 3 discusses the socially optimal investment strategies and solves the social planner's problem. We present our main results in section 4, and section 5 concludes.

2 The Model

Consider a market consisting of two firms producing an identical good. Assume that the two firms are price setters, have no fixed costs and can produce the good at a constant marginal cost of c_1 and c_2, respectively. Both firms have constant return to scale production technology, so neither of them ever faces a binding capacity constraint.

Under the assumption of perfectly inelastic demand, it is well known that Bertrand equilibrium arises in these settings, leading to the lower cost firm to serve the entire market at a price $p(c_1,c_2)$ equal to the marginal cost of production of the higher cost rival, i.e. $p(c_1,c_2) = \max[c_1,c_2]$. In the case where both firms have the same marginal cost of production we obtain the classic result that Bertrand price competition leads to zero profits for both firms at a price equal to their common marginal cost of production. Normalizing the market size to one, we can write the instantaneous profits of firm 1 as

$$r_1(c_1,c_2) = \begin{cases} 0 & \text{if } c_1 \geq c_2, \\ c_2 - c_1 & \text{if } c_1 < c_2, \end{cases} \tag{1}$$

and the profits for firm 2, $r_2(c_1,c_2)$ are defined symmetrically, so we have $r_2(c_1,c_2) = r_1(c_2,c_1)$.

We introduce the dynamics into the model by assuming that at each time period t both firms have the ability to make an investment to acquire a new production facility (plant) to replace their existing technology. Technological progress that drives down the marginal cost of production (while maintaining constant returns to scale) is exogenous and stochastic. Let c denote the current state of the art marginal cost of production, and let $K(c)$ be the cost of investing in the plant that embodies this state of the art production technology. If either one of the firms purchases the state of the art technology, then after a one period lag (constituting the "time to build" the new production facility), the firm can produce at the new marginal cost c.

Assume there are no costs of disposal of an existing production plant, or equivalently, the disposal costs do not depend on the vintage of the existing plant and are embedded as part of the new investment cost $K(c)$. We allow the fixed investment cost $K(c)$ to depend on c to capture different technological possibilities. For example, $K'(c) < 0$ reflects a situation where the cost of a new plant increases as the state of the art marginal cost gets lower. On the other hand, if $K'(c) > 0$, then technological improvements that lower the marginal cost of production c also lower the cost of building the plant that is capable of producing output at this state of the art marginal cost.

Clearly, if investment costs are too high, then there may be a point at which the potential gains from lower costs of production are insufficient to justify incurring the investment cost $K(c)$. Moreover, when the competition between the duopolists leads to leapfrogging behavior, the investing firm will not be able to capture the entire benefit of lowering its cost of production: some of these benefits will be passed on to consumers in the form of lower prices.

Let c_t denote the marginal cost of production under the state of the art production technology at time period $t \in \{0,1,2,..,\infty\}$. Each period t the firms face a simple binary investment decision: firm j can decide not to invest and continue to produce using its existing production facility at the marginal cost $c_{t,j}$. If firm j pays the investment cost $K(c_t)$ and acquires the state of the art production plant with marginal cost c_t, then when this new plant comes on line at $t+1$, firm j will be able to produce at the marginal cost $c_{t+1,j} = c_t < c_{t,j}$.

We consider both a continuous state and finite state formulation of the game depending on how we specify the stochastic process for the state of the art cost, c. If c is a continuous stochastic process, the state space for this model which we denote S, is given by the pyramid $S = \{(c_1, c_2, c) : c_1 \geq c \text{ and } c_2 \geq c \text{ and } 0 \leq c \leq c_0\}$ in R^3, where $c_0 > 0$ is the initial state, and zero represents the lower bound of the state of the art technology. The choice of lower bound is not essential for any of our results. The Bertrand investment game starts at the apex of the pyramid given by (c_0, c_0, c_0). In cases where for computational reasons we restrict c to a finite set of possible values in $[0, c_0]$, the "discretized" state space is a finite lattice subset of S.

We assume both firms believe that the state of the art technology for producing the good evolves stochastically according to a Markov process with transition density $\pi(c_{t+1}|c_t)$. Specifically, suppose that with probability $\pi(c_t|c_t)$ there is no improvement in the state of the art technology, and with probability $1 - \pi(c_t|c_t)$ technology improves to marginal cost c_{t+1} which is a draw from some distribution over the interval $[0, c_t]$. An example of a convenient functional form for such a distribution is the Beta distribution. However, the presentation of the model and neither of our results do not depend on specific functional form assumptions about π.

The feature of the transition density π that turns out to be crucial for the uniqueness of equilibrium is whether $\pi(c|c) > 0$ for some $c > 0$ or not. We single out a special case of *strictly monotonic* technological progress when $\pi(c|c) = 0$ for all $c > 0$, i.e. the state of art improves in every time period until it reaches the absorbing state where $\pi(0|0) = 1$.[1] Under deterministic technological

[1] Throughout the paper we use $\pi(c|c) = 0$ to refer to strictly monotonic progress bearing in mind that it only applies for $c > 0$.

progress we have $\pi(c_{t+1}|c_t) = 1$ if $c_t > c_{t+1} \geq 0$ and $\pi(c_{t+1}|c_t) = 0$ if $c_t = c_{t+1}$ including the case $c_t = c_{t+1} = 0$. Thus, deterministic technological improvement is strictly monotonic, but not vice versa.

For technical reasons, we also need the following continuity assumption in the continuous state version of the game.

Assumption 1 (Continuity)**.** Consider the continuous state version of the game. The investment cost function $K(c)$ is a continuous function of $c \in [0, c_0]$. For each $c \in [0, c_0]$, the transition probability $\pi(c'|c)$ has at most one discontinuity at $c' = c$. The probability of staying at the same state of the art cost, $\pi(c|c)$, is a continuous function of c. Further π is weakly continuous in c, that is, for every continuous, bounded function $f : R \to R$, we have

$$\lim_{c' \to c} Ef(c') = Ef(c) \tag{2}$$

where

$$Ef(c) = \int_0^c f(c')\pi(dc'|c). \tag{3}$$

2.1 Timing of Moves

Let m_t be a state variable that governs which of the two firms are "allowed" to undertake an investment at time t. We will assume that $\{m_t\}$ evolves as an exogenous Markov chain with transition probability $f(m_{t+1}|m_t)$ independent of the other state variables $(c_{t,1}, c_{t,2}, c_t)$. While it is natural to assume firms simultaneously set their prices, their investment choices may or may not be made simultaneously.

In this paper we analyze two variants of the Bertrand investment game: 1) a *simultaneous move* game where the firms make their investment choices simultaneously, denoted by $m_t = 0$, with $f(0|m_t) = 1$ (so $m_t = 0$ with probability 1 for all t), and 2) *alternating move* game, with either deterministic or random alternation of moves, but where there is no chance that the firms could ever undertake simultaneous investments (i.e. where $m_t \in \{1,2\}$ and $f(0|m_t) = 0$ for all t). Here $m_t = 1$ indicates a state where only firm 1 is allowed to invest, and $m_t = 2$ is the state where only firm 2 can invest. Under either the alternating or simultaneous move specifications, each firm always observes the investment decision of its opponent after the investment decision is made. However, in the simultaneous move game, the firms must make their investment decisions based on their assessment of the probability their opponent will invest. In the alternating move game, since only one of the firms can invest at each time t, the mover can condition its decision on the investment decision of its opponent if it was the opponent's turn to move in the previous period. The alternating move specification can potentially reduce some of the strategic uncertainty that arises in a fully simultaneous move specification of the game.

We interpret random alternating moves as a way of reflecting *asynchronicity* of timing of decisions in a discrete time model that occurs in continuous time models where probability of two firms making investment decisions at the exact same instant of time is zero. In some cases there is a unique equilibrium when players move in an alternating fashion (e.g. Lagunoff and Matsui, 1997) and we provide sufficient conditions for uniqueness in our model.

The timing of events in the model is as follows. At the start of period t each firm knows the costs of production $(c_{t,1}, c_{t,2})$, and both learn the current values of c_t and m_t. If $m_t = 0$, then the firms simultaneously decide whether or not to invest. We assume that both firms know each others' marginal cost of production, i.e. there is common knowledge of state $(c_{t,1}, c_{t,2}, c_t, m_t)$. Further, both firms have equal access to the new technology by paying the investment cost $K(c_t)$ to acquire the current state of the art technology with marginal cost of production c_t.

After each firm decides whether or not to invest in the latest technology, the firms then independently and *simultaneously* set the *prices* for their products, where production is done in period t with their existing plant. The Bertrand equilibrium price is the unique Nash equilibrium of the simultaneous move pricing stage game. The one period time-to-build assumption implies that even if both firms invest in new plants at time t, their marginal costs $c_{t,1}$ and $c_{t,2}$ in period t are unchanged, and enter profit formula (1).

We assume that consumer purchases of the good is a purely static decision, and consequently there are no dynamic effects of pricing for the firms, unlike in the cases of durable goods where consumer expectations of future prices affects their timing of new durable purchases as in Goettler and Gordon (2011). Thus in our model, the pricing decision is given by the simple static Bertrand equilibrium in every period. The only dynamic decision is firms' investment decisions.

2.2 Solution concept

Assume that the two firms are expected discounted profit maximizers and have a common discount factor $\beta \in (0,1)$. We adopt the standard concept of *Markov-perfect equilibrium* (MPE) where the firms' investment and pricing decision rules are restricted to be functions of the current state, $(c_{t,1}, c_{t,2}, c_t, m_t)$. When there are multiple equilibria in this game, the Markovian assumption also restricts the “equilibrium selection rule” to depend only on the current value of the state variable. The firms’ pricing decisions only depend on their current production costs $(c_{t,1}, c_{t,2})$ in accordance with the static Bertrand equilibrium. However, the firms’ investment decisions also depend on the value of the state of the art marginal cost of production c_t and the designated mover m_t.

Definition 1. *A Stationary Markov Perfect Equilibrium of the duopoly investment and pricing game consists of a pair of strategies $(P_j(c_1,c_2,c,m), p_j(c_1,c_2))$, $j \in \{1,2\}$ where $P_j(c_1,c_2,c,m) \in [0,1]$ is firm j's probability of investing and $p_j(c_1,c_2) = \max[c_1,c_2]$ is firm j's pricing decision. The investment rules $P_j(c_1,c_2,c,m)$ must maximize the expected discounted value of firm j's future profit stream taking into account the investment and pricing strategies of its opponent.*

We allow the investment strategies of the firms to be probabilistic to allow for the possibility of mixed strategy equilibria. To derive the functional equations characterizing a stationary Markov-perfect equilibrium, suppose the current state is (c_1,c_2,c,m), i.e. firm 1 has a marginal cost of production c_1, firm 2 has a marginal cost of production c_2, and the marginal cost of production using the current best technology is c, and m denotes which of the firms has the right to make a move and invest (or both if $m = 0$) . The firms’ value functions V_j, $j = 1,2$ take the form

$$V_j(c_1,c_2,c,m) = \max[v_{I,j}(c_1,c_2,c,m), v_{N,j}(c_1,c_2,c,m)] \tag{4}$$

where, when $m = 0$, $v_{N,j}(c_1,c_2,c,m)$ denotes the expected value to firm j if it does not invest in the latest technology, and $v_{I,j}(c_1,c_2,c,m)$ is the expected value to firm j if it invests. However, when $m \in \{1,2\}$, the subscripts N and I refer to whether an investment is made in period t by the firm m, who has the right of move. When $m = 1$ (firm 1 has the right to invest), $v_{I,1}(c_1,c_2,c,1)$ and $v_{N,1}(c_1,c_2,c,1)$ denote the expected values to firm 1 from investing and not investing. When $m = 2$

(firm 2 has the right to invest), $v_{I,1}(c_1, c_2, c, 2)$ and $v_{N,1}(c_1, c_2, c, 2)$ denote the expected values to firm 1 from the scenarios when firm 2 makes the investment or does not make the investment.

The formula for the expected profits associated with *not* investing is given by:

$$v_{N,j}(c_1, c_2, c, m) = r_j(c_1, c_2) + \beta EV_j(c_1, c_2, c, m, 0), \tag{5}$$

where $EV_j(c_1, c_2, m, c, 0)$ denotes the conditional expectation of firm j's next period value function $V_j(c_1, c_2, c, m)$ given that it does not invest this period (represented by the last 0 argument in EV_j), conditional on the current state (c_1, c_2, c, m).

The formula for the expected profits associated with investing is given by

$$v_{I,j}(c_1, c_2, c, m) = r_j(c_1, c_2) - K(c) + \beta EV_j(c_1, c_2, c, m, 1), \tag{6}$$

where $EV_j(c_1, c_2, c, m, 1)$ is firm j's conditional expectation of its next period value function given that it invests (the last argument is 1), conditional on (c_1, c_2, c, m).

Let $P_1(c_1, c_2, c, m)$ be firm 2's belief about the probability that firm 1 will invest in state is (c_1, c_2, c, m). Consider the simultaneous move case ($m = 0$) first. It follows from (4) that

$$P_1(c_1, c_2, c, m) = \mathbb{1}\left\{v_{I,1}(c_1, c_2, c, m) > v_{N,1}(c_1, c_2, c, m)\right\}, \tag{7}$$

where $\mathbb{1}\{\cdot\}$ denotes an indicator function, and mixed strategy investment probability arises in the case of equality. A similar formula holds for $P_2(c_1, c_2, c, m)$.

The Bellman equations for firm 1 in the simultaneous move case are as follows.[2]

$$
\begin{aligned}
v_{N,1}(c_1,c_2,c) &= r_1(c_1,c_2)+\beta\int_0^c \big[P_2(c_1,c_2,c)\max(v_{N,1}(c_1,c,c'),v_{I,1}(c_1,c,c'))+ \\
&\quad (1-P_2(c_1,c_2,c))\max(v_{N,1}(c_1,c_2,c'),v_{I,1}(c_1,c_2,c'))\big]\,\pi(dc'|c). \\
v_{I,1}(c_1,c_2,c) &= r_1(c_1,c_2)-K(c)+\beta\int_0^c \big[P_2(c_1,c_2,c)\max(v_{N,1}(c,c,c'),v_{I,1}(c,c,c'))+ \\
&\quad (1-P_2(c_1,c_2,c))\max(v_{N,1}(c,c_2,c'),v_{I,1}(c,c_2,c'))\big]\,\pi(dc'|c).
\end{aligned} \tag{8}
$$

In the alternating move case, the Bellman equations for the two firms lead to a system of eight functional equations for $\{v_{N,j}(c_1,c_2,c,m),v_{I,j}(c_1,c_2,c,m)\}$ for $j,m\in\{1,2\}$. The Bellman equations for firm 1 are given below, similar equations for firm 2 are omitted.

$$
\begin{aligned}
v_{N,1}(c_1,c_2,c,1) &= r_1(c_1,c_2)+\beta f(1|1)\int_0^c \max(v_{N,1}(c_1,c_2,c',1),v_{I,1}(c_1,c_2,c',1))\pi(dc'|c)+ \\
&\quad \beta f(2|1)\int_0^c \rho(c_1,c_2,c')\pi(dc'|c) \\
v_{I,1}(c_1,c_2,c,1) &= r_1(c_1,c_2)-K(c)+\beta f(1|1)\int_0^c \max(v_{N,1}(c,c_2,c',1),v_{I,1}(c,c_2,c',1))\pi(dc'|c)+ \\
&\quad \beta f(2|1)\int_0^c \rho(c,c_2,c')\pi(dc'|c) \\
v_{N,1}(c_1,c_2,c,2) &= r_1(c_1,c_2)+\beta f(1|2)\int_0^c \max(v_{N,1}(c_1,c_2,c',1),v_{I,1}(c_1,c_2,c',1))\pi(dc'|c)+ \\
&\quad \beta f(2|2)\int_0^c \rho(c_1,c_2,c')\pi(dc'|c) \\
v_{I,1}(c_1,c_2,c,2) &= r_1(c_1,c_2)+\beta f(1|2)\int_0^c \max(v_{N,1}(c_1,c,c',1),v_{I,1}(c_1,c,c',1))\pi(dc'|c)+ \\
&\quad \beta f(2|2)\int_0^c \rho(c_1,c,c')\pi(dc'|c).
\end{aligned} \tag{9}
$$

where

$$
\rho(c_1,c_2,c)=P_2(c_1,c_2,c,2)v_{I,1}(c_1,c_2,c,2)+[1-P_2(c_1,c_2,c,2)]v_{N,1}(c_1,c_2,c,2). \tag{10}
$$

[2]Variable $m=0$ is omitted for clarity. The Bellman equations for firm 2 are omitted for space considerations given that they are analogous to those we present for firm 1.

Note that $P_2(c_1,c_2,c,1) = 0$, since firm 2 is not allowed to invest when it is firm 1's turn to invest (i.e. when $m = 1$), and similarly $P_1(c_1,c_2,c,c,2) = 0$.

The equilibria of the Bertrand investment game with simultaneous moves are characterized by the system of non-linear equations composed of equations (7) and (8) written for every combination of (c_1,c_2,c) in the discrete representation of the state space S. Similarly, in the alternating moves game, all equilibria are characterized by the system composed of equations (7) and (9) for every combination of (c_1,c_2,c) and all values of m. In this paper we study the theoretical properties of this system and its solutions, but also solve it for a number of parameter values to provide computed examples and counter-examples.

The key feature of the Bertrand investment game is the *directionality* of its transitions in the state space S implied by the unidirectional evolution of the state of the art cost c (which can only improve), and the fact that c_1 and c_2 will never increase under any feasible strategy. This implies that the Bertrand investment game is in the class of *dynamic directional games* (DDGs) defined in IRS2016. This paper introduced the *recursive lexicographical search* (RLS) algorithm that is guaranteed to find all MPE of finite state DDGs provided certain conditions hold. The key conditions are a) there is a finite number of equilibria in every "stage game" (which in the leapfrogging model corresponds to each unique (c_1,c_2,c) combination), and b) there is an algorithm that can find all MPE of every stage game. IRS2016 show that both of these conditions hold in the finite state version of the leapfrogging game, so the RLS algorithm can be used to find all MPE of the Bertrand investment game. In particular, IRS2016 developed a non-iterative combinatoric algorithm based on the roots of second degree polynomial that finds all of the 1, 3 or 5 possible MPE of every stage game of the simultaneous move version of the investment game, and the 1 or 3 possible MPE of every stage game of the alternating move version.

While RLS can only be applied to compute all MPE in the finite state formulation of the game, the same polynomial equations that characterize the solution to the stage game equilibria applies to both the continuous and the discrete case. We therefore formulate general results that unless explicitly stated applies to both formulations of the game.

Apart from several numerical counter-examples that we provide to the conjecture in Riordan and Salant (1994), we only use the numerical solution of the game to illustrate our general results, which are based on analytic proofs rather than on computed results to specific examples. However the ability to use the RLS algorithm to find all MPE of specific instances of the Bertrand investment game greatly improved our understanding of the structure and the properties of the set of equilibrium outcomes and ultimately facilitated the analytical proofs of the theoretical results we present below.

Before we state our main results in the following sections, we give rigorous definition to the *symmetric MPE* in our game, and discuss the symmetry of equilibrium payoffs.

Definition 2. *A Symmetric Markov Perfect Equilibrium* of the simultaneous move version of the duopoly investment and pricing game satisfies the following restrictions for any (c_1, c_2, c) in the state space:

$$P_1(c_1,c_2,c) = P_2(c_2,c_1,c), \quad v_{N,1}(c_1,c_2,c) = v_{N,2}(c_2,c_1,c), \quad v_{I,1}(c_1,c_2,c) = v_{I,2}(c_2,c_1,c). \tag{11}$$

Thus, symmetry implies that the equilibrium behavior of the firms only depends on their cost values (c_1,c_2) and not on their "identity" as "firm 1" or "firm 2". We can also define symmetry in the alternating move version of the game analogously. We will show that there exist symmetric MPE in both the alternating and simultaneous move games. However using the RLS algorithm (which can find all MPE in finite state versions of these games) IRS2016 have shown that *symmetric MPE constitute only a small fraction of the set of all MPE.*

For example, one of the "asymmetric" MPE of the game (whose existence we establish below) is a *monopoly equilibrium* where only 1 firm (the "monopolist") invests on the equilibrium path whereas its opponent never invests. It is easy to see that this monopoly equilibrium cannot satisfy the definition of symmetry (11) since if firm 1 is the monpolist, then in any state (c_1,c_2,c) where it is optimal for the monopolist to invest in the MPE, we have

$$P_1(c_1,c_2,c) = 1 \neq P_2(c_2,c_1,c) = 0. \tag{12}$$

That is, simply permuting the cost states (c_1,c_2) does not change the fact that firm 1 is the monopolist (and thus is the only firm that invests in equilibrium) whereas firm 2 is the "follower" and never invests. Thus in an asymmetric equilibrium the two firms have "identities" and simply permuting the cost values does not lead the two firms to behave the same way or to earn the same discounted profits from investing or not investing.

Of course there is a wider notion of symmetry in terms of the *set of equilibrium payoffs.* We will show that the payoff set at any symmetric point in the state space, $(\overline{c},\overline{c},c)$ for $\overline{c} \in [c,c_0]$ will itself be a symmetric subset of R^2_+, the set of all possible pairs of payoffs to the two firms in any possible MPE of the game starting from $(\overline{c},\overline{c},c)$. This is a consequence of the overall arbitrariness (and hence symmetry) in terms of how we label the two firms as "firm 1" and "firm 2". For example if we can find one monopoly equilibrium of the simultaneous move game where firm 1 is the monopolist and firm 2 is the follower, it is clear that there will also be a symmetric monopoly equilibrium where firm 2 is the monopolist and firm 1 is the follower. Though neither of these monopoly MPE are symmetric in themselves for the reason discussed above, it is clear that we will have overall symmetry in the set of MPE payoffs, at least when considered from from the symmetric or "diagonal" points in the state space, $(\overline{c}, \overline{c}, c)$.

3 Socially optimal production and investment

We assess the efficiency of the Bertrand investment outcomes relative to a social planning benchmark that maximizes total expected discounted consumer and producer surplus. In a dynamic model, the social planner has to account for the costs of investment in new technologies. Since the production technology has constant returns to scale the social planner will only operate a single plant. Thus, the duopoly equilibrium can be inefficient due to *duplicative investments* but we will show that inefficiency manifests itself through *inefficient timing of investments* as well.

We assume that consumers have quasi-linear preferences so the surplus they receive from consuming the good at a price of p equals their income net of p. The social planning solution entails selling the good at the marginal cost of production and adopting an efficient investment strategy that minimizes the expected discounted costs of production. Let c_1 be the marginal cost of production of the current production plant, and let c be the marginal cost of production of the current state of the art production process, which we continue to assume evolves as an exogenous Markov process with transition probability $\pi(c'|c)$ whose evolution is beyond the purview of the social planner. The social planner's problem reduces to finding an *optimal investment strategy* for the production of the good.

Unlike the optimization problems of the duopolists, we cast the social planner's problem as cost minimization. Let $C(c_1,c)$ be the *smallest* present discounted value of costs of investment and production when the plant operated by the social planner has marginal cost c_1 and the state of the art technology has a marginal cost of $c \le c_1$. The minimization occurs over all feasible investment and production strategies, but subject to the constraint that the planner must produce enough in every period to satisfy the unit mass of consumers in the market. We have

$$C(c_1,c) = \min\left\{ c_1 + \beta \int_0^c C(c_1,c')\pi(dc'|c),\ c_1 + K(c) + \beta \int_0^c C(c,c')\pi(dc'|c) \right\}, \tag{13}$$

where the first component corresponds to the case when investment is not made, and cost c_1 is carried in the future, and the second component corresponds to the case when new state of the art cost c is acquired for additional expense of $K(c)$.

It follows that the optimal investment strategy takes the form of a *cutoff rule* where it is optimal to invest in the state of the art technology if the current cost c_1 is above a cutoff threshold $\overline{c}_1(c)$. Otherwise the drop in expected future operating costs is not sufficiently large to justify undertaking the investment and thus it is optimal to produce the good using the existing plant with marginal cost c_1. The cutoff rule $\overline{c}_1(c)$ is the indifference point in (13), and thus it is the solution to the equation

$$K(c) = \beta \int_0^c \left[C(\overline{c}_1(c),c') - C(c,c') \right] \pi(dc'|c), \tag{14}$$

if it exists, and $\overline{c}_1(c) = c_0$ otherwise.[3]

[3] In problems where the support of $\{c_t\}$ is a finite set, the cutoff $\overline{c}_1(c)$ is defined as the smallest value of c_1 in

We have implicitly assumed that the cost of investment $K(c)$ is not prohibitively high, so that the social planner would always want to invest in a new technology. Theorem 1 provides a bound on the costs of investments that must be satisfied for investment to occur under the socially optimum solution.

Theorem 1 (Necessary and sufficient condition for investment by the social planner)**.** *Let the current costs be* (c_1, c)*. Investment (in the current period or some time in the future) is socially optimal if and only if there exists* $c' \in [0, c]$ *in the support of the Markov process of the state of the art marginal cost such that*

$$\frac{\beta(c_1 - c')}{1 - \beta} > K(c'). \tag{15}$$

The proof of Theorem 1, and all subsequent proofs unless sufficiently short, are provided in Appendix A. The condition under which it is socially optimal to invest plays a central role when we analyze the duopoly investment dynamics in section 4.

Assumption 2. We will say that the investment costs are not prohibitively high, or that investment is socially optimal if the condition (15) in Theorem 1 holds with a value c_1 equal to the smaller of the marginal costs of production of the two firms in the Bertrand investment game.

As we will prove in the next section, the Bertrand investment game with simultaneous moves supports a monopoly outcome. The following lemma establishes the efficiency of a monopoly outcome, which is useful for what follows in the next section.

the support of $\{c_t\}$ such that $K(c) > \beta \int_0^c [C(c_1, c') - C(c, c')]\pi(dc'|c)$.

Lemma 1 (Social optimality of monopoly solution)**.** *The socially optimal investment policy is identical to the profit maximizing investment policy of a monopolist who faces the same discount factor* β *and the same technological process* $\{c_t\}$ *with transition probability* π *as the social planner, assuming that in every period the monopolist can charge a price of* c_0 *equal to the initial value of the state of the art production technology.*

Proof. Since the monopolist is constrained to charge a price no higher than c_0 every period, it follows that the monopolist maximizes expected discounted value of profits by adopting a cost-minimizing production and investment strategy as per social planner. $\square$

4 Duopoly Investment Dynamics

We are now in position to solve the model of Bertrand duopoly investment and pricing and characterize the stationary Markov Perfect equilibria of this model. As mentioned above, we used the RLS algorithm from IRS2016 to compute all MPE of the Bertrand investment game. These computations facilitated the illustrative examples below. Yet the majority of our results are based on analytical *proofs* of the general properties of the equilibria of this game.

In our analysis we consider the case where the support of state of the art cost c is the entire interval $[0, c_0]$, where $c_0 > 0$ is the initial value of the marginal cost of production at the start of the game at $t = 0$. We will show below that in this case the game will have a continuum of MPE. We can also analyze versions of the game where c is restricted to a finite set of states in the interval $[0, c_0]$ and in this case there will be a finite number of MPE which can be computed using the RLS algorithm. If we further restrict the set of possible equilibrium selection rules to be *deterministic* functions of the current state (c_1, c_2, c), we can show that there will only be a finite number of possible equilibria in both the simultaneous and alternating move formulations of the game. The number of the equilibria grows exponentially fast in the number of points in the discretized state space if investment is socially optimal and as the support of the Markov process $\{c_t\}$ becomes dense in the full interval $[0, c_0]$ we conjecture that the finite set of MPE in the simultaneous move Bertrand investment and pricing converges to a dense subset of the continuum of MPE for the game where the support of c is the full interval $[0, c_0]$.

4.1 Equilibria of the Simultaneous Move Game

Provided that the investment cost is not prohibitively high, the set of all MPE in the Bertrand investment game is surprisingly rich. Despite the prevalence of leapfrogging in equilibrium, we show that "monopoly" equilibria is supported in the simultaneous move game.[4] A static Bertrand-like outcome with zero expected payoff for both duopolists is also supported in the simultaneous move game. We summarize these findings in the following theorem that constitutes our main result.

Theorem 2 (Equilibrium payoffs in the simultaneous move game)**.** *If investments are socially optimal (in the sense of Assumption 2) at the initial point $(c_{1,0}, c_{2,0}, c_0) \in S$ of the state space of the Bertrand investment and pricing game with simultaneous moves, the following holds:*

1. *No investments by both firms in all states (c_1, c_2, c) is not supported in any of the MPE equilibria of the game;*

[4] Note that the monopoly equilibria we characterize below are *not* the preemption equilibrium of Riordan and Salant (1994). In contrast to their rent dissipation result, monopoly profits in our model are positive and are equal to the maximum possible profits subject to the limit on price, and by Lemma 1 the monopoly outcome is efficient.

2. *There are two fully efficient "monopoly" equilibria in which either one or the other firm makes all the investments and earns maximum feasible profit while their opponent earns zero profits;*

3. *If Assumption 1 (Continuity) holds, there exists a symmetric mixed strategy equilibrium in the simultaneous move game that results in zero expected payoffs to both firms in the subgames starting at all diagonal states $(c,c,c') \in S$ with $c' \in [0,c]$, and zero expected payoffs to the high cost firm and positive expected payoffs to the low cost firm in the subgames starting in states (c_1,c_2,c) where $c_1 \neq c_2$*

4. *If Assumption 1 (Continuity) holds, the convex hull of the set of the expected discounted equilibrium payoffs to the two firms in all MPE equilibria of simultaneous move game at the apex (c_0,c_0,c_0) is a triangle with vertices $(0,0)$, $(0,V_M)$ and $(V_M,0)$, where $V_M = V_i(c_0,c_0,c_0)$ is the expected discounted payoff of firm i which makes all investments in the monopoly equilibrium.*

When we refer to "mixed strategy equilibrium" it is with the qualification that there may be certain subsets of the state space where neither firm finds it optimal to invest. For example, we can show that the only possible MPE is a pure strategy where there is no investment by either firm in *edge states* — states of the form (c_1,c,c) and (c,c_2,c). Thus, any MPE will have a non-empty *no investment region* — a band of states bordering the edge states where the cost of investment outweighs any gains from investing and leapfrogging the rival firm so neither firm invests. For example there will always be a non-empty no investment region in the bottom layer of the game where $c = 0$, because it is easy to see that neither firm can profit from investing in any state

$(c_1, c_2, 0)$ where $\beta c_1/(1-\beta) < K(0)$ or $\beta c_2/(1-\beta) < K(0)$. For example, if c_2 is the high cost firm, the payoff from investing to leapfrog firm 1 to become the permanent low cost leader is $\beta c_1/(1-\beta) - K(0) < 0$ when c_1 is sufficiently close to the edge state $(0, c_2, 0)$.

We note that all statements in Theorem 2 hold regardless of whether the support of the process $\{c_t\}$ for state of the art cost of production is a finite set or the entire interval $[0, c_0]$ except for Statement 3 and 4. Our proof of the existence of a mixed strategy MPE with the stated properties in statement 3 of Theorem 2 depends on continuity of the state space and continuity of the functions $K(c)$ and $\pi(c'|c)$ as per Assumption 1. We have not yet been able to prove that a mixed strategy MPE with these properties also exists in finite state versions of the game, though we have never found a counter example that violates all the conditions of Statement 3 when we enumerate *all* MPE of finite state versions these games using the RLS algorithm (which as we noted above, provably finds all MPE of the finite state version of the game). The key property of the mixed symmetric MPE that we identify in Statement 3 that we conjecture holds in general is that along the diagonal states $(\overline{c}, \overline{c}, c)$ both firms earn zero expected profits:

$$\max[v_{N,1}(\overline{c}, \overline{c}, c), v_{I,1}(\overline{c}, \overline{c}, c)] = \max[v_{N,2}(\overline{c}, \overline{c}, c), v_{I,2}(\overline{c}, \overline{c}, c)] = 0. \tag{16}$$

However we have not yet been able to prove this result *for finite state games* and for continuous state games we have not yet been able to establish several additional properties of the equilibrium that we conjecture are true and are stated below:

$$\frac{\partial}{\partial c_1} P_1(c_1, c_2, c) \geq 0, \quad \frac{\partial}{\partial c_2} P_1(c_1, c_2, c) \geq 0, \quad \frac{\partial}{\partial c_1} v_1(c_1, c_2, c) \leq 0, \quad \frac{\partial}{\partial c_2} v_1(c_1, c_2, c) \geq 0, \tag{17}$$

where $v_1(c_1, c_2, c) = \max[v_{N,1}(c_1, c_2, c), v_{I,1}(c_1, c_2, c)]$ is the payoff to firm 1 under the mixed strategy MPE. By symmetry of the equilibrium, symmetrical results also hold for firm 2.

In our proof of Theorem 2 we show that in the mixed strategy MPE of the *overall game* there will be subsets of states in S where the firms actually do not play mixed strategies. We show that the *stage game MPE* (i.e. the set of all possible MPE at any given state of the game, (c_1, c_2, c)) can be only one of the following three types:

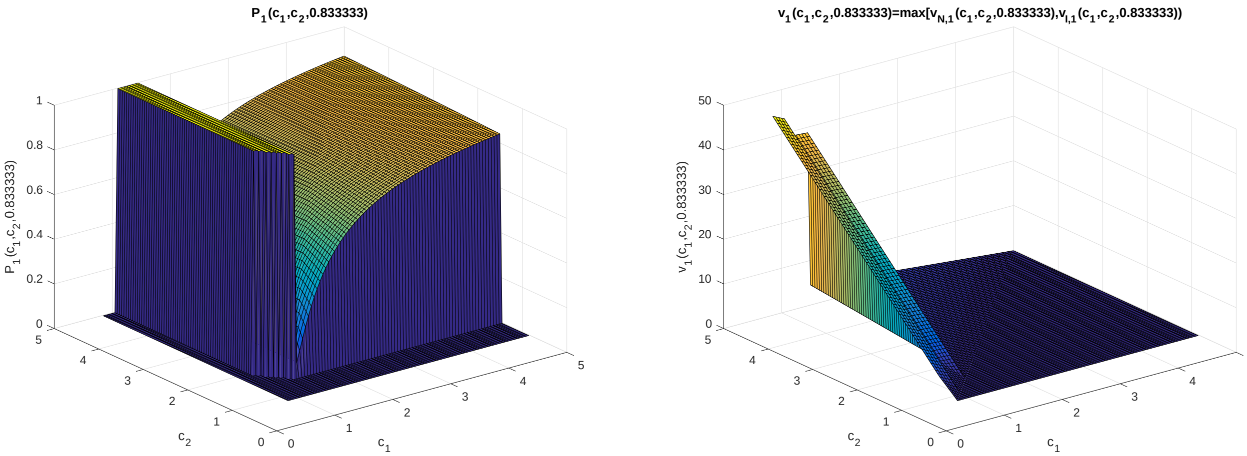
Type 1 three possible MPE, two of which are pure strategy "anti-coordination" MPEs where one firm invests and the other doesn't, and a third mixed MPE where both firms invest with positive probability

Type 2 a single pure strategy "no investment" MPE where neither firm invests

Type 3 a single pure strategy MPE where the low cost firm invests but the high cost firm doesn't

Figure 1 illustrate the three possible types of MPE and the regions in the state space where these MPE exist. The left panel of the figure plots the equilibrium investment probability for firm 1, $P_1(c_1, c_2, c)$ where $c = 0.83333$ and $K = 5.2$ and $\beta = .95$. We see the region where the Type 1 mixed strategy stage game MPE exist in the curved region in the upper northeast orthant in the diagram. The Type 2 no investment region occurs along the edges, and the Type 3 pure strategy MPE occurs along a narrow band for c_1 in the interval $[1.09, 1.5]$ and c_2 is higher than 1.5. The right hand panel shows the value function for firm 1 in this MPE. As predicted by the Theorem, we have $v_1(c_1, c_2, c) = 0$ for all $c_1 \geq c_2$. But $v_1(c_1, c_2, c) > 0$ for all $c_1 < c_2$. Furthermore there are evident discontinuities in v_1 at $c_1 = 1.5$ at the boundary between the Type 1 mixed strategy MPE region and the Type 3 pure strategy region where firm 1 invests and firm 2 doesn't, as well as another discontinuity between the Type 3 region and the Type 2 no investment MPE region, and a third discontinuity along a line where c_1 is equal to the boundary between the Type 1 mixed strategy region and the Type 2 no investment region where the probability firm 1 invests drops discontinuously to zero.

Figure 1: Example of (P_1, v_1) in the mixed strategy MPE of the simultaneous move game

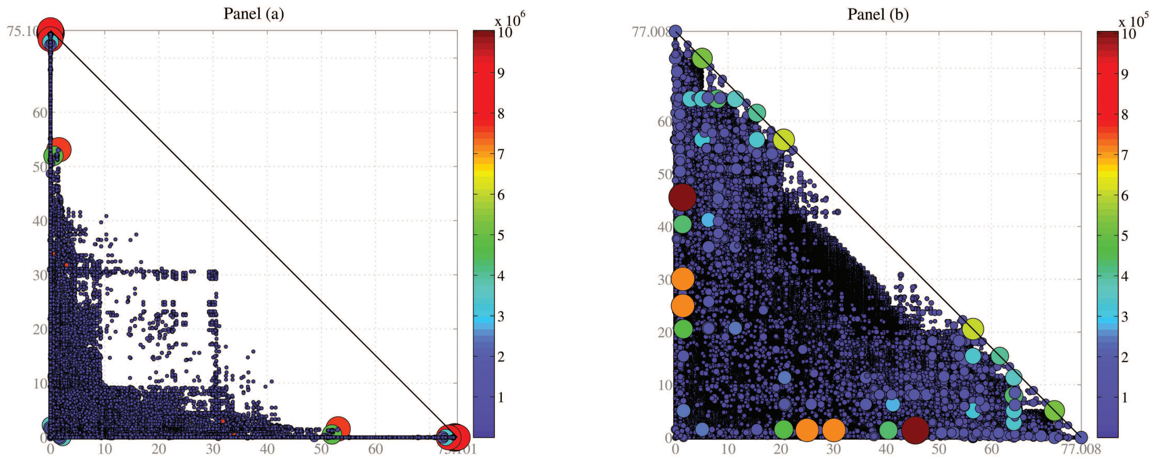


The no investment region is generally larger at higher stages of the game than in the bottom layer of the game, i.e. as the state of the art cost c increases. As we show in the Appendix, the no investment region in the bottom layer of the state space (i.e. where $c = 0$) are the "edge states" $\min(c_1, c_2) \leq K(1-\beta)/\beta$. For our parameters this corresponds to $K(1-\beta)/\beta = 0.2605$. According to Theorem 1 the no-investment region at stage c should be $\min(c_1, c_2) \leq (1-\beta)K/\beta + c = 1.094$ when $c = 0.83333$. By continuity, at sufficiently low state of the art cost values c, the no investment region is approximately equal to the "socially efficient" no investment region, as per Theorem 1.

It is important to note that our result that there is no investment in the edge states in the mixed strategy MPE in statement 3 of Theorem 2 is not a general result, and in particular it is not true in *all MPE* that there is no investment by either firm on all edge states. For example, consider a game which has three possible values for the state of the art marginal cost of production: $\{0, c, 2c\}$. Suppose there is deterministic technological progress so the state of the art marginal cost drops by c each period with probability 1. Consider the edge state $(2c, c, c)$ where firm 1 is the high cost follower when the state of the art marginal cost of production is c. Suppose that in the "diagonal" interior state in the bottom layer $(c,c,0)$ the monopoly equilibrium is selected where firm 1 is the designated monopolist investor, and it is optimal to invest in that state, i.e. $\beta c/(1-\beta) - K(0) > 0$. Suppose that in the bottom layer off-diagonal interior state $(2c,c,0)$ that either firm 2 is the designated monopolist, or alternatively that the mixed strategy MPE is selected. In either case firm 1 will earn zero expected profits in state $(2c,c,0)$. Now consider firm 1's decision at the edge state $(2c,c,c)$. If it invests, the state will transit next period to the diagonal interior of the bottom layer of the state space $(c,c,0)$ where firm 1 is the monopolist, and firm 1's discounted profit from investing in state $(2c,c,0)$ is $-K(c)+\beta[-K(0)+\beta c/(1-\beta)]$. Assume that $K(c)$ is not too large so that $K(c) < \beta[-K(0)+\beta c/(1-\beta)]$. Then it is easy to see that firm 1 earns a positive discounted profit from investing at the edge state $(2c,c,c)$. However if firm 1 does not invest, it earns zero profits in the current period (since it is the high cost firm) and it is easy to see it earns zero profits in all subsequent periods because the state transits to the interior of the bottom layer states $(2c,c,0)$ where, via our choice of equilibrium in this state, firm 1 earns zero expected profits as well.

Figure 2 illustrates Theorem 2 by plotting all apex payoffs to the two firms under all possible deterministic equilibrium selection rules in the simultaneous move game where the support of $\{c_t\}$ is the 5 point set $\{0,1.25,2.5,3.75,5\}$. Panel (a) plots the set of payoffs that occur when technological progress is deterministic, whereas panel (b) shows the much denser set of payoffs that occur when technological progress is stochastic. Though there are actually a greater total number of equilibria (192,736,405) under deterministic technological progress, many of these equilibria are observationally equivalent *repetitions* of the same payoff point which arise due to our treatment of the equilibrium selection rules that only differ off the equilibrium path as distinct. We indicate the

Figure 2: Initial node equilibrium payoffs in the simultaneous move game



Notes: The panels plot payoff maps of the Bertrand investment game with deterministic (a) and random (b) technologies. Parameters are $\beta = 0.9512$, $k_1 = 8.3$, $k_2 = 1$, $n_c = 5$. Parameters of beta distribution for random technology are $a = 1.8$ and $b = 0.4$. Panel (a) displays the initial state payoffs to the two firms in the 192,736,405 equilibria of the game, though there are 63,676 distinct payoff pairs among all of these equilibria. Panel (b) displays the 1,679.461 distinct payoff pairs for the 164,295,079 equilibria that arise under stochastic technology. The color and size of the dots reflect the number of repetitions of a particular payoff combination.

number of repetitions by the size of the payoff point plotted to be proportional to the number of repetitions. Figure 2 shows that when technology is stochastic there are fewer repetitions and so even though there are actually 28 million fewer equilibria, there are actually a substantially greater number (1,679,461 versus 63,676) of distinct payoff points.

Define the leapfrogging equilibria as those where the high cost firm has a positive probability of investing at least once in the support of the equilibrium, and thus a realization of such equilibrium may contain an event when a high cost firm leapfrogs the cost leader. With this definition, leapfrogging equilibria are very typical. Moreover, when using RLS to numerically solve the game we found that the high cost follower has a strictly higher probability of investing in mixed strategy equilibria. In the Appendix we prove this property holds for the points where $c = 0$, but we were not able to prove it for $c > 0$. We can also show (via the RLS algorithm) that leapfrogging may occur in *pure strategy MPE* of the either the simultaneous or alternating move versions of the game, and further *there exist fully efficient leapfrogging MPE.* The mixed strategy MPE will generally be inefficient, both due to *duplicative investments* and due to *suboptimally timed investments.* We will discuss the issue of efficiency of the MPE in subsection 4.3 below.

4.2 Equilibria of the Alternating Move Game

When firms make simultaneous investment decisions in the monopoly equilibrium, the high cost firm has no incentive to deviate from the equilibrium path in which its opponent always invests. However, when the firms move in an alternating fashion, the high cost firm will have an incentive to deviate because it knows that its opponent will not be able to invest at the same time (thereby avoiding the Bertrand investment paradox), and once the opponent sees that the firm has invested, it will not have an incentive to immediately invest to leapfrog for a number of periods until it is once again its turn to invest and there has been a sufficient improvement in the state of the art marginal cost of production. This creates a temptation for each firm to invest and leapfrog their rival that is not present in the simultaneous move game, and the alternating move structure prevents the firms from undertaking inefficient simultaneous investments, though it also generally prevents either firm from being able to time their investments in a socially optimal way.

Theorem 3 (The set of equilibrium payoffs in the Alternating Move Game)**.** *If investments are socially optimal (in the sense of Assumption 2) at the apex (c_0, c_0, c_0) of the state space of the Bertrand investment and pricing game with alternating moves, no investments by both firms is* *not supported in any of the MPE equilibria of the game. The convex hull of the set of expected discounted equilibrium payoffs to the two firms in all possible MPE equilibria at the apex of the alternating move game is a strict subset of the corresponding convex hull of payoffs in the simultaneous move game, i.e. the triangle with vertices defined in Theorem 2.*

While it is perhaps not surprising that when firms move in an alternating fashion neither one of them will be able to attain monopoly payoffs in any equilibrium of the alternating move game, Theorem 3 states that a zero expected profit mixed strategy equilibrium payoffs is not sustainable in the alternating move game either. Though it may seem tempting to conclude that mixed strategies can never arise in the alternating move game, we find that both pure and mixed strategy stage game equilibria are possible in the alternating move game. The intuition as to why this should occur is that even though only one firm invests at any given time, when $\pi(c|c) > 0$ the firms know that there is a positive probability that they will remain in the same state (c_1, c_2, c) for multiple periods until the technology improves. The possibility of remaining in the same state implies that the payoff to each firm from *not investing* depends on their belief about the probability their opponent will invest in this state at its turn.

Thus, by examining the Bellman equations (9) it not hard to see that for firm 1 the value of not investing when it is its turn to invest, $v_{N,1}(c_1, c_2, c, 1)$, depends on $P_2(c_1, c_2, c, 2)$ when $\pi(c|c) > 0$. This implies that $P_1(c_1, c_2, c, 1)$ will depend on $P_2(c_1, c_2, c, 2)$, and similarly, $P_2(c_1, c_2, c, 2)$ will depend on $P_1(c_1, c_2, c, 1)$. This mutual dependency creates the possibility for multiple solutions to the Bellman equations and the firms' investment probabilities and multiple equilibria at various stage games of the alternating move game.

In spite of very large number of MPE we find in the Bertrand investment game, there is a subclass of games for which the equilibrium is unique.

Theorem 4 (Sufficient conditions for uniqueness)**.** *In the dynamic Bertrand investment and pricing game a sufficient condition for the MPE to be unique is that (i) firms move in alternating fashion (i.e.* $m \neq 0$*), and (ii) for each* c *in the support of* π *we have* $\pi(c|c) = 0$*.*

Theorem 4 implies that under strictly monotonic technological improvement the alternating move investment game has a unique Markov perfect equilibrium. This is closely related to, but not identical with an assumption of the *deterministic technological progress* as discussed in section 2. There are specific types of non-deterministic technological progress for which Theorem 4 will still hold, resulting in a unique equilibrium to the alternating move game.

4.3 Efficiency of equilibria

We evaluated the efficiency of duopoly equilibria by calculating their *efficiency score* defined as the ratio of total surplus (i.e. the sum of discounted consumer surplus plus total discounted profits) under the duopoly equilibrium to the maximum total surplus achieved under the social planning solution.[5] We note that the calculation of efficiency is *equilibrium specific* and thus its value depends on the particular equilibrium of the overall game that we select. For example, we have already proved that monopoly investment by one of the firms is an equilibrium in the simultaneous move game, provided the cost of investment is not prohibitively high. This implies immediately

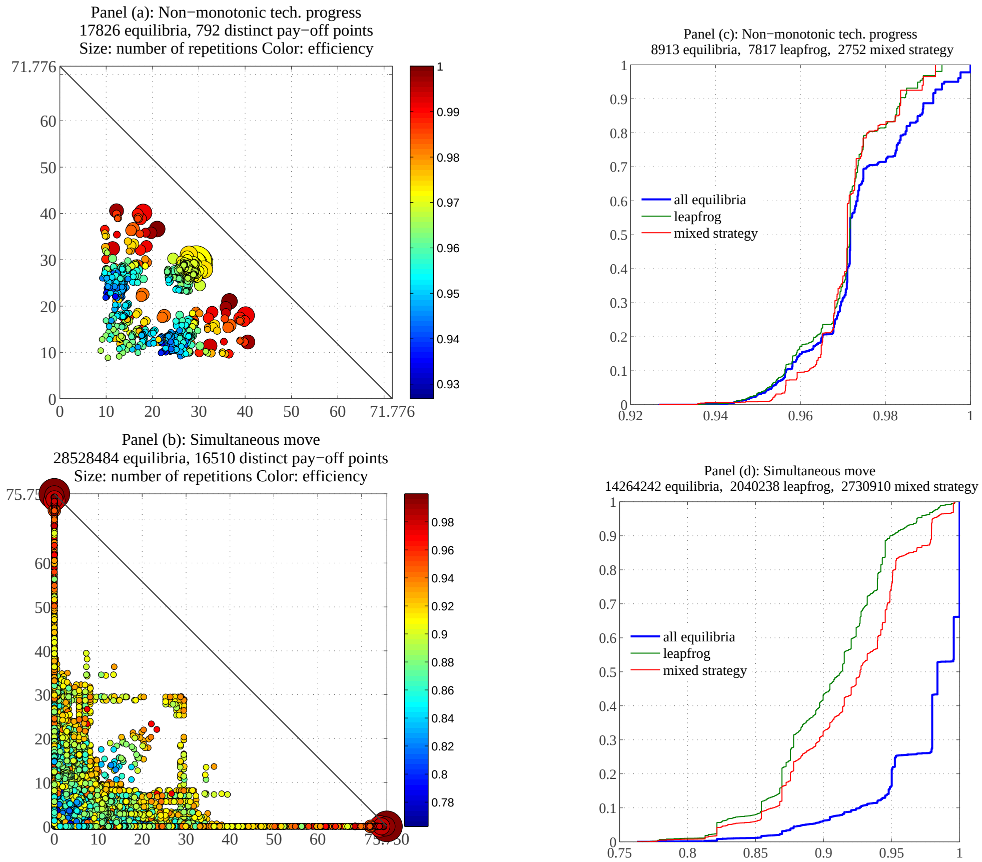
[5]It is not hard to show that the sum of discounted expected consumer surplus and discounted expected profits can be calculated from model parameters and the expected discounted cost function $C(c_1, c_2, c)$ which we compute in the similar procedure as the value functions in Section 2. Further details are available from authors upon request.

that there do exist fully efficient MPE in the simultaneous move game. We now show that the non-monopoly equilibria of either the simultaneous or alternating move investment games are generally *inefficient* and this inefficiency is typically due to two sources a) duplicative investments (only in mixed strategy equilibria in the simultaneous move investment game), and b) excessively frequent investments. Note that it is logically possible that inefficiency could arise from *excessively infrequent investments* and the logic of the Bertrand investment paradox might lead us to conjecture that we should see investments that are too *infrequent* in equilibrium relative to what the social planner would do. Surprisingly, we find that duopoly investments are generally excessively frequent compared to the social optimum, with preemptive investments (when they arise) representing the most extreme form of inefficient excessively frequent investment in the new technology.

The two panels in the left column in Figure 3 illustrate the set of equilibrium payoffs from all MPE equilibria computed by the RLS algorithm. We compute the efficiency of each of the equilibria, and treating the calculated efficiencies as "data", we plot their empirical distribution in the corresponding panels in the right column in Figure 3.

Panels (a) and (c) in Figure 3 represent an alternating move investment game with deterministic alternations of the right to move and technological progress which is not strictly monotonic, i.e. $\pi(c|c) > 0$ for some c. Consistent with Theorem 3 the set of equilibrium payoffs is a strict subset of the triangle, showing that it is not possible to achieve the monopoly payoffs (corners) or the zero profit mixed strategy equilibrium payoff (origin) in this case. As before, we have used the size of the plotted payoff points to indicate the number of repetitions of the payoff points, but now we use the color of plotted equilibrium payoffs to indicate the efficiency. Red indicates high efficiency

Figure 3: Payoff maps and efficiency of MPE in two specifications of the game



Notes: Panel (a)-(b) plots payoff maps and panel (c)-(d) cdf plots of efficiency by equilibrium type for two versions of the Bertrand investment pricing game. In panel (a) and (c) the case of deterministic alternating moves and non-strictly monotonic one step stochastic technological progress. Parameters in this case are $\beta = 0.9512$, $k_1 = 5$, $k_2 = 0$,, $f(1|1) = f(2|2) = 0, f(2|1) = f(1|2) = 1$, $c_{tr} = 1$ $n_c = 4$. In panel (b) and (d) we plot the payoffs and the distribution of efficiency for the simultaneous move game with deterministic one step technology. Leapfrog equilibria are defined as having positive probability to invest by the cost follower along the equilibrium path, mixed strategy equilibria are defined as involving at least one mixed strategy stage equilibrium along the equilibrium path.

payoffs, and blue indicates lower efficiency payoffs.

We see a clear positive correlation between payoff and efficiency in panel (a) — there is a tendency for the points with the highest total payoffs (i.e. points closest to the line connecting the monopoly outcomes) to have higher efficiency indices. The CDFs of efficiency levels in panel (c) shows that 1) overall efficiency is reasonably high, with the median equilibrium having an efficiency in excess of 97%, and 2) the maximum efficiency of the equilibria involving mixed strategies along the equilibrium path is strictly less than 100%.

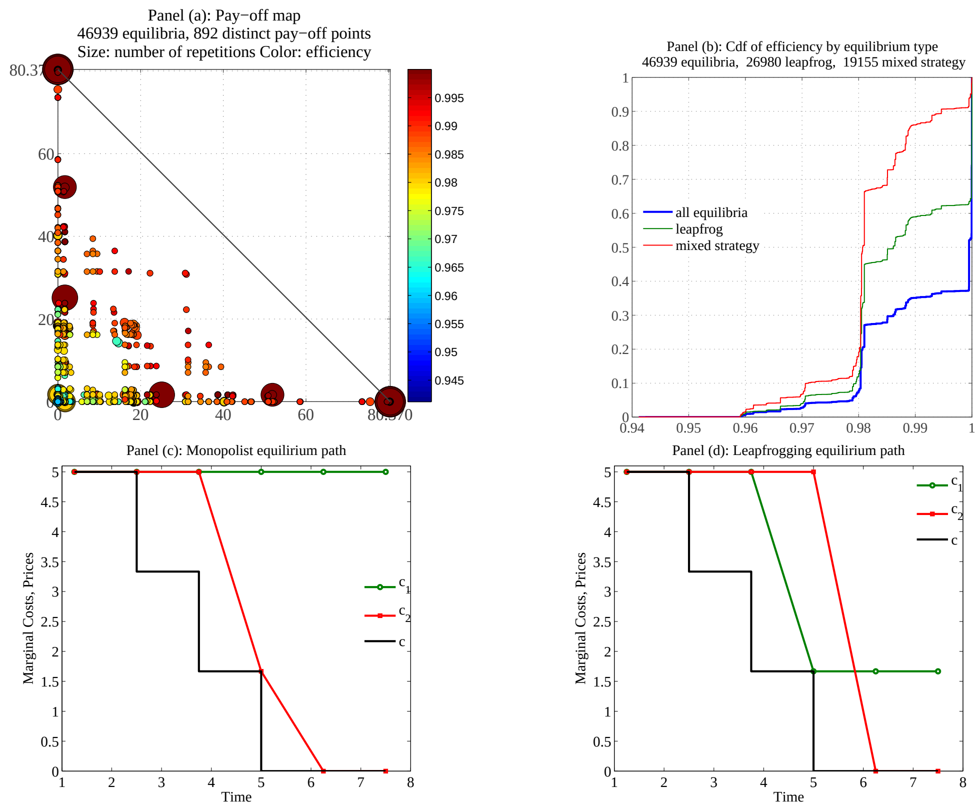
In panels (b) and (d) of Figure 3 we plot the set of equilibrium payoffs and distribution of equilibrium efficiency for a simultaneous move investment under the deterministic technology process. In accordance with Theorem 2 the monopoly and zero profit outcomes are now present among the computed MPE equilibria of the model. Overall, the equilibria in this game are less efficient compared to the equilibria in the alternating move game displayed in the top row panels, but the tendency of more efficient equilibria to be located closer to the "monopoly" frontier remains. An additional source of inefficiency in the simultaneous move game is duplicative investments, which occur in mixed strategy equilibria. The cumulative distribution plot in panel (d) shows that even though more than 30% of the equilibria are approaching full efficiency[6], the mixed strategy equilibria are not among them. Instead, the distribution of their efficiency is stochastically dominated by the distribution of efficiencies in all the equilibria of the game.

Theorem 5 (Inefficiency of mixed strategy equilibria)**.** *A necessary condition for efficiency in the dynamic Bertrand investment and pricing game is that only pure strategis are played on the equilibrium path.*

Figure 4 establishes the existence of *fully efficient leapfrogging equilibria.* Panel (a) of figure 4 plots the set of equilibrium payoffs in a simultaneous move investment game where there are four possible values for state of the art costs $\{0, 1.67, 3.33, 5\}$ and technology improves deterministically. Recall that the payoff points colored in dark red are 100% efficient, so we see that there

[6]To be exact, 15.22% have efficiency of 0.9878 and the same fraction of equilibria is fully efficient.

Figure 4: Efficiency of equilibria



Notes: Panel (a) and panel (b) plots the apex payoff map and distribution of efficiency indices for the simultaneous move game. 25.88% of all equilibria are fully efficient. The most efficient mixed strategy equilibrium has the efficiency index 0.99998 but does not violate Theorem 5. Panel (c) displays the simulated investment profile from a fully efficient "monopoly" equilibrium, while panel (d) displays the example of fully efficient equilibrium that involves leapfrogging.

are a number of other *non-monopoly equilibria* that can achieve *full efficiency.* The significance of this finding is that we have shown that it is possible to obtain competitive equilibria where leapfrogging by the firms ensures that consumers receive some of the surplus and benefits from technological progress without a cost in terms of inefficient investment such as we have observed occurs in mixed strategy equilibria of the game where socially inefficient excessive investment results in lower prices to consumers but at the cost of zero expected profits to firms. Notice, however, that even the least efficient mixed strategy equilibrium still has an efficiency of 96%, so that in this particular example the inefficiency of various equilibria may not be a huge concern.

Panels (c) and (d) of Figure 4 plot the simulated investment profiles of two different equilibria. Panel (c) shows the monopoly equilibria where firm 2 is the monopolist investor. The socially optimal investment policy is to make exactly two investments: the first when costs have fallen from 5 to 1.67, and the second when costs have fallen to the absorbing value of 0. Panel (d) shows the equilibrium realization from a pure strategy equilibrium that involves leapfrogging, yet the investments are made at exact same time as the social planner would do. After firm 1 invests when costs reach 1.67 (consumers continue to pay the price $p_1 = 5$), in time period 5 it is leapfrogged by firm 2 who becomes the permanent low cost producer. At this point a "price war" brings the price down from 5 to 1.67, which becomes new permanent level.

We conclude that the leapfrogging equilibria may be fully efficient if investments are made in the same moments of time as the monopolist would invest, but in these equilibria consumers also benefit from the investments because the price decreases in a series of permanent drops.

Lemma 2 (Existence of efficient non-monopoly equilibria)**.** *In both the simultaneous move and alternating move investment games, there exist fully efficient non-monopoly equilibria.*

Proof. The proof is by example shown in Figure 4. An example of a fully efficient non-monopoly equilibrium when the firms move alternately (in deterministic fashion) can be constructed as well[7].

□

[7] Let the possible cost states be $\{0, 5, 10\}$, assume deterministic technological progress, the cost of investing $K = 4$, and the discount factor $\beta = 0.95$. Then the socially optimal investment strategy is for investments to occur when $c = 5$ and $c = 10$, and these investments will occur at those states in the unique equilibrium of the game, but where one firm makes the first investment at $c = 5$ and the opponent makes the other investment when $c = 0$. These investments clearly involve leapfrogging that is also fully efficient.

While we find that efficient leapfrogging occur generically as equilibria in the simultaneous move investment game, the result that there exist efficient leapfrogging equilibria in the alternating move investment game should be viewed as a special counterexample, and that we typically do not get fully efficient leapfrogging equilibria in alternating move games with a sufficiently fine discretization of the state space and when investment costs are "reasonable" in relation to production costs (i.e. when the cost of building a new plant $K(c)$ significantly different from zero). However, due to the vast multiplicity of equilibria in the simultaneous move investment game, we have no basis for asserting that efficient leapfrogging equilibria are any more likely to arise than other more inefficient equilibria.

In summary inefficiency is generally caused by *excessive frequency of investment* rather than *underinvestment.* In simultaneous move games we already noted that another source of inefficiency is *redundant, duplicative investments* that occur only in mixed strategy equilibria. We noted that while mixed strategy equilibria also exist in the alternating move investment game, duplicative simultaneous investments cannot occur by the assumption that only one firm can invest at any given time. Thus, the inefficiency of the mixed strategy equilibria of the alternating move games is generally a result of excessively frequent investment under the mixed strategy equilibrium. However, it is important to point out that we have constructed examples of inefficient equilibria where there is *underinvestment* relative to the social optimum. Such an example is provided in panel (b) of Figure 5 in the next section.

4.4 Leapfrogging, Rent-dissipation and Preemption

In this section we consider the Riordan and Salant conjecture that was discussed in the introduction. Riordan and Salant (1994) conjectured that regardless of whether the firms move simultaneously or alternately, or whether technological progress is deterministic or stochastic, the general outcome in all of these environments is *rent-dissipating preemptive investments* where only one firm invests sufficiently frequently to deter its opponent from investing. These frequent preemptive investments fully dissipate any profits the investing firm can expect to earn from preempting its rival and hence also dissipate all social surplus. We first confirm their main result stated in terms of our model.

Theorem 6 (Riordan and Salant, 1994). *Consider a continuous time investment game with deterministic alternating moves. Assume that the cost of investment is independent of c, $K(c) = K$ and is not prohibitively high in the sense of Assumption 2). Further, assume that technological progress is deterministic with state of the art costs at time $t \geq 0$ given by the continuous, non-decreasing function $c(t)$ and continuous time interest rate $r > 0$. Assume that the continuous time analog of the condition that investment costs are not too high holds, i.e. $C(0) > rK$. Then there exists a unique MPE of the continuous time investment game (modulo relabeling of the firms) that involve preemptive investments by one or the other of the two firms and no investment in equilibrium by its opponent. The discounted payoffs of both firms in equilibrium is 0, so the entire surplus is wasted on excessively frequent investments by the preempting firm.*

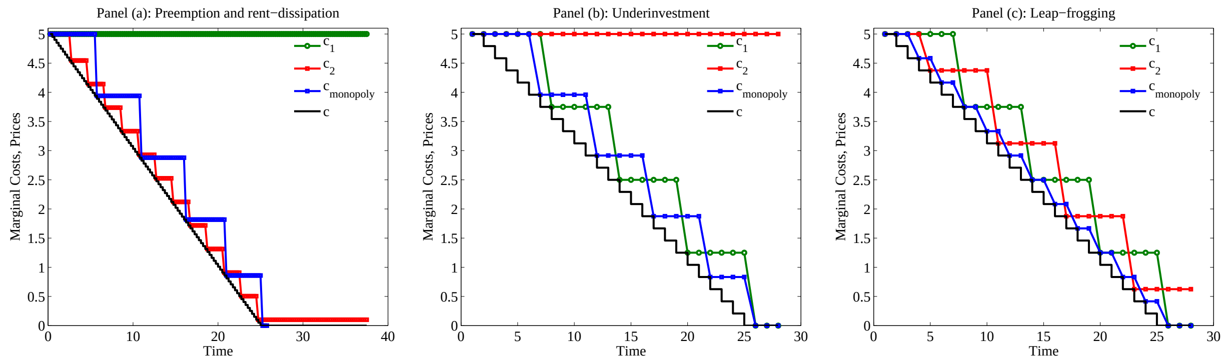
Corollary 6.1 (Riordan and Salant, 1994). *The continuous time equilibrium in Theorem 6 is a limit of the unique equilibria of a sequence of discrete time games where $\beta = \exp\{-r\Delta t\}$ and per period profits of the firms, $r_i(c_1, c_2)$, are proportional to Δt and the order of moves alternates* *deterministically, for a deterministic sequence of state of the art costs given by* $(c_0, c_1, c_2, c_3, \ldots) = (c(0), c(\Delta t), c(2\Delta t), c(3\Delta t), \ldots)$ *as* $\Delta t \to 0$.

The proofs of Theorem 6 and Corollary 6.1 is given in Riordan and Salant (1994) who used a mathematical induction argument to establish the existence of the continuous time equilibrium as the limit of the equilibria of a sequence of discrete time alternating move investment games.

In Figure 5 we plot simulated MPE for three versions of the Bertrand investment pricing game with deterministic alternating moves and deterministic technological progress. In the panel (a) we let the length of the time periods be relatively small to provide a good discrete time approximation to Riordan and Salant's model in continuous time. In panel (b) we decrease the number of points of support of the marginal cost and increase the length of the time period. In panel(c) in addition we lower investment cost. These three examples demonstrate that preemptive rent-dissipating investments indeed can happen in discrete time when the cost of investing in the new technology $K(c)$ is large enough relative to per period profits, but fails when the opposite is true as shown in panels (b) and (c). In discrete time, both duopolist have temporary monopoly power that can lead to inefficient under-investment as shown in the equilibrium realization in panel (b) or leapfrogging as shown in panel (c). Since per period profits are proportional to the length of the time period, the latter increases the value of the temporary cost advantage a firm gains after investment in the state of the art technology. If investment costs are sufficiently low relative to per period profits, it can be optimal for the cost follower to leapfrog the cost leader, in the limiting case even for a one period cost leadership.

While the Riordan and Salant result of strategic preemption with *full rent dissipation* only

Figure 5: Production and state of the art costs in simulated MPE: continuous. vs. discrete time

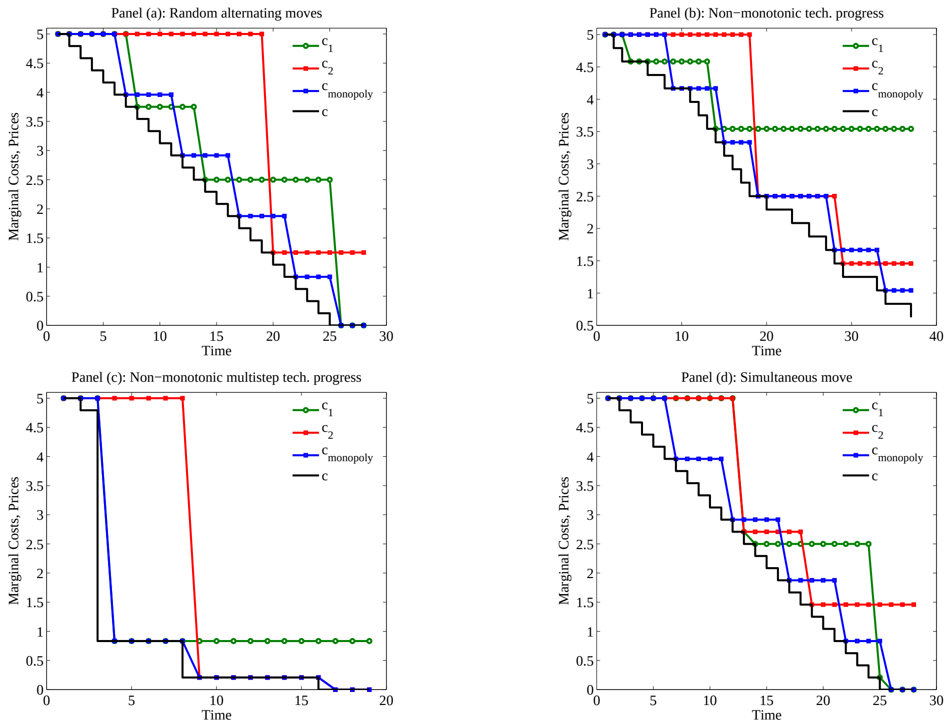


Notes: The figure plots simulated MPE equilibria for three versions of the Bertrand investment pricing game with deterministic alternating move and strictly monotonic technological progress. In panel (a) we present a discrete time approximation to Riordan and Salant's model in continuous time, with parameters $\beta = 0.9512$ $k_1 = 2$, $k_2 = 0$, $\pi(c|c) = 0$, $f(1|1) = f(2|2) = 0, f(2|1) = f(1|2) = 1$, $n_c = 100$, $\Delta t = 0.25$. In panel (b) we decrease the number of discrete support points for c to $n_c = 25$ and increase the length of the time period such that $\Delta t = 1$ adjusting per period values. In panel (c) we in addition lower investment costs by setting $k_1 = 0.5$.

holds in the continuous time limit $\Delta t \to 0$, their conclusion that investment preemption will occur is robust to discreteness of time. We find investment preemption is the unique equilibrium in our discrete time numerical solutions when Δt is sufficiently small. Thus, there is a "neighborhood" of Δt about the limit value 0 for which their unique preemption equilibrium also holds in a discrete time framework. However, the conclusion that preemption is fully inefficient and rent dissipating is not robust to discrete time. In discrete time the preempting firm does earn positive profits so the equilibrium is not completely inefficient.

Allowing for *random alternation* in the right to move, we obtain a unique pure strategy equilibrium, since random alternations does not violate the sufficient conditions for uniqueness given in Theorem 4. Yet, random alternation of the right to move destroys the ability to engage in strategic preemption and creates the opportunity for leapfrogging, since firms cannot have full control. Figure 6, panel (a) gives an example of a simulated equilibrium path when the right to move alternates

Figure 6: Production and state of the art costs in simulated MPE under uncertainty



Notes: The figure plots simulated MPE by type for four stochastic generalizations of the model illustrated in Figure 5.b. In panel (a) we consider random alternating moves where $f(1|1) = f(2|2) = 0.2$ and $f(2|1) = f(1|2) = 0.8$. In panel (b) we allow for non-strictly monotonic one step random technological improvement. In panel (c) we allow technological progress to follow a beta distribution over the interval $[c,0]$ where c is the current best technology marginal cost of production. The scale parameters of this distribution is $a = 1.8$ and $b = 0.4$ so that the expected cost, given an innovation, is $c * a/(a+b)$. Panel (d) plots an equilibrium path from the simultaneous move game. Unless mentioned specifically remaining parameters are as in panel (b) of Figure (5).

randomly. While this equilibrium path depicts a unique pure strategy equilibrium, we clearly see the leapfrogging pattern.

If there is positive probability of remaining with the same state of the art cost c for more that one period of time, i.e. $\pi(c|c) > 0$, the main results of Riordan and Salant (1994) will no longer hold in our model. We may have multiple equilibria, there will be leapfrogging, and full rent dissipation fails.

Figure 6 presents simulated equilibrium paths when we introduce randomness in the evolution of the state of the art technology, the order of moves in the alternating move game, or allow for simultaneous investments. All panels exhibit leapfrogging, reflecting the statement that stochasticity in the model presents the cost follower with more opportunities to leapfrog its opponent and makes it harder for the cost leader to preempt leapfrogging. Overall, in presence of uncertainty, the game becomes much more contestable.

Lemma 3. *(Limits to Riordan and Salant result) Preemption does not hold when (1) cost of investment K(c) is sufficiently small relative to per period profits, (2) investment decisions are made simultaneously, (3) the right to move alternates randomly, (4) the probability of an advance in the state of the art technology is less than one, i.e.* $\pi(c|c) > 0$.

Proof. The proof is by counter examples which are shown in Figure 5 and 6. $\square$

The vast majority of MPE equilibria in the many specifications of the game we have solved using the RLS algorithm exhibited leapfrogging. It appears that Riordan and Salant's results are not robust to any of the mentioned assumptions, at least in the discrete time analog of their model. However, with the exception of the full rent dissipation result, we believe that there is a *neighborhood* about the limiting set of parameter values that Riordan and Salant used to prove Theorem 6 for which their conjectured preemption equilibrium will continue to hold, at least with high probability.

5 Conclusions

The key contribution of this paper is to provide a characterization of *all* equilibria of a dynamic duopoly model of Bertrand price competition in the presence of stochastic technological progress. Contrary to the previous literature which has focused on *investment preemption* as the generic equilibrium outcome, we have shown that equilibrium outcomes typically involve various types of *leapfrogging* that result in some of the benefits of technological progress being passed on to consumers. We have shown that these dynamic equilibria are generally inefficient due to a combination of excessively frequent investments and duplicative investments resulting from coordination failures between the firms. However, in the numerical examples we have solved the efficiency is generally very high and there even exist fully efficient asymmetric monopoly equilibria, as well as efficient non-monopoly equilibria involving perfectly coordinated leapfrogging by the two firms.

Our analysis provides an alternative interpretation of "price wars." In the equilibria of our model, price trajectories are piecewise flat with large discontinuous price drops that occur when a high cost firm leapfrogs its rival to become the new temporary low cost leader. It is via these periodic price drops that consumers benefit from technological progress and the competition between the duopolists. Unlike models of temporary sales or price wars studied in models of tacit collusion, prices never increase in this model: each price drop represents a permanent gain to consumers.

Even with the assumption that the dynamics of the state of the art production technology is independent of the actions of the players, which leads to a relatively simple dynamic model, we find a surprisingly large and complex set of equilibria ranging from pure strategy monopoly outcomes to mixed strategy equilibria where expected profits of both firms are zero. In between are equilibria where leapfrogging investments are relatively infrequent so that consumers see fewer benefits from technological progress in the form of lower prices.

Our analysis also contributes to the long-standing debate about the relationship of market structure and innovation and the adoption of new technologies. Schumpeter (1939) argued a monopolist innovates more rapidly than a competitive industry since the monopolist can fully appropriate the benefits of R&D or other cost-reducing investments, whereas some of these investments would be dissipated in a competitive market. However, Arrow (1962) argued that innovation (or new technology adoption) under a monopolist will be slower than in a competitive market which is in turn slower than the social optimum. Both types of results have appeared in the subsequent literature. For example, in the R&D investment model analyzed by Goettler and Gordon (2011), the rate of innovation under monopoly is higher than under duopoly but still below the rate of innovation that would be chosen by a social planner. These inefficiencies are driven in part by the existence of externalities such as *knowledge spillovers* that are commonly associated with R&D investments.

In a setting where each competing firm can at any time adopt an exogenously improving state of the art technology, we have shown that the rate of adoption of new cost-reducing technologies under the duopoly equilibrium is generally *higher* than the monopoly or socially optimal solution. We showed that equilibria involving both leapfrogging and investment preemption both lead the duopolists to collectively invest more in cost reducing technologies than a social planner, as well as demonstrating the existence of monopoly equilibria where investments coincide with the socially optimal investment strategy. This latter result is rather specialized, and results from some of the restrictive simplifying assumptions we made to keep the analysis mathematically tractable. In particular, it would be important to extend the model to allow more than two firms and allow for entry and exit of firms.[8]

A disturbing aspect of our findings from a methodological standpoint is the plethora of Markov perfect equilibria present in a relatively simple extension of the standard static model of Bertrand price competition, which is reminiscent of the "Folk theorem" for repeated games. Though we have shown that the set of payoffs shrinks dramatically under the alternating move specification of the game and a unique MPE obtains when the probability of technological improvement in every time period is sufficiently close to one, there will generally be a huge multiplicity of equilibria either when firms move simultaneously, or when the probability of technological improvement is sufficiently low. Thus, though we have demonstrated how leapfrogging can be viewed as an endogenous solution to the "anti-coordination problem" and provides a resolution to the "Bertrand investment paradox" our paper leaves unsolved the more general question of how firms coordinate on a single equilibrium when there is a vast multiplicity of possible equilibria.

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A Proofs of Lemmas and Theorems

Theorem 1 (Necessary and sufficient condition for investment by the social planner).

Proof. Note that the left hand side of inequality (15) is the discounted cost savings from adopting the state of the art technology c' when the existing plant has marginal cost c_1. We first prove that if this inequality holds, then investment will be socially optimal at some state $({c_1}', c')$ satisfying ${c_1}' \in [0, c_1]$ and $c' \in [0, c_1']$. Suppose, to the contrary, that investment is not optimal for the social planner for any value $c_1' \leq c_1$ and any $c' \in [0, c_1']$. It follows that $C(c_1', c') = c_1'/(1-\beta)$ for all $c_1' \leq c_1$ and all $c' \in [0, c_1']$. However if the social planner did decide to invest when the state is (c_1, c) the planner's discounted costs would be $c_1 + K(c) + \beta \int_0^c C(c, c')\pi(dc'|c)$. Since we have assumed it is not optimal for the social planner to invest at any state (c_1', c') with $c' \in [0, c_1']$, then it cannot be optimal to invest in particular at any state (c_1, c) with $c \in [0, c_1]$. It follows that $c_1/(1-\beta) \leq c_1 + K(c) + \beta \int_0^c C(c, c')\pi(dc'|c)$ for all $c \in [0, c_1]$. However since $C(c, c') = c/(1-\beta)$ for all $c' \in [0, c]$, it follows that $\beta c_1/(1-\beta) \leq K(c) + \beta c/(1-\beta)$ for all $c \in [0, c_1]$, but this contradicts inequality (15).

Conversely, suppose inequality (15) does not hold. Then it follows that there is no value of $c' \in [0, c_1]$ for which investment is optimal, since in this case $C(c_1, c') = c_1/(1-\beta)$ for all $c' \in [0, c_1]$. This latter result follows by verifying that it is a solution to the Bellman equation (13), where it follows that the cost of replacing a plant with marginal cost c' when the state of the art marginal cost is c' is $c_1 + K(c') + \beta c'/(1-\beta)$ which exceeds the cost of keeping the existing plant $C(c_1, c') = c_1/(1-\beta)$ by our assumption that inequality (15) does not hold for any $c' \in [0, c_1]$. Since the solution to the Bellman equation is unique (via the contraction mapping property) and corresponds to an optimal investment policy, we conclude that there is no state (c_1, c') with $c' \in [0, c_1]$ for which investment in the state of the art technology c' is socially optimal. $\square$

Before proving the main results (Theorems 2 and 3) we preset a number of lemmas needed for these proofs.

Lemma A.1 (Necessary condition for social optimality of no investment at a particular state (c_1, c))**.** *Suppose that it is not optimal for the social planner (or monopolist) to invest at state (c_1, c), $c_1 \geq c$. It must then hold*

$$K(c) > \beta E\{\sum_{t=1}^{\tilde{\tau}-1} \beta^{t-1}(c_1 - c)\} = \frac{(c_1 - c)(\beta - E\{\beta^{\tilde{\tau}}\})}{1-\beta}, \tag{18}$$

where $\tilde{\tau}$ denotes the random number of time periods when investment will remain socially non-optimal.

Proof. Essentially, the Lemma A.1 states that whenever the social planner is optimally choosing not to invest at state (c_1, c), it must be that the expected discounted saving in production cost does not compensate for the cost of investment. From the point of view of the current time period, the potential saving in production cost is given by the difference in cost of production over $\tilde{\tau}$ periods starting from the following one, and up to and including the period when investment is again optimal. In other words, it is given by the partial sum of the geometric series with the constant element equal to $c_1 - c$ and the factor $\beta < 1$ taken over $\tilde{\tau} - 1$ periods, which also has to be discounted once more because of the one period time to build. After $\tilde{\tau}$ periods the state of the art cost c will reach the lower level $\tilde{c}_{\tilde{\tau}} < c$ and the investment by the social planner will again be justified, unless $\tilde{\tau} = \infty$ in which case (18) simplifies to (15), and by Theorem 1 no more investment ever occurs.

When investment is not optimal at (c_1, c), it follows from the social planner Bellman equation (13) that

$$C(c_1, c) = c_1 + \beta \int_0^c C(c_1, c')\pi(dc'|c), \tag{19}$$

and from (14) it follows that

$$K(c) > \beta \int_0^c \left[C(c_1, c') - C(c, c')\right] \pi(dc'|c). \tag{20}$$

It is also trivial to show that the investment at (c, c) is not optimal as it only entails to wasting the investment cost, giving

$$C(c, c) = c + \beta \int_0^c C(c, c')\pi(dc'|c). \tag{21}$$

Combining (19), (20) and (21) we get

$$K(c) > \left[C(c_1, c) - C(c, c)\right] - \left[c_1 - c\right]. \tag{22}$$

Consider then a particular investment policy starting in state (c, c), not investing for $\tilde{\tau}$ periods, then investing in the state of the art technology $\tilde{c}_{\tilde{\tau}}$ in the next period, and then following the socially optimal investment policy thereafter. Denote the cost associated with this strategy by $C_{\tilde{\tau}}(c, c)$. Because $C(c, c)$ is the minimal cost under an optimal investment policy, it follows that $C(c, c) \leq C_{\tilde{\tau}}(c, c)$, and from (22) we have

$$K(c) > \left[C_1(c_1, c) - C_{\tilde{\tau}}(c, c)\right] - \left[c_1 - c\right]. \tag{23}$$

Since $C_{\tilde{\tau}}(c, c)$ is the discounted expected cost of following, with probability 1, the same optimal investment policy that the social planner would follow once the state of the art cost reduces to $\tilde{c}_{\tilde{\tau}}$, it follows that

$$C(c_1, c) - C_{\tilde{\tau}}(c, c) = E\{\sum_{t=1}^{\tilde{\tau}} \beta^{t-1}(c_1 - c)\} = \frac{(c_1 - c)(1 - E\{\beta^{\tilde{\tau}}\})}{1 - \beta}, \tag{24}$$

i.e. the difference in the values is simply the total expected discounted difference in per period costs, $c_1 - c$, from not investing for $\tilde{\tau}$ periods when initial production costs are c_1 and c, respectively, and following the optimal investment policy from the point $(c_1, \tilde{c_\tau})$ thereafter. Substituting (24) into (23) we obtain inequality (18). □

Lemma A.2 (Necessary condition for no investment equilibrium in the whole Bertrand pricing and investment game)**.** *In any Markov perfect equilibrium of the simultaneous and alternative move formulation of the Bertrand investment game where neither firm invests, we must have*

$$K(c) \geq \frac{\beta(\min(c_1, c_2) - c)}{1-\beta} \tag{25}$$

in every state $(c_1, c_2, c) \in S = \{(c_1, c_2, c) : \; c \leq c_1 \leq c_0, c \leq c_2 \leq c_0, 0 \leq c \leq c_0\}$. *That is, there is no state of the art cost c for which a social planner would find it optimal to invest either.*

Proof. Consider the simultaneous move game first, i.e. the case $m = 0$ (omitted below for clarity of exposition). Assume that $c_1 \leq c_2$. This is without loss of generality since if $c_1 > c_2$ we can repeat the argument below from firm 2's perspective. If it is an equilibrium for neither firm to invest, it follows that in any state (c_1, c_2, c) satisfying $c_1 \leq c_2$ we have

$$\begin{aligned} v_{N,1}(c_1, c_2, c) &= \frac{c_2 - c_1}{1-\beta}, \\ v_{I,1}(c_1, c_2, c) &= c_2 - c_1 - K(c) + \frac{\beta(c_2 - c)}{1-\beta}, \end{aligned}$$

and that $v_{N,1}(c_1, c_2, c) \geq v_{I,1}(c_1, c_2, c)$. This implies that the following inequality must hold $\forall c_1 \in [0, c_2]$ and $\forall c \in [0, c_1]$

$$\frac{c_2 - c_1}{1-\beta} \geq (c_2 - c_1) - K(c) + \beta\frac{(c_2 - c)}{1-\beta}. \tag{26}$$

A similar inequality exist for $\forall c_2 \in [0, c_1]$ and $\forall c \in [0, c_2]$. It is easy to see via simple algebra that these inequalities are equivalent to inequality (25).

Now consider the alternating move game, $m \neq 0$. Recall that c_0 is the initial value of the state of the marginal production c. It is not hard to show, using the Bellman equations for the alternating move game (see equation 9 in section 2), that if it is never optimal for either firm to invest, then it follows that for the state $(c_1, c_2, c) = (c_0, c_2, c)$, that $P_2(c_0, c_2, c) = 0$ for all $c_2 \in [0, c_0]$ and for all $c \in [0, c_1]$. But this will be true if and only if

$$\begin{aligned} v_{N,1}(c_1, c_2, c, 1) &= \frac{c_2 - c_1}{1-\beta} \\ v_{I,1}(c_1, c_2, c, 1) &= c_2 - c_1 - K(c) + \frac{\beta(c_2 - c)}{1-\beta} \end{aligned}$$

and $v_{N,1}(c_1, c_2, c, 1) \geq v_{I,1}(c_1, c_2, c, 1)$. It is easy to see that this is equivalent to inequality (26) above and that a similar inequality can be derived from the perspective of firm 2. In turn inequality (25) must hold, thereby proving Lemma A.2. □

Lemma A.3 (Stage game equilibria in the simultaneous move game when $c = 0$)**.** *Consider the stage games of the simultaneous move game ($m = 0$) where $c = 0$. In states where investment is not socially optimal in the sense of Assumption 2, the investment game has a unique pure strategy equilibrium where neither firm invests. When investment is socially optimal, the investment game has three subgame perfect Nash equilibria: two pure strategy "anti-coordination" equilibria and one mixed strategy equilibrium.*

Proof. When $c=0$, it follows trivially from Theorem 1 that the social planner operating the minimum cost plant will invest in the state of the art technology if and only if

$$\frac{\beta \min(c_1,c_2)}{1-\beta} > K(0). \tag{27}$$

Now consider the Nash equilibria of the $(c_1,c_2,0)$ end game. It follows from A.2 that unless (27) holds, i.e investment is optimal for the social planner, then there is only a single "no investment" equilibrium.

Consider the states where $c_1=0$ or $c_2=0$ where one or both firms had already invested in the state of the art technology $c=0$. Condition (27) does not hold in these states, and therefore the only MPE is no investment by either firm. It then follows from (8) for firm 1 that

$$\begin{array}{lll} v_{N,1}(0,0,0)=0, & v_{I,1}(0,0,0)=-K(0), & P_1(0,0,0)=0, \\ v_{N,1}(c_1,0,0)=0, & v_{I,1}(c_1,0,0)=-K(0), & P_1(c_1,0,0)=0, \\ v_{N,1}(0,c_2,0)=\dfrac{c_2}{1-\beta}, & v_{I,1}(0,c_2,0)=\frac{c_2}{1-\beta}-K(0), & P_1(0,c_2,0)=0, \end{array} \tag{28}$$

where $c_1>0$ and $c_2>0$. Similar expressions hold for firm 2:

$$\begin{array}{lll} v_{N,2}(0,0,0)=0, & v_{I,2}(0,0,0)=-K(0), & P_2(0,0,0)=0, \\ v_{N,2}(c_1,0,0)=\dfrac{c_1}{1-\beta}, & v_{I,2}(c_1,0,0)=\frac{c_1}{1-\beta}-K(0), & P_2(c_1,0,0)=0, \\ v_{N,2}(0,c_2,0)=0, & v_{I,2}(0,c_2,0)=-K(0), & P_2(0,c_2,0)=0. \end{array} \tag{29}$$

Now consider the states $\{(c_1,c_2,0):\ c_1>0, c_2>0\}$ in the case when investment is socially optimal, i.e. condition (27) holds.[9] Focusing on firm 1 again, the Bellman equations (8) simplify to

$$\begin{array}{rcl} v_{N,1}(c_1,c_2,0) &=& r_1(c_1,c_2)+\beta\big(1-P_2(c_1,c_2,0)\big)\max\{v_{N,1}(c_1,c_2,0),v_{I,1}(c_1,c_2,0)\}, \\ v_{I,1}(c_1,c_2,0) &=& r_1(c_1,c_2)-K(0)+\beta\big(1-P_2(c_1,c_2,0)\big)\dfrac{c_2}{1-\beta}. \end{array} \tag{30}$$

Consider first the pure strategy equilibria at these states. If firm 2 invests with probability $P_2(c_1,c_2,0)=1$, then the last terms in (30) disappear, leading to $v_{N,1}(c_1,c_2,0)>v_{I,1}(c_1,c_2,0)$. Thus firm 1 does not invest, i.e. $P_1(c_1,c_2,0)=0$, forming the first pure strategy "anti-coordination" equilibrium. Conversely, if firm 2 surely does not invest, i.e. $P_2(c_1,c_2,0)=0$, there are two possibilities. If $v_{N,1}(c_1,c_2,0)<v_{I,1}(c_1,c_2,0)$ and thus $P_1(c_1,c_2,0)=1$, together with (30) this leads to the same inequality as condition (27). This establishes the second pure strategy "anti-coordination" equilibrium where firm 1 invests and firm 2 doesn't. If, on the other hand $v_{N,1}(c_1,c_2,0)>v_{I,1}(c_1,c_2,0)$ we get a contradiction. Similar argument establishes the symmetric result for firm 2.

[9]The rest of the proof follows closely the proof of Theorem 6 in Iskhakov, Rust, Schjerning (2016).

Consider now the mixed strategy equilibrium in the states $\{(c_1,c_2,0): c_1>0, c_2>0\}$, assuming that (27) holds. For some level of $P_2(c_1,c_2,0)$ firm 1 can be indifferent between investing and not investing, in which case any $P_1(c_1,c_2,0)\in[0,1]$ would constitute its best response investment probability. In this case combining (30) with the equality $v_{N,1}(c_1,c_2,0)=v_{I,1}(c_1,c_2,0)$ gives rise to the following quadratic equation in $P_2=P_2(c_1,c_2,0)$

$$Q_1(P_2)=\frac{\beta^2 c_2}{1-\beta}(1-P_2)^2+\beta\big(r_1(c_1,c_2)-K(0)-\frac{c_2}{1-\beta}\big)(1-P_2)+K(0)=0. \tag{31}$$

Conditional on falling to the unit interval, the roots of (31) paired with the roots of the symmetric equation for firm 2 would define the mixed strategy equilibrium of the stage game. Note however, that for $P_2=1$ the left hand side of (31) is equal to $K(0)>0$, while for $P_2=0$ it is equal to

$$(1-\beta)\left(K(0)-\frac{\beta\big(c_2-r_1(c_1,c_2)\big)}{1-\beta}\right)=(1-\beta)\left(K(0)-\frac{\beta\min(c_1,c_2))}{1-\beta}\right)<0,$$

and thus only a single root lies in the unit interval. Similar argument holds for firm 2. This establishes the existence and uniqueness of the mixed strategy equilibrium, and concludes the proof of the lemma. □

Lemma A.4 (Leapfrogging in the simultaneous move game when $c=0$)**.** *In the mixed strategy equilibria in Lemma A.3 the high cost firm has the higher probability of investing, i.e.* $P_1(c_1,c_2,0)>P_2(c_1,c_2,0)$ *when* $c_1>c_2$*, and* $P_1(c_1,c_2,0)<P_2(c_1,c_2,0)$ *when* $c_1<c_2$*.*

Proof. The proof builds on Lemma A.3 where it shows the existence of a single mixed strategy equilibrium in the states $\{(c_1,c_2,0): c_1>0, c_2>0\}$ under condition (27). The only root of the equation (31) that lies in the unit interval defines the equilibrium probability of investment of firm 2, $P_2(c_1,c_2,0)$. The symmetric equation that defines the equilibrium investment probability $P_1=P_1(c_1,c_2,0)$ of firm 1 is

$$Q_2(P_1)=\frac{\beta^2 c_1}{1-\beta}(1-P_1)^2+\beta\big(r_2(c_1,c_2)-K(0)-\frac{c_1}{1-\beta}\big)(1-P_1)+K(0)=0. \tag{32}$$

Evaluating the left hand sides of equations (31) and (32) at the ends of the unit interval we have

$$\begin{aligned}
&Q_1(0)=Q_2(0)=(1-\beta)\left(K(0)-\frac{\beta\min(c_1,c_2))}{1-\beta}\right)<0,\\
&Q_1(1)=Q_2(1)=K(0)>0.
\end{aligned}$$

Because both $Q_1(P_2)$ and $Q_2(P_1)$ are quadratic polynomials and have the same values at 0 and 1, it is sufficient to compare their derivatives on one side of the unit interval to establish the order of their roots on this interval. We have

$$\frac{\partial Q_1(P_2)}{\partial P_2}\Big|_{P_2=1}-\frac{\partial Q_2(P_1)}{\partial P_1}\Big|_{P_1=1}=\beta\big[r_2(c_1,c_2)-r_1(c_1,c_2)+\frac{c_2-c_1}{1-\beta}\big]. \tag{33}$$

For the case $c_1 < c_2$ the difference in (33) evaluates to $\frac{\beta^2(c_2-c_1)}{1-\beta} > 0$ which implies that the root of $Q_1(P_2)$ in the unit interval lies to the right of the root of $Q_2(P_1)$. In other words, $P_2(c_1,c_2,0) > P_1(c_1,c_2,0)$ when $c_1 < c_2$. Conversely, for the case $c_1 > c_2$ the difference in (33) evaluates to $-\frac{\beta^2(c_1-c_2)}{1-\beta} > 0$ which implies that the root of $Q_1(P_2)$ in the unit interval lies to the left of the root of $Q_2(P_1)$. In other words, $P_2(c_1,c_2,0) < P_1(c_1,c_2,0)$ when $c_1 > c_2$. We conclude that in the state of the game $\{(c_1,c_2,0): \ c_1 > 0, c_2 > 0\}$ where the mixed strategy equilibrium exists under condition (27), the firm with higher marginal cost of production has a higher probability of investment. $\square$

Lemma A.5 (Efficiency of the stage game equilibria in the simultaneous move game when $c = 0$)**.** *The pure strategy equilibria in Lemma A.3 are fully efficient. The mixed strategy equilibria in Lemma A.3 are inefficient due to the possibility of over- or underinvestment.*

Proof. If follows from Lemma A.3 that in pure strategy equilibria investment by any firm takes place if and only if the condition (27) is satisfied, i.e. it is socially optimal to invest. Moreover, because in these "anti-coordination" equilibria no more than firm invests at a time, there is no possibility for efficiency loss due to duplicative investment. Thus, the pure strategy equilibria at the states $(c_1,c_2,0)$ are fully efficient.

Consider then the mixed strategy equilibria in states $\{(c_1,c_2,0): \ c_1 > 0, c_2 > 0\}$ where it is socially optimal to invest, i.e. assuming (27) is satisfied. Let $C_m(c_1,c_2,0)$ denote the present discounted value of investment and production costs under this mixed strategy equilibrium at the point $(c_1,c_2,0)$. Suppressing the arguments $(c_1,c_2,0)$ of the investment probabilities, we have

$$\begin{aligned} C_m(c_1,c_2,0) &= \min(c_1,c_2) + 2K(0)P_1P_2 + K(0)P_1(1-P_2) + K(0)(1-P_1)P_2 \\ &\quad + \beta(1-P_1)(1-P_2)C_m(c_1,c_2,0) \\ &= \min(c_1,c_2) + K(0)(P_1+P_2) + \beta(1-P_1)(1-P_2)C_m(c_1,c_2,0). \end{aligned}$$

We will now show that $C_m(c_1,c_2,0) = \big(K(0)(P_1+P_2)+c_1\big)/\big(1-\beta(1-P_1)(1-P_2)\big)$ exceeds the socially optimal production and investment costs $C^*(c_1,c_2,0) = \min(c_1,c_2)+K(0)$ that a social planner can achieve by undertaking only a single investment in the state of the art technology and avoid i) the higher costs due to redundant duplicative investments and ii) the costs due to delayed investment due to the probability $(1-P_1)(1-P_2)$ that neither firm invests under the mixed strategy equilibrium.

We establish this inequality via an indirect argument. Let $p = P_1 + P_2 - P_1P_2 \in (0,1)$ be the probability that *at least one of the firms invests in the mixed strategy equilibrium*, and let $\underline{C}_m(p) < C_m(c_1,c_2,0)$ denote the present value of costs under a mixed strategy equilibrium that ignores the occurrence of redundant investments by the two firms and therefore constitutes the lower bound on $C_m(c_1,c_2,0)$. We write $\underline{C}_m(p)$ to emphasize its dependence on p.

$$\begin{aligned} \underline{C}_m(p) &= \min(c_1,c_2) + pK(0) + \beta(1-p)\underline{C}_m(p) \\ &= \frac{\min(c_1,c_2)+pK(0)}{1-\beta(1-p)}. \end{aligned}$$

Note that $\underline{C}_m(1)$ equals the social optimal level of cost, $C^*(c_1,c_2,0)$. It then suffices to show that $\underline{C}_m(p)$ is a decreasing function:

$$\frac{d\underline{C}_m(p)}{dp} = \frac{K(0)(1-\beta)-\beta\min(c_1,c_2)}{\left(1-\beta(1-p)\right)^2} < 0, \tag{34}$$

where the inequality holds under (27). Since we know that $p \in (0,1)$ in the mixed strategy equilibrium, we have

$$C_m(c_1,c_2,0) > \underline{C}_m(p) > C^*(c_1,c_2,0). \tag{35}$$

which establishes the inefficiency of the mixed strategy equilibrium. □

Theorem 2 (The set of equilibrium payoffs in the Simultaneous Move Game).

We prove Theorem 2 statement by statement. It is helpful to re-write the Bellman equations (8) as

$$\begin{aligned} v_{N,1}(c_1,c_2,c) &= r_1(c_1,c_2)+\beta\Big[P_2(c_1,c_2,c)H_1(c_1,c,c)+\big(1-P_2(c_1,c_2,c)\big)H_1(c_1,c_2,c)\Big], \quad (36)\\ v_{I,1}(c_1,c_2,c) &= r_1(c_1,c_2)-K(c)+\beta\Big[P_2(c_1,c_2,c)H_1(c,c,c)+\big(1-P_2(c_1,c_2,c)\big)H_1(c,c_2,c)\Big], \end{aligned}$$

where the expected value function $H_1(c_1,c_2,c)$ is given by

$$\begin{aligned} H_1(c_1,c_2,c) &= h_1(c_1,c_2,c)+\pi(c|c)\max\left\{v_{N,1}(c_1,c_2,c),v_{I,1}(c_1,c_2,c)\right\}, & (37)\\ h_1(c_1,c_2,c) &= (1-\pi(c|c))\int_0^c \max\left\{v_{N,1}(c_1,c_2,c'),v_{I,1}(c_1,c_2,c')\right\}f(dc'|c). & (38) \end{aligned}$$

where $\pi(c|c)$ is the probability for a cost-reducing innovation not to occur, and $f(c'|c)$ is the conditional density of the new (lower) state of the art marginal cost of production conditional on an innovation having occurred. The Bellman equations for firm 2 have the symmetric form

$$\begin{aligned} v_{N,2}(c_1,c_2,c) &= r_2(c_1,c_2)+\beta\Big[P_1(c_1,c_2,c)H_2(c,c_2,c)+\big(1-P_1(c_1,c_2,c)\big)H_2(c_1,c_2,c)\Big], \quad (39)\\ v_{I,2}(c_1,c_2,c) &= r_2(c_1,c_2)-K(c)+\beta\Big[P_1(c_1,c_2,c)H_2(c,c,c)+\big(1-P_1(c_1,c_2,c)\big)H_2(c_1,c,c)\Big], \end{aligned}$$

where the expected value function $H_2(c_1,c_2,c)$ is given by

$$\begin{aligned} H_2(c_1,c_2,c) &= h_2(c_1,c_2,c)+\pi(c|c)\max\left\{v_{N,2}(c_1,c_2,c),v_{I,2}(c_1,c_2,c)\right\}, & (40)\\ h_2(c_1,c_2,c) &= (1-\pi(c|c))\int_0^c \max\left\{v_{N,2}(c_1,c_2,c'),v_{I,2}(c_1,c_2,c')\right\}f(dc'|c). & (41) \end{aligned}$$

Statement 1. (No investments by both firms in all states (c_1,c_2,c) is not supported in any of the MPE equilibria of the game).

Proof. By Theorem 1, if investment is optimal for the social planner, then inequality (25) cannot hold. By Lemma A.2, it follows that no investment cannot be an MPE outcome. □

Statement 2. (There are two fully efficient "monopoly" equilibria in which either one or the other firm makes all the investments and earns maximum feasible profit while their opponent earns zero profits).

Proof. The two candidate equilibria are where firm 2 never invests in equilibrium and firm 1 does all the investing (whenever it is profit-maximizing for firm 1 to do so), and symmetrically, where firm 1 never invests and firm 2 does all the investing (whenever it is profit-maximizing for firm 2 to do so). Without loss of generality, we only consider the case when firm 1 makes all the investments, a symmetric proof holds for the other case.

The proof proceeds in the following way. We consider the pair of strategies where firm 2 never makes the investments, and firm 1 maximizes its profits acting as a monopolist constrained to charge the price equal to the initial value of the cost c_0. We then show that this indeed constitutes the pure strategy Markov perfect equilibrium of the Bertrand investment game by showing that neither firm has an incentive to deviate. For firm 2 the proof of the latter requires transfinite induction on the state of the art cost of production c. Recall that transfinite induction works as follows. We first show that a set of hypotheses ("the result") holds when the state of the art cost of production is $c = 0$. The "inductive step" is to show that if these hypotheses hold for all $c' < c$, then they hold for c as well. Transfinite induction then guarantees that the result holds for all c, similar to the way the proof by induction would work if c was discretized into a countable set of possible values.

Consider first firm 1 under the assumption that firm 2 follows the strategy of never investing. The fact that firm 1 will adopt a socially optimal investment strategy follows immediately from this assumption and Lemma 1. Since firm 1 knows that firm 2 will not invest, and the price of the good is fixed at c_0, firm 1 maximizes its profits by adopting an investment strategy that minimizes its present discounted costs of production and investment from any given starting node in the game (c_1, c_2, c), in particular the apex, (c_0, c_0, c_0). For some of the points, it may be optimal for firm 1 not to invest, however when this is the case, it would not be socially optimal for investment to occur there as well.

Consider next firm 2, assuming that firm 1 follows the socially optimal investment policy. For the described "monopoly" equilibrium to take place we need to prove that not investing is the optimal response of firm 2 along the equilibrium path. We do this by transfinite induction using inductive hypothesis that

$$v_{N,2}(c_1, c_2, c) = 0 \text{ and } \max\left(v_{N,2}(c_1, c_2, c), v_{I,2}(c_1, c_2, c)\right) = 0 \quad \text{for all states } (c_1, c_2, c) \in S \text{ where } c_1 \leq c_2. \tag{42}$$

It follows from Lemma A.3 that (42) holds in all states $(c_1, c_2, 0)$ where $c_1 \leq c_2$. Namely, it holds in the single no investment equilibria at the states $(0, 0, 0)$ and $(0, c_2, 0)$, $c_2 > 0$, per (29), and in the pure strategy eqilibria where firm 1 invests and firm 2 does not invest in the states $(c_1, c_2, 0)$, $c_1 > 0, c_2 > 0$. This establishes the base for the transfinite induction.

Now for the inductive step, we prove that if the hypothesis holds for $c' < c$, then it also holds at the state of the art cost c, i.e. for all points $(c_1, c_2, c) \in S$ where $c_1 \leq c_2$. Note that under the inductive hypothesis the Bellman equations for firm 2 simplify due to the fact that the function

$h_2(c_1,c_2,c)$ in (41) is zero at these states. To prove the inductive hypothesis for $c'=c$ we consider several subsets of the state points.

First, it is easy to show that in the point (c,c,c) the value functions for firm 2 are $v_{N,2}(c,c,c)=0$ and $v_{I,2}(c,c,c)=-K(c)$, and thus the inductive hypothesis holds.

Second, consider the points (c,c_2,c) where $c_2>c$. In these points firm 1 has already attained the state of the art production cost, and therefore has no incentive to invest. It is easy to show that in these points $v_{N,1}(c,c_2,c)=v_{I,1}(c,c_2,c)+K(c)$, and therefore $P_1(c,c_2,c)=0$. Then from (39) and (40) we have

$$\begin{aligned}
v_{I,2}(c,c_2,c) &= -K(c)+\beta\pi(c|c)\max\big(v_{N,2}(c,c,c),v_{I,2}(c,c,c)\big)=-K(c),\\
v_{N,2}(c,c_2,c) &= \beta\pi(c|c)\max\big(v_{N,2}(c,c_2,c),-K(c)\big).
\end{aligned}$$

We conclude that $v_{N,2}(c,c_2,c)=0$, and therefore the inductive hypothesis holds.

Third, consider the points $(c_1,c_2,c)\in S$ where $c<c_1\leq c_2$ and assume $P_1(c_1,c_2,c)=1$. In this case the Bellman equations for firm 2 give

$$\begin{aligned}
v_{I,2}(c_1,c_2,c) &= -K(c)+\beta\pi(c|c)\max\big(v_{N,2}(c,c,c),v_{I,2}(c,c,c)\big)=-K(c),\\
v_{N,2}(c_1,c_2,c) &= \beta\pi(c|c)\max\big(v_{N,2}(c,c_2,c),v_{I,2}(c,c_2,c)\big)=0.
\end{aligned}$$

Therefore the inductive hypothesis also holds when it is optimal for firm 1 to invest.

Finally, consider the points $(c_1,c_2,c)\in S$ where $c<c_1\leq c_2$ in the case when it is optimal for firm 1 not to invest, i.e. $P_1(c_1,c_2,c)=0$. This is the most complicated case because there is a potential for firm 2 to use the non-investment by firm 1 as an opportunity to sneak in and leapfrog firm 1 to become the new low cost leader. We now show that this "off the equilibrium path deviation" is not optimal for firm 2. While such a deviation will reduce the profits of firm 1 (by permanently lowering the price from c_0 to c_1), by the inductive hypothesis the behavior of the firms will not change in any subgame (c_1,c_2,c') for $c'<c$, so that firm 1 will resume playing the role of the monopolist investor and choose the timing of its subsequent investments in a socially optimal manner whereas firm 2 will not invest in any such subgame. The only effect of the deviation by firm 2 is to lower the revenues firm 1 can expect to get in any subgame following the deviation. However we now show that firm 2 will not find it optimal to deviate because it can only obtain *temporary* cost leadership for a random duration $\tilde{\tau}$ until the state of the art cost c transits to some lower cost $c_{\tilde{\tau}}<c$, at which point the game enters a subgame $(c_1,c,c_{\tilde{\tau}})$ where firm 1 is the monopolist (and thus follows the socially optimal investment strategy) and firm 2 is the high cost follower and earns zero expected discounted profits, i.e. $V_2(c_1,c,c_{\tilde{\tau}})=0$ since the value for firm 2 is zero for all $c'<c$, by our inductive hypothesis. In summary, under a deviation by firm 2 the game will spend $\tilde{\tau}$ periods in the state (c_1,c,c), during which firm 2 wiil earn a per period profit of c_1-c.

Thus, the total expected gain to firm 2 from deviating at (c_1,c_2,c) is $\Delta-K(c)$ where Δ is given by

$$\Delta=\beta E\{\sum_{t=1}^{\tilde{\tau}}\beta^{t-1}(c_1-c)\}=\frac{(c_1-c)(\beta-E\{\beta^{\tilde{\tau}}\})}{1-\beta}. \tag{43}$$

By Lemma A.1 $\Delta < K(c)$, and thus from the Bellman equations (39) that take the form

$$\begin{aligned} v_{I,2}(c_1,c_2,c) &= \Delta - K(c) < 0, \\ v_{N,2}(c_1,c_2,c) &= \beta\pi(c|c)\max\big(v_{N,2}(c_1,c_2,c), \Delta - K(c)\big), \end{aligned}$$

we conclude that $v_{N,2}(c_1,c_2,c) = 0$. Therefore the inductive hypothesis still holds.

We have shown that the inductive hypothesis (42) indeed holds in all points $(c_1,c_2,c) \in S$ where $c_1 \le c_2$ at the level of the state of the art cost c. Then by transfinite induction, it holds for all levels of c. This implies that the best response of firm 2 to the socially optimal investment strategy of firm 1 is not to invest when $c_1 \le c_2$. Because the "monopoly" equilibrium path is fully contained within this subset of the state space due to the fact that only firm 1 makes the cost reducing investments, this suffice to establish the "monopoly" equilibrium as a Markov perfect equilibrium of the whole Bertrand investment game. In this equilibrium firm 1 makes all the investments and follows the socially optimal investment policy, thus making it fully efficient. □

Statement 3. (If Assumption 1 (Continuity) holds, there exists a symmetric mixed strategy equilibrium in the simultaneous move game that results in zero expected payoffs to both firms in the subgames starting at all diagonal states $(c,c,c') \in S$ with $c' \in [0,c]$, and zero expected payoffs to the high cost firm and positive expected payoffs to the low cost firm in the subgames starting in states (c_1,c_2,c) where $c_1 \neq c_2$).

Proof. Recall, that a *mixed strategy equilibrium* is an MPE of the overall game that involves playing mixed strategies at least in some states on the equilibrium path, i.e. the subset of points in the state space which are visited with positive probability under the equilibrium. We also distinguish the MPE for the *full game* from the MPE of its *stage games* — that is, the set of MPE possible at any particular point in the state space (c_1,c_2,c). It may happen that at some points in the support of a mixed strategy equilibrium that it is not socially optimal to invest, and thus the only stage game MPE is a pure strategy equilibrium where neither firm invests. However, we rule out the case when it is not socially optimal to invest at all in the sense of Assumption 2 at the initial point of the game. This implies that at least for some *interior* state $(c_1,c_2,c) \in S$ where $c_1 < c$ and $c_2 < c$, the firms invest with positive probability. As shown below this ensures the existence of a mixed strategy MPE for the full game.

The candidate symmetric mixed strategy equilibrium is the one where expected profits of the firm with strictly lower cost (i.e. firm 1 in states $(c_1,c_2,c) \in S$ where $c_1 < c_2 \ge c$) are positive, and the expected profits in other cases are zero. We also show that in this MPE the value functions and probability of investment are symmetric in their first two arguments, so this MPE is also a symmetric equilibrium.

We prove the result by showing that the described equilibrium is indeed supported as the solution to the system of Bellman equations of the two firm (36) and (39). As for Statement 2 the proof is by transfinite induction using the inductive hypothesis that

$$\begin{aligned} &\max[v_{N,1}(c_1,c_2,c), v_{I,1}(c_1,c_2,c)] = 0 \text{ for all states } (c_1,c_2,c) \in S \text{ where } c_1 \ge c_2, \\ &\max[v_{N,1}(c_1,c_2,c), v_{I,1}(c_1,c_2,c)] > 0 \text{ for all states } (c_1,c_2,c) \in S \text{ where } c_1 < c_2, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{and symmetrically for firm 2.} \end{aligned} \tag{44}$$

We have already established that the inductive hypothesis holds when $c = 0$ and either $c_1 = 0$ or $c_2 = 0$ in the proof of Lemma A.3, see (28) and (29). Consider the *interior* states $(c_1, c_2, 0) \in S$ where $c_1 > 0$ and $c_2 > 0$. If it is socially optimal not to invest, i.e. $\min(c_1, c_2) < (1-\beta)K/\beta$, firm 1 chooses not to invest and earns a positive profit $\frac{c_1 - c_2}{1-\beta}$ in states $c_1 < c_2$ and zero otherwise. If on the other hand, investment is socially optimal, by Lemma A.3 there is a mixed strategy equilibrium where firms are indifferent between investing and not investing, in particular $v_{N,1}(c_1, c_2, 0) = v_{I,1}(c_1, c_2, 0)$. Since current profit is positive for the cost leader, i.e. $r_1(c_1, c_2) > 0$ for $c_1 < c_2$ and zero otherwise, it follows from (30) that $v_{N,1}(c_1, c_2, 0) = v_{I,1}(c_1, c_2, 0) > 0$ for $c_1 < c_2$ and $v_{N,1}(c_1, c_2, 0) = v_{I,1}(c_1, c_2, 0) = 0$ for $c_1 \geq c_2$. A symmetric argument holds for firm 2. Further, it is not difficult to show that symmetry in the payoffs and equilibrium strategies for the two firms holds for $c = 0$: $v_{N,1}(c_1, c_2, 0) = v_{N,2}(c_2, c_1, 0)$ and $v_{I,1}(c_1, c_2, 0) = v_{I,2}(c_2, c_1, 0)$, and $P_1(c_1, c_2, 0) = P_2(c_2, c_1, 0)$. Thus, the inductive hypothesis (44) holds for $c = 0$, which establishes the base case for the transfinite induction.

Now for the inductive step, where we prove that if the hypothesis (44) holds for all $c' < c$, then it also holds at c. Note that the inductive hypothesis implies for (38) $h_1(c_1, c_2, c) = 0$ when $c_1 \geq c_2$, $h_1(c_1, c_2, c) > 0$ when $c_1 < c_2$, and analogously for $h_2(c_1, c_2, c)$ in (41). To prove the inductive hypothesis we consider several subsets of the state points.

Consider first the corner state, (c, c, c). Under the inductive hypothesis the Bellman equations (36) reduce to

$$\begin{aligned} v_{N,1}(c,c,c) &= \beta\pi(c|c)\max[v_{N,1}(c,c,c), v_{I,1}(c,c,c)], \\ v_{I,1}(c,c,c) &= -K(c) + \beta\pi(c|c)\max[v_{N,1}(c,c,c), v_{I,1}(c,c,c)], \end{aligned} \tag{45}$$

which implies that $v_{N,1}(c,c,c) = 0$, $v_{I,1}(c,c,c) = -K(c)$ and $\max[v_{N,1}(c,c,c), v_{I,1}(c,c,c)] = 0$. A symmetric argument holds for firm 2.

Consider next the edge states (c_1, c, c) where firm 2's cost equals the state of the art cost c, and firm 1 is the high cost follower, i.e. $c_1 > c$. From the Bellman equations (39) it is apparent that $v_{N,2}(c_1, c, c) = v_{I,2}(c_1, c, c) + K(c)$ so firm 2 will not invest at these states as long as $K(c) > 0$. Then, under the inductive hypothesis (44) it is easy to see that the Bellman equations (36) for firm 1 reduce to

$$\begin{aligned} v_{N,1}(c_1,c,c) &= \beta\pi(c|c)\max[v_{N,1}(c_1,c,c), v_{I,1}(c_1,c,c)] \\ v_{I,1}(c_1,c,c) &= -K(c) + \beta\pi(c|c)\max[v_{N,1}(c,c,c), v_{I,1}(c,c,c)]. \end{aligned} \tag{46}$$

With the above result for the point (c, c, c), it follows that $v_{I,1}(c_1, c, c) = -K(c)$, $v_{N,1}(c_1, c, c) = 0$, and $\max[v_{N,1}(c_1, c, c), v_{I,1}(c_1, c, c)] = 0$. Symmetric arguments hold for firm 2 at the edge states (c, c_2, c), $c_2 > c$.

Consider next the edge states (c, c_2, c), $c_2 > c$ where firm 1 is the low cost leader. It follows immediately from the Bellman equations (36) that $v_{N,1}(c, c_2, c) = v_{I,1}(c, c_2, c) + K(c)$. We just established above for firm 2 that $v_{N,2}(c, c_2, c) = \max[v_{N,2}(c, c_2, c), v_{I,2}(c, c_2, c)] = 0$, and thus $P_2(c, c_2, c) = 0$. Together with the inductive hypothesis this leads to

$$v_{N,1}(c,c_2,c) = \frac{c_2 - c + \beta h_1(c,c_2,c)}{1 - \beta\pi(c|c)} = \max[v_{N,1}(c,c_2,c), v_{I,1}(c,c_2,c)] > 0. \tag{47}$$

Symmetric arguments hold for firm 2 at the edge statess (c_1,c,c), $c_1 > c$.

Finally, consider the interior states (c_1,c_2,c), where $c_1 > c$ and $c_2 > c$, and again start with firm 1. Combining the inductive hypothesis with the results above, it is easy to show that $H_1(c,c,c) = H_1(c_1,c,c) = 0$, so the Bellman equations (36) can be further simplified to

$$\begin{aligned}
&v_{N,1}(c_1,c_2,c) = r_1(c_1,c_2) \\
&\quad + \beta(1-P_2(c_1,c_2,c))\big(h_1(c_1,c_2,c) + \pi(c|c)\max[v_{N,1}(c_1,c_2,c), v_{I,1}(c_1,c_2,c)]\big), \\
&v_{I,1}(c_1,c_2,c) = r_1(c_1,c_2) - K(c) + \beta(1-P_2(c_1,c_2,c))H_1(c,c_2,c), \\
&H_1(c,c_2,c) = \frac{h_1(c,c_2,c) + \pi(c|c)(c_2-c)}{1-\beta\pi(c|c)}.
\end{aligned} \tag{48}$$

Let $\Upsilon\{\cdot \le \cdot\}$ denote an "indicator correspondence" which takes the values 0 and 1 similar to the usual indicator function, except that Υ takes all values in the unit interval $[0,1]$ when the inequality in the curly brackets is satisfied with equality.[10] The best response of firm 1 to the investment choice probability $P_2 = P_2(c_1,c_2,c)$ of firm 2 is then given by $\Upsilon\{v_{N,1}(c_1,c_2,c,P_2) \le v_{I,1}(c_1,c_2,c,P_2)\}$, where the last argument is added to emphasize the dependence on firm 2's investment probability P_2. All equilibria at the state (c_1,c_2,c) are then the intersection points of $\Upsilon\{v_{N,1}(c_1,c_2,c,P_2) \le v_{I,1}(c_1,c_2,c,P_2)\}$ and $\Upsilon\{v_{N,2}(c_1,c_2,c,P_1) \le v_{I,2}(c_1,c_2,c,P_1)\}$ in the unit square on the plane (P_1,P_2), with mixed strategy equilibria given by the intersection points on the interior of this square.

It follows from (48) that the best response correspondence of firm 1 can be expressed using a quadratic polynomial as

$$P_1(c_1,c_2,c) = \Upsilon\{A_1 + B_1P_2 + C_1P_2^2 \le 0\}, \tag{49}$$

where the coefficients are given by

$$\begin{aligned}
A_1 &= \beta\pi(c|c)r_1(c_1,c_2) + \beta h_1(c_1,c_2,c) \\
&\quad + \big(1-\beta\pi(c|c)\big)\big[K(c) - \beta H_1(c,c_2,c)\big], \\
B_1 &= -\beta h_1(c_1,c_2,c) - \beta\pi(c|c)\big[r_1(c_1,c_2) - K(c) + \beta H_1(c,c_2,c)\big] \\
&\quad + \beta\big(1-\beta\pi(c|c)\big)H_1(c,c_2,c), \\
C_1 &= \beta^2\pi(c|c)H_1(c,c_2,c).
\end{aligned} \tag{50}$$

with analogous expressions holding for firm 2.

Consider first the states (c_1,c_2,c) where $c_1 \le c_2$, i.e. firm 1 is the cost leader and firm 2 is the cost follower. For the latter we have $h_2(c_1,c_2,c) = r_2(c_1,c_2) = 0$, and the polynomial coefficients (50) for firm 2 simplify to

$$\begin{aligned}
A_2 &= \big(1-\beta\pi(c|c)\big)\big[K(c) - \beta H_2(c_1,c,c)\big], \\
B_2 &= \beta\pi(c|c)K(c) + \beta\big(1-2\beta\pi(c|c)\big)H_2(c_1,c,c), \\
C_2 &= \beta^2\pi(c|c)H_2(c_1,c,c).
\end{aligned} \tag{51}$$

[10] Here again the proof follows closely the proof of Theorem 6 in Iskhakov, Rust, Schjerning (2016).

It is not hard to verify that the roots of the corresponding polynomial are

$$P_1^* = 1 - \frac{K(c)}{\beta H_2(c_1,c,c)} \leq 1 \quad \text{and} \quad P_1^{**} = 1 - \frac{1}{\beta\pi(c|c)} \leq 0, \tag{52}$$

where the P_1^{**} clearly does not fall in the unit interval. Depending on the sign of P_1^*, firm 2 either does not invest for any level of P_1 (if $P_1^* < 0$) or to invest with probability one when $P_1 \leq P_1^*$ if P_1^* is positive.

Denote the relevant threshold for the investment cost to distinguish between these two cases as

$$K_1(c_1,c_2,c) = \beta H_2(c_1,c,c) = \beta\frac{h_2(c_1,c,c) + \pi(c|c)(c_1 - c)}{1 - \beta\pi(c|c)}, \tag{53}$$

where the second equality is due to (48). We conclude that the depending on the size of $K(c)$ firm 2's (the high cost follower) best response to firm 1 is given by one of the following two cases

$$\text{Case ① } \quad K(c) \geq K_1(c_1,c_2,c) \quad \Rightarrow \quad P_2(c_1,c_2,c,P_1) = 0 \text{ for all } P_1, \tag{54}$$

$$\text{Case ② } \quad K(c) < K_1(c_1,c_2,c) \quad \Rightarrow \quad P_2(c_1,c_2,c,P_1) = \begin{cases} 1, & \text{if } 0 \leq P_1 < P_1^*, \\ [0,1], & \text{if } P_1 = P_1^*, \\ 0, & \text{if } P_1^* < P_1 \leq 1. \end{cases}$$

Consider now the best response of firm 1 in states (c_1,c_2,c), $c_1 \leq c_2$ when it is the low cost leader. The polynomial coefficients (50) in this case do not simplify in any significant way, as $h_1(c_1,c_2,c) > 0$ and $r_1(c_1,c_2) = c_2 - c_1 > 0$. Consequently, the roots of the polynomial in the best response correspondence are rather involved. We can, however, calculate the values of the polynomial at the ends of the unit interval to get some insights into where the roots may be located. We have

$$\begin{aligned} A_1 + B_1P_2 + C_1P_2^2\Big|_{P_2=1} &= K(c), \\ A_1 + B_1P_2 + C_1P_2^2\Big|_{P_2=0} &= K(c)\Big(1 - \beta\pi(c|c)\Big) - \beta\Big(h_1(c,c_2,c) - h_1(c_1,c_2,c) + \pi(c|c)(c_1 - c)\Big) \end{aligned} \tag{55}$$

As long as $K(c) > 0$, the first line in (55) implies that the best response correspondence of firm 1 is zero at $P_2 = 1$ when firm 2 invests with certainty. But from the second line it follows that for $P_2 < 1$ the best response of firm 1 depends on the cost of investment $K(c)$. Namely, if

$$K(c) \leq K_2(c_1,c_2,c) = \beta\frac{h_1(c,c_2,c) - h_1(c_1,c_2,c) + \pi(c|c)(c_1 - c)}{1 - \beta\pi(c|c)} \tag{56}$$

the polynomial is negative at $P_2 = 0$, ensuring that a single root lies in the unit interval. In the opposite case, additional arguments are needed to differentiate between the case when both roots lie in the unit interval and the case when the polynomial has no real roots, but as it will become obvious below, both of these cases lead to the same set of equilibria. We conclude that the depending on the size of $K(c)$ firm 1 (cost leader's) best response to the investment probability of firm 2 is given by one of the following two cases

$$\begin{aligned} &\text{Case ①} \quad K(c) \geq K_2(c_1,c_2,c) \;\Rightarrow\; P_1(c_1,c_2,c,P_2)\Big|_{P_2=0} = 0, \\ &\text{Case ②} \quad K(c) < K_2(c_1,c_2,c) \;\Rightarrow\; P_1(c_1,c_2,c,P_2) = \begin{cases} 1, & \text{if } 0 \leq P_2 < P_2^*, \\ [0,1], & \text{if } P_2 = P_2^*, \\ 0, & \text{if } P_2^* < P_2 \leq 1, \end{cases} \end{aligned} \tag{57}$$

where P_2^* denotes the single root of the polynomial in the unit interval in the corresponding case.

Assume for the time being, that $K_2(c_1,c_2,c) \geq K_1(c_1,c_2,c)$. Then all possible stage game equilibria at points (c_1,c_2,c), $c_1 \leq c_2$ are given by the combinations of cases in (54) with (57). The following configurations are possible:

Type 1 $K(c) \leq K_1(c_1,c_2,c) \leq K_2(c_1,c_2,c)$: similar to the interior points of the bottom layer game (when $c=0$), there are three stage equilibria, two pure strategy anti-coordination equilibria, and one mixed strategy equilibrium,

Type 2 $K_1(c_1,c_2,c) \leq K_2(c_1,c_2,c) < K(c)$: there is one pure strategy equilibrium in which neither firm invests,

Type 3 $K_1(c_1,c_2,c) < K(c) \leq K_2(c_1,c_2,c)$: there is one pure strategy equilibrium in which firm 1 (cost leader) invests with probability one, and firm 2 (cost follower) does not invest.

It is straightforward to show that the inductive hypothesis holds for all three types of stage game MPE. The argument is trivial for the last two cases where firm 2 remains the cost follower, and the cost leader either continues earning positive profits, or earns additional profits after investing in the state of the art technology and keeping its leader status. For the first configuration choosing the mixed strategy equilibrium leads to the result through the argument based on the equality of the values of investing and not investing.

Similar derivations hold for the points (c_1,c_2,c), $c_1 \geq c_2$ where firm 1 is the cost follower and firm 2 is cost leader.

To complete the proof, it remains to be shown that $K_2(c_1,c_2,c) \geq K_1(c_1,c_2,c)$ in the points (c_1,c_2,c) where $c_1 \leq c_2$. Define a function $G(c_1,c_2,c) = K_2(c_1,c_2,c) - K_1(c_1,c_2,c)$. By simple algebra we have

$$G(c_1,c_2,c) = \frac{\beta}{1-\beta\pi(c|c)}\big[h_1(c,c_2,c) - h_1(c_1,c_2,c) - h_2(c_1,c,c)\big], \tag{58}$$

and thus $G(c_1,c_2,c)$ is the difference in the expected future value between the gains from investing by firm 1 (cost leader) and firm 2 (cost follower) in the point (c_1,c_2,c) (bearing in mind that $h_2(c_1,c_2,c) = 0$).

We prove the claim by a separate transfinite induction argument. The fact that $G(c_1,c_2,0) = 0$ establishes the base case for the induction. The inductive hypothesis is $G(c_1,c_2,c') \geq 0$ for all $c' < c$. Under the assumption that $G(c_1,c_2,c)$ is continuous in its third argument, and using the fact that nonnegative real line is a closed set, it immediately follows that $\lim_{c' \to c} G(c_1, c_2, c') = G(c_1, c_2, c) \geq 0$.

We establish the continuity of $G(c_1, c_2, c)$ by yet another separate transfinite argument that combines the inductive hypothesis (44), and the structure of equilibria that follows from it, with the hypothesis that $\{v_{N,1}(c_1, c_2, c'), v_{I,1}(c_1, c_2, c')\}$ and $\{v_{N,2}(c_1, c_2, c'), v_{I,2}(c_1, c_2, c')\}$ are continuous in c' for almost all points $(c_1, c_2, c') \in S$ with $c' < c$ (except the set of measure zero).

Consider the Bellman equations for the value functions given in equations (36) and (39). Suppose first that there is a Type 2, no investment MPE at (c_1, c_2, c). Suppose without loss of generality, that $c_1 \leq c_2$ (a symmetric argument covers the case where $c_1 > c_2$). By the inductive hypothesis the high cost firm 2 earns zero expected profits at any state (c_1, c_2, c') for $c' < c$, so from equation (41) we see that $h_2(c_1, c_2, c) = 0$ which implies from equation (39) that $v_{N,2}(c_1, c_2, c) = 0$, which is trivially continuous in c. By Assumption 1 (continuity of $K(c)$ and $\pi(c'|c)$), it follows that $h_2(c_1, c, c)$ is continuous in its third argument c, which implies that $v_{I,2}(c_1, c_2, c) = -K(c) + \beta h_2(c_1, c, c)$ is continuous in c. Similarly for the low cost firm, firm 1, we have that Assumption 1 also implies that $h_1(c_1, c_2, c)$ is continuous in c, so we have

$$v_{N,1}(c_1, c_2, c) = \frac{c_2 - c_1 + \beta h_1(c_1, c_2, c)}{1 - \beta\pi(c|c)} \tag{59}$$

from which follows that $v_{N,1}(c_1, c_2, c)$ is continuous in c. Further we can show that

$$v_{I,1}(c_1, c_2, c) = c_2 - c_1 - K(c) + \beta h_1(c, c_2, c) + \beta \frac{c_2 - c + \beta h_1(c, c_2, c)}{1 - \beta\pi(c|c)} \tag{60}$$

which is also continuous in c.

Now consider a Type 1 MPE, where there is a mixed strategy MPE at (c_1, c_2, c). Again, by the inductive hypothesis, it follows that $h_2(c_1, c_2, c) = 0$, which implies from equation (39) that $v_{N,2}(c_1, c_2, c) = v_{I,2}(c_1, c_2, c) = 0$ (which is also trivially continuous in c). Now consider the low cost firm, firm 1. Its expected profits are positive in state (c_1, c_2, c) are given by

$$\begin{aligned} v_{N,1}(c_1, c_2, c) &= \frac{c_2 - c_1 \beta h_1(c_1, c_2, c)}{1 - \beta\pi(c|c)[1 - P_2(c_1, c_2, c)]} \\ v_{I,1}(c_1, c_2, c) &= c_2 - c_1 - K(c) + \beta[1 - P_2(c_1, c_2, c)] H_1(c, c_2, c). \end{aligned} \tag{61}$$

By the induction hypothesis and the continuity assumption it follows that $h_1(c_1, c_2, c)$ and $H_1(c_1, c_2, c)$ are continuous functions of c. The root P_2^* defining the equilibrium probability $P_2(c_1, c_2, c)$ is a function of the coefficients A_1, B_1, C_1 which are continuous functions of c, which implies that $P_2(c_1, c_2, c)$ is continuous in c, which in turn implies that $v_{N,1}(c_1, c_2, c)$ and $v_{I,1}(c_1, c_2, c)$ are continuous functions of c when $c_1 \leq c_2$. A symmetric argument can be used to establish continuity in the case where $c_1 > c_2$. The continuity of $G(c_1, c_2, c)$ follows from the continuity of the value functions.

It remains to establish the symmetry of this mixed strategy MPE for the overall game. Consider the effect of permuting (c_1, c_2) so we now consider the MPE at state (c_2, c_1, c). By assumption $c_1 \leq c_2$ so firm 2 is now the low cost producer and firm 1 is the high cost producer. We can write the Bellman functions for these as we did above using the rewritten Bellman equations in equations (36) and (39) as follows

$$v_{N,2}(c_2,c_1,c) = \frac{c_2 - c_1 + \beta[1 - P_1(c_2,c_1,c))]h_2(c_2,c_1,c)}{1 - \beta[1 - P_1(c_2,c_1,c)]\pi(c|c)}, \tag{62}$$

where h_2 is defined in equation (41) above. By our inductive hypothesis it follows that $h_1(c_1,c_2,c) = h_2(c_2,c_1,c)$. Now consider $v_{I,2}$. We have

$$v_{I,2}(c_2,c_1,c) = c_2 - c_1 - K(c) + \beta[1 - P_1(c_1,c_2,c))]H_2(c_2,c,c), \tag{63}$$

where H_2 is defined in equation (40) above, and again by our inductive hypothesis we have $H_1(c,c_2,c) = H_2(c_2,c,c)$. Using these equivalences we can now see the equations for the values of firm in in state (c_1,c_2,c) are identical to the equations for firm 2 in state (c_2,c_1,c). It follows that the unique value of P_1 that sets $v_{N,2}(c_2,c_1,c) = v_{I,2}(c_2,c_1,c)$ in equations (62) and (63) above is the same as the P_2 that sets $v_{N,1}(c_1,c_2,c) = v_{I,1}(c_1,c_2,c)$ in similar equations for firm 1, so it follows that

$$\begin{aligned} P_1(c_2,c_1,c) &= P_2(c_1,c_2,c) \\ v_{N,1}(c_1,c_2,c) &= v_{N,2}(c_2,c_1,c) \\ v_{I,1}(c_1,c_2,c) &= v_{I,2}(c_2,c_1,c). \end{aligned} \tag{64}$$

Similarly we can write the value for the high cost firm which is now firm 1 when the state is (c_2,c_1,c) and $c_1 < c_2$. We have

$$\begin{aligned} v_{N,1}(c_2,c_1,c) &= 0 \\ v_{I,1}(c_2,c_1,c) &= -K(c) + \beta[1 - P_2(c_2,c_1,c)]H_1(c,c_1,c), \end{aligned} \tag{65}$$

which implies that

$$P_2(c_2,c_1,c) = 1 - \frac{K(c)}{\beta H_1(c,c_1,c)}. \tag{66}$$

Now we appeal again to the inductive hypothesis, which guarantees that $H_2(c_1,c,c) = H_1(c,c_1,c)$ and from this we can conclude

$$\begin{aligned} P_2(c_2,c_1,c) &= P_1(c_1,c_2,c) \\ v_{N,1}(c_2,c_1,c) &= v_{N,2}(c_1,c_2,c) \\ v_{I,1}(c_2,c_1,c) &= v_{I,2}(c_1,c_2,c). \end{aligned} \tag{67}$$

Analogous arguments hold for the case $c_1 \geq c_2$. It follows that the symmetry property of statement 3 of Theorem 2 also holds. $\square$

Statement 4. (If Assumption 1 (Continuity) holds, the convex hull of the set of the expected discounted equilibrium payoffs to the two firms in all MPE equilibria of simultaneous move game at the apex (c_0,c_0,c_0) is a triangle with vertices $(0,0)$, $(0,V_M)$ and $(V_M,0)$, where $V_M = V_i(c_0,c_0,c_0)$ is the expected discounted payoff of firm i which makes all investments in the monopoly equilibrium.).

Proof. Statement 2 of this Theorem ensures the existence of two monopoly equilibria in the simultaneous move game, proving that the two corner payoff points $(V_M, 0)$ and $(0, V_M)$ exist, where $V_M = v_{N,1}(c_0, c_0, c_0) = v_{N,2}(c_0, c_0, c_0)$ is the monopoly payoff at the initial node (apex) $(c_0, c_0, c_0) \in S$. Since the monopoly profit equals the full social surplus and is efficient, it is infeasible to obtain any payoff higher than the line segment joining these two monopoly payoff points,and thus all payoffs for all equilibria in the simultaneous move game (which are generally less than 100% efficient) must lie below the line segment joining the two monopoly payoff points. Finally, Statement 3 of this Theorem guarantees the existence of the zero payoff point at the origin $(0,0)$. Obviously the convex hull of these three payoff points equals the full triangle, and thus any point in this triangle can be an expected payoff to the two firms if we allow *stochastic* equilibrium selection rules (i.e. selecting one of these three possible "extremal equilibria" with probabilities (p_1, p_2, p_3) with $p_1 + p_2 + p_3 = 1$ and $p_i \geq 0$, $i \in \{1,2,3\}$). $\square$

Theorem 3 (The set of equilibrium payoffs in the Alternating Move Game).

Proof. By Theorem 1, if investment is optimal for the social planner, then inequality (25) cannot hold. By Lemma A.2, it follows that no investment cannot be an MPE outcome in the alternating move game similarly to the simultaneous move game.

Then it is sufficient to show that the origin is not an equilibrium payoff pair at the apex of the alternating move game if investment costs are not too high. We have already shown that no investment cannot be a MPE of the full alternating move game at the initial point $(c_0, c_0, c_0) \in S$. However if it is optimal for one of the firms to invest at some point on the equilibrium path, it must be because the firm expects a positive profit from doing so. However from the Bellman equation for the alternating move game, equation (9), if one or the other of the firms expects a positive profit from investing in some stage game on the equilibrium path, the expected profit for that firm at the initial apex of the game $(c_0, c_0, c_0) \in S$ cannot be zero. Thus there can be no zero expected profit mixed strategy MPE in the alternating move game.

Also, monopoly outcomes (for firm 1 and 2, respectively) cannot be supported as MPEs in the alternating move game since the firm that does not play the role of the monopolist would have to have no incentive to invest in any state, even when it is was its turn to invest. But as just one counterexample, consider the end game state $(c_1, c_2, 0)$ where it is socially efficient for the social planner to invest (i.e. $\beta \min(c_1, c_2)/(1-\beta) > K(0)$). Consider a monopoly equilibrium where firm 1 is the monopolist. However if it is firm 2's turn to invest, firm 2 will earn zero profit if it does not invest but a positive profit equal to $-K(0) + \beta c_1/(1-\beta)$ if it does invest, contradicting the hypothesis that this is a monopoly equilibrium.

We note that Theorem 6 implies that a zero payoff for both firms is approached in the limit as $\Delta t \to 0$ when $\pi(c_t|c_t) = 0$ and the order of moves alternates deterministically. However in that case, since the equilibrium is unique, it follows that the monopoly payoff vertices are not supportable in the limit as $\Delta t \to 0$. Thus, even in limiting cases, the set of equilibrium payoffs in the alternating move game will be a strict subset of the payoff triangle that excludes the origin (zero expected profit mixed strategy payoff) and the two monopoly payoffs.

$\square$

Theorem 4 (Sufficient conditions for uniqueness).
The proof requires some intermediary results.

Lemma A.6 (Efficiency of the alternating move end game)**.** *In the alternating move ($m \neq 0$) end game ($c = 0$) in every state $(c_1, c_2, 0)$ there is a unique efficient equilibrium, i.e. both firms invest when it is their turn to invest if and only if investment would be optimal from the point of view of the social planner.*

Proof. Consider the case where $c_1 < c_2$. The proof for the case $c_1 \geq c_2$ is symmetric to the one provided below for $c_1 < c_2$ and is omitted for brevity. Suppose that it is socially optimal to undertake investment, i.e. $\beta c_1/(1-\beta) - K(0) > 0$. We now show that in the unique equilibrium to the alternating move end game, both firms 1 and 2 would want to invest when it is their turn to invest, where uniqueness of equilibrium is a consequence of the uniqueness of the firms' best responses, and the fact that only one of the firm moves at a time. Consider firm 2's decision in this unique equilibrium. If firm 2 chooses to invest, its payoff is $v_{I,2}(c_1, c_2, 0, 2) = \beta c_1/(1-\beta) - K(0)$ and if it chooses not to invest its payoff is $v_{N,2}(c_1, c_2, 0, 2) = 0$ since it believes that firm 1 will invest at its turn with probability 1, which we will verify is true below. Thus, firm 2 will invest in equilibrium if and only if $\beta c_1/(1-\beta) - K(0) > 0$, which is the same condition for optimal investment by the social planner.

Now consider firm 1. At it's turn to move the payoff to investing is

$$v_{I,1}(c_1, c_2, 0, 1) = c_2 - c_1 + \beta c_2/(1-\beta) - K(0). \tag{68}$$

Since $c_2 > c_1$ and by assumption $\beta c_1/(1-\beta) - K(0) > 0$, it is easy to see that the payoff to investing is strictly positive for firm 1. However we must also show that this is higher than the payoff it would get from not investing. Since firm 1 knows that firm 2 will invest when it gets a chance to move, the value to firm 1 to not investing is given by

$$v_{N,1}(c_1, c_2, 0, 1) = c_2 - c_1 + \beta f(1|1)\left[c_2 - c_1 + \frac{\beta}{1-\beta} c_2 - K(0)\right] + \beta f(2|1)[c_2 - c_1]. \tag{69}$$

If the posited equilibrium holds (i.e. it is optimal for firm 1 to invest), then we must have $v_{I,1}(c_1, c_2, 0, 1) > v_{N,1}(c_1, c_2, 0, 1)$, and using the formulas for these values given above, this is equivalent to

$$\frac{\beta}{1-\beta} c_2 - K(0) > \frac{\beta(c_2 - c_1)}{1 - \beta f(1|1)}. \tag{70}$$

Notice that the right hand side of inequality (70) above is maximized when $f(1|1) = 1$ (i.e. when it is always firm 1's turn to invest) and in this case this inequality is equivalent to $\beta c_1/(1-\beta) - K(0) > 0$, confirming that for all $f(1|1) \in [0,1]$ it is strictly optimal for firm 1 to invest when it is its turn to invest when it is socially optimal for this investment to occur.

Now consider the converse situation where it is not socially optimal to invest, and $\beta c_1/(1-\beta) - K(0) < 0$. Following the same reasoning as above, it is easy to see that it is not optimal for firm 2 to invest when it is its turn to invest since firm 2's payoff to investing is $v_{I,2}(c_1, c_2, 0, 2) = \beta c_1/(1-\beta) - K(0) < 0$ and its payoff to not investing is $v_{N,2}(c_1, c_2, 0, 2) = 0$. Now we must show that firm 1, knowing that firm 2 will not want to invest at its turn, will also not want to invest when it is its turn. If firm 1 never invests, its payoff is

$$v_{N,1}(c_1, c_2, 0, 1) = \frac{c_2 - c_1}{1 - \beta}, \tag{71}$$

and if it invests, its payoff is given by the same formula for $v_{I,1}(c_1, c_2, 0, 1)$ as given in equation (68) above. So the condition for investment not to be optimal for firm 1 is $v_{N,1}(c_1, c_2, 0, 1) > v_{I,1}(c_1, c_2, 0, 1)$ which is algebraically equivalent to $\beta c_1/(1-\beta) - K(0) < 0$, the condition for when it is not socially optimal for investment to occur.

□

Now we have the key result needed to prove Theorem 4.

Proof. When $\pi(c|c) = 0$, the probability of remaining in any given state $(c_1, c_2, c) \in S$ is also zero. Using the Bellman equations (9) defining the firms' value functions for investing and not investing when it is their turn to invest, it is not difficult to see that each firm's values are independent of the probability that their opponent will invest in this case. That is, for firm 1 we have $v_{N,1}(c_1, c_2, c, 1)$ and $v_{I,1}(c_1, c_2, c)$ are independent of $P_2(c_1, c_2, c, 2)$, the probability that firm 2 will invest when it is its turn to invest. This implies that the probability that firm 1 will invest, $P_1(c_1, c_2, c, 1)$, is also independent of $P_2(c_1, c_2, c, 2)$, as it is given by formula (7) of section 2, which shows that $P_1(c_1, c_2, c, 1)$ is a logistic function of $v_{N,1}(c_1, c_2, c, 1)$ and $v_{I,1}(c_1, c_2, c, 1)$, both of which are independent of $P_2(c_1, c_2, c, 2)$. Similar arguments hold for firm 2, so that $P_2(c_1, c_2, c, 2)$ is independent of $P_1(c_1, c_2, c, 1)$. Since the value functions $(v_{N,1}, v_{I,1}, v_{N,2}, v_{I,2})$ can be calculated recursively using the Bellman equations (7), and since Lemma A.6 establishes that there is always a unique (efficient) equilibrium in the end game states (c_1, c_2, c), it follows that at every state $(c_1, c_2, c) \in S$ there is a unique stage game equilibrium with probabilities of investing given by $(P_1(c_1, c_2, c, 1), P_2(c_1, c_2, c, 2))$, which depend on the value functions $(v_{N,1}, v_{I,1}, v_{N,2}, v_{I,2})$ that are defined recursively via the Bellman equations (9).

□