STATIONARY AND NONSTATIONARY EQUILIBRIUM IN THE AUTOMOBILE MARKET*

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Abstract

This paper describes a simple infinite horizon model for constructing equilibria in a stationary economy (i.e. with no macro or fuel price shocks) and in a nonstationary economy (where we allow for macro and fuel price shocks) with and without transactions costs. The model provides a test bed for methods used for solving and estimating dynamic structural equilibrium models for automobile markets. We establish the existence of multiple Pareto-ranked equilibria, and provide algorithms for computing equilibria of these models in a variety of increasingly realistic scenarios, demonstrating that this is a rich class of models that are a flexible basis for empirical work, and for studying issues of identification of structural dynamic equilibrium models of the automobile market.

KEYWORDS: Structural estimation, Durable assets, Dynamic Equilibrium models, Transaction cost, NPL, NFXP, MPEC.

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1 Introduction

Since Berkovec (1985) and Rust (1985c) economists have estimated and numerically computed equilibria automobile markets and demonstrated the importance of secondary markets for the allocation of new and used durable goods [1]. A central aspect of the dynamics of vehicle markets is the existence of transaction cost. While Rust (1985c) computes the stationary equilibrium in a model without transaction cost, Konishi and Sandfort (2002) prove the existence of a stationary equilibrium in the presence of transaction and trading costs. Yet, our theoretical understanding and ability to "exactly" solve and estimate dynamic general equilibrium for automobile markets is still far from perfect.

This paper describes a simple infinite horizon model for constructing equilibria in a stationary economy (i.e. with no macro or fuel price shocks) with and without transactions costs to provide a test bed to check that the equilibrium solver we develop finds the correct equilibrium. In the process of doing this, we discovered the possibility of multiple Pareto-ranked equilibria in the simplest model with homogeneous consumers and zero transactions costs. We consider this model first, and then describe a model with heterogeneous consumers and when transactions costs are allowed. This second model encompasses the first as a special case in the limit as we allow transactions costs to tend to zero and the degree of consumer heterogeneity to become a degenerate distribution corresponding to a homogeneous consumer economy.

The remainder of the paper is structured as follows. The next section provides review of the previous literature. Section 3 develops a simple dynamic equilibrium model for automobile markets both with and without transactions costs. Section 5 describes how we solve for equilibria in a stationary economy. Section 7 discusses our estimation approach. Section 8 presents a series of simulation results and Section 9 concludes.

2 Previous literature

This paper builds on and contributes to three different literatures: 1) discrete/continuous choice of durable goods, where there is a discrete choice of type of durable (including attributes such as the durable's energy efficiency) and where the continuous choice represents usage of the durable (such as driving in the case of automobiles), 2) numerical and theoretical models of equilibrium in automobile markets, and 3) structural estimation of dynamic choice models, in-

[1](Rust, 1985c; Anderson and Ginsburgh, 1994; Hendel and Lizzeri, 1999a,b; Stolyarov, 2002; Gavazza, Lizzeri and Roketskiy, 2014), as well as the influence of durability on the dynamics of vehicle demand (Adda and Cooper, 2000a; Stolyarov, 2002; Esteban and Shum, 2007; Chen, Esteban and Shum, 2013).

cluding dynamic discrete choice models applied to choice of automobiles. We provide reviews of each of these literatures below. Most of these literatures emerged after the oil price shocks and concern about permanently higher fuel prices in the late 1970s. Since that time fuel prices have increased but not as dramatically as once feared. Instead attention has refocused more recently on concerns about the effects of vehicle emissions on the environment, with particular concern about CO_2 emissions and its impact on global warming.

2.1 Discrete-continuous Models of Durables

This literature goes back to Dubin and McFadden (1984), where households choose electrical appliances taking into account their future usage of the durable. The key insight is that the usage falls out of Roy's identity. Models of this type place strict cross-equation restrictions on the parameters of the model in the sense that they force the consumer to be time-consistent in treating money in the same way when making the purchase decision and the usage decision.[2] Earlier work on discrete-continuous choice models tended to use two-step approaches (Mannering and Winston, 1985; Goldberg, 1998; West, 2004). More recently, applications to car choice and usage have featured simultaneous estimation of both choice margins (Feng, Fullerton and Gan, 2005; Bento, Goulder, Jacobsen and von Haefen, 2009; Jacobsen, 2013). For example, Bento, Goulder, Jacobsen and von Haefen (2009) use their model to analyze the distributional impacts of fuel taxes in the US. In their model, the discrete choice is the car choice and the continuous choice is how much to drive the car. Gillingham (2012) also uses a discrete-continuous model applied to car choice and use and focuses on the selection of consumers based on anticipated driving and allowing for selection on observed and unobserved factors. Munk-Nielsen (2015) applies a similar model to new car sales in Denmark to study the costs of environmental taxation. The model admits an estimate of the so-called "rebound effect", the effect on driving from an exogenous increase in fuel efficiency. This important policy parameter has been widely discussed and estimated (e.g. Small and Van Dender, 2007; Hymel and Small, 2015).

Engers, Hartmann and Stern (2009) study the interrelationship between vehicle usage and price depreciation in the used car market. They argue that "changes in a vehicles net benefits, proxied by annual miles, explain the observed decline in used car prices over the vehicle's life." (p. 29). They find that households drive fewer miles per year the older their car is, and estimate a structural model of household choice of driving and vehicle type that differs from

[2] Whether consumers accurately take into account future savings in fuel costs is widely discussed in recent empirical work (Allcott and Wozny, 2012; Busse, Knittel and Zettelmeyer, 2013).

the literature surveyed above. They conclude that "the structural model of household mileage decisions better explains the observed price decline in used car prices." and "the observed decline in used car prices as a vehicle ages is best explained by decomposing the age effect into three components: the direct aging effect, the household portfolio effect, and the household demographics (or car turnover) effect." (p. 30).

2.2 Models of Equilibrium in Automobile Markets

This paper builds on a theoretical and empirical literature for modeling equilibrium in the market for automobiles. The earliest work that we are aware was by Manski (see Manski (1980), Manski and Sherman (1980) and Manski (1983)). We believe Manski's original work stimulated the subsequent chain of research on micro-econometrically estimable equilibrium models of the automobile market, and his work provided both theoretical models of equilibrium in new and secondhand auto markets, and numerical calculation of equilibrium prices and quantities that demonstrated how these models could be used for policy forecasting of a wide range of policies of interest. Manski and Sherman (1980) did their pioneering work in an environment around the first large oil price shocks in the late 1970s when it first became clear that gasoline prices would inevitably rise and there would be a demand for increasingly fuel efficient vehicles. They concluded that "our initial research on developing and applying a disaggregate modeling approach to forecasting future motor-vehicle sales and holdings has proved highly encouraging. Our results are really the beginning of an ongoing need to analyze and monitor the motor vehicle market through the 1980s. … With an eye toward improvement of our models, future work should seek to further illuminate the linkages that connect household behavior in choosing motor vehicles and other vehicle-related decisions. In particular, a joint analysis of ownership level, the composition of holdings, and vehicle use would be a valuable contribution." (p. 103).

The contributions of Manski and coauthors inspired further work such as the 1983 PhD thesis research by James A. Berkovec at MIT (subsequently published as Berkovec (1985)) who followed the footsteps of Manski and Sherman (1980) and developed the second microeconometrically estimated and numerically solved large scale equilibrium model of the new and used car markets that we are aware of. The contributions of Manski and coauthors, and Berkovec was extremely advanced given the limits of computing power at the time, and still represents the closest point of departure and template for our own work in this area.

Berkovec described his model as a "short run" equilibrium model as it was based on a model where consumer *expectations* about depreciation rates of their vehicles was estimated econometrically using data on new and used car prices in 1978. Berkovec assumed that consumers choose vehicles based on a quasi-linear utility function that is an additively separable sum of a utility for car attributes (with declining utility for cars of older ages) less the disutility of the "expected capital cost" of owning the vehicle. The expected capital cost is essentially the expected depreciation of holding the vehicle plus maintenance costs, using the econometrically estimated depreciation rates.

Berkovec assumed that consumers choose a vehicle that maximizes their utility where the price of the vehicle enters via the expected capital cost. He used a nested logit discrete choice model that allow for patterns of correlation in the unobserved components of the utility of a vehicle that capture patterns of similarity in the unobserved characteristics of vehicles in 13 different car classes that he used in his analysis (e.g. luxury cars, compact cars, vans, pickups, etc). He developed and estimated separate microeconometric nested logit discrete choice models for households that own 1, 2 and 3 cars, respectively.

Using the microeconometrically estimated choice model, Berkovec constructed an "expected demand function" for vehicles of different ages and classes by summing the estimated discrete choice probabilities for cars of each age and class. He defined an equilibrium to be a vector of prices (with one price for each possible age and price of car) that equates the expected demand for vehicles of each car age and type to the actual supply of such vehicles, net of scrappage. Berkovec used a probabilistic model of vehicle scrappage due to Manski and Goldin (1983) where the probability a vehicle is scrapped is a decreasing function of the difference between the second-hand price of the car (net of any repair costs) and an exogenously specified scrap value for the vehicle. This implies that, except for random accidents, there is very little chance that new cars are scrapped, but the probability a used car is scrapped increases monotonically with the age of the car.

Berkovec used Newton's method to compute the equilibrium prices in the market. For the problem he analyzed there were 131 vehicle class/age price categories. At the time Berkovec did his work, inversion of the 131×131 Jacobian matrix of excess demands necessary to implement Newton's method was a much bigger computational challenge than it is today. Berkovec showed that the Jacobian matrix had special structure he called "identity outer product" that enabled him to invert the Jacobian via inverting a smaller 48×48 matrix and doing some additional matrix vector multiplications. Though Berkovec's paper did not discuss the equilibrium prices implied by his model, he concluded that "Overall, the simulation model forecasts appear to do reasonably well for the 1978- 1982 period. Although there are discrepancies in specific areas (as would be expected because of underlying macroeconomic fluctuations), the general trends evident in the data would seem to be captured in the forecasts." (Berkovec (1985), p. 213).

Subsequent work on empirical equilibrium modeling of the auto market includes the paper by Bento, Goulder, Jacobsen and von Haefen (2009) that estimated a micro level model of automobile driving and model/age choice using a sample of 20,429 U.S. households from the 2001 National Transportation Survey. Using microaggregated demands from the estimated discrete choice model, they numerically solved for equilibrium in the new and used car markets for a total of 284 composite age/model vehicle classes. They used their model to predict the impact of a 25 cent increase in the U.S. gasoline tax. Their model predicts that most of the response to this tax increase is via reduced driving: they found negligible longer run substitution to more fuel efficient vehicles or to non-automobile modes of transportation: "the size of the vehicle fleet falls about 0.5 percent" but "The impacts on new and used car ownership differ substantially over time. In the first year of the policy, the reduction in vehicle ownership comes largely by way of a decline in new car purchases. However, the ratio of fuel economy of new to old vehicles increases over time, and the increased gasoline tax gives greater importance to fuel economy. As a result, the decline in new car ownership is attenuated over time, and by year 10 the reduction in car ownership applies nearly uniformly to new and used vehicles." (p. 697).

A separate more theoretically oriented thread of the literature focused on modeling the role and benefits of the secondary market for automobiles (or more generally for other durable goods) in frameworks where the dynamics of trading were more explicitly modeled relative to the work surveyed above. Rust (1985c) established the existence of a stationary equilibrium in a market for new and used durable assets that provided a theoretical rationale for the conditions under which consumer choice of a stochastically deteriorating durable good (e.g. an automobile) involves a trade-off between the utility provided by the durable and its expected price depreciation. He showed that the key condition for this to hold is that there are *zero transactions costs* in the market. When this holds, the optimal strategy for each consumer in a stationary equilibrium (i.e. one where there are no macro shocks or other time-varying factors altering the prices or quantities of vehicles in the market) involves trading each period for the preferred age/condition of car $x^*(\tau)$ where τ is a parameter that indexes heterogeneity among consumers, e.g. differential preferences for "newness" or different degrees of wealth that affect consumer willingness to pay for newer/better condition durables. Similar to Berkovec, Rust assumed that per period preferences for durables are quasi-linear in the attributes of the durable and in income, which is a simple way of representing preferences for all other goods without explicitly modeling them.

Unlike Berkovec, who considered a discrete set of car classes and ages, Rust modeled a durable as having a *state* x, where $x = 0$ corresponds to a brand new durable good and higher values of x correspond to more deteriorated, less desirable older durable goods. For example in the case of automobiles, x might be the odometer on the car, and consumers may be more concerned about the level of wear/tear on a car as represented by the odometer value x than the discrete age of the car. In this framework an equilibrium requires finding a *price function* $P(x)$ that clears the market (i.e. sets the demand for durables of each condition x equal to the supply). The supply of durables is represented by another function $S(x)$ that Rust called a *holdings distribution* — it is the fraction of durables in the economy with condition less than or equal to x. If each vehicle deteriorates stochastically, according to a Markov transition probability $f(x'|x)$ (where x' is the condition of the durable next period given that its condition is x this period), then in a stationary equilibrium with a continuum of agents, Rust showed that there will be a stationary distribution $S(x)$ that is related to the *invariant distribution* of the Markov transition probability $f(x'|x)$.

Rust assumed that there was an infinitely elastic supply of new durables at an exogenously fixed price $\overline{P}$ and an infinitely elastic demand for scrap at an exogenously fixed price $\underline{P} < \overline{P}$, and provided sufficient conditions for a *stationary equilibrium* $(P(x), S(x))$ that satisfies the conditions 1) $P(0) = \overline{P}$, 2) there is a *scrap threshold* $\gamma > 0$ such that $P(x) = \underline{P}$ if $x \geq \gamma$, and 3) there is equilibrium for all conditions $x \in (0, \gamma)$, i.e. the fraction of consumers who wish to hold a durable with condition less than or equal to x equals the stationary holdings distribution $S(x)$. Rust called condition 3) *stock equilibrium* i.e. it amounts to the usual condition that the demand for every condition of car equals the supply in the case of a continuum of goods x. Rust also showed that a stationary equilibrium also implies a condition he called *flow equilibrium* i.e. the fraction of used durables that are scrapped each period equals the fraction of the population that buys new durables. This implies that the overall stock of durables in the economy is not changing over time.

In subsequent work Rust (1985a) showed that when applied to the automobile market, a calibrated version of the stationary holdings distribution implied by Rust (1985c) provides a good approximation to the joint distribution of ages and odometer values (x) in the US economy using data from the 1970s. He also showed that for a range of plausible utility functions for consumers, the stationary equilibrium resulted in *convex price functions* $P(x)$, which implies the rapid early depreciation for new cars and the slower depreciation for older cars that we observe in most auto markets. However the assumption of zero transactions costs is an unrealistic feature of his model as it implies that it is optimal for consumers to trade every period for their preferred condition $x^*(\tau)$ and this is something we definitely do not observe in real world automobile markets. When there are transactions costs (which are separate from *trading costs*, i.e. the difference between the list price of a car x a consumer wishes to buy, $P(x)$ and the list price $P(x')$ of an older car x' that the consumer wishes to trade in exchange for the newer car x), Rust (1985c) showed that the optimal strategy generally involves keeping the current durable for multiple periods. In a stationary market the optimal trading strategy in the presence of transactions costs consists of two thresholds $(\underline{x}^*(\tau), \overline{x}^*(\tau))$ where $\underline{x}^*(\tau) < \overline{x}^*(\tau)$ and $\underline{x}^*(\tau)$ is the condition of the optimal replacement durable that a consumer will choose whenever he/she replaces their current durable x, but $\overline{x}^*(\tau)$ is a *selling threshold* and it is not optimal to replace the current durable x until x exceeds the selling threshold $\overline{x}^*(\tau)$. When $x > \overline{x}^*(\tau)$ the consumer of type τ sells their current durable x for $P(x)$ and buys a replacement durable of condition $\underline{x}^*(\tau)$ for price $P(\underline{x}^*(\tau))$. Notice that generally $\underline{x}^*(\tau) > 0$, so the replacement durable is generally not a brand new durable good $\underline{x}^*(\tau) = 0$. However consumers who are sufficiently rich or who have a sufficiently strong preference for "newness" will replace their used durable with a brand new one.

Establishing the existence of a stationary equilibrium in the presence of transactions costs is a much more daunting undertaking due to the possibility that there may be consumer types τ who desire to buy a slightly used but not completely brand new durable $\underline{x}^*(\tau)$ yet, there may not be any consumer type τ' whose optimal strategy involves buying a brand new durable whenever they replace their old one who has a selling threshold $\overline{x}^*(\tau') < \underline{x}^*(\tau)$. That is, there is no automatic guarantee that there will be someone willing to sell a sufficiently new durable good to another consumer who wishes to buy a very new but not brand new durable good — perhaps to try to take advantage of the the rapid early depreciation in durables and buy an "almost new" durable good for a price that is much lower than the price of a new durable good $\overline{P}$. However Konishi and Sandfort (2002) did prove that a stationary equilibrium can exist in the presence of transactions costs under certain conditions. Their proof shows that it is possible for the equilibrium price function $P(x)$ to adjust to prevent any of the coordination failures of the type discussed above, i.e. where some consumer type τ' wishes to buy some sufficiently new durable good $\underline{x}^*(\tau')$ but no other consumer type τ is willing to sell their used durable to that consumer.

There is also a growing literature on the interaction between the market for new and used durable goods. Generally, the secondary market increases the lifetime of a durable good by facilitating a string of trades from customers who prefer newer durables and sell to a sequence of customers who are either poorer or who have weaker preferences for new durable goods relative to older ones. If the secondary market does not exist, each consumer can of course simply "buy and hold" — that is, buy brand new durable goods and hold them until they decide to scrap the old one and then buy another brand new replacement. Rust (1985c) showed that if consumers are homogeneous, they are indifferent between following such a buy an hold strategy, or trading each period for a preferred durable good in the secondary market. Thus, the existence of a secondary market does not produce any net welfare gain when consumers are homogeneous. However if consumers are heterogeneous, there is a welfare gain from the existence of a secondary market, and durable goods will have a longer lifespan on average when there is a secondary market than when it does not exist. Intuitively, the secondary market enables a chain of "hand me downs" of an aging durable good that would not be possible in the absence of a secondary market, and hence durables will live longer before they are scrapped when a secondary market exists, and consumers will be strictly better off compared to situation where there is no secondary market. In fact, Figure **??** indicates that the most common a 15 year old car is to have had five owners.

However a secondary market is not necessarily desired by producers of new durable goods, because it allows consumers to keep used durable goods longer. Since used durable goods serve as a substitute for new durable goods, the existence of a secondary market limits a firm's ability to extract rents from consumers via sales of new durable goods, i.e. in the "primary market." A standard solution to this problem is, in the case of a monopolist producer of durable goods, for the monopolist to rent rather than sell durable goods. Then the monopolist has the ability to control when durables are scrapped and extract rents from consumers without the distortions caused when durables are sold. When a monopolist sells new durables but attempts to set a high price, consumers react by keeping their used durables longer to reduce how frequently they have to replace their durables and thus pay the high price to the monopolist. However if a limitation to rental contracts only is not feasible, Rust (1985d) showed that a monopolist producer of new durable goods has an incentive to limit competition provided by the existence of a market for used durable goods by engaging in "planned obsolescence" — i.e. selling new durable goods that deteriorate more quickly than would be optimal under a social planning solution. In extreme cases the monopolist might even find it optimal to kill off the secondary market by producing goods with zero durability.

Esteban and Shum (2007) and Chen, Esteban and Shum (2013) developed empirically implementable models of equilibrium in new and used automobile markets and used these models to study the effect of the existence of a secondary market on oligopoly competition between new car producers in the primary market. Esteban and Shum (2007) formulated a model of oligopoly competition in new car markets under the assumption that a secondary market exists and there are zero transactions costs. Under their assumptions, demand for various new car models are linear functions of price, and the firms' profit functions are quadratic functions of current and future production levels, which implies that a Markov perfect equilibrium exists in strategies that specify the auto companies' production quantity decisions that are linear functions of a vector of the stock of cars produced prior to the current period that are still traded in secondary markets. Though the authors reported "difficulty of the theoretical model in generating price patterns similar to those observed in the data" (p. 345), they are able to use their model to study the effect of a temporary elimination of the secondary market on production decisions in the new car market. "Overall, we find that aggregate new-car production would increase by 12.08% for the 1987–1990 time frame were the secondary market to disappear temporarily." (p. 349).

Chen, Esteban and Shum (2013) estimated a model of dynamic oligopolistic competition in the new car market allowing for the existence of a secondary market in each car brand (make/model) sold in the primary market, and allowing for the possibility of positive transactions costs. Their econometrically estimated transaction cost was $4,400, and they note that "This is corroborated by the Kelley Blue Book, which indicates that, typically, the difference between the trade-in value of a used car (seller's price for consumers) and its suggested retail value (buyer's price) — which may serve as a proxy for the transactions cost — is in the $3,000 to $4,000 range." (p. 2922). They conduct counterfactual experiments by varying the transactions cost parameter from a value large enough to result in the closure of all secondary markets (for transactions costs larger than $8,000), to a value of $0, which corresponds to the case of a "frictionless" and active secondary market with no transactions costs. They find that relative to the equilibrium where there are no secondary markets, the case where there are active secondary markets with no transactions costs lowers firms' profits in the primary market by 35 percent. They also find that "when the secondary market becomes more active, firms have a stronger incentive to make their cars less durable." (p. 2929).

The final study on equilibrium in automobile markets that is most relevant to this paper is Gavazza, Lizzeri and Roketskiy (2014). The focus of their analysis is to quantify the welfare benefit of the secondary market and to investigate the effect of transactions costs on consumer trading and welfare. They formulated and numerically solved a dynamic model of vehicle holding that allows for the presence of transactions costs, and similar to the Chen, Esteban and Shum (2013) study, they used a discrete state model where automobiles are distinguished by their discrete age t rather than the continuous state framework that Rust (1985c) and Konishi and Sandfort (2002) used. Rather than focusing on the effects on profits of firms in the primary market, the focus of Gavazza, Lizzeri and Roketskiy (2014) was on welfare of consumers in the secondary market, and how welfare is affected by changes in transactions costs. They find that a calibrated version of their model "successfully matches several aggregate features of the US and French used car markets." and that "Counterfactual analyses show that transactions costs have a large effect on the volume of trade, allocations, and the primary market. Aggregate effects on consumers surplus and welfare are relatively small, but the effect on lower-valuation households can be large." (p. 3668).

While our review of the literature shows that there has been tremendous progress in both theoretical and empirical modeling of equilibrium in automobile markets, one of the gaps in the literature is the absence of work on modeling the effects of macroeconomic shocks. Since automobiles are among the most expensive durable goods outside of housing, it should not be surprising that macroeconomic fluctuations can have a huge effect on the timing of household purchases of new cars. In particular, when the economy is in a recession or about to go into recession, households worry about heightened risks of unemployment if they have not experienced unemployment already. Precautionary motives as well as tightened budget constraints appear to induce customers to hold onto their existing durables longer and wait to replace them until better times when they start to have more optimistic expectations about their employment prospects and earnings potential. We have already seen evidence of this in the descriptive graphs in section 2.3. Other analyses that have found similar effects include Adda and Cooper (2000a) and Adda and Cooper (2000b). However these studies have not modeled equilibrium in the primary and secondary markets in the presence of macro shocks. The cyclical variations in purchases of new cars generate slowly evolving "waves" in the stock of used cars as we illustrated in our descriptive analysis of the Danish data in section 2.3. Prices in the secondary market must adjust dynamically to enable the wave in the "supply" of used cars from a previous macroeconomic boom period to match the demand. Thus, both quantities and prices in an automobile market that is subject to macroeconomic shocks do not satisfy the conditions for "stationary equilibrium" that has been the focus of analysis in virtually all of the existing literature that we are aware of.

A major reason why there has been little work on modeling equilibrium in a non-stationary environment with macro shocks and other time-varying factors affecting consumer demand for automobiles is due to the complexity in modeling the dynamics of equilibrium prices in the presence of a dynamically evolving stock of vehicles in the economy. Since the stock of vehicles that have not been scrapped that have been been inherited from the previous period affects the supply of various ages of vehicles that will be supplied to the market, it follows that potentially consumers would need to know the entire age distribution of the vehicle stock to help predict market prices and how they will co-evolve over time along with the macro economic shocks and other time varying variables such as fuel prices that affect new car purchase, scrappage of old cars, and decisions on whether to sell or keep existing used cars. In principle, a high dimensional object — the entire age distribution of the automobile stock — needs to be on of the "state variables" that individuals need to keep track of to improve their forecasts of future auto prices. However due to the well known "curse of dimensionality" of dynamic programming, it becomes computationally infeasible to incorporate such high dimensional state variables in consumers' optimization problems.

In this paper we follow an approach of Krusell and Smith (1998) that avoids the curse of dimensionality of carrying the entire age distribution of cars as a state variable in the model and instead using "summary statistics" to capture movements in this distribution over time. In the problem Krusell and Smith (1998) studied, consumer heterogeneity implies that it is generally necessary to know the entire distribution of wealth in the economy to determine interest rates, which in turn affect individual consumers' savings decisions. However they showed that consumers can make highly accurate forecasts of future interest rates if they only keep track of the *mean value of wealth* (i.e. the mean of the distribution of wealth). Specifically, they found that the R^2 of regressions of current interest rate on mean wealth holdings in the economy was very high: typically over 97% in the numerical solutions and simulations of their model. This suggests that it is not necessary to confront the huge computational burden of carrying the entire distribution of wealth as a state variable to provide good forecasts of future interest rates.

In our paper we follow their insight and do not attempt to carry the entire distribution of car types and ages as a state variable in our dynamic programming model that we assume consumers solve to determine their holdings and trading decisions for vehicles. In fact, we go a step further and do not even attempt to use the mean ages of different vehicle types (in analogy to what Krusell and Smith did) as state variables that consumers use to forecast future automobile prices. Instead we assume that sufficiently good forecasts can be obtained using a flexibly parameterized price function of the form $P(\tau, a, p, m)$ where τ is the type of car, a is the age of a vehicle, and (p, m) capture the current fuel price and macro state (which are assumed to evolve as an exogenous Markov process). We use a flexibly parameterized price forecasting function and find that it enables consumers to provide very good forecasts of future auto prices for different ages and types. It appears that there are high substitution elasticities for demands for vehicles of different ages of a given type, as well as high substitution elasticities for the decision to sell existing used cars, so even when there are pronounced "waves" in the stock of vehicles caused by macro shocks, these waves do not result in pronounced waves in the prices of vehicles due to the high substitution elasticities. That is, it is not necessary for prices to adjust dramatically over time to equate supply and demand for cars of different ages in response to various shocks and dynamic factors that lead to bunching and waves in the stock of vehicles.

2.3 Estimation of Dynamic Discrete Choice Models

This paper extends the literature by using fully dynamic models of individual households' decisions about which vehicles to hold and to trade. As we noted above, most of the previous models in this literature ignored the fact that consumer decisions about automobiles are inherently dynamic choices. Previous empirical models of household choices such as as Berry, Levinsohn and Pakes (1995), Goldberg (1998), or Petrin (2002) focused on household choice of new vehicles only, and did so using a static discrete choice modeling approach. As we noted, the earliest empirical, disaggregate discrete choice models of equilibrium in the automobile market such as Manski and Sherman (1980) and Berkovec (1985) did estimate discrete choice models of holdings that allowed consumers to choose both new or used cars, but they also adopted a static choice perspective that treated consumers as making these choices every period, which would potentially result in excessive amount of trading of cars relative to what actually occurs.

As we noted above, when there are zero transactions costs, the assumption that consumers trade their existing cars for another new or used car every period can be rigorously justified, but this is clearly not an empirically realistic assumption. In the presence of transactions costs, households face a decision of whether to keep their current vehicle versus to trade for another new or used one. A literature on dynamic discrete choice, originating in the late 1980s (see, e.g. Rust (1985b)) provided the econometric methods for structural estimation of dynamic discrete choice models. This is a very flexible class of models that model probabilistic discrete dynamic choice models where the values or discounted utilities of choosing various discrete alternatives in each period are computed from the solution to a dynamic programming problem. These models can readily accommodate transactions costs and result in predicted behavior that is much closer to what we actually observe, specially with regard to the frequency at which households trade their existing vehicle for another one.

Schiraldi (2011) is an example of the application of a micro-based dynamic discrete choice modeling approach to study holding and tradings decisions of Italian households, but using *aggregate data.* Shiraldi takes prices of new and used cars in the Italian market as exogenously determined from the standpoint of individual households, and formulates and solves an individual households' optimal holding and trading strategy for vehicles to maximize their discounted expected lifetime utility. Using microaggregation of the individual consumer decision rules implied by the dynamic programming problem, Schiraldi was able to predict the aggregate vehicle holdings and trading patterns for the Italian economy as a whole, and he estimated the parameters of model using a simulated method of moments estimation strategy that finds parameter values for household preferences that enable the predicted, simulated moments to best match a set of actual moments characterizing aggregate holdings and trading of different types of vehicles over the period 1994 to 2004.

A novel feature of Schiraldi's analysis is to allow households to be "uncertain about future product attributes but rationally expect them to evolve, based on the current market structure." He captures this uncertainty using a variable he calls the "mean net augmented utility flow" arguing that "In a durable-goods setting, where the quality of the goods changes over time and there is the possibility of reselling, consumers maximize the utility derived from the good in any particular period net of the implicit rental price paid in that period to keep the good. Hence, the net augmented utility flow seems a natural index that captures the per-period quality adjusted by the price that consumers take into account to make their decisions." (p. 274). Schiraldi estimates significant transaction costs, with mean transactions cost equal to about 3200 Euros in 1994 that slowly decline over time. It is interesting that these estimates are in the same ballpark as those provided by Chen, Esteban and Shum (2013) for the U.S. market.

We are not aware of any dynamic discrete choice model of household-level holdings and trading of vehicles that has been estimated using disaggregate household-level choice data, and believe this is one of the contributions of this paper. With aggregate data, it is impossible to observe how long individual households keep their vehicles before they are traded, nor it is possible to say much about the heterogeneity in vehicle choices, such as which types of households choose to hold newer cars and which choose older ones.

Cho and Rust (2010) provide an analysis of the vehicle trading behavior of a large rental car company. Unlike most households, rental car companies typically buy brand new vehicles and sell them very quickly, typically once the car is one or two years old. Due to the rapid initial price depreciation of vehicles that we observe in most car markets, this strategy would prove to be very expensive one and this is why we see few households except the very wealthiest ones following this type of trading strategy. Another interesting feature of the rental car market that Cho and Rust (2010) point out is that rental car prices are typically *flat* as a function of age or odometer value, whereas they argue that predictions of most models of equilibrium in a competitive auto market are that rental prices should be declining functions of age or odometer value, reflecting the decline in prices and price depreciation rates in the used car market as a function of these variables. Cho and Rust (2010) argue that the trading strategy of the rental car company they analyze is also "too expensive" in the sense that it is suboptimal from a profit maximization perspective. Cho and Rust (2010) perform counterfactual analyses using a microeconometrically estimated dynamic programming model that show that the car rental company could significantly increase its profits by keeping its rental cars longer and discounting the rental prices of older rental vehicles to induce its customers to rent them. Their findings caused the rental car company to undertake a controlled experiment to verify the predictions of their model and the company did indeed find that profits did increase significantly from shifting to the recommended policy of discounting rental prices of older cars and keeping rental cars roughly twice as long as the company keeps its cars under its *status quo* operating policy.

There has been comparatively little work on solution and estimation of dynamic models of discrete and continuous choice beyond some recent work in this area such as Iskhakov, Jorgensen, Rust and Sch (2015) that is not directly applicable to our problem. A final contribution of this paper is to provide an estimable dynamic model with both discrete and continuous choices, where households make an optimal short run continuous choice of how much to drive their vehicle each period in response to their characteristics, the type of car they own, and the price of fuel, as well as a longer run dynamic choice of the type of car to own, which takes into account expectations of future driving and fuel prices, the household's future income and age-varying life cycle needs for driving (e.g. the presence of children, retirement, etc) as well as future macro shocks that can affect both car prices and the household's income.

3 A Simple Dynamic Equilibrium Models for Automobile Markets

3.1 Homogeneous consumer economy, no transactions costs

Consider a simplified model where there is only one type of car (though of different ages) and consumers live forever. We assume any utility from driving is subsumed into the quasi-linear specification where the utility of owning a car of age a is given by a function $u(a)$ which we can also (for notational simplicity) consider as net of the cost of maintenance (converted to utils, assuming a coefficient of 1, so the net utility is $u(a) - \mu m(a)$ once maintenance expenditures $m(a)$ are deducted, where $\mu > 0$ is the "marginal utility of money". Thus, we assume that the maintenance costs are already incorporated into the function $u(a)$ and we assume it is a decreasing function of a (this will be the case whenever utility is decreasing in a and maintenance costs are non-decreasing in a).

When there are no transactions costs, it will be optimal for the consumer to trade every period for a preferred vehicle age $d^* \in \{0, 1, \ldots, \overline{a} - 1\}$ where $\overline{a}$ is the age at which cars in this economy are scrapped, since we assume that consumers are not allowed to buy and drive cars that are destined for the scrap yard (i.e. we assume that consumers are not allowed to own any car of age $\overline{a}$ or older).

Suppose consumers can buy new cars at an exogenously fixed price $\overline{P}$ and there is an infinitely elastic demand for cars for scrap metal at price $\underline{P} < \overline{P}$. In addition, assume there is a secondary market where used cars are traded and consumers can buy or sell a car of age a at a price $P(a)$ with no transtions cost, where $a \in \{1, \ldots, \overline{a} - 1\}$. Note that consumers do incur *trading costs* of $P(d^*) - P(a)$ when they sell their car of age a to buy a desired car of age d^* but we assume there are no taxes or additional transactions costs in addition to this trading cost.

The Bellman equation for a consumer who owns a car of age $a \in \{1, \ldots, \overline{a} - 1\}$ is given by

$$V(a) = \max\left[u(a) + \beta V(a+1), \max_{d \in \{0,1,\ldots,\overline{a}-1\}} [u(d) - \mu[P(d) - P(a)] + \beta V(d+1)]\right]. \quad (1)$$

Notice that we consider the decision at the *start of each period* and assume that a consumer always owns a car, so we exclude the possibility of selling an existing car and not replacing it with another one. For this reason we exclude the possibility of owning a new car $a = 0$ at the start of the period. The reason this is excluded is that if the consumer had purchased a new car in the previous period, that car would be of age $a = 1$ at the start of the current period. While we do allow the consumer to buy a new car (by selling their existing car and purchasing a new car at price $P(0) = \overline{P}$), due to our timing convention and definition of a as the age of the *current car, at the start of the current period, before the consumer has made a decision on whether to trade it or keep it,* it is not possible for the age of the existing car to be $a = 0$, at least at the start of the period before the consumer has made their decision to trade their existing car. As a result, the possible values of the age state variable are $a \in \{1, 2, \ldots, \overline{a}\}$.

We also assume that consumers are not allowed to hold cars that are age $\overline{a}$ or older so the value function for these consumers excludes the value of keeping the car and the only option is to sell their existing car. Thus for $a \geq \overline{a}$ the Bellman equation is given by

$$V(a) = \max_{d \in \{0,\ldots,\overline{a}-1\}} \left[u(d) - \mu[P(d) - P(a)] + \beta V(d+1)\right]. \tag{2}$$

It is easy to see from the Bellman equation (1) that is always optimal for consumers to trade for a preferred car age d^*, equal to the value of d that attains the maximum in the sub-maximization problem in the second term in brackets in the Bellman equation (1). This implies that

$$V(a) = u(d^*) - \mu[P(d^*) - P(a)] + \beta V(d^* + 1) \tag{3}$$

and in particular,

$$V(d^*) = u(d^*) + \beta V(d^* + 1) \tag{4}$$

and

$$V(d^* + 1) = V(d^*) - \mu[P(d^*) - P(d^* + 1)]. \tag{5}$$

so these equations imply that

$$V(d^*) = \frac{u(d^*) - \beta\mu[P(d^*) - P(d^* + 1)]}{1 - \beta}. \tag{6}$$

Note that the value in equation (6) above describes the case where the consumer already has the optimal age car, d^*. So the consumer does not have to trade this car, and will receive a present value of utilities from owning a future sequence of cars of age d^* of $u(d^*)/(1 - \beta)$. If the first period is period $t = 0$ then the consumer does not incur any cost of buying car d^* in period $t = 0$ since the consumer already owns it. But starting in period $t = 1$ and continuing for every $t \in \{1, 2, 3, \ldots\}$ the consumer will incur future trading costs in order to trade back to their preferred car age d^*. These trading costs are $[P(d^*) - P(d^* + 1)]$ in every period $t \in \{1, 2, 3, \ldots\}$

and the present value of these costs (translated from dollars to utils) equals $\beta\mu[P(d^*) - P(d^* + 1)]/(1-\beta)$. The extra factor β is necessary to discount these trading costs back to period $t = 0$. So this is the intuitive explanation for the expression of the value function in equation (6).

Now consider a consumer with a car of age $a \neq d^*$. This consumer will have to sell this car in period $t = 0$ and buy the optimal age car d^*. The trading cost for this is $[P(d^*) - P(a)]$ (which could be negative if $a < d^*$). The Bellman equation (1) implies that

$$V(a) = V(d^*) - \mu[P(d^*) - P(a)] = \frac{u(d^*) - \mu[P(d^*) - \beta P(d^* + 1)]}{1-\beta} + \mu P(a). \tag{7}$$

This equation tells us that the discounted utility of a consumer from buying their optimal choice of car d^* equals the discounted stream of utilities (net of maintenance cost) $u(d^*)/(1-\beta)$, less the discounted stream of *depreciation costs* (converted to utils) $\mu[P(d^*) - \beta P(d^* + 1)]/(1-\beta)$. Notice that equation for $V(a)$ in equation (7) has a subtle difference with respect to the equation for $V(d^*)$ in equation (6) even though equation (7) results in the same value for $V(d^*)$ when $a = d$. The subtle difference in equation (7) is that when the consumer does not own a car of age d^* in the initial period $t = 0$, they will have to buy it, and this involves a cash outlay of $P(d^*)$. Similarly in every future period the consumer will be trading back to a car of age d^* and thus will be paying out $P(d^*)$ in every future period. The present value of these outlays is $P(d^*)/(1-\beta)$. In period 0 the consumer will obtain a cash inflow of $P(a)$ from selling their initial car of age a, but in periods $t \in \{1, 2, 3, \dots\}$ the consumer will be selling a car of age $d^* + 1$ because of the consumer's decision to always trade for a car of the optimal age d^*. So the present value of these cash inflows is $\beta P(d^* + 1)/(1-\beta)$. Thus, the formula $V(a)$ equals the present value of the stream of utilities from always owning a car of the optimal age d^*, $u(d^*)/(1-\beta)$, less the present value of the costs of purchasing these cars in every period $t \in \{0, 1, 2, \dots\}$, $P(d^*)/(1-\beta)$, plus the present value of the cash inflows from selling cars of age $d^* + 1$ in periods $t \in \{1, 2, 3, \dots\}$, $\beta P(d^* + 1)/(1-\beta)$, plus the proceeds from the sale of the initial car of age a at time $t = 0$, $P(a)$. All of these latter dollar present values must be multiplied by μ to convert them to discounted utilities.

If all consumers have the same discount factor β and have homogeneous preferences, then in equilibrium all consumers must be indifferent between holding any of the available ages of vehicles. That is, there should be no consumer who has a strict preference for any particular age $d^* \in \{0, 1, \dots, \overline{a} - 1\}$. Given that we assume new car prices and scrap prices are exogenously fixed at values $\overline{P}$ and $\underline{P}$, respectively, there are only $\overline{a} - 1$ "free prices" left to equilibrate supply and demand for used cars of ages $a \in \{1, \dots, \overline{a} - 1\}$, and their prices are $(P(1), \dots, P(\overline{a} - 1))$.

In a homogeneous consumer economy, these prices must adjust to make consumers indifferent about holding any of the ages of vehicles, $d \in \{0, 1, \dots, \overline{a}-1\}$. From equation (7) we see that the discounted utility from a policy of trading for a car of age d in every period $t \in \{0, 1, 2, \dots\}$ is given by $U(d)$ given by

$$U(d) = \frac{u(d) - \mu[P(d) - \beta P(d+1)]}{1-\beta} \tag{8}$$

If consumers are indifferent between all available ages of vehicles, then $U(d) = K$ for all $d \in \{0, 1, \dots, \overline{a}-1\}$ for some constant K, or

$$u(d) - \mu[P(d) - \beta P(d+1)] = K(1-\beta), \tag{9}$$

for $d \in \{0, 1, \dots, \overline{a}-1\}$. These indifference restrictions imply a system of $\overline{a}-1$ linear equations in the $\overline{a}-1$ unknowns $P(1), \dots, P(\overline{a}-1)$. This system can be written in matrix form as

$$X \times P = Y \tag{10}$$

where $P' = (P(1), \dots, P(\overline{a}-1))$ and X is the $\overline{a}-1 \times \overline{a}-1$ matrix given by

$$X = \begin{bmatrix} -\mu(1+\beta) & \mu\beta & 0 & \cdots & 0 & 0 \\ \mu & -\mu(1+\beta) & \mu\beta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \mu & -\mu(1+\beta) & \mu\beta \\ 0 & 0 & \cdots & 0 & \mu & -\mu(1+\beta) \end{bmatrix}, \tag{11}$$

and Y is a $\gamma - 1 \times 1$ vector given by

$$Y = \begin{bmatrix} u(0) - u(1) - \mu\overline{P} \\ u(1) - u(2) \\ \cdots \\ u(\overline{a}-3) - u(\overline{a}-2) \\ u(\overline{a}-2) - u(\overline{a}-1) - \mu\beta\underline{P} \end{bmatrix}. \tag{12}$$

Notice that we do not impose a) monotonicity or b) the restriction that $P(a) \in [\underline{P}, \overline{P}]$ on the solution P to the linear system (10). Thus, we need to check if the solution has these properties. If it does, it is an equilibrium since the price vector results in all consumers being indifferent between holding any one of the available vehicles that are traded in the new or secondary markets, $a \in \{0, 1, \dots, \overline{a}-1\}$. The equilibrium "quantities" are the holdings of vehicles of these different ages. It is easy to see that without any accidents or "endogenous scrappage" of cars prior to the scrappage threshold age $\overline{a}$, then the equilibrium or steady state age distribution of cars will be uniform on the interval $\{0, \dots, \overline{a}-1\}$, so that a fraction $1/\overline{a}$ of the total vehicle stock will be of age a at the *beginning of each period* just *after* the consumers have made their trading decisions (the distribution will be uniform on the set $\{1, \dots, \overline{a}\}$ at the start of the period, but just *before* they have traded their vehicles).

This implies in particular that (assuming all consumers hold just one car) that the fraction $1/\overline{a}$ of the population will buy a new car each period, and the corresponding fraction will scrap their cars, so the market will be in "flow equilibrium". It will also be in "stock equilibrium" since the fact that consumers are indifferent about which age vehicle they own and hold, they can be arranged so that their demand for the different ages is also uniform, matching the supply. Thus there will be zero excess demand for any vehicle age $a \in \{0, 1, \dots, \overline{a}-1\}$ for the price function given above.

Note that multiple equilibria appear possible in this model. That is, we can find different values of $\overline{a}$ and different corresponding price vectors P (one for each conjectured value of $\overline{a}$) that satisfy the linear system (10) and are monotonically decreasing from $\overline{P}$ to $\underline{P}$. In the examples we have computed these equilibria can be Pareto-ranked, with the equilibria corresponding to larger values of $\overline{a}$ being Pareto-preferred by consumers to equilibria with smaller values of $\overline{a}$. That is, consumer welfare is lower in equilibria where cars are scrapped "prematurely". However $\overline{a}$ cannot be increased to arbitrarily large values for a fixed utility function $u(a)$. Eventually for large enough $\overline{a}$, the solution P to the linear system (10) is no longer monotonically decreasing from $\overline{P}$ to $\underline{P}$ and thus no longer constitutes an equilibrium.

Thus, there is a range of potential homogeneous consumer equilibria that can be ranked by the scrappage age $\overline{a}$, i.e. the smallest age a for which $P(a) = \underline{P}$. We will now show that there is a largest possible value of $\overline{a}$ equal to the scrappage threshold a social planner would choose in an appropriately specified social planning problem. However will show below that for all other values of $\overline{a}$, the conjectured Pareto-subdominant equilibria cannot exist in the absence of a legal restriction to "scrap all cars of age $\overline{a}$ or older", because they are dominated by "autarky" — i.e. consumers can obtain higher utility by buying new cars and keeping them until they scrap them then in participating in a secondary market where cars are scrapped (in aggregate) prematurely. In a homogeneous consumer economy without transactions costs, consumers can obtain the same discounted utility in autarky as they can from participating in the Pareto-dominant equilibrium with a secondary market. However we will show that consumers strictly prefer autarky to any of the other Pareto subdominant secondary equilibria, and hence they can only exist if there is a restriction (e.g. a government regulation) that forces all cars of age $\overline{a}$ or older to be scrapped.

The consumer autarky problem turns out to be equivalent to a "social planning problem" where a social planner determines the optimal age to scrap cars in order to maximize consumer utility, but when there is no secondary market. In this social planning problem a single representative consumer simply chooses the optimal age threshold at which to replace their current car with a brand new one. The value function $W(a)$ to this problem is given by

$$W(a) = \max\left[u(0) + \beta W(1) - \mu[\overline{P} - \underline{P}], u(a) + \beta W(a+1)\right]. \tag{13}$$

It is easy to see from the Bellman equation above that $W(0) = u(0) + \beta W(1)$ and thus, any consumer who is "endowed" with a brand new car would never immediately replace it with another new one since this would involve the additional replacement cost $\overline{P} - \underline{P}$. However if $u(a)$ is decreasing sufficiently rapidly there will be a finite age, $\overline{a}$, for which we have

$$W(\overline{a}) = u(0) + \beta W(1) - \mu[\overline{P} - \underline{P}] \geq u(\overline{a}) + \beta W(\overline{a}+1) \tag{14}$$

and we define $\overline{a}$ as the smallest integer satisfying the inequality above and it is easy to show that for this $\overline{a}$ we have

$$W(\overline{a}) = W(0) - \mu[\overline{P} - \underline{P}]. \tag{15}$$

Using the value function to the social planning problem, we can define a *shadow price function* $P(a)$ by

$$P(a) = \overline{P} - [W(0) - W(a)]/\mu. \tag{16}$$

Notice that this shadow price function statisfies $P(0) = \overline{P}$, $P(\overline{a}) = \underline{P}$, and $P(a)$ is monotonically declining in a for the values of a for which $W(a)$ is monotonically decreasing in a, which is the set of $a \in \{0, 1, \ldots, \overline{a}-1\}$. However it is not hard to see from the Bellman equation (13) that for $a < \overline{a}$ we have

$$W(a) = u(a) + \beta W(a+1), \tag{17}$$

which simply says that it is optimal for the consumer to keep their car if its age is younger than the optimal scrappage age $\overline{a}$. However using this condition, it is then easy to verify that the shadow price function (16) makes consumers indifferent between all car ages $a \in \{0, 1, \ldots, \overline{a} - 1\}$. Specifically, we want to show that

$$u(a) - \mu[P(a) - \beta P(a+1)] = K \tag{18}$$

for $a \in \{0, 1, \ldots, \overline{a} - 1\}$ where $\overline{a}$ is the optimal scrapping threshold from the solution to social planning problem (13) and $P(a)$ is given by the "shadow prices" in equation (16). Notice that equation (16) implies that

$$P(a) - \beta P(a+1) = [\overline{P} - W(0)/\mu](1 - \beta) + [W(a) - \beta W(a+1)]/\mu. \tag{19}$$

However the Bellman equation (13) implies that for all $a < \overline{a}$ we have

$$W(a) - \beta W(a+1) = u(a). \tag{20}$$

Substituting equation (20) into equation (19) we obtain

$$P(a) - \beta P(a+1) = [\overline{P} - W(0)/\mu](1 - \beta) + u(a)/\mu \tag{21}$$

and substituting this expression into the left hand side of the indifference condition (18) we obtain

$$u(a) - \mu[P(a) - \beta P(a+1)] = u(a) - \mu[\overline{P} - W(0)/\mu](1 - \beta) - u(a) = [W(0) - \mu\overline{P}](1 - \beta). \tag{22}$$

Since $W(0) - \mu\overline{P}$ does not depend on a, it follows that the shadow prices in equation (16) do result in a homogeneous consumer equilibrium as claimed.

We can also solve for the value function $W_{\overline{a}}(a)$ for a potentially suboptimal policy of "keep the car until age $\overline{a}$ and then scrap it" for some other arbitrary scrappage age $\overline{a}$ not necessarily equal to the optimal scrappage age from the solution to equation (14) above. We have

$$\begin{aligned} W_{\overline{a}}(a) &= u(a) + \beta W_{\overline{a}}(a+1), \quad a \in \{0, 1, \ldots, \overline{a} - 1\} \\ W_{\overline{a}}(\overline{a}) &= W_{\overline{a}}(0) - \mu[\overline{P} - \underline{P}]. \end{aligned}$$

This is a linear system of equations that has a unique solution $W_{\overline{a}}$ and clearly we have $W_{\overline{a}}(a) \leq W(a)$ since $W(a)$ constitutes the optimal car replacement policy. It is easy to follow the same steps as above to show that we can define shadow prices $P_{\overline{a}}(a) = \overline{P} - [W_{\overline{a}}(0) - W_{\overline{a}}(a)]/\mu$ analogous to the shadow price function (16) defined in terms of the optimal value function W. By construction, these shadow prices will make the consumer indifferent between choosing to buy different ages of cars available in the secondary market and hence are the same as the solutions to equation (10). However for choices of $\overline{a}$ that exceed the socially optimal scrappage threshold, the solutions will not necessarily be monotonically decreasing in a and prices may dip below $\underline{P}$ for some values of a, and for this reason these solutions will not constitute valid equilibria. For values of $\overline{a}$ that are less than the socially optimal scrappage threshold, the value functions $W_{\overline{a}}(a)$ will be monotonically decreasing in a and hence the shadow prices $P_{\overline{a}}(a)$ will also be monotonically decreasing in a (and thus lie in the interval $[\underline{P},\overline{P}]$ since by construction we have $P_{\overline{a}}(0)=\overline{P}$ and $P_{\overline{a}}(\overline{a})=\underline{P}$), however even these candidate solutions cannot be equilibria in the secondary market absent an arbitrary restriction or government regulation that mandates that all cars of age $\overline{a}$ or older must be scrapped. Without this restriction, then the "autarky solution" is for consumers to keep cars until the optimal scrappage age $\overline{a}$ given by the smallest integer satisfying inequality (14), and no consumer would choose to participate in a secondary market equilibrium that involved scrapping cars earlier than this socially optimal scrappage age $\overline{a}$.

Proposition *The consumer's discounted utility in the Pareto dominant homogeneous consumer equilibrium equals the welfare the consumer obtains from the solution to the social planning problem. That is,*

$$W(a)=V(a), \quad a=1,\dots,\overline{a} \tag{23}$$

where $\overline{a}$ is the optimal scrappage threshold and W is the value function from the solution to the social planning problem (13), and V is the solution to the consumer problem (1) in the homogeneous consumer equilibrium where prices P are given by the shadow prices in equation (16) and also for $a=0$ we have

$$u(0)+\beta V(1)=W(0) \tag{24}$$

so that the discounted utility of a consumer in the homogeneous consumer equilibrium who is endowed with a new car at time $t=0$ is the same as the welfare the consumer would receive under the social planner problem, which is equivalent to a "buy and hold" strategy where the consumer never trades the car until its age exceeds the scrappage threshold $\overline{a}$, at which age the consumer scraps their existing car and buys a new one.

Proof It should be intuitively clear that consumers are indifferent between trading every period or holding their car until its age reaches the scrappage threshold in a homogeneous consumer equilibrium with no transactions costs. The value function W to the social planning problem can be interpreted as the welfare a consumer would receive in "autarky" where there is no secondary market and each consumer simply buys a new car when the age of their existing car exceeds the optimal scrappage threhold $\overline{a}$ given in equation (14).

When a secondary market exists, consumers can trade, but with zero transactions costs and homogeneous consumers, the market price must adjust to make consumers indifferent between trading for any of the available ages of cars that are traded in the secondary market. In the Pareto dominant such equilibrium, the consumer can buy any used car of age $a \in \{0,\ldots,\overline{a}-1\}$, and the prices are such that the consumer will be indifferent between buying any of them, so we have

$$u(d) - \mu[P(d) - \beta V(d+1)] = K, \quad d \in \{0,1,\ldots,\overline{a}-1\}, \tag{25}$$

for some constant K. From the consumer's Bellman equation (1), it follows that

$$V(a) = K + \mu P(a), \quad a \in \{1,\ldots,\overline{a}\}. \tag{26}$$

Further, the Bellman equation (1) implies that the consumer is indifferent between keeping their current car or trading it, for each age $a \in \{1,\ldots,\overline{a}\}$. Now, using the shadow prices given in equation (16) and substituting them into equation (26), we have

$$V(a) = W(a) + K - W(0) + \mu\overline{P}. \tag{27}$$

But we have

$$K = u(0) - \mu\overline{P} + \beta V(1) = V(0) - \mu\overline{P}, \tag{28}$$

where $V(0) = u(0) + \beta V(1)$ is the discounted utility of a consumer who has been endowed with a brand new car at the start of period $t = 0$. But since the consumer is indifferent between holding this car until it reaches the scrappage threshold $\overline{a}$ and then trading it for a new one, or trading it every period for any other age a of car in the set of cars the consumer can purchase in the market, $a \in \{0,1,\ldots,\overline{a}-1\}$, it follows that $W(0) = V(0)$ since $W(0)$ is the discounted utility of a consumer who follows a buy and hold strategy. It follows from equations (28) and (27) that $V(a) = W(a)$ for $a \in \{1,2,\ldots,\overline{a}\}$. $\square$

It is also easy to see, using the same argument as above, that the following Corollary holds.

Corollary *Let $\overline{a}$ be an integer that does not equal the optimal scrappage threshold given by the solution to the social planning problem in (13) and (14). Let the shadow price function $P_{\overline{a}}(a)$*

be given by

$$P_{\overline{a}}(a) = \overline{P} - [W_{\overline{a}}(0) - W_{\overline{a}}(a)]/\mu, \quad a \in \{0, 1, \ldots, \overline{a}\} \tag{29}$$

where the value function $W_{\overline{a}}(a)$ is given by the solution to equation (23). Define the value function $V_{\overline{a}}(a)$ as the value a consumer would obtain in a hypothesized secondary market where the trading strategy involves selling their current car of age a for a replacement car of age $d \in \{0, 1, \ldots, \overline{a} - 1\}$ each period, and the consumer is indifferent about which age car d to replace their current car with

$$V_{\overline{a}}(a) = \frac{u(d) - \mu[P_{\overline{a}}(d) - \beta P_{\overline{a}}(d+1)]}{1 - \beta} + \mu P_{\overline{a}}(a). \tag{30}$$

Then we have:

$$V_{\overline{a}}(a) = W_{\overline{a}}(a), \quad a \in \{0, 1, \ldots, \overline{a}\} \tag{31}$$

From the Corollary it is now straightforward to see why there cannot exist any Pareto suboptimal equilibria in a market with homogeneous consumers and no transactions costs — absent some arbitrary restriction that forces consumers to scrap cars prior to the optimal scrappage threshold $\overline{a}$ equal to the smallest solution to equation (14). The reason is that if such an equilibrium existed, the welfare that a consumer would obtain in such a market from owning a car of age a is $V_{\overline{a}}(a)$ but by the Corollary this equals $W_{\overline{a}}(a) < W(a)$ where the latter is the *welfare a consumer can secure in autarky.* That is, $W(a)$ is the welfare a consumer can obtain from following a "buy and hold" trading strategy and never trading in the secondary market. Thus, we conclude that all consumers would abandon the secondary market if they had the freedom to choose when to scrap their autos and a Pareto-suboptimal equilibrium were posited so exist. Thus, only the Pareto dominant equilibrium will exist if there are no arbitrary restrictions on when consumers can scrap their autos. In the Pareto-dominant equilibrium consumers are indifferent between avoiding the secondary market and following a buy and hold strategy, or a frequent trading strategy of trading every period for a preferred car of age d. Since they are indifferent between holding any car of age d traded in the market, whether the secondary market exists or not is of no consequence to them. Thus, in order to provide an adequate explanation of why secondary markets for cars exist, we need to extend our model to allow for *consumer heterogeneity* which will provide a source of *gains from trade.*

3.2 Heterogeneous consumer economy with transactions costs

Similar to the homogeneous consumer model already presented, our heterogeneous consumer model is based on discrete model of automobiles where each automobile is identical and indexed only by its age a, where $a = 0$ is a new car. We will assume that there is an exogenously specified infinitely elastic supply of new cars at price $\overline{P}$ and an infinitely elastic demand for cars for scrap metal at price $\underline{P}$ where $\underline{P} < \overline{P}$ just as in the previous section.

We now allow for heterogeneity in consumers and we will adopt a slight change in notation. We assume consumers have quasi-linear utility functions and that $u(a)$ represents the utility of a consumer for a car of age a net of any maintenance costs. We assume that it is a strictly monotonically decreasing function of a so that if $a' > a$ we have $u(a') < u(a)$. We introduce heterogeneity via extreme value shocks to these utilities. So if a consumer decides to keep their currently owned car of age a their utility is given by $u(a) + \varepsilon(-1)$ where we use the -1 index on the extreme value error term to denote the decision to keep the current car, for reasons that will be clearer shortly.

We also assume that a consumer cannot keep a car that is older than some threshold $\overline{a}$ where all cars are scrapped. The consumer who holds a car of this age is forced to trade it for a car of age $a \in \{0, 1, \ldots, \overline{a} - 1\}$ and the consumer receives the scrap price $\underline{P}$ as the "trade-in" value when this happens. We also assume that every consumer must hold a car in every period of their infinite lifespan, so that whenever a consumer sells or scraps a car, they must immediately replace it with another one (either brand new, $a = 0$ or used, $a \in \{1, \ldots, \overline{a} - 1\}$).

We assume that there are a continuum of consumers in the economy and that there are no accidents or scrappage decisions of cars except once the car reaches the scrappage threshold $\overline{a}$ (we will discuss below how this scrappage threshold can be determined endogeneously as part of the equilibrium solution). We will assume that there is a stationary equilibrium with a price function $P(a)$ satisfying $P(0) = \overline{P}$, $P(a) = \underline{P}$ for $a \geq \overline{a}$, and for each $a \in \{1, \ldots, \overline{a} - 1\}$ $P(a)$ sets the supply of cars of age a being sold by existing consumers in the secondary market to the demand for cars of age a by other consumers in this economy. We assume that in any secondary market transaction the seller of a car pays an exogenously fixed transaction cost $T \geq 0$.

Let $V(a, \varepsilon)$ denote the value function of a consumer who is indexed by a vector of *IID* extreme value "taste shocks" ε (to be detailed shortly) and who owns a car of age a. All consumers are infinitely lived and discount the future with common discount factor $\beta \in (0, 1)$. For $a < \overline{a}$, the consumer has a total of $\overline{a} + 1$ choices, $D(a) = \{-1, 0, 1, \ldots, \overline{a} - 1\}$ where the choice $d = -1$ corresponds to the decision to keep the current car of age a, and the choices

$d \in \{0,1,\ldots,\overline{a}-1\}$ correspond to the decision to trade the current car for a car of age d. Notice that we do not allow consumers to buy cars from the scrapper at the scrap price $\underline{P}$, i.e. they cannot buy any car that as old or older than the scrappage threshold $\overline{a}$.

However if a consumer has a car (or buys a car) of age $\overline{a}-1$ and keeps it, then next period it will be of age $\overline{a}$ and we assume that the consumer is forced to scrap it and buy another car, so the choice set in this state is $D(\overline{a}) = \{0,1,\ldots,\overline{a}-1\}$.

Now consider the Bellman equation for the consumer's optimization problem, which takes the price function $P(a)$ as given. If $a < \overline{a}$ the consumer has a choice set that has the $\overline{a}+1$ elements $D(a) = \{-1,0,1,\ldots,\overline{a}-1\}$ and ε is a conformable $(\overline{a}+1\times 1)$ vector with *IID* extreme value components corresponding to each of the consumer's possible choices

$$V(a,\varepsilon) = \max\left[v(-1,a)+\varepsilon(-1), \max_{d\in\{0,1,\ldots,\overline{a}-1\}}[v(d,a)+\varepsilon(d)]\right], \tag{32}$$

where we define $v(d,a)$ as the value of trading the current car of age $a \in \{1,2,\ldots,\overline{a}\}$ for a replacement car of age $d \in \{0,1,\ldots,\overline{a}-1\}$ by

$$v(d,a) = u(d) - \mu[P(d) - P(a) - T(P,a,d)] + \beta EV(d+1) \tag{33}$$

where $\mu \geq 0$ is the "marginal utility of money" parameter of the quasi-linear consumer preference function, and $T(P,a,d)$ is a transaction cost function that reflects other costs (both monetary and psychic equivalent) "search and transactions costs" involved in trading one's existing car of age a for another car of age d. We assume here that transactions costs are imposed on the buyer of a car but not on the seller. In particular, in an extended version of the model we consider subsequently, we allow for the option not to have a car. Then with no transaction cost on the seller side, the seller would receive the price $P(a)$ for selling their age a car, either for the purpose of buying another one, or to go into the "no car" state (in the extended model we wil consider shortly). But for a buyer, the buyer of a car of age d must pay the market price $P(d)$ plus an additional cost of $T(P,a,d)$. In principle this could be any function that could depend on both the age or price of the car that is bought, d and $P(d)$ respectively. If we extend T to allow it to also depend on the age and price of the car that is sold in this transaction (if any), a and $P(a)$ repectively, then we can think of this is a transaction cost that is ultimately incured by the seller, but is "picked up" by the buyer. There is no particular reason why we could not allow for a seller side transction cost too, but at some point it is hard to know how the data could identify how transactions and search costs are actually apportioned between buyer and seller in a trade in any event. Some "transactions costs" might also be capturing for relative bargaining strength of buyer and seller that is relevant in real world situations but not relevant in our continuum agent model where any individual buyer or seller is just atomistic and takes all prices as given.

We let $v(-1,a)$ denote the value of keeping the currently held car of age a

$$v(-1,a) = u(a) + \beta EV(a+1), \quad a \in \{1,\ldots,\overline{a}-1\}. \tag{34}$$

If $a = \overline{a}$ then the consumer's choice set has only $\overline{a}-1$ elements since keeping the car is no longer allowed and in this case ε is a $(\overline{a}-1 \times 1)$ dimensional *IID* extreme value vector and we have

$$V(\overline{a},\varepsilon) = \max_{d \in \{0,1,\ldots,\overline{a}-1\}} \left[v(d,a) + \varepsilon(d)\right]. \tag{35}$$

We assume that the ε state variable is also serially independent as well as independent across consumers in the economy and each of its components are mutually independently distributed. Let $\sigma \geq 0$ denote the (common) scale parameter parameter of each of the components of the extreme value ε vector. Then as is well known, we have

$$\begin{aligned} EV(\overline{a}) &= E\{V(\overline{a},\varepsilon)\} \\ &= \int_{\varepsilon} V(\overline{a},\varepsilon) f(\varepsilon) d\varepsilon \\ &= \sigma \log\left(\sum_{d=0}^{\overline{a}-1} \exp\{v(d,\overline{a})/\sigma\}\right) \end{aligned} \tag{36}$$

and for $a \in \{1,\ldots,\overline{a}-1\}$ we have

$$\begin{aligned} EV(a) &= E\{V(a,\varepsilon)\} \\ &= \int_{\varepsilon} V(a,\varepsilon) f(\varepsilon) d\varepsilon \\ &= \sigma \log\left(\exp\{v(-1,a)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,a)/\sigma\}\right). \end{aligned} \tag{37}$$

The system of equations (36) and (37) represents a system of $\overline{a}$ equations in the $\overline{a}$ unknowns $(EV(1),\ldots,EV(\overline{a}))$. The solution to this system can be shown to define a fixed point to a contraction mapping, so it has a unique solution that can be calculated by the method of successive approximations, though we actually use the Newton-Kantorovich algorithm (to be described shortly) to speed up the numerical solution to this system. Once we have found the solution

$(EV(1),\ldots,EV(\overline{a}))$, we can construct the values $v(-1,a)$ and $v(d,a)$, $d \in \{0,1,\ldots,\overline{a}-1\}$ and using these we also use the fact that the extreme value specification implies the following multinomial choice probabilities for $a < \overline{a}-1$

$$\begin{aligned} \Pi(-1|a) &= \frac{\exp\{v(-1,a)/\sigma\}}{\exp\{v(-1,a)/\sigma\} + \sum_{a'=0}^{\overline{a}-1}\exp\{v(a',a)/\sigma\}} \\ \Pi(d|a) &= \frac{\exp\{v(d,a)/\sigma\}}{\exp\{v(-1,a)/\sigma\} + \sum_{a'=0}^{\overline{a}-1}\exp\{v(a',a)/\sigma\}}, \quad d \in \{0,1,\ldots,\overline{a}-1\} \end{aligned} \tag{38}$$

and for $a = \overline{a}$ we have

$$\Pi(d|a) = \frac{\exp\{v(d,a)/\sigma\}}{\sum_{a'=0}^{\overline{a}-1}\exp\{v(a',a)/\sigma\}}, \quad d \in \{0,1,\ldots,\overline{a}-1\} \tag{39}$$

As we noted above, since $P(0) = \overline{P}$ and $P(\overline{a}) = \underline{P}$ are treated as exogenously fixed parameters (i.e. we treat the auto market as a "small open economy" where prices of new cars are given exogenously on the world market, and the demand by consumers in this small country have negligible impact on the prices of new cars, and similarly we assume there is an exogenous given infinitely elastic demand for cars for their scrap value $\underline{P}$). Thus, there are actually $\overline{a}-1$ "endogenous" values $P(a)$, $a = 1,\ldots,\overline{a}-1$, i.e. these are the prices that are free to move in order to find equilibrium in the market, i.e. to set supply of each age of used cars to the demand for these cars. Hereafter we let $P = (P(1),\ldots,P(\overline{a}-1))$ be the $\overline{a}-1\times 1$ vector of endogenous prices to be solved for.

To emphasize the dependence of the value functions on P we write $v(d,a,P)$, and note that the solutions to the expected value functions in equations (36) and (37) above imply that $v(d,a,P)$ is a smooth implicit function of P whose gradients with respect to P we can calculate. Since the choice probabilities in equations (38) and (39) above are smooth functions of $v(d,a,P)$, it follows that they are also smooth functions of P. We emphasize this dependence by writing the consumer choice probabilities as $\Pi(d|a,P)$, which also indicates that these probability respond to the prices in the secondhand market for autos. Now consider the equations for equilibrium in the market

$$
\begin{aligned}
\sum_{a=1}^{\overline{a}} \Pi(1|a,P) &= 1-\Pi(-1|1,P) \\
\sum_{a=1}^{\overline{a}} \Pi(2|a,P) &= 1-\Pi(-1|2,P) \\
&\cdots \\
\sum_{a=1}^{\overline{a}} \Pi(\overline{a}-2|a,P) &= 1-\Pi(-1|\overline{a}-2,P) \\
\sum_{a=1}^{\overline{a}} \Pi(\overline{a}-1|a,P) &= 1-\Pi(-1|\overline{a}-1,P). \qquad (40)
\end{aligned}
$$

The left hand side represents the fraction of consumers in our continuum of consumer economy who wish to buy used cars of ages $a \in \{1,\ldots,\overline{a}-1\}$, and thus represents the demand for each car age. The right hand side is the fraction of consumers who hold cars of each of these ages who wish to sell their car, and thus the right hand side represents the supply of cars of each age.

Thus the equilibrium condition (40) is a system of $\overline{a}-1$ smooth nonlinear equations in $\overline{a}-1$ unknowns, $P(a)$, $a \in \{1,\ldots,\overline{a}\}$ which we can expect to have at least one solution under fairly weak restrictions. Let's write (40) in a more usual form as a set of prices that sets excess demand in each of the $\overline{a}-1$ second hand markets to zero

$$E(P) = 0 \qquad (41)$$

by subtracting the left hand side of equation (40) from both sides of the equilibrium conditions. Since E is differentiable, we can solve it using Newton's method, iteratively as follows

$$P_{t+1} = P_t - [\nabla E(P_t)]^{-1} E(P_t) \qquad (42)$$

where $\nabla E(P_t)$ is the $(\overline{a}-1)\times(\overline{a}-1)$ Jacobian matrix of $E(P)$. Use of Newton's method requires us to show that the value functions $v(d,a,P)$ are smooth functions of P, but this follows from the Implicit Function Theorem using equations (36) and (37). Given these values (and their gradients with respect to P) it is straightforward to implement the Newton iteration (42) to search for equilibrium in this economy with heterogeneous consumers and transactions costs.

Appendix 1 provides formulas for the gradient of $EV(a)$ with respect to P using the Implicit Function Theorem, then successively (using the Chain Rule of Calculus) calculates the gradients of the choice probabilities $\Pi(d|a,P)$ and Jacobian matrix $\nabla E(P)$ with respect to P.

3.3 Equilibria with Stochastic Accidents

In the models discussed so far, the only way cars are scrapped (and thus removed from the economy) is when they reach the threshold age $\overline{a}$ where all cars are scrapped. In the absence of stochastic accidents, this implies a uniform stationary age distribution of cars over the set of ages $\{1,2,\ldots,\overline{a}\}$. Now consider the possibility of exogenous stochastic accidents that can also serve to remove cars younger than age $\overline{a}$ from the economy. Let $\alpha(a)$ be the probability that a car of age a is involved in an accident that results in a total loss and its subsequent scappage and removal from the car stock. We can represent this as a probability $\alpha(a)$ of transiting to the scrappage age $\overline{a}$ next period, where it is scrapped with probability 1. Thus, with probability $1-\alpha(a)$ a car of age a becomes age $a+1$ next period, and with probability $\alpha(a)$ it transits to the scrappage state $\overline{a}$. If a car is at the scrappage state $\overline{a}$, in a "flow equilibrium" in the economy, the number of cars that are scrapped will equal the number of new cars sold, since otherwise the stock of cars will either decrease or increase over time if scrappage of cars does not equal the production of new cars. We can represent flow equilibrium as a transition from state $a=\overline{a}$ to $a=1$ with probability 1.

It is important to be clear on timing of events in our model. At the start of any time period t the economy inherits a stock of cars carried over from the previous period $t-1$. This means there can be no new cars in the stock of cars at time t, since any new cars sold to consumers (that were not involved in an accident that totalled them) would have aged to become a 1 year old car at period t. Thus *there can be brand new cars at the start of each period t, at least prior to any trading of vehicles, including sales of new cars to consumers.* We assume this trading process is instantaneous, so we will observe new cars in the stock of cars *immediately after consumers have conducted their trades, including any purchases of new cars at period t.* But at the start of period $t+1$, prior to any trading, it should be clear that there will not be any new cars in the stock of cars either. It will only be for an instant – immediately *after* the trading of cars in each period t (where old cars are scrapped and new cars are sold to consumers) that we will observe brand new cars in the economy, but technically if we measure the stock of cars at the start of each period (prior to any trade) there will never be any brand new cars in the economy.

We can represent the aging of cars in the presence of stochastic age-dependent accidents via an $\overline{a}+1 \times \overline{a}+1$ transition probability matrix Ω given by

$$\Omega = \begin{bmatrix} 0 & 1-\alpha(0) & \cdots & 0 & \alpha(0) \\ 0 & 0 & 1-\alpha(1) & \cdots & \alpha(1) \\ 0 & 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1-\alpha(\overline{a}-2) & \alpha(\overline{a}-2) \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 1-\alpha(0) & \cdots & 0 & \alpha(0) \end{bmatrix}. \tag{43}$$

This transition probability matrix represents car aging and the impact of stochastic accidents. For example the first row of Ω indicates that a brand new car will become a 1 year old car next period with probability $1-\alpha(0)$ (i.e. if there is no accident) or a car of age $\overline{a}$ with probability $\alpha(0)$. Thus we represent an accident as a transition of the car age directly to the scrappage age, indicating that after the accident the car is considered "totalled" and the car is scrapped. The last row of Ω indicates that for cars that have reached the scrappage age, $\overline{a}$, these are all replaced by new cars. However due to stochastic accidents, only a fraction of $1-\alpha(0)$ of all of these scrappged cars will avoid accidents and become age $a = 1$ year old cars next period, whereas the fraction $\alpha(0)$ of these new cars will have an accident and return to the scap state at the start of period $t+1$.

The invariant age distribution implied by the transition probablity matrix Ω is a $(1 \times \overline{a}+1)$ vector q satisfying

$$q = q\Omega \tag{44}$$

We can see immediately that $q(0) = 0$, i.e. the first component of the invariant probability vector q that provides the fraction of new car holdings in stead state, will be zero, as we discussed above. The other components of q give the invariant probabilities for ages $a = 1,\ldots,\overline{a}$ given by

$$q(\overline{a}) = \frac{1}{2+\alpha(0)+(1-\alpha(0))\sum_{a=2}^{\overline{a}-1}\prod_{i=1}^{a=1}(1-\alpha(i))} \tag{45}$$

and

$$q(a) = q(\overline{a})\prod_{i=1}^{a}(1-\alpha(i)) \quad a = 1,\ldots,\overline{a}-1 \tag{46}$$

Since there will be no cars of age $a = 0$ in the invariant holdings distribution at the start of any time t, it is convenient to work with the reduced $\overline{a} \times \overline{a}$ transition probabiity matrix, given by the lower $\overline{a} \times \overline{a}$ submatrix of Ω in equation (43):

$$\Omega = \begin{bmatrix} 0 & 1-\alpha(1) & \cdots & \alpha(1) \\ 0 & \cdots & \cdots & \cdots \\ \cdots & 0 & 1-\alpha(\overline{a}-2) & \alpha(\overline{a}-2) \\ 0 & \cdots & 0 & 1 \\ 1-\alpha(0) & \cdots & 0 & \alpha(0) \end{bmatrix}. \tag{47}$$

and it is easy to see that the last $\overline{a}$ components of the invariant probability q from equation (43) is also an invariant probability distribution for the reduced transition matrix Ω given in equation (47).

Notice that in a stationary equilibrium in the auto market, we will have *flow equilibrium* — the fraction of cars that are scrapped in any period will equal the production/sales of new cars to consumers in the economy immediately after trading in each period t. This implies that $q(0) = q(\overline{a})$ and $q(\overline{a}) = 0$ right at the instant after old cars are scrapped and replaced with new ones in this economy. However this only holds at that instant and at the start of new period, $t+1$, we will have $q(0) = 0$ and $q(\overline{a}) > 0$ just before trading in cars starts again at period $t+1$.

When there are accidents, we have to modify the Bellman equations to account for the "forced replacement" of a car that the own might have wanted to keep, but for a severe auto accident that "totalled" the vehicle. We model this as a stochastic transition for the next period age of the car. For a consumer who chooses to keep their current car of age $a < \overline{a}$, instead of becoming a car of age $a+1$ with probability 1 next period, the car will be age $a+1$ with probability $1-\alpha(a)$ and age $\overline{a}$ with probability $\alpha(a)$. Similarly if the consumer decides to trade their current car of age a for another car of age d, it will become a car of age $d+1$ next period with probability $1-\alpha(d)$ (i.e. if there is no accident), or a car of age $\overline{a}$ with probability $\alpha(d)$ (if there is an accident that totals it).

Thus, let $v(-1,a)$ be the value of keeping a car of age a. Now the Bellman equation is given by

$$v(-1,a) = u(a) + \beta\left[(1-\alpha(a))EV(a+1) + \alpha(a)EV(\overline{a})\right]. \tag{48}$$

Simiarly, the value for trading a car of age a for a car of age d is given by

$$v(d,a) = u(d) - \mu[P(d) - P(a) - T(P,a,d)] + \beta\left[(1-\alpha(d))EV(d+1) + \alpha(d)EV(\overline{a})\right]. \tag{49}$$

The equation for the value of starting the period owning a car of age $\overline{a}$ is not affected by the presence of accidents, and is given by equation (36) above. The reason is that the owner of a car of age $\overline{a}$ is forced to trade for some other car (or purge the car when there is an outside good) and given this, if the consumer does choose some other car of age d, the possibility of accidents have already been accounted for via the formula for $v(d,a)$ in equation (49) above. Thus, it is relatively straightforward to extend the Bellman equation recursions (and smoothed Bellman formula) to account for accidents, which amount to a stochastically and exogenously "forced replacement" of one's car.

Using the steady state distribution q of cars, we can write the equation for equilibrium in the market in the presence of exogenous accidents as follows

$$\begin{aligned}
\sum_{a=1}^{\overline{a}} \Pi(1|a,P)q(a) &= q(1)\,[1-\Pi(-1|1,P)] \\
\sum_{a=1}^{\overline{a}} \Pi(2|a,P)q(a) &= q(2)\,[1-\Pi(-1|2,P)] \\
&\cdots \\
\sum_{a=1}^{\overline{a}} \Pi(\overline{a}-2|a,P)q(a) &= q(\overline{a}-2)\,[1-\Pi(-1|\overline{a}-2,P)] \\
\sum_{a=1}^{\overline{a}} \Pi(\overline{a}-1|a,P)q(a) &= q(\overline{a}-1)\,[1-\Pi(-1|\overline{a}-1,P)]\,.
\end{aligned} \tag{50}$$

We also have the following result:

Proposition 1 *Let P be an equilibrium price vector with equilibrium scrappage threshold $\overline{a}$ from a solution to (**??**) and let $\Delta(P)$ be the mapping from $R^{\overline{a}-1} \to R^{\overline{a}} \times R^{\overline{a}}$, i.e. the $\overline{a} \times \overline{a}$ transition probability matrix with elements given by:*

$$\begin{bmatrix}
\Pi(1|1)+\Pi(-1|1) & \Pi(2|1) & \cdots & \Pi(\overline{a}-1|1) & \Pi(0|1) \\
\Pi(1|2) & \Pi(2|2)+\Pi(-1|2) & \cdots & \Pi(\overline{a}-1|2) & \Pi(0|2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\Pi(1|\overline{a}-1) & \Pi(2|\overline{a}-1) & \cdots & \Pi(\overline{a}-1|\overline{a}-1)+\Pi(-1|\overline{a}-1) & \Pi(0|\overline{a}-1) \\
\Pi(1|\overline{a}) & \Pi(2|\overline{a}) & \cdots & \Pi(\overline{a}-1|\overline{a}) & \Pi(0|\overline{a})
\end{bmatrix}, \tag{51}$$

Then the invariant probability distribution q given in (44) is also the invariant probability distribution for the transition probability matrix Δ

$$q = q\Delta(P). \tag{52}$$

Proof First notice that $\Delta(P)$ is indeed a transition probability matrix since all of its elements are in the unit interval and all of its rows sum to 1. In particular the last row sums to 1 because the option of keeping a car that has reached the scrappage threshold $\overline{a}$ is not allowed: $\Pi(-1|\overline{a},P) = 0$ for all P. Further, since the auto market is in flow equilibrium, the demand for new cars must be equal to the number of old cars that are scrapped in steady state, so we have the condition

$$q(\overline{a}) = \sum_{a=1}^{\overline{a}} \Pi(0|a)q(a). \tag{53}$$

Notice that this equation is equivalent to $q * D(:,\overline{a}) = q(\overline{a})$, where $D(:,\overline{a})$ is the $\overline{a}^{\text{th}}$ column of the matrix D. For the other columns in equation (52), these equalities follow from the equilibrium conditions in equation (**??**). $\square$

We can think of the transition probability matrix Δ as representing the "trade transition probability matrix". That is, the economy enters the start of some period t with an initial distribution q of cars. These cars are held by the various consumers in the population who immediately trade them among each other as well as scrapping all of the cars that have reached the scrappage threshold $\overline{a}$. So $q\Delta(P)$ represents the distribution of the stock of cars that are held immediately after this scrappage and trading occurs. There are no cars of age $\overline{a}$ after this trading: they have been replaced by newly produced cars of age 0. The equation $q\Delta(P) = q$ therefore tells us that trading obeys the supply restriction on the stock of all cars, namely that all existing used cars of ages $a = 1,\ldots,\overline{a}-1$ must be held by the consumers, and the fraction $q(\overline{a})$ of cars that are scrapped are replaced by brand new cars of age 0, which are also held collectively by the consumers following this instantaneous round of trading. Thus we can view Proposition 1 as a confirmation that the supply of cars does indeed equal the demand for new and used cars in any stationary equilibrium.

We can combine the two equations $q = \Omega q$ and $q = \Delta(P)q$ into a single equation for the invariant distribution

$$q = q\Delta(P)\Omega \tag{54}$$

and we can consider the $\overline{a}\times\overline{a}$ transition probability matrix $\Delta(P)\Omega$ to reflect of both trading and aging/accidents. That is, if we enter period t with a distribution of cars q, then $q\Delta(P)$ represents the distribution of cars immediately after trading has occurred. As we have noted, there are no longer any cars of age $\overline{a}$ in this post-trade distribution of cars since all these cars have been scrapped and replaced by new cars. Then $q\Omega$ represents the effects of aging and accidents that occur between t and $t+1$, and thus $q\Delta(P)\Omega$ represents both the "reshuffling" in holdings of cars among different consumers due to trading (as well as scrappage and sales of new cars) and

Ω captures the effect of aging and accidents.

Notice the recursive nature of our algorithm to compute the equilibria of this model. We first "guess" a scrappage level $\overline{a}$ (often using the $\overline{a}$ from the social planning problem as the initial guess for $\overline{a}$) then we solve equation (44) for $q(\cdot|\overline{a})$, i.e. the invariant distribution of cars conditional on this initial guess of $\overline{a}$. Then we use this disribution to evaluate excess demand $ED(P,q(\cdot|\overline{a}))$ in equation (**??**). We note that excess demand is itself roughly an inner product of the vector q and a vector valued function of the choice probabilities which we write schematically as $ED(\Pi(P),q(\cdot|\overline{a}))$. So given the guess of $\overline{a}$ our Newton algorithm searches for a vector P that sets excess demand to zero, $ED(\Pi(P),q(\cdot|\overline{a}))=0$. We can test if a solution to this is a "valid equilibrium" by checking three additional things: 1) $P(0)=\overline{P}$, 2) $P(\overline{a})=\underline{P}$, 3) P is non-increasing. Actually we can potentially relax condition 3) though in all solutions we have calculated so far, the solutions are non-increasing in a.

There may be multiple equilibria, i.e. multiple values of P that set excess demand to zero for any given $\overline{a}$. So far we have not looked for, or found by chance, this former type of multiplicity of equilibrium. Instead the type of mulitiplicity we easily find is *different equilibria corresponding to different guesses of $\overline{a}$.* Typically we find an interval of values of $\overline{a}$ for which an equilibrium can be calculated that satisfy the three key properties above.

3.4 Allowing an Outside Good

The model has so far forced consumers to own exactly one car in every period. However it is straightforward to extend the model, and our definition of stationary equilibrium, to allow for an "outside good" — i.e. the possibility that consumers can use other transport modes as an alternative to owning a car. Let $\emptyset$ denote the choice of not owning a car, and let $u(\emptyset,\tau)$ be the utility that a type τ consumer obtains from the choice of not owning a car in a single period. So we now allow a consumer who has a car of age a the choice to sell their current car but *not replace it,* a choice we denote as $d=\emptyset$. We also let $a=\emptyset$ denote the state that a customer does not own a car, and for such a consumer $d=\emptyset$ represents the decision to continue not owning a car for another period, and $d=a$ denotes the decision to buy a car of age a.

Let $V(\emptyset,\varepsilon,\tau)$ be the discounted utility of a consumer τ expects if they do not own a car. Since τ is a time invariant type of consumer, we will suppress it to space in the equations below. The Bellman equation for this state is given by

$$V(\emptyset,\varepsilon)=\max\left[v(\emptyset,\emptyset)+\varepsilon(\emptyset),\max_{d\in\{0,1,\ldots,\overline{a}-1\}}[v(d,\emptyset)+\varepsilon(d)]\right],\tag{55}$$

where

$$\begin{aligned} v(d,\emptyset) &= u(d) - \mu[P(d)+T] + \beta\left[(1-\alpha(d))EV(d+1) + \alpha(d)EV(\overline{a})\right] \\ v(\emptyset,\emptyset) &= u(\emptyset) + \beta EV(\emptyset), \end{aligned} \tag{56}$$

where $EV(\emptyset)$ is the conditional expectation of $V(\emptyset,\varepsilon)$, which represents the expectation of future utility for a consumer who does not currently own a car. When ε is a vector of *IID* Extreme valued preference shocks, following equation (36) we have

$$EV(\emptyset) = \sigma \log\left(\exp\{v(\emptyset,\emptyset)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,\emptyset)/\sigma\}\right). \tag{57}$$

Similarly, we extend the Bellman equation for a consumer who owns a car of age a to allow for the option to "purge" their car, i.e. sell the car but not buy another to replace it:

$$V(a,\varepsilon) = \max\left[v(\emptyset,a) + \varepsilon(\emptyset), v(-1,a) + \varepsilon(-1), \max_{d\in\{0,1,\ldots,\overline{a}-1\}}[v(d,a)+\varepsilon(d)]\right], \tag{58}$$

where $v(\emptyset,a)$ is the value of selling one's current car of age a and not replacing it

$$v(\emptyset,a) = u(\emptyset) + \mu P(a) + \beta EV(\emptyset). \tag{59}$$

and $v(d,a)$ is given by the same equation (33) above, i.e. the same as in the case where we ignored the outside good but where we modify equation (37) to allow $EV(d)$ to account for the outside good as follows

$$\begin{aligned} EV(d) &= E\{V(d,\varepsilon)\} \\ &= \int_{\varepsilon} V(d,\varepsilon) f(\varepsilon) d\varepsilon \\ &= \sigma \log\left(\exp\{v(\emptyset,d)/\sigma\} + \exp\{v(-1,d)/\sigma\} + \sum_{d'=0}^{\overline{a}-1} \exp\{v(d',d)/\sigma\}\right). \end{aligned} \tag{60}$$

As we discussed in the previous subsections the system of equations (98) and (101) are the equivalent of the Bellman equation, but after the *IID* extreme value random variable preference shocks ε have been integrated out. So as we previously discussed the system defines EV as the unique fixed point of a contraction mapping, and EV can also be calculated using Newton iterations exactly as described in (42) above. We can also use the implicit function theorem to show that EV is a smooth implicit function of P which we denote as $EV(P)$. From the solution

EV we can construct the choice-specific values $v(d,a)$ which are also implicit functions of P and we write $v(d,a,P)$ to emphasize this. Then following the previous sections we can calculate the logit choice probabilities which are the same as written above except for the new choice probabilities for a consumer who does not own a car, $a = \emptyset$,

$$\Pi(d|\emptyset,P) = \frac{\exp\{v(d,\emptyset,P)/\sigma\}}{\exp\{v(\emptyset,\emptyset,P)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,\emptyset,P)/\sigma\}}, \tag{61}$$

and for consumers who do own a car, $a \in \{1,2,\ldots,\overline{a}\}$ we expand their choice sets to include the "purge" option, $d = \emptyset$ (i.e. to sell their current car and not replace it with another one) which has probability

$$\Pi(\emptyset|a,P) = \frac{\exp\{v(\emptyset,a,P)/\sigma\}}{\exp\{v(\emptyset,a,P)/\sigma\} + \exp\{v(-1,a,P)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,a,P)/\sigma\}}, \tag{62}$$

and as above we constrain any consumer who holds a car of age $a = \overline{a}$ to scrap it, i.e. $\Pi(-1|\overline{a},P) = 0$ for all $a \in \{1,\ldots,\overline{a}\}$, so we have

$$\Pi(d|\overline{a},P) = \frac{\exp\{v(d,\overline{a},P)/\sigma\}}{\exp\{v(\emptyset,\overline{a},P)/\sigma\} + \sum_{d=0}^{\overline{a}-1} \exp\{v(d,\overline{a},P)/\sigma\}}. \tag{63}$$

Now consider the transition probability matrix Ω representing both the aging of cars and the "holding of the outside good" $\emptyset$ (i.e. the decision not to own a car, represented by the choice $d = \emptyset$). This is a straightforward generalization of the matrix Ω given in (47) above in the case where we excluded the possibility of an outside good. When there is an outside good, we defin Ω as the $(\overline{a}+1) \times (\overline{a}+1)$ transition probability given by

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1-\alpha(1) & \cdots & 0 & \alpha(1) \\ 0 & 0 & 0 & 1-\alpha(2) & \cdots & \alpha(2) \\ 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1-\alpha(\overline{a}-2) & \alpha(\overline{a}-2) \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1-\alpha(0) & 0 & \cdots & 0 & \alpha(0) \end{bmatrix}. \tag{64}$$

Notice that Ω in equation (64) includes the $(\overline{a} \times \overline{a})$ transition matrix S in equation (47) as its lower right $\overline{a} \times \overline{a}$ submatrix. This submatrix accounts for the aging of cars among the fraction of the population that own cars. However the fraction of the population that do not own cars at time t will also enter the next period $t+1$ not owning a car, and hence the transition probability in the $(1,1)$ element of Ω in equation (64) is 1 and the other elements in the first row of Ω are zero. For similar reasons, all of the elements in rows 2 to $\overline{a}+1$ of the first column of Ω are zero. This simply tells us that there is zero probability that consumers in this economy who owned a car at time t will transit to the state of not owning a car at the *start* of time $t+1$, i.e. before they have a chance to trade (and potentially sell their current vehicle without replacing it with another one).

Thus Ω in equation (64) represents the transition dynamics for the supply of vehicles in the economy, as well as the (trivial) transition dynamics for the fraction of the consumers in the economy who choose the outside good (i.e. choose not to own a car). Let $q_\emptyset$ be the $1\times(\overline{a}+1)$ invariant distribution corresponding to the transition probability matrix Ω in equation (64) and let q be the $1\times\overline{a}$ invariant distribution for the lower right $\overline{a}\times\overline{a}$ submatrix of Ω given in equation (47). Then it is easy to see that we have

$$q_\emptyset\Omega = q_\emptyset, \tag{65}$$

where

$$q_\emptyset = (q_\emptyset(\emptyset), (1-q_\emptyset(\emptyset))q). \tag{66}$$

Thus, $q_\emptyset$ reflects a *scaling* of the invariant distribution q from equation (44) to reflect the fact that only $1-q_\emptyset(\emptyset)$ of the population owns cars. However the the *conditional invariant distribution q* (i.e. the distribution of car ages in steady state for the fraction of the population that does own cars) is the same distribution (44) as in the case without an outside good, since allowing for the option not to own cars does not affect the aging dynamics of cars for the people who do own them, and thus cannot affect its steady state distribution.

It is easy to see there is a continuum of invariant distributions in this case, i.e. there are a continuum of solutions to equation (65) corresponding to each possible choice of $q_\emptyset(\emptyset)\in[0,1]$. However we now show how $q_\emptyset(\emptyset)$ is "tied down" by the equilibrium in the car market, when consumers have choices over whether to own a car or not, and if so, which age of car to own.

These "choice dynamics" are given by the $(\overline{a}+1\times\overline{a}+1)$ transition probability matrix $\Delta(P)$

given by

$$\begin{bmatrix} \Pi(\emptyset|\emptyset) & \Pi(1|\emptyset) & \cdots & \Pi(\overline{a}-1|\emptyset) & \Pi(0|\emptyset) \\ \Pi(\emptyset|1) & \Pi(1|1)+\Pi(-1|1) & \cdots & \Pi(\overline{a}|1) & \Pi(0|1) \\ \Pi(\emptyset|2) & \Pi(1|2) & \cdots & \Pi(\overline{a}-1|2) & \Pi(0|2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Pi(\emptyset|\overline{a}-1) & \Pi(1|\overline{a}-1) & \cdots & \Pi(\overline{a}-1|\overline{a}-1)+\Pi(-1|\overline{a}-1) & \Pi(0|\overline{a}-1) \\ \Pi(\emptyset|\overline{a}) & \Pi(1|\overline{a}) & \cdots & \Pi(\overline{a}-1|\overline{a}) & \Pi(0|\overline{a}) \end{bmatrix}, \tag{67}$$

Once again, the transition probability matrix $\Delta(P)$ is the *within period post-trade transition probability matrix* — i.e. it represents the dynamics of how consumers change the status of owning or not owning cars during the instantaneous trades that take place just after the start of each period t but before the economy transits to the next period $t+1$.

Take care to notice the subtle notational distinction between the empty set symbol $\emptyset$ and the numeral 0: for example $\Pi(\emptyset|\emptyset)$ represents the "self-transition" from the no car state to itself — i.e. the probability that a consumer who does not own a car at the start of period t chooses to remain without a car (i.e. to choose the "outside good" again), whereas $\Pi(0|\emptyset)$ represents the fraction of consumers who do not have a car at the start of period t who choose to purchase a brand new car, $d=0$. Clearly the transition probabilities in the first row of $\Delta(p)$ must add up to 1, since all consumers who do not own a car face the same choice set $d \in \{\emptyset, 0, 1, \dots, \overline{a}-1\}$.

If a consumer owns a car, they have the choice set $d \in \{\emptyset, -1, 0, 1, \dots, \overline{a}-1\}$ that includes the option $d=\emptyset$ of "purging" their current car (i.e. selling it but not replacing it), or keeping their current car, $d=-1$, or trading their car for another car of age $d \in \{0, 1, \dots, \overline{a}-1\}$. Thus for ages $a \in \{1, 2, \dots, \overline{a}-1\}$ the diagonal elements of the matrix $\Delta(P)$ includes the probability of keeping the currently held car, $\Pi(-1|a)$. However at the scrappage age, $a=\overline{a}$, we impose the restriction that consumers who own these cars are forced to scrap them, hence $\Pi(-1|\overline{a})=0$ in the lower right corner element in position $(\overline{a}+1, \overline{a})$ of $\Delta(P)$. It is easy to see that all rows of $\Delta(P)$ are in the $[0,1]$ interval and sum to 1, and hence $\Delta(P)$ can be regarded as a sort of a transition probability matrix, though it represents transitions in the holding of cars and the outside good in this economy after trade, within the same period t rather that a between-period transition probability matrix such as the transition probability matrix Ω in (64).

It is easy to see from our definition of choice probabilities given above that $\Delta(P)$ is a transition probability matrix for *any* $P \in R^{\overline{a}-1}$. As above, we ignore the prices $P(0)=\overline{P}$ and $P(\overline{a})=\underline{P}$, due to our assumption that new car prices and scrap prices are exogenously fixed, so there are only $\overline{a}-1$ "endogenous" prices that clear the secondary market, $(P(1),P(2),\ldots,P(\overline{a}-1))$. We now wish to define the conditions for equilibrium in the car market allowing for the presence of an outside good. Similar to our definition of equilibirum in the case where we ruled out the possibility of an outside good, there will be $\overline{a}-1$ equations that set the excess demand for cars of each age $a \in \{1,2,\ldots,\overline{a}-1\}$ to zero. However when there is an outside good, we need to add an additional equation to ensure that the "excess demand for the outside good" is zero as well. This extra equation serves to "tie down" the fraction of the economy that choose the outside good, $q_\emptyset(\emptyset)$, that we have shown is not determined in steady state using the equation (65) alone.

Unlike the case where we excluded the possibility of an outside good, when consumers have the option to choose an outside good the equilibrium quantities $q_\emptyset$ will generally depend on both the conjectured scrappage threshold $\overline{a}$ and the equilibrium price vector P, which we denote by $q_\emptyset(P,\overline{a})$. We will derive a formula for this dependence on P shortly, but intuitively it arises because prices in the secondary market for cars affect the desirability of cars relative to the outside good, and hence the fraction of consumers who choose the outside good, $q_\emptyset(\emptyset)$, will depend on both $\overline{a}$ and P. However to keep notation compact, we will suppress the dependence of $q_\emptyset$ on $(\overline{a},P)$ and only denote its dependence on the age/outside good index a, $q_\emptyset(a)$ for $a \in \{\emptyset,1,\ldots,\overline{a}-1\}$ below, where $q_\emptyset(a)$ is the fraction of the population that owns cars of age a in a stationary competitive equilibrium, in the model where we allow for the possibility of an outside good $\emptyset$.

As we noted in equation (66) above, once we determine $q_\emptyset(\emptyset)$, the other values of $q_\emptyset(a)$ for $a \in \{1,\ldots,\overline{a}\}$ are fully determined, so the addition of the outside good only results in one additional "unknown" $q_\emptyset(\emptyset,P,\overline{a})$ and once this is determined, then all of the other values of $q_\emptyset$ are determined.

When we add the extra equation to enforce the constraint that excess demand for the outside good is zero as well, we obtain a system of $\overline{a}$ equations in the $\overline{a}$ unknowns $(P,q_\emptyset(\emptyset))$ $ED(P,q_\emptyset(\emptyset))=0$ where $ED(P,q_\emptyset(\emptyset))$ is given by

$$\begin{bmatrix} \Pi(\emptyset|\emptyset,P)q_\emptyset(\emptyset)+[1-q_\emptyset(\emptyset)]\left[\sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a)\right]-q_\emptyset(\emptyset) \\ \Pi(1|\emptyset,P)q_\emptyset(\emptyset)+[1-q_\emptyset(\emptyset)]\left[\sum_{a=1}^{\overline{a}}\Pi(1|a,P)q(a)-q(1)[1-\Pi(-1|1,P)]\right] \\ \Pi(2|\emptyset,P)q_\emptyset(\emptyset)+[1-q_\emptyset(\emptyset)]\left[\sum_{a=1}^{\overline{a}}\Pi(2|a,P)q(a)-q(2)[1-\Pi(-1|2,P)]\right] \\ \cdots \\ \Pi(\overline{a}-2|\emptyset,P)q_\emptyset(\emptyset)+[1-q_\emptyset(\emptyset)]\left[\sum_{a=1}^{\overline{a}}\Pi(\overline{a}-2|a,P)q(a)-q(\overline{a}-2)[1-\Pi(-1|\overline{a}-2,P)]\right] \\ \Pi(\overline{a}-1|\emptyset,P)q_\emptyset(\emptyset)+[1-q_\emptyset(\emptyset)]\left[\sum_{a=1}^{\overline{a}}\Pi(\overline{a}-1|a,P)q(a)-q(\overline{a}-1)[1-\Pi(-1|\overline{a}-1,P)]\right] \end{bmatrix} \quad (68)$$

where $q(a)$, $a = 1,\ldots,\overline{a}$ is given by formulas (45) and (46) above. The system of equations (68) is basically the same as the set of excess demands for the case where there is no outside good, except for the addition of the first equation, which can be rewritten as

$$q_{\emptyset}(\emptyset)\left[1-\Pi(\emptyset|\emptyset)\right] = \left[1-q_{\emptyset}(\emptyset)\right]\left[\sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a)\right]. \tag{69}$$

The left side of equation (69) is the "supply" of the outside good, i.e. the fraction of the population who did not own a car at the start of period t but who choose to buy a car. The right hand side of equation (69) is the "demand" for the outside good, i.e. the fraction of the population who do own cars, but which to sell them and "purchase the outside good" instead. Another way to say this is that equation (69) states that the fraction of the population who don't have a car and which to buy a new or used one must equal the fraction of the population who have a new or used car and want to sell it and not replace it. This keeps the market in "flow equilibrium" similar to the way we showed how the fraction of population who scrap cars each period must equal the fraction of the population who purchase new cars in order to keep the car market in flow equilibrium.

We can solve equation (69) for this expression for $q_{\emptyset}(\emptyset)$

$$q_{\emptyset}(\emptyset) = \frac{\sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a)}{1-\Pi(\emptyset|\emptyset,P)+\sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a)}. \tag{70}$$

This formula has an intuitive interpretation: it is the invariant probability of the state $\emptyset$ for a two state Markov chain with 2×2 transition probability matrix M given by

$$M = \begin{bmatrix} \Pi(\emptyset|\emptyset,P) & 1-\Pi(\emptyset|\emptyset,P) \\ \sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a) & 1-\sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a) \end{bmatrix}. \tag{71}$$

Thus, $\Pi(\emptyset|\emptyset,P)$ is the transition probability for staying in the "outside good state" whereas the lower right $(2,2)$ element is the transition probability for staying in the "car state", and $q_{\emptyset}(\emptyset)$ is the invariant probability of the no-car state implied by the two state Markov chain M. Once we know this probability, the invariant probabilities for the various car ages are $[1-q_{\emptyset}(\emptyset)]q(a)$, $a = 1,\ldots,\overline{a}$ where $q(a)$ are given by formulas (45) and (46) above.

We can substitute the expression for $q_{\emptyset}(\emptyset,P,\overline{a})$ back into the excess demand system (68), eliminating the first equation, resulting in a reduced system of $\overline{a}-1$ equations in the $\overline{a}-1$ unknowns P. Any solution to this reduced system of equations constitutes an equilibrium in the economy that accounts for the existence of the outside good.

The analog of Proposition 1 also holds when we allow for an outside good, i.e. $q_\emptyset$ is the invariant distribution of the the demand or post trade transition probability matrix $\Delta(P)$ given in equation (67) as well as the invariant distribution of the "supply transition probability matrix" $S_\emptyset$ given in equation (64).

Proposition 2 *Let P be an equilibrium price vector with equilibrium scrappage threshold $\overline{a}$ from a solution to (68) and let $\Delta(P)$ be the mapping from $R^{\overline{a}-1} \to R^{\overline{a}} \times R^{\overline{a}}$, i.e. the $\overline{a} \times \overline{a}$ transition probability matrix with elements given in equation (67). Then the invariant probability distribution $q_\emptyset$ given in (65) is also the invariant probability distribution for the transition probability matrix Ω in equation (64) and the demand or post trade transition probability matrix $\Delta(P)$ given in equation (67)*

$$q_\emptyset = q_\emptyset \Delta(P) = q_\emptyset \Omega = q_\emptyset \Delta(P)\Omega \tag{72}$$

where $q_\emptyset$ has the representation given in (66) where $q(a)$, $a = 1, \ldots, \overline{a}$ is given by formula (45) and (46) and $q_\emptyset(\emptyset)$ is given by equation (70).

Appendix 2 provide some formulas that are useful for computing equilibria in the case of an outside good.

3.5 Equilibria with Time-Invariant Heterogeniety

The model outlined above has consumer heterogeneity of a particular type, namely only transient, idiosyncratic heterogeneity that is due to the *IID* preference shocks. This idiosyncratic heterogeneity is sufficient to generate trade and equilibrium even in the presence of transactions costs, but it does not enable the model to capture the larger gains from tade from the operation of a secondary market that come from trade between consumers when there are persistent differences in consumer preferences for cars. These persistent differences in the extreme case can be captured by allowing for different *types* of consumers, which we denote by τ. The types can index permanent differences in consumer preferences, which we denote by utility functions indexed both by the age of the car a and the type of the consumer, τ, given by $u(a, \tau)$.

In our continuum consumer economy let there be a finite number of types $\{\tau_1, \ldots, \tau_n\}$ and let $f(\tau_i)$ be the fraction of consumers of type τ_i, $i = 1, \ldots, n$. Let $v(d, a, \tau)$ be the decision-specific value function for a type τ consumer, and $P(d|a, \tau)$ be the corresponding choice probability for these consumers. In addition, let $\Delta(P, \tau)$ be the trade transition probability matrix for a consumer of type τ when the price vector is P, given by formula (67) above. In a stationary equilibrium, consumers will end up "specializing" in their holdings of different ages of automobiles, and there may be differences across types of consumers in the fraction of each type that holds the outside good.

To account for this and define an equilibrium with a finite number of different types of consumers, let q_τ be the invariant probability distribution for the transition probability matrix $\Delta(P,\tau)\Omega$, i.e.

$$q_\tau = q_\tau \Delta(P,\tau)\Omega. \tag{73}$$

That is, in a stationary equilibrium, the distribution of holdings of consumers of type τ will be given by the solution q_τ to equation (73). The overall stationary holdings distribution q will be given by

$$q = \sum_\tau q_\tau f(\tau). \tag{74}$$

We will need to show that in a stationary equilibrium we have $q = q\Omega$, i.e. the overall distribution of holdings is an invariant distribution with respect to the aging/accident transition probability Ω. But before demonstrating this, we define the equilibrium price vector P as the value of P that sets aggregate excess demand, $ED(P) = 0$, similar to the previous sections. However with different types of consumers we must calculate supplies and demands weighting both by the probability $f(\tau)$ of each type of consumer, and to recognize that different consumers have different stationary holdings distributions in equilibrium. Let $S(a,P)$ be the aggregate supply of cars of age a, $a = 1,\ldots,\overline{a}-1$ when the price is P. This is given by

$$S(P)(a) = \sum_\tau q_\tau(a)[1-\Pi(-1|a,\tau,P)]f(\tau) \tag{75}$$

where $q_\tau(a)$ is the fraction of cars of age a held by consumers of type τ implied by the invariant distribution q_τ given in equation (73). Similarly let $D(a,P)$ be the aggregate demand for cars of age $a = 1,\ldots,\overline{a}-1$ given by

$$D(P)(a) = \sum_\tau \left[\Pi(a|\emptyset,\tau,P)q_\tau(\emptyset) + \sum_{a'=1}^{\overline{a}} \Pi(a|a',\tau,P)q_\tau(a')\right] f(\tau) \tag{76}$$

Then $S(P)$ and $D(P)$ are the $\overline{a}-1\times 1$ vectors whose elements are given by formulas (75) and (76), respectively. Note that we can use Walras Law or the conditions for the invariant distributions in equation (73) to show that if we set excess demand to zero in the markets for used cars of ages $a = 1,2,\ldots,\overline{a}-1$, then the market for the outside good will also automatically clear, i.e.

$$\sum_{\tau} q_{\tau}(\emptyset) f(\tau) = \sum_{\tau} \left[\Pi(\emptyset|\emptyset,\tau,P) q_{\tau}(\emptyset) + \sum_{a=1}^{\overline{a}-1} \Pi(\emptyset|a,\tau,P) q_{\tau}(a) \right] f(\tau). \tag{77}$$

Proposition 3 below verifies that our definition of stationary equilibrium with time-invariant heterogeneity is well defined in the sense that if such an equilibrium exists, the aggregate holdings distribution must be an invariant distribution of Ω.

Proposition 3 *If a stationary equilibrium exists to the economy with consumers with time invariant heterogeneity given above, then we have*

$$q = \sum_{\tau} q_{\tau} f(\tau) = \sum_{\tau} q_{\tau} \Delta(\tau,P) \Omega f(\tau) = q\Omega \tag{78}$$

Proof: The condition that $ED(P) = 0$ is mathematically equivalent to the equation

$$q = \sum_{\tau} q_{\tau} f(\tau) = \sum_{\tau} q_{\tau} \Delta(\tau,P) f(\tau) \tag{79}$$

Using this equation, multiply q by Ω to get

$$q\Omega = \sum_{\tau} q_{\tau} \Delta(\tau,P) f(\tau) \Omega = \sum_{\tau} q_{\tau} \Delta(\tau,P) \Omega f(\tau) = \sum_{\tau} q_{\tau} f(\tau) = q \tag{80}$$

□Our approach to solving an equilibrium changes in the presence of time-invariant heterogeneity. We must verify smoothness of q_{τ} in P, i.e. that $q_{\tau}(P) = q_{\tau}(P)\Delta(P,\tau)\Omega$ is a well-defined implicit function of P that is continuously differentiable in P. We provided these details in appendix 3 along with formulas for the Jacobian matrices, $\nabla_P q_{\tau}(P)$. Basically, to apply Newton's method to solve the system $ED(P) = 0$ we must first calculate the invariant distributions in equation (73) for each type of consumer, as well as the Jacobian $\nabla_P q_{\tau}(P)$. Then we use these to calculate the excess demand vector $ED(P)$ as well as the Jacobian of $ED(P)$, $\nabla_P ED(P)$ and iterate using the usual Newton formula.

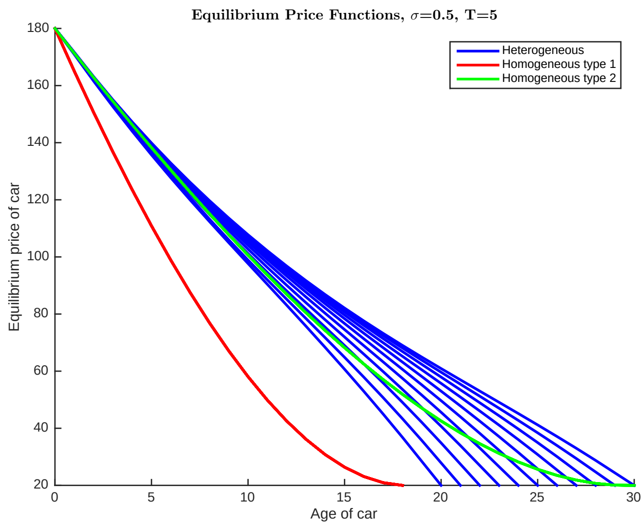
Note that it will not generally be the case that $q_{\tau} = q_{\tau}\Delta(\tau,P)$, even in equilibrium. This condition is equivalent to the condition that *excess demand for each type* τ *equals zero* and this will not generally hold if there is net trade between different types of consumers in the economy. The reason is that in equilibrium, there is no need for supply to equal demand for used cars within each consumer types: the only requirement that equilibrium imposes is that aggregate demand (for all types) equals the aggregate supply of each type of car of each age $a = 1,\ldots,\overline{a}-1$. For example, we may observe patterns of specialization and gains from trade across types of consumers with "poor" consumers "specializing" in older cars and richer consumers specializing in newer cars. We will illustrate such an equilibrium below, and in this equilibrium the rich consumers will be net demanders of newer cars and net suppliers of older cars, and the poorer consumers will be net demanders of older cars but will not be suppliers of newer cars. Thus, we will see trade between rich and poor consumers of cars of roughtly middle ages: the rich supply their middle aged cars to the poor consumers and supply and demand will be equal in aggregate, but the supply of middle aged cars of the rich will not equal the demand by the rich for these cars, and similarly the demand for middle aged cars by poor consumers will not equal the supply of these cars by other poor consumers.

Figure 1 illustrates some of the equilibrium in a heterogeneous agent economy with two types of consumers. Consumers of type τ_1 have utility functions $u(a,y,\tau_1) = 70 - a/10 - .08y$ where y represents non-car consumption. Their utility for the outside good, i.e. not having a car, is $u(\emptyset,y,\tau_1) = 69 - .08y$. Type τ_2 consumers have utility function $u(a,y,\tau_2) = 40 - a/20 + .09y$ and have a utility for the outside good of $u(\emptyset,y,\tau_2) = 39.3 + .09y$. Thus, type τ_1 consumers have a lower "marginal utility of money" (i.e. $\mu = .08$) than type τ_2 consumers, and they also have a stronger preference for cars and their utility for older cars decreases more quickly as cars age compared to type τ_2 consumers. However type τ_1 consumers have a higher utility for the outside good, i.e. the non-car alternative, compared to type τ_2 consumers.

Thus it seems reasonable to conjecture that type τ_1 consumers are akin to "richer" consumers who would be more likely to own newer cars in equilibrium, whereas type τ_2 consumers can be considered as "poorer" consumers who are likely to own older cars in equilibrium. However it is not clear *a priori* whether what relative fractions of type τ_1 and τ_2 consumers will choose to have no car. It might appear that since the type τ_1 consumers have a higher utility of having no car than type τ_2 consumers, there will be fewer type τ_1 consumers who own cars overall. However we must also keep in mind that type τ_1 consumers also obtain higher utility from owning cars, so the question depends on the equilibrium in the car market and the relative *discounted utilities* of owning a car versus not owning a car. This question can only really be settled once we have calculated the equilibria of this model.

In figure 1 we plot some of the equilibria we calculated using a hybrid algorithm that starts with the Matlab `fsolve` function for solving systems of nonlinear equations and then switches to a pure Newton algorithm after `fsolve` converges with a loose convergence tolerance. The idea is that `fsolve` employs more robust trust region algorithm to find a value of P that is in a

Figure 1: Equilibrium Price Functions in a Two Type Economy



domain of attraction of an equilibrium of the model. Once in a suitable domain of attraction, switching to Newton's method results in faster quadratic convergence to the equilibrium.

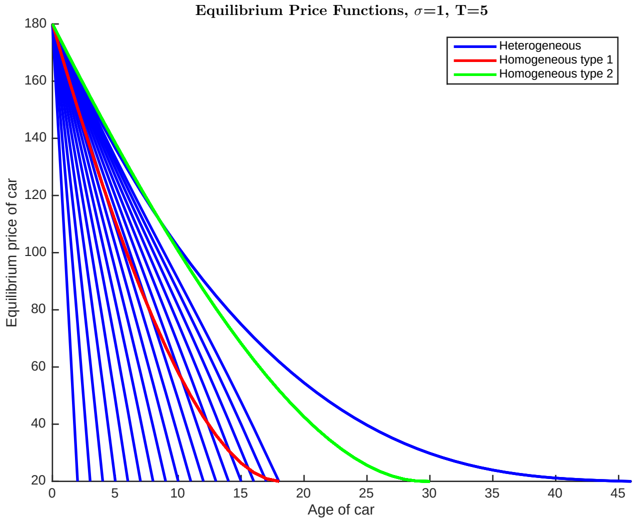
We were able to find multiple equilibria using the following "bootstrap" method. We started by fixing a guess for $\overline{a}$, the equilibrium scrapping age in the model. In our illustrative economy, 20% of the population are type τ_1 consumers and 80% are type τ_2's, and both types have common discount factor $\beta = .95$ and a fixed transaction cost of $T = 5$. The figure plots the Pareto dominant homogeneous agent equilibria as the red and green lines. The scrappage ages for these homogeneous agent equilibria were used to determine an initial guess of the "lowest" (Pareto inferior) equilibrium of the heterogeneous agent model.

Thus, in figure 1 the red line plots the equilibrium price function for an economy with only the rich τ_1 consumers and under the assumption that all of these consumers own cars (i.e. we ruled out the outside good option). This equilibrium has a value of $\overline{a} = 18$. We then guess that a heterogeneous agent equilibrium would have to have a value of $\overline{a} > 18$ since with heterogeneity, there should be strict gains from trade that enable the type τ_2 consumers to "specialize" in holding older cars, thereby extending the age at which they are scrapped in equilibrium.

However we could also find equilibria for values of $\overline{a}$ equal to 18 and lower, and these other equilibria are illustrated in figure 2 below.

In summary we were able to find multiple Pareto-ranked equilibria for values of $\overline{a}$ ranging from $\overline{a} = 3$ to $\overline{a} = 46$ in the model under the assumption of a regulatory constraint that consumers are not allowed to hold any car of age $\overline{a}$ or older. In the absence of such a constraint, we

Figure 2: Equilibrium Price Functions in a Two Type Economy



conjecture there will only be a single equilibrium — the Pareto dominant equilibrium where $\overline{a} = 46$. However we have not established whether there might be mulitiple (non-Pareto ranked) equilibria for any given value of $\overline{a}$ including $\overline{a} = 46$.

As you can see from figure 1 the lowest equilibrium in the figure (lowest of the blue lines) is close to the homogeneous agent equilibrium for type τ_2 consumers (green line) for ages 1 to 12, but results in lower prices for ages 13 to 20. Given this equilibrium we then used it as a starting guess for an equilibrium with $\overline{a} = 21$ and our polyalgorithm was able to converge and find an equilibrium for this scrappage threshold as well. The equilibrium with $\overline{a} = 21$ Pareto dominates the equilibrium with $\overline{a} = 20$ similar to what we found in the case of homogeneous agent equilibria. The Pareto dominance of the equilibrium with $\overline{a} = 21$ over the equilibrium with $\overline{a} = 20$ is reflected by a lower fraction of both type τ_1 and τ_2 consumers choosing the outside good (whose utility is fixed, independent of the conjectured value of $\overline{a}$) in the latter equilibrium.

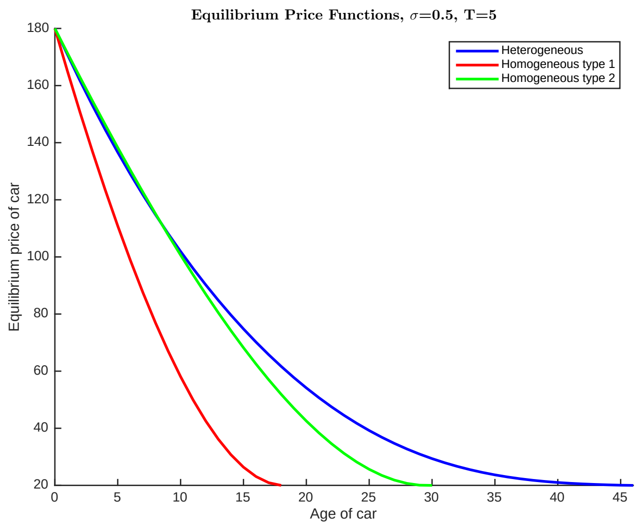
Continuing in this way we were able to use this "bootstrap" process to find a seqeunce of equilibria to the model with increasing values of $\overline{a}$ until we reached $\overline{a} = 47$ and our polyalgorithm produced a solution that violated monotonicity and the requirement that for each age of used car a we have $P(a) \in [\underline{P}, \overline{P}]$. It follows that the "largest" possible equilibrium, which is also the Pareto dominant equilibrium in this model (at least the best equilibrium we were able to find) is one where $\overline{a} = 46$. This equilibrium reflects substantial gains from trade between type τ_1 and τ_2 consumers that also has the effect of substantially extending the useful lives of cars in this economy. If the economy consisted only of the rich type τ_1 consumers and no type τ_2 consumers and there was no idiosyncratic heterogeneity, then there would be no trade in used cars and these consumers would either a) not hold cars at all (if the discounted utility of the outside good is sufficiently high), or b) all consumers would follow a "buy and hold" strategy of buying brand new cars at price $\overline{P} = 180$ and holding them until $\overline{a} = 18$ and then selling them as scrap for price $\overline{P} = 20$ and then buying another new car to replace the scrapped car.

When there are two persistent types of heterogeneity, this creates the gains from trade that enable an active secondary market in used cars despite the relatively high transactions costs of $T = 5$. In the Pareto dominant heterogeneous agent equilibrium cars are not scrapped until age $\overline{a} = 46$, which is more than twice the lifespan that type τ_1 would find optimal if they were to follow the buy and hold strategy they would adopt in the absence of a secondary market (a strategy which is suboptimal in the presence of a secondary market), and 16 years longer than the optimal replacement threshold of $\overline{a} = 30$ that even the poorer type τ_2 agents would adopt in "autarky".

Figure 3 plots the Pareto dominant equilibrium price function with $\overline{a} = 46$ that we found from this bootstrapping procedure. The Pareto dominant equilibrium starts out very close to the homogeneous agent equilibrium with only type τ_2 agents (the green line in the figure) but ultimately results in significantly higher prices for car ages $a \geq 10$, and of course, all cars would be scrapped at age $\overline{a} = 30$ in the homogeneous agent equilibrium (which is an autarky outcome where consumers all buy new cars and hold them until age $\overline{a} = 30$ before scrapping them and buying another new car). The useful life of cars has been extended by 16 years in the heterogeneous agent equilibrium, and the welfare of both type τ_1 and τ_2 consumers is significantly greater than what they would each obtain in autarky, and thus we can directly calculate the "gains from trade" from the existence of a secondary market in used autos.

Figure 4 plots the equilibrium holdings distributions $q_\emptyset$, $q_\emptyset(\tau_1)$ and $q_\emptyset(\tau_2)$ that obtain in the Pareto dominant equilibrium of the model. About 15% of the type τ_1 consumers choose not to own a car, whereas about 10% of the poorer type τ_2 consumers choose not to own a car. Given that 20% of the population are type τ_1 and 80% are τ_2, the overall fraction of the population that does not own a car is $q_\emptyset(\emptyset) = 0.115$. These shares are reflected in figure 4 as the value of the lines at the $a = -1$, and for $a \in \{0, 1, \ldots, 46\}$ the curves plot the respective fractions of type τ_1, τ_2 consumers and the overall population that hold a car of age *a just after trading is complete at the start of each period.* Recall that the flow equilibrium condition guarantees that the fraction of population that scraps a car is equal to the fraction of the population that buys

Figure 3: Pareto Dominant Equilibrium in a Two Type Economy

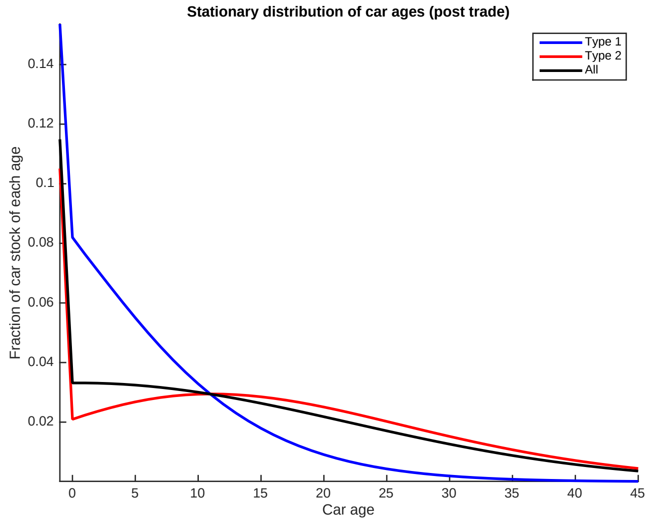


a new car. So $q_{\emptyset}(0)$ also represents the fraction of the population that scraps their cars each period in a stationary equilibrium.

The self-sorting of the two types of consumers into the ages of cars they hold in equilibrium is obvious from figure 4. The "rich" type τ_1 consumers hold the newest cars and in particular are much more likely to buy new cars than the poorer type τ_2 consumers. However due to the much higher utility that type τ_1 consumers receive from not owning a car, the fraction of type τ_1 consumers who do not have a car is also higher as well.

We believe that very rich patterns of holding cars and trading cars can be obtained from this relatively simple dynamic equilibrium model. We believe it can be extended in various interesting directions, such as allowing for multiple types of cars and oligopoly competition in the new car market, where oligopolists consider not only the competition from other car manufacturers, but also the "competition" provided by their own used cars, similar to the lines of Esteban and Shum (2007). The model could be extended further by allowing oligopolists to make product quality and durability choices as well as pricing decisions, and it is potentially possible to allow for the presence of rental intermediaries as well. We leave these extensions for future work and turn now to extending the model to allow for macroeconomic shocks that affect consumers willingness to purchase new cars, or to economize by moving to the no car state during recessionary periods. This extension raises important new questions about how to define a *non stationary equilibrium* where car holdings and prices change of time in response to macroeconomic shocks.

Figure 4: Equilibrium holdings distribution in a Two Type Economy



Before we do, we note that there is more work to be done on characterizing the multiplicity of equilibria in this model. We have shown that multiple equilibria exist from our computational examples but we have not established whether our inability to find an equilibrium below some minimum value of $\overline{a}$ (i.e. $\overline{a} = 20$ in the example above) and beyond some upper bound value $\overline{a}$ ($\overline{a} = 46$ in the example above, which we have called the "Pareto dominant equilibrium") means that there actually are no such equilibria or it simply indicates that our bootstrapping procedure using or equilibrium polyalgorithm simply failed to find an equilibrium for higher values of $\overline{a}$ that may actually exist but may require a more sophisticated algorithm to find them. In addition, we do not know that even for the values of $\overline{a}$ for which we have found an equilibrium, whether they may be other values of P for the same values of $\overline{a}$ which are also equilibria.

3.6 Equilibrium with persistent, time-varying heterogeneity

The previous sections have covered the two polar cases: a) all heterogeneity in consumers is of the form of *IID* preference shocks, and b) in addition to the *IID* extreme value heterogeneity there is time invariant heterogeneity in the form of a finite number of fixed consumer types. This section covers the intermediate case where there is heterogeneity that is time-varying and persistent. To fix ideas, introduce a Markovian state variable income y that takes two possible values, low income y_l and high income y_h. We assume that income evolves in a serially correlated manner with a transition probabilty $\pi(y'|y)$ and corresponding 2×2 transition probability matrix M

$$M = \begin{bmatrix} \pi(y_l|y_l) & \pi(y_h|y_l) \\ \pi(y_l|y_n) & \pi(y_h|y_h) \end{bmatrix}. \tag{81}$$

Let π also denote the invariant probability distribution for the income process, i.e. π is the unique solution to

$$\pi = \pi M \tag{82}$$

so $\pi(y_l)$ represents the fraction of consumers with low income in the economy in steady state and $\pi(y_h) = 1 - \pi(y_l)$ represents the fraction of high income consumers. In a stationary equilibrium these fractions do not vary over time as we have not allowed for any macro shocks that could could cause correlation across consumers in their idiosyncratic indpendently evolving income processes.

We assume that the level of income could affect car choices in two different ways: 1) y could enter the utility of owning a car $u(a,y)$, or 2) income could enter the marginal utility of money $\mu(y)$. For example if $\mu(y)$ is lower when y is high, this can generate an idiosyncratic motivation for either purchasing a car (for consumers who do not hold a car) or trading an existing car for a newer one.

Though we do not repeat the Bellman equations to conserve on space, let $v(d,a,y)$ denote the value function for a consumer who makes choice d in car state a when their income is y. In general, income will affect the values of different choices in different states, and this will induce different choice probabilities. Let $\Pi(d|a,y)$ be the choice probability of a consumer whose car state is a and whose current income is y. Since y affects utilities, and the values $v(d,a,y)$ it will clearly affect the choice probabilities for consumers as well.

Clearly all consumers who currently have high income will behave differently from those who have low income, and this will affect their post-trade holdings of cars. Let $\Delta(y,P)$ be the $\overline{a}+1 \times \overline{a}+1$ transition probability matrix for consumers in income state y given the current price vector P. This is the same *post-trade transition probability matrix* given in equation (67) except that we now allow for the effect of income y on this transition probability matrix.

We now claim there will be a stationary equilibrium in this market that takes the following form. Let q be the overall distribution of car holdings for the overall population. As in the case of time-invariant heterogeneity analyzed in the previous section, q must be be time invariant in the stationary equilibrium and statisfy the key condition,

$$q = q\Omega \tag{83}$$

However at any time, just after trading there will be two different holdings distributions for low and high income consumers. Let q_y be given by

$$q_y = q\Delta(y,P)\Omega. \tag{84}$$

Thus, q_y represents the holdings of cars for consumers whose income is currently equal to y just after trade occurs. If $\pi(y)$ is the invariant probability of income equalling level y, then we have

$$q = \sum_y \pi(y)q_y = \sum_y q\Delta(y,P)\Omega\pi(y) = q\Delta(P)\Omega \tag{85}$$

where $\Delta(P)$ is the the averaged post-trade transition probability matrix given by

$$\Delta(P) = \sum_y \pi(y)\Delta(y,P) \tag{86}$$

So we see that in a stationary equilibrium q is also an invariant probability of the average post-trade transition probability matrix $\Delta(P)$ just as we found in the case of an economy with no persistent heterogeneity (see equation (52) in Proposition 1). Equilibrium prices will be given by the solution to the same set of excess demand equations as in the case of of the "homogeneous agent equilibrium" (or more specifically the equilibrium with no persistent heteorgeneity amongst consumers). That is P will solve $ED(P) = 0$ where ED is the mapping given in equation (50) above, with the only difference being that the choice probabilities entering the definition of equilibrium are averaged by the invariant probabilities $\pi(y)$ of the different possible income states y,

$$\Pi(d|a,P) = \sum_y \Pi(d|a,y,P)\pi(y). \tag{87}$$

We conclude this section by discussing the difference between equation (84) which provides the distribution of holdings of cars at the start of period $t+1$ for consumers who had income y in period t when they executed their trades at the equilibrium prices P versus the type-specific holding distribution q_τ given by the invariant distribution in equation (73) in the previous section. The key difference is that in the case of q_y we use the overall averaged holdings distribution q as the distribution of holdings multiplying the $\Delta(y,P)\Omega$ transition matrix in equation (84) whereas in the case of time-invariant heterogeneity we used q_τ as the holdings distribution multiplying the matrix $\Delta(\tau,P)\Omega$ in equation (**??**).

The reason for this difference is that when there is time-invariant heterogeneity, a consumer who is type τ in period t will remain a type τ consumer in period $t+1$ with probability 1, and thus, the relevant distribution of holdings for consumers of type τ is q_τ in both periods t and $t+1$, and it follows that q_τ must be an invariant distribution for the transition probability matrix $\Delta(\tau,P)\Omega$. However when there is persistent by not completely time invariant heteorgeneity, the relevant distribution of holdings corresponding to the set of consumers who have income y at time t is the overall average distribution q rather than the income y specific holdings distribution q_y.

To see this, note that at time t the fraction of individuals who have a given income y (say low income y_l) is equal to a weighted average of consumers who previously had high income and those who previously had low income:

$$\pi(y_l) = \pi(y_l)\pi(y_l|y_l) + \pi(y_h)\pi(y_l|y_h) \tag{88}$$

Similarly, the distribution of holdings for all consumers who have low income at time t will be a weighted average of the income-specific holdings distributions q_y given in equation (84) above

$$q = \pi(y_l)q_{y_l} + \pi(y_h)q_{y_h} \tag{89}$$

Thus in a stationary equilibrium the "mixing" between consumers of different incomes generates a stationarity in overall holdings even though we will see in every time period differences in the holdings of cars of rich and poor consumers, both before and after trade occurs.

It is easy to see, then, from the forgoing discussion how to define equilibrium in cases where we have both time varying and time-invariant heterogeneity. For example if τ indexes time-invariant heterogeneity in consumers (i.e. different "types" of consumers) and y indexes time-varying state variables that result in time varying heterogeneity in consumers, then we can combine the equilibrium conditions in this section and the previous section to define a stationary equilibrium as a price vector P that sets excess demand $ED(P)$ to zero, where

$$ED(P) = D(P) - S(P) \tag{90}$$

and the supply and demand functions are given in equations (75 and (76) above respectively, where q_τ is given by the invariant distribution to equation (73) but with the main difference is that the choice probabilities entering equations (75) and (76) are weighted average of the y-specific transition probabilities

$$\Pi(d|a,\tau,P) = \sum_y \Pi(d|a,\tau,y,P)\pi(y) \tag{91}$$

where π is the invariant distribution of the $\{y_t\}$ process (which is again an idiosyncratic process that evolves indpendently over different consumers), and the $\Delta(\tau, P)$ matrix that constitutes the post-trade transition probability matrix is also a weighted average of the y-specific transition matrices

$$\Delta(\tau, P) = \sum_y \Delta(\tau, y, P)\pi(y). \tag{92}$$

Thus, we can define $q_{\tau,y}$ to be the holdings of type τ consumers who had income y in the previous period by

$$q_{\tau,y} = q_\tau D(\tau, y, P)\Omega \tag{93}$$

and clearly we will have

$$q_\tau = \sum_y q_{\tau,y}\pi(y) \tag{94}$$

and finally we have the overall adding up condition of Proposition 3 continuing to hold, i.e.

$$q = \sum_\tau q_\tau f(\tau) = q\Omega. \tag{95}$$

Finally it is possible to further extend this to allow the transition dynamics for the time-varying persistent heterogeneity y to depend on τ. So if $\pi(y'|y, \tau)$ is the transition probability for income for a type τ consumer, then following our previous abuse of notation, then let π_τ also denote the invariant distribution for income. Then it is easy to see all of the equations defining equilibrium hold except with π replaced by π_τ in the equations above.

3.7 Equilibrium with many types of cars

Suppose there are J different types of cars (e.g. makes/models), with corresponding new car prices $\overline{P}_j$, $j = 1, \ldots, J$. For simplicity, assume the scrap price $\underline{P}$ is the same for all J types of car. In this section we provide the equations for equilibrium in a market with $J > 1$ different types of cars.

Let $V(\emptyset, \varepsilon, \tau)$ be the discounted utility of a consumer τ who does not own a car. Since τ denotes the type of consumer which is time-invariant, we will suppress it to space in the equations below. The Bellman equation for this state is given by

$$V(\emptyset, \varepsilon) = \max\left[v(\emptyset, \emptyset) + \varepsilon(\emptyset), \max_{j \in \{1, \ldots, J\}} \max_{d \in \{0, 1, \ldots, \overline{a}_j - 1\}} [v(d, j, \emptyset) + \varepsilon(d, j)]\right], \tag{96}$$

where $\overline{a}_j$ is the scrappage age for car type j and

$$\begin{aligned} v(d,j,\emptyset) &= u(d,j) - \mu[P(d,j) + T(P,d,j)] + \beta\left[(1-\alpha(d,j))EV(d+1,j) + \alpha(d,j)EV(\overline{a}_j,j)\right] \\ v(\emptyset,\emptyset) &= u(\emptyset) + \beta EV(\emptyset), \end{aligned} \tag{97}$$

where $u(d,j)$ and $EV(d,j)$ are the current period utility and expected future utility that a consumer obtains from owning a car of type j and age d, and $EV(\emptyset)$ is the conditional expectation of $V(\emptyset,\varepsilon)$, which represents the expectation of future utility for a consumer who does not currently own a car. When ε is a vector of *IID* Extreme valued preference shocks, following equation (36) we have

$$EV(\emptyset) = \sigma \log\left(\exp\{v(\emptyset,\emptyset)\sigma\} + \sum_{j=1}^{J}\sum_{d=0}^{\overline{a}_j-1}\exp\{v(d,j,\emptyset)/\sigma\}\right). \tag{98}$$

Similarly, we extend the Bellman equation for a consumer who owns a car of type j and age a to allow for the option to "purge" their car, i.e. sell the car but not buy another to replace it:

$$\begin{aligned} V(a,j,\varepsilon) &= \max\Big[v(\emptyset,a,j) + \varepsilon(\emptyset), v(-1,a,j) + \varepsilon(-1,j), \\ &\quad \max_{j'\in\{1,\ldots,J\}}\ \max_{d\in\{0,1,\ldots,\overline{a}_{j'}-1\}}[v(d,j',a,j) + \varepsilon(d,j')]\Big], \end{aligned}$$

where $v(\emptyset,a,j)$ is the value of selling one's current car of type j age a and not replacing it

$$v(\emptyset,a,j) = u(\emptyset) + \mu P(a,j) + \beta EV(\emptyset). \tag{99}$$

and $v(d,j',a,j)$ is given by a similar equation to (33) above, i.e. the value of trading one's current car (a,j) for another car (d,j').

$$\begin{aligned} v(d,j',a,j) &= u(d,j') - \mu[P(d,j') - P(a,j) - T(P,a,j,d,j')] + \\ &\quad \beta\left[(1-\alpha(d,j'))EV(d+1,j') + \alpha(d,j')EV(\overline{a}_{j'},d)\right] \end{aligned} \tag{100}$$

but where we modify equation (37) to allow $EV(d,j)$ to account for the outside good as follows

$$
\begin{aligned}
EV(d,j) &= \sigma \log\left(\exp\{v(\emptyset,d,j)/\sigma\} + \sum_{j'=1}^{J}\sum_{d'=0}^{\overline{a}_{j'}-1}\exp\{v(d',j',d,j)/\sigma\}\right), \quad d=\overline{a}_j \\
EV(d,j) &= \sigma \log\Bigg(\exp\{v(\emptyset,d,j)/\sigma\} + \exp\{v(-1,d,j)/\sigma\} + \\
&\qquad \sum_{j'=1}^{J}\sum_{d'=0}^{\overline{a}_{j'}-1}\exp\{v(d',j',d,j)/\sigma\}\Bigg), \quad d<\overline{a}_j
\end{aligned}
\tag{101}
$$

where the first equation for $EV(d,j)$ in (101) above is for the current car is at the scrappage age threshold, $d=\overline{a}_j$, so keeping this car is assumed to no longer be an option.

As we discussed in the previous subsections the system of equations (98) and (101) are the equivalent of the Bellman equation, but after the *IID* extreme value random variable preference shocks ε have been integrated out. So as we previously discussed the system defines EV as the unique fixed point of a contraction mapping, and EV can also be calculated using Newton iterations exactly as described in (42) above. We can also use the implicit function theorem to show that EV is a smooth implicit function of P which we denote as $EV(P)$. From the solution EV we can construct the choice-specific values $v(d,j',a,j)$ which are also implicit functions of P and we write $v(d,j',a,j,P)$ to emphasize this. Then following the previous sections we can calculate the logit choice probabilities which are the same as written above except for the new choice probabilities for a consumer who does not own a car, $a=\emptyset$,

$$
\Pi(d,j|\emptyset,P) = \frac{\exp\{v(d,j,\emptyset,P)/\sigma\}}{\exp\{v(\emptyset,\emptyset,P)/\sigma\} + \sum_{j'=1}^{J}\sum_{d=0}^{\overline{a}_{j'}-1}\exp\{v(d,j',\emptyset,P)/\sigma\}}, \tag{102}
$$

and for consumers who do own a car of type j and age $a \in \{1,2,\ldots,\overline{a}_j\}$ we expand their choice sets to include the "purge" option, $d=\emptyset$ (i.e. to sell their current car and not replace it with another one) which has probability

$$
\Pi(\emptyset|a,j,P) = \frac{\exp\{v(\emptyset,a,j,P)/\sigma\}}{\exp\{v(\emptyset,a,j,P)/\sigma\} + \exp\{v(-1,a,j,P)/\sigma\} + \sum_{j'=1}^{J}\sum_{d=0}^{\overline{a}_{j'}-1}\exp\{v(d,j',a,j,P)/\sigma\}}, \tag{103}
$$

and as above we constrain any consumer who holds a car of type j and age $a=\overline{a}_j$ to scrap it, i.e. $\Pi(-1|\overline{a}_j, j, P) = 0$ so we have

$$\Pi(d, j'|\overline{a}_j, j, P) = \frac{\exp\{v(d, j', \overline{a}_j, j, P)/\sigma\}}{\exp\{v(\emptyset, \overline{a}_j, j, P)/\sigma\} + \sum_{j'=1}^{J} \sum_{d=0}^{\overline{a}_{j'}-1} \exp\{v(d, j', \overline{a}_j, j, P)/\sigma\}}. \tag{104}$$

To write the equation for excess demand, we first note with mulitple types of cars, each car may have different accident probabilities, so we let $\alpha_j(a)$ denote the accident probability for a car of type j and age a (which leads to the scrappage of the car). Then let $q_j(a)$ be the *conditional invariant distribution* of ages of cars of type j, following the same formula (**??**) as we derived for the single type car case above. Let q be a vector formed by concatenating the conditional invariant distributions for each of the J car types,

$$q = (q_1, q_2, \ldots, q_J) \tag{105}$$

Let S_j be the $\overline{a}_j \times \overline{a}_j$ transition probability matrix representing the aging of a car of type j allowing for accidents, i.e. the transition probability matrix given by formula (47) but with the age-specific accident probabilities $\alpha_j(a)$ given by those specific to car type j. Then q_j is the invariant distribution corresponding to S_j

$$q_j = q_j * S_j. \tag{106}$$

Define the constant $\overline{a}$ by

$$\overline{a} \equiv \sum_{j=1}^{J} \overline{a}_j \tag{107}$$

where $\overline{a}_j$ is the scrappage age for car type j. Now consider an $(\overline{a}+1) \times (\overline{a}+1)$ matrix M given by

$$M = \begin{bmatrix} 1 & z_1' & z_2' & \cdots & z_{J-1}' & z_J \\ z_1 & S_1 & 0_2 & \cdots & 0_{J-1} & 0_J \\ z_2 & 0_1 & S_2 & \cdots & 0_{J-1} & 0_J \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z_{J-2} & 0_1 & 0_2 & \cdots & 0_{J-1} & 0_J \\ z_{J-1} & 0_1 & 0_2 & \cdots & S_{J-1} & 0_J \\ z_J & 0_1 & 0_2 & \cdots & 0_{J-1} & S_J \end{bmatrix}, \tag{108}$$

where O_j is a $\overline{a}_j \times \overline{a}_j$ matrix of zeros and z_j is a $\overline{a}_j \times 1$ vector of zeros. Similar to the case where there is only one type of car, there will be a continuum of stationary distributions to

M, i.e. a continuum of solutions satisfying $q * M = q$. We will need to impose equilibrium restrictions in order to pick out a single solution, a single invariant distribution $q_\emptyset * M = q_\emptyset$ from the continuum of possible invariant distributions for the "supply transition probability transition matrix" M.

To do this, we follow the approach of the analysis of a single car type, but generalize it. Define the $(J+1) \times (J+1)$ transition probability matrix $L(P)$ by

$$L(P) = \begin{bmatrix} L(\emptyset|\emptyset,P) & L(1|\emptyset,P) & \cdots & L(J-1|\emptyset,P) & L(J|\emptyset,P) \\ L(\emptyset|1,P) & L(1|1,P) & \cdots & L(J-1|1,P) & L(J|1,P) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ L(\emptyset|J-1,P) & L(1|J-1,P) & \cdots & L(J-1|J-1,P) & L(J|J-1,P) \\ L(\emptyset|J,P) & L(1|J,P) & \cdots & L(J-1|J,P) & L(J|J,P) \end{bmatrix} \quad (109)$$

where $L(\emptyset|\emptyset,P)$ is the probability that a consumer who does not own a car at time t will continue not to own a car at time $t+1$ and is given by

$$L(\emptyset|\emptyset,P) = \Pi(\emptyset|\emptyset,P) \quad (110)$$

where again, $\Pi(\emptyset|\emptyset,P)$ is a weighted average of type-specific transition probabilities $\Pi(\emptyset|\emptyset,P,\tau_i)$, $i = 1,\ldots,n$ is a weighted average of type-specific transition probabilities in the case where there are n types of consumers, where the weights are $f(\tau_i)$, the fraction of consumers who are of type τ_i. Similarly $L(j|\emptyset,P)$ is the probability that a consumer who has no car at time t will choose to have a car of type j (but without caring about which specific age this car might be) at time $t+1$, and is given by

$$L(j|\emptyset,P) = \sum_{d=0}^{\overline{a}_j-1} \Pi(d,j|\emptyset,P). \quad (111)$$

Conversely, consider a consumer who currently owns a car of type j (again without regard for the specific age of this car). For this consumer, $L(\emptyset|j,P)$ represents the probability of "purging" this car, i.e. selling it and choosing to have no car at time $t+1$. This probability is given by

$$L(\emptyset|j,P) = \sum_{a=1}^{\overline{a}_j} \Pi(\emptyset|a,j,P)q_j(a), \quad (112)$$

where $q_j(a)$ is the fraction of cars of type j that are age a. Similarly $L(j|l,P)$ is the probability that a consumer who owns a car of type j at time t and chooses to trade it for a car of type l at time $t+1$ (again without regard to the ages of either car l or car j) and is given by

$$L(j|l,P) = \sum_{a=1}^{\overline{a}_l} \sum_{d=0}^{\overline{a}_j-1} \Pi(d,j|a,l,P) q_l(a), \tag{113}$$

if $j \neq l$ and

$$L(j|j,P) = \sum_{a=1}^{\overline{a}_j} \sum_{d=0}^{\overline{a}_j-1} \Pi(d,j|a,j,P) q_j(a) + \sum_{a=1}^{\overline{a}_j} \Pi(-1|a,j,P) q_j(a) \tag{114}$$

if $j = l$ where P is the concatenated vector of car-type-specific prices,

$$P = (P_1', P_2', \ldots, P_J')' \tag{115}$$

where P_j is the $\overline{a}_j - 1 \times 1$ vector of age specific prices of car type j, i.e. $P_j(a)$ is the price of a car of type j of age a.

Let $q_\emptyset$ be the invariant distribution for the transition probability matrix $L(P)$, and assume it is unique. It is an implicit function of the price vector P so we write it as $q_\emptyset(P)$ given by the solution to

$$q_\emptyset(P) = q_\emptyset(P) L(P) \tag{116}$$

with the interpretation that $q_\emptyset(\emptyset, P)$ is the fraction of the population who choose the outside good in steady state when the price vector is P and $q_\emptyset(j,P)$ is the fraction of the population who choose to own cars of type j, $j \in \{1,2,\ldots,J\}$. Thus $q_\emptyset$ provides the overall "market shares" for the outside good and the J car types. However to determine the more detailed breakdown of the J car types into their specific ages, we define the $\overline{a}_j \times 1$ vector given by

$$q_\emptyset(j,P) q_j = q_\emptyset(j,P)(q_j(1), \ldots, q_j(\overline{a}_j)) \tag{117}$$

which provides the fraction of the population holding cars of type j and each possible age a in a steady state equilibrium. So we can define the overall stationary distribution $q_\emptyset$ corresponding to a stationary equilibrium in this economy as the $\overline{a}+1 \times 1$ vector $q_\emptyset$ given by

$$q_\emptyset = (q_\emptyset(\emptyset), q_\emptyset(1) q_1, q_\emptyset(2) q_2, \ldots, q_\emptyset(J-1) q_{J-1}, q_\emptyset(J) q_J). \tag{118}$$

This is the generalization of equation (66) that we derived above for an economy with only one type of car.

We are now ready to define a stationary equilibrium in the case of multiple car types. It is simply a solution P^* to the system of equations

$$ED(P^*) = 0 \tag{119}$$

where $ED(P) = (ED_1(P), \ldots, ED_J(P))$ where

$$ED_j(P) = D_j(P) - S_j(P) \tag{120}$$

where $D_j(P)$ is the $\overline{a}_j - 1 \times 1$ vector given by

$$D_j(P) = \begin{bmatrix} \Pi(1,j|\emptyset,P)q_\emptyset(\emptyset,P) + \sum_{l=1}^{J}\sum_{a=1}^{\overline{a}_l}\Pi(1,j|a,l,P)q_\emptyset(l,P)q_l(a) \\ \Pi(2,j|\emptyset,P)q_\emptyset(\emptyset,P) + \sum_{l=1}^{J}\sum_{a=1}^{\overline{a}_l}\Pi(2,j|a,l,P)q_\emptyset(l,P)q_l(a) \\ \cdots \\ \Pi(\overline{a}_j-2,j|\emptyset,P)q_\emptyset(\emptyset,P) + \sum_{l=1}^{J}\sum_{a=1}^{\overline{a}_l}\Pi(\overline{a}_j-2,j|a,l,P)q_\emptyset(l,P)q_l(a) \\ \Pi(\overline{a}_j-1,j|\emptyset,P)q_\emptyset(\emptyset,P) + \sum_{l=1}^{J}\sum_{a=1}^{\overline{a}_l}\Pi(\overline{a}_j-1,j|a,l,P)q_\emptyset(l,P)q_l(a) \end{bmatrix} \tag{121}$$

and where $S_j(P)$ is given by

$$S_j(P) = q_\emptyset(j,P)\begin{bmatrix} [1-\Pi(-1|1,j,P)]q_j(1) \\ [1-\Pi(-1|2,j,P)]q_j(2) \\ \cdots \\ [1-\Pi(-1|\overline{a}_j-1,j,P)]q_j(\overline{a}_j-1) \\ [1-\Pi(-1|\overline{a}_j,j,P)]q_j(\overline{a}_j) \end{bmatrix}. \tag{122}$$

We find an equilibrium by searching for a price vector P that sets $ED(P) = 0$ where 0 is an $\overline{a} \times 1$ vector of zeros. Similar to our strategy above, we use the Matlab `fsolve` function that employs a more robust trust region algorithm to find an initial esitmate of P that is in a "domain of attraction" of an equilibrium point P^*, and then we switch to Newton's method to rapidly converge on the equilibrium.

4 Non-Stationary Equilibrium in the Automobile Market

Now, consider extending the previous framework to allow for the possibility of "macro shocks" that affect consumers' willingness to buy new cars, hold used cars, or hold no car at all. These shocks could also include fuel price shocks that affect the cost of operating autos, but for simplicity we will just refer to these as "macro shocks" in this section — i.e. shocks that are common to all consumers in the economy (unlike the idiosyncratic shocks such as accidents or idiosyncratic taste shocks which we assume are household-specific and distributed independently across housholds in the economy.

It is well known that as an economy goes into a recession, consumers are more fearful of losing their job and are likely to cut back on spending for precautionary reasons, and thus less likely to replace their old car with a new one. Or in other cases of consumers living near public transportation or in walking/biking distance of shopping or work locations, these may be more likely to purge their existing car as a way to raise precautionary cash reserves and enter the no car state during the recession. However once the economy goes back into recovery, many of the consumers without cars may choose to buy a car again (either a new or used vehicle) and many consumers who held on to their cars may find their existing cars have gotten rather old and broken down during the recession, and now with increased optimism of coming into a period of economic growth, lower unemployment and higher earnings, they may feel confident enought to trade their used car for a new car, or possibly a newer version of their car.

We can mimic these sorts of effect in our modeling framework by introducing a *macro state variable* m_t which for simplicity takes two possible values, $m_t = 1$ denoting a "boom" period for the macroeconomy, and $m_t = 0$ denoting a "bust" or recession period for the macroeconomy. Let $f(m'|m)$ denote the transition probability for the macro state, which can be represented by a 2×2 *macro state transition probability matrix* F

$$F = \begin{bmatrix} f(0|0) & f(1|0) \\ f(0|1) & f(1|1) \end{bmatrix} \tag{123}$$

and when the diagonal elements of F are closer to 1, this simple economy will have a higher degree of persistence for both boom and bust periods, respectively. We assume all consumers know F. Though our toy model does not include employment and earnings, we can capture the effect of macro shocks by allowing the marginal utility of money to depend on m, so we have $\mu(m, \tau_i)$ is the marginal utility of money of a consumer of type τ_i when the macro state is m. Just as we noted above that we can model "rich" consumers as those who have a lower marginal utility of money, we can also capture many of the effects of macro recessions and booms by allowing $\mu(m_t, \tau_i)$ to depend on the macro state m_t, so that for any type of consumer the marginal utility of money is higher in the bust macro state $m_t = 0$ and lower in the boom macro state $m_t = 1$. This will generate behavior that is similar to the "precautionary savings" motive discussed above, since consumers will be more likely to purge their cars in recession periods when the marginal utility of money is high, and buy cars (including new cars) in boom periods when the marginal utility of money is low.

To keep notation simple, we will consider the car where there is only one type of car, $J = 1$. When we consider the effect of adding this macro shock, the consumer problem becomes much more complex for the following reason: *there will no longer be a time-invariant stationary distribution for car holdings q_0 when their are macro shocks.* Instead this distribution will be continually changing in response to the shocks, which will create *waves* in the age distribution that will propogate over time in response to macro-induced changes in trading behavior of consumers.

For example, consider an economy that emerges from a long recession where there is "pent-up demand" for cars on the part of consumers. Then when the economy makes the transition from bust to boom, many consumers with used cars will choose to sell their used cars and buy new ones, and in addition a larger fraction of consumers who do not own a car will also choose to buy a new or used car. This pent up demand will lead to a larger flow of new car sales compared to the depressed level of car sales during the recession, and thus lead to a "crest" of a wave of new cars that will propogate over time. Thus if the economy transits from bust to boom at time t, then there will be a relatively large number of new car sales at time t and perhaps the next few additional periods (assuming the boom is persistent). This leads to a "wave" of new cars that will become a corresponding wave of cars of age $a = 1$ at time $t+1$ and a wave of cars of age $a = 2$ at time $t+2$ and so on. Due to stochastic accidents, the size of this wave will tend to attenuate over time, so the size or amplitude of the wave peak will tend to decline over time similar to what happens to a water wave as spreads out in a large body of water.

Conversely when the economy goes into a recession then new car sales will fall and more consumers will choose to purge their cars and enter the no car state. This will generate a "trough" that will also propogate over time through the age distribution of cars: if there are relatively few new cars purchased due to a recession that starts at time t, then there will be relatively fewer cars of age $a = 1$ at time $t+1$, and so on. So it is should be clear that in the presence of macro shocks there will be no stationary distribution of car holdings, but instead this distribution will be continually evolving. We use the term *non-stationary equilibrium* to account for this situation, where we use "non-stationary" as a synonym for an equilibrium that is *not time-invariant.*[3]

[3]However in the language of stochastic processes, a random process can change over time and still be described as "stationary" if the probability distribution of the random variables is time-invariant even if the realizations of the random variable

Thus, once we realize that the age distribution of cars will be continually changing, it follows that we must index this age distribution by the time index, which we denote by q_t, the probability distribution of the holdings of cars of all ages and the outside good at time t. Given this distribution and the macro shock m_t consumers will make trading/holding decisions at the start of period t and given these decisions, prices will have to adjust so that the supply and demand for the ages of all used ages are equated. However in a non-stationary equilibrium we relax the *flow equilibrium* condition that the fraction of the population who scrap their cars at time t must equal the fraction of consumers who buy a new car. Similarly, we drop the condition that the fraction of consumers who do not own a car is time-invariant: this probability can change over time in response to the macroeconomic shocks.

It follows that the equation for equilibrium will change: it now will consist only of equations that set the excess demand for all used cars to zero, but given our "small open economy" assumptions, the flows of new cars will vary over time since we continue to assume that the supply of new cars is infinitely elastic at a price $\overline{P}(m)$ that may depend on the macro state, and also we continue to assume that there is an infinitely elastic demand for cars at a scrap value $\underline{P}(m)$ which may also depend on the macro state. Assume that no matter what the age distribution of cars q_t might be, or the macro state might be, there is a finite upper bound $\overline{a}$ on the scrappage age in any equilibrium. Let $P(q,m)$ be the prices that equates the supply and demand for all used cars when the car age distribution is q and m. We will define the excess demand equations shortly and show that equilibrium prices will depend only on (q,m), and we will assume that in the case of multiple equilibria in any given (q,m) state, we select the Pareto dominant equilibrium that results in the highest state-dependent scrappage age $\overline{a}(q,m)$. Specifically $\overline{a}(q,m)$ is the age at which $P(q,m)(a) = \underline{P}(m)$ for $a > \overline{a}(q,m)$. We have assumed that $\overline{a}(q,m)$ is bounded above by some finite integer $\overline{a}$ so we have

$$\sup_{q,m} \overline{a}(q,m) = \overline{a} \tag{124}$$

where q is an element of the $\overline{a}$-dimensional unit simplex (i.e. q is a $\overline{a}$ vector whose elements are non-negative and sum to 1).

Thus, for a given state (q,m) $P(q,m)$ can be regarded as an $\overline{a} \times 1$ vector that gives the prices of all cars of age $a \in \{1,\dots,\overline{a}\}$ with the additional condition that $P(q,m)(a) = \underline{P}(m)$ and $q(a) = 0$ for $a > \overline{a}(q,m)$. The condition that $q(a) = 0$ for $a > \overline{a}(q,m)$ means that all cars

are time-varying. We will discuss below whether in this more general sense the "non-stationary equilibrium" in the automobile market subject to macro shocks might still be described as a "stationary equilibrium" in this generalized sense.

that are older than the scrappage age relevant for state (q,m) are scrapped, and thus not held by anyone in this economy.

Since we need both (q,m) to solve for equilibrium prices, it follows that *consumers need to take both of these variables into account in solving their optimal car holding/trading strategies.* Using (q,m) and knowledge of the equilibrium price function $P(q,m)$, consumers would have the information they need to predict what prices will be in any possible future state of the economy. So we assume that consumers can observe the aggregate holdings/age distribution in the overall economy, q_t, at any point in time t as well as the current macro state m_t. Further we assume they have *rational expectations* and know the equilibrium price function $P(q,m)$, i,e. they can predict the prices of new and used cars in any possible state of the economy.

Though the macro state evolves stochastically, we now show that if the current aggregate age/holdings distribution is q_t, consumers will be able to perfectly predict q_{t+1} given m_t. That is, we now show there is are $(\overline{a}+1\times\overline{a}+1)$ transition probability matrices $D(q,m)$ and $S(q,m)$ such that

$$q_{t+1}=q_tD(q_t,m_t)S(q_t,m_t) \tag{125}$$

where $D(q,m)$ is the "demand transition matrix" similar to the transition probability matrix defined in equation (67) and $S(q,m)$ is the "supply transition probability matrix" similar to the matrix defined in equation (64) above. Note that while consumers can perfectly predict q_{t+1} given (q_t,m_t) using equation (125) above, it is important to realize that they cannot perfectly predict *all* future values of $\{q_t\}$ including all q_s for $s\geq t+2$ since they cannot perfectly predict the future macro states $\{m_t\}$ as long as the transition probability matrix F is not degenerate (i.e. all of its elements are strictly between 0 and 1). For example q_{t+2} is given by $q_{t+2}=q_{t+1}S(q_{t+1},m_{t+1})$ and if the consumer cannot perfectly predict m_{t+2} then it follows they will be unable to perfectly predict q_{t+2} as well.

In order to define the matrices $D(q,m)$ and $S(q,m)$ we need to start by describing a modification of the consumer's dynamic programming problem that accounts for the non-stationarity due to the macro shocks. Since (q,m) helps the consumer predict future auto prices, these become "payoff relevant state variables" that consumers take into account in solving their dynamic programming problems. The state variables (including the idiosyncratic state variables ε) are now given by (a,q,m,ε) where we use $a=\emptyset$ to denote the state of not owning any car. We continue to keep the simplifying assumption that all consumers scrap cars once the age of the car equals the state-dependent scrappage age $\overline{a}(q,m)$. Assume that $a\neq\emptyset$ so the consumer currently holds a car of age a and assume that $a<\overline{a}(q,m)$ so that the consumer choose to keep the current car rather than being obligated to scrap it. The Bellman equation for this state is given by

$$\begin{aligned} V(a,q,m,\varepsilon) &= \max\Big[v(\emptyset,a,q,m)+\varepsilon(\emptyset), v(-1,a,q,m)+\varepsilon(-1), \\ &\qquad \max_{d\in\{0,1,\ldots,\overline{a}(q,m)-1\}}[v(d,a,q,m)+\varepsilon(d)]\Big], \end{aligned} \tag{126}$$

where

$$\begin{aligned} v(\emptyset,\emptyset,q,m) &= u(\emptyset)+\beta\sum_{m'\in\{0,1\}} EV(\emptyset,q',m')f(m'|m) \\ v(d,\emptyset,q,m) &= u(d)-\mu(m)[P(d)+T]+\beta\sum_{m'\in\{0,1\}} EV(d+1,q',m')f(m'|m) \\ v(\emptyset,a,q,m) &= u(\emptyset)+\mu(m)P(a)+\beta\sum_{m'\in\{0,1\}} EV(\emptyset,q',m')f(m'|m) \\ v(-1,a,q,m) &= u(a)+ \\ &\quad \beta\sum_{m'\in\{0,1\}}\left[(1-\alpha(a,m))EV(a+1,q',m')+\alpha(a,m)EV(\overline{a},q',m')\right]f(m'|m) \\ v(d,a,q,m) &= u(d)-\mu(m)[P(d,q,m)-P(a,q,m)+T]+ \\ &\quad \beta\sum_{m'\in\{0,1\}}\left[(1-\alpha(d,m))EV(d+1,q',m')+\alpha(d,m)EV(\overline{a},q',m')\right]f(m'|m) \end{aligned} \tag{127}$$

where $EV(\emptyset,q',m')$ is the expected discounted utility for a consumer who does not have a car and the economy is in (next period's) state (q',m') where $q'=qD(q,m)S(q,m)$ and $EV(\emptyset,q,m)$ is given by

$$EV(\emptyset,q,m)=\sigma\left(\exp\{v(\emptyset,\emptyset,q,m)/\sigma\}+\sum_{d=0}^{\overline{a}(q,m)-1}\exp\{v(d,\emptyset,q,m)/\sigma\}\right) \tag{128}$$

and $EV(a,q,m)$ is the expected value of a consumer who owns a car of age $a<\overline{a}(q,m)$ and is given by

$$EV(a,q,m)=\sigma\left(\exp\{v(\emptyset,a,q,m)/\sigma\}+\exp\{v(-1,a,q,m)/\sigma\}+\sum_{d=0}^{\overline{a}(q,m)-1}\exp\{v(d,a,q,m)/\sigma\}\right) \tag{129}$$

for $a<\overline{a}(q,m)$ and if $a\geq\overline{a}(q,m)$ we have

$$EV(a,q,m)=\sigma\left(\exp\{v(\emptyset,a,q,m)/\sigma\}+\sum_{d=0}^{\overline{a}(q,m)-1}\exp\{v(d,a,q,m)/\sigma\}\right) \tag{130}$$

since we continue with the assumption that consumers do not keep a vehicle whose age equals or exceeds the scrappage thresold age $\overline{a}(q,m)$ beyond which the value of the car equals the scrap price $\underline{P}(m)$.

The choice probabilities implied from the solution to the consumer's dynamic programming problem in equation (126) are of the form $\Pi(d|a,q,m,\tau)$ that depend on the new macro state variables (q,m) that are relevant in the case of a non-stationary equilibrium. We assume that the consumer knows the equilibrium price function $P(q,m)$ which is a mapping from the cartesian product of the $\overline{a}$-dimensional simplex and the set $\{0,1\}$ into $R^{\overline{a}}$ with the interpretation that $P(a,q,m)$ is the price of a car of age a when the state of the car market is (q,m). As we noted above, for each state (q,m) we assume that the Pareto dominant equilibrium is selected and define $\overline{a}(q,m)$ as

$$\overline{a}(q,m) = \underset{a\leq\overline{a}}{argmin}\, P(a,q,m) = \underline{P}(m) \tag{131}$$

i.e. it is the smallest age of car at which the price equals the infinitely elastic demand price for scrap $\underline{P}(m)$ in state m.

Define the *conditional excess demand function* as the function $ED(P,q,m,\overline{a})$ that depends on the state (q,m) and a conjectured scrappage age $\overline{a}$ as

$$\begin{bmatrix} \Pi(1|\emptyset,P,q,m)q(\emptyset) + \left[\sum_{a=1}^{\overline{a}}\Pi(1|a,P,q,m)q(a) - q(1)[1-\Pi(-1|1,P,q,m)]\right] \\ \Pi(2|\emptyset,P,q,m)q(\emptyset) + \left[\sum_{a=1}^{\overline{a}}\Pi(2|a,P,q,m)q(a) - q(2)[1-\Pi(-1|2,P,q,m)]\right] \\ \cdots \\ \Pi(\overline{a}-2|\emptyset,P,q,m)q(\emptyset) + \left[\sum_{a=1}^{\overline{a}}\Pi(\overline{a}-2|a,P,q,m)q(a) - q(\overline{a}-2)[1-\Pi(-1|\overline{a}-2,P,q,m)]\right] \\ \Pi(\overline{a}-1|\emptyset,P,q,m)q(\emptyset) + \left[\sum_{a=1}^{\overline{a}}\Pi(\overline{a}-1|a,P,q,m)q(a) - q(\overline{a}-1)[1-\Pi(-1|\overline{a}-1,P,q,m)]\right] \end{bmatrix} \tag{132}$$

We can calculate the Pareto-dominant equilibrium via an iterative process of starting with small guesses of the state-specific scrappage age $\overline{a}(q,m)$ and finding a solution $P(q,m,\overline{)}$ to the equilibrium condition

$$ED(P(q,m,\overline{a}),q,m,\overline{a}) = 0 \tag{133}$$

where we define $\overline{a}(q,m)$ to be the *largest* such solution to equation (133) subject to the constraint that $P(a,q,m)$ is strictly decreasing in a for $a < \overline{a}$.

Now we define the $(\overline{a}+1 \times \overline{a}+1)$ transition probability matrix $D(q,m)$ given by

$$\begin{bmatrix} \Pi(\emptyset|\emptyset) & \Pi(0|\emptyset) & \Pi(1|\emptyset) & \cdots & \Pi(\overline{a}-1|\emptyset) \\ \Pi(\emptyset|1) & \Pi(0|1) & \Pi(1|1)+\Pi(-1|1) & \cdots & \Pi(\overline{a}|1) \\ \Pi(\emptyset|2) & \Pi(0|2) & \Pi(1|2) & \cdots & \Pi(\overline{a}-1|2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Pi(\emptyset|\overline{a}-1) & \Pi(0|\overline{a}-1) & \Pi(1|\overline{a}-1) & \cdots & \Pi(\overline{a}-1|\overline{a}-1)+\Pi(-1|\overline{a}-1) \\ \Pi(\emptyset|\overline{a}) & \Pi(0|\overline{a}) & \Pi(1|\overline{a}) & \cdots & \Pi(\overline{a}-1|\overline{a}) \end{bmatrix}, \tag{134}$$

where $\Pi(d|a) = \Pi(d|a,P(q,m),q,m)$ is the transition probability implied in state (q,m) under the equilibrium prices $P(q,m)$, where the choice index d ranges over the index set $\{\emptyset,0,1,\ldots,\overline{a}-1\}$ and the age index a ranges over the index set $\{\emptyset,1,\ldots,\overline{a}\}$. Note that in the Pareto dominant equilibrium in state (q,m) if $\overline{a}(q,m) < \overline{a}$ (where $\overline{a}$ is the least upper bound of $\overline{a}(q,m)$ over all possible states (q,m) as discussed above), then element $D(d,a)$ (i.e. the element in row d and column a of $D(q,m)$) will equal 0 if $d > \overline{a}(q,m)$ since consumers are not allow to purchase cars that are over the scrappage age $\overline{a}(q,m)$. Note that by construction $D(q,m)$ is a transition probability matrix and it differs from the demand transition probability matrix given in formula (67) above by reordering the last column of the latter and moving it to column 2. Also not that if $a,a' \geq \overline{a}(q,m)$ then we assume that $\Pi(d|a,P(q,m),q,m) = \Pi(d,a',P(q,m),q,m)$ for all possible choices d, and $P(-1|a,P(q,m),q,m) = 0$ (it is not allowed to keep any car that is older than the scrappage age $\overline{a}(q,m)$), and $P(d|a,P(q,m),q,m) = 0$ for $d > \overline{a}(q,m)$ (it is not allowed to purchase any car whose age is the scrappage age or older). So this implies that columns $\overline{a}(q,m),\ldots,\overline{a}$ of $D(q,m)$ are identically zero.

Thus $qD(q,m)$ represents the "post trade" distribution of auto holdings, and this represents a distribution that holds at time t since we assume that trading is instantaneous. By definition there can be no car older than $\overline{a}(q,m)$ after the trading has taken place, so if we write $q' = qD(q,m)$ then q' is a vector satisfying $q'(a) = 0$ for $a \geq \overline{a}(q,m)$. This means that just after trade at any period t, there will be no car older than the scrappage threshold age $\overline{a}(q_t,m_t)$ applicable at period t.

Now we define the "Supply transition probability matrix" $S(q,m)$ that specifies how the distribution of car holdings evolves between t and $t+1$. If $q' = qD(q,m)$ is the post-trade distribution of car holdings at time t, then $q'S(q,m)$ is the distribution of car holdings at the start of the next period $t+1$ that reflects the aging of cars between t and $t+1$ including the possibility of stochastic accidents. $S(q,m)$ is the $(\overline{a}+1\times\overline{a}+1)$ matrix given by

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\alpha(0,m) & 0 & 0 & \cdots & 0 & \alpha(0,m) \\ 0 & 0 & 0 & 1-\alpha(1,m) & 0 & \cdots & 0 & \alpha(1,m) \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1-\alpha(\overline{a}(q,m)-1,m) & \cdots & 0 & \alpha(\overline{a}(q,m)-1,m) \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{135}$$

Note that the supply transition probability matrix for the nonationary economy, $S(q,m)$ in equation (135), differs from the supply transition matrix for the stationary economy in equation (64) in that $S(q,m)$ has a 1 in the first column for all rows after the $\overline{a}(q,m)-1$ which is the oldest age of used car that can be purchased in in equilibrium when the economy is in state (q,m), whereas $S(P)$ in the stationary equilibrium case has a 1 in the *second* column of $S(P)$ in equation (64). Why the difference? Recall that the first column of $S(q,m)$ corresponds to the outside of having no car. So for any car of age $a\geq\overline{a}(q,m)$ we represent scrappage of these cars as a transition from a car of age a to the state $\emptyset$, i.e. this indicates that the car has been disposed of so the car transits to the "no car state". But the second column of both $S(q,m)$ and $S(P)$ represents a car of age $a=1$, and by putting a 1 in the second column of last row of $S(P)$ we are explicitly enforcing the concept of *flow equilibrium* — i.e. every car that is scrapped will be replaced by a new car.

Thus via construction of $S(P)$ and by requiring the holdings distribution to be an invariant distribution of $S(P)$, we are enforcing both flow equilibrium and time stationarity in the holdings distribution q. In particular, the invariant distribution satisifies $q(1)=q(\overline{a})$ and in a stationary equilibrium with a zero probability of accidents for new cars, the number of 1 year old cars, $q(1)$ equals the number of new cars purchased the previous period, which in turn equals the number of cars that are scrapped each period, which equals $q(\overline{a})$.

In a non-stationary equilibrium we are no longer enforcing a flow equilibrium: the number of new cars that are sold in any period t does not necessarily have to equal the number of cars that were scrapped in in period t so that the market is not necessarily in flow equilibrium in any period t even if there are zero accidents for new cars

$$q_t D(q_t,m_t)(0) = \sum_{a=0}^{\overline{a}(q_t,m_t)} \Pi(0|a,P(q_t,m_t),q_t,m_t)q_t(a) \neq \sum_{a=\overline{a}(q_t,m_t)}^{\overline{a}} q_t(a) \tag{136}$$

Thus, we are not even enforcing that there is a *post trade* balance between the number of new cars demanded (first equation in (136) and the number of of cars scrapped in period t (the total fraction of cars that are age $\overline{a}(q_t,m_t)$ or older given in the last sum of equation (136)). The only equilibrium condition that we are enforcing is that supply and demand for all traded used cars, i.e. cars of age $a \in \{1,\dots,\overline{a}(q_t,m_t)\}$ is zero, i.e that $P(q,m)$ satisfies $ED(P(q,m),q,m) = 0$ for all possible values of (q,m).

Notice that it is a much more complex problem to find an equilibrium in the nonstationary case. Instead of simply searching for a vector P of dimension $\overline{a}-1$ that sets excess demand $ED(P)$ to zero, in the nonstationary case we must find an entire *function* $P(q,m)$ that depends on the $\overline{a}+1$-dimensional vector q in the $\overline{a}+1$-dimensional simplex and the two valued binary macro state indicator m. Any numerical method will only be able to approximate $P(q,m)$ and it is not clear that we can employ a Newton algorithm similar to what we developed to find equilibria in the stationary case. We leave the question of existence and computationa/approximation of non-stationary equilibrium in the auto market to future work, but note that this is a long standing unsolved problem for which simpler heuristic methods have been used to find approximate solution (e.g. the approach of Krusell and Smith (1998)) have been used, but without any formal justification as to whether these methods any really do approximate a well defined equilibrium.

We believe that with further work and some additional assumptions and restrictions it will be possible not only to prove that our definition of non-stationary equilibrium given above exists, we believe it will be possible to approximate the equilibria to this model using numerical methods arbitrarily closely. It will then be of interest to see how the "correct" equilibrium solution compares to some of the heuristic solutions such as the method of Krusell and Smith (1998).

5 Model Identification

In this section we consider the identification of the structural parameters of the model in two main cases: a) where the model is correctly specified, and b) under various types of model misspecification. Our goal is to determine whether there are various types of "serious" model misspecifications that significantly bias the structural parameter estimates and make counterfactual policy forecasts from this model unreliable. This work is motivated by the high estimated transactions costs we obtained from maximum likelihood of an overlapping generations, finite lived agent version of this model using Danish register data. We observed that the model tended to produce very low estimated marginal utilities of money, μ, and what we felt were unreasonably high estimated transactions costs T. We noticed that the product, $\mu * T$, which we can interpret as the transaction cost measured in utility units, is a much more stably estimated quantity, and in estimations where we fixed μ at higher values then T was estimated to be correspondingly lower. We are not yet sure which aspect of the data is tending to drive the unrestricted maximum likelihood estimates of μ to values that we regard as unreasonably low, and thus corresponding estimated values of T to be unreasonably high. We conjectured that the problem may be a manifestation of some sort of "aggregation bias" that results from approximating the underlying reality where consumers have a very large choice set with very many makes and models of cars, into a model where there is only a single car type, or very few car types. In particular, such a model may have difficulty explaining why both low income consumers and high income consumers buy new cars if the model is treating both as buying a single "representative new car" when in fact richer consumers are buying expensive new cars and poorer consumers are buying inexpensive new cars. From the models' perspective both rich and poor consumer are both buying the single "representative new car" and this may be difficult for the model to "explain" if the estimated value of the marginal utility of money is high.

However in our attempts to estimated a model with two car types (expensive ones and inexpensive ones) we have so far been unable to see a pattern where disaggregating the types of cars into more types (which is computationally expensive for us) results in increases in the estimated marginal utility of money. It may be that there are other factors and other types of model misspecification that are at play, so this motivated our creation of the toy model as a laboratory where we could know the "true model" and study how our maximum likelihood estimator performs, particularly in the case where there are various types of model misspecification present. We can study this by solving and simulating data from a richer, more complex model than the one we actually estimate. For example we could solve and simulate data from a model where there are multiple types of cars but estimate a model where we only allow a single type of "representative car" and see if this does indeed tend to downward bias our estimates of μ and thus lead to spuriously upward biased estimates of transactions costs T as we have conjectured above.

Identification of consumer preferences for autos may involve subtle issues even with the restrictive assumption of quasi linear utility functions. These utility functions affect consumer choices of autos in two different ways: 1) via the utility function for cars as a function of their age, $u(a)$, and 2) via the replacement costs $\mu[P(d) - P(a) - T]$ that the consumer incurs when they sell their current car of age a to replace it with another car of age d, or to "purge" their car and choose the outside good. We find that consumers substitute quite elastically across different ages of cars in response to the market price signals as provides the equilibrium price function $P(a)$, but consumers also are strongly affected by the level and slope of the utility function $u(a)$. If we choose smaller values of μ keeping $u(a)$ fixed, this causes consumers to choose to replace their cars sooner and choose newer cars, *cetaris paribus.* However this can be partially counteracted by increases in T which causes consumers to tend to keep their cars longer before selling them. If we also scale down $u(a)$ at the same time, we can also reduce the "preference for newness" and make consumers less willing to buy new cars and more willing to hold older cars. In summary, it seems that it may be possible to find "observationally similar" equilibria of the model, in one case where μ is relatively high, T is relatively low and $u(a)$ relatively high, with equilibrium outcomes that are very similar to case where consumers have lower marginal utilities μ, higher transactions costs T and lower preference for newness $u(a)$.

As is usual in any discrete choice model, cardinal utilities cannot be identified from the data and therefore we must impose both a location and scale normalization on the utility function. The location normalization involves fixing the mean of the discrete choice error terms, $\varepsilon(a)$, to zero. The scale normalization involves fixing the scale parameter of the Type 3 extreme value shocks, σ, to 1. Once we have normalized $\sigma = 1$, if we solve for equilibrium with scaled down of utilities and marginal utility of money, then random preference shocks will play a bigger role in choice of autos, and we will need higher values of transactions costs T to counteract the effect of these shocks and get the degree of persistence in holdings of cars that we observe in the Danish register data. When we scale up μ and $u(a)$, we conjecture that we can reduce T to get the same degree of persistence in holdings, but the main difference is that the random $\varepsilon(a)$ shocks will play a less important role in the choice of when to replace an existing car and which new or used vehicle to replace it with.

As is well known, if we multiply all agent payoffs and the scale of extreme value shocks by a common positive scale factor $\lambda > 0$, there is no change to any of the resulting choice probabilities. Since the equilibrium is defined in terms of choice probabilities this implies that an equilibrium to the model calculated for a fixed scale normalization such as $\sigma = 1$ and $\lambda = 1$ is observationally equivalent to an equilibrium with any other scale normalization $\lambda \neq 1$. However there is one subtlety here: since the transactions cost function T is a parameter of the model, and transactions costs (in utility terms) enter as the product μT, if we increase the parameter μ as part of a parameter rescaling of the model, we *do not* also scale transactions costs T by the same factor, otherwise the resulting model would not be scale-invariant. To see this more clearly, consider the utility of trading an existing or "base" utility function $u(a,d,\varepsilon)$ given by

$$u(a,d,\varepsilon) = u(d) - \mu[P(d) - P(a) - T] + \varepsilon(d) \tag{137}$$

under a scale normalization for the extreme value shock that its scale parameter $\sigma = 1$. Then $u(a,d,\varepsilon)$ will be scale invariant to the utility function $\lambda u(a,d,\varepsilon)$ for any $\lambda > 0$ (i.e. it will imply the same choice probabilities and thus result in the same set of equilibria), but if we were also to multiply T by λ too, then the utility function $\tilde{u}(a,d,\varepsilon)$ given by

$$\tilde{u}(a,d,\varepsilon) = \lambda u(d) - \lambda\mu[P(d) - P(a) - \lambda T] + \lambda\varepsilon(d) \tag{138}$$

is not scale invariant to either $u(a,d,\varepsilon)$ or $\lambda u(a,d,\varepsilon)$ because for $\tilde{u}(a,d,\varepsilon)$ transactions costs have been scaled up by a factor λ^2 rather than λ and for sufficiently large λ such a utility function will imply much more persistence in holding of cars, and/or fewer people buying cars.

When we fix a particular scale normalization of the extreme value shocks, say $\sigma = 1$, then the scale of the other components of utility will be identified relative to this normalization. If the scale of $u(d)$ and the value of μ are estimated to be rather small, this results in a model that is driven more by "noise" in the Type 3 extreme value taste shocks than by the values of the "systematic" components of consumer utility $u(d) + \mu[P(d) - P(a) - T]$. However if the scale of the systematic components of uility is large, then the effect of extreme value noise is moderated and the choices by the model are predicted mainly by the values calculated from the dynamic programming problem, i.e. the $v(d,a)$ function defined in equation (33) above. We noted however as the scale of $v(d,a)$ increases, if the transactions cost parameter T is not *scaled down* at an inverse rate then transactions costs effectively increase and result in more persistence in holdings of automobiles.

However if we do scale down transactions costs proportionately so that the product of μ and T remains approximately invariant to different scalings of the utility function, we find behavior to be *relatively* invariant, though as we show below behavior and equilibrium is not strictly invariant because the different scalings imply differences in the relative importance of the random "noise" tht is captured by the extreme value utility shocks $\varepsilon(d)$ (where d varies over the choices available to individuals depending on their state), and these shocks have a scale factor σ that is normalized to 1 regardless of the scaling of the other components of utility.

Though we do not have an analytical proof, our numerical computations suggest that it may be possible to identify the structural objects of the model from the equilibrium price vector and holdings distribution. In particular, we show below that the equilibria for different scalings of the utility discussed above (but where the σ scale parameter of the extreme value distribution is fixed at 1, a minimal condition, along with a location normalization on consumer utility functions that we do know is necessary for identification) are not "observationally equivalent" given our parametric assumptions on consumer utility.

Figure 5 shows two different equilibrium price functions for two scalings of consumer utility function in an economy with two types of consumers, "rich" consumers τ_1 that make up $f(\tau_1) = 0.2$ of the population and "poor" consumers τ_2 that make up the remaining $f(\tau_2) = 0.8$ of the population. Both types of consumers have a commmon discount factor $\beta = .95$ and there are accidents given by $\alpha(a) = a/(10\overline{a})$. We normalized the utility of the outside good for both comsumers to 0.

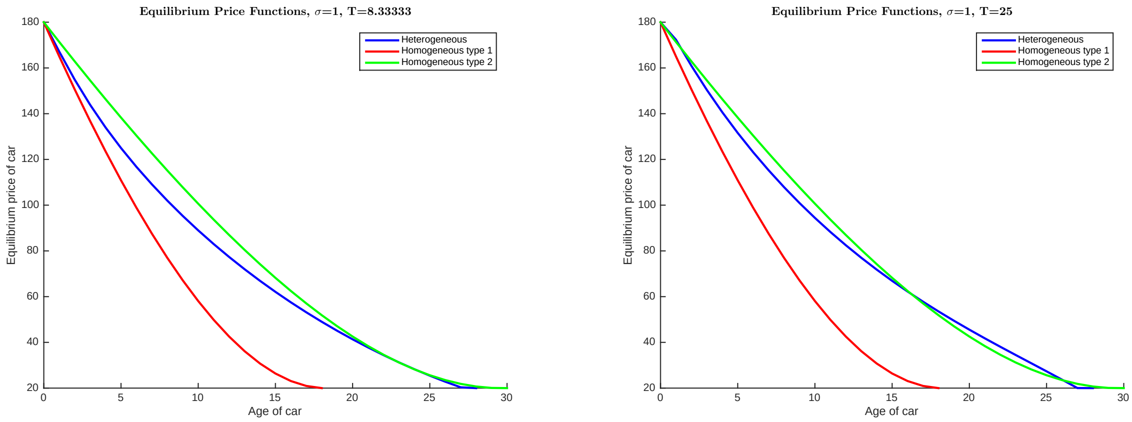
The right panel of figure 5 plot the Pareto dominant equilibrium price function where we set transactions costs $T = 25$ and the utility of of the two consumers as follows

$$\begin{aligned} u(a,\tau_1) &= 3.7 - .2a \\ u(a,\tau_2) &= 3.2 - .1a \\ \mu(\tau_1) &= .16 \\ \mu(\tau_2) &= .18 \end{aligned} \tag{139}$$

Thus, the type τ_1 consumer obtains higher utility from new cars (reflected by the higher intercept term, 3.7) but their utility for older cars declines more rapidly than a type τ_2 consumer. We assume that the marginal utility of a type τ_1 consumer is $\mu(\tau_1) = .16$ whereas the marginal utility of money for a type τ_2 consumer is $\tau_2 = .18$, so this justifies our classifying the type τ_1 consumers as "rich" consumers and the type τ_2 consumers as "poor".

The left panel of figure 5 plots the equilibrium for a rescaling of the utility function where we multiplied the utilities and marginal utilities of money of both consumers by 3 but divided the transaction cost parameter $T = 25$ by 3. Thus, the product of the marginal utility of

Figure 5: Equilibrium price functions for two scalings of utility in a two type economy

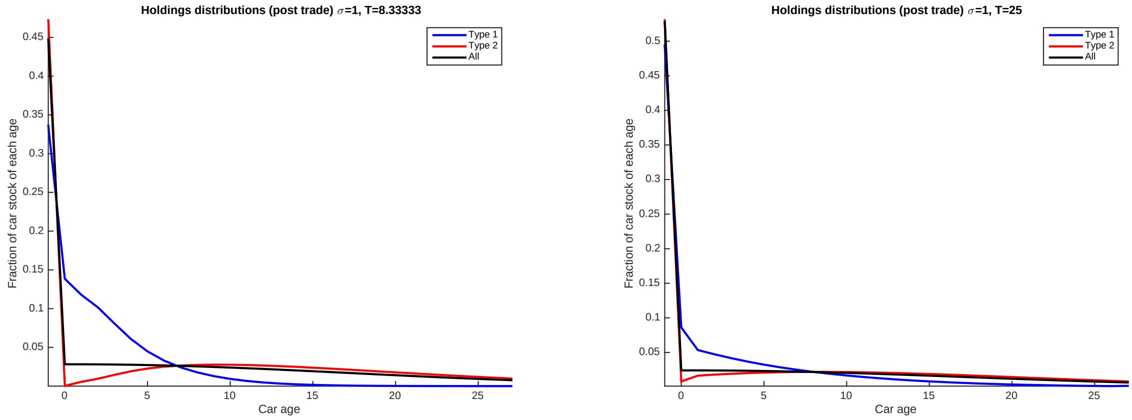


money times the monetary transactions cost was the same for both consumers in both cases: $\mu(\tau_1)T = 4.5$ and $\mu(\tau_2)T = 4$, respectively, for the two consumer types in both scalings of the utility function and marginal utility of money. We see from figure 5 that to the naked eye the price functions look very similar in both cases, yet there are significant differences in the price functions when we compute their differences. In particular equilibrium prices for the low scaling of utility but high transaction cost $T = 25$ given in equation (139) and in the right panel of figure 5 are noticeably higher than the equilibrium price function for the case where utilities are scaled by 3 but transactions cost is scaled down by one third in the left panel of figure 5.

Evidently, in the equilibrium where utility is scaled up by a factor of 3, with the extreme value scale parameter fixed at $\sigma = 1$ the role of random, idiosyncratic heterogeneity in trading is lower. Even though the cost of trading (in utility terms) is the same in both scalings of utility, the scale of the idiosyncratic shocks relative to the deterministic components of the value function, i.e. $v(d,a)$, is smaller under the higher scaling of utility. This means that there will be lower gains from trade in the case of the high scaling of utility compared to the low scaling of utility. The lower gains from trade is reflected lower equilibrium prices under the high scaling of utility relative to the low scaling.

Figure 6 plots the implied holdings distributions for cars of different ages by the two types of consumers as well as the overall distribution of holdings q. We see from the left panel that under the high scaling of utility, there are more consumers holding cars than in the right panel where 52.7% of consumers in the economy choose not to own a car. Evidently, the relatively high contribution of idiosyncratic shocks not only affects trading of cars, but it also affects the number of consumers who choose the outside good. Under the low scaling of utility there is

Figure 6: Equilibrium holdings distributions for two scalings of utility in a two type economy

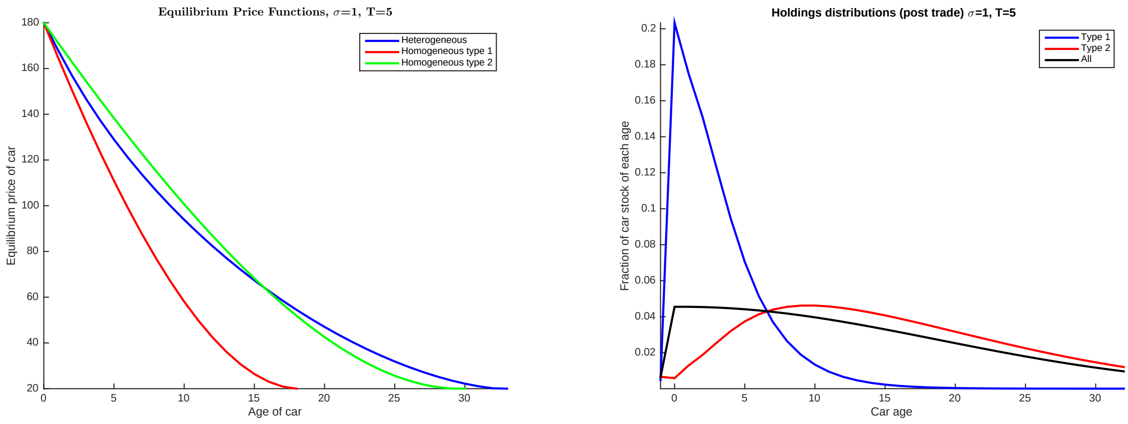


high persistence in the no car state (with probabilities of remaining without a car given no car equalling .75 and .84 for the rich and poor consumers, respectively) but lower persistence in keeping cars. For example the probability of keeping a one year old car is approximately 79% for a rich consumer who has a one year old car, whereas the probability of buying a new car is 3.6% and the probability of purging the car and going to the no car state is 7.6%. A poor consumer also has a low probability of buying a new car, or keeping their old car. For exmaple a poor consumer with a 15 year old car has a 66% probability of keeping this car and a 16% probability of purging this car and entering the no car state. Thus, trading of cars is more driven by stochastic "noise" in the low scaling of utility, and this noise generates lower persistence in auto holdings. However due to the relatively higher value of being in the no car state, the probability of remaining in the no car state is relatively high.

For the high scaling of utility we observe much more persistence in holdings. The rich consumer holding a one year old car will continue to hold it (and not trade it) with probability 91%, and a poor consumer holding a 15 year old car will continue to hold it with probability 78%. A poor consumer who does not have a car chooses to continue not having a car with 98% probability, whereas a rich consumer without a car chooses to continue not having a car with a 73% probability. Thus, it is clear that the idiosyncratic shocks to preferences play a less important role in the higher scaling of utility, and this generates a higher degree of persistence of holdings of cars and remaining in the no car state.

We conclude this section by considering the effect of reducing transactions costs under the high scaling of utility (i.e. where we multiply the utility function and marginal utility of money by the scale factor 3 in the utility function in equation (139) above. At the same time we

Figure 7: Effect of lowering T from 25/3 to 5 on equilibrium for high scaling of utility



reduced the transaction cost from $T = 25/3$ to $T = 5$ so that in utility terms the transaction cost of trading an existing car for another one falls from 4.5 to 2.4 for a type τ_1 (rich) consumer and from 4 to 2.7 for a type τ_2 (poor) consumer.

Due to the reduction in transactions cost, we see there is a dramatic shift that causes virtually all consumers to prefer owning a car to the outside good. Further the equilibrium price function shifts upward and the scrappage age increases from $\overline{a} = 28$ to $\overline{a} = 33$. The left panel of figure 7 shows the last in a sequence of price functions, where we started with an initial guess equal to the equilibrium price function when $T = 8/3$ and calculated successive equilibria for $\overline{a} \in \{29, 30, 31, 32, 33\}$ and stopping at $\overline{a} = 33$ since an attempt to calculate an equilibrium for $\overline{a} = 34$ resulted in an invalid solution (a price function that falls below $\underline{P}$).

The decrease in transaction costs from $T = 25/3$ to $T = 5$ caused the fraction of consumers who do not own a car to fall from nearly 45% when $T = 25/3$ to under 1% when $T = 5$. Part of this is due to the fact that the degree of persistence of the state of not having a car is lower when transactions costs are lower: for the rich consumers who do not have a car there is only a 30% probability of staying without a car, and for poor consumers without a car the probability of remaining without a car is 56%.

The degree of persistence of holdings of cars is lower when the transaction cost is lowered, but still reasonably high. Rich consumers who own a 1 year old car have a 66% probability of keeping it, and poor consumers who hold a 10 year old car have a 40% chance of keeping it. The poor consumers do relatively more trading and hold their cars for shorter durations compared to rich consumers, and we can see from the right panel of figure 7 that their distribution of car holdings is more uniform over ages than for rich consumers: the modal car age for poor consumers is 10 years old. Rich consumers are much more concentrated in holding the newest cars. They typically follow trading/replacement cycles involving buying a new car, keeping it for 4 or 5 years, and then selling it to buy another new car.

6 Adding Driving to the Model

So far we assumed that a consumer of type τ obtains a utility of $u(a)$ from owning a car of age a. In this section we develop this utility function a bit further by recognizing that consumers obtain this utility principally via the act of driving their cars, so we provide a simple model of the continuous driving decision. We assume that consumers believe that the future resale value of their cars is only a function of the car type and age, and not a function of the odometer value on their car. This implies that the optimal amount of driving will be the solution to a static driving subproblem that can be solved independently of the overall dynamic programming problem governing vehicle trading.

Let p_f be the price of fuel and let x be the consumption of "driving" each period, measured in kilometers. Given current fuel prices in Denmark (about €1.40 per liter) the price of fuel per kilometer driven will be about $p_f = .11$ for a car with a fuel efficiency of 12 kilometers per liter. Let $u(a,x,\tau)$ be the utility function for driving x kilometers and assume it is quadratic in x given by

$$u(a,x,\tau) = u_0(a,\tau) + u_1(a,\tau)x - \frac{1}{2}x^2 - \mu(\tau)p_f x \tag{140}$$

Maximizing with respect to driving, x, we obtain the following linear demand function for driving

$$x(a,p_f,\tau) = u_1(a,\tau) - \mu(\tau)p_f \tag{141}$$

and substituting this back into the utility function we obtain the following indirect utility function for owning a car of age a

$$u(a,x(a,p_f,\tau),\tau) = u_0(a,\tau) - \mu(\tau)p_f u_1(a,\tau) + \frac{1}{2}\left[u_1(a,\tau)^2 + [\mu(\tau)p_f]^2\right] \tag{142}$$

Note that

$$\frac{\partial}{\partial p_f}u(a,x(a,p_f,\tau),\tau) = -\mu(\tau)x(a,p_f,\tau) < 0 \tag{143}$$

so we see that consumers who drive more, or "poorer" consumers with a higher marginal utility of money are the ones who are hurt the most by a rise in fuel prices.

Keeping with out assumption of stationarity, we will assume the price of fuel p_f is time invariant. However the model with driving is useful to evaluate policy changes such as a reduction in the new car tax that is compensated by a corresponding rise in p_f (e.g. via a tax on fuel) so that the policy change is approximately revenue.

We provide a illustrative calculation below starting with a situation where the new car price, gross of taxation, is $\overline{P} = 280$ reflecting a 180% tax on new cars, similar to the new car tax in effect in Denmark. If the new car tax were repealed then the price of new cars would fall to $\overline{P} = 100$, which we assume is the exogenously fixed international price of cars in the world market that are imported into the domestic market, assuming that Denmark is a "small open economy."

Use specification for $u_1(a,\tau) = 60 - 1.2a$ calibrate $u_0(a,\tau)$ to yield 40% share of Danish population who have no car.

7 Structural Estimation

8 Monte Carlo Evidence

9 Conclusion

Appendix 1: Gradients of EV with respect to P

Let $\Gamma(EV,P)$ be the contraction mapping on the right hand side of equations (36) and (37), so that EV can be expressed as the unique fixed point of Γ

$$EV = \Gamma(EV,P). \tag{144}$$

The *Newton-Kantorovich algorithm* is simply the application of Newton's algorithm to find the fixed point above, but converted to the problem of finding a zero to a non-linear operator

$$F(EV) = EV - \Gamma(EV,P) = 0, \tag{145}$$

where F is a nonlinear mapping from $R^{\overline{a}}$ to $R^{\overline{a}}$. Application of Newton's method to finding a zero of F results in iterations of the form

$$EV_{t+1} = EV_t - [\nabla F(EV_t)]^{-1} F(EV_t) \equiv EV_t - [I - \nabla_{EV}\Gamma(EV_t,P)]^{-1}[EV_t - \Gamma(EV_t,P)] \tag{146}$$

Since Γ is differentiable in both EV and P, the Implicit Function Theorem holds and enables us to show that EV is a smooth implicit function of P which we denote by $EV(P)$, with a $(\overline{a}\times\overline{a}-1)$ Jacobian matrix given by

$$\nabla EV(P) \equiv \nabla_P EV(P) = [I - \nabla_{EV}\Gamma(EV,P)]^{-1}\nabla_P\Gamma(EV,P). \tag{147}$$

where we use the subscript for the gradient operator, ∇_P to indicate gradient with respect to the price vector P and ∇_{EV} to indicate gradient with respect to the expected value vector, EV.

The Jacobian matrix $\nabla_{EV}\Gamma(EV,P)$ equals β times the transition probability matrix M given by

$$M = \begin{bmatrix} \Pi(0|1) & \Pi(1|1) & \Pi(2|1) & \cdots & \Pi(\overline{a}-1|1) \\ \Pi(0|2) & \Pi(1|2) & \Pi(2|2) & \cdots & \Pi(\overline{a}-1|2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Pi(0|\overline{a}-2) & \Pi(1|\overline{a}-2) & \cdots & \Pi(\overline{a}-2|\overline{a}-2) & \Pi(\overline{a}-1|\overline{a}-2) \\ \Pi(0|\overline{a}-1) & \Pi(1|\overline{a}-1) & \cdots & \Pi(\overline{a}-2|\overline{a}-1) & \Pi(\overline{a}-1|\overline{a}-1) \\ \Pi(0|\overline{a}) & \Pi(1|\overline{a}) & \cdots & \Pi(\overline{a}-2|\overline{a}) & \Pi(\overline{a}-1|\overline{a}) \end{bmatrix}. \tag{148}$$

It is important to note that as a short hand, the "self-transition" probabilities $\Pi(a|a)$ in equation (148) are actually equal to $\Pi(a|a)+\Pi(-1|a)$, $a = 1,\ldots,\overline{a}-1$. This is the sum of the probabilities that consumers will either a) trade their current car age a for another car of age a, $\Pi(a|a)$, b) keep their current car of age a. This ensures that the rows of M sum to 1 and hence M can be viewed as a Markov transition probability matrix, whose rows describe the probabilities of the age a car a consumer will hold at the start of each period, but *after* making their instantaneous decision about whether to hold or trade their current vehicle. This implies that $\nabla_{EV}\Gamma(EV,P) = \beta M$ which in turn guarantees that $[I-\nabla_{EV}\Gamma(EV,P)]^{-1}$ exists and has the geometric series representation

$$[I-\nabla_{EV}\Gamma(EV,P)]^{-1} = \sum_{t=0}^{\infty}\beta^t M^t \tag{149}$$

and thus, the Newton iteration can be used to calculate the fixed point $EV = \Gamma(EV,P)$

$$EV_{t+1} = EV_t - [I-\nabla_{EV}\Gamma(EV_t,P)]^{-1}[EV_t - \Gamma(EV_t,P)] \tag{150}$$

where EV_t is the approximation to the fixed point $EV = \Gamma(EV,P)$ in the t^{th} iteration of the

Newton iteration.

We can also calculate the Jacobian matrix $\nabla_P \Gamma(EV,P)$, which is given by μ times the matrix

$$\nabla_P \Gamma(EV,P) = \begin{bmatrix} \sum_{d\neq 1}\Pi(d|1) & -\Pi(2|1) & \cdots & -\Pi(\overline{a}-2|1) & -\Pi(\overline{a}-1|1) \\ -\Pi(1|2) & \sum_{d\neq 2}\Pi(d|2) & \cdots & -\Pi(\overline{a}-2|2) & -\Pi(\overline{a}-1|2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\Pi(1|\overline{a}-1) & -\Pi(2|\overline{a}-1) & \cdots & -\Pi(\overline{a}-2|\overline{a}-1) & \sum_{d\neq \overline{a}-1}\Pi(d|\overline{a}-1) \\ -\Pi(1|\overline{a}) & -\Pi(2|\overline{a}) & \cdots & -\Pi(\overline{a}-2|\overline{a}) & -\Pi(\overline{a}-1|\overline{a}) \end{bmatrix}. \tag{151}$$

To simplify notation we assumed that the transction cost function $T(P,a,d)$ is not a function of prices, P, otherwise there would be additional terms to the elements of the matrix above to account for the gradient of T with respect to P, i.e. $\nabla_P T$.

Using the definition of the discounted utilities of trading the current car of age a for a car of age d, $v(d,a)$ in equation (33), we have

$$\frac{\partial}{\partial P(a')} v(d,a) = \begin{cases} \beta \frac{\partial}{\partial P(a')} EV(d+1) & \text{if } a = d \\ \beta \frac{\partial}{\partial P(a')} EV(d+1) & \text{if } a \neq d, a' \neq a, a' \neq d \\ \mu + \beta \frac{\partial}{\partial P(a')} EV(d+1) & \text{if } a' = a \neq d \\ -\mu + \beta \frac{\partial}{\partial P(a')} EV(d+1) & \text{if } a' = d \neq a \end{cases} \tag{152}$$

and

$$\frac{\partial}{\partial P(a')} v(-1,a) = \beta \frac{\partial}{\partial P(a')} EV(a+1), \quad a \in \{1,\dots,\overline{a}-1\}. \tag{153}$$

Now we calculate the derivatives of the choice probabilities with respect to $P = (P(1),\dots,P(\overline{a}-1))$. Let's call the gradient with respect to P operator ∇_P. Then we have

$$\nabla_P \Pi(-1|a,P) = \frac{\Pi(-1|a,P)}{\sigma} \left[[1-\Pi(-1|a,P)]\nabla_P v(-1,a) - \sum_{d'=0}^{\overline{a}-1} \Pi(d'|a,P)\nabla_P v(d',a) \right], \tag{154}$$

and

$$\nabla_P \Pi(d|a,P) = \frac{\Pi(d|a,P)}{\sigma} \left[\nabla_P v(d,a) - \sum_{d'=0}^{\overline{a}-1} \Pi(d',a,P)\nabla_P v(d',a) - \Pi(-1|a,P)\nabla_P v(-1,a) \right]. \tag{155}$$

For age $a = \overline{a}$ we have

$$\nabla_P \Pi(d,a,P) = \frac{\Pi(d|\overline{a},P)}{\sigma} \left[\nabla v(d,a) - \sum_{d'=0}^{\overline{a}-1} \Pi(d'|\overline{a},P)\nabla_P v(d',a) \right]. \tag{156}$$

Using all of these gradient formulas, we are finally ready to write the formula for the Jacobian matrix for excess demand:

$$\nabla_P ED(P) = \begin{bmatrix} \sum_{a=1}^{\overline{a}} \nabla_P \Pi(1|a,P) + \nabla_P \Pi(-1|1,P) \\ \sum_{a=1}^{\overline{a}} \nabla_P \Pi(2|a,P) + \nabla_P \Pi(-1|2,P) \\ \cdots \\ \sum_{a=1}^{\overline{a}} \nabla_P \Pi(\overline{a}-2|a,P) + \nabla_P \Pi(-1|\overline{a}-2,P) \\ \sum_{a=1}^{\overline{a}} \nabla_P \Pi(\overline{a}-1|a,P) + \nabla_P \Pi(-1|\overline{a}-1,P) \end{bmatrix}. \tag{157}$$

Appendix 2: Jaobian of Excess Demand, Outside Good Case

We start by providing derivatives of the expected value function EV with respect to the $\overline{a}-1\times 1$ vector of free price parameters, $P = (P(1),\dots,P(\overline{a}-1))$, which we denoted as $\nabla_P EV(P)$. Following the argument in equation (147), we can derive $\nabla_P EV(P)$ from (147) and the following formulas for $\nabla_{EV}\Gamma(EV,P)$ and $\nabla_P\Gamma(EV,P)$ which are extensions of the corresponding formulas (148) and (151) in the case without an outside good.

If we let the last component of the vector EV be expected value of the outside good, $EV(\emptyset)$, then the Jacobian matrix $\nabla_{EV}\Gamma(EV,P)$ equals β times the transition probability matrix M given by

$$M = \begin{bmatrix} \Pi(0|1) & \Pi(1|1) & \Pi(2|1) & \cdots & \Pi(\overline{a}-1|1) & \Pi(\emptyset|1) \\ \Pi(0|2) & \Pi(1|2) & \Pi(2|2) & \cdots & \Pi(\overline{a}-1|2) & \Pi(\emptyset|2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Pi(0|\overline{a}-2) & \Pi(1|\overline{a}-2) & \cdots & \Pi(\overline{a}-2|\overline{a}-2) & \Pi(\overline{a}-1|\overline{a}-2) & \Pi(\emptyset|\overline{a}-2) \\ \Pi(0|\overline{a}-1) & \Pi(1|\overline{a}-1) & \cdots & \Pi(\overline{a}-2|\overline{a}-1) & \Pi(\overline{a}-1|\overline{a}-1) & \Pi(\emptyset|\overline{a}-1) \\ \Pi(0|\overline{a}) & \Pi(1|\overline{a}) & \cdots & \Pi(\overline{a}-2|\overline{a}) & \Pi(\overline{a}-1|\overline{a}) & \Pi(\emptyset|\overline{a}) \\ \Pi(0|\emptyset) & \Pi(1|\emptyset) & \Pi(2|\emptyset) & \cdots & \Pi(\overline{a}-1|\emptyset) & \Pi(\emptyset|\emptyset) \end{bmatrix}. \tag{158}$$

Thus, with the outside good, the $(\overline{a}+1\times\overline{a}+1)$ Jacobian matrix $\nabla_{EV}\Gamma(EV,P)$ has a leading $(\overline{a}\times\overline{a})$ submatrix equal to the same formula as in the case of no outside good, equation (148), which is bordered in column $\overline{a}+1$ with the probabilities of choosing the outside good, $\Pi(\emptyset|a)$ for $a\in\{1,\dots,\overline{a}\}$ and with a last row equal to the choice probabilities of a consumer who currently has the outside good, $\Pi(d|\emptyset)$, $d\in\{0,1,\overline{a}-1,\emptyset\}$. Thus, $\Pi(\emptyset|\emptyset)$ is the probability that a consumer who does not have a car this period chooses not to have a car next period.

We also calculate the Jacobian matrix $\nabla_P\Gamma(EV,P)$ in the presence of and outside good, and this Jacobian is given by the $(\overline{a}+1 \times \overline{a}-1)$ matrix

$$\begin{bmatrix} Q(1,1) & Q(2,1) & \cdots & Q(\overline{a}-2,1) & Q(\overline{a}-1,1) \\ Q(1,2) & Q(2,2) & \cdots & Q(\overline{a}-2,2) & Q(\overline{a}-1,2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ Q(1,\overline{a}-1) & Q(2,\overline{a}-1) & \cdots & Q(\overline{a}-2,\overline{a}-1) & Q(\overline{a}-1,\overline{a}-1) \\ Q(1,\overline{a}) & Q(2,\overline{a}) & \cdots & Q(\overline{a}-2,\overline{a}) & Q(\overline{a}-1,\overline{a}) \\ -\Pi(1|\emptyset) & -\Pi(2|\emptyset) & \cdots & -\Pi(\overline{a}-2|\emptyset) & -\Pi(\overline{a}-1|\emptyset) \end{bmatrix}. \tag{159}$$

where $\rho \geq 0$ is a proportional transactions cost parameter and the elements $Q(d,a)$ are given by

$$Q(d,a) = \begin{cases} -\Pi(d|a)(1+\rho) & \text{if } d \neq a \\ \Pi(\emptyset|a) + \sum_{d' \neq d} \Pi(d'|d) - \Pi(d|d)\rho & \text{if } d = a \end{cases} \tag{160}$$

Now let's consider the strategy for solving for an equilibrium in the model with outside goods. Recall there are $\overline{a}-1$ unknowns, $P = (P(1),\ldots,P(\overline{a}-1))$ and after we have eliminated the first equation in the system of $\overline{a}$ equations that equate supply and demand for all ages of cars plus for the outside good (i.e. the equation that the fraction of the population who "demand" the outside good is equal to the fraction of the population who "supply" the outside good), then we have a reduced system of only $\overline{a}-1$ equations in the $\overline{a}-1$ unknowns P. That is, we have substituted out one of the unknowns in the overall system, $q_\emptyset(\emptyset)$, the fraction of the population who choose not to own a car in steady state. We see from equation (70) that this fraction depends on P so we can write it more explicitly as $q_\emptyset(\emptyset,P)$ to emphasize its dependence on P.

When we substitute this solution into the last $\overline{a}-1$ equations for excess demand, i.e. the excess demand for all traded vintages of used cars aged $a = 1,\ldots,\overline{a}-1$ we have this system of $\overline{a}-1$ equations in $\overline{a}-1$ unknowns P to solve, $ED(P,q_\emptyset(\emptyset,P)) = 0$ where $ED(P,q_\emptyset(\emptyset,P))$ is given by

$$\begin{bmatrix} \Pi(1|\emptyset,P)q_\emptyset(\emptyset,P) + [1-q_\emptyset(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}} \Pi(1|a,P)q(a) - q(1)[1-\Pi(-1|1,P)]\right] \\ \Pi(2|\emptyset,P)q_\emptyset(\emptyset,P) + [1-q_\emptyset(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}} \Pi(2|a,P)q(a) - q(2)[1-\Pi(-1|2,P)]\right] \\ \cdots \\ \Pi(\overline{a}-2|\emptyset,P)q_\emptyset(\emptyset,P) + [1-q_\emptyset(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}} \Pi(\overline{a}-2|a,P)q(a) - q(\overline{a}-2)[1-\Pi(-1|\overline{a}-2,P)]\right] \\ \Pi(\overline{a}-1|\emptyset,P)q_\emptyset(\emptyset,P) + [1-q_\emptyset(\emptyset,P)]\left[\sum_{a=1}^{\overline{a}} \Pi(\overline{a}-1|a,P)q(a) - q(\overline{a}-1)[1-\Pi(-1|\overline{a}-1,P)]\right] \end{bmatrix} \tag{161}$$

Similar to the case without an outside good, (see equation (42) above), we will also solve the system (161) by Newton's method, possibly starting from an initial guess that is computed by a more robust initial zero finder (e.g. the equivalent of `fsolve` in Matlab) with relatively large solution tolerances. To implement Newton's method we need the Jacobian matrix of $ED(P,q_{\emptyset}(\emptyset,P))$ which is the $\overline{a}-1\times\overline{a}-1$ matrix $\nabla_P ED(P,q_{\emptyset}(\emptyset,P))$ with row a given by

$$\begin{aligned}
&\nabla_P\Pi(a|\emptyset,P)q_{\emptyset}(\emptyset,P)+\\
&[1-q_{\emptyset}(\emptyset,P)]\left(\sum_{a'=1}^{\overline{a}}\nabla_P\Pi(a|a',P)q(a')+q(a)\nabla_P\Pi(-1|a,P)\right)\\
&\nabla_p q_{\emptyset}(\emptyset,P)\left[\Pi(a|\emptyset,P)-\sum_{a'=1}^{\overline{a}}\Pi(a|a',P)q(a')+q(a)[1-\Pi(-1|a,P)]\right] \qquad (162)
\end{aligned}$$

where $\nabla_P q_{\emptyset}(\emptyset,P)$ is given by

$$\begin{aligned}
\nabla_P q_{\emptyset}(\emptyset,P) &= \frac{\sum_{a=1}^{\overline{a}}\nabla_P\Pi(\emptyset|a,P)q(a)}{1-\Pi(\emptyset|\emptyset,P)+\sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a)}-\\
&\quad q_{\emptyset}(\emptyset,P)\left[\frac{-\nabla_P\Pi(\emptyset|\emptyset,P)+\sum_{a=1}^{\overline{a}}\nabla_P\Pi(\emptyset|a,P)q(a)}{1-\Pi(\emptyset|\emptyset,P)+\sum_{a=1}^{\overline{a}}\Pi(\emptyset|a,P)q(a)}\right] \qquad (163)
\end{aligned}$$

where

$$\begin{aligned}
\nabla_P\Pi(d|\emptyset,P) &= \frac{\Pi(d|\emptyset,P)}{\sigma}\Bigg[\nabla_P v(d,\emptyset,P)-\\
&\quad \Pi(\emptyset|\emptyset,P)\nabla_P v(\emptyset,\emptyset,P)-\sum_{d'=0}^{\overline{a}-1}\Pi(d'|\emptyset,P)\nabla_P v(d',\emptyset,P)\Bigg], \qquad (164)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_P\Pi(\emptyset|a,P) &= \frac{\Pi(\emptyset|a,P)}{\sigma}\Bigg[\nabla_P v(\emptyset,a,P)-\Pi(\emptyset|a,P)\nabla_P v(\emptyset,a,P)-\\
&\quad \Pi(-1|a,P)\nabla_P v(-1,a,P)-\sum_{d'=0}^{\overline{a}-1}\Pi(d'|a,P)\nabla_P v(d',a,P)\Bigg]. \qquad (165)
\end{aligned}$$

The gradients of the other choice probabilities $\nabla_P\Pi(d|a,P)$ are given by formulas analogous to those given above in the case where $a=\emptyset$ or $d=\emptyset$ simply by adding an exponentiated value of choosing the outside good, $\exp\{v(\emptyset,a,P)/\sigma\}$, into the denominator of the choice probabilities in equation (38) thereby extending the probabilities for the case where there is no outside good to the one where there is an outside good.

Appendix 3: Gradient of invariant distribution

In this appendix we consider the general problem of calculating the derivative of an invariant distribution with respect to parameters affecting a Markov transition matrix. Let the parameters be θ (in our application θ is a vector of prices of cars in a secondary market equilibrium) and consider a Markov transition probability matrix $P(\theta)$ that depends on these parameters in a continuously differentiable fashion. Thus, we assume that the mapping $\nabla_\theta P(\theta)$ from R^k to R^{k*n*n} (where the latter can be interpreted as the space of k-tuples of $n \times n$ matrices) exists and is a continuous function of θ. To make things easier to understand, assume initially that $k = 1$ so we are considering $P(\theta)$ and $q(\theta)$ as functions of a single parameter θ. If θ has k components (i.e. $\theta \in R^k$) we simply "stack" the formulas we provide below in the univariate case into a k-tuple.

We are interested in determining the conditions under which $q(\theta)$, the unique invariant distribution of $P(\theta)$, is a continuously differentiable function of θ and, if so, to find an expression for $\nabla_\theta q(\theta)$. The invariant distribution $q(\theta)$ satisfies the equation

$$q(\theta) = q(\theta)P(\theta), \tag{166}$$

which can be recast as $q(\theta)$ being a left zero of the matrix $I - P(\theta)$, $q(\theta)[I - P(\theta)] = 0$. The usual application of the Implicit Function Theorem applies when $q(\theta)$ can be written as a zero of some continuously differentiable nonlinear mapping $F(q,\theta) = 0$ with the added condition that $\nabla_q F(q,\theta)$ is nonsingular at a zero of F. Then the Implicit Function Theorem guarantees that there is a continuously differentiable function $q(\theta)$ in a neighborhod of this zero, and we have

$$\nabla_\theta q(\theta) = -[\nabla_q F(q(\theta),\theta)]^{-1} \nabla_\theta F(q(\theta),\theta). \tag{167}$$

However this usual application of the Implicit Function Theorem is inapplicable because in this case $\nabla_q F(q,\theta) = I - P(\theta)$ and this matrix is singular (note that if e is a vector of ones, then $[I - P(\theta)]e = 0$ where 0 is a vector of zeros). Thus, we have to approach this problem from a different angle.

When the invariant distribution is unique, it can be shown that $q(\theta)'$, the $n \times 1$ transpose of $q(\theta)$, is the unique solution to the expanded $(n+1) \times (n+1)$ linear system given by

$$\begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix} \begin{bmatrix} q(\theta)' \\ 1 \end{bmatrix} = \begin{bmatrix} e \\ 2 \end{bmatrix} \tag{168}$$

where e is an $n \times 1$ vector all of whose elements equal 1. Thus, the matrix on the right hand side of equation (168) is invertible and we can write

$$\begin{bmatrix} q(\theta)' \\ 1 \end{bmatrix} = \begin{bmatrix} I - P(\theta)' & e \\ e' & 1 \end{bmatrix}^{-1} \begin{bmatrix} e \\ 2 \end{bmatrix}. \tag{169}$$

Let $A(\theta)$ be the $(n+1) \times (n+1)$ matrix on the right hand side of equation (168). Then we have that $\nabla_\theta q(\theta)'$ is the upper left $n \times n$ submatrix of the product of $\nabla_\theta A^{-1}(\theta)$ times the vector $(e'\ 2)'$. Further, we use the following formula for the gradient of $A^{-1}(\theta)$ with respect to θ

$$\nabla_\theta A^{-1}(\theta) = -A^{-1}(\theta)\,[\nabla_\theta A(\theta)]\,A^{-1}(\theta). \tag{170}$$

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