Adaptive Simplicial Interpolation

Johannes Brumm
Karlsruhe Institute of Technology (KIT)
johannes.brumm@kit.edu

January 31, 2017

part of
ZICE 2017

based on
Johannes Brumm and Michael Grill (2014),
Computing Equilibria in Dynamic Models with Occasionally Binding Constraints,
Journal of Economic Dynamics and Control, Volume 38, 142-160, January 2014

Motivation

A global solution method
for dynamic general equilibrium models with

- occasionally binding constraints,
- several continuous state variables.

$\Rightarrow$ Computational challenge: Kinks in policy functions

Main components of the algorithm:

- Adaptive grid scheme to add interpolation nodes at kinks
- Flexible interpolation technique based on Simplicial Interpolation

$\Rightarrow$ Called Adaptive Simplicial Interpolation (ASI).

Main Findings

The major advantage of ASI is that

- kinks are matched accurately, thus
- less grid points are needed for a good fit, and
- computation time is reduced up to 200 times compared to equidistant grids.

Furthermore,

- ASI is very stable
- and is applicable to n-dimensional state spaces.

Overview

Adaptive Simplicial Interpolation

We explain ASI using a simple 2-period endowment economy:

- Three agents receive stochastic endowments in periods 1 and 2.
- In period 1, agents trade in a one period bond subject to ad hoc borrowing limits.
- We are interested in how policies of period 1 depend on the wealth distribution.
- We show how kinks induced by borrowing limits are identified by ASI.

2-Period Bond Economy

Individual Problem:

Each agent chooses consumption c_1 and bond holding b_1 to solve

$$\max_{c_1, b_1} \frac{c_1^{1-\gamma}}{1-\gamma} + \beta \mathbb{E}\left[\frac{c_2^{1-\gamma}}{1-\gamma}\right]$$

s.t. the budget constraint

$$b_0 + e_1(x_1) = p_1 b_1 + c_1, \qquad b_1 + e_2(x_2) = c_2$$

and the borrowing limit

$$b_1 \geq \underline{b} \quad \text{with} \quad \underline{b} \in \mathbb{R}^-,$$

where $x_1, x_2 \in X = \{1, ..., K\}$ are realizations of random shocks.

State Space:

$$S = X \times Y, \text{ where } Y \equiv \left\{ (b_0^1, b_0^2) \in [\underline{b}, -2\underline{b}]^2 \,\middle|\, \sum_{i=1}^{2} b_0^i \in [\underline{b}, -2\underline{b}] \right\}.$$

2-Period Bond Economy

Equilibrium Problem:

Given state $s = \left(x_1, b_0^1, b_0^2\right)$,

find policies and prices $f(s) = \left(\{c_1^h, b_1^h\}_{h=1,2,3}, p_1\right)$ that satisfy

$$
\begin{aligned}
&\sum_{h \in \{1,2,3\}} b_1^h = 0, \\
&c_1^h + b_1^h p_1 - e_1^h - b_0^h = 0 \;\; \forall h \in \{1,2,3\}, \\
&-u'(c_t^h) p_1 + \mu^h + \mathbb{E}\left[\beta u'(e_2^h + b_1^h)\right] = 0 \;\; \forall h \in \{1,2,3\}, \\
&0 \leq b_1^h - \underline{b} \perp \mu^h \geq 0 \;\; \forall h \in \{1,2,3\}.
\end{aligned}
$$

Parametric Equilibrium Problem:

Find a policy function $f : S \to \mathbb{R}^7$

s.t. for all $s \in S$, $f(s)$ solves the equilibrium problem.

Overview of ASI

Adaptive Simplicial Interpolation:

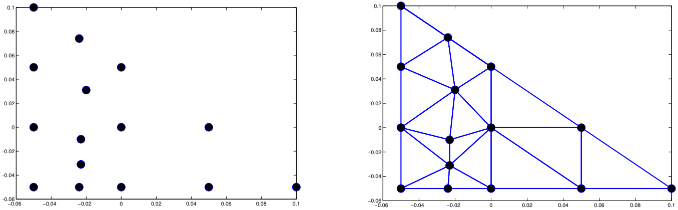
1. Initialization:
   Start with an initial grid G_{init} and solve for $\{f(g)\}_{g \in G_{init}}$.
2. Grid Adaptation:
   Use the information $\{f(g)\}_{g \in G_{init}}$ to solve jointly for adapted points G_{adapt} that lie on the kinks and for the solutions $\{f(k)\}_{k \in G_{adapt}}$ at these points.
3. Simplicial Interpolation:
   Interpolate f on $G = G_{init} \cup G_{adapt}$.
   To interpolate on such an irregular grid, use Delaunay interpolation.

Delaunay Interpolation

Delaunay Interpolation consists of two steps:

1. Delaunay triangulation:

Covers an arbitrary set of grid points with triangles:



- Maximizes the minimal angle within the triangulation:
  $\Rightarrow$ Desirable for Interpolation.
- Popular in many areas, e.g. engineering:
  $\Rightarrow$ Code for n dimensions available.

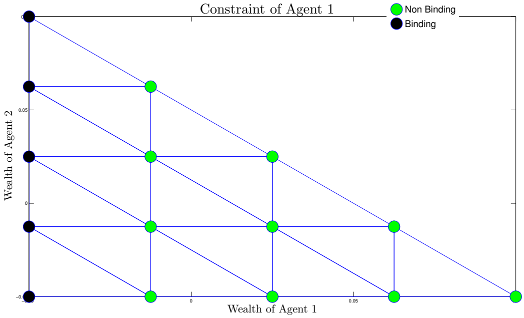
2. Simplicial interpolation:

On each triangle, interpolate linearly between the corners.

Grid Adaptation 1: How to Detect Kinks

For each constraint do the following:

1. From the solutions on the initial grid, observe where the constraint is binding or non-binding.
2. Consider each edge of the triangulation and check whether the constraint changes from binding to non-binding
3. If yes, put an adapted point on the edge.

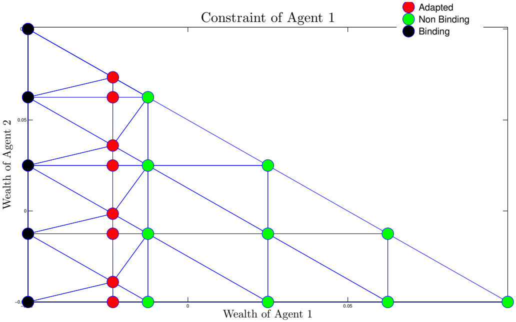


Grid Adaptation 2: How to Put Points Precisely on the Kink

Note: The Kink is where the constraint is just binding!

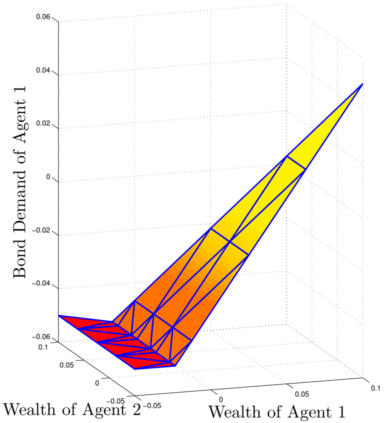
To search on an edge for the point that lies exactly on the kink, solve a modified version of the equilibrium problem:

- Let the state variable vary on the edge;
- fix the Lagrange multiplier associated with the kink at zero: $\mu^h = 0$;
- and force the associated inequality constraint to be binding: $b^h = \underline{b}$.

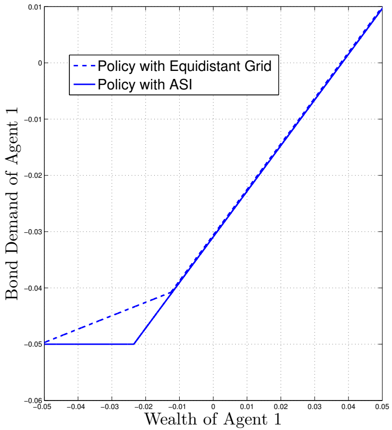


A Policy Function

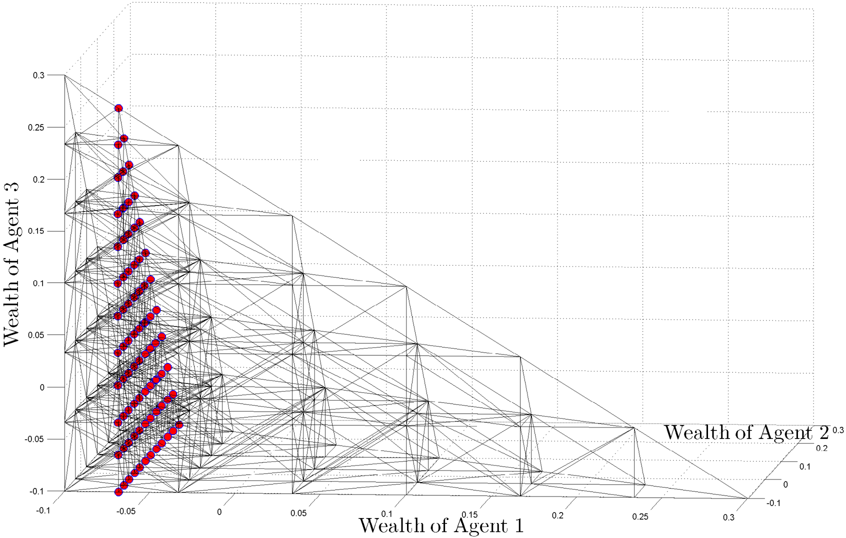
Bond demand of Agent 1:



Cut at Zero Wealth of Agent 2:



Adapted Grid with 3 Continuous State Variables



Overview

ASI and Time Iteration

We consider the Bond economy with infinite horizon:
Given next period's policy function $f^{next} = \left(\{c^h_{t+1}, b^h_{t+1}\}, p_{t+1}\right)$, the Period-to-Period Equilibrium Problem has the same structure as in the 2-period economy (Details).

$$\Downarrow$$

We may solve this model using a time iteration algorithm, and use ASI for each time iteration step.

The Algorithm

Time Iteration with Adaptive Simplicial Interpolation:

1. Select a grid G_{init}, and a policy function f^{start}. Set $f^{next} \equiv f^{start}$.
2. Make one time iteration step using ASI:

   1. For all $g \in G_{init}$, find $f(g)$ that solves the Period-to-Period Equilibrium Problem given f^{next}.

   2. Use $\{f(g)\}_{g \in G_{init}}$ to solve for kink points $k \in G_{adapt}$ and for the respective policies $\{f(k)\}_{k \in G_{adapt}}$.

   3. Use solutions at all grid points $G = G_{init} \cup G_{adapt}$ to interpolate f by simplicial interpolation.

   If $\left\|f - f^{next}\right\|_\infty < \epsilon$, go to step 3. Else set $f^{next} \equiv f$ and repeat step 2.
3. Set the numerical solution to the infinite horizon problem: $\tilde{f} = f$.

Euler Error Table

- To check accuracy we calculate relative errors in Euler equations (EEs) as proposed by Judd (1992 JET)
- For each exogenous state we draw 5000 random points and get the following EE-results (in log10-scale) for ASI:

	grid points	max EE	average EE	time (min)
3 agents	**40**	-3.0	**-4.1**	**0.5**
4 agents	**112**	-2.7	**-3.3**	**4.5**

- Using equidistant grids we try to match the maximum EE:

	grid points	max EE	average EE	time (min)
3 agents	**20301**	-2.8	**-5.3**	**79**
4 agents	**20825**	-2.0	**-3.6**	**895**

- Hence, with respect to the max EE, the adaptive grid needs up to 500 times less points for the same accuracy!

Overview

Endogenous Collateral Constraints

We extend the above model by including the following features:

- Additional asset: Lucas tree in unit net supply paying a fixed fraction of aggregate endowment as dividends.
- Short selling constraints on the Lucas Tree.
- Endogenous collateral constraints: Short positions in the bond need to be collateralized by stock holdings.

Details Endogenous Collateral Constraints

Individual Problem:

Choose c_t, b_t, and Lucas tree holding, l_t, to maximize lifetime utility s.t. the *budget constraint*

$$b_{t-1} + l_{t-1}\left(q_t + d_t\right) + e_t = p_t b_t + q_t l_t + c_t,$$

the *short-selling constraint* and the *collateral constraint*

$$l_t \geq 0, \quad -b_t \leq \min_{s_{t+1} > s_t} \left\{ \left(q_{t+1}(s_{t+1}) + d_{t+1}(s_{t+1})\right) l_t \right\},$$

where the *min* is over all states that succeed s_t with positive probability.

State Space:

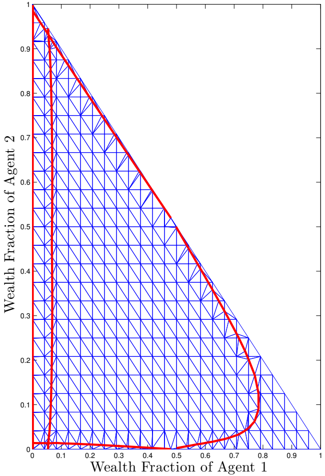
Define wealth $w_t^h \equiv l_{t-1}^h\left(q_t + d_t\right) + b_{t-1}^h$, and fraction of wealth: $y_t^h = w_t^h / q_t$.

$$S = X \times Y, \text{ where } Y \equiv \left\{ \left(y_t^1, ..., y_t^{H-1}\right) \in \mathbb{R}_+^{H-1} \middle| \sum_{i=1}^{H-1} y^i \leq 1 \right\}.$$

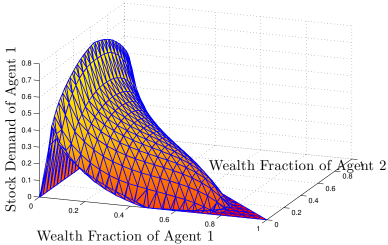
Parametrization: Like Bond economy, but $\alpha = 0.1$.

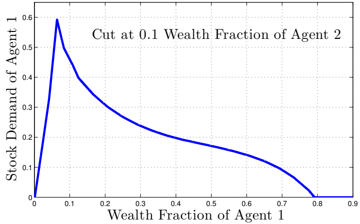
Grid and Stock Demand

Adapted Grid with 5 Kinks:



Stock Demand of Agent 1:





Euler Error Table

- For each exogeneous state we draw 5000 random points and get the following EE-results (in log10-scale) for ASI:

α	grid points	max EE	average EE	time (min)
0.1	**1235**	-2.5	**-3.7**	**310**

- Using equidistant grids we try to match the maximum EE:

α	grid points	max EE	average EE	time (min)
0.1	**25400**	-2.4	**-4.0**	**4500**

- Hence, with respect to the max EE, the adaptive grid needs at least 20 times less points for the same accuracy
- We improved this performance further by adapting the grid to non-linearities.

Conclusion

- We develop ASI, an algorithm that solves dynamic models with occasionally binding constraints and several continuous state variables.
- ASI uses Delaunay Interpolation, which allows for flexible grids.
- ASI includes an adaptive grid scheme that adjusts precisely to kinks.

Appendix: Parameterization

$x_t \in X = \{1, ..., K\}$ follows a Markov process with $2 \times H$ states:

$$\underset{(\text{ahigh/alow} = \nu_{agg})}{\text{2 aggregate endowments}} \quad \times \quad \underset{(\text{ihigh/ilow} = \nu_{idio})}{H \text{ endowment distributions}}$$

Parametrization:

γ	ν_{idio}	ν_{agg}	ρ_{idio}	ρ_{agg}	β	$\underline{b}$
1.5	1.6	1.06	0.9	0.65	0.95	-0.10

BondEcon

Infinite Horizon Bond Economy

Period-to-Period Equilibrium Problem:

Given next period's policy function $f^{next} = \left(\{c^h_{t+1}, b^h_{t+1}\}_{h=1,..,H}, p_{t+1}\right)$, and given state $s_t = \left(x_t, b^1_{t-1}, .., b^{H-1}_{t-1}\right)$, find policies $f(s_t) = \left(\{c^h_t, b^h_t\}_{h=1,..,H}, p_t\right)$ that satisfy

$$\sum_{h=1}^{H} b^h(s) = 0,$$

$$c^h_t(s) + b^h_t(s)p_t(s) - e^h_t(s) - b^h_{t-1}(s) = 0, \ \ \forall h \in \{1,..,H\},$$

$$-u'(c^h_t(s))p_t(s) + \mu^h_t(s) + \mathbb{E}\left[\beta u'\left(c^h_{t+1}\left(s_{t+1}\right)\right)\right] = 0, \ \ \forall h \in \{1,..,H\},$$

$$0 \leq b^h_t(s) - \underline{b} \perp \mu^h_t(s) \geq 0, \ \ \forall h \in \{1,..,H\},$$

BondEcon