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Stability analysis of velocity-aided attitude observers for accelerated vehicles[☆]

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ABSTRACT

An advanced nonlinear observer for rigid bodies evolving in 3D-space relying on measurement data provided by an IMU and linear velocity measured in the body-fixed frame has been recently proposed, but without specifying its domain of convergence and stability. With respect to conventional attitude observers, which do not involve linear velocity measurements, this observer is better adapted for vehicles subjected to sustained accelerations. This paper presents further modifications yielding two velocity-aided attitude observers, with rigorous almost global convergence and stability analysis. Moreover, the estimation of the gravity direction is globally decoupled from magnetometer measurements.

1. Introduction

The development of a robust and reliable attitude estimator is the key technology enabler for robotic vehicles. Nonlinear attitude observers have become an alternative to extended Kalman filters over the last two decades (Bonnabel, Martin, & Rouchon, 2008; Dukan & Sorensen, 2013; Grip, Fossen, Johansen, & Saberi, 2012; Hua, 2010; Hua, Ducard, Hamel, Mahony, & Rudin, 2014; Hua, Martin, & Hamel, 2014; Mahony, Hamel, & Pflimlin, 2008; Martin & Salaun, 2008, 2010; Roberts & Tayebi, 2011; Troni & Whitcomb, 2013). The performance of recent attitude observers is comparable to modern nonlinear filtering techniques. Moreover, they are often endowed with provable strong stability (e.g., almost global asymptotical stability[2]) and robustness with respect to measurement noise; they are also simple to tune and to implement (cf. Hua, Ducard et al., 2014 and Mahony et al., 2008). Most existing attitude estimators make use of an Inertial Measurement Unit (IMU) and are based on the assumption of weak accelerations of the vehicle so that the gravity direction estimate can be approximated by the accelerometer measurements (Hua, Ducard et al., 2014; Mahony et al., 2008; Martin & Salaun, 2010). However, the accuracy of the estimated attitude using this assumption is far from satisfactory when the vehicle undergoes sustained accelerations (cf. Hua, 2010). To get rid of this unsatisfactory assumption some authors have considered attitude observers "aided" by complementary linear velocity measurements in either an inertial frame (Grip et al., 2012; Hua, 2010; Martin & Salaun, 2008; Roberts & Tayebi, 2011) or a body-fixed frame (Bonnabel et al., 2008; Dukan & Sorensen, 2013; Troni & Whitcomb, 2013). This paper focuses on the latter category. For instance, an observer is proposed and tested experimentally on an underwater vehicle in Dukan and Sorensen (2013), but without convergence and stability analysis; in the same context Troni and Whitcomb (2013) use the numerical derivative of the linear velocity to recover the gravity direction estimate, which is sensitive to measurement noise. Bonnabel et al. proposed in Bonnabel et al. (2008) a more advanced observer, endowed with some nice properties such as the *local* exponential stability around any trajectory of the system and the *local* decoupling of roll and pitch estimation from magnetometer measurements. In view of convincing simulation results, they suspect that the stability is much stronger than local and conjecture that this observer is almost global convergent. In fact, the almost global asymptotical

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[2] Stability with convergence of the observer error to zero for all initial conditions and system trajectories other than on a set of measure zero.

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stability domain is the largest domain one can obtain for an observer designed on the rotation group SO(3) (or the group of unit quaternions). This is due to the associated topological obstruction (Bhat & Bernstein, 2000). However, all results in Bonnabel et al. (2008) are only based on local analysis. Determining the stability domain for a body-fixed velocity-aided attitude observer is very challenging due to the underlying nonlinearity of the system equations; and to our knowledge no existing works (even Bonnabel et al., 2008) have addressed this issue. On the other hand, it is worth noticing that magnetometer measurements are often corrupted by on-board magnetic fields generated by motors and currents, as well as external magnetic disturbances. Therefore, decoupling of input signals to ensure that the estimation of the gravity direction (i.e. roll and pitch) is not disturbed by magnetic disturbances is particularly important in practice (cf. Hua, Ducard et al., 2014 and Martin & Salaun, 2010).

In this paper, we propose some modifications on the innovation terms of the observer proposed in Bonnabel et al. (2008), yielding two nonlinear invariant observers. The main interest is that we can now establish rigorous stability analysis for (a) *almost global asymptotical stability* of the observer error dynamics and (b) *global decoupling* of the roll and pitch estimation from magnetometer measurements, while ensuring local exponential stability. A preliminary version of this paper has been presented at a conference (Hua, Martin et al., 2014).

2. Preliminary material

2.1. Notation

- $\{e_1, e_2, e_3\}$ denotes the canonical basis of $\mathbb{R}^3$. Let $(\cdot)_\times$ denote the skew-symmetric matrix associated with the cross product, i.e., $u_\times v = u \times v,\ \forall u, v \in \mathbb{R}^3$. Let I denote the identity element of the rotation group SO(3).
- Let $\{\mathcal{I}\} = \{O; \vec{\imath}_0, \vec{\jmath}_0, \vec{k}_0\}$ denote an inertial frame attached to the earth, typically chosen as the north-east-down frame. Let $\{\mathcal{B}\} = \{G; \vec{\imath}, \vec{\jmath}, \vec{k}\}$ be a body-fixed frame attached to the vehicle's center of mass G.
- Let $v \in \mathbb{R}^3$ and $\omega \in \mathbb{R}^3$ denote the vehicle's linear and angular velocities expressed in $\{\mathcal{B}\}$. The vehicle's attitude is represented by a rotation matrix $R \in \mathrm{SO}(3)$ of the frame $\{\mathcal{B}\}$ relative to $\{\mathcal{I}\}$. The column vectors of R correspond to the vectors of coordinates of $\vec{\imath}, \vec{\jmath}, \vec{k}$ expressed in the basis of $\{\mathcal{I}\}$. By representing the gravity direction by the unit vector $\gamma \triangleq R^T e_3 \in S^2$ (the unit 2-sphere), one deduces that roll and pitch Euler angles can be (locally) uniquely determined from γ (see, e.g., Hua, Ducard et al., 2014 for technical details).

2.2. System equations and measurements

The attitude satisfies the differential equation

$$\dot{R} = R\omega_\times, \tag{1}$$

from which one deduces the dynamics of γ as

$$\dot{\gamma} = \gamma \times \omega. \tag{2}$$

Assume that the vehicle is equipped with a velocity sensor (e.g., a Doppler sensor) to measure v. It is also equipped with an IMU consisting of a 3-axis gyrometer and a 3-axis accelerometer to provide the measurements of the angular velocity ω and the *specific acceleration* $a_{\mathcal{B}} \in \mathbb{R}^3$, expressed in $\{\mathcal{B}\}$. Using the flat non-rotating Earth assumption, one has (Bonnabel et al., 2008)

$$\dot{v} = v \times \omega + a_{\mathcal{B}} + gR^T e_3 = v \times \omega + a_{\mathcal{B}} + g\gamma, \tag{3}$$

where g is the gravity constant. In many IMUs, a 3-axis magnetometer is also integrated to provide the measurement of the Earth's magnetic field vector $m_{\mathcal{B}} \in \mathbb{R}^3$, expressed in $\{\mathcal{B}\}$. One verifies that $m_{\mathcal{B}} = R^T m_{\mathcal{I}}$, with $m_{\mathcal{I}}$ the Earth's magnetic field vector expressed in $\{\mathcal{I}\}$.

3. Observer design

3.1. Nonlinear invariant observer design

Let $\hat{v} \in \mathbb{R}^3$ and $\hat{R} \in \mathrm{SO}(3)$ denote the estimates of v and R, respectively. In view of (1) and (3), the nonlinear invariant observer should take the following form

$$\begin{cases} \dot{\hat{v}} = \hat{v} \times \omega + a_{\mathcal{B}} + g\hat{R}^T e_3 + \sigma_v^i & \text{(a)} \\ \dot{\hat{R}} = \hat{R}(\omega + \sigma_R^i)_\times & \text{(b)} \end{cases} \tag{4}$$

with $\sigma_v^i, \sigma_R^i \in \mathbb{R}^3$ (for some index i) the innovation terms to be designed.

The observer proposed in Bonnabel et al. (2008) (that we term "Observer 0") consists in defining the innovation terms as

$$\begin{cases} \sigma_v^0 \triangleq \hat{R}^T(\mathcal{L}_v^v e_v + \mathcal{L}_m^v e_m) & \text{(a)} \\ \sigma_R^0 \triangleq \hat{R}^T(\mathcal{L}_v^R e_v + \mathcal{L}_m^R e_m) & \text{(b)} \end{cases} \tag{5}$$

with $e_v \triangleq \hat{R}(v - \hat{v})$ and $e_m \triangleq m_{\mathcal{I}} - \hat{R}m_{\mathcal{B}}$ the invariant errors; and $\mathcal{L}_v^v, \mathcal{L}_m^v, \mathcal{L}_v^R, \mathcal{L}_m^R$ the gain matrices chosen as

$$\begin{aligned} &\mathcal{L}_v^v = \mathrm{diag}(N_{11}, N_{22}, N_{33}), \qquad \mathcal{L}_m^v = 0_{3\times 3}, \\ &\mathcal{L}_v^R = \begin{bmatrix} 0 & M_{12} & 0 \\ -M_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathcal{L}_m^R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda m_2 & \lambda m_1 & 0 \end{bmatrix} \end{aligned} \tag{6}$$

with m_1, m_2 the first and second components of $m_{\mathcal{I}}$; and $N_{11}, N_{22}, N_{33}, M_{12}, M_{21}, \lambda$ some positive numbers.

Motivated by the fact that the convergence and stability along with the decoupling results of the observer proposed in Bonnabel et al. (2008) are only proved on the basis of *local* analysis, we propose instead the following innovation terms

$$\begin{cases} \sigma_v^1 \triangleq k_1^v \tilde{v} - k_2^v \hat{\gamma} \times (\hat{\gamma} \times \tilde{v}) & \text{(a)} \\ \sigma_R^1 \triangleq k_1^r \tilde{v} \times \hat{\gamma} + k_2^r ((m_{\mathcal{B}} \times \hat{m}_{\mathcal{B}})^T \hat{\gamma})\hat{\gamma} & \text{(b)} \end{cases} \tag{7}$$

with $\tilde{v} \triangleq v - \hat{v}$, $\hat{\gamma} \triangleq \hat{R}^T e_3$, $\hat{m}_{\mathcal{B}} \triangleq \hat{R}^T m_{\mathcal{I}}$, and positive gains $k_1^v, k_2^v, k_1^r, k_2^r$, so that the analysis for (a) *almost global asymptotical stability* and (b) *global decoupling of roll and pitch estimation from magnetometer measurements* can be established hereafter. Let us call this observer by "Observer 1".

Remark. The innovation terms σ_v^0 and σ_R^0 defined by (5), with gains given by (6), respectively coincide with the proposed innovation terms σ_v^1 and σ_R^1 defined by (7) if $N_{11} = N_{22} = k_1^v + k_2^v$, $N_{33} = k_1^v$, $M_{12} = M_{21} = k_1^r$, and $\lambda = k_2^r$ (the proof is left to the interested reader).

Defining the invariant errors $\bar{v} \triangleq R(v - \hat{v})$ and $\bar{R} \triangleq R\hat{R}^T$, one easily deduces for Observer 1 that

$$\begin{cases} \dot{\bar{v}} = g(I - \bar{R})e_3 - \bar{\sigma}_v^1 \\ \dot{\bar{R}} = -\bar{\sigma}_{R\times}^1 \bar{R}, \end{cases} \tag{8}$$

with

$$\bar{\sigma}_v^1 = k_1^v \bar{v} - k_2^v (\bar{R}e_3) \times ((\bar{R}e_3) \times \bar{v}), \tag{9a}$$

$$\bar{\sigma}_R^1 = k_1^r \bar{v} \times \bar{R}e_3 + k_2^r ((m_{\mathcal{I}} \times \bar{R}m_{\mathcal{I}})^T \bar{R}e_3)\bar{R}e_3. \tag{9b}$$

Therefore, the dynamics of $(\bar{v}, \bar{R})$ are autonomous.

3.2. Reduction to gravity direction estimation

The estimate $\hat{\gamma} \in S^2$ of γ can be calculated from $\hat{R}$ as $\hat{\gamma} = \hat{R}^T e_3$. But it can also be obtained from an observer directly designed on $\mathbb{R}^3 \times S^2$ as follows.

Lemma 1. *Observer 1 can be reduced to the following observers of v and γ (that we term "γ-observer"):*

$$\begin{cases} \dot{\hat{v}} = \hat{v} \times \omega + a_{\mathcal{B}} + g\hat{\gamma} + \sigma_v^1 & \text{(a)} \\ \dot{\hat{\gamma}} = \hat{\gamma} \times (\omega + \sigma_\gamma^1) & \text{(b)} \\ \sigma_v^1 \triangleq k_1^v \tilde{v} - k_2^v \hat{\gamma} \times (\hat{\gamma} \times \tilde{v}) & \text{(c)} \\ \sigma_\gamma^1 \triangleq k_1^r \tilde{v} \times \hat{\gamma} & \text{(d)} \end{cases} \tag{10}$$

with positive gains k_1^v, k_2^v and k_1^r. In addition, this γ-observer is independent of $m_{\mathcal{B}}$.

Proof. The expression (10)(a) is directly obtained from (4)(a) by replacing $\hat{R}^T e_3$ by $\hat{\gamma}$. Differentiating $\hat{\gamma} = \hat{R}^T e_3$ and using (4)(b) one deduces

$$\dot{\hat{\gamma}} = \hat{\gamma} \times (\omega + \sigma_R^1) = \hat{\gamma} \times (\omega + \sigma_\gamma^1),$$

where the latter equality is obtained using

$$\hat{\gamma} \times [k_2^r((m_{\mathcal{B}} \times \hat{m}_{\mathcal{B}})^T \hat{\gamma})\hat{\gamma}] = 0.$$

Finally, the statement about the independence of this γ-observer on $m_{\mathcal{B}}$ is straightforward. ■

Defining $\bar{\gamma} \triangleq R\hat{\gamma} = \bar{R}e_3$, one can easily verify from (2), (3), (10) that the dynamics of the invariant errors $(\bar{v}, \bar{\gamma})$ of the γ-observer (10) are autonomous:

$$\begin{cases} \dot{\bar{v}} = g(e_3 - \bar{\gamma}) - k_1^v \bar{v} + k_2^v \bar{\gamma} \times (\bar{\gamma} \times \bar{v}) \\ \dot{\bar{\gamma}} = -k_1^r \bar{\gamma} \times (\bar{\gamma} \times \bar{v}). \end{cases} \tag{11}$$

Lemma 2. *Consider the error dynamics (11) and assume that the positive gains k_1^v, k_2^v and k_1^r are chosen such that:*

$$k_1^r \le k_1^v k_2^v / g. \tag{12}$$

Then, the following properties hold:

(1) *System (11) has only two isolated equilibrium points $(\bar{v}, \bar{\gamma}) = (0, e_3)$ and $(\bar{v}, \bar{\gamma}) = ((2g/k_1^v)e_3, -e_3)$.*
(2) *The equilibrium $(\bar{v}, \bar{\gamma}) = (0, e_3)$ is almost globally asymptotically stable and locally exponentially stable.*
(3) *The equilibrium $(\bar{v}, \bar{\gamma}) = ((2g/k_1^v)e_3, -e_3)$ is unstable.*

Proof. Consider the Lyapunov function candidate

$$\mathcal{L}_0 \triangleq \frac{1}{2}|\bar{v}|^2 + \frac{gk_2^v}{2k_1^v k_1^r}|e_3 - \bar{\gamma}|^2 - \frac{g}{k_1^v}\bar{v}^T(e_3 - \bar{\gamma}), \tag{13}$$

which is positive-definite under condition (12). From (11) and (13), one verifies that

$$\begin{aligned} \dot{\mathcal{L}}_0 &= -k_1^v|\bar{v}|^2 - (g^2/k_1^v)|e_3 - \bar{\gamma}|^2 + 2g\bar{v}^T(e_3 - \bar{\gamma}) \\ &\quad + (k_2^v - gk_1^r/k_1^v)\bar{v}^T(\bar{\gamma} \times (\bar{\gamma} \times \bar{v})) \\ &= -k_1^v|\bar{v} - g/k_1^v(e_3 - \bar{\gamma})|^2 - (k_2^v - gk_1^r/k_1^v)|\bar{\gamma} \times \bar{v}|^2, \end{aligned}$$

which is negative-(semi)definite under condition (12). By application of LaSalle's theorem, one deduces the convergence of $\dot{\mathcal{L}}_0$ to zero. This in turn implies the convergence of $\delta \triangleq \bar{v} - (g/k_1^v)(e_3 - \bar{\gamma})$ to zero, and additionally the convergence of $\bar{\gamma} \times \bar{v}$ to zero if $k_1^r < k_1^v k_2^v/g$. However, if $k_1^r = k_1^v k_2^v/g$, only the convergence of δ to zero can be deduced. Let us now prove the convergence of $e_3 \times \bar{\gamma}$ to zero for the two possible cases satisfying (12).

- *Case $k_1^r < k_1^v k_2^v/g$*: The convergence of $\bar{v}$ to $(g/k_1^v)(e_3 - \bar{\gamma})$ is deduced from the definition of δ and its convergence to zero (proved previously). This implies that $\bar{v} \times \bar{\gamma}$ converges to $(g/k_1^v)e_3 \times \bar{\gamma}$, which must converge to zero since $\bar{v} \times \bar{\gamma}$ converges to zero (proved previously).
- *Case $k_1^r = k_1^v k_2^v/g$*: One deduces from (11) and the definition of δ that $\dot{\delta} = -k_0\delta$. The definition of δ implies $\bar{v} = \delta + (g/k_1^v)(e_3 - \bar{\gamma})$. Then, one verifies from (11) that

$$\begin{aligned} \frac{d}{dt}(1 - e_3^T\bar{\gamma}) &= -k_1^r e_3^T(\bar{\gamma} \times (\bar{v} \times \bar{\gamma})) \\ &= -k_1^r(\delta \times \bar{\gamma})^T(e_3 \times \bar{\gamma}) - \frac{gk_1^r}{k_1^v}|e_3 \times \bar{\gamma}|^2 \\ &\le k_1^r|\delta||e_3 \times \bar{\gamma}| - \frac{gk_1^r}{k_1^v}|e_3 \times \bar{\gamma}|^2. \end{aligned} \tag{14}$$

Now, consider the Lyapunov function candidate

$$\mathcal{L}_1 \triangleq (1/2)|\delta|^2 + (2g/k_1^r)(1 - e_3^T\bar{\gamma}). \tag{15}$$

One deduces from (14), (15) and $\dot{\delta} = -k_0\delta$ that

$$\dot{\mathcal{L}}_1 \le -k_1^v|\delta|^2 - (2g^2/k_1^v)|e_3 \times \bar{\gamma}|^2 + 2g|\delta||e_3 \times \bar{\gamma}|.$$

Since $k_1^v|\delta|^2 + (2g^2/k_1^v)|e_3 \times \bar{\gamma}|^2 \ge 2\sqrt{2}g|\delta||e_3 \times \bar{\gamma}|$, there exist two positive numbers α_1, α_2 such that $\dot{\mathcal{L}}_1 \le -\alpha_1|\delta|^2 - \alpha_2|e_3 \times \bar{\gamma}|^2$. By application of LaSalle's theorem, one deduces the convergence of $\dot{\mathcal{L}}_1$ and, subsequently, of δ and $e_3 \times \bar{\gamma}$ to zero.

Therefore, for both cases $\bar{\gamma}$ converges to either e_3 (desired) or $-e_3$ (undesired), implying that $\bar{v}$ converges to zero (desired) or $(2g/k_1^v)e_3$ (undesired). Now, to prove that the "desired" equilibrium $(\bar{v}, \bar{\gamma}) = (0, e_3)$ is locally exponentially stable, it suffices to study the stability of the linearized system about this point, i.e.

$$\begin{cases} \dot{\bar{v}} = g(e_3 - \bar{\gamma}) - k_1^v\bar{v} + k_2^v e_3 \times (e_3 \times \bar{v}) \\ \dot{\bar{\gamma}} = -k_1^r e_3 \times (e_3 \times \bar{v}) \end{cases}$$

which can be decomposed into three subsystems

$$\begin{bmatrix} \dot{\bar{v}}_i \\ \dot{\bar{\gamma}}_i \end{bmatrix} = \begin{bmatrix} -(k_1^v + k_2^v) & -g \\ k_1^r & 0 \end{bmatrix} \begin{bmatrix} \bar{v}_i \\ \bar{\gamma}_i \end{bmatrix}, \quad i = 1, 2 \tag{16a}$$

$$\dot{\bar{v}}_3 = -k_1^v\bar{v}_3. \tag{16b}$$

By application of Hurwitz criteria, one deduces that the origin of these three subsystems is stable for any set of positive constant gains (k_1^v, k_2^v, k_1^r). Similarly, one can easily prove that the "undesired" equilibrium $(\bar{v}, \bar{\gamma}) = ((2g/k_1^v)e_3, -e_3)$ is unstable by analyzing the linearized system about this equilibrium. Therefore, the equilibrium $(\bar{v}, \bar{\gamma}) = (0, e_3)$ is almost globally asymptotically stable and locally exponentially stable. ■

3.3. Stability analysis

To analyze the asymptotic stability of Observer 1 along the system trajectory, it is more convenient to consider the dynamics of the errors $(\bar{v}, \bar{R})$ and prove that their trajectory converges to $(0, I)$.

Theorem 1. *Consider Observer 1 with positive gains k_1^v, k_2^v, k_1^r, and k_2^r satisfying condition (12). Assume that $m_{\mathcal{I}} \times e_3 \neq 0$. Then, the following properties hold:*

(1) *The dynamics of $(\hat{v}, \hat{R}^T e_3)$ are independent of $m_{\mathcal{B}}$.*
(2) *The dynamics of the estimate errors $(\bar{v}, \bar{R})$ have only four isolated equilibria, one of which is $(\bar{v}, \bar{R}) = (0, I)$.*
(3) *The equilibrium $(\bar{v}, \bar{R}) = (0, I)$ is locally exponentially stable and almost globally asymptotically stable; and the other three equilibria of $(\bar{v}, \bar{R})$ are unstable.*

Proof. Property 1 is a direct result of Lemma 1. We now prove Property 2. First, recall that the dynamics of $(\bar{v}, \bar{R})$ are given by (8). As a result of Lemma 2, $(\bar{R}e_3, \bar{v})$ converges to $(e_3, 0)$ or $(-e_3, (2g/k_1^v)e_3)$. For both cases, the term $\bar{\sigma}_R^1$ given by (9b) converges exponentially to

$$\begin{aligned}\bar{\sigma}_R^1 &\to k_2^r((m_\jmath \times \bar{R}m_\jmath)^T e_3)e_3\\ &\to k_2^r|\pi_{e_3}m_\jmath|^2((\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3)e_3,\end{aligned}$$

where $\pi_x \triangleq I - xx^T$, $\forall x \in S^2$, and $\bar{m}_\jmath \triangleq \frac{\pi_{e_3}m_\jmath}{|\pi_{e_3}m_\jmath|} \in S^2$. Consequently, the dynamics of $\bar{R}$ can be written as

$$\dot{\bar{R}} = -\bar{k}_2^r((\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3)e_{3\times}\bar{R} + \epsilon(\bar{v}, \bar{R})_\times \bar{R}, \tag{17}$$

where $\bar{k}_2^r \triangleq k_2^r|\pi_{e_3}m_\jmath|^2$ and the term $\epsilon(\bar{v}, \bar{R}) \triangleq -\bar{\sigma}_R^1 + k_2^r|\pi_{e_3}m_\jmath|^2 ((\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3)e_3$ remains bounded and converges exponentially to zero since $(\bar{v}, \bar{R}e_3)$ converges exponentially to $(0, e_3)$. Using (17), one verifies that the time-derivative of the positive function $\mathcal{V} \triangleq 1 - \bar{m}_\jmath^T\bar{R}\bar{m}_\jmath$ satisfies $\dot{\mathcal{V}} \leq -\bar{k}_2^r((\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3)^2 + |\epsilon(\bar{v}, \bar{R})|$. Then, by integration one deduces

$$\int_0^\infty \bar{k}_2^r((\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3)^2 d\tau \leq \int_0^\infty |\epsilon(\bar{v}, \bar{R})| d\tau + \mathcal{V}(0) - \mathcal{V}(\infty).$$

Thus, $\int_0^\infty ((\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3)^2 d\tau$ remains bounded since $\mathcal{V}$ is bounded and $|\epsilon(\bar{v}, \bar{R})|$ converges exponentially to zero. Then, the application of Barbalat's lemma yields the convergence of $(\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3$ to zero.

From its definition, $\bar{m}_\jmath$ belongs to Span(e_1, e_2). Thus, there exists a constant angle α such that

$$\bar{m}_\jmath = \cos\alpha\, e_1 + \sin\alpha\, e_2 = \begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix} e_1 = R_\alpha e_1.$$

Since $R_\alpha e_3 = R_\alpha^T e_3 = e_3$, one writes

$$(\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3 = (e_1 \times R_\alpha^T \bar{R} R_\alpha e_1)^T e_3 = e_2^T(R_\alpha^T \bar{R} R_\alpha)e_1,$$

which implies that $e_2^T(R_\alpha^T \bar{R} R_\alpha)e_1$ converges to zero using the fact that $(\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3 \to 0$. One also verifies from the convergence $\bar{R}e_3 \to \pm e_3$ that $R_\alpha^T \bar{R} R_\alpha e_3 \to \pm e_3$. From here, one deduces that $R_\alpha^T \bar{R} R_\alpha$ converges to one of the four following rotation matrices: $R_1^* \triangleq I$, $R_2^* \triangleq \text{diag}(-1, -1, 1)$, $R_3^* \triangleq \text{diag}(-1, 1, -1)$, $R_4^* \triangleq \text{diag}(1, -1, -1)$, where the first two matrices correspond to the case $\bar{R}e_3 \to e_3$, and the last two correspond to the case $\bar{R}e_3 \to -e_3$. This in turn implies that $\bar{R}$ converges to one of the four matrices $R_\alpha R_i^* R_\alpha^T$ $(i = 1, \ldots, 4)$, with the first one equal to I.

We now prove Property 3. It is straightforward that the last two equilibria $(\bar{v}, \bar{R}) = ((2g/k_1^v)e_3, R_\alpha R_{3,4}^* R_\alpha^T)$ are unstable, since the corresponding equilibria of the subsystem $(\bar{v}, \bar{R}e_3)$ are unstable (as a result of Lemma 2). Denoting $\eta \triangleq R_\alpha^T \bar{R} R_\alpha e_1$, one obtains $(\bar{m}_\jmath \times \bar{R}\bar{m}_\jmath)^T e_3 = e_2^T \eta$ and verifies from (17) that

$$\dot{\eta} = -\bar{k}_2^r(e_2^T\eta)\, e_3 \times \eta. \tag{18}$$

The linearized system of (18) about the "undesired" equilibrium $\eta = R_1^* e_1 = -e_1$ satisfies: $\dot{\eta}_2 = \bar{k}_2^r \eta_2$, $\dot{\eta}_1 = \dot{\eta}_3 = 0$, which clearly indicates that this "undesired" equilibrium is unstable. On the other hand, the linearized system of (18) about the "desired" equilibrium $\eta = R_0^* e_1 = e_1$ is given by: $\dot{\eta}_2 = -\bar{k}_2^r \eta_2$, $\dot{\eta}_1 = \dot{\eta}_3 = 0$. From here, the local exponential stability of this equilibrium is directly deduced. This along with the local exponential stability of $(\bar{v}, \bar{R}e_3) = (0, e_3)$ proved in Lemma 2 yields the local exponential stability of the "desired" equilibrium $(\bar{v}, \bar{R}) = (0, I)$. ■

Note that there is a filtering justification for the proposed observer that can only be formulated in the linearization of the error dynamics, similarly to the ones described in Hua, Ducard et al. (2014) and Mahony et al. (2008). When the gain condition (12) is not satisfied, no proof for the almost global asymptotical stability of Observer 1 is available. What is the meaning of this sufficient condition? It indicates that the determinant of the characteristic polynomial $P(\lambda) = \lambda^2 + (k_1^v + k_2^v)\lambda + gk_1^r$ of the linearized subsystems (16a) is positive, which in turn implies that this linearized subsystems can only possess negative real poles. To get rid of this condition on gains, we propose to slightly modify the innovation term σ_1^v defined by (5)(a) by adding a "quadratic" term $-k_1^r\tilde{v} \times (\tilde{v} \times \hat{\gamma})$. This yields the second observer (that we term "Observer 2"), with the innovation terms

$$\sigma_v^2 \triangleq \sigma_v^1 - k_1^r\tilde{v} \times (\tilde{v} \times \hat{\gamma}), \qquad \sigma_R^2 \triangleq \sigma_R^1. \tag{19}$$

Identical results of Theorem 1 are stated next.

Proposition 1. *Consider Observer 2 with any positive gains $k_1^v, k_2^v, k_1^r, k_2^r$. Assume that $m_\jmath \times e_3 \neq 0$. Then, all the properties stated in Theorem 1 hold.*

Proof. Similarly to the case of Observer 1, we compute first the dynamics of the invariant errors $(\bar{v}, \bar{\gamma})$ as (compared to (11))

$$\begin{cases}\dot{\bar{v}} = g(e_3 - \bar{\gamma}) - k_1^v\bar{v} + k_2^v\bar{\gamma} \times (\bar{\gamma} \times \bar{v}) + k_1^r\bar{v} \times (\bar{v} \times \bar{\gamma})\\ \dot{\bar{\gamma}} = -k_1^r\bar{\gamma} \times (\bar{\gamma} \times \bar{v}).\end{cases} \tag{20}$$

We will show next that all properties stated in Lemma 2 also hold for Observer 2. Using (20) and the identity $u \times (v \times w) = v(u^Tw) - w(u^Tv)$, $\forall u, v, w \in \mathbb{R}^3$, one verifies that

$$\frac{d}{dt}(\bar{v} \times \bar{\gamma}) = g\, e_3 \times \bar{\gamma} - (k_1^v + k_2^v)\, \bar{v} \times \bar{\gamma}. \tag{21}$$

From (20) and (21), it is clear that the dynamics of $(\bar{v} \times \bar{\gamma}, \bar{\gamma})$ are autonomous. Consider the following function

$$\mathcal{S}_0 \triangleq 1/2|\bar{v} \times \bar{\gamma}|^2 + g/k_1^r(1 - e_3^T\bar{\gamma}). \tag{22}$$

Using (20)–(22), one deduces

$$\dot{\mathcal{S}}_0 = -(k_1^v + k_2^v)\, |\bar{v} \times \bar{\gamma}|^2. \tag{23}$$

From LaSalle's theorem, one deduces the convergence of $\dot{\mathcal{S}}_0$ and, thus, of $\bar{v} \times \bar{\gamma}$ to zero. From (22) and (23), the boundedness of $\bar{v} \times \bar{\gamma}$ can be deduced. It is verified that $\dot{\bar{\gamma}}$ is also bounded, which implies the uniform continuity of $\bar{\gamma}$. Then, by application of the extended Barbalat's lemma (Micaelli & Samson, 1993) to (21) one deduces the convergence of $\frac{d}{dt}(\bar{v} \times \bar{\gamma})$ to zero. This in turn implies the convergence of $e_3 \times \bar{\gamma}$ to zero. Therefore, $\bar{\gamma}$ converges to either e_3 or $-e_3$.

Since $\bar{v} \times \bar{\gamma}$ converges to zero, the zero-dynamics of $\dot{\bar{v}}$ are

$$\dot{\bar{v}} = g(e_3 - \bar{\gamma}) - k_1^v\bar{v}. \tag{24}$$

The convergence of $\bar{\gamma}$ to either e_3 or $-e_3$ associated with the zero-dynamics (24) ensures the convergence of $\bar{v}$ to either zero or $(2g/k_1^v)e_3$. Then, the proof of almost global asymptotical stability and local exponential stability of the "desired" equilibrium $(\bar{v}, \bar{\gamma}) = (0, e_3)$ and the proof of instability of the "undesired" one $(\bar{v}, \bar{\gamma}) = ((2g/k_1^v)e_3, -e_3)$ proceed identically to the proof of Lemma 2. From here, the remainder of proof proceeds identically to the proof of Theorem 1. ■

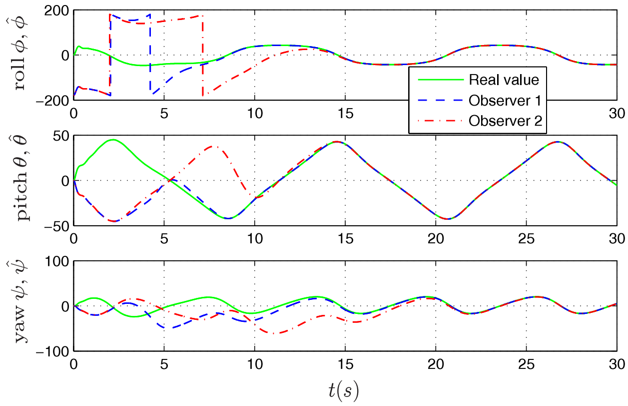

Fig. 1. Estimated and real attitude represented by roll, pitch and yaw Euler angles (deg) versus time (s).

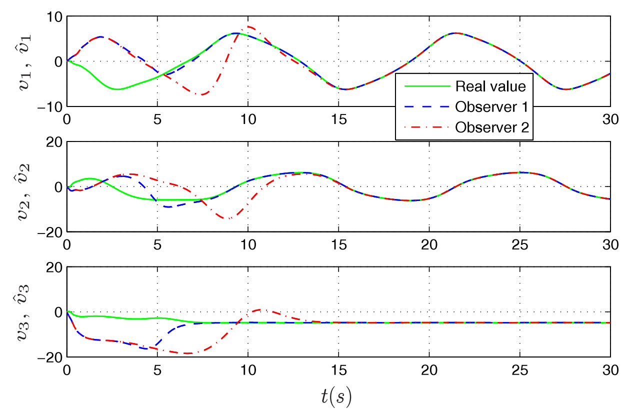

Fig. 2. Estimated and real velocity (m/s) versus time (s), where v_i (resp. $\hat{v}_i$) is the ith component of v (resp. $\hat{v}$).

4. Simulation results

Simulations are conducted on a model of a ducted-fan VTOL aerial drone, also used in Hua (2010). The vehicle is stabilized along a circular reference trajectory, with the linear velocity expressed in the inertial frame $\{\mathcal{I}\}$ given by $\dot{x}_r = [-15\alpha \sin(\alpha t); 15\alpha \cos(\alpha t); 0]$ (m/s), with $\alpha = 2/\sqrt{15}$. Due to aerodynamic forces acting on the vehicle, its orientation constantly varies in large proportions. The normalized earth's magnetic field and the gravity constant are respectively taken as $m_{\mathcal{I}} = [0.434; -0.0091; 0.9008]$ and $g = 9.81$ (m/s^2).

The initial conditions are chosen such that the initial error variables are very close to an "undesired" equilibrium, i.e. $\tilde{v}(0) = [0; 0; 0]$ (m/s) and $\tilde{R}(0) \approx \mathrm{diag}(-1, 1, -1)$. The gains are chosen so that condition (12) is satisfied: $k_1^v = 1.2$, $k_2^v = 1.2$, $k_1^r = 1.44/g$, $k_2^r = 2.764$. The values of k_1^v, k_2^v, k_1^r ensure that the linearized subsystem (16a) has a double pole equal to -1.2, while the value of k_2^r ensures that a pole of the linearized system of (18) is also equal to -1.2.

The evolutions of the estimated and real attitudes, represented by roll, pitch and yaw Euler angles, along with the estimated and real linear velocity expressed in the body-fixed frame $\{\mathcal{B}\}$ are shown in Figs. 1 and 2, respectively. Both Observers 1 and 2 ensure the asymptotic convergence of the estimated variables to the real values despite the large initial estimation errors and strong reference acceleration. The convergence rate provided by Observer 1 is slightly faster than Observer 2, but the performance of both observers is quite satisfactory.

5. Conclusions and perspectives

Two novel velocity-aided nonlinear attitude observers for accelerated vehicles have been proposed, with rigorous Lyapunov-based analyses of convergence and stability showing that they are almost globally asymptotically stable and locally exponentially stable. Moreover, the estimation of the gravity direction (i.e. roll and pitch) is globally decoupled from magnetometer measurements. To our knowledge, no other works on the topic have achieved this. Perspectives include the extension of the proposed observers to compensate for gyrometer bias and/or accelerometer bias.

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