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# Optimal Motion Planning for Differentially Flat Systems Using Bernstein Approximation

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***Abstract*—This letter presents a computational framework to efficiently generate feasible and optimal trajectories for differentially flat autonomous vehicle systems. We formulate the optimal motion planning problem as a continuous-time optimal control problem, and approximate it by a discrete-time formulation using Bernstein polynomials. These polynomials allow for efficient computation of various constraints along the entire trajectory, and are particularly convenient for generating trajectories for safe operation of multiple vehicles in complex environments. The advantages of the proposed method are investigated through theoretical analysis and numerical examples.**

***Index Terms*—Optimal motion planning, autonomous vehicles, Bézier curves, Bernstein polynomial, discrete approximation.**

## I. Introduction

MOTION planning is a challenging problem in operation of autonomous vehicles engaged in complex missions. The past few decades have produced a vast number of methods, including potential field methods, roadmap path planners, cell decomposition methods, and optimal control based motion planning (see [1] and references therein). Motion planning based on optimal control problem formulation is particularly suitable for applications that require the trajectory to minimize (or maximize) some cost function while satisfying a complex set of vehicle and problem constraints.

In general, finding a closed-form solution to a nonlinear constrained optimal control problem is hard. Direct methods can be used to approximate optimal control problems to simpler problems, which are easier to solve [2]–[5]. Direct methods based on discretization, for example, approximate the states of the dynamic system, or its inputs or both, thus reducing the original problem into a nonlinear programming problem (NLP) [5], which can then be solved by nonlinear optimization solvers. An important role in the literature on direct methods based on discretization is played by the work of Polak on *consistency of approximation theory* (see [6, Sec. 3.3]). Borrowing tools from variational analysis, Polak provides a theoretical framework to assess the convergence properties of discretization schemes for optimal control problems. Motivated by the consistency of approximation theory, a wide range of methods that use different discretization schemes have been developed. Few examples include Euler [6], Runge-Kutta [7], Pseudospectral [8] methods, as well as the method presented in this letter.

When solving constrained optimal control problems using discretization methods, the constraints can be enforced only at the discretization nodes, and not in between the nodes. As pointed out in [9] and [10], this implies a major drawback in applications such as motion planning, where both satisfaction of constraints along the trajectories and real-time execution of the algorithms could be essential. Depending on the complexity of the problem (vehicles' dynamics, number of vehicles and obstacles, etc.) low order of approximations (number of discretization nodes) might be required in order to enable real-time computation of trajectories. However, this could potentially result in generation of trajectories that are neither feasible nor safe [11]. On the other hand, to avoid violation of the constraints, the order of approximation can be increased, leading to higher dimensional NLPs, which become computationally expensive and inefficient for real-time applications. Additionally, discretization methods also suffer from spatial and temporal scalability issues.

This letter proposes a discretization method that uses Bézier curves as a special tool to overcome the above issues. In particular, we present a direct method based on Bernstein approximation of the trajectories. Bernstein approximants and Bézier curves have several important features. First, Bernstein basis possesses optimal numerical stability properties [12], [13], and can handle large order of approximations without suffering from numerical instability issues. Second, Bernstein approximants converge uniformly to the functions that they approximate – and so do their derivatives [14]. This is useful for proving convergence properties of the proposed method. Third, due to their geometric properties, Bézier curves afford computationally efficient algorithms for the computation of constraints such as minimum and maximum velocity, acceleration, minimum distance between paths, etc., for the entire

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trajectory, and not only at the discretization points [15]. Hence, the trajectories can be guaranteed to be dynamically feasible and collision-free for all times.

As we will point out later, Bernstein approximation converges slower than other interpolation or approximation techniques. This implies that the approach proposed in this letter is outperformed by, for example, pseudospectral methods in terms of accuracy of the approximation of the optimal solution. This is not surprising, since the choice of nodes and interpolating polynomials in pseudospectral methods is dictated by approximation accuracy and convergence speed, while sacrificing constraints satisfaction in between the nodes. On the other hand, the approach in this letter prioritizes safety and constraint satisfaction, at the expense of a slower convergence rate.

A growing number of papers exploit the properties of Bézier curves for motion planning (e.g., [11], [16], and [17]). Using the notion of consistency of approximation introduced by Polak [6], this letter provides a theoretical foundation for the use of Bézier curves in optimal motion planning. Similar consistency results have been demonstrated for various discretization schemes, including pseudospectral methods [18], [19]. However, these results are limited to collocation methods [20]. The contribution of this letter is an extension of these results to a class of non-collocation methods.

Finally, this letter is focused on differentially flat systems [21]. This class of systems is particularly suited for motion planning, since the trajectory can be planned in (flat) output space, and the states and inputs can be computed through algebraic mappings. Thus, the optimal motion planning problem reduces to a simpler calculus of variations problem. Moreover, the majority of vehicle systems of our interest have been shown to be differentially flat [22], [23], making this approach applicable to a wide range of applications.

This letter is structured as follows: in Section II we present the notation and mathematical results which will be used later in this letter. Section III introduces the problem of optimal motion planning for differentially flat systems. Section IV presents the NLP based on Bernstein polynomials, and demonstrates consistency results for the proposed method. Numerical examples are discussed in Section V. This letter ends with conclusions in Section VI.

## II. Notation and Mathematical Preliminaries

In what follows, vectors are denoted by bold letters, e.g., $\boldsymbol{x} = [x_1, \dots, x_n]$. The symbol $\mathcal{C}^r$ denotes the space of functions with $r$ continuous derivatives, while $\mathcal{C}^r_n$ denotes the space of $n$-vector valued functions in $\mathcal{C}^r$. Finally, $\|\cdot\|$ denotes the Euclidean norm, i.e., $\|\boldsymbol{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$.

Let $\boldsymbol{f}_N : [0, t_f] \to \mathbb{R}^n$ denote a vector of $N$th order Bézier curves defined as

$$\boldsymbol{f}_N(t) = \sum_{j=0}^{N} \boldsymbol{c}_j b_{j,N}(t) \,, \tag{1}$$

where $\boldsymbol{c}_j$, $j = 0, \dots, N$, are *control points*, and $b_{j,N}(t) = \binom{N}{j} \frac{t^j (t_f - t)^{N-j}}{t_f^N}$ is the *generalized* Bernstein polynomial basis of degree $N$, with $\binom{N}{j} = \frac{N!}{j!(N-j)!}$. The $r$th derivative of $\boldsymbol{f}_N(t)$ can be easily computed as follows

$$\boldsymbol{f}_N^{(r)}(t) = \sum_{j=0}^{N} \left( \sum_{i=0}^{N} \boldsymbol{c}_i D^r_{ij} \right) b_{j,N}(t) \,, \tag{2}$$

where $D^r_{ij}$ denotes the $ij$ element of the matrix $\boldsymbol{D} \in \mathbb{R}^{(N+1)\times(N+1)}$ elevated to the $r$th power, and $\boldsymbol{D}$ is a constant square differentiation matrix, which can be easily computed using the degree elevation and derivative of Bézier curve properties [24, Ch. 5].

Bézier curves and Bernstein polynomial basis can be used to approximate smooth functions.

*Definition 1 (Bernstein Approximation):* Consider a $n$-vector valued function $\boldsymbol{f} : [0, t_f] \to \mathbb{R}^n$. The $N$th order *Bernstein approximation* of $\boldsymbol{f}(t)$ is a Bézier curve $\boldsymbol{f}_N(t)$ computed as in (1) with $\boldsymbol{c}_j = \boldsymbol{f}(t_j)$, $t_j = j\frac{t_f}{N}$, for all $j = 0, \dots, N$, i.e.,

$$\boldsymbol{f}_N(t) = \sum_{j=0}^{N} \boldsymbol{f}\left( j \frac{t_f}{N} \right) b_{j,N}(t) \,. \tag{3}$$

The following result is true for Bernstein approximations.

*Lemma 1 (Uniform Convergence):* Assume $\boldsymbol{f}(t) \in \mathcal{C}^{r+2}_n$, $r \geq 0$, and let $\boldsymbol{f}_N(t)$ be the Bernstein approximation of $\boldsymbol{f}(t)$. Then, the following inequalities hold:

$$\begin{aligned} \left\| \boldsymbol{f}_N(t) - \boldsymbol{f}(t) \right\| &\leq \frac{C_0}{N} \\ &\vdots \\ \left\| \boldsymbol{f}_N^{(r)}(t) - \boldsymbol{f}^{(r)}(t) \right\| &\leq \frac{C_r}{N} \,, \end{aligned}$$

for all $t \in [0, t_f]$, where $C_0, \dots, C_r$ are independent of $N$.

*Proof:* The proof of Lemma 1 is a trivial extension of the proof given in [14, Sec. 3] for $f : [0, 1] \to \mathbb{R}$.

To approximate the integral of functions, the following quadrature approximation can be used:

$$\int_0^{t_f} \boldsymbol{f}(t) dt \approx \sum_{j=0}^{N} w \boldsymbol{f}(t_j) \,, \quad w = \frac{t_f}{N+1} \,, \tag{4}$$

with $t_j = j\frac{t_f}{N}$, for all $j = 0, \dots, N$. ■

*Lemma 2 (Quadrature Approximation):* Assume $\boldsymbol{f}(t) \in \mathcal{C}^2_n$. Then, the following inequality holds:

$$\left\| \int_0^{t_f} \boldsymbol{f}(t) dt - \sum_{j=0}^{N} w \boldsymbol{f}(t_j) \right\| \leq \frac{C}{N} \,, \tag{5}$$

where $C$ is independent of $N$.

*Proof:* The right-hand side of Equation (4) is equal to the integral $\int_0^{t_f} \boldsymbol{f}_N(t) dt$, with $\boldsymbol{f}_N(t)$ computed as in Equation (3) (see [24, Ch. 5]). Then, Lemma 2 follows easily by applying Lemma 1 to the left-hand side of Equation (5). ■

Finally, the minimum distance between two Bézier curves can be efficiently computed by exploiting the *convex hull* property of Bézier curves and the *de Casteljau* algorithm [24], in combination with the Gilbert-Johnson-Keerthi (GJK) distance algorithm [25]. The latter is widely used in computer graphics and video games to compute the minimum distance between convex shapes. The same properties and algorithms can also be employed to compute the extrema (maximum and minimum) of a Bézier curve. We refer the reader to [15], where an efficient implementation of the minimum distance algorithm is presented.

## III. Problem Formulation

The problem of optimal control for differentially flat systems can be formulated as follows.

*Problem 1 (Problem $P^{OC}$):* Determine $\boldsymbol{x}(t) : [0, t_f] \to \mathbb{R}^{n_x}$ and $\boldsymbol{u}(t) : [0, t_f] \to \mathbb{R}^{n_u}$ (and possibly $t_f$) that minimizes

$$\tilde{I}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = \tilde{E}\big(\boldsymbol{x}(0), \boldsymbol{x}(t_f)\big) + \int_0^{t_f} \tilde{F}(\boldsymbol{x}(t), \boldsymbol{u}(t))dt\,, \quad (6)$$

subject to

$$\boldsymbol{x}'(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))\,, \quad \forall t \in [0, t_f] \quad (7)$$
$$\tilde{\boldsymbol{e}}\big(\boldsymbol{x}(0), \boldsymbol{x}(t_f)\big) = \boldsymbol{0}\,, \quad (8)$$
$$\tilde{\boldsymbol{h}}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \le \boldsymbol{0}\,, \quad \forall t \in [0, t_f]\,, \quad (9)$$

where $\tilde{E} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}$, $\tilde{F} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$, $\boldsymbol{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$, $\tilde{\boldsymbol{e}} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_e}$, and $\tilde{\boldsymbol{h}} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_h}$.

The system given by Equation (7) is differentially flat by assumption. Thus, there exists a flat output $\boldsymbol{y} : [0, t_f] \to \mathbb{R}^{n_y}$,

$$\boldsymbol{y}(t) = \boldsymbol{\varphi}\Big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{u}'(t), \ldots, \boldsymbol{u}^{(s)}(t)\Big)\,,$$

such that

$$\boldsymbol{x}(t) = \boldsymbol{\varphi}_1\Big(\boldsymbol{y}(t), \boldsymbol{y}'(t), \ldots, \boldsymbol{y}^{(r-1)}(t)\Big),$$
$$\boldsymbol{u}(t) = \boldsymbol{\varphi}_2\Big(\boldsymbol{y}(t), \boldsymbol{y}'(t), \ldots, \boldsymbol{y}^{(r)}(t)\Big), \quad (10)$$

see [21]. It follows that the optimal control problem, Problem $P^{\mathrm{OC}}$, can be transcribed as a calculus of variations problem, here referred to as Problem $P^{\mathrm{CV}}$. Letting

$$z(t) = \Big[\boldsymbol{y}(t)^\top, \boldsymbol{y}'(t)^\top, \ldots, \boldsymbol{y}^{(r)}(t)^\top\Big]^\top \in \mathbb{R}^{(r+1)n_y}\,, \quad (11)$$

Problem $P^{\mathrm{CV}}$ can be stated as follows:

*Problem 2 (Problem $P^{CV}$):* Determine $\boldsymbol{y}(t)$ (and possibly $t_f$) that minimizes

$$I(\boldsymbol{y}(t)) = E\big(z(0), z(t_f)\big) + \int_0^{t_f} F(z(t))dt\,, \quad (12)$$

subject to

$$\boldsymbol{e}\big(z(0), z(t_f)\big) = \boldsymbol{0}\,, \quad (13)$$
$$\boldsymbol{h}(z(t)) \le \boldsymbol{0}\,, \quad \forall t \in [0, t_f]\,, \quad (14)$$

where $E(z(0), z(t_f))$, $F(z(t))$, $\boldsymbol{e}(z(0), z(t_f))$, and $\boldsymbol{h}(z(t))$ are obtained by expressing the functions $\tilde{E}(\boldsymbol{x}(0), \boldsymbol{x}(t_f))$, $\tilde{F}(\boldsymbol{x}(t), \boldsymbol{u}(t))$, $\tilde{\boldsymbol{e}}(\boldsymbol{x}(0), \boldsymbol{x}(t_f))$, and $\tilde{\boldsymbol{h}}(\boldsymbol{x}(t), \boldsymbol{u}(t))$ in terms of the flat output using the maps $\boldsymbol{\varphi}_1(\cdot)$ and $\boldsymbol{\varphi}_2(\cdot)$ introduced in Equation (10).

Imposed onto the above problem are the following set of assumptions.

*Assumption 1:* $E$, $F$, $\boldsymbol{e}$, and $\boldsymbol{h}$ are Lipschitz continuous with respect to their arguments; $F \in \mathcal{C}^2$.

*Assumption 2:* An optimal solution $\boldsymbol{y}^*(t)$ to Problem $P^{\mathrm{CV}}$ exists and satisfies $\boldsymbol{y}^*(t) \in \mathcal{C}^{r+2}_{n_y}$.

## IV. Bernstein Approximation

Let $0 = t_0 < t_1 < \cdots < t_N = t_f$ be a set of equidistant *time nodes*, i.e., $t_j = j\frac{t_f}{N}$. Consider the following $N$th order Bézier curve:

$$\boldsymbol{y}_N(t) = \sum_{j=0}^{N} \boldsymbol{c}_j b_{j,N}(t)\,, \quad (15)$$

with derivatives $\boldsymbol{y}'_N(t), \ldots, \boldsymbol{y}^{(r)}_N(t)$ computed as in Equation (2). Define $z_N(t) = [\boldsymbol{y}_N(t)^\top, \ldots, \boldsymbol{y}^{(r)}_N(t)^\top]^\top$, and $\boldsymbol{c} = [\boldsymbol{c}_0, \ldots, \boldsymbol{c}_N]$. Then, Problem $P^{\mathrm{CV}}$ can be approximated as follows.

*Problem 3 (Problem $P^{CV}_N$):* Let $0 < \delta_P < 1$. Determine $\boldsymbol{c}$ (and possibly $t_f$) that minimizes

$$I_N(\boldsymbol{c}) = E(z_N(0), z_N(t_N)) + w \sum_{j=0}^{N} F\big(z_N(t_j)\big)\,, \quad (16)$$

subject to

$$\|\boldsymbol{e}(z_N(0), z_N(t_N))\| \le N^{-\delta_P}\,, \quad (17)$$
$$\boldsymbol{h}\big(z_N(t_j)\big) \le N^{-\delta_P}\boldsymbol{1}\,, \quad \forall j = 0, \ldots, N\,, \quad (18)$$

where $w$ is defined in Equation (4).

The outcome of Problem $P^{\mathrm{CV}}_N$ is a set of optimal control points $\boldsymbol{c}^* = [\boldsymbol{c}^*_0, \ldots, \boldsymbol{c}^*_N]$, which determine the trajectories (Bézier curves)

$$\boldsymbol{y}^*_N(t) = \sum_{j=0}^{N} \boldsymbol{c}^*_j b_{j,N}(t)\,. \quad (19)$$

In the remainder of this section we address the following theoretical concerns:

1) the existence of a feasible solution to Problem $P^{\mathrm{CV}}_N$,
2) the convergence of $\boldsymbol{y}^*_N(t)$ to the optimal solution of Problem $P^{\mathrm{CV}}$, $\boldsymbol{y}^*(t)$.

The following analysis assumes that the final time in the original optimal control problem is fixed; however, the results can be easily extended to the case where $t_f$ is a decision variable. The main results of this letter are summarized in Theorems 1 and 2 below.

*Theorem 1:* There exists $N_1$ such that for any order of approximation $N \ge N_1$ Problem $P^{\mathrm{CV}}_N$ is feasible.

*Proof:* To prove Theorem 1 it suffices to show that there exists $\boldsymbol{c} = [\boldsymbol{c}_0, \ldots, \boldsymbol{c}_N]$ that satisfies the constraints of Problem $P^{\mathrm{CV}}_N$, namely Equations (17) and (18). Let $\boldsymbol{y}(t) \in \mathcal{C}^{r+2}_{n_y}$ be a feasible solution to Problem $P^{\mathrm{CV}}$, which exists by assumption, and define $\boldsymbol{c}_j = \boldsymbol{y}(t_j)$, $t_j = j\frac{t_f}{N}$, $j = 0, \ldots, N$. Then, let

$$\boldsymbol{y}_N(t) = \sum_{j=0}^{N} \boldsymbol{c}_j b_{j,N}(t)\,.$$

From Lemma 1 and Assumption 2 it follows that

$$\|z_N(t) - z(t)\| \leq \frac{C}{N},$$

where $z(t) = [\boldsymbol{y}(t)^\top, \ldots, \boldsymbol{y}^{(r)}(t)^\top]^\top$, and $z_N(t) = [\boldsymbol{y}_N(t)^\top, \ldots, \boldsymbol{y}_N^{(r)}(t)^\top]^\top$, for some $C$ independent of $N$. Now consider the inequality constraint given by (18). We have

$$\boldsymbol{h}\big(z_N(t_j)\big) \leq \boldsymbol{h}\big(z(t_j)\big) + \big\|\boldsymbol{h}\big(z_N(t_j)\big) - \boldsymbol{h}\big(z(t_j)\big)\big\| \leq L_h \frac{C}{N},$$

where $L_h$ is the Lipschitz constant of $\boldsymbol{h}(\cdot)$ (see Assumption 1). Thus, using the properties of exponential growth, there exists $N_1$ such that for all $N \geq N_1$ the inequality in (18) holds. Following a similar argument it can be shown that the equality constraint given by Equation (17) is also satisfied, thus proving Theorem 1. ■

*Theorem 2:* Assume that $\boldsymbol{y}_N^*(t)$ has a uniform accumulation point, i.e., there exists an infinite subset of indices $V \subset \mathbb{N}$ such that

$$\lim_{N\in V} \boldsymbol{y}_N^*(t) = \boldsymbol{y}^\infty(t),$$

and assume $\mathbf{y}^\infty(t) \in \mathcal{C}_{n_y}^{r+2}$. Then, $\boldsymbol{y}^\infty(t)$ is an optimal solution to Problem $P^{\mathrm{CV}}$.

*Proof:* This proof is divided into three steps: (1) we show that $\boldsymbol{y}^\infty(t)$ is a feasible solution to Problem $P^{CV}$; (2) we prove that

$$\lim_{N\in V} I_N(\boldsymbol{c}^*) = I(\boldsymbol{y}^\infty(t)); \tag{20}$$

(3) finally, we show that $I(\boldsymbol{y}^\infty(t)) = I(\boldsymbol{y}^*(t))$.

*Step (1):* We need to show that $\boldsymbol{y}^\infty(t)$ satisfies the constraints of Problem $P^{CV}$, namely Equations (13) and (14). We start by demonstrating that Equation (14) holds, and we do so in a proof by contradiction. Assume that $\boldsymbol{y}^\infty(t)$ does not satisfy (14). Then, there exists $t' \in [0, t_f]$ such that

$$\boldsymbol{h}\big(z^\infty(t')\big) > 0. \tag{21}$$

Since the nodes $\{t_k\}_{k=0}^N$ are dense in $[0, t_f]$, for any infinite set $V$ there exists a sequence of indices $\{k_N\}_{N\in V}$ such that

$$\lim_{N\in V} \big\|z_N^*(t') - z_N^*\big(t_{k_N}\big)\big\| = 0.$$

Then, we have

$$\begin{aligned}\boldsymbol{h}\big(z^\infty(t')\big) &\leq \lim_{N\in V} \big\|\boldsymbol{h}\big(z_N^*(t')\big) - \boldsymbol{h}\big(z_N^*\big(t_{k_N}\big)\big)\big\| \\ &\quad + \lim_{N\in V} \boldsymbol{h}\big(z_N^*\big(t_{k_N}\big)\big) \\ &\leq \lim_{N\in V} L_h \big\|z_N^*(t') - z_N^*\big(t_{k_N}\big)\big\| + \lim_{N\in V} N^{-\delta_P} = 0,\end{aligned}$$

where we used the fact that $z_N^*(t_{k_N})$ satisfies the constraints in (18), and $\boldsymbol{h}(\cdot)$ is Lipschitz. This contradicts (21), and in doing so proves that $\boldsymbol{y}^\infty(t)$ satisfies the inequality constraint in (14). By using an identical argument it can be shown that $\boldsymbol{y}^\infty(t)$ satisfies also the equality constraint in (13).

*Step (2):* We need to show that the following equalities hold

$$E\big(z^\infty(0), z^\infty(t_f)\big) = \lim_{N\in V} E\big(z_N^*(0), z_N^*(t_N)\big),$$

$$\int_0^{t_f} F\big(z^\infty(t)\big)dt = \lim_{N\in V} w \sum_{j=0}^N F\big(z_N^*(t_j)\big).$$

The first relationship above follows easily from $z^\infty(0) = \lim_{N\in V} z_N^*(0)$ and $z^\infty(t_f) = \lim_{N\in V} z_N^*(t_N)$. To prove the second equality, we notice that from Lemma 2 we have

$$\int_0^{t_f} F\big(z^\infty(t)\big)dt = \lim_{N\in V} w \sum_{j=0}^N F\big(z^\infty(t_j)\big),$$

which combined with the following result

$$\lim_{N\in V} w \sum_{j=0}^N F\big(z^\infty(t_j)\big) = \lim_{N\in V} w \sum_{j=0}^N F\big(z_N^*(t_j)\big),$$

proves Equation (20).

*Step (3):* Finally, we need to demonstrate that $I(\boldsymbol{y}^\infty(t)) = I(\boldsymbol{y}^*(t))$. First, define

$$\tilde{\boldsymbol{y}}_N(t) = \sum_{j=0}^N \tilde{\boldsymbol{c}}_j b_{j,N}(t),$$

with $\tilde{\boldsymbol{c}}_j = \boldsymbol{y}^*(t_j)$, $j = 0, \ldots, N$, $t_j = j\frac{t_f}{N}$. Similarly to the proof of Theorem 1, one can show that $\tilde{\boldsymbol{c}}$ is a feasible solution of Problem $P_N^{\mathrm{CV}}$. Furthermore, Lemma 2 and an argument similar to the one presented in *Step* (2) of this proof yield

$$I\big(\boldsymbol{y}^*(t)\big) = \lim_{N\in V} I_N(\tilde{\boldsymbol{c}}). \tag{22}$$

Recall that $\boldsymbol{c}^*$ is an optimal solution of Problem $P_N^{\mathrm{CV}}$. Then, we can write

$$I\big(\boldsymbol{y}^*(t)\big) \leq I\big(\boldsymbol{y}^\infty(t)\big) = \lim_{N\in V} I_N(\boldsymbol{c}^*) \leq \lim_{N\in V} I_N(\tilde{\boldsymbol{c}}).$$

The combination of the above expression with Equation (22) completes the proof of Theorem 2. ■

*Remark 1:* By virtue of the differential flatness property of the systems under consideration, Problem $P^{\mathrm{CV}}$ is equivalent to Problem $P^{\mathrm{OC}}$. Therefore, Theorem 2 proves the convergence of the approximate solutions to optimal solutions of the original control problem, Problem $P^{\mathrm{OC}}$.

*Remark 2:* This letter focuses on Bernstein approximation of the trajectories. However, the results reported in Theorems 1 and 2 and their proofs apply to any approximation or interpolation method that satisfies Lemmas 1 and 2.

## V. Implementation and Numerical Results

This section describes the benefits of the proposed approach through a simulation example. The results are obtained using MATLAB's built in *fmincon* function. The motion of the vehicle is governed by the following differential equations

$$\begin{cases} \dot{x}_1(t) = V(t)\cos(x_3(t)) \\ \dot{x}_2(t) = V(t)\sin(x_3(t)) \\ \dot{x}_3(t) = \omega(t), \end{cases} \tag{23}$$

with input $\boldsymbol{u}(t) = [V(t), \omega(t)]^\top$, and flat output $\boldsymbol{y}(t) = [x_1(t), x_2(t)]^\top$. The vehicles is subject to input constraints $V_{\min}^2 \leq V^2(t) \leq V_{\max}^2$ and $-\omega_{\max} \leq \omega(t) \leq \omega_{\max}$. Additional constraints must be imposed to avoid collisions with two static obstacles positioned at $\boldsymbol{p}_{o_i}$, $i = 1, 2$. The objective at hand is to generate a trajectory that, starting from a given initial position $\boldsymbol{y}_0$, arrives at the desired final destination $\boldsymbol{y}_f$, satisfies the

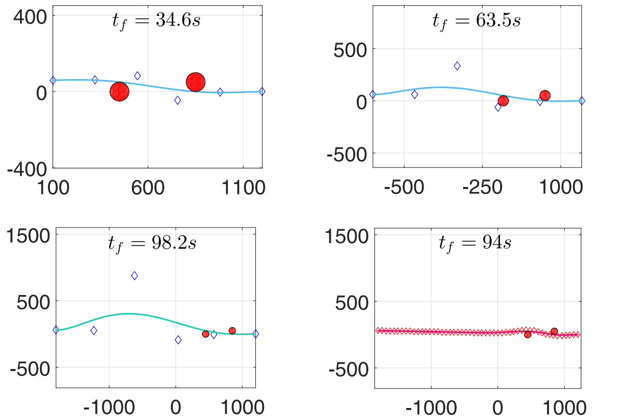



Fig. 1. Motion planning for 1 vehicle: trajectories (solid lines), control points (diamonds), and obstacles (red circles). Each plot depicts a mission with different initial positions and number of nodes: $\boldsymbol{y}(0) = [100, 60]^\top$, $N = 5$ (top-left); $\boldsymbol{y}(0) = [-800, 60]^\top$, $N = 5$ (top-right); $\boldsymbol{y}(0) = [-1500, 60]^\top$, $N = 5$ (bottom-left); $\boldsymbol{y}(0) = [-1500, 60]^\top$, $N = 50$ (bottom-right).

above constraints, while minimizing the time of arrival. The Bernstein approximation of the flat output $\boldsymbol{y}(t)$ is defined as

$$\boldsymbol{y}_N(t) = \sum_{j=0}^{N} \boldsymbol{c}_j b_{j,N}(t) = \left[x_{1_N}(t)\,,\; x_{2_N}(t)\right]^\top . \tag{24}$$

The above problem is transcribed as follows: find $\boldsymbol{c} = [\boldsymbol{c}_0, \ldots, \boldsymbol{c}_N]$ and $t_f$ that minimize $J = \int_0^{t_f} dt$ subject to

$$V_{\min}^2 \le \dot{x}_{1_N}^2(t) + \dot{x}_{2_N}^2(t) \le V_{\max}^2\,, \tag{25}$$

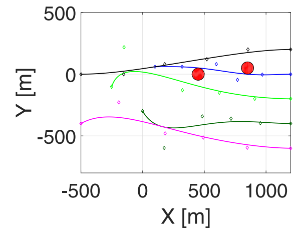
$$-\omega_{\max} \le \frac{\dot{x}_{1_N}(t)\ddot{x}_{2_N}(t) - \ddot{x}_{1_N}(t)\dot{x}_{2_N}(t)}{\dot{x}_{1_N}^2(t) + \dot{x}_{2_N}^2(t)} \le \omega_{\max}\,, \tag{26}$$

$$\left\|\boldsymbol{y}_N(t) - \boldsymbol{p}_{o_i}\right\| \ge E\,, \quad \forall t \in [0, t_f]\,, \tag{27}$$

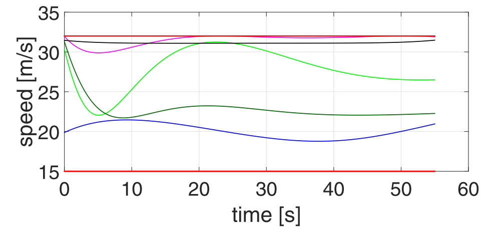
$$\boldsymbol{y}_N(0) = \boldsymbol{y}_0, \qquad \boldsymbol{y}_N(t_f) = \boldsymbol{y}_f\,. \tag{28}$$

It can be verified that the expression for the square of the speed in Equation (25) is a Bézier curve, and the angular rate in Equation (26) is a *rational* Bézier curve [24]. The properties of Bézier curves given in Section II carry over *rational* Bézier curves [26]. Thus, the continuous-time expressions in Equations (25), (26), and (27) can be computed by means of the minimum distance algorithm, and the above problem can be solved as a finite dimensional problem. Finally, the constraints in Equation (28) can be enforced directly on the first and last control points, since $\boldsymbol{c}_0 = \boldsymbol{y}_N(0)$ and $\boldsymbol{c}_N = \boldsymbol{y}_N(t_f)$ (end-point values property of Bézier curves [24]).

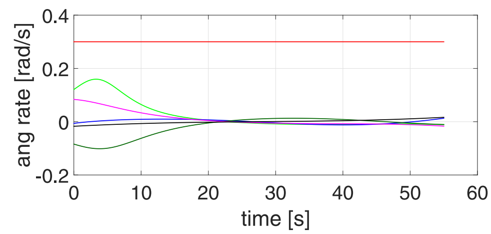
Figure 1 illustrates the results of the proposed approach with $V_{\min} = 15\text{m/s}$, $V_{\max} = 32\text{m/s}$, $\omega_{\max} = 0.3\text{rad/s}$, $E = 50\text{m}$, and $\boldsymbol{y}_f = [1200\,,\,0]^\top$ and for three different initial positions. The objective is to demonstrate that the collision avoidance constraints, for example, are satisfied independently of the length of the trajectory, for fixed (small) order of approximation, $N = 5$. If a more accurate approximation to the optimal solution is required, the number of nodes can be increased (see bottom-left and right plots in Figure 1). This is one aspect of the present approach that differs from other discretization methods, such as pseudospectral methods. When using




(a) 2D Plot. $N = 5$, $t_f \approx 55.01$s.




(b) Vehicles' speed.




(c) Vehicles' angular rate.

Fig. 2. Motion planning for 5 vehicles. Temporal Separation.

pseudospectral methods, collision avoidance can be guaranteed only at the collocation points. As the length of the path increases, for example, the order of approximation must grow to guarantee separation with the obstacles.

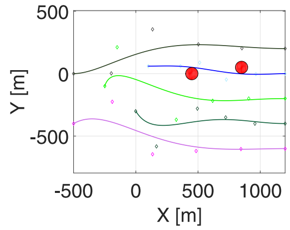
In the next scenario, the same time-optimal motion planning problem for one vehicle depicted in the top-left plot of Figure 1 is augmented with four additional vehicles. Each vehicle is required to satisfy the constraints given by Equations (25), (26), and (27), plus *temporal separation* between each pair of trajectories for inter-vehicle safety, i.e.,

$$\left\|\boldsymbol{y}_{i_N}(t) - \boldsymbol{y}_{j_N}(t)\right\| \ge E\,, \tag{29}$$

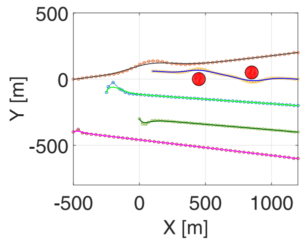
$\forall i, j = 1, \ldots, 5,\ i \ne j\ \forall t \in [0, t_f]$, where $\boldsymbol{y}_{i_N}(t)$ is the Bernstein approximant of the flat output of vehicle $i$ computed as in Equation (24). Similarly to the previous example, the above constraints can be efficiently computed using the minimum distance algorithm, and inter-vehicle safety can be guaranteed for the entire trajectories for any order of approximation. Figure 2 depicts the results of the proposed method with $N = 5$. Temporal separation between each trajectory can be inferred in Figure 2a. Figure 2b and 2c depict the speed and angular rates, respectively, which remain within the lower and upper limits. The optimal time of arrival is $t_f \approx 55.01$s.

The advantages of the proposed method become even more evident when *spatial separation* constraints, i.e.,

$$\left\|\boldsymbol{y}_{i_N}(t_i) - \boldsymbol{y}_{j_N}(t_j)\right\| \ge E\,, \tag{30}$$




(a) 2D Plot. $N = 5$, $t_f \approx 55.62$s.




(b) 2D Plot. $N = 40$, $t_f \approx 53.73$s.

Fig. 3. Motion planning for 5 vehicles. Spatial separation.

$\forall i, j = 1, \dots, 5,\ i \neq j\ \forall t_i, t_j \in [0, t_f]$, must be enforced instead of temporal separation. Notice that with the above constraints, the trajectories are required to be separated by $E$ for all times and cannot intersect. If one had to enforce (30) using pseudospectral methods, separation should be enforced between each node of each trajectory, resulting in $10N^2$ spatial separation constraints. It is clear that an increase of the number of nodes for safety could jeopardize the computational appeal of the method. On the other hand, when using the method proposed in this letter, not only spatial separation can be guaranteed with low orders of approximation, but also, to obtain more accurate approximations, the order of approximation can be scaled up without drastically increasing the complexity of the NLP, i.e., the number of constraints is independent of $N$. Figure 3 depicts the results obtained by enforcing the spatial separation constraints instead of the less conservative temporal separation constraints. In Figure 3a the order of approximation is set to $N = 5$, while $N = 40$ in Figure 3b. The optimal time of arrivals are $t_f \approx 55.62$s and $t_f \approx 53.73$s, respectively.

## VI. Conclusion

This letter proposed a numerical method to generate feasible and safe trajectories for differentially flat autonomous vehicle systems. The method is based on direct approximation of a continuous-time optimal control problem by a discrete-time formulation using Bernstein polynomials. These polynomials have favorable geometric properties which allow to efficiently compute the minimum distance between curves along the entire trajectory. Thus, the proposed approach is particularly convenient for generating trajectories for safe operation of multiple autonomous vehicles in complex environments. A rigorous analysis is provided that shows convergence of the discrete solution to the solution of the continuous-time problem. The benefits of the proposed method are discussed through numerical examples.

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