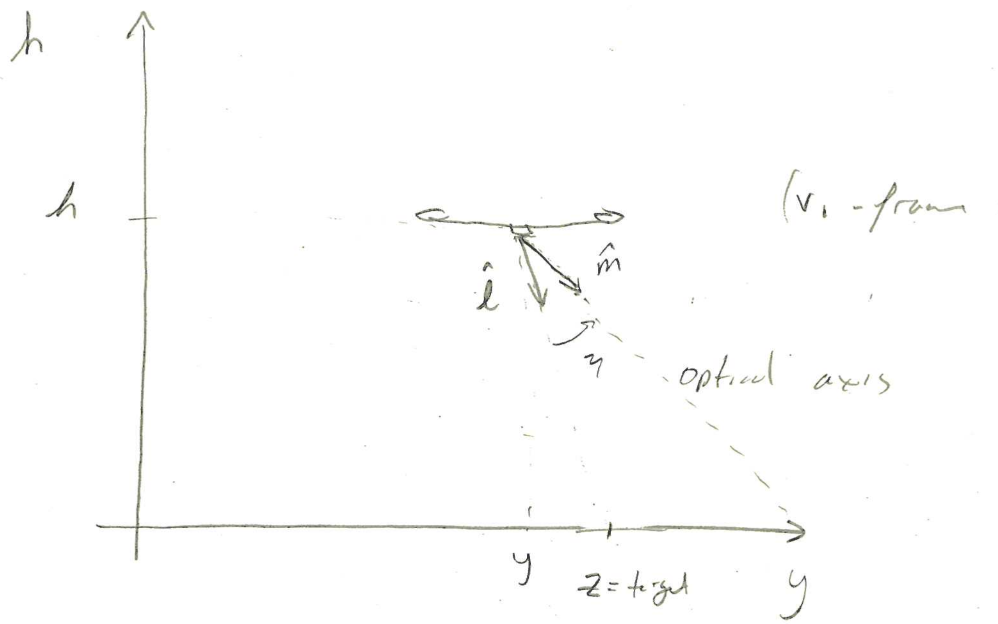


Assume dynamics

$$\ddot{y} = u_1$$
$$\ddot{h} = u_2$$

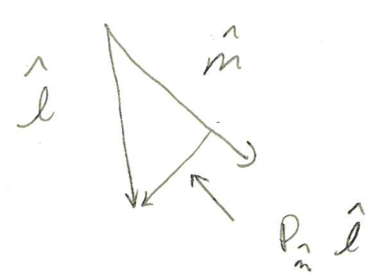
Assume 1. - h is unknown

2. - controller measures $\hat{\ell}$ (unit vector) and $\dot{\hat{\ell}}$ and $\ddot{\hat{\ell}}$ (by numerically differentiating $\hat{\ell}$)

3. $\hat{m}$ is fixed in v_1 - frame

Define the projection matrix (onto the null space of $\hat{m}$)

$$P_{\hat{m}} = (I - \hat{m}\hat{m}^T)$$



$$P_{\hat{m}} \hat{\ell} = (I - \hat{m}\hat{m}^T)\hat{\ell}$$
$$= \hat{\ell} - (\hat{\ell}^T \hat{m})\hat{m}$$

The idea is to drive $P_{\hat{m}}\hat{\ell}$ to zero

- we really only need to drive the 1st component to zero (horizontal direction)

Define $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then the horizontal error is given by

$$e_x = \hat{e}_1^T P_{\hat{m}} \hat{\ell} \qquad (2.1)$$

Note that since $\hat{m}$, $\hat{e}_1$ are fixed and known, and since $\hat{\ell}$ is measured, e_x is a measurable quantity.

Also since $\dot{\hat{\ell}}$ can be approximated by numerical differentiation $\dot{e}_x = \hat{e}_1^T P_{\hat{m}} \dot{\hat{\ell}}$ is also measurable.

The line of sight vector is

$$\ell = \underbrace{\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}}_{\text{target position}} - \underbrace{\begin{pmatrix} y \\ h \\ 0 \end{pmatrix}}_{\text{UAV position}}$$

and note that

$$\dot{\ell} = \begin{pmatrix} \dot{z} - \dot{y} \\ \dot{h} \\ 0 \end{pmatrix} \quad \text{and}$$

$$\ddot{\ell} = \begin{pmatrix} \ddot{z} - \ddot{y} \\ \ddot{h} \\ 0 \end{pmatrix}$$

Assuming that the acceleration of the target is

$$\ddot{z} = a_t, \quad \text{then}$$

$$\ddot{\ell} = \begin{pmatrix} a_t - u_1 \\ u_2 \\ 0 \end{pmatrix}$$

let $L = \|\ell\|$

Then we measure

$$\hat{\ell} = \frac{\ell}{L}$$

Differentiating gives

$$\dot{\hat{\ell}} = \frac{L\dot{\ell} - \ell\dot{L}}{L^2} = \frac{\dot{\ell}}{L} - \left(\frac{\ell}{L}\right)\left(\frac{\dot{L}}{L}\right) = \frac{\dot{\ell}}{L} - \hat{\ell}\left(\frac{\dot{L}}{L}\right) \qquad 3.1$$

Differentiating again gives

$$\ddot{\hat{\ell}} = \frac{L\ddot{\ell} - \dot{\ell}\dot{L}}{L^2} - \dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{L\ddot{L} - (\dot{L})^2}{L^2}\right)$$

$$= \frac{\ddot{\ell}}{L} - \left(\frac{\dot{\ell}}{L}\right)\left(\frac{\dot{L}}{L}\right) - \dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{\ddot{L}}{L}\right) + \hat{\ell}\left(\frac{\dot{L}}{L}\right)^2$$

plugging in for $\frac{\dot{\ell}}{L} = \dot{\hat{\ell}} + \hat{\ell}\left(\frac{\dot{L}}{L}\right)$ from (3.1) gives

$$\ddot{\hat{\ell}} = \frac{\ddot{\ell}}{L} - \left(\dot{\hat{\ell}} + \hat{\ell}\left(\frac{\dot{L}}{L}\right)\right)\left(\frac{\dot{L}}{L}\right) - \dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{\ddot{L}}{L}\right) + \hat{\ell}\left(\frac{\dot{L}}{L}\right)^2$$

$$= \frac{\ddot{\ell}}{L} - 2\dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{\ddot{L}}{L}\right)$$

Differentiating (2.1) twice gives

$$\ddot{e}_x = \hat{e}_1^T P_{\hat{m}} \ddot{\hat{\ell}}$$

$$= \hat{e}_1^T P_{\hat{m}} \left[\frac{\ddot{\ell}}{L} - 2\dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{\ddot{L}}{L}\right)\right]$$

$$= \frac{1}{L}\left(\hat{e}_1^T P_{\hat{m}} \ddot{\ell}\right) + \left(\frac{\dot{L}}{L}\right)\left(-2\hat{e}_1^T P_{\hat{m}} \dot{\hat{\ell}}\right) + \left(\frac{\ddot{L}}{L}\right)\left(-\hat{e}_1^T P_{\hat{m}} \hat{\ell}\right)$$

Note that

$$\hat{e}_1^T P_{\hat{m}} \ddot{\ell} = (1\ 0\ 0)\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}m_1\\m_2\\0\end{pmatrix}(m_1\ m_2\ 0)\right]\begin{pmatrix}a_t - u_1\\u_2\\0\end{pmatrix}$$

$$= (1\ 0\ 0)\begin{pmatrix}1-m_1^2 & -m_1 m_2 & 0\\ -m_1 m_2 & 1-m_2^2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_t - u_1\\u_2\\0\end{pmatrix}$$

$$= (1-m_1^2 \quad -m_1 m_2 \quad 0)\begin{pmatrix}a_t - u_1\\u_2\\0\end{pmatrix}$$

$$= (1-m_1^2)(a_t - u_1) - m_1 m_2 u_2$$

Assuming constant altitude, ie $u_2 = 0$, then

$$\hat{e}_1^T P_{\hat{m}} \ddot{\ell} = (1-m_1^2)(a_t - u_1)$$

$\therefore$

$$\ddot{e}_x = \left(\frac{1}{L}\right)(1-m_1^2)(a_t - u_1) + \left(\frac{\dot{L}}{L}\right)(-2\hat{e}_1^T P_{\hat{m}} \dot{\hat{\ell}}) + \left(\frac{\ddot{L}}{L}\right)(-\hat{e}_1^T P_{\hat{m}} \hat{\ell})$$

unknown, unmeasurable.

Define

$$\theta_1 = \frac{1}{L} \qquad \phi_1 = (1-m_1^2)$$

$$\theta_2 = \frac{\dot{L}}{L} \qquad \phi_2 = -2\hat{e}_1^T P_{\hat{m}} \dot{\hat{\ell}}$$

$$\theta_3 = \frac{\ddot{L}}{L} \qquad \phi_3 = -\hat{e}_1^T P_{\hat{m}} \hat{\ell}$$

We will assume that $\theta_1, \theta_2\ \theta_3$ are roughly constant, or slowly varying

Then

$$\ddot{e}_x = \theta_1 (1-m_1^2)(a_t - u_1) + \theta_2 \phi_2 + \theta_3 \phi_3$$

We want $e_x \to 0$.

Define $s = \dot{e}_x + k\, e_x$ where $k > 0$

and note that we can measure s.

Then

$$\dot{s} = \ddot{e}_x + k\,\dot{e}_x$$

$$= \theta_1 (1-m_1^2)(a_t - u_1) + \theta_2 \phi_2 + \theta_3 \phi_3 + k\,\dot{e}_x$$

Assume a constant velocity target, i.e. $a_t = 0$

Then

$$\dot{s} = -\theta_1 (1-m_1^2) u_1 + \theta_2 \phi_2 + \theta_3 \phi_3 + k\,\dot{e}_x$$

Let

$$u_1 = \frac{1}{\hat{\theta}_1 (1-m_1^2)} \left[+ \hat{\theta}_2 \phi_2 + \hat{\theta}_3 \phi_3 + k\,\dot{e}_x - \xi \right]$$

Then

$$\dot{s} = -\theta_1 (1-m_1^2) u_1 - \hat{\theta}_1 (1-m_1^2) u_1 + \hat{\theta}_1 (1-m_1^2) u_1 + \theta_2 \phi_2 + \theta_3 \phi_3 + k\,\dot{e}_x$$

$$= -(\theta_1 - \hat{\theta}_1)(1-m_1^2) u_1 + (\theta_2 - \hat{\theta}_2)\phi_2 + (\theta_3 - \hat{\theta}_3)\phi_3 + \xi$$

Define

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \quad \hat{\Theta} = \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \end{pmatrix}, \quad \tilde{\Theta} = \Theta - \hat{\Theta}, \quad \Phi = \begin{pmatrix} -(1-m_1^2) u_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

Define the Lyapunov equation

$$V = \frac{1}{2} s^2 + \frac{1}{2} \tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta}$$

then

$$\dot{V} = s\dot{s} + \tilde{\Theta}^T \Gamma^{-1} \dot{\tilde{\Theta}}$$

$$= s\left(\tilde{\Theta}^T \Phi + \xi \right) + \tilde{\Theta}^T \Gamma^{-1} \dot{\tilde{\Theta}}$$

Selecting $\xi = -\alpha s = -\alpha(\dot{e}_x + k e_x)$ where $\alpha > 0$

then

$$\dot{V} = -\alpha s^2 + \tilde{\Theta}^T \left[s\Phi + \Gamma^{-1} \dot{\tilde{\Theta}} \right]$$

Now assuming that Θ is constant we get

$$\dot{\tilde{\Theta}} = \dot{\Theta} - \dot{\hat{\Theta}} = -\dot{\hat{\Theta}}$$

$$\Rightarrow \quad \dot{V} = -\alpha s^2 + \tilde{\Theta}^T \left[s\Phi - \Gamma^{-1} \dot{\hat{\Theta}} \right]$$

let $\dot{\hat{\Theta}} = s \Gamma \Phi$

$$\Rightarrow \quad \dot{V} = -\alpha s^2$$

Summarizing

$$u_2 = 0$$

$$u_1 = \frac{1}{\hat{\theta}_1 (1-m_1^2)} \left[\hat{\theta}_2 \phi_2 + \hat{\theta}_3 \phi_3 + k \dot{e}_x - \alpha s \right]$$

$$\dot{\hat{\theta}} = s \Gamma \Phi$$

$$s = (\dot{e}_x + k e_x)$$

$$\Phi = \begin{pmatrix} -(1-m_1^2) u_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

$$e_x = \hat{e}_1^T P_{\hat{m}} \hat{l} \quad , \quad \dot{e}_x = \hat{e}_1^T P_{\hat{m}} \dot{\hat{l}}$$

$$\phi_2 = -2 \hat{e}_1^T P_m \dot{\hat{l}}$$

$$\phi_3 = -e_x$$

$\hat{l}$, $\dot{\hat{l}}$ - measured

$\hat{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$ - known

Control gains: k, α, Γ

Assumptions

- constant velocity target
- unknown but constant altitude
- $\frac{1}{L}$, $\frac{\dot{L}}{L}$, $\frac{\ddot{L}}{L}$ - constant