Robust Adaptive Attitude Tracking on SO(3) With an Application to a Quadrotor UAV

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***Abstract*— This brief describes robust adaptive tracking control systems for the attitude dynamics of a rigid body. Both the attitude dynamics and the proposed control system are globally expressed on the special orthogonal group, to avoid complexities and ambiguities associated with other attitude representations, such as Euler angles or quaternions. By designing an adaptive law for the inertia matrix of a rigid body, the proposed control system can asymptotically follow an attitude command without the knowledge of the inertia matrix, and it is extended to guarantee boundedness of tracking errors in the presence of unstructured disturbances. These are illustrated by the experimental results of the attitude dynamics of a quadrotor unmanned aerial vehicle.**

***Index Terms*— Attitude tracking, robust adaptive control, special orthogonal group.**

I. Introduction

The ATTITUDE dynamics of a rigid body appear in various engineering applications, such as aerial and underwater vehicles, robotics, and spacecraft (see [1]–[3]). One distinct feature of attitude dynamics is that its configuration manifold is not linear; it evolves on a nonlinear manifold, referred to as the special orthogonal group, SO(3). This yields important and unique properties that cannot be observed from dynamic systems evolving on a linear space. For example, it has been shown that there exists no continuous feedback control that asymptotically stabilizes an attitude globally [4].

However, most of the prior work on the attitude control is based on minimal representations of an attitude, or quaternions [5]–[7]. It is well known that any minimal attitude representations are defined only locally, and exhibit kinematic singularities for large angle rotational maneuvers. Quaternions do not have singularities, but as the three-sphere double-covers the SO(3), the ambiguity in representing an attitude should be carefully resolved in any quaternion-based attitude control system. Otherwise, it may yield unwinding behaviors, where a rigid body rotates unnecessarily through large angles, or it may become sensitive to small measurement noise [8]–[10].

Geometric control is concerned with the development of control systems for dynamic systems evolving on nonlinear manifolds that cannot be globally identified with Euclidean spaces [11]–[13]. By characterizing the geometric properties of nonlinear manifolds intrinsically, geometric control techniques completely avoid singularities and ambiguities that are associated with local coordinates or improper characterizations of a configuration manifold. This approach has been applied to fully actuated rigid body dynamics on Lie groups to achieve almost global asymptotic stability [13]–[18].

In this brief, we develop a robust adaptive controller on SO(3) to track an attitude and angular velocity command without the knowledge of the inertia matrix of a rigid body. An estimate of the inertia matrix is updated online to provide an asymptotic tracking property when the inertia matrix is not available. The proposed adaptive control system is developed directly on SO(3) to avoid the complexity, singularity, ambiguity, and discontinuity associated with Euler angles or quaternions. It is also extended to a robust adaptive attitude tracking control system. Stable adaptive control schemes designed without consideration of uncertainties may become unstable in the presence of small disturbances [19], [20]. The presented robust adaptive scheme guarantees the boundedness of the attitude tracking error and the inertia matrix estimation error even if there exist unstructured modeling errors or disturbances. These properties are verified experimentally for the attitude dynamics of a quadrotor unmanned aerial vehicle (UAV).

Compared with [21], the proposed adaptive tracking control system has simpler controller structures, and the proposed robust adaptive tracking control system is applied to a general class of unstructured or nonharmonic uncertainties. Compared to the other robust adaptive attitude control systems studied in [22]–[25], the proposed robust adaptive scheme does not rely on sliding model controls. Instead, several properties of an attitude error function are found, and a strict Lyapunov analysis is presented to show the boundedness of errors. In short, the main contribution of this brief is the development of a new adaptive attitude tracking system on SO(3), which can eliminate the effects of both parametric and nonparametric uncertainties.

II. Attitude Dynamics of a Rigid Body

We consider the rotational attitude dynamics of a fully actuated rigid body. We define an inertial reference frame and a body-fixed frame whose origin is located at the center of mass of the rigid body. The configuration of the rigid body is the orientation of the body-fixed frame with respect to the inertial frame, and it is represented by a rotation matrix $R \in \mathrm{SO}(3)$, where SO(3) is the group of 3×3 orthogonal matrices with determinant 1, i.e., $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \,|\, R^T R = I, \det R = 1\}$. The equations of motion are given by

$$J\dot{\Omega} + \Omega \times J\Omega = u + \Delta \tag{1}$$

$$\dot{R} = R\hat{\Omega} \tag{2}$$

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where $J \in \mathbb{R}^{3\times 3}$ is the inertia matrix in the body-fixed frame, and $\Omega \in \mathbb{R}^3$ and $u \in \mathbb{R}^3$ are the angular velocity of the rigid body and the control moment, represented with respect to the body-fixed frame, respectively. The vector $\Delta \in \mathbb{R}^3$ represents uncertainties.

The hat map $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms a vector in $\mathbb{R}^3$ to a 3×3 skew-symmetric matrix such that $\hat{x}y = x \times y$ for any $x, y \in \mathbb{R}^3$. In case the argument of the hat map is too wide, the hat map is denoted by $(\ )^\wedge$. The inverse of the hat map is denoted by the vee map $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$. Several properties of the hat map are summarized as follows [26]:

$$\hat{x}y = x \times y = -y \times x = -\hat{y}x \tag{3}$$

$$\mathrm{tr}\Big[A\hat{x}\Big] = \frac{1}{2}\mathrm{tr}\Big[\hat{x}(A - A^T)\Big] = -x^T(A - A^T)^\vee \tag{4}$$

$$\hat{x}A + A^T\hat{x} = (\{\mathrm{tr}[A]\, I_{3\times 3} - A\}\, x)^\wedge \tag{5}$$

$$R\hat{x}R^T = (Rx)^\wedge \tag{6}$$

for any $x, y \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 3}$, and $R \in \mathrm{SO}(3)$. Throughout this brief, the 2-norm of a matrix A is denoted by $\|A\|$, and its Frobenius norm is denoted by $\|A\|_F = \sqrt{\mathrm{tr}[A^T A]}$. We have $\|A\| \leq \|A\|_F \leq \sqrt{r}\|A\|$, where r is the rank of A. The dot product of two vectors is denoted by $x \cdot y = x^T y$ for $x, y \in \mathbb{R}^3$.

III. Geometric Tracking Control on SO(3)

We develop adaptive control systems to follow a given smooth attitude command $R_d(t) \in \mathrm{SO}(3)$. The kinematic equation for the attitude command can be written as

$$\dot{R}_d = R_d\hat{\Omega}_d \tag{7}$$

where $\Omega_d \in \mathbb{R}^3$ is the desired angular velocity of the given attitude command.

A. Attitude Error Dynamics

One of the important procedures in constructing a control system on a nonlinear manifold Q is choosing a proper configuration error function, which is a smooth positive-definite function $\Psi : \mathrm{Q} \times \mathrm{Q} \to \mathbb{R}$ that measures the error between a current configuration and a desired configuration. Once a configuration error function is chosen, a configuration error vector and a velocity error vector can be defined in the tangent $\mathrm{T}_q\mathrm{Q}$ by using the derivatives of Ψ [13]. Then, the remaining procedure is similar to nonlinear control system design in Euclidean spaces; control inputs are carefully designed as a function of these error vectors through a Lyapunov analysis on Q.

The following form of a configuration error function has been used in [13] and [16]. Here, we summarize its properties developed in the above papers, and show several additional facts required for an adaptive control system.

Proposition 1: For a given tracking command (R_d, Ω_d), and the current attitude and angular velocity (R, Ω), we define an attitude error function $\Psi : \mathrm{SO}(3) \times \mathrm{SO}(3) \to \mathbb{R}$, an attitude error vector $e_R : \mathrm{SO}(3) \times \mathrm{SO}(3) \to \mathbb{R}^3$, and an angular velocity error vector $e_\Omega : \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

$$\Psi(R, R_d) = \frac{1}{2}\mathrm{tr}\Big[G(I - R_d^T R)\Big] \tag{8}$$

$$e_R(R, R_d) = \frac{1}{2}\Big(GR_d^T R - R^T R_d G\Big)^\vee \tag{9}$$

$$e_\Omega(R, \Omega, R_d, \Omega) = \Omega - R^T R_d \Omega_d \tag{10}$$

where the matrix $G \in \mathbb{R}^{3\times 3}$ is given by $G = \mathrm{diag}[g_1, g_2, g_3]$ for distinct positive constants $g_1, g_2, g_3 \in \mathbb{R}$. Then, the following statements hold.

1) Ψ is locally positive definite about $R = R_d$.
2) The left-trivialized derivative of Ψ is given by
$$\mathrm{T}_I^*\mathrm{L}_R\left(\mathbf{D}_R\Psi(R, R_d)\right) = e_R. \tag{11}$$
3) The critical points of Ψ, where $e_R = 0$, are $\{R_d\} \cup \{R_d \exp(\pi\hat{s}) \,|\, s \in \{e_1, e_2, e_3\}\}$.
4) Ψ is locally quadratic since
$$b_1\|e_R(R, R_d)\|^2 \leq \Psi(R, R_d) \tag{12}$$
where the constant b_1 is given by $b_1 = (h_1/h_2 + h_3)$ for
$$h_1 = \min\{g_1 + g_2, g_2 + g_3, g_3 + g_1\}$$
$$h_2 = \max\Big\{(g_1 - g_2)^2, (g_2 - g_3)^2, (g_3 - g_1)^2\Big\}$$
$$h_3 = \max\Big\{(g_1 + g_2)^2, (g_2 + g_3)^2, (g_3 + g_1)^2\Big\}.$$
5) Let ψ be a positive constant that is strictly less than h_1. If $\Psi(R, R_d) < \psi < h_1$, then
$$\Psi(R, R_d) \leq b_2\|e_R(R, R_d)\|^2 \tag{13}$$
where the constant b_2 is given by $b_2 = (h_1 h_4/h_5(h_1 - \psi))$ for
$$h_4 = \max\{g_1 + g_2, g_2 + g_3, g_3 + g_1\}$$
$$h_5 = \min\{(g_1 + g_2)^2, (g_2 + g_3)^2, (g_3 + g_1)^2\}.$$

Proof: The proofs of 1)–3) are available in [13, Ch. 11]. To show 4) and 5), let $Q = R_d^T R = \exp\hat{x} \in \mathrm{SO}(3)$ for $x \in \mathbb{R}^3$ from Rodrigues' formula. Using the MATLAB symbolic computation tool, we find

$$\Psi = \frac{1 - \cos\|x\|}{2\|x\|^2}\sum_{(i,j,k)\in\mathcal{C}}(g_i + g_j)x_k^2$$

$$\|e_R\|^2 = \frac{(1 - \cos\|x\|)^2}{4\|x\|^4}\sum_{(i,j,k)\in\mathcal{C}}(g_i - g_j)^2 x_i^2 x_j^2 + \frac{\sin^2\|x\|}{4\|x\|^2}\sum_{(i,j,k)\in\mathcal{C}}(g_i + g_j)^2 x_k^2$$

where $\mathcal{C} = \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. When $\Psi = 0$, (12) is trivial. Assuming that $\Psi \neq 0$, therefore $\|x\| \neq 0$, an upper bound of $(\|e_R\|^2/\Psi)$ is given by

$$\frac{\|e_R\|^2}{\Psi} \leq \frac{1}{2h_1}(1 - \cos\|x\|)h_2 + \frac{1}{2h_1}(1 + \cos\|x\|)h_3 \leq \frac{h_2 + h_3}{h_1}$$

which shows (12).

Next, we consider 5). When $\Psi = 0$, (13) is trivial. Hereafter, we assume $\Psi \neq 0$, therefore $R \neq R_d$. At the three remaining critical points of Ψ, the values of Ψ are given by $g_1 + g_2$,

$g_2 + g_3$, or $g_3 + g_1$. So, from the given bound $\Psi < \psi$, these three critical points are avoided, and we can guarantee that $e_R \neq 0$ and $\|x\| < \pi$. An upper bound of $(\Psi/\|e_R\|^2)$ is given by

$$\frac{\Psi}{\|e_R\|^2} \leq \frac{2(1-\cos\|x\|)}{\sin\|x\|^2}\frac{\sum_{\mathcal{C}}(g_i+g_j)x_k^2/\|x\|^2}{\sum_{\mathcal{C}}(g_i+g_j)^2x_k^2/\|x\|^2} \leq \frac{2}{1+\cos\|x\|}\frac{h_4}{h_5}. \tag{14}$$

Also, an upper bound of $h_1 - \psi$ is given by

$$h_1 - \psi < h_1 - \Psi \leq h_1 - \frac{1-\cos\|x\|}{2}h_1 = \frac{h_1}{2}(1+\cos\|x\|).$$

Substituting this into (14), we obtain (13). ■

The corresponding attitude error dynamics for the attitude error function Ψ, the attitude error vector e_R, and the angular velocity error e_Ω are summarized as follows.

Proposition 2: The error dynamics for Ψ, e_R, e_Ω satisfies

$$\frac{d}{dt}(R_d^T R) = R_d^T R\hat{e}_\Omega \tag{15}$$

$$\frac{d}{dt}(\Psi(R, R_d)) = e_R \cdot e_\Omega \tag{16}$$

$$\dot{e}_R = E(R, R_d)e_\Omega \tag{17}$$

$$\dot{e}_\Omega = J^{-1}(-\Omega \times J\Omega + u + \Delta) - \alpha_d \tag{18}$$

where the matrix $E(R, R_d) \in \mathbb{R}^{3\times3}$, and the angular acceleration $\alpha_d \in \mathbb{R}^3$, which is caused by the attitude command and measured in the body fixed frame, are given by

$$E(R, R_d) = \frac{1}{2}\left(\mathrm{tr}[R^T R_d G]I - R^T R_d G\right) \tag{19}$$

$$\alpha_d = -\hat{\Omega}R^T R_d\Omega_d + R^T R_d\dot{\Omega}_d. \tag{20}$$

Furthermore, the matrix $E(R, R_d)$ is bounded by

$$\|E(R, R_d)\| \leq \frac{1}{\sqrt{2}}\mathrm{tr}[G]\,. \tag{21}$$

Proof: From the kinematic equations (2) and (7), and the property, the time derivative of $R_d^T R$ can be written as

$$\frac{d}{dt}(R_d^T R) = R_d^T R\left(-(R^T R_d\Omega_d)^\wedge + \hat{\Omega}\right)$$

which shows (15). From this, the time derivative of the attitude error function is given by

$$\frac{d}{dt}\Psi(R, R_d) = -\frac{1}{2}\mathrm{tr}\Big[GR_d^T R\hat{e}_\Omega\Big].$$

Applying (4) and (9) into this, we obtain (16). Next, the time derivative of the attitude error vector is given by

$$\dot{e}_R = \frac{1}{2}\left(GR_d^T R\hat{e}_\Omega + \hat{e}_\Omega R^T R_d G\right)^\vee.$$

Using the properties of the hat map, given by (5), this can be further reduced to (17) and (19).

To show (21), we find the Frobenius norm $\|E\|_F$

$$\|E(R, R_d)\|_F = \sqrt{\mathrm{tr}\Big[E^T E\Big]} = \frac{1}{2}\sqrt{\mathrm{tr}[G^2] + \mathrm{tr}\Big[R^T R_d G\Big]^2} \tag{22}$$

where we use the facts that $\mathrm{tr}[A] = \mathrm{tr}[A^T]$ and $\mathrm{tr}[cA] = c\mathrm{tr}[A]$ for any matrix A and a constant c. Let $Q = R_d^T R = \exp\hat{x} \in \mathrm{SO}(3)$ for $x \in \mathbb{R}^3$ from Rodrigues' formula. Using the MATLAB symbolic computation tool, we find

$$\mathrm{tr}\Big[R^T R_d G\Big] = \cos\|x\|\sum_{i=1}^3 g_i(1 - \frac{x_i^2}{\|x\|^2}) + \sum_{i=1}^3 g_i\frac{x_i^2}{\|x\|^2}.$$

Since $0 \leq x_i^2/\|x\|^2 \leq 1$, we have $\mathrm{tr}[R^T R_d G] \leq \sum_{i=1}^3 g_i = \mathrm{tr}[G]$. Substituting this into (22), we get

$$\|E(R, R_d)\|_F^2 \leq \frac{1}{4}(\mathrm{tr}\Big[G^2\Big] + \mathrm{tr}[G]^2) \leq \frac{1}{2}\mathrm{tr}[G]^2$$

which shows (21), since $\|E\| \leq \|E\|_F$.

From (1), (2), and (7), and using the fact that $\hat{\Omega}_d\Omega_d = \Omega_d \times \Omega_d = 0$ for any $\Omega_d \in \mathbb{R}^3$, the time derivative of the angular velocity error e_Ω is given by

$$\dot{e}_\Omega = J^{-1}(-\Omega \times J\Omega + u + \Delta) - \alpha_d$$

where α_d is given by (20). ■

B. Adaptive Attitude Tracking

Attitude tracking control systems require the knowledge of an inertia matrix when the given attitude command is not fixed. But, it is difficult to measure the value of an inertia matrix exactly. In general, there is an estimation error $\tilde{J} \in \mathbb{R}^{3\times3}$, defined as

$$\tilde{J} = J - \bar{J} \tag{23}$$

where the exact inertia matrix and its estimate are denoted by the matrices J and $\bar{J} \in \mathbb{R}^{3\times3}$, respectively. All matrices, J, $\bar{J}$, and $\tilde{J}$ are symmetric.

Here, an adaptive tracking controller for the attitude dynamics of a rigid body is developed to follow a given attitude command without the knowledge of its inertia matrix assuming that there is no disturbance, and the bounds of the inertia matrix are given.

Assumption 1: The lower bound λ_m and the upper bound λ_M of the eigenvalue of the true inertia matrix are known, i.e., for given constants λ_m, λ_M, we have $\lambda_m \leq \lambda(J) \leq \lambda_M$.

Proposition 3: Assume that there is no disturbance in the attitude dynamics, i.e., $\Delta = 0$ at (1), and Assumption 1 is satisfied. For a given attitude command $R_d(t)$, and positive constants $k_R, k_\Omega, k_J \in \mathbb{R}$, we define a control input $u \in \mathbb{R}^3$, and an update law for $\bar{J}$ as follows:

$$u = -k_R e_R - k_\Omega e_\Omega + \Omega \times \bar{J}\Omega + \bar{J}\alpha_d \tag{24}$$

$$\dot{\bar{J}} = \frac{k_J}{2}\left(-\alpha_d e_A^T - e_A\alpha_d^T + \Omega\Omega^T\hat{e}_A - \hat{e}_A\Omega\Omega^T\right) \tag{25}$$

where $e_A \in \mathbb{R}^3$ is an augmented error vector given by

$$e_A = e_\Omega + ce_R \tag{26}$$

for a positive constant c satisfying

$$c < \min\left\{\sqrt{\frac{2b_1k_R\lambda_m}{\lambda_M^2}},\ \frac{\sqrt{2}k_\Omega}{\lambda_M\mathrm{tr}[G]},\ \frac{4k_Rk_\Omega}{k_\Omega^2 + \frac{1}{\sqrt{2}}k_R\lambda_M\mathrm{tr}[G]}\right\}. \tag{27}$$

Then, the following properties hold.

1) The zero equilibrium of the tracking errors (e_R, e_Ω) and the estimation error $\tilde{J}$ is stable, and those errors are uniformly bounded.

2) The tracking errors e_R and e_Ω asymptotically converge to zero, i.e., $R \to \{R_d\} \cup \{R_d \exp(\pi \hat{s}) \mid s \in \{e_1, e_2, e_3\}\}$, and $e_\Omega \to 0$ as $t \to \infty$.
3) The undesired equilibria where $R \in \{R_d \exp(\pi \hat{s}) \mid s \in \{e_1, e_2, e_3\}\}$ are unstable.
4) the inertia matrix estimation $\bar{J}$ satisfies that $\dot{\bar{J}} \to 0$ as $t \to \infty$, and $\|\bar{J}\|_F$ converges to a constant as $t \to \infty$.

Proof: Consider the following Lyapunov function:

$$\mathcal{V} = \frac{1}{2} e_\Omega \cdot J e_\Omega + k_R \Psi(R, R_d) + c J e_\Omega \cdot e_R + \frac{1}{2k_J} \|\tilde{J}\|_F^2. \tag{28}$$

From (12), we obtain

$$z^T W_{11} z \leq \mathcal{V} \tag{29}$$

where $z = [\|e_R\|;\ \|e_\Omega\|;\ \|\tilde{J}\|_F] \in \mathbb{R}^3$, and the matrix $W_{11} \in \mathbb{R}^{3\times 3}$ is given by

$$W_{11} = \begin{bmatrix} b_1 k_R & -\frac{1}{2} c \lambda_M & 0 \\ -\frac{1}{2} c \lambda_M & \frac{1}{2} \lambda_m & 0 \\ 0 & 0 & \frac{1}{2k_J} \end{bmatrix}. \tag{30}$$

Equation (27) for the constant c guarantees that the matrix W_{11} is positive definite. Substituting (24) into (18) with $\Delta = 0$, we obtain

$$J \dot{e}_\Omega = -k_R e_R - k_\Omega e_\Omega - \tilde{J} \alpha_d - \Omega \times \tilde{J} \Omega. \tag{31}$$

Using (16), (17), and (31), the time derivative of $\mathcal{V}$ is given by

$$\begin{aligned} \dot{\mathcal{V}} = &-k_\Omega \|e_\Omega\|^2 - c k_R \|e_R\|^2 + c J e_\Omega \cdot E e_\Omega - c k_\Omega e_\Omega \cdot e_R \\ &-(e_\Omega + c e_R) \cdot (\tilde{J} \alpha_d + \Omega \times \tilde{J} \Omega) + \frac{1}{k_J} \mathrm{tr}\Big[\tilde{J} \dot{\tilde{J}}\Big]. \end{aligned}$$

From (26), and using the fact that $x \cdot y = \mathrm{tr}[x y^T] = \mathrm{tr}[y x^T]$ and $x \cdot (y \times z) = y \cdot (z \times x) = z \cdot (x \times y)$ for any $x, y, z \in \mathbb{R}^3$, this can be written as

$$\begin{aligned} \dot{\mathcal{V}} = &-k_\Omega \|e_\Omega\|^2 - c k_R \|e_R\|^2 + c J e_\Omega \cdot E e_\Omega - c k_\Omega e_\Omega \cdot e_R \\ &+\mathrm{tr}\left[\tilde{J} \left\{ -\alpha_d e_A^T - \Omega (e_A \times \Omega)^T + \frac{1}{k_J} \dot{\tilde{J}} \right\}\right]. \end{aligned}$$

Since $\dot{\tilde{J}} = -\dot{\bar{J}}$, we can substitute (25) into this. Using the facts that $\mathrm{tr}[\tilde{J} A] = \mathrm{tr}[\tilde{J} A^T]$ for any $A \in \mathbb{R}^{3\times 3}$, and $(e_A \times \Omega)^T = (\hat{e}_A \Omega)^T = -\Omega^T \hat{e}_A$, it reduces to

$$\dot{\mathcal{V}} = -k_\Omega \|e_\Omega\|^2 - c k_R \|e_R\|^2 + c J e_\Omega \cdot E e_\Omega - c k_\Omega e_\Omega \cdot e_R. \tag{32}$$

From (21), it is bounded by

$$\begin{aligned} \dot{\mathcal{V}} \leq &-\left(k_\Omega - \frac{c}{\sqrt{2}} \lambda_M \mathrm{tr}[G]\right) \|e_\Omega\|^2 - c k_R \|e_R\|^2 \\ &+ c k_\Omega \|e_\Omega\| \|e_R\| = -\zeta^T W_2 \zeta \end{aligned} \tag{33}$$

where $\zeta = [\|e_R\|;\ \|e_\Omega\|] \in \mathbb{R}^2$, and the matrix $W_2 \in \mathbb{R}^{2\times 2}$ is given by

$$W_2 = \begin{bmatrix} c k_R & -\frac{c k_\Omega}{2} \\ -\frac{c k_\Omega}{2} & k_\Omega - \frac{c}{\sqrt{2}} \lambda_M \mathrm{tr}[G] \end{bmatrix}. \tag{34}$$

Equation (27) for the constant c guarantees that the matrices W_{11}, W_2 are positive definite.

This implies that the Lyapunov function $\mathcal{V}(t)$ is bounded from below and it is nonincreasing, which shows 1). Therefore, it has a limit, $\lim_{t\to\infty} \mathcal{V}(t) = \mathcal{V}_\infty$, and $e_R, e_\Omega, \bar{J} \in \mathcal{L}_\infty$.[1] From (17) and (31), we have $\dot{e}_R, \dot{e}_\Omega \in \mathcal{L}_\infty$. Furthermore, $e_R, e_\Omega \in \mathcal{L}_2$ since $\int_0^\infty \zeta(\tau)^T W_2 \zeta(\tau) d\tau \leq \mathcal{V}(0) - \mathcal{V}_\infty < \infty$. According to Barbalat's lemma [19, Lemma 3.2.5], we have $e_R, e_\Omega \to 0$ as $t \to \infty$, which yields 2). This also implies $\dot{\bar{J}}$ converges to zero from (25), and $\|\bar{J}\|_F$ asymptotically converges to a constant from (28), which implies 4).

Now we show 3). At the first undesired equilibrium $R = R_d \exp(\pi \hat{e}_1)$, we have $\Psi\,(R_d \exp(\pi \hat{e}_1), R_d) = g_2 + g_3$. Define $\mathcal{W} = k_R(g_2 + g_3) - \mathcal{V}$. Then, $\mathcal{W} = 0$ at the undesired equilibrium. Due to the continuity of Ψ, in an arbitrarily small neighborhood of $R_d \exp(\pi \hat{e}_1)$ in SO(3), there exists R such that $k_R(g_2 + g_3) - \Psi(R, R_d) > 0$. For such attitudes, we can guarantee that $\mathcal{W} > 0$ if $\|e_\Omega\|$ and $\|\tilde{J}\|_F$ are sufficiently small, since

$$\begin{aligned} \mathcal{W} \geq &\ k_R(g_2 + g_3 - \Psi) - \frac{\lambda_M}{2} \|e_\Omega\|^2 - c \lambda_M \|e_\Omega\| \|e_R\| \\ &-\frac{1}{2k_J} \|\tilde{J}\|_F^2. \end{aligned}$$

In other words, at any arbitrarily small neighborhood of the undesired equilibrium, there exists a domain, namely U, such that $\mathcal{W} > 0$ in U. And we have $\dot{\mathcal{W}} = -\dot{\mathcal{V}} > 0$ from (33) since U excludes the equilibrium where $e_R = e_\Omega = 0$. According to [27, Th. 3.3], the undesired equilibrium is unstable. The instability of the other two equilibrium configurations can be shown by the similar way. This shows 3). ■

Remark 1: This proposition guarantees that the attitude error vector e_R asymptotically converges to zero. But, this implies that R asymptotically converges either to the desired attitude R_d, or to one of the three undesired attitudes $\{R_d \exp(\pi \hat{s}) \mid s \in \{e_1, e_2, e_3\}\}$. These three undesired equilibria cannot be avoided for any continuous attitude control system because of a topological property of SO(3) [4], [9]. However, we have shown that those undesired equilibria are unstable, which implies that the region of attraction to the undesired equilibria has zero measure. Therefore, for almost all initial conditions, the attitude converges to the desired attitude R_d. More detailed discussions on almost-global stability properties of attitude control systems are available in [28].

Remark 2: Although $\dot{\bar{J}}$ and $\|\bar{J}\|_F$ converge to zero and a positive constant, respectively, these do not guarantee that the estimate $\bar{J}$ asymptotically converges to a fixed matrix. But, if there is an additional condition on persistent excitation, we can show that $\bar{J} \to J$ as $t \to \infty$ by following an approach presented in [7].

C. Robust Adaptive Attitude Tracking

The adaptive tracking control system developed in the previous section is based on the assumption that there is no disturbance in the attitude dynamics. But, it has been discovered that adaptive control schemes might become unstable in the presence of small disturbances [19]. Robust adaptive control

[1] A function $f : \mathbb{R} \to \mathbb{R}$ belongs to the $\mathcal{L}_p$ space for $p \in [1, \infty)$ if the following p-norm of the function exits, $\|f\|_p = \left\{\int_0^\infty |f(\tau)|^p\, d\tau\right\}^{1/p}$.

deals with redesigning or modifying adaptive control schemes to make them robust with respect to unmodeled dynamics or bounded disturbances. In this section, we develop a robust adaptive attitude tracking control system assuming that the bounds of disturbances are given.

Assumption 2: The disturbance term in the attitude dynamics at (1) is bounded by a known constant, i.e., $\|\Delta\| \leq \delta$, for a given positive constant δ.

Proposition 4: Suppose that Assumptions 1 and 2 hold. For a given attitude command $R_d(t)$, and positive constants $k_R, k_\Omega, k_J, \sigma, \epsilon \in \mathbb{R}$, we define a control input $u \in \mathbb{R}^3$, and an update law for $\bar{J}$ as follows:

$$u = -k_R e_R - k_\Omega e_\Omega + \Omega \times \bar{J}\Omega + \bar{J}\alpha_d + v \tag{35}$$

$$v = -\frac{\delta^2 e_A}{\delta\|e_A\| + \epsilon} \tag{36}$$

$$\dot{\bar{J}} = \frac{k_J}{2}\left(-\alpha_d e_A^T - e_A\alpha_d^T + \Omega\Omega^T\hat{e}_A - \hat{e}_A\Omega\Omega^T - 2\sigma\bar{J}\right) \tag{37}$$

where $e_A \in \mathbb{R}^3$ is an augmented error vector given at (26) for a positive constant c satisfying (27). Then, if σ and ϵ are sufficiently small, the zero equilibrium of the tracking errors (e_R, e_Ω) and the estimation error $\tilde{J}$ are uniformly bounded.

Proof: Consider the Lyapunov function $\mathcal{V}$ at (28). For a positive constant $\psi < h_1$, define $D \subset \mathrm{SO}(3)$ as

$$D = \{R \in \mathrm{SO}(3) \,|\, \Psi < \psi < h_1\}.$$

From Proposition 1, the Lyapunov function is bounded in D by

$$z^T W_{11} z \leq \mathcal{V} \leq z^T W_{12} z \tag{38}$$

where $z = [\|e_R\|;\ \|e_\Omega\|;\ \|\tilde{J}\|_F] \in \mathbb{R}^3$, the matrix $W_{11} \in \mathbb{R}^{3\times3}$ is given by (30), and the matrix W_{12} is given by

$$W_{12} = \begin{bmatrix} b_2 k_R & \frac{1}{2}c\lambda_M & 0 \\ \frac{1}{2}c\lambda_M & \frac{1}{2}\lambda_M & 0 \\ 0 & 0 & \frac{1}{2k_J} \end{bmatrix}.$$

The time derivative of $\mathcal{V}$ along the presented control inputs is written as

$$\begin{aligned}\dot{\mathcal{V}} = &-k_\Omega\|e_\Omega\|^2 - ck_R\|e_R\|^2 + cJe_\Omega \cdot Ee_\Omega - ck_\Omega e_\Omega \cdot e_R \\ &+ e_A \cdot (\Delta + v) + \sigma\,\mathrm{tr}\left[\tilde{J}\bar{J}\right].\end{aligned} \tag{39}$$

Compared to (32), this has three additional terms caused by Δ, v, and σ. From Assumption 2 and (36), the second last term of (39) is bounded by

$$e_A \cdot (\Delta + v) \leq \delta\|e_A\| - \frac{\delta^2\|e_A\|^2}{\delta\|e_A\| + \epsilon} = \frac{\delta\|e_A\|}{\delta\|e_A\| + \epsilon}\epsilon \leq \epsilon. \tag{40}$$

The last term of (39) is bounded by

$$\begin{aligned}\mathrm{tr}\left[\tilde{J}\bar{J}\right] &= \mathrm{tr}\left[\tilde{J}(J - \tilde{J})\right] \\ &= \sum_{1\leq i,j\leq 3}(-\tilde{J}_{ij}^2 + J_{ij}\tilde{J}_{ij}) \\ &\leq \sum_{1\leq i,j\leq 3}\left(-\frac{1}{2}\tilde{J}_{ij}^2 + \frac{1}{2}J_{ij}^2\right) \\ &= -\frac{1}{2}\mathrm{tr}\left[\tilde{J}^2\right] + \frac{1}{2}\mathrm{tr}\left[J^2\right] \\ &= -\frac{1}{2}\|\tilde{J}\|_F^2 + \frac{1}{2}\|J\|_F^2.\end{aligned}$$

(a) (b)

Fig. 1. Attitude control experiment for a quadrotor UAV. (a) Hardware configuration. (b) Attitude control testbed.

Using the relation between a Frobenius norm and a matrix 2-norm, we have $\|J\|_F \leq \sqrt{3}\|J\| = \sqrt{3}\lambda_M$. Therefore

$$\mathrm{tr}\left[\tilde{J}\bar{J}\right] \leq -\frac{1}{2}\|\tilde{J}\|_F^2 + \frac{3}{2}\lambda_M^2. \tag{41}$$

Substituting (40) and (41) into (39), we obtain

$$\dot{\mathcal{V}} \leq -z^T W_3 z + \frac{3}{2}\sigma\lambda_M^2 + \epsilon \tag{42}$$

where the matrix $W_3 \in \mathbb{R}^{3\times3}$ is given by

$$W_3 = \begin{bmatrix} ck_R & -\frac{ck_\Omega}{2} & 0 \\ -\frac{ck_\Omega}{2} & k_\Omega - \frac{c}{\sqrt{2}}\lambda_M\mathrm{tr}[G] & 0 \\ 0 & 0 & \frac{1}{2}\sigma \end{bmatrix}. \tag{43}$$

Equation (27) for the constant c guarantees that the matrices W_{11}, W_{12}, and W_3 become positive definite. Then, we have

$$\dot{\mathcal{V}} \leq -\frac{\lambda_{\min}(W_3)}{\lambda_{\max}(W_{12})}\mathcal{V} + \frac{3}{2}\sigma\lambda_M^2 + \epsilon \tag{44}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ represent the minimum eigenvalue and the maximum eigenvalue of a matrix, respectively. This implies that $\dot{\mathcal{V}} < 0$ when $\mathcal{V} > (\lambda_{\max}(W_{12})/\lambda_{\min}(W_3))(3/2\sigma\lambda_M^2 + \epsilon) \triangleq d_1$. From now on, we will show the ultimate boundedness of the tracking error and the estimation error. More explicitly, we will show that the region where $\dot{\mathcal{V}}$ is negative lies in the given domain D, using the bounds of the Lyapunov function [27].

Let a sublevel set of $\mathcal{V}$ be $L_\gamma = \{(R, \Omega, \bar{J}) \in \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^{3\times3}\} \,|\, \mathcal{V} \leq \gamma\}$ for a constant $\gamma > 0$. If the inequality for γ

$$\gamma < \frac{\psi}{b_2}\lambda_{\min}(W_{11}) \triangleq d_2$$

is satisfied, we can guarantee that $L_\gamma \subset D \times \mathbb{R}^3 \times \mathbb{R}^{3\times3}$, since it implies that $\|z\|^2 < \psi/b_2$, which leads $\Psi \leq b_2\|e_R\|^2 \leq b_2\|z\|^2 < \psi$.

Then, from (44), a sublevel set L_γ is a positively invariant set when $d_1 < \gamma < d_2$, and it becomes smaller until $\gamma = d_1$. In order to guarantee the existence of such L_γ, the following inequality should be satisfied:

$$d_1 = \frac{\lambda_{\max}(W_{12})}{\lambda_{\min}(W_3)}\left(\frac{3}{2}\sigma\lambda_M^2 + \epsilon\right) < \frac{\psi}{b_2}\lambda_{\min}(W_{11}) = d_2 \tag{45}$$

which can be achieved by choosing sufficiently small σ and ϵ. Then, according to [27, Th. 5.1], for any initial condition

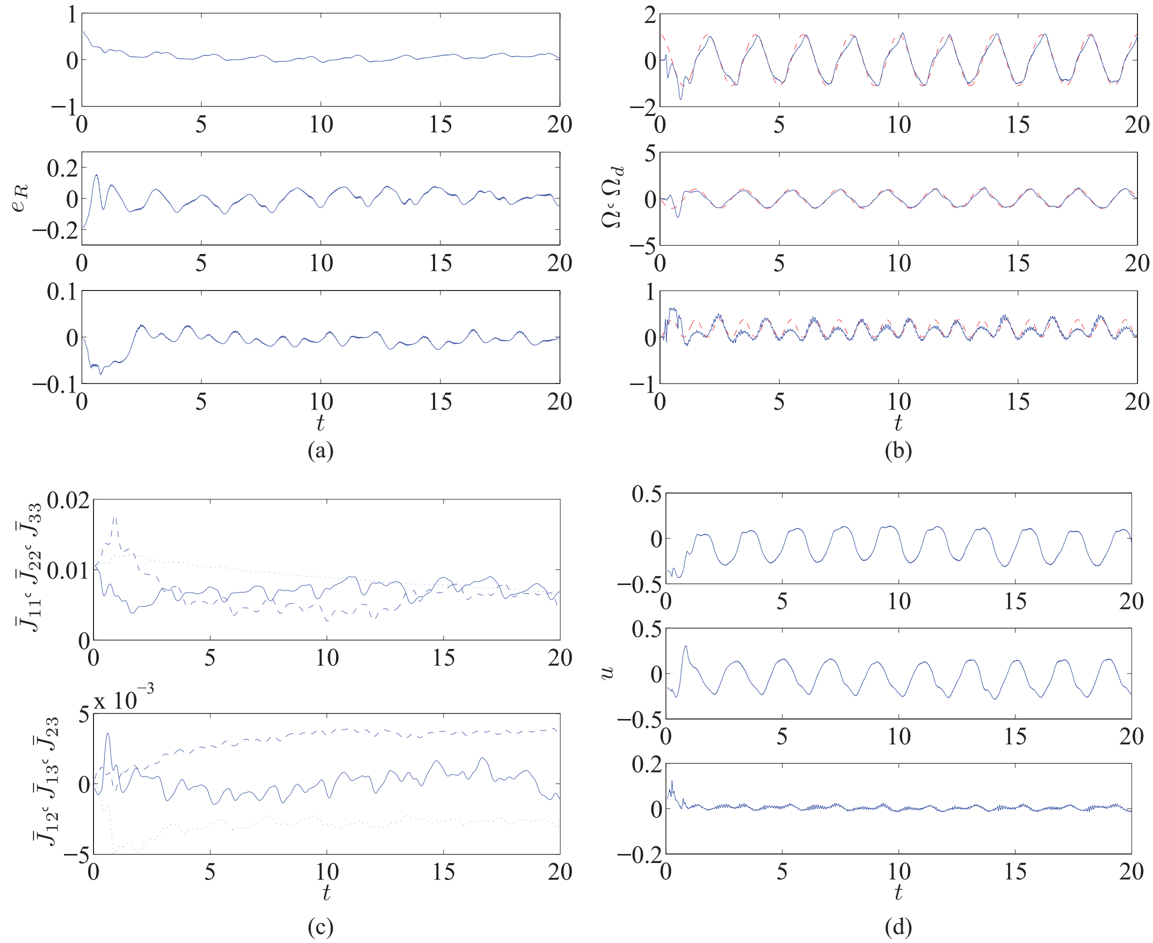

Fig. 2. Robust adaptive attitude tracking experiment. (a) Attitude error vector e_R. (b) Angular velocity (Ω: blue, Ω_d: red). (c) Inertia estimate $\bar J$ [$\bar J_{11}$, $\bar J_{12}$: solid, $\bar J_{22}$, $\bar J_{13}$: dashed, $\bar J_{33}$, $\bar J_c 23$: dotted (kgm^2)]. (d) Control input u.

satisfying $\mathcal{V}(0) < d_2$, its solution exponentially converges to the following set:

$$L_{d_1} \subset \left\{ \|z\|^2 \leq \frac{\lambda_{\max}(W_{12})}{\lambda_{\min}(W_{11})\lambda_{\min}(W_3)} \left(\frac{3}{2}\sigma \lambda_M^2 + \epsilon \right) \right\}.$$

■

Remark 3: The robust adaptive control system in Proposition 4 is referred to as the fixed σ-modification [19], where robustness is achieved at the expense of replacing the asymptotic tracking property of Proposition 3 by boundedness. This property can be improved by the following approaches: 1) the leakage term $-2\sigma \bar J$ at (37) can be replaced by $-2\sigma(\bar J - J^\star)$, where J^* denotes the best possible prior estimate of the inertia matrix, which shifts the tendency of $\bar J$ from zero to $J^\star$, thereby reducing the ultimate bound; 2) a switching σ-modification or ϵ_1-modification can be used to improve the convergence properties in the expense of discontinuities; and 3) the constant ϵ at (36) can be replaced by $\epsilon \exp(-\beta t)$ for any $\beta > 0$ to reduce the ultimate bound. The corresponding stability analyses are similar to the presented case, and they are deferred to a future study.

IV. Experiment on a Quadrotor UAV

A quadrotor UAV is composed of two pairs of counterrotating rotors and propellers. Because of its simple mechanical structure, it has been envisaged for various applications such as surveillance or mobile sensor networks as well as for educational purposes.

We have developed a hardware system for a quadrotor UAV (see Fig. 1). To test the attitude dynamics only, it is attached to a spherical joint. As the center of rotation is below the center of gravity, there exists a destabilizing gravitational moment, and the resulting attitude dynamics is similar to that of an inverted rigid body pendulum.

We apply the robust adaptive attitude control system at Proposition 4 to this quadrotor UAV. The control input at (35) is augmented with an additional term to eliminate the gravitational moment. The disturbances are mainly due to the error in canceling the gravitational moment, the friction in the spherical joint, as well as the sensor and thrust measurement errors.

The controller parameters are given as follows: $k_R = 0.0424$, $k_\Omega = 0.0296$, $k_J = 0.01$, $c = 1.0$, $\sigma = 0.01$, $\epsilon = 0.35$, and $\delta = 0.2$. The bound δ of the uncertainty is conservatively chosen such that it is sufficiently larger than the friction at the joint. Other parameters are chosen by trial and error. Initial conditions are given by $\bar J(0) = 0.001I$, $R(0) = I$, and $\Omega(0) = 0$.

The desired attitude command is described by using 3-2-1 Euler angles [26], i.e., $R_d(t) = R_d[\phi(t), \theta(t), \psi(t)]$, and these angles are chosen as

$$\phi(t) = \frac{\pi}{9}\sin(\pi t), \quad \theta(t) = \frac{\pi}{9}\cos(\pi t), \quad \psi(t) = 0.$$

The corresponding experimental results are illustrated in Fig. 2. Overall, it exhibits a good attitude command tracking performance, while the second component of the attitude error vector e_R and the third component of the angular velocity tracking error are relatively large. These errors may be due to the fact that the control moment about the third body-fixed axis is based on the reaction torques of propellers, and they are assumed to be directly proportional to thrust in the quadrotor dynamics model. The actual dynamics for the third control moment is possibly more complicated. The estimates of the inertia matrix are bounded. Further numerical results illustrating properties of the proposed robust adaptive control system are available in [29].

V. Conclusion

We developed an adaptive tracking control system on SO(3). The proposed control system was constructed directly on SO(3) to avoid singularities and ambiguities that are inherent to other attitude representations. The control system was developed to asymptotically follow a given attitude tracking command without the knowledge of an inertia matrix, in the absence of disturbances. A robust adaptive control system was proposed to eliminate the effects of unstructured disturbances. These properties were illustrated by an experiment for the attitude dynamics of a quadrotor UAV.

References

[1] J. Wen and K. Kreutz-Delgado, "The attitude control problem," *IEEE Trans. Autom. Control*, vol. 36, no. 10, pp. 1148–1162, Oct. 1991.

[2] M. Sidi, *Spacecraft Dynamics and Control*. Cambridge, U.K.: Cambridge Univ. Press, 1997.

[3] P. Hughes, *Spacecraft Attitude Dynamics*. New York: Wiley, 1986.

[4] D. Koditschek, "Application of a new Lyapunov function to global adaptive attitude tracking," in *Proc. IEEE Conf. Decision Control*, Dec. 1998, pp. 63–68.

[5] S. Joshi, A. Kelkar, and J. Wen, "Robust attitude stabilization of spacecraft using nonlinear quaternion feedback," *IEEE Trans. Autom. Control*, vol. 40, no. 10, pp. 1800–1803, Oct. 1995.

[6] B. Wie, H. Weiss, and A. Arapostathis, "Quaternion feedback regulator for spacecraft eigenaxis rotation," *J. Guid., Control, Dyn.*, vol. 12, no. 3, pp. 375–380, 1989.

[7] J. Ahmed, V. Coppola, and D. Bernstein, "Asymptotic tracking of spacecraft attitude motion with inertia matrix identification," in *Proc. IEEE Conf. Decision Control*, Dec. 1997, pp. 2471–2476.

[8] C. Mayhew, R. Sanfelice, and A. Teel, "Robust global asymptotic attitude stabilization of a rigid body by quaternion-based hybrid feedback," in *Proc. IEEE Conf. Decision Control*, Dec. 2009, pp. 2522–2527.

[9] S. Bhat and D. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Syst. Control Lett.*, vol. 39, no. 1, pp. 66–73, 2000.

[10] C. Mayhew, R. Sanfelice, and A. Teel, "Quaternion-based hybrid control for robust global attitude tracking," *IEEE Trans. Autom. Control*, vol. 56, no. 11, pp. 2555–2566, Nov. 2011.

[11] V. Jurdjevic, *Geometric Control Theory*. Cambridge, U.K.: Cambridge Univ. Press, 1997.

[12] A. Bloch, *Nonholonomic Mechanics and Control* (Interdisciplinary Applied Mathematics), vol. 24. New York: Springer-Verlag, 2003.

[13] F. Bullo and A. Lewis, "Modeling, analysis, and design for simple mechanical control systems," in *Geometric Control of Mechanical Systems* (Texts in Applied Mathematics), vol. 49. New York: Springer-Verlag, 2005.

[14] D. Maithripala, J. Berg, and W. Dayawansa, "Almost-global tracking of simple mechanical systems on a general class of Lie groups," *IEEE Trans. Autom. Control*, vol. 51, no. 2, pp. 216–225, Feb. 2006.

[15] D. Cabecinhas, R. Cunha, and C. Silvestre, "Output-feedback control for almost global stabilization of fully-actuated rigid bodies," in *Proc. IEEE Conf. Decision Control*, Dec. 2008, pp. 3583–3588.

[16] N. Chaturvedi, N. H. McClamroch, and D. Bernstein, "Asymptotic smooth stabilization of the inverted 3-D pendulum," *IEEE Trans. Autom. Control*, vol. 54, no. 6, pp. 1204–1215, Jun. 2009.

[17] T. Lee, M. Leok, and N. McClamroch, "Geometric tracking control of a quadrotor UAV on SE(3)," in *Proc. IEEE Conf. Decision Control*, Dec. 2010, pp. 5420–5425.

[18] T. Lee, "Exponential stability of an attitude tracking control system on SO(3) for large-angle rotational maneuvers," *Syst. Control Lett.*, vol. 61, no. 1, pp. 231–237, 2012.

[19] P. Ioannou and J. Sun, *Robust Adaptive Control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[20] K. Narendra and A. Annaswamy, *Stable Adaptive Systems*. New York: Dover, 2005.

[21] A. Sanyal, A. Fosbury, N. Chaturvedi, and D. Bernstein, "Inertia-free spacecraft attitude tracking with disturbance rejection and almost global stabilization," *J. Guid., Control, Dyn.*, vol. 32, no. 4, pp. 1167–1178, 2009.

[22] A. Dando, "Robust adaptive control of rigid spacecraft attitude maneuvers," Ph.D. dissertation, Dept. Aerosp. Avionics, Queensland Univ. Technol., Brisbane, Australia, 2008.

[23] X. Liao, B. Li, and Y. Song, "Robust adaptive attitude control of crew exploration vehicles (CEVs) with guaranteed performance," in *Proc. Amer. Control Conf.*, 2007, pp. 3849–3854.

[24] Z. Weitai, Y. Changqing, C. Fuzhong, Z. Hao, and Y. Goujun, "Robust adaptive sliding mode attitude tracking control for spacecraft formation flying," in *Proc. IEEE Conf. Meas. Technol., Mechatron. Autom.*, Jan. 2011, pp. 175–178.

[25] Q. Hu, "Robust adaptive attitude tracking control with L_2-gain performance and vibration reduction of an orbiting flexible spacecraft," *ASME J. Dyn. Syst., Meas., Control*, vol. 133, no. 1, pp. 011009-1–011009-11, 2011.

[26] M. Shuster, "Survey of attitude representations," *J. Astronaut. Sci.*, vol. 41, no. 4, pp. 439–517, 1993.

[27] H. Khalil, *Nonlinear Systems*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[28] N. Chaturvedi, A. Sanyal, and N. McClamroch, "Rigid-body attitude control," *IEEE Control Syst. Mag.*, vol. 31, no. 3, pp. 30–51, Jun. 2011.

[29] T. Lee. (2011). *Robust Adaptive Geometric Tracking Control on SO(3) with an Application to the Attitude Dynamics of a Quadrotor UAV* [Online]. Available: http://arxiv.org/abs/1108.6031