Polynomial Trajectory Optimization Using Dual Approximation

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1 Introduction

Because quadrotor dynamics are differentially flat, it is desireable to be able to produce arbitrary trajectories for quadrotors are frequently represented using piecewise polynomials. Finding a trajectory is typically posed as an optimization problem where the objective is to minimize a certain derivative of the trajectory, given constraints such as waypoint locations and continuity of derivatives between segments. In [1, 3] this optimization is shown to be a constrained quadratic program and is solved using a conventional solver. In this paper, I take a different approach, using dual approximation and least-squares to satisfy both segment endpoint constraints and to minimize the k^{th} derivative along a single-dimensional, piecewise polynomial.

When generating a piecewise polynomial trajectory for a differentially flat system, we have three different priorities to consider.

1. Knot point constraints. Position specified, and need to ensure that the spline is continuous up to the highest derivative taken in the flat map. For a quadrotor, this corresponds to the fourth derivative for the position splines, and to the second derivative for the yaw spline [1]. We use a minimum norm method to acheive these constraints.

2. Smoothness constraints. In [1], the integral of the fourth derivative (snap) of the position splines and the second derivative of the yaw spline were minimized to produce quadrotor trajectories. We acheive this constraint by This has a similar effect as minimizing the integral of the control effort. ?

3. Vehicle dynamic constraints. It is possible to formulate the trajectory generation problem as a nonlinear optimization which computes the system states and control inputs along the trajectory in order to meet constraints or optimize an objective. This, however, adds additional complexity to the trajectory generation, so we instead rely on the smoothness constraints described above, and time-scaling to acheive dynamic constraints.

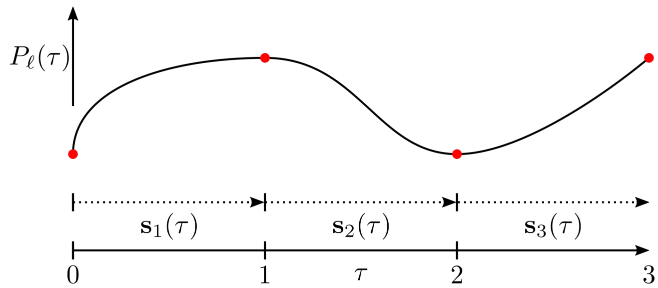

Figure 1: Notation used for a single spline.

2 Notation

$P(\tau) = (P_1(\tau), \ldots, P_r(\tau))^T$

Let $P(\tau)$ be a set of r splines, denoted $P_\ell(\tau)$, each of which is made up of m segments, denoted $P_{\ell,i}(\tau)$, where $\tau \in [0, m]$. Each segment is parameterized by $s \in [0, 1]$ and consists of the inner product of a vector of n basis functions,

$$\boldsymbol{\psi}(s) = [\psi_1(s), \psi_2(s), \ldots, \psi_n(s)]^\top \tag{1}$$

may be use \varphi instead of \psi

with n coefficients $p_{\ell,i,j}$.

standard polynomials

perhaps

Remark. *Typical choices for basis functions are the ~~set of functions~~ $\{\cancel{1, s, s^2, \ldots, s^n}\}$, or the Legendre polynomials.*

$\psi(s) = [1, s, s^2, \ldots s^n]^T$

We use a uniform spacing between segments, switching when $\tau \bmod 1 = 0$. To handle switching between segments, we define the basis vector for the i^{th} segment as

$$\mathbf{s}_i(\tau) = \begin{cases} \boldsymbol{\psi}(s) & \text{for } i \leq \tau < i+1, \text{ with } s = \tau \bmod i \\ \mathbf{0}_n & \text{otherwise} \end{cases} \tag{2}$$

$\leq$

where $\mathbf{0}_n$ is an $n \times 1$ vector of zeros. Similarly, we define the k^{th} derivative of $\mathbf{s}_i(\tau)$ with respect to τ as where

$\leq$

$$\mathbf{s}_i^{(k)}(\tau) \triangleq \frac{d^k \mathbf{s}(\tau)}{d\tau^k} = \begin{cases} \boldsymbol{\psi}^{(k)}(s) & \text{for } i \leq \tau < i+1, \text{ with } s = \tau \bmod i \\ \mathbf{0}_n & \text{otherwise} \end{cases} \tag{3}$$

where

$$\boldsymbol{\psi}^{(k)} = [\psi_1^{(k)}(s), \psi_2^{(k)}(s), \ldots, \psi_n^{(k)}(s)]^\top \tag{4}$$

and

$$\psi_j^{(k)} \triangleq \frac{d^k \psi}{ds^k}. \tag{5}$$

Stacking these basis vectors, we get the $nm \times 1$ τ-dependent vector

$$\mathbf{S}(\tau) = \left[\mathbf{s}_1(\tau)^\top, \quad \mathbf{s}_2(\tau)^\top, \quad \ldots, \quad \mathbf{s}_m(\tau)^\top\right]^\top \tag{6}$$

and

$$\mathbf{S}^{(k)}(\tau) = \left[\mathbf{s}_1^{(k)}(\tau)^\top, \quad \mathbf{s}_2^{(k)}(\tau)^\top, \quad \ldots, \quad \mathbf{s}_m^{(k)}(\tau)^\top\right]^\top. \tag{7}$$

Define the $n \times 1$ vector of segment coefficients for the i^{th} segment of the ℓ^{th} spline to be

$$\mathbf{p}_{\ell,i} = \left[p_{\ell,i,1}, \quad p_{\ell,i,2}, \quad \ldots, \quad p_{\ell,i,n}\right]^\top, \tag{8}$$

the $nm \times 1$ vector of all segment coefficients for the ℓ^{th} spline to be

$$\mathbf{p}_\ell = \left[\mathbf{p}_{\ell,1}^\top, \quad \mathbf{p}_{\ell,2}^\top, \quad \ldots, \quad \mathbf{p}_{\ell,n}^\top\right]^\top, \tag{9}$$

and the $r \times nm$ matrix of segment coefficients for all splines to be

$$\mathbf{P} = \left[\mathbf{p}_1^\top, \quad \mathbf{p}_2^\top, \quad \ldots, \quad \mathbf{p}_r^\top\right]^\top. \tag{10}$$

Then, we can write

$$P_{\ell,i}(\tau) = \mathbf{p}_{\ell,i}^\top \mathbf{s}_i(\tau), \tag{11}$$

$$P_{\ell,i}^{(k)}(\tau) = \mathbf{p}_{\ell,i}^\top \mathbf{s}_i^{(k)}(\tau), \tag{12}$$

$$P_\ell(\tau) = \mathbf{p}_\ell^\top \mathbf{S}(\tau), \tag{13}$$

$$P_\ell^{(k)}(\tau) = \mathbf{p}_\ell^\top \mathbf{S}^{(k)}(\tau), \tag{14}$$

$$P(\tau) = \mathbf{P}^\top \mathbf{S}(\tau), \tag{15}$$

and

$$P^{(k)}(\tau) = \mathbf{P}^\top \mathbf{S}^{(k)}(\tau). \tag{16}$$

3 Satisfying knot point constraints

Knot point constraints ensure that the trajectory passes through desired waypoints, and that it is continuous at the desired knot points. This leads to two different types of constraints: ensuring continuity of a derivative between two connecting segments, and fixing the value of a derivative at the endpoint of the segment. These can be written as inner products between $\mathbf{p}$ and $mn \times 1$ vectors consisting of zeros and one or more $\boldsymbol{\psi}^{(k)}(s)$, evaluated at 0 or 1.

First, we define the $mn \times 1$ vector with $\boldsymbol{\psi}^{(k)}(s)$ in the i^{th} position and zeros elsewhere as

$$\Psi_i^{(k)}(s) = [\mathbf{0}_{n(i-1)}^\top, \boldsymbol{\psi}^{(k)}(s)^\top, \mathbf{0}_{m(n-i)}^\top]^\top. \tag{17}$$

Theorem 1. *Derivative continuity and knot point value constraints for a trajectory member $P_\ell(\tau)$ can be written as an inner product between $\Psi_i^{(k)}(s)$ and $\mathbf{p}_\ell$.*

Proof: The value of the starting knot points of segments in $P_\ell(\tau)$ are

$$\mathbf{p}_\ell^\top \Psi_i^{(k)}(0) \text{ for } i \in [1, n] \tag{18}$$

and the value of the ending knot points of segments in $P_\ell(\tau)$ are

$$\mathbf{p}_\ell^\top \Psi_i^{(k)}(1) \text{ for } i \in [1, n]. \tag{19}$$

Thus, the k^{th} derivative at the initial knot point of a segment i can be set to a value a by enforcing the constraint

$$\mathbf{p}_\ell^\top \Psi_i^{(k)}(0) = a, \tag{20}$$

and the k^{th} derivative at the final knot point of a segment i can be set to a value a by enforcing the constraint

$$\mathbf{p}_\ell^\top \Psi_i^{(k)}(1) = a. \tag{21}$$

Continuity of the k^{th} derivative between segments i and $i+1$ can be ensured while allowing the value to vary by enforcing

$$\mathbf{p}_\ell^\top \Psi_i^{(k)}(1) = \mathbf{p}_\ell^\top \Psi_{i+1}^{(k)}(0) \tag{22}$$

$$\iff \mathbf{p}_\ell^\top \Psi_i^{(k)}(1) - \mathbf{p}_\ell^\top \Psi_{i+1}^{(k)}(0) = 0 \tag{23}$$

$$\iff \mathbf{p}_\ell^\top (\Psi_i^{(k)}(1) - \Psi_{i+1}^{(k)}(0)) = 0 \tag{24}$$

■

Letting $\bar{\mathbf{a}}_\ell$ be the vector of desired values and derivatives at the knot points and k_ℓ to be the highest order of derivative we wish to enforce continuity, we define

$$\mathbf{a}_\ell = \begin{bmatrix} \bar{\mathbf{a}}_\ell \\ \mathbf{0}_{mk_\ell} \end{bmatrix} \tag{25}$$

to be a vector of length c. We can then formulate the constraint inner products in matrix form as

$$\mathbf{D}\mathbf{p}_\ell = \mathbf{a}_\ell \tag{26}$$

where

start & end positions plus ~~wa~~ intermediate waypoints

Doesn't this also encode 0-order derivative constraints?

$$\underset{c \times mn}{\mathbf{D}} = \begin{bmatrix} (\Psi_1^{(k)}(0))^\top \\ \vdots \\ (\Psi_m^{(k)}(0))^\top \\ (\Psi_m^{(k)}(1))^\top \\ (\Psi_1^{(k)}(1) - \Psi_2^{(k)}(0))^\top \\ \vdots \\ (\Psi_{m-1}^{(k)}(1) - \Psi_m^{(k)}(0))^\top \end{bmatrix} \in \mathbb{R}^{c \times mn}. \tag{27}$$

derivatives

This needs work but idea is right

Since we enforce constraints on the continuity of the spline we can constrain the value at the starting knot point for each segment but the last, where we constrain both the start and end points.

By choosing n such that $\cancel{mn} < c$ $mn > c$, $\mathbf{D}$ will be wide, and Eq. 26 will be an ~~over~~ under determined system. Note that if $\operatorname{rank}(\mathbf{D}) < c$, there are linearly dependent constraints that can be removed from $\mathbf{D}$.

with $mn-c$ degrees of freedom

As shown in [2], the solution to Eq. 26 of minimum norm is

gives

$$\mathbf{p}_\ell^\star = \mathbf{D}^\top(\mathbf{D}\mathbf{D}^\top)^{-1}\mathbf{a}_\ell. \tag{28}$$

$D[p_1 p_2 p_3 p_4] = [a_1 a_2 a_3 a_4]$

$\boxed{P^{*T} = D^T(DD^T)^{-1}A}$ or $DP^T = A$

P

which exactly satisfies the knot value and continuity constraints.

Thus, a polynomial satisfying the constraints given by $\mathbf{D}$ and $\mathbf{a}_\ell$ is

$$\cancel{P_\ell(\tau) = (\mathbf{p}_\ell^\star)^\top}\mathbf{S}(\tau). \qquad P(\tau) = p^{*\top} S(\tau) \tag{29}$$

It is significant that, assuming the same ordering in $\mathbf{a}_\ell$, the pseudoinverse of $\mathbf{D}$ only needs to be found once to solve Eq. 26 for any $\mathbf{a}_\ell$.

4 Minimizing the k^{th} derivative

Beyond satisfying continuity and value constraints, it is common [3, 1] to minimize some derivative k_r of $P(\tau)$ over the trajectory,

$$\begin{aligned}
&\min_{p_\ell} \int_0^m \|P^{(k_r)}(\tau)\|^2 d\tau \\
&\cancel{\text{with respect to } \mathbf{p}_\ell} \\
&\text{subject to } \; \mathbf{D}\mathbf{p}_\ell = \mathbf{a}_\ell \; \text{ for } \ell \in 1, \ldots, r.
\end{aligned} \tag{30}$$

The choice of k_r is typically determined by the highest derivative needed to perform control. For a quadroter, snap ($k_r = 4$) is often minimized [1].

First, we show that Eq. 30 can be satisified by considering each trajectory member individually.

Theorem 2. *Minimizing $\int_0^m \|P^{(k_r)}(\tau)\|^2 d\tau$ with respect to $\mathbf{P}$ is equivalent to minimizing $\int_0^m \|P_\ell^{(k_r)}(\tau)\|^2 d\tau$ with respect to $\mathbf{p}_\ell$ for all $\ell \in 1, \ldots, r$.*

Proof:

$$\min \int_0^m \|P^{(k_r)}(\tau)\|^2 d\tau \tag{31}$$

$$= \min \int_0^m \sum_{\ell=1}^r (P_\ell^{(k_r)}(\tau))^2 d\tau \tag{32}$$

$$= \min \sum_{\ell=1}^r \int_0^m (P_\ell^{(k_r)}(\tau))^2 d\tau \tag{33}$$

$$= \min \sum_{\ell=1}^r \int_0^m (\mathbf{p}_\ell^\top \mathbf{S}^{(k_r)}(\tau))^2 d\tau \tag{34}$$

Each member of the sum in Eq. 34 is positive and independent in $\mathbf{p}_\ell$, so it can be written as

$$\sum_{\ell=1}^r \min \int_0^m (\mathbf{p}_\ell^\top \mathbf{S}^{(k_r)}(\tau))^2 d\tau. \tag{35}$$

■

It is possible to vary $\mathbf{p}^\star$ while still satisfying Eq. 26.

Theorem 3. *The solution* $\mathbf{p}_\ell^\star$*, given by Eq. 28, resides in* $\mathcal{R}(\mathbf{D})$*, and for any* $\mathbf{q} \in \mathcal{N}(\mathbf{D})$*,* $\mathbf{r}_\ell = \mathbf{p}_\ell^\star + \mathbf{q}_\ell$ *is also a valid solution to Eq. 26.*

Proof:

$$\mathbf{D}\mathbf{r}_\ell \tag{36}$$
$$=\mathbf{D}\mathbf{p}_\ell^\star + \mathbf{D}\mathbf{q}_\ell \tag{37}$$
$$=\mathbf{D}\mathbf{p}_\ell^\star + \mathbf{0} \tag{38}$$
$$=\mathbf{a}_\ell \tag{39}$$

■

The fact that $\mathbf{p}_\ell^\star$ is of minimum norm has little practical meaning for the application of finding trajectories, so we formulate another optimization over $\mathbf{q}_\ell \in \mathcal{N}(\mathbf{D})$ to satisfy additional constraints on the trajectory. Recall that for the singular value decomposition,

$$\mathbf{D} = U\Sigma V^H = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix}, \tag{40}$$

an orthonormal basis for the nullspace of $\mathbf{D}$ is given by span (V_2), where V_2 is $mn \times (mn - c)$ [2]. By picking $\mathbf{q}_\ell = V_2\mathbf{b}_\ell$, we write $\mathbf{r}_\ell = \mathbf{p}_\ell^\star + V_2\mathbf{b}$, and formulate the unconstrained optimization

$$\begin{aligned} &\text{minimize} \quad \int_0^m ((\mathbf{p}_\ell^\star + V_2\mathbf{b}_\ell)^\top \mathbf{S}^{(k_r)}(\tau))^2 d\tau \\ &\text{with respect to } \ \mathbf{b}_\ell. \end{aligned} \tag{41}$$

Theorem 4. *The optimization given by Eq. 41 can be solved analytically.*

Proof: We begin by considering the integrand of Eq. 41. For readability, let $\mathbf{S} = \mathbf{S}^{(k_r)}(\tau)$.

$$(\mathbf{r}_\ell^\top \mathbf{S})^2 = (\mathbf{r}_\ell^\top \mathbf{S})(\mathbf{r}_\ell^\top \mathbf{S})^\top = \mathbf{r}_\ell^\top \mathbf{S}\mathbf{S}^\top \mathbf{r}_\ell \tag{42}$$

Then the objective in Eq. 41 can be written as

$$\int_0^m (\mathbf{r}_\ell^\top \mathbf{S}^{(k_r)}(\tau))^2 d\tau = \mathbf{r}_\ell^\top \left(\int_0^m \mathbf{S}\mathbf{S}^\top d\tau \right) \mathbf{r}_\ell. \tag{43}$$

We define $\mathbf{W} \in \mathbb{R}^{nm \times nm}$ to be the integral on the right of Eq. 43. Letting $\mathbf{s}_i^{(kr)}(\tau) = \mathbf{s}_i$ we have

$$\mathbf{W} \triangleq \int_0^m \mathbf{S}\mathbf{S}^\top d\tau \tag{44}$$
$$= \int_0^m \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \vdots \\ \mathbf{s}_m \end{bmatrix} \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \dots & \mathbf{s}_m \end{bmatrix} d\tau \tag{45}$$
$$= \int_0^m \begin{bmatrix} \mathbf{s}_1\mathbf{s}_1^\top & \mathbf{s}_1\mathbf{s}_2^\top & \dots & \mathbf{s}_1\mathbf{s}_m^\top \\ \mathbf{s}_2\mathbf{s}_1^\top & \mathbf{s}_2\mathbf{s}_2^\top & \dots & \mathbf{s}_2\mathbf{s}_m^\top \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{s}_m\mathbf{s}_1^\top & \mathbf{s}_m\mathbf{s}_2^\top & \dots & \mathbf{s}_m\mathbf{s}_m^\top \end{bmatrix} d\tau. \tag{46}$$

Since $\mathbf{s}_i^{(k_r)}(\tau) = \mathbf{0}_n$ for $\tau \notin [i, i+1]$, and $\mathbf{s}_i^{(k_r)}(\tau) = \boldsymbol{\psi}^{(k_r)}(s)$ for $\tau \in [i, i+1]$ with $s = \tau \bmod i$, the cross terms $\mathbf{s}_i \mathbf{s}_j^\top$ for $i \neq j$ in Eq. 46 evaluate to zero. This makes $\mathbf{W}$ block diagonal. Additionally, each of the diagonal terms is identical, giving us $\mathbf{W} = \mathrm{diag}(W, W, \dots, W)$, where $W \in \mathbb{R}^{n \times n}$ is the integral of a diagonal term of Eq. 46:

$$W \triangleq \int_0^m \mathbf{s}_i^{(k_r)}(\tau)(\mathbf{s}_i^{(k_r)}(\tau))^\top d\tau \tag{47}$$

$$= \int_0^1 \boldsymbol{\psi}^{(k_r)}(s)(\boldsymbol{\psi}^{(k_r)}(s))^\top ds \tag{48}$$

$$= \int_0^1 \begin{bmatrix} (\psi_1^{(k_r)}(s))^2 & \psi_1^{(k_r)}(s)\psi_2^{(k_r)}(s) & \dots & \psi_1^{(k_r)}(s)\psi_n^{(k_r)}(s) \\ \psi_2^{(k_r)}(s)\psi_1^{(k_r)}(s) & (\psi_2^{(k_r)}(s))^2 & \dots & \psi_2^{(k_r)}(s)\psi_n^{(k_r)}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_n^{(k_r)}(s)\psi_1^{(k_r)}(s) & \psi_n^{(k_r)}(s)\psi_2^{(k_r)}(s) & \dots & (\psi_n^{(k_r)}(s))^2 \end{bmatrix} ds. \tag{49}$$

It can be seen that W is symmetric, and positive semi-definite. If the basis members ψ_j are orthogonal, then W is diagonal. If $k_r > 0$ (which, in general, will be true), then $\psi_j^{(k_r)} = 0$ for $j < k_r$, making W singular.

We can now write the objective in Eq. 41 as

$$\int_0^m ((\mathbf{p}_\ell^\star + V_2\mathbf{b}_\ell)^\top \mathbf{S}^{(k_r)}(\tau))^2 d\tau \tag{50}$$

$$=(\mathbf{p}_\ell^\star + V_2\mathbf{b}_\ell)^\top \mathbf{W}(\mathbf{p}_\ell^\star + V_2\mathbf{b}_\ell) \tag{51}$$

$$=(\mathbf{p}_\ell^\star)^\top \mathbf{W}\mathbf{p}_\ell^\star + \mathbf{b}_\ell^\top V_2^\top \mathbf{W} V_2 \mathbf{b}_\ell + 2\mathbf{b}_\ell^\top V_2^\top \mathbf{W}\mathbf{p}_\ell^\star. \tag{52}$$

Since this is an unconstrained optimization, it can be solved analytically by setting $\frac{\partial}{\partial \mathbf{b}_\ell} = \mathbf{0}$.

This yields

$$2V_2^\top \mathbf{W} V_2 \mathbf{b}_\ell + 2V_2^\top \mathbf{W}\mathbf{p}_\ell^\star = 0 \tag{53}$$

$$=V_2^\top \mathbf{W}(V_2\mathbf{b}_\ell + \mathbf{p}_\ell^\star) = 0 \tag{54}$$

$$=V_2^\top \mathbf{W}\mathbf{r}_\ell = 0 \tag{55}$$

JBW: This is a really interesting result (assuming my math is correct). What we really care about is finding $\mathbf{r}_\ell$, since that is what will actually be applied to the trajectory. In general, $\mathbf{W}$ will be singular. Additionally, $V_2^T\mathbf{W}$ will be $(mn-c) \times mn$ (tall), so solving here for $\mathbf{r}_\ell$ will be approximate - or will we just get $\mathbf{r} = 0$?. In that case, $\mathbf{W}$ is irrelevant, and we only need to find $\mathbf{b}_\ell$ such that $\|V_2\mathbf{b}_\ell + \mathbf{p}_\ell^\star\|$ is minimized, ie $\mathbf{b}_\ell = -V_2^\dagger \mathbf{p}$. That's a pretty cool result if that is the case, as it would also imply that it is true for minimizing any derivative - including the zeroth, so the minimum snap trajectory would also be minimum accel, velocity, and length. ■

5 Ensuring dynamic feasibility using time scaling

$$\underbrace{P(\tau)}_{4\times 1} = \underbrace{P^T}_{4\times nm} \underbrace{S(\tau)}_{nm\times 1}$$

End point + intermediate constraints are written as

$$D p_1 = a \qquad D p_2 = a_2 \quad \cdots \quad D p_4 = a_4$$

or $\underbrace{D}_{c\times nm}\underbrace{P}_{nm\times 4} = \underbrace{A}_{c\times 4}$ where $P = [p_1\ p_2\ p_3\ p_4]$ and $A = [a_1\ a_2\ a_3\ a_4]$

min norm solution is $\boxed{P^* = D^T(DD^T)^{-1}A}$

Assume $\mathrm{rank}(DD^T) = c$

i.e. D is full rank

SVD of D gives $D = \underbrace{U}_{c\times c}(\underbrace{\Sigma_1}_{c\times c}\ \underbrace{\Sigma_2}_{c\times(nm-c)})\begin{pmatrix} V_1^T \\ V_2^T\end{pmatrix}$ where $\mathcal{N}(D) = \mathrm{span}(V_2)$

$\underbrace{V_1}_{nm\times c}$ $\underbrace{V_2}_{nm\times(nm-c)}$

Therefore all solutions of $DP = A$ are given by

$$\boxed{P = P^* + P_{null} = P^* + V_2 Z = V_1 \Sigma_1^{-1} U^T A + V_2 Z}$$

$$Z \in \mathbb{R}^{(nm-c)\times 4}$$

In addition we want to minimize

$$\min_Z \int_0^m \| P^{(k)}(\tau)\|^2 d\tau = \min_Z \int_0^m P^{(k)T}(\tau)\, P^{(k)}(\tau)\, d\tau$$

$$= \min_Z \int_0^m S^{(k)T}(\tau)\, P P^T S^{(k)}(\tau)\, d\tau$$

$$= \min_Z \mathrm{tr}\left\{ \int_0^m P^T S^{(k)}(\tau)\, S^{(k)T}(\tau)\, P\, d\tau \right\}$$

$$= \min_Z \mathrm{tr}\left\{ P^T\left(\int_0^m S^{(k)}(\tau)\, S^{(k)T}(\tau)\, d\tau\right) P\right\}$$

$$= \min_Z \mathrm{tr}(P^T W P)$$

where $W = \int_0^m S^{(k)}(\tau)\, S^{(k)T}(\tau)\, d\tau$

$$= \min_Z \mathrm{tr}\left\{(P^* + V_2 Z)^T W (P^* + V_2 Z)\right\}$$

$$= \min_Z \mathrm{tr}\left\{ P^{*T} W P^* + P^{*T} W V_2 Z + Z^T V_2^T W P^* + Z^T V_2^T W V_2 Z\right\}$$

$$(1) \qquad = \min_Z \mathrm{tr}\left\{ P^{*T} W P^* + 2 Z^T V_2^T W P^* + Z^T V_2^T W V_2 Z\right\}$$

use the fact that

$$\frac{\partial\, \mathrm{tr}(ABA^T)}{\partial A} = A(B + B^T)$$

$$\frac{\partial\, \mathrm{tr}(AB)}{\partial A} = B^T$$

so Z^* minimizing (1) must satisfy (necessary cond)

$$\frac{\partial}{\partial Z^T}\left\{ P^{*T} W P^* + 2 Z^T V_2^T W P^* + Z^T V_2^T W V_2 Z\right\} = 0$$

$$\Longleftrightarrow \quad Z^{*T}\left(V_2^T W V_2 + (V_2^T W V_2)^T\right) + 2 P^{*T} W V_2 = 0$$

$$\Longleftrightarrow \quad 2 V_2^T W V_2 Z^* + 2 V_2^T W P^* = 0$$

$$\Longleftrightarrow \quad \underbrace{Z^*}_{(nm-c)\times 4} = -(\underbrace{V_2^T}_{(nm-c)\times nm} \underbrace{W}_{nm\times nm} \underbrace{V_2}_{nm\times nm-c})^{-1} \underbrace{V_2^T}_{(nm-c)\times nm} \underbrace{W}_{nm\times nm} \underbrace{P^*}_{nm\times 4}$$

using $P^* = V_1 \Sigma_1^{-1} U^T A$ gives

$$\boxed{Z^* = -(V_2^T W V_2)^{-1} V_2^T W V_1 \Sigma_1^{-1} U^T A}$$

The ~~opt~~ min snap spline coefficients are therefore

$$\boxed{P = \left(I - (V_2^T W V_2)^{-1} V_2^T W\right) V_1 \Sigma_1^{-1} U^T A}$$

* Let $H = \left(I - (V_2^T W V_2)^{-1} V_2 W\right) V_1 \Sigma_1^{-1} U^T$

and note that H depends only on the basis functions

and can be computed once off line

Then $P = HA$.

6 old

6.1 Notation

Let $P(\tau)$ be a piecewise polynomial made up of m n^{th} order segments, with $\tau \in [0, m]$. Let $P_i(t) = p_{i,0} + p_{i,1}t + \cdots + p_{i,n}t^n$ denote the i^{th} segment of $P(\tau)$ with $i \in 0 \ldots m$ and $t = \tau$ mod 1.0. Then let $\mathbf{p}_i = [p_{i,0}, p_{i,1}, p_{i,2}, \ldots, p_{i,n}]^\top$ and let $\mathbf{p} = [\mathbf{p}_0^\top, \mathbf{p}_1^\top, \ldots, \mathbf{p}_m^\top]^\top$ be the vector of parameters of $P(t)$.

We can write $P_i(t) = \mathbf{t}^\top \mathbf{p}_i$ where $\mathbf{t} = [1, t^1, t^2, \ldots, t^n]^\top$. Extending this to all τ, we let $\mathbf{T} = [0 \ldots 0, \mathbf{t}^\top, 0 \ldots 0]^\top$ where $\mathbf{T}$ is zero everywhere except it has $\mathbf{t}$ in the ni through $n(i+1)$ positions, with $i = \text{floor}\,(\tau)$. This gives $P(\tau) = \mathbf{T}^\top \mathbf{p}$.

Let $P^{(k)}(t) = \frac{d^k P(t)}{dt^k}$ be the k^{th} derivative of $P(t)$. Then we write $P_i^{(k)}(t) = \mathbf{t}_k^\top \mathbf{p}_i$, with $\mathbf{t}_k = \frac{d^k \mathbf{t}}{dt^k} = [\frac{d^k 1}{dt^k}, \frac{d^k t}{dt^k}, \frac{d^k t^2}{dt^k}, \ldots, \frac{d^k t^n}{dt^k}]^\top$. Note that this is also valid for the 0^{th} derivative, $k = 0$. We also write $P^{(k)}(t) = \mathbf{T}_k^\top \mathbf{p}_i$, with $\mathbf{T}_k = [0 \ldots 0, \mathbf{t}_k, 0 \ldots 0]^\top$.

7 Satisfying endpoint constraints

To form a dynamically feasible trajectory, constraints on the endpoints of segments in $P(\tau)$ must be met. These constraints have two forms: continuity of a derivative between two connecting segments, and the value of a derivative at the endpoint of the segment. These can be written as inner products between $\mathbf{p}$ and $mn \times 1$ vectors consisting of zeros and one or more $\mathbf{t}_k$, evaluated at 0 or 1.

For example, setting the k^{th} derivative at the end of segment i equal to the k^{th} derivative at the beginning of segment $i + 1$ can be formulated as follows:

$$
\begin{aligned}
&P_i^{(k)}(1) = P_{i+1}^{(k)}(0) && (56)\\
&\implies \mathbf{t}_k^\top \mathbf{p}_i \big|_{t=1} = \mathbf{t}_k^\top \mathbf{p}_{i+1} \big|_{t=0} && (57)\\
&\implies \left\langle \mathbf{p}, [0 \ldots 0, \mathbf{t}_k^\top|_{t=1}, -\mathbf{t}_k^\top|_{t=0}, 0 \ldots 0]^\top \right\rangle = 0. && (58)
\end{aligned}
$$

Similarly, we can fix the location ($k = 0$) or derivative ($k > 0$) of an endpoint to a value α:

$$
\begin{aligned}
&P_i^{(k)}(t) = \alpha && (59)\\
&\implies \mathbf{t}_k^\top \mathbf{p}_i \big|_{t=0 \text{ or } 1} = \alpha && (60)\\
&\implies \left\langle \mathbf{p}, [0 \ldots 0, \mathbf{t}_1^\top|_{t=0 \text{ or } 1}, 0 \ldots 0]^\top \right\rangle = \alpha. && (61)
\end{aligned}
$$

Because the endpoint constraints can be written as inner products, we can represent the problem of exactly satisfying $c < mn$ of these constraints as a dual approximation problem. If $c > mn$, then there are not enough degrees of freedom in the polynomial to meet the constraints exactly. Choosing n so that $mn > c$ will add the necessary degrees of freedom to meet the constraints.

Let D be the $c \times mn$ matrix consisting of rows of vectors as on the right side of the inner products in Eqs. 58 and 61, and let $\mathbf{a}$ be a $c \times 1$ vector of constraints. If $\operatorname{rank}(D) < c$, there are linearly dependent constraints that can be removed from D.

The constraints established by D and $\mathbf{a}$ are satisfied by a solution $\mathbf{p}$ to

8 Minimizing the Derivatives of $P(\tau)$

From [2] we know that $\mathbf{p}^*$, given in Eq. 28, resides in $\mathcal{R}(D)$, and for any $\mathbf{q} \in \mathcal{N}(D)$, $\mathbf{r} = \mathbf{p}^* + \mathbf{q}$ is also a solution to Eq. **??**. The fact that $\mathbf{p}^*$ is of minimum norm has little practical meaning for the application of finding trajectories, so we can formulate another optimization over $\mathbf{q} \in \mathcal{N}(D)$ to satisfy additional constraints on the trajectory. In particular, we are interested in minimizing the k^{th} derivative of $P(\tau)$.

Recall that for the singular value decomposition,

$$D = U\Sigma V^H = [U_1 U_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^H \\ V_2^H \end{bmatrix}, \tag{62}$$

an orthonormal basis for the nullspace of D is given by $\operatorname{span}(V_2)$ [2]. By picking $\mathbf{q} = V_2\mathbf{b}$, we can adjust $P(\tau) = \mathbf{T}^\top \mathbf{r}$ while ensuring that the endpoint constraints are still met.

The cost to minimize the k^{th} derivative of $P(\tau)$ can then be represented as

$$\mathbf{J}_k(\mathbf{r}) = \int_0^m \left(P^{(k)}(\tau)\right)^2 d\tau \tag{63}$$

$$= \int_0^m \left(\mathbf{T}_k^\top \mathbf{r}\right)^2 d\tau \tag{64}$$

$$= \int_0^m \left(\mathbf{T}_k^\top \mathbf{p}^* + \mathbf{T}_k^\top V_2 \mathbf{b}\right)^2 d\tau \tag{65}$$

$$= \int_0^m \left(\mathbf{T}_k^\top \mathbf{p}^*\right)^2 + 2\mathbf{T}_k^\top \mathbf{p}^* \mathbf{T}_k^\top V_2 \mathbf{b} + \left(\mathbf{T}_k^\top V_2 \mathbf{b}\right)^2 d\tau. \tag{66}$$

Since $\left(\mathbf{T}_k^\top \mathbf{p}^*\right)^2$ is independent of $\mathbf{b}$, it has a fixed cost and can be ignored while minimizing $\mathbf{J}_k(\mathbf{r})$. So, we wish to find

$$\underset{\mathbf{b}}{\operatorname{argmin}} \int_0^m 2\mathbf{T}_k^\top \mathbf{p}^* \mathbf{T}_k^\top V_2 \mathbf{b} + \mathbf{b}^\top V_2^\top \mathbf{T}_k \mathbf{T}_k^\top V_2 \mathbf{b}\, d\tau. \tag{67}$$

Let

$$\mathbb{T} = \int_0^m \mathbf{T}_k \mathbf{T}_k^\top dt \tag{68}$$

then, let

$$s = 2{\mathbf{p}^*}^\top \mathbb{T} V_2 \tag{69}$$

and

$$S = V_2^\top \mathbb{T} V_2. \tag{70}$$

Then the optimization program can be written as

$$\underset{\mathbf{b}}{\operatorname{argmin}}\, s\mathbf{b} + \mathbf{b}^\top S\mathbf{b}. \tag{71}$$

Taking the gradient with respect to $\mathbf{b}$, we have

$$\frac{\partial}{\partial \mathbf{b}} = s + 2S\mathbf{b}. \tag{72}$$

If S is invertible, this can be solved in closed form as

$$\mathbf{b}^* = -\frac{1}{2}S^{-1}s. \tag{73}$$

However, S is not guaranteed to be full rank. The solution can be approximated [2] using $S^\dagger$, the pseudoinverse of S. We then have the approximation

$$\mathbf{b}^* \approx \hat{\mathbf{b}} = -\frac{1}{2}S^\dagger s. \tag{74}$$

Letting $\mathbf{r}^* = \mathbf{p}^* + V_2\hat{\mathbf{b}}$, the polynomial with the k^{th} derivative minimized is

$$P(\tau) = \mathbf{T}^\top \mathbf{r}^*. \tag{75}$$

9 Example

To demonstrate the method described above, consider a polynomial with $m = 3$ segments and of order $n = 6$. We set the endpoint constraints listed in Tab. 1. Note that the location of the endpoint between segment one and segment two is not fixed, but its velocity is fixed to $-3\frac{\text{m}}{\text{s}}$. This illustrates the flexibility of the endpoint constraint method. In total, there are $c = 13$ endpoint constraints on the trajectory. We create the 13×18 matrix D, and the 13×1 vector $\mathbf{a}$ as described in Sec. 7, and solve for $\mathbf{p}^*$. The resulting polynomial is shown in Fig. 2.

With $mn = 18$ and $c = 13$, there are five additional degrees of freedom that reside in $\mathcal{N}(D)$. To illustrate the minimization of the k^{th} derivative as described in Sec. 8, we will minimize the second derivative of $P(\tau)$. The resulting minimum second derivative solution is compared with the minimum norm solution in Fig. 2. The cost $\mathbf{J}_2(\mathbf{p}^*) = 31.97$, and $\mathbf{J}_2(\mathbf{r}^*) = 25.00$. In contrast, $\|\mathbf{p}^*\|_2 = 12.40$ and $\|\mathbf{r}^*\|_2 = 13.02$. This illustrates that while the norm of $\mathbf{r}^*$ is greater than that of $\mathbf{p}^*$, its cost is lower when considering the integral of the polynomial's second derivative as the cost metric.

10 Conclusion

In this paper I have presented a means of optimizing polynomial trajectories using dual approximation and least-squares techniques. This shows that segment endpoint constraints can be met by formulating them as inner products and performing dual approximation.

Table 1: Example trajectory endpoint constraints

Constraint	Interpretation
$P_i(1) = P_{i+1}(0)$	0^{th} derivative continuity
$P_i'(1) = P_{i+1}'(0)$	1^{st} derivative continuity
$P_i''(1) = P_{i+1}''(0)$	2^{nd} derivative continuity
$P_0(0) = 5$	
$P_1(0) = 7$	Fixed endpoint locations
$P_2(1) = 3$	
$P_0'(0) = 5$	
$P_1'(1) = -3$	Fixed endpoint velocities
$P_2'(1) = 0$	
$P_2''(1) = 0$	Fixed endpoint acceleration

The derivatives of the polynomial can be minimized by performing a quadratic optimization over the nullspace of the dual approximation. In future work, I will extend this method to multi-dimensional polynomial trajectories and compare the time complexity to traditional quadratic programming-based trajectory optimization techniques. In addition, I will examine the impact of my choice of polynomial bases on numerical stability and the implications of the invertibility of S on the optimality of the solution $\hat{\mathbf{b}}$.

References

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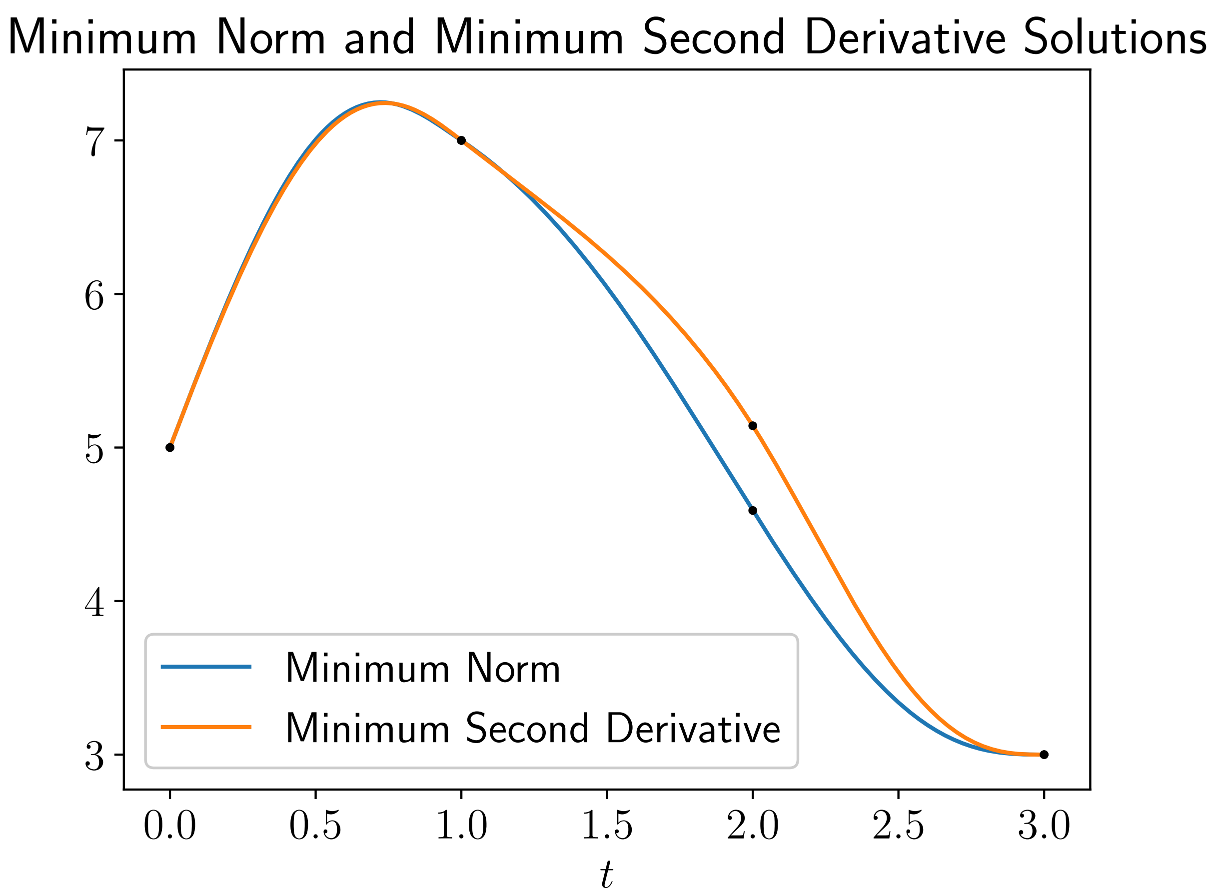

Figure 2: Plot of the minimum norm solution $P(\tau) = \mathbf{T}^{\top}\mathbf{p}^{*}$ and the minimum acceleration solution $P(\tau) = \mathbf{T}^{\top}\mathbf{r}^{*}$. Both curves meet the endpoint constraints listed in Tab. 1.