Visually Following a Ground Target from a Multi-rotor UAV using Geometric Control

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Abstract

This is the abstract.

1 Introduction

2 Mathematical Models

2.1 Preliminaries

Throughout this document, we will use bold face to denote a vector, and a superscript on the vector to denote the coordinate frame where the vector is expressed. For example, $\mathbf{v}^r \in \mathbb{R}^3$ denotes the vector $\mathbf{v}$ expressed relative to coordinate frame $\mathcal{F}^r$. The rotation matrix that transforms vectors expressed relative to frame $\mathcal{F}^r$ into vectors expressed relative to frame $\mathcal{F}^s$ is denoted $R_r^s \in SO(3)$.

Let $\boldsymbol{\omega}_{r/s}$ denote the angular velocity of frame $\mathcal{F}^r$ relative to frame $\mathcal{F}^s$. Then the kinematic equations of motion for R_r^s is given by

$$\dot{R}_r^s = R_r^s(\boldsymbol{\omega}_{r/s}^r)^\wedge, \tag{1}$$

where the *wedge* operator is defined as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}^\wedge \triangleq \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

Alternatively, the *vee* operator is defined as

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}^\vee \triangleq \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

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Recall that the cross product is invariant under rotation, or in other words

$$R(\mathbf{v} \times \mathbf{w}) = (R\mathbf{v}) \times (R\mathbf{w}),$$

when $R \in SO(3)$. Therefore

$$\begin{aligned} (R\mathbf{v})^\wedge \mathbf{w} &= (R\mathbf{v}) \times \mathbf{w} \\ &= R\left(\mathbf{v} \times (R^\top \mathbf{w})\right) \\ &= R\mathbf{v}^\wedge R^\top \mathbf{w}, \end{aligned}$$

which implies that

$$(R\mathbf{v})^\wedge = R\mathbf{v}^\wedge R^\top. \tag{2}$$

Noting that $\boldsymbol{\omega}^s_{r/s} = R^s_r \boldsymbol{\omega}^r_{r/s}$, then from Equations (1) and (2) we have that

$$\begin{aligned} \dot{R}^s_r &= R^s_r (R^s_r \boldsymbol{\omega}^s_{r/s})^\wedge \\ &= R^s_r ((R^s_r)^\top \boldsymbol{\omega}^s_{r/s})^\wedge \\ &= R^s_r (R^s_r)^\top (\boldsymbol{\omega}^s_{r/s})^\wedge R^s_r \\ &= (\boldsymbol{\omega}^s_{r/s})^\wedge R^s_r \end{aligned}$$

The Frobenius norm of matrix $A \in \mathbb{R}^{n \times n}$ is defined as

$$\|A\| \triangleq \sqrt{tr[A^\top A]},$$

where $tr(M)$ is the trace of M. We will have need of the following properties of the trace:

T.1. $tr\left[A^\top\right] = tr\,[A]$,

T.2. $tr\,[AB] = tr\,[BA]$,

T.3. $tr\,[\alpha A + \beta B] = \alpha tr\,[A] + \beta tr\,[B]$ where α and β are scalars,

T.4. $tr\,[AB] = 0$ when A is a symmetric matrix and B is a skew-symmetric matrix,

T.5. $tr\,[a^\wedge b^\wedge] = -2a^\top b$, where $a, b \in \mathbb{R}^3$.

When $\tilde{R} \in SO(3)$, properties T.1 and T.3 imply that if $V \triangleq \frac{1}{2}\left\|I - \tilde{R}\right\|^2$, then

$$\begin{aligned} V &= \frac{1}{2}\left\|I - \tilde{R}\right\|^2 \\ &= \frac{1}{2} tr\left[(I - \tilde{R})^\top (I - \tilde{R})\right] \\ &= \frac{1}{2} tr\left[I - \tilde{R} - \tilde{R}^\top + \tilde{R}^\top \tilde{R}\right] \\ &= tr\left[I - \tilde{R}\right]. \end{aligned}$$

Furthermore, if $\tilde{R} = \breve{R}R^\top$, where $\breve{R}, R \in SO(3)$ and

$$\dot{R} = \boldsymbol{\omega}^\wedge R$$
$$\dot{\breve{R}} = \breve{\boldsymbol{\omega}}^\wedge \breve{R},$$

then

$$\begin{aligned}
\dot{V} &= -tr\left[\dot{\tilde{R}}\right] \\
&= -tr\left[\dot{\breve{R}}R^\top + \breve{R}\dot{R}^\top\right] \\
&= -tr\left[\breve{\boldsymbol{\omega}}^\wedge \breve{R}R^\top + \breve{R}R^\top(\boldsymbol{\omega}^\top)^\wedge\right] \\
&= -tr\left[\breve{\boldsymbol{\omega}}^\wedge \tilde{R} - \tilde{R}\boldsymbol{\omega}^\wedge\right]. \qquad (3)
\end{aligned}$$

Define the symetric and skew-symmetric operators as

$$\mathbb{P}_s(A) \triangleq \frac{1}{2}(A + A^\top)$$
$$\mathbb{P}_a(A) \triangleq \frac{1}{2}(A - A^\top),$$

and note that $A = \mathbb{P}_s(A) + \mathbb{P}_a(A)$. Equation (3) then become

$$\begin{aligned}
\dot{V} &= -tr\left[\breve{\boldsymbol{\omega}}^\wedge(\mathbb{P}_s(\tilde{R}) - \mathbb{P}_a(\tilde{R})) + (\mathbb{P}_s(\tilde{R}) + \mathbb{P}_a(\tilde{R}))\boldsymbol{\omega}^\wedge\right] \\
&= -tr\left[\breve{\boldsymbol{\omega}}^\wedge \mathbb{P}_a(\tilde{R}) - \mathbb{P}_a(\tilde{R})\boldsymbol{\omega}^\wedge\right] \\
&= -tr\left[\mathbb{P}_a(\tilde{R})(\boldsymbol{\omega} - \breve{\boldsymbol{\omega}})^\wedge\right] \qquad (4)
\end{aligned}$$

where the second line follows from property T.4, and the third line follows from property T.2.

2.2 Equations of Motion

Let $\mathbf{p}^i_{b/i}$ denote the position of the vehicle/UAV with respect to the inertial frame, as expressed in inertial coordinate, and let $\mathbf{v}^i_{b/i}$ denote the velocity of the UAV with respect to the inertial frame, expressed in inertial coordinates. Then the translational kinematics are given by

$$\dot{\mathbf{p}}^i_{b/i} = \mathbf{v}^i_{b/i}.$$

Let $R^i_b \in SO(3)$ denote the rotation matrix from body coordinates to inertial coordinates, and let $\boldsymbol{\omega}^b_{b/i}$ denote the angular velocity of the UAV with respect to inertial coordinates, expressed in the body frame of the UAV. Then the rotational kinematics are given by

$$\dot{R}^i_b = R^i_b\left(\boldsymbol{\omega}^b_{b/i}\right)^\wedge.$$

Let m and J denote the mass and inertia of the vehicle respectively, g the gravitational force exerted on a unit mass at sea level, $T \in \mathbb{R}^+$ the total thrust on the UAV, $\mathbf{f}_{\text{drag}}$ the drag force on the vehicle, and $M \in \mathbb{R}^3$ the applied moments, then Newton's law implies the following dynamic equations of motion:

$$\begin{aligned} m\dot{\mathbf{v}}_{b/i}^i &= mg\mathbf{e}_3^i + \mathbf{f}_{\text{drag}}^i + TR_b^i\mathbf{e}_3^b \\ J\dot{\boldsymbol{\omega}}_{b/i}^b &= -\boldsymbol{\omega}_{b/i}^b \times J\boldsymbol{\omega}_{b/i}^b + \mathbf{M}^b, \end{aligned}$$

where $\mathbf{e}_i^*$ is the three dimensional column vector with one in the i^{th} row and zero in the other elements, and the superscript is added to emphasize the frame in which the unit vector is defined. Note that $mg\mathbf{e}_3^i$ is the gravity term that is fixed in the inertial frame and points towards the center of the earth, and that $TR_b^i\mathbf{e}_3^b$ is the thrust vector that is fixed in the UAV body frame.

As shown in [**?**], the drag term is most easily described in the multirotor body frame as

$$\mathbf{f}_{\text{drag}}^b = \mu\Pi_{\mathbf{e}_3}\mathbf{v}_{b/i}^b,$$

where μ is the drag coefficient and

$$\Pi_{\mathbf{x}} \triangleq I - \mathbf{x}\mathbf{x}^\top,$$

is the projection matrix onto the two dimensional subspace that is orthogonal to the unit vector $\mathbf{x} \in \mathbb{R}^3$. Therefore, the drag force always acts orthogonal to the thrust vector and is contained in the body fixed plane $x - y$ plane of the multirotor. The drag force in inertial coordinates is given by

$$\mathbf{f}_{\text{drag}}^i = \mu R_b^i \Pi_{\mathbf{e}_3}(R_b^i)^\top \mathbf{v}_{b/i}^i.$$

We will assume in this paper that the camera is mounted on a gimbal and that the center of the camera and gimbal frame are both located at the center of the UAV body frame, which coincides with its center of mass. Let $\boldsymbol{\ell}_o$ denote the unit vector that is aligned with the optical axis of the camera, and let $R_c^b \in SO(3)$ denote the rotation matrix from the camera frame to the body frame. Then the optical axis in the body frame is given by

$$\boldsymbol{\ell}_o^b = R_c^b\mathbf{e}_3^c.$$

We will assume the ability to command the angular rates of the gimbal with respect to the body. Therefore, the kinematics of the gimbal are given by

$$\dot{R}_c^b = R_c^b\left(\boldsymbol{\omega}_{c/b}^c\right)^\wedge$$

where $\boldsymbol{\omega}_{c/b}^c$ are the commanded angular rates of the gimbal.

In summary, the equations of motion for the multirotor with gimbal are given by

$$\dot{\mathbf{p}}^i_{b/i} = \mathbf{v}^i_{b/i} \tag{5}$$
$$m\dot{\mathbf{v}}^i_{b/i} = mg\mathbf{e}^i_3 + \mu R^i_b \Pi_{\mathbf{e}_3} (R^i_b)^\top \mathbf{v}^i_{b/i} + T R^i_b \mathbf{e}^b_3 \tag{6}$$
$$\dot{R}^i_b = R^i_b \left(\boldsymbol{\omega}^b_{b/i}\right)^\wedge \tag{7}$$
$$J\dot{\boldsymbol{\omega}}^b_{b/i} = -\boldsymbol{\omega}^b_{b/i} \times J\boldsymbol{\omega}^b_{b/i} + \mathbf{M}^b \tag{8}$$
$$\dot{R}^b_c = R^b_c \left(\boldsymbol{\omega}^c_{c/b}\right)^\wedge. \tag{9}$$

Note that if the UAV velocity vector is expressed in body coordinates, then the translational equations of motion become

$$\begin{aligned}
\dot{\mathbf{p}}^i_{b/i} &= R^i_b \mathbf{v}^b_{b/i} \\
m\dot{\mathbf{v}}^b_{b/i} &= -\boldsymbol{\omega}^b_{b/i} \times \mathbf{v}^b_{b/i} + mg(R^i_b)^\top \mathbf{e}^i_3 + \mu \Pi_{\mathbf{e}_3} \mathbf{v}^b_{b/i} + T\mathbf{e}^b_3.
\end{aligned}$$

Let $\mathbf{p}_{t/i} \in \mathbb{R}^3$ and $\mathbf{v}_{t/i} \in \mathbb{R}^3$ be the position and velocity of the target relative to the inertial frame. We will assume a constant velocity model where

$$\begin{aligned}
\dot{\mathbf{p}}^i_{t/i} &= \mathbf{v}^i_{t/i} \\
\dot{\mathbf{v}}^i_{t/i} &= 0.
\end{aligned}$$

The camera measures the normalized line-of-sight vector in camera coordinate

$$\boldsymbol{\ell}^c_{t/c} = \frac{\mathbf{p}^c_{t/i} - \mathbf{p}^c_{b/i}}{\left\|\mathbf{p}_{t/i} - \mathbf{p}_{b/i}\right\|}$$

3 The Body-Level Frame

The target-following problem will be cast in the body-level frame. The basic idea is that the body-level frame is the un-rolled and un-pitched body frame. The heading direction for the body frame and the body-level frame will be identical, but the z-axis of the body level frame will always point down along the gravity vector. Letting ℓ denote the body-level frame, we have that

$$R^i_b = R^i_\ell R^\ell_b,$$

or

$$R^i_\ell = R^i_b (R^\ell_b)^\top.$$

To make things concrete, if ϕ, θ, and ψ are the roll, pitch, and yaw Euler angles, then

$$\begin{aligned}
R^i_b &= R_z(\psi) R_y(\theta) R_x(\phi) \\
&\triangleq \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.
\end{aligned}$$

In this case $R^i_\ell = R_z(\psi)$ and $R^\ell_b = R_y(\theta) R_x(\phi)$.

3.1 Equations of Motion of the Body-Level Frame

Since the origin of the body-level frame is coincident with the origin of the body frame, we have that

$$
\begin{aligned}
\mathbf{p}_{\ell/i} &= \mathbf{p}_{b/i} \\
\mathbf{v}_{\ell/i} &= \mathbf{v}_{b/i}.
\end{aligned}
$$

Therefore, using Equation (5) and (6) we get that the translational equations of motion in the body-level frame are given by

$$
\begin{aligned}
\dot{\mathbf{p}}^i_{\ell/i} &= \mathbf{v}^i_{\ell/i} \\
m\dot{\mathbf{v}}^i_{\ell/i} &= mg\mathbf{e}^i_3 + \mu R^i_\ell R^\ell_b \Pi_{\mathbf{e}_3} (R^\ell_b)^\top (R^i_\ell)^\top \mathbf{v}^i_{\ell/i} + T R^i_\ell R^\ell_b \mathbf{e}^b_3.
\end{aligned}
$$

The angular velocity of the body, resolved in the body-level frame, is given by

$$
\boldsymbol{\omega}^\ell_{b/i} = R^\ell_b \boldsymbol{\omega}^b_{b/i}.
$$

Since the body-level frame only rotates about its own $\mathbf{e}_3$-axis, we have that

$$
\boldsymbol{\omega}^\ell_{\ell/i} = \mathbf{e}_3\mathbf{e}_3^\top \boldsymbol{\omega}^\ell_{b/i} = \mathbf{e}_3\mathbf{e}_3^\top R^\ell_b \boldsymbol{\omega}^b_{b/i}. \tag{10}
$$

From Equation (1), the kinematic equation of motion for R^i_ℓ is given by

$$
\dot{R}^i_\ell = R^i_\ell (\boldsymbol{\omega}^\ell_{\ell/i})^\wedge.
$$

Differentiating Equation (10) gives

$$
\dot{\boldsymbol{\omega}}^\ell_{\ell/i} = \mathbf{e}_3\mathbf{e}_3^\top \left[R^\ell_b \dot{\boldsymbol{\omega}}^b_{b/i} + \dot{R}^\ell_b \boldsymbol{\omega}^b_{b/i} \right].
$$

From Equation (1) we have that

$$
\dot{R}^\ell_b = R^\ell_b (\boldsymbol{\omega}^b_{b/\ell})^\wedge,
$$

where

$$
\boldsymbol{\omega}^b_{b/\ell} = \boldsymbol{\omega}^b_{b/i} - \boldsymbol{\omega}^b_{\ell/i},
$$

which implies that

$$
\begin{aligned}
\dot{\boldsymbol{\omega}}^\ell_{\ell/i} &= \mathbf{e}_3\mathbf{e}_3^\top \left[R^\ell_b \dot{\boldsymbol{\omega}}^b_{b/i} + R^\ell_b (\boldsymbol{\omega}^b_{b/i} - \boldsymbol{\omega}^b_{\ell/i})^\wedge \boldsymbol{\omega}^b_{b/i} \right] \\
&= \mathbf{e}_3\mathbf{e}_3^\top \left[R^\ell_b \dot{\boldsymbol{\omega}}^b_{b/i} - R^\ell_b (\boldsymbol{\omega}^b_{\ell/i})^\wedge \boldsymbol{\omega}^b_{b/i} \right] \\
&= \mathbf{e}_3\mathbf{e}_3^\top R^\ell_b \left[J^{-1}\mathbf{M}^b - J^{-1}(\boldsymbol{\omega}^b_{b/i})^\wedge (J\boldsymbol{\omega}^b_{b/i}) + (\boldsymbol{\omega}^b_{b/i})^\wedge \boldsymbol{\omega}^b_{\ell/i} \right]
\end{aligned}
$$

Using Equation (1) and the fact that $(Rv)^\wedge = Rv^\wedge R^\top$ we get that

$$
\dot{R}^s_r = (\boldsymbol{\omega}^s_{r/s})^\wedge R^s_r,
$$

which implies that

$$\dot{R}_b^\ell = (\boldsymbol{\omega}_{b/\ell}^\ell)^\wedge R_b^\ell.$$

Using the facts that $\boldsymbol{\omega}_{b/\ell}^\ell = \boldsymbol{\omega}_{b/i}^\ell - \boldsymbol{\omega}_{\ell/i}^\ell$ and $\boldsymbol{\omega}_{b/i}^\ell = R_b^\ell \boldsymbol{\omega}_{b/i}^b$, we have that

$$\begin{aligned}
\dot{\boldsymbol{\omega}}_{b/\ell}^\ell &= \dot{\boldsymbol{\omega}}_{b/i}^\ell - \dot{\boldsymbol{\omega}}_{\ell/i}^\ell \\
&= R_b^\ell \dot{\boldsymbol{\omega}}_{b/i}^b + R_b^\ell (\boldsymbol{\omega}_{b/\ell}^b)^\wedge \boldsymbol{\omega}_{b/i}^b - \dot{\boldsymbol{\omega}}_{\ell/i}^\ell \\
&= R_b^\ell \left[J^{-1}\mathbf{M}^b - J^{-1}(\boldsymbol{\omega}_{b/i}^b)^\wedge (J\boldsymbol{\omega}_{b/i}^b) + (\boldsymbol{\omega}_{b/i}^b)^\wedge \boldsymbol{\omega}_{\ell/i}^b \right] - \dot{\boldsymbol{\omega}}_{\ell/i}^\ell \\
&= (I - \mathbf{e}_3\mathbf{e}_3^\top) R_b^\ell \left[J^{-1}\mathbf{M}^b - J^{-1}(\boldsymbol{\omega}_{b/i}^b)^\wedge (J\boldsymbol{\omega}_{b/i}^b) + (\boldsymbol{\omega}_{b/i}^b)^\wedge \boldsymbol{\omega}_{\ell/i}^b \right].
\end{aligned}$$

Summarizing, the dynamics in the body-level frame are given by

$$\begin{aligned}
\dot{\mathbf{p}}_{\ell/i}^i &= \mathbf{v}_{\ell/i}^i && (11) \\
m\dot{\mathbf{v}}_{\ell/i}^i &= mg\mathbf{e}_3^i + \mu R_\ell^i R_b^\ell \Pi_{\mathbf{e}_3} (R_b^\ell)^\top (R_\ell^i)^\top \mathbf{v}_{\ell/i}^i + T R_\ell^i R_b^\ell \mathbf{e}_3^b && (12) \\
\dot{R}_\ell^i &= R_\ell^i (\boldsymbol{\omega}_{\ell/i}^\ell)^\wedge && (13) \\
\dot{\boldsymbol{\omega}}_{\ell/i}^\ell &= \mathbf{e}_3\mathbf{e}_3^\top R_b^\ell \left[J^{-1}\mathbf{M}^b - J^{-1}(\boldsymbol{\omega}_{b/i}^b)^\wedge (J\boldsymbol{\omega}_{b/i}^b) + (\boldsymbol{\omega}_{b/i}^b)^\wedge \boldsymbol{\omega}_{\ell/i}^b \right] && (14) \\
\dot{R}_b^\ell &= R_b^\ell (\boldsymbol{\omega}_{b/\ell}^b)^\wedge && (15) \\
\dot{\boldsymbol{\omega}}_{b/\ell}^\ell &= \Pi_{\mathbf{e}_3} R_b^\ell \left[J^{-1}\mathbf{M}^b - J^{-1}(\boldsymbol{\omega}_{b/i}^b)^\wedge (J\boldsymbol{\omega}_{b/i}^b) + (\boldsymbol{\omega}_{b/i}^b)^\wedge \boldsymbol{\omega}_{\ell/i}^b \right]. && (16)
\end{aligned}$$

3.2 Feedback Projecting Control

In this section we develop a feedback linearizing control that will facilitate tracking in the local level frame. The first step is to let

$$\mathbf{M}^b = (\boldsymbol{\omega}_{b/i}^b)^\wedge (J\boldsymbol{\omega}_{b/i}^b - J(\boldsymbol{\omega}_{b/i}^b)^\wedge \boldsymbol{\omega}_{\ell/i}^b + J(R_b^\ell)^\top \begin{pmatrix} u_\phi \\ u_\theta \\ u_\psi \end{pmatrix}.$$

Substituting into Equations (14) and (16) gives

$$\begin{aligned}
\dot{\boldsymbol{\omega}}_{\ell/i}^\ell &= \mathbf{e}_3 u_\psi \\
\dot{\boldsymbol{\omega}}_{b/\ell}^\ell &= \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} \begin{pmatrix} u_\phi \\ u_\theta \end{pmatrix} \triangleq E_{12} u_{12}.
\end{aligned}$$

Ignoring the drag term, i.e., setting $\mu = 0$, Equations (11) and (12) become

$$\ddot{\mathbf{p}}_{\ell/i}^i = g\mathbf{e}_3 + \frac{T}{m} R_\ell^i R_b^\ell \mathbf{e}_3.$$

Throughout the paper, we will use the "breve" mark to denote a desired quantity. Accordingly, by selecting $\breve{\mathbf{u}}_{\ell/i}^i$ as the desired acceleration in the body level frame, $\breve{\mathbf{R}}_b^\ell \in SO(3)$ as the desired rotation from body to body-level frames, and $\breve{T} \in \mathbb{R}$ as the desired thrust, we get that

$$\frac{\breve{T}}{m}\breve{R}_b^\ell \mathbf{e}_3 = (R_\ell^i)^\top \left(\breve{\mathbf{u}}_{\ell/i}^i - g\mathbf{e}_3\right).$$

Letting

$$\breve{R}_b^\ell = \begin{bmatrix}\breve{\mathbf{r}}_1, & \breve{\mathbf{r}}_2, & \breve{\mathbf{r}}_3\end{bmatrix}$$

we get that

$$\begin{aligned}\breve{T} &= m\left\|\breve{\mathbf{u}}_{\ell/i}^i - g\mathbf{e}_3\right\| \\ \breve{\mathbf{r}}_3 &= \frac{(R_\ell^i)^\top(\breve{\mathbf{u}}_{\ell/i}^i - g\mathbf{e}_3)}{\left\|\breve{\mathbf{u}}_{\ell/i}^i - g\mathbf{e}_3\right\|}.\end{aligned}$$

Since R_b^ℓ represents only the roll and pitch angles of the body, the first column of $\breve{R}_b^\ell$ is defined so that it is in the $x - z$ plane of the local-level frame, i.e., perpendicular to $\mathbf{e}_2$ and a 90 degree rotation of $\breve{\mathbf{r}}_3$. Therefore

$$\breve{\mathbf{r}}_1 = \frac{R_y(\frac{\pi}{2})(I - \mathbf{e}_2\mathbf{e}_2^\top)\breve{\mathbf{r}}_3}{\left\|(I - \mathbf{e}_2\mathbf{e}_2^\top)\breve{\mathbf{r}}_3\right\|}.$$

The second column of $\breve{R}_b^\ell$ is selected to form a right handed coordinate system as

$$\breve{\mathbf{r}}_2 = \breve{\mathbf{r}}_3 \times \breve{\mathbf{r}}_1.$$

Note that since

$$R_y(\frac{\pi}{2})(I - \mathbf{e}_2\mathbf{e}_2^\top) = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},$$

then if $\breve{\mathbf{r}}_3 = (a, b, c)^\top$, then $\breve{\mathbf{r}}_1 = (c, 0, -a)^\top/\sqrt{a^2 + c^2}$, and $\breve{\mathbf{r}}_2 = (-ab, a^2 + c^2, -bc)^\top/\sqrt{a^2 + c^2}$, which implies that

$$\breve{R}_b^\ell = \begin{pmatrix}\frac{c}{\alpha} & \frac{-ab}{\alpha} & a\\ 0 & \alpha & b\\ \frac{-a}{\alpha} & \frac{-bc}{\alpha} & c\end{pmatrix},$$

where $\alpha = \sqrt{a^2 + c^2}$.

Let $\dot{\breve{R}}_b^\ell$ be the time derivative of $\breve{R}_b^\ell$, which is assumed to be computed numerically. Then since $\dot{\breve{R}}_b^\ell = (\breve{\boldsymbol{\omega}}_{b/\ell}^\ell)^\wedge \breve{R}_b^\ell$ we have that

$$\breve{\boldsymbol{\omega}}_{b/\ell}^\ell = \left[\dot{\breve{R}}_b^\ell(\breve{R}_b^\ell)^\top\right]^\vee.$$

Define the Lyapunov function

$$
\begin{aligned}
V &= \frac{1}{2}\left\| I - \breve{R}_b^\ell (R_b^\ell)^\top \right\|^2 + \frac{1}{2}\left\| \boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right\|^2 \\
&= tr\left[I - \breve{R}_b^\ell (R_b^\ell)^\top \right] + \frac{1}{2}\left\| \boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right\|^2.
\end{aligned}
$$

Then from Equation (4), differentiation with respect to time gives

$$
\begin{aligned}
\dot{V} &= -tr\left[\mathbb{P}_a\left(\breve{R}_b^\ell (R_b^\ell)^\top \right)\left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right)^\wedge \right] + \left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right)^\top \left(\dot{\boldsymbol{\omega}}_{b/\ell}^\ell - \dot{\breve{\boldsymbol{\omega}}}_{b/\ell}^\ell \right) \\
&= 2\left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right)^\top \left(\mathbb{P}_a(\breve{R}_b^\ell (R_b^\ell)^\top) \right)^\vee + \left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right)^\top \left(\dot{\boldsymbol{\omega}}_{b/\ell}^\ell - \dot{\breve{\boldsymbol{\omega}}}_{b/\ell}^\ell \right) \\
&= \left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right)^\top \left(E_{12}u_{12} - \dot{\breve{\boldsymbol{\omega}}}_{b/\ell}^\ell + 2\left(\mathbb{P}_a(\breve{R}_b^\ell (R_b^\ell)^\top) \right)^\vee \right).
\end{aligned}
$$

Therefore, select

$$
u_{12} = E_{12}^\top \left[\dot{\breve{\boldsymbol{\omega}}}_{b/\ell}^\ell - 2\left(\mathbb{P}_a(\breve{R}_b^\ell (R_b^\ell)^\top) \right)^\vee - K_d\left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right) \right],
$$

and note that $E_{12}E_{12}^\top = \Pi_{\mathbf{e}_3}$ to get

$$
\begin{aligned}
\dot{V} = &-\left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right)^\top \Pi_{\mathbf{e}_3} K_d \left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right) \\
&+ \left(\boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell \right)^\top \mathbf{e}_3\mathbf{e}_3^\top \left(\dot{\breve{\boldsymbol{\omega}}}_{b/\ell}^\ell - 2\left(\mathbb{P}_a(\breve{R}_b^\ell (R_b^\ell)^\top) \right)^\vee \right).
\end{aligned}
$$

RWB: Need to show that second term on RHS is zero

RWB: Need to revise the stuff below to include Π_{e_3}.

Define $\tilde{R} = \breve{R}_b^\ell (R_b^\ell)^\top$ and $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}_{b/\ell}^\ell - \breve{\boldsymbol{\omega}}_{b/\ell}^\ell$, and define $E = \{(\tilde{R}, \tilde{\boldsymbol{\omega}}) | \tilde{\boldsymbol{\omega}} = 0\}$, and let M be the largest invariant set in E. Then for all trajectories in M we have that

$$
\frac{d}{dt}\left\| I - \tilde{R} \right\|^2 = -tr\left[\mathbb{P}_a(\tilde{R})\tilde{\boldsymbol{\omega}}^\wedge \right] \equiv 0,
$$

which implies that $\tilde{R}$ is a constant matrix. Therefore for all trajectories in M

$$
\begin{aligned}
&\dot{\tilde{R}} = \tilde{R}(\boldsymbol{\omega}_{b/\ell}^b)^\wedge - (\breve{\boldsymbol{\omega}}_{b/\ell}^b)^\wedge \tilde{R} = 0 \\
\Longrightarrow\ &\tilde{R}(\boldsymbol{\omega}_{b/\ell}^b)^\wedge = (\breve{\boldsymbol{\omega}}_{b/\ell}^b)^\wedge \tilde{R} \\
\Longrightarrow\ &\tilde{R}(\boldsymbol{\omega}_{b/\ell}^b)^\wedge \tilde{R}^\top = (\breve{\boldsymbol{\omega}}_{b/\ell}^b)^\wedge \\
\Longrightarrow\ &(\tilde{R}\boldsymbol{\omega}_{b/\ell}^b)^\wedge = (\breve{\boldsymbol{\omega}}_{b/\ell}^b)^\wedge \\
\Longrightarrow\ &\tilde{R}\boldsymbol{\omega}_{b/\ell}^b = \breve{\boldsymbol{\omega}}_{b/\ell}^b,
\end{aligned}
$$

but since $\boldsymbol{\omega}_{b/\ell}^b = \breve{\boldsymbol{\omega}}_{b/\ell}^b$ on M, it must be that $\tilde{R} = I$ on M. Therefore, by the LaSalle invariance principle, $R_b^\ell \to \breve{R}_b^\ell$ and $\boldsymbol{\omega}_{b/\ell}^b \to \breve{\boldsymbol{\omega}}_{b/\ell}^b$.

RWB: The goal is to simplify the dynamics to the following:

$$\ddot{\mathbf{p}}^i_{\ell/i} = \mathbf{u}^i_{\ell/i}$$
$$\dot{R}^i_\ell = R^i_\ell(\boldsymbol{\omega}^\ell_{\ell/i})^\wedge$$
$$\dot{\boldsymbol{\omega}}^\ell_{\ell/i} = \mathbf{e}_3 u_\psi.$$

4 Target Following in 2D

We begin with the simplified scenario of following a target in a two-dimensional world. The following scenario is shown in Figure 1.

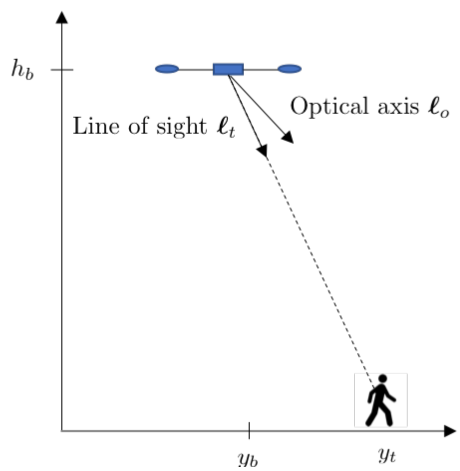


Figure 1: The following scenario in two dimensions.

We will assume in this section that the body-level dynamics of the camera are given by

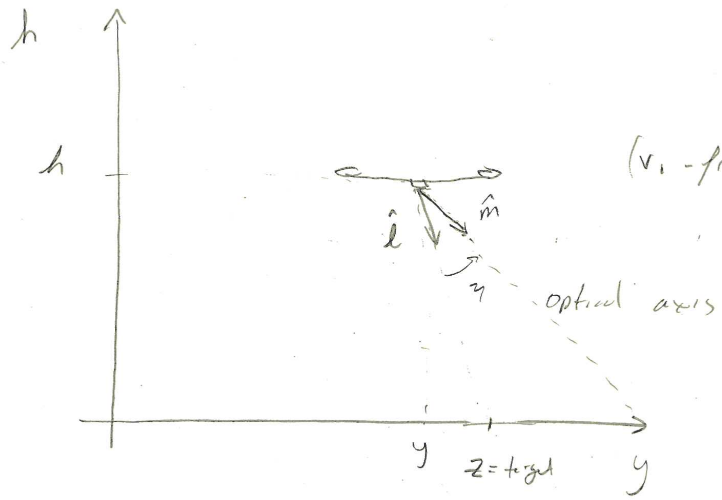
$$\ddot{y}_b = u_1$$
$$\ddot{h}_b = u_2,$$

where the height above ground h_b is in general unknown. We assume that the optical axis $\boldsymbol{\ell}_o$ is fixed in the body level frame, and that the camera measures

$$\boldsymbol{\ell}_t = \frac{\mathbf{p}_{t/\ell}}{\left\|\mathbf{p}_{t/\ell}\right\|}$$

where $\mathbf{p}_{t/\ell} = \left(y_t - y_b, -h_b\right)^\top$. The controller also has access to $\dot{\boldsymbol{\ell}}_t$ and $\ddot{\boldsymbol{\ell}}_t$ by numerically differentiating $\boldsymbol{\ell}_t$.

use "body-level" frame $\mathbf{p}_{\ell/i}$



Assume dynamics

$$\ddot{y} = u_1$$
$$\ddot{h} = u_2$$

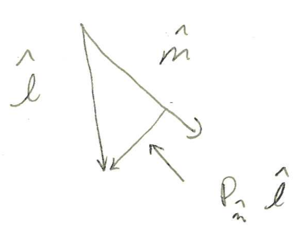
Assume 1. - h is unknown

2. - controller measures $\hat{\ell}$ (unit vector) and $\dot{\hat{\ell}}$ and $\ddot{\hat{\ell}}$ (by numerically differentiating $\hat{\ell}$)

3. $\hat{m}$ is fixed in v_1-frame

Define the projection matrix (onto the null space of $\hat{m}$)

$$P_{\hat{m}} = \left(I - \hat{m}\hat{m}^T\right)$$



$$P_{\hat{m}}\hat{\ell} = (I - \hat{m}\hat{m}^T)\hat{\ell}$$
$$= \hat{\ell} - (\hat{\ell}^T\hat{m})\,\hat{m}$$

The idea is to drive $P_{\hat{m}} \hat{\ell}$ to zero

- we really only need to drive the 1st component to zero (horizontal direction)

Define $\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then the horizontal error is given by

$$e_x = \hat{e}_1^T P_{\hat{m}} \hat{\ell} \tag{2.1}$$

Note that since $\hat{m}$, $\hat{e}_1$ are fixed and known, and since $\hat{\ell}$ is measured, e_x is a measurable quantity.

Also since $\dot{\hat{\ell}}$ can be approximated by numerical differentiation $\dot{e}_x = \hat{e}_1^T P_{\hat{m}} \dot{\hat{\ell}}$ is also measurable.

The line of sight vector is

$$\ell = \underbrace{\begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}}_{\text{target position}} - \underbrace{\begin{pmatrix} y \\ h \\ 0 \end{pmatrix}}_{\text{UAV position}}$$

and note that

$$\dot{\ell} = \begin{pmatrix} \dot{z} - \dot{y} \\ \dot{h} \\ 0 \end{pmatrix} \quad \text{and}$$

$$\ddot{\ell} = \begin{pmatrix} \ddot{z} - \ddot{y} \\ \ddot{h} \\ 0 \end{pmatrix}$$

(3)

Assuming that the acceleration of the target is

$$\ddot{z} = a_t, \quad \text{then}$$

$$\ddot{\ell} = \begin{pmatrix} a_t - u_1 \\ u_2 \\ 0 \end{pmatrix}$$

let $L = \|\ell\|$

Then we measure

$$\hat{\ell} = \frac{\ell}{L}$$

Differentiating gives

$$\dot{\hat{\ell}} = \frac{L\dot{\ell} - \ell\dot{L}}{L^2} = \frac{\dot{\ell}}{L} - \left(\frac{\ell}{L}\right)\left(\frac{\dot{L}}{L}\right) = \frac{\dot{\ell}}{L} - \hat{\ell}\left(\frac{\dot{L}}{L}\right) \qquad 3.1$$

Differentiating again gives

$$\ddot{\hat{\ell}} = \frac{L\ddot{\ell} - \dot{\ell}\dot{L}}{L^2} - \dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{L\ddot{L} - (\dot{L})^2}{L^2}\right)$$

$$= \frac{\ddot{\ell}}{L} - \left(\frac{\dot{\ell}}{L}\right)\left(\frac{\dot{L}}{L}\right) - \dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{\ddot{L}}{L}\right) + \hat{\ell}\left(\frac{\dot{L}}{L}\right)^2$$

plugging in for $\frac{\dot{\ell}}{L} = \dot{\hat{\ell}} + \hat{\ell}\left(\frac{\dot{L}}{L}\right)$ from (3.1) gives

$$\ddot{\hat{\ell}} = \frac{\ddot{\ell}}{L} - \left(\dot{\hat{\ell}} + \hat{\ell}\left(\frac{\dot{L}}{L}\right)\right)\left(\frac{\dot{L}}{L}\right) - \dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{\ddot{L}}{L}\right) + \hat{\ell}\left(\frac{\dot{L}}{L}\right)^2$$

$$= \frac{\ddot{\ell}}{L} - 2\dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{\ddot{L}}{L}\right)$$

Differentiating (2.1) twice gives

$$\ddot{e}_x = \hat{e}_1^T P_{\hat{m}} \ddot{\hat{\ell}}$$

$$= \hat{e}_1^T P_{\hat{m}} \left[\frac{\ddot{\ell}}{L} - 2\dot{\hat{\ell}}\left(\frac{\dot{L}}{L}\right) - \hat{\ell}\left(\frac{\ddot{L}}{L}\right)\right]$$

$$= \frac{1}{L}\left(\hat{e}_1^T P_{\hat{m}} \ddot{\ell}\right) + \left(\frac{\dot{L}}{L}\right)\left(-2\hat{e}_1^T P_{\hat{m}} \dot{\hat{\ell}}\right) + \left(\frac{\ddot{L}}{L}\right)\left(-\hat{e}_1^T P_{\hat{m}} \hat{\ell}\right)$$

Note that

$$\hat{e}_1^T P_{\hat{m}} \ddot{\ell} = (1\ 0\ 0)\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}m_1\\m_2\\0\end{pmatrix}(m_1\ m_2\ 0)\right]\begin{pmatrix}a_t-u_1\\u_2\\0\end{pmatrix}$$

$$= (1\ 0\ 0)\begin{pmatrix}1-m_1^2 & -m_1m_2 & 0\\ -m_1m_2 & 1-m_2^2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a_t-u_1\\u_2\\0\end{pmatrix}$$

$$= (1-m_1^2\quad -m_1m_2\quad 0)\begin{pmatrix}a_t-u_1\\u_2\\0\end{pmatrix}$$

$$= (1-m_1^2)(a_t-u_1) - m_1m_2u_2$$

Assuming constant altitude, ie $u_2 = 0$, then

$$\hat{e}_1^T P_{\hat{m}} \ddot{\ell} = (1-m_1^2)(a_t-u_1)$$

$\therefore$

$$\ddot{e}_x = \left(\frac{1}{L}\right)(1-m_1^2)(a_t-u_1) + \left(\frac{\dot{L}}{L}\right)(-2\hat{e}_1^T P_{\hat{m}} \dot{\hat{\ell}}) + \left(\frac{\ddot{L}}{L}\right)(-\hat{e}_1^T P_{\hat{m}} \hat{\ell})$$

unknown, unmeasurable.

Define

$$\theta_1 = \frac{1}{L} \qquad \phi_1 = (1-m_1^2)$$

$$\theta_2 = \frac{\dot{L}}{L} \qquad \phi_2 = -2\hat{e}_1^T P_{\hat{m}} \dot{\hat{\ell}}$$

$$\theta_3 = \frac{\ddot{L}}{L} \qquad \phi_3 = -\hat{e}_1^T P_{\hat{m}} \hat{\ell}$$

We will assume that $\theta_1, \theta_2\ \theta_3$ are roughly constant, or slowly varying

Then

$$\ddot{e}_x = \theta_1 (1-m_1^2)(a_t - u_1) + \theta_2 \phi_2 + \theta_3 \phi_3$$

We want $e_x \to 0$.

Define $s = \dot{e}_x + k\, e_x$ where $k > 0$

and note that we can measure s.

Then

$$\dot{s} = \ddot{e}_x + k\,\dot{e}_x$$

$$= \theta_1 (1-m_1^2)(a_t - u_1) + \theta_2 \phi_2 + \theta_3 \phi_3 + k\,\dot{e}_x$$

Assume a constant velocity target, i.e. $a_t = 0$

then

$$\dot{s} = -\theta_1 (1-m_1^2) u_1 + \theta_2 \phi_2 + \theta_3 \phi_3 + k\,\dot{e}_x$$

Let

$$u_1 = \frac{1}{\hat{\theta}_1 (1-m_1^2)} \left[+ \hat{\theta}_2 \phi_2 + \hat{\theta}_3 \phi_3 + k\,\dot{e}_x - \xi \right]$$

Then

$$\dot{s} = -\theta_1 (1-m_1^2) u_1 - \hat{\theta}_1 (1-m_1^2) u_1 + \hat{\theta}_1 (1-m_1^2) u_1 + \theta_2 \phi_2 + \theta_3 \phi_3 + k\,\dot{e}_x$$

$$= -(\theta_1 - \hat{\theta}_1)(1-m_1^2) u_1 + (\theta_2 - \hat{\theta}_2)\phi_2 + (\theta_3 - \hat{\theta}_3)\phi_3 + \xi$$

Define

$$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}, \quad \hat{\Theta} = \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \end{pmatrix}, \quad \tilde{\Theta} = \Theta - \hat{\Theta}, \quad \underline{\Phi} = \begin{pmatrix} -(1-m_1^2) u_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

(6)

Define the Lyapunov equation

$$V = \frac{1}{2} s^2 + \frac{1}{2} \tilde{\Theta}^T \Gamma^{-1} \tilde{\Theta}$$

then
$$\dot{V} = s\dot{s} + \tilde{\Theta}^T \Gamma^{-1} \dot{\tilde{\Theta}}$$

$$= s\left(\tilde{\Theta}^T \Phi + \xi \right) + \tilde{\Theta}^T \Gamma^{-1} \dot{\tilde{\Theta}}$$

Selecting $\xi = -\alpha s = -\alpha(\dot{e}_x + k e_x)$ where $\alpha > 0$

then
$$\dot{V} = -\alpha s^2 + \tilde{\Theta}^T \left[s\Phi + \Gamma^{-1} \dot{\tilde{\Theta}} \right]$$

Now assuming that Θ is constant we get

$$\dot{\tilde{\Theta}} = \dot{\Theta} - \dot{\hat{\Theta}} = -\dot{\hat{\Theta}}$$

$$\Rightarrow \quad \dot{V} = -\alpha s^2 + \tilde{\Theta}^T \left[s\Phi - \Gamma^{-1} \dot{\hat{\Theta}} \right]$$

let
$$\dot{\hat{\Theta}} = s \Gamma \Phi$$

$$\Rightarrow \quad \dot{V} = -\alpha s^2$$

Summarizing

$$u_2 = 0$$

$$u_1 = \frac{1}{\hat{\theta}_1 (1-m_1^2)} \left[\hat{\theta}_2 \phi_2 + \hat{\theta}_3 \phi_3 + k \dot{e}_x - \alpha s \right]$$

$$\dot{\hat{\theta}} = s \Gamma \Phi$$

$$s = (\dot{e}_x + k e_x)$$

$$\Phi = \begin{pmatrix} -(1-m_1^2) u_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

$$e_x = \hat{e}_1^T P_{\hat{m}} \hat{l} \quad , \quad \dot{e}_x = \hat{e}_1^T P_{\hat{m}} \dot{\hat{l}}$$

$$\phi_2 = -2 \hat{e}_1^T P_{\hat{m}} \dot{\hat{l}}$$

$$\phi_3 = -e_x$$

$\hat{l}$, $\dot{\hat{l}}$ - measured

$\hat{m} = \begin{pmatrix} m_1 \\ m_3 \end{pmatrix}$ - known

Control gains: k, α, Γ

Assumptions

- constant velocity target
- unknown but constant altitude
- $\frac{1}{L}$, $\frac{\dot{L}}{L}$, $\frac{\ddot{L}}{L}$ - constant

5 Conclusion

Acknowledgments

Funded by...

References