Indeed, consider the action of the one-parameter subgroup of Z in $H^{\overset{r}{T}}$ given for instance by matrices

$$z = \begin{bmatrix} 1 & & & 0 \\ z & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}. \tag{15}$$

By virtue of eq. (1), $z\overset{k}{x} = (\overset{k}{x}_1, z\overset{k}{x}_1 + \overset{k}{x}_2, \overset{k}{x}_3, \ldots, \overset{k}{x}_n)$, $k = 1, 2, \ldots, r$. This gives

$$(T_z e)_{123\ldots r} = e_{123\ldots r} + z e_{113\ldots r} = e_{123\ldots r}. \tag{16}$$

Using the representation (9) for the component (8) and eq. (16) we conclude that the component (8) is invariant under the action of the subgroup (15). Using eq. (11) one immediately verifies that the action of any other one-parameter subgroup of Z also leaves the component $e_{12\ldots r}$ as well as the component $T_{\boxed{1}\boxed{\cdots}\boxed{1}}$ unchanged. Consequently the component (8) is Z-invariant and therefore the element u_0^m represents the highest vector in $H^{\overset{r}{T}}$. The inspection of the action of the projector Y_λ shows that for any element T in $H^{\overset{r}{T}}$ other than u_0^m one can easily find a one-parameter subgroup of Z, which changes this element. Hence, the linear space $H^{\overset{r}{T}}$ has only one Z-invariant element and therefore is irreducible according to corollary 2 to th. 8.2.2.

For $\overset{k}{x} \in C^n$, $k = 1, 2, \ldots, r$, and $\delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix} \in D$, we have

$$\delta\overset{k}{x} = (\delta_1 \overset{k}{x}_1, \delta_2 \overset{k}{x}_2, \ldots, \delta_n \overset{k}{x}_n).$$

Thus, by virtue of eqs. (8) and (14) the action of T_δ on the component (8) of the tensor T is given by

$$T_\delta u_0^m = \delta_1^{\lambda_1}\delta_2^{\lambda_2}\ldots\delta_n^{\lambda_n} u_0^m. \tag{17}$$

Consequently, by eq. 8.2(18) the highest weight m associated with the irreducible representation $g \to T_g$ in $H^{\overset{r}{T}}$ has the form $m = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. ▼

Clearly, if $m = \overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$, then th. 3 reduces to th. 2. If $m = (f, 0, \ldots, 0)$, the representation T^{L^m} can be realized in the space of totally symmetric tensors of order f.

It is interesting that in quantum physics we use totally symmetric representations for a collection of bosons, and totally antisymmetric ones for a collection of fermions. Representations with other symmetry types are used in the so-called *parastatistics*, or in the description of systems with 'hidden' variables.

Remark: All irreducible tensor representations of $GL(n, C)$ remain irreducible, when restricted to the subgroups $GL(n, R)$; $U(p, q)$, $p+q = n$, $U(n)$, $SL(n, C)$, $SL(n, R)$, $SU^*(2n)$, $SU(p, q)$, $p+q = n$, or $SU(n)$ as a result of the Weyl unitary trick.

C. *Tensor Representations of* SO(n, C), SO(n), SO(p, q) *and* SO*(n)

The elements of the orthogonal group $SO(n, C)$ are matrices that satisfy the additional condition

$$gg^T = g^T g = I, \quad \text{or} \quad g_{ij} g_{ik} = \delta_{jk}. \tag{18}$$

The condition (18) implies that there is a new operation which commutes with the group action in the linear space $\overset{r}{H^T}$ spanned by the components $T_{i_1 i_2 \ldots i_r}$ of a tensor T. Indeed, consider a contraction (trace) of the tensor given by the formula

$$T^{(12)}_{i_3 i_4 \ldots i_r} \equiv T_{iii_3 i_4 \ldots i_r} = \delta_{i_1 i_2} T_{i_1 i_2 i_3 \ldots i_r}. \tag{19}$$

This operation of contraction and the group transformation by an element $g \in SO(n, C)$ commute:

$$\begin{aligned}(T_g T)^{(12)}_{i_3 i_4 \ldots i_r} &= g_{ik_1} g_{ik_2} g_{i_3 k_3} \cdots g_{i_r k_r} T_{k_1 k_2 \ldots k_r} = \delta_{k_1 k_2} g_{i_3 k_3} \cdots g_{i_r k_r} T_{k_1 k_2 k_3 \ldots k_r} \\ &= g_{i_3 k_3} \cdots g_{i_r k_r} T^{(12)}_{k_3 k_4 \ldots k_r} = (T_g T^{(12)})_{i_3 i_4 \ldots i_r}.\end{aligned} \tag{20}$$

The operation of contraction can be applied to any pair of indices of a tensor $\{T_{i_1 i_2 \ldots i_r}\}$. A tensor $\{T_{i_1 i_2 \ldots i_r}\}$ is said to be *traceless* if the contraction of any pair of indices vanishes. By virtue of eq. (20) the traceless tensors transform among themselves under $SO(n, C)$, hence form an invariant subspace. Moreover, we have

PROPOSITION 4. *Every tensor* $\{T_{i_1 i_2 \ldots i_r}\}$ *can be decomposed uniquely into a traceless tensor* $\mathring{T}$ *and a tensor* Q *with components*

$$Q_{i_1 i_2 \ldots i_r} = \delta_{i_1 i_2} R^{(12)}_{i_3 i_4 \ldots i_r} + \ldots + \delta_{i_p i_q} R^{(pq)}_{i_1 \ldots i_{p-1} i_{p+1} \ldots i_{q-1} i_{q+1} \ldots i_r} + \ldots +$$
$$+ \delta_{i_{r-1} i_r} R^{(r-1, r)}_{i_1 i_2 \ldots i_{r-2}} \quad \left(\frac{r(r-1)}{2} \ \textit{terms}\right), \tag{21}$$

i.e.,

$$T = \mathring{T} + Q.$$

This decomposition is invariant under $SO(n, C)$. ▼

PROOF: We introduce in the space $\overset{r}{H^T}$ of all tensors of rank r a scalar product $(\cdot, \cdot)$ given by the formula

$$(T, T') = T_{i_1 i_2 \ldots i_r} \overline{T'}_{i_1 i_2 \ldots i_r}. \tag{22}$$

Let $\mathscr{K}$ be the subspace of $\overset{r}{H^T}$ consisting of all tensors of the form (21). A tensor T is orthogonal to the subspace $\mathscr{K}$, i.e.,

$$(T, Q) = T_{i_1 i_2 \dots i_r} \bar{Q}_{i_1 i_2 \dots i_r} = 0, \tag{23}$$

if all traces of T vanish. Indeed, for a particular $Q \in \mathscr{K}$ such that only $R^{(12)} \neq 0$, eq. (23) implies that $T^{(12)}$ must be zero. By taking successive non-vanishing terms in eq. (21) we see that all traces $T^{(pq)}$ of T must be zero. Consequently, a linear set of all traceless tensors $\mathring{T} \in \overset{r}{H^T}$ forms a subspace orthogonal to $\mathscr{K}$. Because the whole space $\overset{r}{H^T}$ is the sum of $\mathscr{K}$ and $\mathscr{K}^{\perp}$, an arbitrary tensor T can be uniquely represented in the form

$$T = \mathring{T} + Q, \qquad \mathring{T} \in \mathscr{K}^{\perp},\ Q \in \mathscr{K}. \tag{24}$$

Because $\mathring{T}$ form an invariant subspace, the decomposition (24) is invariant under SO(n, C) by the Weyl theorem on full reducibility. ▼

Applying successively the proposition 4 to the tensors $R^{(pq)}$, etc., we obtain the invariant decomposition of any tensor T onto traceless tensors of rank r, $r-2$, $r-4$, and so on.

Under the action of the symmetric group S_r a traceless tensor is transformed into another traceless tensor. Hence, we expect that irreducible representations of SO(n, C) are realized in the linear subspace of $\overset{r}{H^T}$ spanned by the components of a traceless tensor $\{T_{i_1 i_2 \dots i_r}\}$ with a given Young symmetry defined by the partition $(\lambda_1, \lambda_2, \dots, \lambda_n)$ of the number $r = \lambda_1 + \lambda_2 + \dots + \lambda_n$.

We first find the induced representations corresponding to the polyvector representations.

THEOREM 5. *The tensor representation of* SO(n, C), $n = 2\nu$ *or* $n = 2\nu+1$, *realized in the linear space* $\overset{r}{H^T}$ *of all polyvectors of rank r associated with the Young frame* $\overset{r}{\lambda} = (\underbrace{1, 1, \dots, 1}_{(r)}, 0, \dots, 0)$ *is equivalent to the representation* $T^{L^{\overset{r}{m}}}$ *associated with the highest weight* $\overset{r}{m} = (\underbrace{1, 1, \dots, 1}_{(r)}, 0, \dots, 0)$. *The tensor representations* $\overset{r}{\lambda}$ *and* $\overset{n-r}{\lambda}$ *are equivalent.*

PROOF: The space $\overset{r}{H^T}$ is spanned by the basis vectors $\hat{e}_{i_1 i_2 \dots i_r}$ defined in the proof of th. 2. Because $\delta x = (\delta_1 x_1, \delta_2 x_2, \dots, \delta_n x_n)$, we get

$$(T_\delta \hat{e})_{i_1 i_2 \dots i_r} = \delta_{i_1} \delta_{i_2} \dots \delta_{i_r} \hat{e}_{i_1 i_2 \dots i_r}, \tag{25}$$

i.e., every vector $\hat{e}_{i_1 i_2 \dots i_r}$ is a weight vector. The group SO(n, C) conserves the form

$$x_1 x_n + x_2 x_{n-1} + \dots + x_n x_1 \tag{26}$$

so that the one-parameter subgroups of Z have now the form, e.g.,

$$z = \begin{bmatrix} 1 & 0 & & & & & 0 \\ z & 1 & & & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & 1 & 0 \\ 0 & & & & & -z & 1 \end{bmatrix} \tag{27}$$

and all other one-parameter subgroups of Z are obtained by shifting properly the 'active blocks' $\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ -z & 1 \end{bmatrix}$. The action of (27) on a vector $x \in C^n$ follows from (11). This implies, as in (16),

$$(T_z e)_{123\dots r} = e_{123\dots r} + z e_{113\dots r} - z e_{123\dots n-1, n-1} = e_{123\dots r} \tag{28}$$

i.e., T_z reproduces $e_{123\dots r}$. An analogous result holds for all other one-parameter subgroups of Z acting of the vector $e_{12\dots r}$. Thus $e_{12\dots r}$ is the invariant of the subgroup Z. One can verify that any vector in $\overset{r}{H^T}$ other than $e_{12\dots r}$ is not invariant under Z. Hence $e_{12\dots r}$ is the highest vector. By virtue of eq. (25) the corresponding highest weight has the form

$$m = \overset{r}{m} = (\underbrace{1, 1, \dots, 1}_{(r)}, 0, \dots, 0).$$

Now let $\{e_{i_1 i_2 \dots i_{n-r}}\}$, $r < \nu$, be the polyvector associated with the Young frame $\overset{n-r}{\lambda} = (\underbrace{1, 1, \dots, 1}_{(n-r)}, 0, \dots, 0)$. Then $e_{12\dots n-r}$ is the highest vector, by the previous argument. The corresponding integral highest weight, according to eq. (25), has the form $L^m = \delta_1 \delta_2 \dots \delta_{n-r}$. But for SO$(n, C)$ the element δ of the subgroup D has to conserve the form (26); this implies the restriction $\delta_i = \delta_{n-i+1}$ for arbitrary n, and the additional restriction $\delta_{\nu+1} = 1$, for $n = 2\nu+1$. Hence $\overset{n-r}{L^m} = \delta_1 \delta_2 \dots \delta_r = \overset{r}{L^m}$, $r < \nu$. Consequently the representations determined by the Young frames $\overset{r}{\lambda}$ and $\overset{n-r}{\lambda}$, $r < \nu$, are equivalent, by virtue of th. 8.2.2. ▼

Remark 1: Notice that for SO$(2\nu+1)$ the highest weight $\overset{r}{m}$, $r < \nu$, corresponding to the polyvector representation $\overset{r}{\lambda}$, coincides with the fundamental highest weight $\overset{r}{m}$ given by eq. 8.6(4). However, the polyvector representation $\overset{(\nu)}{\lambda} = (1, 1, \dots, 1) \equiv \overset{(\nu)}{m}$ is given by the Young product of spinor representations—indeed, from eq. 8.6 (4), we have

$$\overset{(\nu)}{L^m} = \overset{\nu}{L^m} \cdot \overset{\nu}{L^m}. \tag{29}$$

In the case of the group SO(2ν) the polyvector representation $\overset{r}{\lambda}$ coincides with the fundamental representation 8.6(5) for $r < \nu-1$. The representations $\overset{(\nu-1)}{\lambda}$ and $\overset{(\nu)}{\lambda}$ are Young products of spinor representations: indeed, according to 8.6(5) we have

$$\overset{(\nu-1)}{L^{m}} = \overset{\nu-1}{L^{m}}\overset{\nu}{L^{m}} \quad \text{and} \quad \overset{(\nu)}{L^{m}} = \overset{\nu-1}{L^{m}}\overset{\nu-1}{L^{m}}. \tag{30}$$

Remark 2: The equivalence of the polyvector representations $\overset{r}{\lambda}$ and $\overset{n-r}{\lambda}$, $r < \nu$, is a generalization of the fact, well-known in physics, of equivalence under rotation between a three-dimensional vector $\{T_i\}$ $(\sim \overset{1}{\lambda} = (1\ 0\ 0))$, and the skew-symmetric tensor $\{T_{ij}\}(\sim \overset{2}{\lambda} = (1\ 1\ 0))$. ▼

The following theorem gives the connection between the tensor representations and the induced irreducible representations:

THEOREM 6. *The representation* T^{L^m} *of* SO(n, C), $n = 2\nu$ *or* $n = 2\nu+1$, *determined by the highest weight* $m = (m_1, m_2, \ldots, m_\nu)$ (*with integer components*) *is equivalent to the tensor representation realized in the space of traceless tensors*:

$$\{T_{i_1 i_2 \ldots i_r}\}, \qquad r = \sum_{i=1}^{\nu} m_i,$$

with the Young symmetry defined by the partition $\lambda = (m_1, m_2, \ldots, m_\nu)$.

PROOF: The representation T^{L^m} for which m_i are positive integers can be realized as the Young product of representations $T^{\overset{k}{L^m}}$ with $\overset{k}{m} = (\underbrace{1, 1, \ldots, 1}_{(k)}, 0, \ldots, 0)$, $k = 1, 2, \ldots, \nu$. The carrier space of a Young product of representations is spanned by the product of basis vectors of $T^{\overset{(k)}{L^m}}$ (cf. remark below eq. 8.6(2)). Hence the component

$$\underbrace{e_{12\ldots\nu} \ldots e_{12\ldots\nu}}_{m_\nu \text{ factors}} \ldots \underbrace{e_{12} \ldots e_{12}}_{m_2-m_3} \ldots T_{\boxed{1\,|\,2\,|\ldots|\,m_1-m_2}} \tag{31}$$

where $e_{12\ldots\nu}$ are given by eq. (7), is the highest vector u_0^m with $m = (m_1, m_2, \ldots, m_r)$. This vector has the Young symmetry defined by the partition $\lambda = (m_1, m_2, \ldots, m_r)$. The tensor space associated with the representation T^{L^m} is obtained from the highest vector (31) by the formula 8.2(21). Since the group action commutes with permutations, the obtained tensor has the symmetry defined by partition $\lambda = (m_1, m_2, \ldots, m_r)$. Finally, since the carrier space is irreducible, the obtained tensor must be traceless. ▼

The representations of SO(n, C) determined by the highest weights with half-integer coefficients are spinor representations and cannot be described in the language of Young diagrams. The remaining highest weights have integer components. Hence th. 6 gives the description of all tensor representations of SO(n, C). This provides also, by the th. 8.3.1, the description of all irreducible tensor representations of SO(n), SO(p, q), $p+q = n$ and SO*(n).

§ 3. The Method of Harmonic Functions

We have shown that for the rotation group SO(3) the basis vectors e^J_M of the carrier space H^J of an irreducible representation can be realized in terms of spherical harmonic functions $Y^J_M(\vartheta, \varphi)$ defined over a symmetric space $X = \mathrm{SO}(3)/\mathrm{SO}(2)$. This realization is extremely useful in the solution of various problems of representation theory and in physical applications. We expect therefore that a realization of the basis vectors of the carrier space H as general harmonic functions* which are the eigenfunctions of a maximal set of commuting operators in H, will also be useful for other groups.

A. *Harmonic Functions for* SO(p).

We consider in detail in this subsection the representation theory of the group SO(p) in the framework of harmonic functions. The approach is in fact a direct extension of the harmonic functions method for SO(3).

Symmetric Spaces and Representation Theory

It follows from Table I in ch. 4.2 that there are two series of symmetric spaces X which can be related to the group SO(n):

X	rank	dimension of X	
$\mathrm{SO}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(p)$	$\min(p, q)$	pq	(1)
$\mathrm{SO}(2n)/U(n)$	$[\frac{1}{2}n]$	$n(n-1)$	

According to the Gel'fand–Chevalley theorem (cf. 9.6.B.(v)) the number of generators of the ring of invariant operators in the enveloping algebra in the space $H = L^2(X, \mu)$ is equal to the rank of the symmetric space X. Thus construction of the representations will be simplest in the symmetric spaces of rank one. The inspection of Table I shows that we obtain symmetric spaces of rank one for SO(p) only in the following cases:

$$X = \mathrm{SO}(p)/\mathrm{SO}(p-1), \quad X_1 = \mathrm{SO}(4)/U(2) \quad \text{and} \quad X_2 = \mathrm{SO}(6)/U(3). \tag{2}$$

The th. 9.6.2 states that in symmetric spaces of rank one the ring of invariant operators in the enveloping algebra is generated by the *Laplace–Beltrami operator* only. This operator has the form

$$\Delta(x) = |\bar{g}|^{-1/2}\partial_\alpha g^{\alpha\beta}(x)|\bar{g}|^{1/2}\partial_\beta, \tag{3}$$

where $g_{\alpha\beta}(x)$ is the left-invariant metric tensor on X and $\bar{g}(x) = \det[g_{\alpha\beta}(x)]$.

Let $\psi_\lambda(x)$ be the eigenfunctions of $\Delta(x)$, i.e.,

$$\Delta(x)\psi_\lambda(x) = \lambda\psi_\lambda(x). \tag{4}$$

* The term 'harmonic functions' is sometimes used (in mathematical literature) as eigenfunctions of the invariant operators $C_1, \ldots, C_N$ with zero eigenvalue, i.e. as solutions of the equation $C_p u = 0$, $p = 1, 2, \ldots, N$ [e.g. $\Delta u = 0$]. We shall use this term for all eigenfunctions of C_p.

Then because any generator Y of SO(p) commutes with $\Delta(x)$.

$$[\Delta(x), Y] = 0,$$

the linear hull H^λ of all functions $\psi_\lambda(x)$ forms an invariant subspace of $H(X)$. Thus the construction of irreducible representations in $H(X)$ may be performed along the following steps.

(i) Construction of a convenient model of the abstract quotient space (2), and the selection of a proper coordinate system on X such that the metric tensor $g_{\alpha\beta}(x)$ is diagonal.

(ii) Solution of the eigenfunction problem (4) for the Laplace–Beltrami operator $\Delta(x)$.

(iii) Proof of the irreducibility and unitarity of the representations $T_g\psi_\lambda(x) = \psi_\lambda(g^{-1}x)$ associated to with the set of harmonic functions $\{\psi_\lambda(x)\}$, λ fixed.

The Construction of Harmonic Functions

First we shall introduce a convenient model of the abstract quotient space $X = \mathrm{SO}(p)/\mathrm{SO}(p-1)$. A model must be a transitive manifold with respect to the group and have the same dimension and stability group as the given symmetric space X. We choose the sphere S^{p-1} embedded in the p-dimensional Euclidean space R^p, given by the equation

$$(x^1)^2+(x^2)^2+\ldots+(x^p)^2 = 1. \tag{5}$$

The transitivity of the sphere S^{p-1} with respect to the group SO(p) follows from the fact that any real vector $x = (x^1, x^2, \ldots, x^p)$ fulfilling eq. (5) can be attained from the vector $e^1 = [1, 0, 0, \ldots, 0]$ by the rotation matrix $g(x)$ for which the first column $g^i{}_1(x) = x^i$. Therefore any two vectors x' and x'' obeying eq. (5) can be related by the rotation matrix $g = g(x')g(x'')^{-1}$. The stability group of the point $e^1 = [1, 0, 0, \ldots, 0]$ is the group SO($p-1$). The stability group of any other point x of the hypersurface S^{p-1} is the group

$$G_0 \simeq g(x)\mathrm{SO}(p-1)g^{-1}(x)$$

which, for fixed $g(x)$, is isomorphic to SO($p-1$). Consequently the stability group of the sphere S^{p-1} is the same as the stability group of the abstract quotient space (2). The dimension of X defined by (2) is

$$\dim X = \dim \mathrm{SO}(p) - \dim \mathrm{SO}(p-1) = p-1$$

and equals to the dimension of the sphere S^{p-1}.

Generally there exists a large number of different coordinate systems on the sphere S^{p-1}. It turns out, however, that the most convenient one is the *biharmonic coordinate system* because in this system, not only can the Laplace–Beltrami operator be separated but also the Cartan subalgebra is diagonal and therefore the harmonic functions can be expressed solely in terms of well-known exponentials and $d^J_{N,M}$ $(\cos\vartheta)$-functions.

We shall construct the biharmonic coordinate system on the sphere S^{p-1} by means of a recursion formula. Suppose that p is even ($p = 2n$) and that we have constructed a coordinate system for $x'^1, \dots, x'^{2k-2}$ ($k \leqslant n$). Then the expression for the variables $x^1, \dots, x^{2k}$ obeying eq. (5) is given by

$$\begin{aligned} x^i &= x'^i \sin\vartheta^k, & i &= 1, 2, \dots, 2k-2, \\ x^{2k-1} &= \cos\varphi^k \cos\vartheta^k, & \varphi^k &\in [0, 2\pi),\ k = 2, 3, \dots, n, \\ x^{2k} &= \sin\varphi^k \cos\vartheta^k, & \vartheta^k &\in \left[0, \frac{\pi}{2}\right),\ k = 2, 3, \dots, n. \end{aligned} \tag{5a}$$

Therefore, putting

$$\begin{aligned} x'^1 &= \cos\varphi^1, \\ x'^2 &= \sin\varphi^1, \quad \varphi^1 \in [0, 2\pi), \end{aligned}$$

and applying successively the procedure (5a), we obtain the parametrization of all coordinates for the sphere S^{p-1} for an arbitrary even dimension p.

If p is odd ($p = 2n+1$) we first construct the coordinates x'^i ($i = 1, 2, \dots, 2n$) by means of the method described above for $p = 2n$; we then obtain the corresponding x^k, $k = 1, 2, \dots, 2n+1$, by

$$\begin{aligned} x &= x'^i \sin\vartheta^{n+1}, & i &= 1, 2, \dots, 2n, \\ x^{2n+1} &= \cos\vartheta^{n+1}, & \vartheta^{n+1} &\in [0, 2\pi). \end{aligned} \tag{6}$$

In what follows we denote the set of angles $(\varphi^1, \varphi^2, \dots, \varphi^{[p/2]}, \vartheta^2, \vartheta^3, \dots, \vartheta^{\{p/2\}})$ by ω, where the brackets around the indices are defined as follows:

$$\left[\frac{p}{2}\right] = \begin{cases} \dfrac{p}{2} & \text{if } p = 2n, \\ \dfrac{p-1}{2} & \text{if } p = 2n+1, \end{cases} \quad \text{and} \quad \left\{\frac{p}{2}\right\} = \begin{cases} \dfrac{p}{2} & \text{if } p = 2n, \\ \dfrac{p+1}{2} & \text{if } p = 2n+1, \end{cases} \tag{7}$$
$$n = 1, 2, \dots$$

The metric tensor $g_{\alpha\beta}(S^{p-1})$ on the sphere S^{p-1} is induced by the metric tensor $g_{ab}(R^p)$ on the Euclidean space R^p in which the sphere S^{p-1} is embedded and is given by

$$g_{\alpha\beta}(S^{p-1}) = g_{ik}(R^p)\, \partial_\alpha x^i(\omega)\, \partial_\beta x^k(\omega), \qquad \alpha, \beta = 1, 2, \dots, p-1, \tag{8}$$

where

$$g_{ik}(R^p) = \delta_{ik}, \qquad i, k = 1, 2, \dots, p,$$

and ∂_α denotes partial differentiation with respect to an angle φ^α, for $\alpha = 1, 2, \dots, \left[\frac{p}{2}\right]$, and with respect to $\vartheta^2, \dots, \vartheta^{\{p/2\}}$ for $\alpha = \left[\frac{p}{2}\right], \left[\frac{p}{2}\right]+1, \dots$
$\dots, \left[\frac{p}{2}\right] + \left\{\frac{p}{2}\right\} - 1$ respectively.

Using formulae (8), we find that the metric tensor $g_{\alpha\beta}(S^{p-1})$ is diagonal when expressed in the biharmonic coordinate system and has the following form:

(i) SO(2n)

$$g_{\varkappa\lambda}(S^{2n-1}) = \sin^2\vartheta^n g_{\varkappa\lambda}(S^{2n-3}), \qquad \varkappa, \lambda = 1, 2, \ldots, 2n-3,$$

and

$$\begin{aligned} g_{\alpha,2n-1}(S^{2n-1}) &= \cos^2\vartheta^n \delta_{\alpha,2n-1}, \qquad \alpha = 1, 2, \ldots, 2n, \\ g_{\varepsilon,2n-2}(S^{2n-1}) &= \delta_{\varepsilon,2n-2}, \qquad \varepsilon = 1, 2, \ldots, 2n-2. \end{aligned} \tag{9}$$

(ii) SO(2n+1)

$$g_{\varkappa\lambda}(S^{2n}) = \sin^2\vartheta^{n+1} g_{\varkappa\lambda}(S^{2n-1}), \qquad \varkappa, \lambda = 1, 2, \ldots, 2n-1,$$

and

$$g_{\alpha,2n}(S^{2n}) = \delta_{\alpha,2n}, \qquad \alpha = 1, 2, \ldots, 2n. \tag{10}$$

Because the metric tensor $g_{\alpha\beta}(S^n)$ is diagonal in this coordinate system, the Laplace–Beltrami operator (3) can be decomposed into two or three parts:

(i) SO(2n):

for $n = 1$:

$$\Delta(S^1) = \partial^2/\partial(\varphi^1)^2,$$

and for $n = 2, 3, \ldots$

$$\Delta(S^{2n-1}) = (\cos^2\vartheta^n)^{-1}\frac{\partial^2}{\partial\varphi^{n2}} + (\sin^{2n-3}\vartheta^n\cos\vartheta^n)^{-1}\frac{\partial}{\partial\vartheta^n}(\sin^{2n-3}\vartheta^n\cos\vartheta^n)\frac{\partial}{\partial\vartheta^n} + (\sin^2\vartheta^n)^{-1}\Delta(S^{2n-3}).$$

(ii) SO(2n+1), $n = 1, 2, \ldots$

$$\Delta(S^{2n}) = (\sin^{2n-1}\vartheta^{n+1})^{-1}\frac{\partial}{\partial\vartheta^{n+1}}(\sin^{2n-1}\vartheta^{n+1})\frac{\partial}{\partial\vartheta^{n+1}} + \frac{\Delta(S^{2n-1})}{\sin^2\vartheta^{n+1}}.$$

Here, $\Delta(S^{2n-3})$ and $\Delta(S^{2n-1})$ are again the invariant Laplace–Beltrami operators for the groups SO(2n−2) and SO(2n), respectively, which can be decomposed further in a similar manner. Therefore, by the method of separation of variables we can express any eigenfunction of the invariant operator $\Delta(S^p)$ as a product of functions of one variable only. Because the Laplace–Beltrami operator $\Delta(S^{p-1})$ is equal to the second-order Casimir operator 9.6(48) of SO(p), up to a factor -2 its eigenvalues $\lambda_{\{\frac{p}{2}\}}$ are given by eq. 9.4(64). Using this formula one obtains*

$$\lambda_{\{\frac{p}{2}\}} = -l_{\{\frac{p}{2}\}}(l_{\{\frac{p}{2}\}} + p - 2), \qquad \begin{aligned} p &= 3, 4, 5, \ldots, \\ l_{\{\frac{p}{2}\}} &= 0, 1, 2, \ldots \end{aligned} \tag{11}$$

Due to the inductive construction of the Laplace–Beltrami operator we can separate variables in the eigenvalue problem for the operator $\Delta(S^{p-1})$ and finally

* For the proof that this representation is associated with the highest weight $m = (l_{\{\frac{p}{2}\}}, 0, \ldots, 0)$ see subsec. B.

obtain the second-order ordinary differential equation:

$$\Big[\frac{1}{\sin^{(2n-3)}\vartheta^n\cos\vartheta^n}\frac{\mathrm{d}}{\mathrm{d}\vartheta^n}\sin^{(2n-3)}\vartheta^n\cos\vartheta^n\frac{\mathrm{d}}{\mathrm{d}\vartheta^n}-\frac{m_n^2}{\cos^2\vartheta^n}-$$

$$-\frac{l_{n-1}(l_{n-1}+2n-4)}{\sin^2\vartheta^n}+l_n(l_n+2n-2)\Big]\psi^{l_n}_{m_n,l_{n-1}}(\vartheta^n)=0, \tag{12}$$

if $p=2n$, or

$$\Big[\frac{1}{\sin^{(2n-1)}\vartheta^{n+1}}\frac{\mathrm{d}}{\mathrm{d}\vartheta^{n+1}}\sin^{(2n-1)}\vartheta^{n+1}\frac{\mathrm{d}}{\mathrm{d}\vartheta^{n+1}}-\frac{l_n(l_n+2n-2)}{\sin^2\vartheta^{n+1}}+$$

$$+l_{n+1}(l_{n+1}+2n-1)\Big]\psi^{l_{n+1}}_{l_n}(\vartheta^{n+1})=0, \tag{13}$$

if $p=2n+1$.

Because the spectrum (11) of $\Delta(S^{p-1})$ is purely discrete solutions of eqs. (12) or (13) belong to the Hilbert space of square integrable functions with respect to the measure

$$\mathrm{d}\mu(S^{p-1})=|\bar{g}(S^{p-1})|^{1/2}\mathrm{d}\omega=\begin{cases}\displaystyle\prod_{k=2}^{n}\cos\vartheta^k\sin^{(2k-3)}\vartheta^k\,\mathrm{d}\vartheta^k\prod_{l=1}^{n}\mathrm{d}\varphi^l, & p=2n,\\ \displaystyle\sin^{2n-1}\vartheta^{n+1}\mathrm{d}\vartheta^{n+1}\prod_{k=2}^{n}\cos\vartheta^k\sin^{(2k-3)}\vartheta^k\,\mathrm{d}\vartheta^k\times & \\ \displaystyle\qquad\times\prod_{l=1}^{n}\mathrm{d}\varphi^l, & p=2n+1.\end{cases} \tag{14}$$

Setting in eq. (12)

$$\psi^{l_n}_{m_n,l_{n-1}}(\vartheta^n)=\tan^{|l_{n-1}|}\vartheta^n\cos^{l_n}\vartheta^n u(\vartheta^n)$$

and introducing the new variable $y=-\tan^2\vartheta^n$ we reduce eq. (12) to the standard hypergeometric equation and we obtain

$$\psi^{l_n}_{m_n,l_{n-1}}(\vartheta^n)=\tan^{|l_{n-1}|}\vartheta^n\cos^{l_n}\vartheta^n\times$$

$$\times\,{}_2F_1\Big[\frac{1}{2}(|l_{n-1}|-l_n+m_n),\frac{1}{2}(|l_{n-1}|-l_n-m_n),\,l_{n-1}+n-1;\,-\tan^2\vartheta^n\Big], \tag{15}$$

for $p=2$

$$\psi_{m_1}(\varphi^1)=(2\pi)^{-1/2}\exp(\mathrm{i}m_1\varphi^1),$$

where l_n, l_{n-1}, m_n are restricted by the condition that $\psi^{l_n}_{m_n,l_{n-1}}(\vartheta^n)$ is a square integrable function with respect to the measure (14), i.e.,

$$|m_n|+|l_{n-1}|=l_n-2s,\qquad s=0,1,\dots,[\tfrac{1}{2}l_n]. \tag{16}$$

Similarly setting in eq. (13) $\psi_{l_n}^{l_{n+1}}(\vartheta^{n+1}) = \tan^{l_n}(\vartheta^{n+1})\cos l_{n+1}(\vartheta^{n+1})u(\vartheta^{n+1})$ we reduce it to the hypergeometric equation and we obtain:

$$\psi_{l_n}^{l_{n+1}}(\vartheta^{n+1}) = \tan^{l_n}\vartheta^{n+1}\cos^{l_{n+1}}\vartheta^{n+1} \times \\ \times {}_2F_1[\tfrac{1}{2}(l_n-l_{n+1}), \tfrac{1}{2}(l_n-l_{n+1}+1), l_n+n; -\tan^2\vartheta^{n+1}] \tag{17}$$

with the restriction

$$l_n = l_{n+1}-k, \quad k = 0, 1, \dots, l_{n+1}. \tag{18}$$

Both solutions (15) and (17) can be expressed in terms of $d^J_{\Lambda M}$ — functions of the ordinary rotation group SO(3) (cf. eq. 5.8(1)). The orthonormal basis of the corresponding Hilbert spaces $H^{l_n}(S^{2n-1})$ and $H^{l_{n+1}}(S^{2n})$ are then given by the expressions:

(i) SO(2n):

$$Y^{l_2,\dots,l_n}_{m_1,\dots,m_n}(\omega) = N_n^{-1/2}\prod_{k=2}^{n}\sin^{2-k}\vartheta^k d^{J_k}_{M_k M'_k}(2\vartheta^k)\prod_{s=1}^{n}\exp(im_s\varphi^s), \tag{19}$$

(ii) SO(2n+1)

$$Y^{l_2,\dots,l_{n+1}}_{m_1,\dots,m_n}(\omega) = N_{n+1}^{-1/2}\sin^{1-n}\vartheta^{n+1}d^{J_{n+1}}_{M_{n+1},0}(\vartheta^{n+1}) \times \\ \times\prod_{k=2}^{n}\sin^{2-k}(\vartheta^k)d^{J_k}_{M_k M'_k}(2\vartheta^k)\prod_{s=1}^{n}\exp(im_s\varphi^s), \tag{20}$$

where N_n, N_{n+1} are normalization factors given by

$$\begin{aligned} N_n &= 2\pi^n\prod_{k=2}^{n}(l_k+k-1)^{-1}, \\ N_{n+1} &= 4\pi^n[2(l_{n+1}+n)-1]^{-1}\prod_{k=2}^{n}(l_k+k-1)^{-1}, \end{aligned} \tag{21}$$

and the indices $J_{k'}$, $M_{k'}$, M'_k are defined as

$$\begin{aligned} J_k &= \frac{1}{2}(l_k+k-2), \\ M_k &= \frac{1}{2}(m_k+l_{k-1}+k-2), \quad l_1 \equiv m_1, \\ M'_k &= \frac{1}{2}(m_k-l_{k-1}-k+2), \quad k = 2, 3, \dots, n, \\ J_{n+1} &= l_{n+1}+n-1, \quad M_{n+1} = l_n+n-1. \end{aligned} \tag{22}$$

Here l_k, $k = 2, 3, \dots, n+1$, are non-negative integers, m_k, $k = 1, 2, \dots, n$ are integers restricted by the conditions (16) and (18).

B. *Irreducibility and Unitarity*

Let $H = L^2(S^{p-1}, \mu)$. The global transformation of SO(p) in H are defined by means of the left translations, i.e.,

$$T_g u(x) = u(g^{-1}x). \tag{23}$$

The generators of one-parameter subgroups $g_{(ik)}(\vartheta)$, defined by eq. 1(41), have, according to eq. (23), the form

$$L_{ik} = x_i \partial_k - x_k \partial_i, \qquad i, k = 1, 2, \ldots, p. \tag{24}$$

The set of all generators L_{ik} span the Lie algebra of SO(p) with the following commutation relations:

$$[L_{ij}, L_{rs}] = \delta_{is} L_{jr} + \delta_{jr} L_{is} - \delta_{ir} L_{js} - \delta_{js} L_{ir}.$$

Clearly,

$$[\Delta(S^p), L_{ij}] = 0, \qquad i, j = 1, 2, \ldots, p, \tag{25}$$

so that the harmonic functions associated with a definite eigenvalue of $\Delta(S^p)$ span an invariant subspace of H. We shall show now that this subspace is irreducible; the set $\left\{L_{2k, 2k-1}, \ k = 1, 2, \ldots, \left[\frac{1}{2}p\right]\right\}$ of commutative generators forms a Cartan subalgebra of $o(p)$. Using eq. (24) and introducing biharmonic coordinates, one obtains

$$\mathrm{i}L_{2k, 2k-1} = \frac{1}{\mathrm{i}} \frac{\partial}{\partial \varphi^k}, \qquad k = 1, 2, \ldots, \left[\frac{1}{2}p\right]. \tag{26}$$

Consider first the group SO($2n$) and denote the space spanned by vectors (19) by H^{l_n}. By virtue of eqs. (19) and (26), every basis vector in H^{l_n} is the weight vector with the weight $m = (m_n, m_{n-1}, \ldots, m_1)$. Using eq. (16) we find that m_n is maximal (and then $m_n = l_n$) if $l_{n-1} = 0$; this in turn implies by successive applications of eq. (16) that $m_{n-1} = m_{n-2} = \ldots = m_1 = 0$. Hence the weight

$$m = (l_n, 0, \ldots, 0) \tag{27}$$

represents the highest weight in H^{l_n}. Any other admissible highest weight in H^{l_n}, by eq. (16), would have the form

$$m' = (m_n, l_n - m_n, 0, 0, \ldots, 0). \tag{28}$$

However, the inspection of eq. 9.4 (57) shows that only the weight (27) gives the eigenvalue (11) for the second-order Casimir operator $\Delta(S^{p-1})$. Thus the highest weight (27) is unique. Consequently the carrier space H^{l_n} is irreducible.

Using similar arguments and eqs. (16) and (18), one can show that the space $H^{l_{n+1}}$ spanned by basis vectors (20) carries the irreducible representation of SO($2n+1$), which is defined by the unique highest weight m of the form

$$m = (l_{n+1}, 0, \ldots, 0). \tag{29}$$

The global representations of SO(p) defined in H^{l_n} or $H^{l_{n+1}}$ by eq. (23) are unitary because of the invariance of the measure μ on S^{p-1}.

Remark 1: The irreducible representations of SO(p), $p = 2n$ or $p = 2n+1$, determined by the highest weight (27) or (29) correspond by virtue of th. 2.6 to the tensor representations determined by the Young partition $\lambda = (\lambda_1, 0, \ldots, 0)$ with $\lambda_1 = l_n$ or l_{n+1}. Hence the representations in H^{l_n} or in $H^{l_{n+1}}$ are equivalent to tensor representations realized in the space of symmetric tensors of rank l_n or l_{n+1}, respectively.

Remark 2: In general the maximal set of commuting operators for SO(p), by virtue of eqs. 1(55) and (56), contains $\frac{p^2}{4}$ (p even), or $\frac{p^2-1}{4}$ (p odd) operators.

For these special representations, the maximal set of commuting operators is smallest and consists of

$$Q_p \equiv \begin{cases} \Delta[\mathrm{SO}(p)],\ \Delta[\mathrm{SO}(p-2)], \ldots, \Delta[\mathrm{SO}(4)] & \text{for } p \text{ even}, \\ \Delta[\mathrm{SO}(p)], \Delta[\mathrm{SO}(p-1)], \Delta[\mathrm{SO}(p-3)], \ldots, \Delta[\mathrm{SO}(4)] & \text{for } p \text{ odd}, \\ & p > 3, \\ \Delta[\mathrm{SO}(3)] \quad \text{for } p = 3, \end{cases}$$

and

$$H = \left\{L_{2k,2k-1} = -\frac{\partial}{\partial\varphi^k},\ k = 1, 2, \ldots, \left[\frac{1}{2}p\right]\right\}.$$

The set H contains the operators of the Cartan subalgebra. The spectra of the eigenvalues $l_2, \ldots, l_{\left[\frac{p}{2}\right]}$, $m_1, \ldots, m_{\left[\frac{p}{2}\right]}$ are determined by eqs. (16) and (18).

The biharmonic coordinate system seems to be most convenient from the point of view of physical applications: If we associate with each operator appearing in the maximal set of commuting operators a physical observable then, because every operator of the Cartan subalgebra is diagonal in the biharmonic coordinate system, we have in fact thus exhibited the maximal number of linear conservation laws.

§ 4. The Method of Creation and Annihilation Operators

It is well known in the elementary quantum mechanics of a single particle with conjugate dynamical variables p and q, $[p, q] = -i$, that the operators defined by

$$a = \frac{1}{\sqrt{2}}(q+ip), \qquad a^* = \frac{1}{\sqrt{2}}(q-ip), \tag{1}$$

called *annihilation and creation operators*, respectively, satisfy the commutation relations

$$[a, a^*] = 1. \tag{2}$$

Their names are due to the fact that the Hamiltonian $H = \frac{p^2}{2} + \frac{q^2}{2}$ of the linear harmonic oscillator takes the form (in units $\hbar\omega = 1$, $m = 1$)

$$H = a^* a + \tfrac{1}{2} \tag{3}$$

so that, if $|0\rangle$ $(=\pi^{1/4}\exp[-\frac{1}{2}x^2])$ is the ground eigenstate of H with energy 1/2, $a^*|0\rangle$ is another eigenstate of H with energy $(1+1/2)$, $a^*a^*|0\rangle$ another with energy $(2+1/2)$, etc. Moreover, $a(a^{*n})|0\rangle$ is proportional to $a^{*n-1}|0\rangle$ and $a|0\rangle = 0$. Thus a^* creates one unit of 'excitation' (i.e., increases the energy), and a annihilates one unit of 'excitation' (decreases the energy).

The Lie algebra (2) is clearly equivalent to the Heisenberg algebra $[p, q] = -\mathrm{i}$.

We easily generalize (1), (2) and (3) to the set of n independent creation and annihilation operators

$$[a_i, a_j] = [a_i^*, a_j^*] = 0, \qquad [a_i, a_j^*] = \delta_{ij},\ i, j = 1, 2, \ldots, N \tag{4}$$

and

$$H = \sum_{i=1}^{N} a_i^* a_i + N/2 \tag{5}$$

so that the eigenstates of H contain n_1 excitation of type 1, n_2 excitations of type 2, etc., i.e.,

$$a_1^{*n_1} a_2^{*n_2} \ldots a_N^{*n_N}|0\rangle. \tag{6}$$

The operators a_i, a_i^* are also called *boson operators*. In order to explain this nomenclature, we reinterpret the states (6) as follows. Consider the quantum mechanics of N identical particles. Let i be an index counting the set of quantum numbers characterizing the states of a single particle (these quantum numbers may be discrete or continuous with appropriate ranges); in other words, the complete set of one-particle states are φ_i, $i = 1, 2, \ldots$ According to a general postulate of quantum theory for indistinguishable particles, the distinct states of the system of N identical particles are only those characterized by the number of particles n_i in the state φ_i, $i = 1, 2, \ldots$ Thus the state (6) can be interpreted as n_1 particles in the state φ_1, n_2 particles in the state φ_2, etc. Hence a_k^* creates a particle in the state k, a_k annihilates one.

Let H be the space of states of a system of bosons with the basis vectors $|n_1, n_2, n_3, \ldots\rangle$ labelled by the occupation numbers, and V the space of one-particle states.

In H the action of a_i and a_j^* are expressed as follows:

$$\begin{aligned} a_i|n_1 n_2 \ldots n_n\rangle &= \sqrt{n_i}\,|n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_n\rangle, \\ a_i^*|n_1 n_2 \ldots\rangle &= \sqrt{n_i + 1}\,|n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots\rangle. \end{aligned} \tag{7}$$

Then, by direct computation, we indeed obtain from these equations the commutation relations (4). We will show in 20.2 that within equivalence there exists only one irreducible (integrable to the group) representation of the canonical commutation relations (4).

The following example shows another interesting realization of canonical commutation relations (4).

EXAMPLE 1. Let H be a Hilbert space of functions of complex variables $z_1, \ldots, z_n$ with the scalar product

$$(u, v) = \int u(z)\bar{v}(z)\exp(-\bar{z}z)\mathrm{d}\bar{z}\mathrm{d}z, \tag{8}$$

where

$$\mathrm{d}\bar{z}\mathrm{d}z = \pi^{-n}\prod_{k=1}^{n}\mathrm{d}x_k\mathrm{d}y_k, \qquad z_k = x_k+\mathrm{i}y_k. \tag{9}$$

Then the map

$$a_i \to \frac{\partial}{\partial z_i}, \qquad a_i^* \to z_i, \tag{10}$$

defined on analytic functions $u(z)$ in H, gives the representation of the Lie algebra (4). ▼

Construction of Lie Algebras from Bilinear Combinations of Creation and Annihilation Operators

The physical quantities like energy (3) angular momentum $M_{ij} = q_ip_j-p_jq_i$, etc. are bilinear forms of the form

$$c_{ij}a_i^*a_j.$$

This suggests to construct all basis elements X_i of a Lie algebra L in terms of such bilinear combinations. Indeed, let a_i, $i = 1, 2, \ldots, n$, be a set of boson operators in a carrier Hilbert space H. Set $A_{ij} = a_i^*a_j$. Then using eq. (4) we obtain

$$[A_{ij}, A_{kl}] = \delta_{jk}A_{il}-\delta_{il}A_{jk}. \tag{11}$$

Hence, by virtue of 9.4 (2) the set $\{A_{ij}\}_{i,j=1}^n$ forms the set of generators of the Lie algebra gl(n, c). Because any Lie algebra is a subalgebra of gl(n, C) by Ado's Theorem any other complex or real Lie algebra is generated by a subset of $\{A_{ij}\}_{i,j=1}^n$. In particular using eq. 1 (6) we find that operators

$$\begin{aligned} M_{kk} &= a_k^*a_k, \quad k = 1, \ldots, n, \\ M_{kl} &= a_k^*a_l+a_l^*a_k, \\ \tilde{M}_{kl} &= \mathrm{i}(a_k^*a_l-a_l^*a_k), \end{aligned} \qquad k < l \leqslant n \tag{12}$$

generate the Lie algebra $u(n)$.

Similarly using eq. 1 (42) we find that the operators

$$X_{ik} = a_i^*a_k-a_k^*a_i \tag{13}$$

generate the Lie algebra so(n). The explicit construction of generators of sp(n)-algebra is given in exercise 6.4.5.

One may construct also with the help of annihilation and creation operators non-compact Lie algebras like $u(p, q)$, so(p, q), sp(p, q) etc. We give for an illustration the construction of generators for $u(p, q)$ Lie algebras, which are often used in particle physics. Let a_i, a_j^*, $i, j = 1, 2, \ldots, p$ and $b_{\hat{i}}$, $b_{\hat{j}}^*$, $\hat{i}, \hat{j} = p+1, \ldots$ $\ldots, p+q$, be the sets of annihilation and creation operators satisfying the relations:

$$[a_i, a_j^*] = \delta_{ij}, \quad [b_{\hat{i}}, b_{\hat{j}}^*] = \delta_{\hat{i}\hat{j}} \tag{14}$$

and all other commutators equal to zero. Define the set A of operators by the following array of bilinear products

$$A = \begin{bmatrix} A_{ij} & A_{i\hat{j}} \\ A_{\hat{j}\hat{i}}^* & A_{\hat{i}\hat{j}} \end{bmatrix} = \begin{bmatrix} -a_i^* a_j + r\delta_{ij} & a_i^* b_{\hat{j}}^* \\ -b_{\hat{i}} a_j & b_{\hat{i}} b_{\hat{j}}^* + r\delta_{\hat{i}\hat{j}} \end{bmatrix}, \tag{15}$$

where r is any real number. One readily verifies that elements of the set A satisfy the commutation relations for the Lie algebra gl$(p+q, R)$, whereas the operators

$$\begin{gathered} M_{kk} = A_{kk}, \quad k = 1, 2, \ldots, n, \\ M_{kl} = A_{kl} + A_{lk}, \quad \tilde{M}_{kl} = \mathrm{i}(A_{kl} - A_{lk}), \quad k \leqslant l \leqslant p \text{ or } p < k < l, \\ N_{k\hat{l}} = A_{k\hat{l}} - A_{\hat{l}k}, \quad \tilde{N}_{k\hat{l}} = \mathrm{i}(A_{k\hat{l}} + A_{\hat{l}k}), \quad k \leqslant p < l, \end{gathered} \tag{16}$$

generate the Lie algebra $u(p, q)$.

Using (7) we can then construct representations of all these Lie algebras.

§ 5. Comments and Supplements

(a) The algebraic method of construction of irreducible representations considered in sec. 1 was elaborated by Gel'fand and Zetlin in 1950a, b. They only gave the final formulas such as 1(14)–(19). An interesting derivation of these formulas, based on the Weyl theory of tensor representations, was given by Baird and Biedenharn in 1963; they also corrected some formulas in the original work of Gel'fand–Zetlin. In 1965 Gel'fand and Graev extended and improved this formalism and succeeded in calculating the matrix elements of the global finite-dimensional representations of GL(n, C); they also extended the algebraic approach to the representation theory of non-compact Lie algebras. We present this theory in ch. 11. The detailed analysis of representations of $u(n)$, so(n), $u(n, 1)$, and so$(n, 1)$ algebras in terms of Gel'fand–Zetlin patterns was given by Ottoson 1967. Holman and Biedenharn gave the alternative derivation of various results for representations of $u(n)$ using Gel'fand–Zetlin technique 1971.

There is a one-to-one correspondence between the Gel'fand–Zetlin basis vectors and the tensor components. In fact, consider the Gel'fand–Zetlin patterns 1(11) associated with a highest weight $m = (m_{1n}, m_{2n}, \ldots, m_{nn})$ and form the Young tableau which contains:

in the first row: m_{11} entries 1 followed by

$$(m_{12}-m_{11}) \text{ entries } 2, \ldots, (m_{1n}-m_{1,n-1}) \text{ entries } n,$$

in the second row m_{22} entries 2 followed by

$$(m_{23}-m_{22}) \text{ entries } 3, \ldots, (m_{2n}-m_{2n-1}) \text{ entries } n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

in the kth row m_{kk} entries 1 followed by

$$(m_{k,k+1}-m_{kk}) \text{ entries } (k+1), \ldots, (m_{kn}-m_{kn-1}) \text{ entries } n.$$

It is evident that this prescription gives a one-to-one correspondence between the Gel'fand–Zetlin basis vectors and the tensor components; the tensor component 2(14), corresponding to the highest vector m obtained in this manner-has the highest weight $m = (m_{1n}, m_{2n}, \ldots, m_{nn})$ because of eq. 2(17). The correspondence between Gel'fand–Zetlin patterns and basic vectors $|n_1, \ldots, n_n\rangle$ defined by eq. 4 (6) was given by Holman and Biedenharn 1971.

(b) The theory of tensor representations of simple Lie groups is due mainly to Weyl 1939. The Weyl theory was based on the connection between the permutation group and the linear groups. Here we presented an approach based on the concept of induced representations. This approach was originated by Godement 1956, app. and finally elaborated by Želobenko 1962.

(c) The theory of harmonic functions on compact symmetric spaces was also originated by Weyl in 1934. Later on Godement 1952, 1956 and Harish-Chandra 1958 made this one of the main tools in the analysis of irreducible representations of semisimple Lie groups. The case of SO(n) harmonic functions presented in § 3 was elaborated by Rączka, Limić and Niederle 1966a. They also extended this approach to non-compact groups SO(p, q) (cf. 15, § 3).

One can construct the harmonic functions and the corresponding irreducible representations also for other classical Lie groups. The case of $U(n)$ was studied by Rączka and Fischer 1966b. They constructed a class of harmonic functions determined by one and two invariant numbers. The case of symplectic groups Sp(n) was treated by Pajas and Rączka 1968.

The harmonic functions for non-compact unitary groups $U(p, q)$ were constructed by Fischer and Rączka 1965c and for Sp(p, q) by Pajas 1969.

Wigner showed in 1955 that a representation theory of Euclidean group provides a basis for the theory of Bessel functions; in particular, he demonstrated that the composition law of group elements imply various functional relations for Bessel functions. Later on Vilenkin in a series of papers extended this approach to other special functions. The results of Vilenkin are collected in his monograph 1965. Recently other monographs appeared devoted to the study of the special functions of mathematical physics from the point of view of group representation: Miller 1968, and Talman 1968 based on Wigner's lectures of 1955.

(d) The origin of the creation and annihilation operators goes back to the quantum theory of oscillators and of radiation (cf. Dirac 1928). The Fock-space was introduced in 1932 (Fock). The representations of the infinite-dimensional Fermi and Bose operators were first given by Gårding and Wightman 1954 a, b and there has been considerable work on this case since then. (See the reviews by Berezin and Golodets 1969.)

The Hilbert space of entire analytic functions goes back apparently to London, but was introduced in a complete form by Bargmann 1961 and extended to the infinitely many creation and annihilation operators by Segal 1965.

The construction of the representations of the compact Lie algebra su(2) by creation and annihilation operators is due to Schwinger 1952. The non-compact case was discussed by Barut and Fronsdal 1965 for su(1, 1) and by Anderson, Fischer and Rączka 1968 for $u(p, q)$.

§ 6. Exercises

§ 1.1.** Construct the irreducible representations of the Lie algebra sp(n) using Gel'fand–Zetlin method

Hint: Elaborate the structure of Gel'fand–Zetlin patterns using the decomposition of irreducible representations of sp(n) with respect to a sequence of successive maximal subalgebras of sp(n).

§ 2.1. Show that the defining representation L of the Lorentz group in R^4:

$$x \to x' = Lx$$

is equivalent to the $D^{(1/2,1/2)}$ representation, i.e.

$$D^{(1/2,1/2)} = TLT^{-1}.$$

§ 2.2. Show that the generators of the Lorentz group for $D^{(1/2,0)}$ and $D^{(0,1/2)}$ representation can be taken in the form ($k = 1, 2, 3$):

$$\begin{aligned} J_k^{(1/2,0)} &= -\frac{1}{2}\mathrm{i}\sigma_k, \quad N_k^{(1/2,0)} = \frac{1}{2}\sigma_k, \\ J_k^{(0,1/2)} &= -\frac{1}{2}\mathrm{i}\sigma_k, \quad N_k^{(0,1/2)} = -\frac{1}{2}\sigma_k. \end{aligned} \tag{1}$$

§ 2.3. Show that the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of SL(2, C) can be written in the form

$$(D^{(1/2,0)} \oplus D^{(0,1/2)})(A) = \begin{cases} \exp\left(\frac{\mathrm{i}}{2}\,\boldsymbol{w}\cdot\boldsymbol{\Sigma}\right) \equiv \cos\frac{w}{2} + \mathrm{i}\hat{\boldsymbol{w}}\cdot\boldsymbol{\Sigma}\sin\frac{w}{2} \\ \qquad\qquad \text{for special rotations,} \\ \exp\left(-\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{\alpha}\right) \equiv \cosh\frac{u}{2} - \hat{\boldsymbol{u}}\cdot\boldsymbol{\alpha}\sinh\frac{u}{2} \\ \qquad\qquad \text{for special Lorentz transformations,} \end{cases} \tag{2}$$

where $\hat{w}$ and $\hat{u}$ are unit vectors in the direction of w and u, $w = |w|$, $u = |u|$ and

$$\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad \alpha = \begin{bmatrix} -\sigma & 0 \\ 0 & \sigma \end{bmatrix}. \tag{3}$$

§ 2.4. Show that the irreducible representation $D^{(i,j)}$ of SL(2, C) can be realized in the space of symmetric traceless tensors $T_{\mu_1 \mu_{2j}}$.

§ 2.5. Show that the representations $D^{(j,0)}$ and $D^{(0,j)}$ of SL(2, C) satisfy the following equalities:

$$D^{(j,0)}\left(\frac{\sigma p}{m}\right) = m^{-2j}(\sigma p)\otimes(\sigma p)\otimes \dots \otimes(\sigma p) \quad (2j \text{ times}),$$

$$D^{(0,j)}\left(\frac{\sigma p}{m}\right) = m^{-2j}(\tilde{\sigma} p)\otimes(\tilde{\sigma} p)\otimes \dots \otimes(\tilde{\sigma} p) \quad (2j \text{ times}),$$

where $\tilde{\sigma} = -\sigma$.

§ 4.1. Let

$$a_k = 2^{-1/2}(x_k + i\partial_k), \quad a_k^* = 2^{-1/2}(x_k - i\partial_k), \quad k = 1, 2, 3$$

and

$$X_{kl} = a_k^* a_l + \tfrac{1}{2}\delta_{ij}. \tag{4}$$

Show that the operators (4) span the Lie algebra $u(3)$.

Show that in the carrier space $H = L^2(R^3)$ only the most degenerate representations of $u(3)$ characterized by highest weights $m = (m, 0, 0)$ are realized. Show that lowest dimensions of these representations are 1, 6, 10 and 15.

Hint: Show that higher order Casimir operators $C_2, C_3, C_4, \dots$, are functions of $C_1 = \sum_1^3 X_{ll}$.

§ 4.2. Let $H = L^2(R^n)$ and let a_l and a_l^*, $l = 1, 2, \dots, n$, be the creation and annihilation operators given by

$$a_l = 2^{-1/2}(q_l + ip_l), \quad a_l^* = 2^{-1/2}(q_l - ip_l). \tag{5}$$

Show that the vector

$$|0\rangle = \pi^{-n/4}\exp\left(-\frac{x^2}{2}\right) \tag{6}$$

satisfies the condition

$$a_l|0\rangle = 0$$

and

$$(a_l^*)^p|0\rangle = H_p(x_l)|0\rangle, \quad p = 1, 2, \dots, \tag{7}$$

where $H_p(x_l)$ are the normalized Hermite polynomials.

§ 4.3. Set

$$e_a(z) = \exp(\bar{a}z). \tag{8}$$

Show that functions (8) satisfy the conditions (with respect to the scalar product $(\cdot,\cdot)$ given by 4(8))

$$(e_a, e_b) = e_b(a), \quad (e_a, u) = u(a) \tag{9}$$

so they play the role of Dirac δ-functions.

§ 4.4. Let M, N and R be $n\times n$ matrices. Show that the following identities are satisfied

$$\begin{aligned}
[a^*Ma, a^*Na] &= a^*[M,N]a,\\
[aNa, a^*Ra^*] &= a^*[R^TN^T+R^TN+RN^T+RN)a+\\
&\quad +\mathrm{Tr}(NR^T)+\mathrm{Tr}(NR),\\
[a^*Na, aRa] &= -a[R^TN+RN)a,\\
[a^*Ma, a^*Ra^*] &= a^*(R^TM^T+RM^T)a^*,
\end{aligned} \tag{10}$$

where $a^*Ma = a_i^*M_{ik}a_k$, etc.

§ 4.5. Let a_i, a_i^*, $i = 1, 2, \ldots, n$, be a set of boson operators. Show that the bilinear combinations

$$F_{ij} = a_ia_j, \quad G_{ij} = a_i^*a_j+\tfrac{1}{2}\delta_{ij}, \quad H_{ij} = a_i^*a_j^* \tag{11}$$

satisfy the following commutation relations

$$[G_{ij}, G_{kl}] = \delta_{jk}G_{il}-\delta_{il}G_{jk}, \tag{12}$$

$$[F_{ij}, G_{kl}] = \delta_{jk}F_{il}+\delta_{ik}F_{il} \tag{13}$$

and generate the Lie algebra sp(n, C).

§ 4.6. Find the set M^{nm} of matrices in order that the bilinear combinations $a^*M^{nm}a$ form

(i) Lie algebra $u(n)$,

(ii) Lie algebra so(n).

§ 4.7. Let the operators b_k satisfy the following anticommutation relations

$$[b_l, b_k]_+ = 0 = [b_l^*, b_k^*]_+, \quad [b_l, b_k^*]_+ = \delta_{lk}I, \quad l, k = 1, 2, \ldots, n \tag{14}$$

Show that the representations of the algebra (14) is equivalent to those of a finite group of order 2^n.

§ 4.8. Denote by Z^+ the set of positive integers and by A_k, $k\in Z^+$ a copy of the algebra of all matrices of rank two with complex entries. Let $A = \bigotimes_{k\in Z} A_k$ and denote by σ_k^μ, $\mu = 0, 1, 2, 3$ the canonical imbedding of the identity σ^0 and Pauli matrices σ^l, $l = 1, 2, 3$ from A_k into A. Show that

$$\begin{aligned}
b_k &= \tfrac{1}{2}\sigma_1^3\sigma_2^3\ldots\sigma_{k-1}^3(\sigma_k^1+i\sigma_k^2),\\
b_k^* &= \tfrac{1}{2}\sigma_1^3\sigma_2^3\ldots\sigma_{k-1}^3(\sigma_k^1-i\sigma_k^2),
\end{aligned} \tag{15}$$

$k = 1, 2, \ldots$, satisfy canonical anticommutation relations (14) with $n = \infty$.

Chapter 11

Representation Theory of Lie and Enveloping Algebras by Unbounded Operators: Analytic Vectors and Integrability

We present in this chapter the general theory of representations of Lie and enveloping algebras by linear unbounded operators in a Hilbert space. This is one of the most interesting and at the same time difficult branches of modern mathematics. It requires a knowledge of algebra, topology, functional analysis and differential manifolds. The theory provides also a rigorous framework for various problems in quantum theory and particle physics.

Even in non-relativistic quantum mechanics the observables such as position, momenta and angular momenta are represented by partial differential operators which are unbounded in the space of physical states.

In sec. 1 we discuss Gårding's representation theory of Lie algebras by unbounded operators. In sec. 2 we extend this theory to the enveloping algebra E of a Lie algebra. In particular, we derive the fundamental theorem which determines when an element Y of E is essentially self-adjoint.

The basic concept of analytic vectors and analytic dominance of operators is introduced in sec. 3. We derive here a series of important theorems on analytic vectors for self-adjoint operators and analytic dominance in Lie and enveloping algebras.

In sec. 4 we introduce the concept of analytic vectors for a representation T of a group G and show that analytic vectors for Lie algebras and for group representations coincide. We show also that analytic vectors for the Nelson operator $\Delta = X_1^2 + \ldots + X_d^2$, $d = \dim L$, are also analytic vectors for the group representations.

Sec. 5 contains the Nelson's criterion for a skew-symmetric representation of a Lie algebra L to be integrable to a global unitary representation of the corresponding simply-connected Lie group G.

In sec. 6 the beautiful integrability theory of Lie algebras representations of Flato, Simon, Snellman and Sternheimer is presented. This theory is based on the concept of weak analyticity and allows to express the integrability conditions in term of properties of Lie generators of the Lie algebra. In contrast to Nelson's theory it reduces in most practical cases the problem of integrability to the problem of verification of simple properties of first order differential operators. The derived criteria of integrability can be easily verified in applications and are rather important in quantum physics.

We present in sec. 7 an elegant method of an explicit construction of a dense set of analytic vectors for a representaton T of G using the solutions of the heat equation on G. This set represents a common dense invariant domain for the Lie and enveloping algebras of G.

Finally in sec. 8 we present the Gel'fand–Zetlin technique for the construction of irreducible representations of $u(p, q)$ using diagrammatic method.

The applications to quantum theory will be considered in chs. 12, 13, 17, 20, and 21.

§ 1. Representations of Lie Algebras by Unbounded Operators

A. *General Properties of Representations of Lie Algebras*

In the finite-dimensional case, a representation $X \to T(X)$ of a Lie algebra L was defined as a homomorphism of L into $\mathrm{gl}(n, C)$, i.e., for X, Y in L and α, β in C^1, we have

$$\alpha X+\beta Y \to \alpha T(X)+\beta T(Y), \tag{1}$$

$$[X, Y] \to [T(X), T(Y)] = T(X)T(Y)-T(Y)T(X), \tag{2}$$

where $T(\cdot)$ is an element of $\mathrm{gl}(n, C)$ (cf. ch. 1.1.C). The equality $[T(X), T(Y)] = T(X)T(Y)-T(Y)T(X)$ follows from the fact that every Lie algebra L is a subalgebra of $\mathrm{gL}(n, C)$ in which the commutation relations are $[X, Y] = XY-YX$.

One of the principal difficulties in the general representation theory of Lie algebras is due to the fact that, in many important cases, representatives $T(X)$ of elements of a Lie algebra are given by unbounded operators (cf. example 1). Hence, we have to consider the problem of the selection of a proper common domain D for a set of unbounded operators. This is a fundamental problem in functional analysis. A review of basic results in functional analysis is given in appendix B. We refer readers not familiar with these results to consult this appendix.

Because we want to consider the adjoint $T(X)^*$ together with a representative $T(X)$, a common domain D has to be dense in the carrier space H. The domain D cannot, however, be the whole space, because on such a domain only bounded operators would be defined (cf. appendix B, lemma 1.2) Moreover, because we want to define the commutator $T(X)T(Y)-T(Y)T(X)$ for representatives $T(X)$ and $T(Y)$, the range $R(T(X))$ has to be in D for any X in L. Hence, D must be invariant. We, therefore, arrive at the following general definition of a representation of an abstract Lie algebra L:

DEFINITION 1. *A representation T of L in a Hilbert space H* is any homomorphism $X \to T(X)$, $X \in L$, of L into a set of linear operators having a common linear dense invariant domain D. ▼

Definition 1 means that for arbitrary X, Y in L, α, β in C^1 and u in D, we have

$$T(\alpha X+\beta Y)u = \alpha T(X)u+\beta T(Y)u, \tag{3}$$

$$T([X, Y])u = [T(X), T(Y)]u = \big(T(X)T(Y)-T(Y)T(X)\big)u. \tag{4}$$

Note that by eq. (4)

$$[T(X), [T(Y), T(Z)]]u+[T(Y), [T(Z), T(X)]]u+[T(Z), [T(X), T(Y)]]u = 0,$$

i.e., the Jacobi identity is automatically satisfied.

Because the domain D is invariant, any representation T of a Lie algebra L can be extended to a representation of the enveloping algebra of L.

The set $N = T^{-1}(0) \subset L$ is an ideal of L. Indeed, if $X \in N$ and $Y \in L$, then $T([X, Y]) = [T(X), T(Y)] = 0$, i.e., $[X, Y] \in N$. Hence, in particular, non-trivial representations of simple Lie algebras are faithful, i.e., the map $X \to T(X)$ is one-to-one.

A representation T of L is said to be *skew-adjoint* (*skew-symmetric*), if the homomorphism $X \to T(X)$ maps L into a set of skew-adjoint (skew-symmetric) operators. Clearly, in a skew-adjoint representation T, the operators $iT(X)$ are self-adjoint (hermitian) and satisfy the commutation relations with purely imaginary structure constants.

A representation T is said to be *topologically irreducible* if there is no proper closed subspace $H' \subset H$ containing a common, linear invariant domain $D' \subset H'$ of L which is dense in H'.

EXAMPLE 1. Let L be the Poincaré–Lie algebra, and let $H = L^2(\Omega)$, where Ω is the four-dimensional Minkowski space. The commutation relations for L are given by eqs. 1.1 (23a–c). We verify that these commutation relations are satisfied by the following formal differential operators

$$M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad P_\mu = \partial_\mu, \quad \nu, \mu = 0, 1, 2, 3, \tag{5}$$

where $\partial_\mu = \partial/\partial x^\mu$. In order to obtain a representation T of L, we have to determine a common, dense, linear invariant domain D for operators (5). We can take either one of the following two dense subspaces of $L^2(\Omega)$

1° $D = C_0^\infty(\Omega)$, (6)

2° $D = S(\Omega)$, (7)

where $S(\Omega)$ is the Schwartz space of $C^\infty(\Omega)$-functions $\varphi(x)$ with

$$\sup_x |x^\alpha D^\beta \varphi(x)| < \infty, \tag{8}$$

where

$$x^\alpha \equiv x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}, \tag{9}$$

$$D^\beta \equiv \partial^{|\beta|}/\partial x_0^{\beta_0}\partial x_1^{\beta_1}\partial x_2^{\beta_2}\partial x_3^{\beta_3}, \quad |\beta| = \beta_0+\beta_1+\beta_2+\beta_3, \tag{10}$$

$$\alpha_\mu, \beta_\mu = 0, 1, 2, \ldots, \quad \mu = 0, 1, 2, 3.$$

It is easy to verify that all generators (5) are skew-symmetric on these domains, and that these domains are invariant under the operators (5).

B. *Gårding's Theory*

We now discuss the standard method for the construction of a common linear invariant dense domain for a representation $X \to T(X)$ of the algebra L, given a representation $x \to T_x$ of the associated Lie group G. Let $x(t) = \exp(tX)$, $X \in L$, be a one-parameter subgroup of G and $T_{x(t)}$ the corresponding one-parameter subgroup of operators. If for $u \in H$, $\lim_{t\to 0} t^{-1}(T_{x(t)} - I)u$ exists, then the action of the generator $T(X)$ of the subgroup $T_{x(t)}$ is defined by the formula

$$T(X)u = \lim_{t\to 0} t^{-1}(T_{x(t)} - I)u, \quad x(t) = \exp(tX). \tag{11}$$

The set of all $u \in H$, for which the right hand side of eq. (11) is defined, is said to be the *domain* of $T(X)$.

Let $C_0^\infty(G)$ be the set of all infinitely differentiable functions with compact support in the group space G, and let $T(\varphi)$, $\varphi \in C_0^\infty(G)$, be a new 'smeared out' operator defined by (cf. appendix B.2 for the theory of integration of operator functions)

$$T(\varphi)u = \int_G \varphi(x) T_x u \mathrm{d}x, \quad u \in H. \tag{12}$$

Denote by D_G the linear subspace spanned by all vectors $u(\varphi) \equiv T(\varphi)u$, $u \in H$. For $u(\varphi) \in D_G$ we have therefore

$$T_y u(\varphi) = u(L_y \varphi).$$

Indeed,

$$T_y u(\varphi) = \int_G \varphi(x) T_{yx} u \mathrm{d}x = \int_G \varphi(y^{-1}z) T_z u \mathrm{d}z = \int (L_y\varphi)(z) T_z u \mathrm{d}z. \tag{13}$$

THEOREM 1. *Let T be a representation of a Lie group G in a Hilbert space H. Then*

1° *The subspace D_G is dense in H.*

2° *The subspace D_G is a common linear invariant domain for the generators of one-parameter subgroups of G.*

PROOF: *ad* 1°. Let $\varphi \in C_0^\infty(G)$ with support K be such that

$$\varphi \geqslant 0, \qquad \int_K \varphi(x) \mathrm{d}x = 1.$$

Then, for any u in H, by eq. (12), we have

$$u(\varphi) - u = \int_G \varphi(x)(T_x - I)u \mathrm{d}x.$$

Hence,

$$\|u(\varphi) - u\| \leqslant \max_{x \in K} \|T_x u - u\|,$$

Consequently, if K shrinks to the identity e in G, the vector $u(\varphi) \to u$ by continuity of T_x. Because u is an arbitrary vector in H, this shows that the set D_G of vectors (2) is dense in H.

ad 2°. Let $y(t) = \exp(tY)$ be a one-parameter subgroup of G. Due to the invariance of the Haar measure dx on G, we obtain

$$\int_G \varphi\big(y^{-1}(t)x\big)T_x u\,dx = \int_G \varphi(x)\,T_{y(t)x}u\,dx = T_{y(t)}\int \varphi(x)\,T_x u\,dx.$$

Hence,

$$t^{-1}(T_{y(t)}-I)u(\varphi) = \int t^{-1}[\varphi\big(y^{-1}(t)x\big)-\varphi(x)]\,T_x u\,dx. \tag{14}$$

For every v in H the function $|t^{-1}[\varphi\big(y^{-1}(t)x\big)-\varphi(x)](T_x u, v)|$, is integrable on G and the limit $t \to 0$ is in $C_0^\infty(G)$; hence, using Lebesgue theorem (app. A.6) we can interchange $\lim\limits_{t\to 0}$ with the integral sign. Thus for $t \to 0$ one obtains

$$T(Y)u(\varphi) = u(\tilde{Y}\varphi), \tag{15}$$

where

$$(\tilde{Y}\varphi)(x) = \lim_{t\to 0}\frac{\varphi\big(y^{-1}(t)x\big)-\varphi(x)}{t} \in C_0^\infty(G) \tag{16}$$

gives the action of a left regular representation of L in the space $C_0^\infty(G)$. Hence, for any $u(\varphi)$ in D_G and any generator $T(Y)$, $T(Y)u(\varphi)$ is also in D_G. This means that D_G is the common invariant dense domain for all elements of the Lie algebra L of the Lie group G. It is evident that this domain is linear. ▼

The domain D_G in question is called the *Gårding subspace*.

Remark 1: The map $Y \to \tilde{Y}$ given by eq. (16) is a representation of the Lie algebra L by means of right invariant first order differential operators acting on $C_0^\infty(G)$.

COROLLARY 1. *Let L be the Lie algebra of G and let $x \to T_x$ be a representation of G. Then, the map $X \to T(X)$, $X \in L$, given by eq.* (15), *is the representation of L.*

PROOF: Because D_G is the common, linear, dense invariant subspace of H, and the condition (3) is obviously satisfied, it suffices to verify the condition (4). Indeed, we have by eq. (15)

$$\begin{aligned} T[X, Y]u(\varphi) &= u([\tilde{X}, \tilde{Y}]\varphi) = u[(\tilde{X}\tilde{Y}-\tilde{Y}\tilde{X})\varphi] \\ &= T(X)u(\tilde{Y}\varphi)-T(Y)u(\tilde{X}\varphi) \\ &= \big(T(X)T(Y)-T(Y)T(X)\big)u(\varphi) = [T(X), T(Y)]u(\varphi). \end{aligned}$$

Remark 2: Due to the invariance of D_G, we can define by means of eq. (15) the action of any element

$$M = \sum_{\substack{n \\ i_1 \dots i_n}} a_{i_1\dots i_n}X_{i_1}\dots X_{i_n}, \qquad X_{i_r} \in L,$$

of the enveloping algebra E of the Lie algebra L by the formula

$$T(M)u(\varphi) = u(\tilde{M}\varphi), \qquad M \in E, \tag{17}$$

where $\tilde{M}$ is the 'left' differential operator on $C_0^\infty(G)$ corresponding to the element M in E. Hence, the representation $X \to T(X)$ of L can be extended to a representation $M \to T(M)$ of the enveloping algebra E of L.

PROPOSITION 2. *Let $x \to T_x$ be a unitary representation of a Lie group G. Then, the operators* $\mathrm{i}T(X)$, $X \in L$, *are symmetric.*

PROOF: Let $u, v \in D_G$. Then,

$$\big(\mathrm{i}T(X)u, v\big) = \lim_{t\to 0} t^{-1}\big((\mathrm{i}T_{x(t)} - I)u, v\big) = \lim_{t\to 0} t^{-1}\big(u, -\mathrm{i}(T^*_{x(t)} - I)v\big).$$

Because for a unitary representation $T^*_{x(t)} = T^{-1}_{x(t)} = T_{x(t)^{-1}} = T_{x(-t)}$, we obtain

$$\big(\mathrm{i}T(X)u, v\big) = \lim_{t\to 0} t^{-1}\big(u, -\mathrm{i}(T_{x(-t)} - I)v\big) = \lim_{s\to 0} s^{-1}\big(u, \mathrm{i}(T_{x(s)} - I)v\big)$$
$$= \big(u, \mathrm{i}T(X)v\big).$$

Let T be a representation of G in H. A vector u in H is said to be an *infinitely differentiable* or *regular* vector for T if the mapping $x \to T_x u$ of G into H is of class C^∞. A vector u in H is said to be *analytic* for T if the mapping $x \to T_x u$ of G into H is analytic. Every element $u(\varphi)$ in D_G is a regular vector for T. In fact, using the same arguments as in the proof ad 2° of th. 1, for every $n = 1, 2, \ldots$, we obtain

$$\partial_i^{(n)} T_x u(\varphi) = \partial_i^{(n)} \int_G \varphi(y) T_{xy} u \,\mathrm{d}y = \partial_i^{(n)} \int_G \varphi(x^{-1}y) T_y u \,\mathrm{d}y = \int_G \partial_i^{(n)} \varphi(x^{-1}y) T_y u \,\mathrm{d}y. \tag{18}$$

Because $\partial_i^{(n)}\varphi(x^{-1}y) \in C_0^\infty$, partial mixed derivatives of all orders are well defined and therefore D_G is a dense set of regular vectors for T. In secs. 4 and 6, we describe the construction, due to Nelson and Gårding, of a dense set of analytic vectors for T.

Sometimes it is convenient to take as the domain of the operators representing a given Lie algebra L, a subspace D in H other than the Gårding subspace D_G. For example

1° If T is a quasi-regular representation of G on a homogeneous space G/H, then $D = C_0^\infty(G/H)$ is the natural domain.

2° If T is the restriction to G of a representation of a larger group, then the Gårding subspace of the larger representation might be taken as the domain. We use this domain in the derivation of the Nelson–Stinespring results (cf. sec. 2, corollaries 1–5).

3° If $H = L^2(\Omega)$ and L is given by formal differential operators, then the subspace $C_0^\infty(\Omega)$ or Schwartz's space S can be taken as the domain for a representation T of L.

4° The space of analytic vectors for T of G, associated with the operator $T(\Delta) = T(X_1)^2 + \ldots + T(X_d^2)$, $d = \dim L$, can be used as the domain (cf. sec. 4). We use this domain for the solution of the problem of integrability of a given skew-symmetric representation of a Lie algebra to a global unitary representation of the corresponding Lie group (cf. sec. 5).

5° The space of analytic vectors for T of G associated with Lie generators of $T(L)$. This space is most convenient in applications (cf. sec. 6).

6° The space of analytic vectors for T of G, associated with the solutions of the so-called heat equation on the Lie group, can be taken as the domain D (cf. sec. 7).

§ 2. Representations of Enveloping Algebras by Unbounded Operators

We have stated that most of the observables in quantum theory and in particle physics are elements of enveloping algebras. In order to insure a proper interpretation of measurements, we require that these observables be represented by at least essentially self-adjoint operators. There is a widespread belief among physicists that in a unitary representation of a Lie group the elements of an enveloping algebra are always essentially self-adjoint operators. The following counter-example, due to von Neumann (unpublished), shows that this is not true.

EXAMPLE 1. Let G be the three-dimensional nilpotent group of all real matrices of the form

$$\begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}, \qquad \alpha, \beta, \gamma \in R^1. \tag{1}$$

The composition law in G is given by the formula

$$[\alpha, \beta, \gamma][\alpha', \beta', \gamma'] = [\alpha+\alpha', \beta+\beta', \gamma+\alpha\beta'+\gamma']. \tag{2}$$

The subgroup $[0, 0, \gamma]$, $\gamma \in R^1$, is the center of G.

Let $H = L^2(-\infty, \infty)$. The map

$$T_{[\alpha,\beta,\gamma]}u(x) = \exp[\mathrm{i}\lambda(\gamma+x\beta)]u(x+\alpha), \qquad u \in H, \tag{3}$$

defines a unitary representation of G in H. By definition 1 (11) we find that the generators of the one-parameter subgroups of G corresponding to the parameters α, β and γ have the form

$$T(X) = \mathrm{d}/\mathrm{d}x, \qquad T(Y) = \mathrm{i}\lambda x, \qquad T(Z) = \mathrm{i}\lambda. \tag{4}$$

For example, for the subgroup $[0, \beta, 0]$, we have by eq. 1 (11) and eq. (3),

$$T(Y)u = \lim_{\beta\to 0} \frac{\exp(\mathrm{i}\lambda x\beta)u-u}{\beta} = \mathrm{i}\lambda xu. \tag{5}$$

The generators (4) satisfy $[T(X), T(Y)] = T(Z)$ which for $\lambda = -1$ is equivalent to the Heisenberg commutation relations $[p, q] = -\mathrm{i}$. Therefore, the enveloping algebra of G is mapped onto all ordinary differential operators with polynomial coefficients. It is well known that many of these operators are symmetric but not essentially self-adjoint. ▼

This example shows that the unitarity of a group representation does not guarantee that images of elements of the enveloping algebra E are represented by essentially self-adjoint operators. Hence, we have to find some additional criteria which will allow us to determine when an element M of E is represented by an essentially self-adjoint operator $T(M)$.

We have seen that representatives $iT(X)$, $X \in L(G)$, associated with a unitary representation T of G are symmetric operators on the Gårding domain D_G (cf. sec. 1.B., proposition 2). We now extend this result to certain elements of the enveloping algebra E of L. In the following we deal with the universal enveloping algebra of the right invariant real Lie algebra L (cf. ch. 3.3.F).

We define in E a $+$-operation by

$$M = \sum a_{i_1 \ldots i_n} X_{i_1} \ldots X_{i_n} \to M^+ \equiv \sum_{\substack{n \\ i_1 \ldots i_n}} \bar{a}_{i_1 \ldots i_n} X_{i_n}^+ \ldots X_{i_1}^+, \tag{6}$$

where for every X in L

$$X^+ \equiv -X. \tag{7}$$

The map $M \to M^+$ defines an involution in E. An element M is said to be *symmetric* in E if $M^+ = M$.

PROPOSITION 1. *Let*

$$M = \sum_{\substack{n \\ i_1 \ldots i_n}} a_{i_1 \ldots i_n} X_{i_1} \ldots X_{i_n} \in E. \tag{8}$$

The operator $T(M)$ *defined by eq.* 1 (17) *satisfies*

$$(T(M)u, v) = (u, T(M^+)v), \quad u, v \in D_G. \tag{9}$$

In particular if $M = M^+$ *in* E *then* $T(M)$ *is a symmetric operator in* H.

PROOF: We repeat the derivation of proposition 1.2 for a product $X_{i_1}^+ \ldots X_{i_n}^+$, and obtain

$$\begin{aligned}(T(M^+)u, v) &= \Big(\sum_{\substack{n \\ i_1 \ldots i_n}} \bar{a}_{i_1 \ldots i_n} T(X_{i_n}^+) \ldots T(X_{i_1}^+)u, v\Big) \\ &= \Big(u, \sum_{\substack{n \\ i_1 \ldots i_n}} a_{i_1 \ldots i_n} T(X_{i_1}) \ldots T(X_{i_n})v\Big) = (u, T(M)v).\end{aligned} \tag{10}$$

Hence, if $M^+ = M$, then $T(M)^* = T(M)$, i.e., $T(M)$ is symmetric on D_G. ▼

We now derive the Nelson–Stinespring criteria which will determine when a symmetric representative $T(M)$ of an element M of the enveloping algebra E is given by an essentially self-adjoint operator.

In these considerations an important role is played by the so-called elliptic elements of the enveloping algebra. An element L in E is said to be *elliptic* if it is elliptic as a partial differential operator on G. We recall that a formal differential operator

$$L(x, D) = \sum_{0 \leqslant \alpha \leqslant \sigma} a_\alpha(x) D^\alpha, \quad x = \{x_1, \ldots, x_n\} \in G, \tag{11}$$

where

$$D^\alpha \equiv \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n \tag{12}$$

is elliptic if for any vector $\xi = (\xi_1, \ldots, \xi_n) \in R^n$ the σ-linear form

$$L(x, \xi) = \sum_{|\alpha| = \sigma} a_\alpha(x) \xi^\alpha, \quad \xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}, \ \xi \neq 0, \tag{13}$$

is different from zero.

THEOREM 2. *Let G be a Lie group and $x \to T_x$ be a unitary representation of G. If L is an elliptic element of the right invariant enveloping algebra of G, then*

$$\overline{T(L^+)} = \big(T(L)\big)^*. \tag{14}$$

In particular, if L is elliptic and symmetric, then $T(L)$ is essentially self-adjoint (e.s.a.).

PROOF: Consider first the special case of an elliptic element L represented in the form $L = K^+K$ for some $K \in E$. We combine the following two observations to prove that $A \equiv T(L)$ is e.s.a.

(i) $(A+I)^{-1}$ is bounded. To show this notice that $(Au, u) \geqslant 0$ for all u in Gårding subspace D_G. In fact, because the map $X \to T(X)$ defines a representation of E, we have $A = T(K^+K) = T(K^+)T(K)$. Thus it follows from proposition 1 that

$$(Au, u) = \big(T(K^+)T(K)u, u\big) = \big(T(K)u, T(K)u\big) \geqslant 0,$$

for all u in D_G. Consequently,

$$\big((A+I)u, (A+I)u\big) = (Au, Au) + 2(Au, u) + (u, u) \geqslant (u, u) > 0$$

hence $(A+I)$ is positive definite, and on the domain on the operator $(A+I)^{-1}$ consisting of vectors $v = (A+I)u$,

$$\|(A+I)^{-1}v\|^2 \leqslant \|v\|^2,$$

i.e., $(A+I)^{-1}$ is bounded.

(ii) $A+I$ has a dense range. To show this, suppose that $u \in H$ is orthogonal to the range of $A+I$. Then,

$$\big((A+I)T(\varphi)u, u\big) = 0 \quad \text{for all } \varphi \in C_0^\infty(G). \tag{15}$$

It follows from eq. 1 (17) that for $M \in E$, $T(M)T(\varphi)u = T(\tilde{M}\varphi)u$. Hence, eq. (15) can be written in the form

$$\int_G [(\tilde{L}+1)\varphi](x)(T_x u, u)\mathrm{d}x = 0. \tag{16}$$

This means that the function $f(x) = (T_x u, u)$ is a weak solution of the partial differential equation $(\tilde{L}+1)f = 0$ on G. Because $\tilde{L}+1$ is elliptic, $(T_x u, u)$ is analytic (cf. F. John 1951) and

$$(\tilde{L}+1)(T_x u, u) = 0 \tag{17}$$

in the ordinary pointwise sense. Because the function $x \to (T_x u, u)$ is positive definite from exercise 3.11.3.4* we have, for $u \neq 0$,

$$((\tilde{L}+1)f)(e) = (\tilde{L}f)(e) + (u, u) > 0$$

which contradicts eq. (17). Hence, $u = 0$ and thus $(A+I)^{-1}$ is bounded and densely defined. These two observations imply that A is e.s.a. by lemma 5.3 of app. B.

Now let L be a general elliptic element. From the previous case, we deduce that $T(L^+L) = T(L^+)T(L)$ is e.s.a. This implies

$$\overline{T(L^+)} = T(L)^*$$

according to lemma 5.4 of app. B. Consequently, if an elliptic element $L \in E$ is symmetric (i.e., $L^+ = L$), then

$$\overline{T(L)} = (T(L))^*,$$

i.e., $T(L)$ is essentially self-adjoint. ▼

A more general criterion for the essential self-adjointness of elements of the enveloping algebra is provided by the following important theorem.

THEOREM 3. *Let G be a Lie group and let T be a unitary representation of G. Let L be an elliptic element of the right invariant enveloping algebra E of G, such that $L^+ = L$. If M is any arbitrary element of E such that $T(M^+M)$ commutes with $T(L)$, then*

$$\overline{T(M^+)} = T(M)^*. \tag{18}$$

In particular, if M is in addition symmetric, then $T(M)$ is essentially self-adjoint.

PROOF: Let r be a positive integer greater than the order of the differential operator $\tilde{M}$ corresponding to an element M in E, and let $A = T(L^{2r})$, $B = T(\tilde{M}^+\tilde{M})\big(=T(\tilde{M}^+)T(\tilde{M})\big)$ and $C = A+B$. Then, L^{2r} is elliptic because L is elliptic; and $L^{2r}+\tilde{M}^+\tilde{M}$ is elliptic, because the order of L^{2r} is greater than the order of $\tilde{M}^+\tilde{M}$. Consequently, A and C are representatives of elliptic symmetric operators in E and consequently are e.s.a. by th. 2. Moreover, these operators commute on the Gårding domain.

We now show that the closures $\bar{A}$ and $\bar{C}$ also commute, i.e., these operators have mutually commuting spectral resolutions. To see this, notice that the bounded operators $(1+A)^{-1}(1+C)^{-1}$ and $(1+C)^{-1}(1+A)^{-1}$ coincide on their common domain, i.e. on the range of $(1+A)(1+C) = (1+C)(1+A)$. Moreover, the operator $(1+A)(1+C)$ is the representative of a symmetric elliptic operator and so is e.s.a., by th. 2. It has also a dense range. This follows from the fact that the operator $D \equiv A+C+AC$ is positive definite. Hence, the operator $I+\bar{D}$ is positive definite and self-adjoint; and, therefore $(I+\bar{D})^{-1}$ is bounded. This implies that the range $R(I+\bar{D}) = R(\overline{I+D}) = H$, i.e., $R(I+D)$ is dense in H. Consequently, $\bar{A}$ and $\bar{C}$ commute.

We now prove the main part of the theorem, namely, $\overline{T(M^+)} = T(M)^*$. Notice first that $T(M^+) \subset (T(M))^*$ by proposition 1. Hence, if we can show that the operator $B = T(M^+)T(M)$ is e.s.a., then the assertion of the theorem follows from lemma 5.4, app. B. Set $B_1 = \bar{C}-\bar{A}$. To show that B is e.s.a., we first prove that $B_1 \subset \bar{B}$. Let $v \in D(B_1) = D(\bar{A}) \cap D(\bar{C})$. We can select a sequence $\{v_n\}_1^\infty$ such that

$$D_G \in v_n \to v \quad \text{and} \quad Cv_n \to \bar{C}v, \tag{19}$$

because $\bar{C}$ is the closure of C. For an arbitrary u in D_G, we have $||Au|| \leqslant ||Cu||$. In fact, because $A = T(L^r)T(L^r)$; and the fact that $T(L)$ and $T(M^+M)$ commute, we have

$$(Cu, Cu) = (Au, Au)+(Bu, Bu)+2\big(BT(L^r)u, T(L^r)u\big).$$

Here the second and the third terms are positive (because B is positive definite) and consequently $||Cu|| \geqslant ||Au||$. Setting $u = v_n - v_m$, we obtain

$$||Av_n - Av_m|| \leqslant ||Cv_n - Cv_m|| \to 0 \quad \text{for} \quad m > n \to \infty.$$

Hence, the sequence $\{Av_n\}$ is convergent and

$$Av_n \to \bar{A}v, \tag{20}$$

because $\bar{A}$ is closed. Using eqs. (19) and (20), we obtain

$$Bv_n = Cv_n - Av_n \to \bar{C}v - \bar{A}v = B_1 v.$$

Therefore, $B_1 v = \bar{B}v$ for $v \in D(B_1)$, i.e., $B_1 \subset \bar{B}$. The operator B_1 is e.s.a., by lemma 5.4 of app. B

$$\bar{B}_1 = B_1^* \supset \bar{B}^* \supset \bar{B} \supset \bar{B}_1.$$

Consequently, we have $\bar{B}^* = \bar{B}$, i.e. B is e.s.a. ▼

Th. 3 implies a series of corollaries to determine when an element M in the enveloping algebra E is represented by an essentially self-adjoint operator.

In the following, an important role is played by the so-called *Nelson operator* Δ,

$$\Delta = X_1^2 + \dots + X_d^2, \quad d = \dim L, \tag{21}$$

where X_i, $i = 1, 2, \dots, d$, are generators of G. This operator is elliptic. Indeed, because $X_i = a_{ik}(x)\partial^k$, then by eq. (13) we have

$$\Delta(x, \xi) = a_{ik}(x)\, a_{ik'}(x)\xi^k \xi^{k'} = b^2 > 0, \tag{22}$$

where $b_i(x) = a_{ik}(x)\xi^k$.

COROLLARY 1. *Let G be an abelian or a compact Lie group. Then the representative $T(M)$ of an arbitrary element M of the enveloping algebra E satisfies the condition* (18), *i.e.*,

$$\overline{T(M^+)} = (T(M))^*.$$

In particular, if $M^+ = M$, then $T(M)$ is essentially self-adjoint.

PROOF: The enveloping algebra of every abelian Lie algebra contains the elliptic, symmetric element of the form

$$\Delta_0 = X_1^2 + \dots + X_d^2,$$

which obviously is the center of E. Hence, every M in E is e.s.a. by th. 3.

Every compact Lie group is a direct product of its center G_0 and of invariant simple subgroups G_i, $i = 1, 2, \dots, N$ (cf. th. 3.8.2). The operator

$$\Delta^{(i)} = (X_1^{(i)})^2 + (X_2^{(i)})^2 + \dots + (X_{d_i}^{(i)})^2, \quad i = 0, 1, 2, \dots, N, \quad d_i = \dim G_i,$$

in the basis in which the Cartan metric tensor of the Lie algebra of G_i is diagonal, is the central symmetric elliptic element of the enveloping algebra E_i of G_i, $i = 0, 1, \dots, N$. Thus, the operator $\Delta = \sum_{i=0}^{N} \Delta^{(i)}$ is the central symmetric elliptic element of the enveloping algebra E of G. Consequently, for every M in E, we have

$$\overline{T(M^+)} = (T(M))^*,$$

by th. 3. Then, if $M^+ = M$, $T(M)$ is e.s.a. ▼

Next we consider the case of noncompact, semisimple Lie groups. In this case, we have

COROLLARY 2. *Let G be a noncompact semisimple Lie group, K a maximal compact subgroup of G and let*

$$\Delta_k = \sum_{i=1}^{\dim K} X_i^2$$

be the second-order Casimir operator of K. Then, the representative $T(M)$ of any element M of the enveloping algebra E of G, which commutes with Δ_k, satisfies the condition $\overline{T(M^+)} = (T(M))^*$.

Remark: In particular, all symmetric Casimir operators of G or of a subgroup $G_l \supset K$ are essentially self-adjoint.

PROOF OF COROLLARY 2: It follows from th. 1.2.7 that the Cartan metric tensor of the Lie algebra of G can be diagonalized. Hence the second-order Casimir operator of G has the form $C_2 = -\Delta_K + \Delta_P$, where $\Delta_p = \sum_{\dim K+1}^{\dim L} X_i^2$. The corresponding Nelson operator $\Delta = (\Delta_K + \Delta_P)$ is elliptic and symmetric on G (cf. eq. (22)). Because $-\Delta_K + \Delta_P$ is central in E, any M in the enveloping algebra E

of G, which commutes with Δ_K, also commutes with $(\Delta_K+\Delta_P) = C_2+2\Delta_K$. Consequently,

$$[M^+,(\Delta_K+\Delta_P)] = [M,(\Delta_K+\Delta_P)]^+ = 0.$$

Hence,

$$\overline{T(M^+)} = (T(M))$$

by th. 3. ▼

In the general case we have

COROLLARY 3. *Let G be an arbitrary Lie group and let M be a central element of E. Then, $\overline{T(M^+)} = (T(M))^*$. In particular if $M^+ = M$, then $T(M)$ is essentially self-adjoint.*

PROOF: The element M in E commutes with the symmetric elliptic element Δ given by eq. (21). Consequently, $[M^+,\Delta] = [\Delta, M]^+ = 0$. Hence, the assertion follows by th. 3. ▼

Remark: If M and N are central and symmetric elements of E, then the operators $T(M)$ and $T(N)$ are e.s.a. by corollary 3 and by virtue of the formula 1 (17).

However, it is not *a priori* evident that the self-adjoint operators $\overline{T(M)}$ and $\overline{T(N)}$ strongly commute and whether $\overline{T(M)}$ and T_x, $x \in G$, commute. These important problems are solved in sec. 5 after the elaboration of a criterion of commutativity (cf. th. 5.3). ▼

The next corollary shows that representatives of all generators associated with a unitary representation of G are essentially self-adjoint, i.e., they have proper spectral properties. The fact is of importance in physical applications.

COROLLARY 4. *Let $x \to T_x$ be a unitary representation of an arbitrary Lie group G. Let X be an arbitrary element of the Lie algebra of G and $p(X)$ any real polynomial. Then the operator $T(p(\mathrm{i}X))$ is essentially self-adjoint on D_G. In particular, $T(\mathrm{i}X)$ is essentially self-adjoint.*

PROOF: The element $p(\mathrm{i}X)$ is the symmetric element, $((p(\mathrm{i}X))^+ = p(\mathrm{i}X)$, of the enveloping algebra of the one-parameter (and therefore abelian) subgroup G_1 generated by X; hence, the representative $T(p(\mathrm{i}X))$ is e.s.a. on the Gårding domain D_G, by corollary 1. In particular, the representative of $p(\mathrm{i}X) = \mathrm{i}X$ is e.s.a. ▼

It should be emphasized that in a unitary representation of a Lie group the symmetric elements of the enveloping algebra are not necessarily represented by essentially self-adjoint operators on the Gårding subspace. To illustrate this important fact, we consider the following simple counter-example.

EXAMPLE 2. Let G be the two-parameter group of transformations $x' = ax+b$, $a > 0$ of the real line R and let $H = L^2(-\infty, +\infty)$. The unitary representation of G in H is given by the formula

$$(T_g u)(x) = a^{-1/2} u\left(\frac{x+b}{a}\right), \quad u \in H, \tag{23}$$

The generators of one-parameter subgroups are given by

$$T(X) = \mathrm{d}/\mathrm{d}x, \qquad T(Y) = \mathrm{i}\exp x. \tag{24}$$

The symmetric element $M = XY+YX$ in the enveloping algebra E of G has the representative $T(XY+YX) = 2\mathrm{i}e^x\mathrm{d}/\mathrm{d}x+\mathrm{i}e^x$, which is a symmetric operator by proposition 2.1. In order to verify that M is e.s.a. or may be extended to an e.s.a. operator we have find its deficiency index according to th. 1.4 of app. B. Solving the first-order differential equation $M^*u_\pm = \pm\mathrm{i}u_\pm$ we find

$$\begin{aligned} D_+ &= \left\{C\exp\left[-\frac{1}{2}(x+\exp(-x))\right]\right\}. \\ D_- &= \{0\}. \end{aligned} \tag{25}$$

The set D_- is trivial because the second solution

$$u_- = C\exp\left[-\frac{1}{2}(x-\exp(-x))\right]$$

does not belong even to $L^2(-\infty, +\infty)$. Hence, the deficiency indices are $(1,0)$ and consequently the operator $T(XY+YX)$ has no self-adjoint extension by th. of app. B. 1.4. ▼

It should also be emphasized that all the above results are true under the assumption that the representation of the Lie algebra was derived from a given unitary representation of the corresponding Lie group according to eq. 1 (11). These results might not be true if we have a representation of a Lie algebra which cannot be integrated to a global-unitary representation of the corresponding Lie group. An example is considered in ch. 21, § 5.

We now give an interesting application of group representation theory and of Nelson operator to quantum mechanics. It is well known that one of the basic problems of quantum mechanics is the construction of a domain $D \subset H$ on which the energy operator is self-adjoint. We give an example which provides a solution of this problem using group theoretical technique.

EXAMPLE 3. Let $X_1 = \partial/\partial x$, $X_2 = \mathrm{i}x$, $X_3 = \mathrm{i}\lambda x^2$ and $X_4 = \mathrm{i}I$ be skew-symmetric bases of a Lie algebra L on $L^2(R^1)$ with all $\lambda \in R^1$. It is evident that L is nilpotent with obvious commutation relations and that the subalgebra $\tilde{L}$ generated by elements X_2, X_3 and X_4 is invariant. By virtue of th. 3.5.1 we know that every group element of nilpotent Lie group G associated with L can be written as the product of one-parameter subgroups. Since all one-parameter subgroups in $L^2(R^1)$ generated by X_k are unitary the map

$$g(\alpha) \to T_{g(\alpha)} = \mathrm{e}^{\alpha_1 X_1}\mathrm{e}^{\alpha_2 X_2}\mathrm{e}^{\alpha_3 X_3}\mathrm{e}^{\alpha_4 X_4} \tag{26}$$

provides a unitary representation of G in $L^2(R^1)$.

By virtue of eqs. (22) and (6) the Nelson operator Δ is elliptic and symmetric. Hence by virtue of th. 2 the operator

$$T(\Delta) = \sum_{k=1}^{4} T(X_k)^2 = \frac{d^2}{dx^2} - x^2 - \lambda x^4 - I$$

is essentially self-adjoint on the Gårding domain D_G for the representation (26). This implies that the energy operator for the anharmonic oscillator which coincides with $-T(\Delta)-I$ is also essentially self-adjoint on D_G. ▼

It is evident that the above method can be generalized to a large class of energy operators $H = H_0 + V$ with the potential $V(x) = \sum_{k=1}^{n} c_k x^{2k}$.

§ 3. Analytic Vectors and Analytic Dominance

In this section we introduce two fundamental concepts: the concept of analytic vectors and the concept of analytic dominance.

In subsec. A, we discuss the basic properties of the analytic vectors for an unbounded operator and, in particular, prove that every self-adjoint operator has a dense set of analytic vectors. In subsec. B we develop a calculus of absolute values of operators, which provides the basis for the theory of analytic dominance of operators.

In subsec. C we prove the main theorem of the calculus of absolute values and we introduce, by means of it, the concept of analytic dominance.

Finally, we develop the theory of analytic dominance for a Lie algebra of operators.

A. *Analytic Vectors*

Let A be an operator in a Hilbert space H. An element $u \in H$ is said to be an *analytic vector for* A if the series expansion of $\exp(As)u$ has a positive radius of absolute convergence, i.e.,

$$\sum_{n=0}^{\infty} \frac{||A^n u||}{n!} s^n < \infty, \tag{1}$$

for some real $s > 0$. If A is bounded, i.e., $||Au|| \leqslant C||u||$ for every $u \in H$, then

$$\sum_{n=0}^{\infty} \frac{1}{n!} ||A^n u|| s^n \leqslant \sum_{n=0}^{\infty} \frac{1}{n!} C^n s^n = \exp(Cs) < \infty.$$

Hence, for a bounded operator A, every vector u in H is an analytic vector for A. Thus, only analytic vectors for unbounded operators will be of interest.

If u and v in H are analytic vectors for A, then $\alpha u + \beta v$, $\alpha, \beta \in C^1$, is also an analytic vector. Indeed, using the inequality

$$||A(\alpha u + \beta v)|| \leqslant |\alpha|\,||Au|| + |\beta|\,||Av||,$$

we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|A^n(\alpha u+\beta v)\| s^n \leqslant |\alpha| \sum_{n=0}^{\infty} \frac{\|A^n u\|}{n!} s^n + |\beta| \sum_{n=0}^{\infty} \frac{\|A^n v\|}{n!} s^n < \infty .$$

Hence, analytic vectors for an operator A form a linear subspace in H.

The concept of an analytic vector is useful because for some class of unbounded operators they form a dense set in the Hilbert space H. Indeed, we have:

LEMMA 1. *If an operator A is self-adjoint, then A has a dense set of analytic vectors.*

PROOF: This lemma is a direct consequence of the spectral theorem. Indeed, let $E(\lambda)$ be the spectral resolution of the identity, associated with A, and let $\delta = [a, b]$ be a bounded interval of the real line. Then, any vector in the range of $E(\delta)$ is an analytic vector for A. In fact, if $A = \int_{\mathrm{Sp}A} \lambda \mathrm{d}E(\lambda)$, then by eq. 3(16) of app. B we obtain

$$\|A^n E(\delta) u\| \leqslant |c|^n \|E(\delta)u\| \leqslant |c|^n \|u\|,$$

where $c = \max(|a|, |b|)$. Hence,

$$\sum_{n=0}^{\infty} \frac{\|A^n E(\delta) u\| s^n}{n!} \leqslant \|u\| \sum_{n=0}^{\infty} \frac{|c|^n s^n}{n!} = \|u\| \exp(|c|s) < \infty . \tag{2}$$

Consequently, $E(\delta)u$ is an analytic vector for A. According to the corollary 2 to the spectral theorem (cf. app. B.3), the set of all vectors of the form $v = E(\delta)u$, where $\delta = [a, b]$ runs over all finite intervals of R and u runs over H is dense in H. Hence, A has a dense set of analytic vectors. ▼

Remark 1: The inverse of lemma 1 is also true: If a closed symmetric operator has a dense set of analytic vectors, then it is self-adjoint (cf. Nelson 1959, lemma 5.1).

Remark 2: Clearly, iX has the same analytic vectors as X. Hence, lemma 1 and remark 1 remain true if 'symmetric' is replaced by 'skew-symmetric' and 'self-adjoint' by 'skew-adjoint'.

EXAMPLE 1. Let A be a self-adjoint unbounded operator in H. If A has a discrete spectrum (as for instance for

$$A = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\varphi} \qquad \text{in } L^2(0, 2\pi)),$$

then every eigenvector u_n of A with eigenvalue λ_n is an analytic vector. Indeed,

$$\sum_{k=0}^{\infty} \frac{\|A^k u_n\| s^k}{k!} = \sum_{k=0}^{\infty} \frac{\|u_n\| (\lambda_n s)^k}{k!} = \|u_n\| \exp(\lambda_n s) < \infty .$$

The linear envelope of $\{u_n\}$ forms the dense set of analytic vectors for A.

If A has only continuous spectrum (e.g.,

$$A = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x} \quad \text{in } L^2(-\infty, \infty)),$$

then, by the proof of lemma 1, the dense set of analytic vectors consists of all vectors of the form

$$v = E(\delta)u, \quad E(\delta) = E(\lambda)-E(\mu), \tag{3}$$

where $E(\lambda)$ is the spectral resolution of the identity associated with A, δ runs over all bounded subsets of R and u runs over the space H. ▼

B. *The Absolute Value of an Operator*

We now develop the calculus of the so-called absolute values of operators. Let $O(H)$ be the set of all linear operators in H. We recall that for A, B in $O(H)$ the sum $A+B$ has as domain $D(A)\cap D(B)$ and the product AB has as domain the set of all vectors $u \in D(B)$ such that Bu is in $D(A)$. If $u \in D(A)$, then $||Au||$ is well defined. If $u \notin D(A)$, then we set $||Au|| = \infty$.

Let A, B, $C \in O(H)$. If the relation

$$||Cu|| \leqslant ||Au||+||Bu|| \tag{4}$$

is satisfied for all $u \in H$, we *symbolically* represent it in the form

$$|C| \leqslant |A|+|B|. \tag{5}$$

The following relations are true for all operators A, B, C in $O(H)$:

1° $$|A+B| \leqslant |A|+|B|. \tag{6}$$

2° $$\text{If } |A| \leqslant |B|, \quad \text{then } |AC| \leqslant |BC|. \tag{7}$$

Indeed, if $u \in D(A)\cap D(B)$, then $||(A+B)u|| \leqslant ||Au||+||Bu||$. On the other hand if $u \notin D(A)\cap D(B)$, then $||Au||+||Bu|| = \infty$ by our convention, i.e., eq. (3) is still valid. If $||Au|| \leqslant ||Bu||$ for all u, then in particular for a vector $v = Cu$, we have $||ACu|| \leqslant ||BCu||$, i.e., inequality (7) is true.

By analogy with the absolute values of ordinary numbers, the symbol $|A|$ is called the *absolute value of A*. Formally, we can define the absolute value $|A|$ of A as the set consisting of A alone. Let $|O(H)|$ be the free abelian semigroup with the set of all $|A|$, with A in $O(H)$ as generators (cf. app. A.3). By definition of a free abelian semigroup, an element $\alpha \in |O(H)|$ is a finite formal sum of the form

$$\alpha = |A_1|+ \dots +|A_l|. \tag{8}$$

We used the concept of a free semigroup in the definition of $|O(H)|$ because we do not consider an inverse element to an element α in $|O(H)|$.

An element $\beta \in |O(H)|$ of the form

$$\beta = |B_1|+ \dots +|B_m| \tag{9}$$

is equal to α if the summands are identical, except possibly for their order. If a is a positive number, we shall identify a with $|aI|$, where I is the identity operator on H. We define the product $\alpha\beta$ of α and β given by eqs. (8) and (9) by the formula

$$\alpha\beta \equiv \sum_{i=1}^{l} \sum_{j=1}^{m} |A_i B_j|. \tag{10}$$

The set $|O(H)|$ with operation (10) is a semiring, and because of the identification $a \equiv |aI|$, it is a semialgebra. For $\alpha \in |O(H)|$ given by eq. (8), we define $||\alpha u||$ for all u in H by the formula

$$||\alpha u|| \equiv ||A_1 u|| + \ldots + ||A_l u||. \tag{11}$$

We set $\alpha \leqslant \beta$ if $||\alpha u|| \leqslant ||\beta u||$ for all u in H. In the following, we shall also use the elements

$$\varphi = \sum_{n=0}^{\infty} \alpha_n s^n \tag{12}$$

of the semialgebra consisting of all power series in some variable s with coefficients in $|O(H)|$. If $\psi = \sum_{n=0}^{\infty} \beta_n s^n$, we define $\varphi \leqslant \psi$ if $\alpha_n \leqslant \beta_n$, $n = 0, 1, 2, \ldots$, and we define $||\varphi u||$ for u in H by the formula

$$||\varphi u|| \equiv \sum_{n=0}^{\infty} ||\alpha_n u|| s^n. \tag{13}$$

We further define $\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)\varphi(s)$ and $\int_0^s \varphi(t)\mathrm{d}t$ by formal differentiation and integration

With these conventions we have

$$||\exp(|A|s)u|| = \sum_{n=0}^{\infty} \frac{||A^n u||}{n!} s^n. \tag{14}$$

Thus, with the concept $|A|$ we are able to give a closed symbol for the sum in eq. (1). Note the difference between the symbols $\exp(As)$ and $\exp(|A|s) = \sum \frac{|A|^n}{n!} s^n$ (implied, e.g., by eq. (11)).

For α given by eq. (8), we obtain by eqs. (13), (12) and (10)

$$\begin{aligned} ||\exp(\alpha s)u|| &= \sum_{n=0}^{\infty} \frac{1}{n!} ||(|A_1| + \ldots + |A_l|)^n u|| s^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqslant i_1, \ldots, i_n \leqslant l} ||A_{i_1} \ldots A_{i_n} u|| s^n. \end{aligned} \tag{15}$$

Comparing the last expression with the expression (1), we see that $\|\exp(\alpha s)u\| < \infty$ if, and only if, the series expansion of $\exp(A_1 s_1 + A_2 s_2 + \ldots + A_l s_l)u$ is absolutely convergent for any $(s_1, \ldots, s_l)$ sufficiently small. We see, therefore, that by means of the notion of absolute value we can conveniently describe the properties of absolute convergence of the series expansion of $\exp(A_1 s_1 + + \ldots + A_l s_l)u$. A vector u in H is said to be an *analytic vector for* α in $|O(H)|$ if $\|\exp(\alpha s)u\| < \infty$ for some $s > 0$. Notice that by virtue of eq. (15), if u is an analytic vector for α, then it is also an analytic vector for any A_i, $i = 1, 2, \ldots, l$.

Let us remark that the above definition of analytic vectors in the case of Lie algebras of operators gives the following

DEFINITION 1. A vector $u \in H$ is said to be *analytic vector* for the whole Lie algebra L if for some $s > 0$ and some linear basis $\{X_1, \ldots, X_d\}$ of the Lie algebra, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqslant i_1, \ldots, i_n \leqslant d} \|X_{i_1} \ldots X_{i_n} u\| s^n$$

is absolutely convergent. ▼

The last condition is equivalent to $\|X_{i_1}, \ldots, X_{i_n} u\| \leqslant C^n n!$ for some constant $C > 0$.

We had defined in a Lie algebra the operation $(\mathrm{ad}X)Y = XY - YX$. Clearly, this operation can be defined also in $O(H)$. We now extend this operation to elements $\xi = |X_1| + \ldots + |X_d|$ and $\alpha = |A_1| + \ldots + |A_m|$ in $|O(H)|$ by the formula

$$(\mathrm{ad}\,\xi)\,\alpha = \sum_{i=1}^{d} \sum_{j=1}^{m} |X_i A_j - A_j X_i| = \sum_{i=1}^{d} \sum_{j=1}^{m} |\mathrm{ad} X_i(A_j)|. \tag{16}$$

Hence,

$$(\mathrm{ad}\,\xi)^n \alpha = \sum_{1 \leqslant i_1, \ldots, i_n \leqslant d} \sum_{j=1}^{m} |\mathrm{ad} X_{i_n} \ldots \mathrm{ad} X_{i_1} A_j|. \tag{17}$$

We shall need the notion of commutators for absolute values. For this purpose we discuss first a commutator calculus for operators, namely we evaluate the commutator of an operator A with a product of n other operators $X_n \ldots X_1$. The following lemma provides a convenient algorithm for the solution of this problem.

LEMMA 2. *If* $A, X_1, \ldots, X_n$ *are operators on* H, *then**

$$AX_n \ldots X_1 \supset X_n \ldots X_1 A - Q_n, \tag{18}$$

where

$$Q_n = \sum_{k=1}^{n} \sum_{\sigma \in (n,k)} (\mathrm{ad} X_{\sigma(k)} \ldots \mathrm{ad} X_{\sigma(1)} A) X_{\sigma(n)} \ldots X_{\sigma(k+1)}, \tag{19}$$

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* Recall that $A \supset B$, means that $D(A) \supset D(B)$, and $Au = Bu$ for all $u \in D(B)$.

and (n, k) *denotes the set of all* $\binom{n}{k}$ *permutations* σ *of* $1, 2, \ldots, n$ *such that* $\sigma(n) > \sigma(n-1) > \ldots > \sigma(k+1)$ *and* $\sigma(k) > \sigma(k-1) > \ldots > \sigma(1)$.

If $X_1 \ldots, X_n, A$ *have a common invariant domain, then equality holds in eq.* (18) *and we have*

$$[X_n \ldots X_1, A] = Q_n. \tag{20}$$

Remark: The equality (20) can be written in another form which is often used in quantum theory. In fact, for $n = 2$, and for operators having a common invariant domain, we obtain from (20)

$$\begin{aligned}[X_2 X_1, A] &= Q_2 = \mathrm{ad}X_1(A)X_2 + \mathrm{ad}X_2(A)X_1 + \mathrm{ad}X_2\,\mathrm{ad}X_1(A) \\ &= \mathrm{ad}X_2(A)X_1 + X_2\,\mathrm{ad}X_1(A) = [X_2, A]X_1 + X_2[X_1, A],\end{aligned} \tag{21}$$

and in general

$$\begin{aligned}[X_n \ldots X_1, A] = Q_n &= [X_n, A]X_{n-1} \ldots X_1 + \\ &+ X_n[X_{n-1}, A]X_{n-2} \ldots X_1 + \ldots + X_n \ldots X_2[X_1, A].\end{aligned} \tag{22}$$

This equality can be easily proved by the method of induction using eq. (21).

PROOF OF LEMMA 2: We prove formula (18) by the method of induction. We first show that

$$X_n \ldots X_1 A \supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} (\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma(1)}A)\, X_{\sigma(n)} \ldots X_{\sigma(k+1)}. \tag{23}$$

For $n = 0$, eq. (23) states that $A \supset A$ and for $n = 1$ it states that

$$X_1 A \supset AX_1 + (\mathrm{ad}X_1)A, \tag{24}$$

which is also true.

Suppose now that eq. (23) holds for n and let X_{n+1} be an operator in H. Then, by eq. (24), we obtain

$$\begin{aligned}X_{n+1}X_n \ldots X_1 A &\supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} X_{n+1}(\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma_1}A)X_{\sigma(n)} \ldots X_{\sigma(k+1)} \\ &\supset \sum_{k=0}^{n} \sum_{\sigma \in (n,k)} \{(\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma(1)}A)\, X_{\sigma(n)} \ldots X_{\sigma(k+1)}X_{n+1} + \\ &\qquad + (\mathrm{ad}X_{n+1}\,\mathrm{ad}X_{\sigma(k)} \ldots \mathrm{ad}X_{\sigma(1)}A)X_{\sigma(n)} \ldots X_{\sigma(k+1)}\}.\end{aligned} \tag{25}$$

We now show that a term

$$(\mathrm{ad}X_{\tau(k)} \ldots \mathrm{ad}X_{\tau(1)}A)X_{\tau(n+1)} \ldots X_{\tau(k+1)}, \tag{26}$$

corresponding to a permutation τ in $(n+1, k)$, occurs either before the + sign in the curly brackets of eq. (25) or after the + sign. In fact, it occurs before the + sign in the brackets (corresponding to a σ in (n, k)) if $\tau(n+1) = n+1$, and as a term after the + sign (corresponding to σ in $(n, k-1)$), if $\tau(n+1) \neq n+1$, since either $\tau(n+1)$ or $\tau(k)$ must be equal to $n+1$, by the definition of (n, k). Because $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, the correspondence is one-to-one. Consequently, eq. (23) holds for $n+1$.

In order to prove eq. (18), we note that it is true for $n = 0$ and $n = 1$. Assume that it is true for n and let X_{n+1} be an operator in H. Then eq. (18) holds with $X_n, \ldots, X_1$ replaced by $X_{n+1}, \ldots, X_2$ and with Q_n modified accordingly. Setting Q_n' for the expression which replaces Q_n and multiplying both sides by X_1, we obtain

$$AX_{n+1} \ldots X_2X_1 \supset X_{n+1} \ldots X_2AX_1 - Q_n'X_1 \supset X_{n+1} \ldots \\ \ldots X_2X_1A - X_{n+1} \ldots X_2(\mathrm{ad}X_1)A - Q_n'X_1.$$

Applying eq. (23) to $X_{n+1} \ldots X_2(\mathrm{ad}X_1)A$ {with $(\mathrm{ad}X_1)A$ playing the role of A in eq. (23)}, we see that $X_{n+1} \ldots X_2(\mathrm{ad}X_1)A + Q_n'X_1 \supset Q_{n+1}$, where each permutation τ in $(n+1, k)$ with $\tau(1) = 1$ corresponds to a term in $X_{n+1} \ldots X_2(\mathrm{ad}X_1)A$ and each τ with $\tau(1) \neq 1$ corresponds to a term in $Q_n'X_1$.

Finally, if $X_1, \ldots, X_n$, A have a common invariant domain, then both sides of eq. (15) have the same domain and, therefore, they are equal. ▼

We now prove the analog of lemma 2 for elements in $|O(H)|$.

LEMMA 3. *Let ξ and α be in $|O(H)|$. Then*

$$\alpha\xi^n \leqslant \xi^n\alpha + \sum_{k=1}^{n} \binom{n}{k} [(\mathrm{ad}\,\xi)^k\alpha]\xi^{n-k}. \tag{27}$$

PROOF: Let $\xi = |X_1| + \ldots + |X_d|$, $\alpha = |A_1| + \ldots + |A_l|$ and let Σ^* denote the sum over all $1 \leqslant i \leqslant l$, $1 \leqslant g_1 \leqslant d, \ldots, 1 \leqslant g_n \leqslant d$. Then, using eqs. (10) and (18), we obtain

$$\begin{aligned} \alpha\xi^n &= \Sigma^*|A_iX_{g_n} \ldots X_{g_1}| \leqslant \Sigma^*|X_{g_n} \ldots X_{g_1}A_i| + \\ &\quad + \Sigma^*\Big|\sum_{k=1}^{n} \sum_{\sigma\in(n,k)} (\mathrm{ad}X_{g_{\sigma(k)}} \ldots \mathrm{ad}X_{g_{\sigma(1)}}A_i)X_{g_{\sigma(n)}} \ldots X_{g_{\sigma(k+1)}}\Big| \\ &= \xi^n\alpha + \Sigma^*\Big|\sum_{k=1}^{n} \binom{n}{k}(\mathrm{ad}X_{g_k} \ldots \mathrm{ad}X_{g_1}A_i)X_{g_n} \ldots X_{g_{k+1}}\Big|. \end{aligned} \tag{28}$$

In the last step, we took advantage of the fact that there are $\binom{n}{k}$ permutations in (n, k). Hence, by virtue of the summation over g_i we find that each term $(\mathrm{ad}X_{g_n} \ldots \mathrm{ad}X_{g_1}A_i)X_{g_n} \ldots X_{g_{k+1}}$ occurs $\binom{n}{k}$ times. The second term of the last equality is equal to the second term of eq. (27). ▼

C. *Analytic Dominance*

We now introduce the fundamental concept of analytic dominance for elements α and ξ in $|O(H)|$.

THEOREM 4. *Let* ξ and α *be in* $|O(H)|$. *Let* $\xi \leqslant c\alpha$, $(\mathrm{ad}\,\xi)^n\alpha \leqslant c_n\alpha$ *and*

$$\nu(s) = \sum_{n=1}^{\infty} \frac{c_n s^n}{n!}, \tag{29}$$

$$\varkappa(s) = \int_0^s \frac{\mathrm{d}t}{1-\nu(s)}. \tag{30}$$

Then $\exp(\xi s) \leqslant \exp[c\alpha\varkappa(s)]$. ▼

If c and c_n are such that $c < \infty$ and $\nu(s) < \infty$ *for some* $s > 0$, we shall say that α *analytically dominates* ξ.

In order to clarify the content of the theorem we first prove a corollary.

COROLLARY 1. *Let* ξ *and* α *be in* $|O(H)|$. *If* α *analytically dominates* ξ, *then every analytic vector for* α *is an analytic for* ξ.

PROOF: If α analytically dominates ξ, then ν has a positive radius of convergence and so does $\varkappa$. Consequently, by definition every analytic vector for α is an analytic vector for ξ. ▼

PROOF OF TH. 4: Let us define the elements $\pi_n \in |O(H)|$ by the recursion formulae

$$\pi_0 = |I|, \quad \pi_{n+1} = c\pi_n\alpha + \sum_{k=1}^{n} \binom{n}{k} c_k \pi_{n+1-k}. \tag{31}$$

Clearly, $\pi_1 = c\alpha$, $\pi_2 = c^2\alpha^2 + c_1 c\alpha$ and each π_n is a polynomial in α. We first show by the method of induction that

$$\alpha\xi^{n-1} \leqslant \frac{1}{c}\pi_n. \tag{32}$$

Notice that $\xi \leqslant c\alpha$ implies $\xi^n \leqslant c\alpha\xi^{n-1}$. Hence, eq. (32) implies

$$\xi^n \leqslant \pi_n. \tag{33}$$

For $n = 1$, eq. (32) says that $\alpha \leqslant \alpha$. Suppose that eq. (32) is satisfied for all $k \leqslant n$. Then, using lemma 3 the assumption of the theorem, eqs. (32) and (33), we obtain

$$\begin{aligned}\alpha\xi^n &\leqslant \xi^n\alpha + \sum_{k=1}^{n} \binom{n}{k} [(\mathrm{ad}\,\xi^k)\,\alpha]\,\xi^{n-k} \\ &\leqslant \xi^n\alpha + \sum_{k=1}^{n} \binom{n}{k} c_k\,\alpha\xi^{n-k} \\ &\leqslant \pi_n\ddot{\alpha} + \sum_{k=1}^{n} \binom{n}{k} c_k \frac{1}{c}\pi_{n+1-k} \leqslant \frac{1}{c}\pi_{n+1}.\end{aligned}$$

Thus, eq. (32) and consequently eq. (33) hold for any n. Let $\pi(s)$ be the power series

$$\pi(s) = \sum_{n=0}^{\infty} \frac{\pi_n}{n!} s^n. \tag{34}$$

By eq. (31) and the relation $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ we have

$$(n+1)\frac{\pi_{n+1}}{(n+1)!} = c\frac{\pi_n \alpha}{n!} + \sum_{k=1}^{n} \frac{c_k}{k!}(n+1-k)\frac{\pi_{n+1-k}}{(n+1-k)!}. \tag{35}$$

Using the definitions (29) and (30) of $\pi(s)$ and $\nu(s)$ and eq. (35), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\pi(s) = c\pi(s)\alpha + \nu(s)\frac{\mathrm{d}}{\mathrm{d}s}\pi(s), \tag{36}$$

i.e.,

$$\frac{\mathrm{d}\pi(s)}{\mathrm{d}s} = c\alpha\pi(s)/\big(1-\nu(s)\big). \tag{37}$$

Hence, from the definition of $\varkappa(s)$, we obtain

$$\pi(s) = \exp\big(c\alpha\varkappa(s)\big). \tag{38}$$

In fact, differentiating formula (38), we obtain eq. (37), and setting $s = 0$, we get $\pi(0) = |I|$, in agreement with the definition (34). Using eq. (32), we finally obtain

$$\exp(\xi s) \leqslant \pi(s) = \exp\big(c\alpha\varkappa(s)\big). \;\blacktriangledown$$

Notice that nowhere have we used the fact that the operators $A_1, \ldots, A_l$ are linear or that the carrier space H is complete.

The next useful corollary of th. 4 can be stated without using the terminology of the theory of absolute values.

COROLLARY 2. *Let $X_1, \ldots, X_d$, and A be operators on a Hilbert space H. Let k and k_n, $n = 1, 2, \ldots$, be positive numbers such that for all u in the domain of A*

$$||X_i u|| \leqslant k(||Au|| + ||u||), \quad 1 \leqslant i \leqslant d, \tag{39}$$

and

$$||\mathrm{ad}X_{i_1} \ldots \mathrm{ad}X_{i_n} Au|| \leqslant k_n(||Au|| + ||u||) \quad \textit{for } 1 \leqslant i_1, \ldots, i_n \leqslant d. \tag{40}$$

Suppose that $k < \infty$ and that

$$\sum_{n=1}^{\infty} (k_n/n!)s^n < \infty$$

for some $s > 0$. If there is an $s > 0$ such that

$$\sum_{n=0}^{\infty} \frac{||A^n u||}{n!} s^n < \infty, \tag{41}$$

then for $(s_1, \ldots, s_d)$ *sufficiently close to* $(0, \ldots, 0)$, *we have*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leqslant i_1, \ldots, i_n \leqslant d} \|X_{i_1} \ldots X_{i_n} u\| s_{i_1} \ldots s_{i_n} < \infty . \blacktriangledown \tag{42}$$

PROOF: Let $\xi = |X_1| + \ldots + |X_d|$ and $\alpha = |A| + |I|$. By eq. (16), $(\operatorname{ad}\xi)^n \alpha = (\operatorname{ad}\xi)^n |A|$. By virtue of eq. (39), $\xi \leqslant dk\alpha$ and by eq. (40), $(\operatorname{ad}\xi)^n \alpha \leqslant \mathrm{d}^n k_n \alpha$. Because $v(s) = \sum \frac{\mathrm{d}^n k_n}{n!} s^n < \infty$ for some $s > 0$, we obtain, using th. 4, that α analytically dominates ξ. Consequently, by corollary 1, every analytic vector for α is an analytic vector for ξ, i.e., eq. (41) implies eq. (42). $\blacktriangledown$

LEMMA 5. *Let* $X_1, \ldots, X_d$, *and* A *be symmetric operators on a Hilbert space* H *with a common invariant domain* D, *and let* A *be essentially self-adjoint. Let* $\xi = |X_1| + \ldots + |X_d|$, $\alpha = |A| + |I|$, $\xi \leqslant c\alpha$ *and* $(\operatorname{ad}\xi)^n \alpha \leqslant c_n \alpha$ *with* $c < \infty$ *and* $c_n < \infty$ *for all* $n \geqslant 1$. *Then, for all finite sequences* $i_1, \ldots, i_n$, *we have*

$$D(\bar{A}^n) \subset D(\bar{X}_{i_1} \ldots \bar{X}_{i_n}). \tag{43}$$

Let $\tilde{D} = \bigcap_{n=1}^{\infty} D(\bar{A}^n)$ *and let* $\tilde{X}_1, \ldots, \tilde{X}_d$ *and* $\tilde{A}$ *be the restriction of* $\bar{X}_1, \ldots, \bar{X}_d$ *and* $\bar{A}$ *respectively, to* $\tilde{D}$. *Let* $\tilde{\xi} = |\tilde{X}_1| + \ldots + |\tilde{X}_d|$, $\tilde{\alpha} = |\tilde{A}| + |I|$. *Then*

$$\tilde{\xi} \leqslant c\tilde{\alpha}, \quad (\operatorname{ad}\tilde{\xi})^n \tilde{\alpha} \leqslant c_n \tilde{\alpha} \quad \textit{for all } n \geqslant 1. \tag{44}$$

If α *analytically dominates* ξ, *then there is an* $s > 0$ *such that the set of* $u \in \tilde{D}$ *for which* $\|\exp(\tilde{\xi}s)u\| < \infty$ *is dense in* H, *and each* X_i *is essentially self-adjoint.*

PROOF: We prove the relation (43) by the method of induction. Let $n = 1$, If $u \in D(\bar{A})$, then by definition of the closure $\bar{A}$ there is a sequence u_j in $D(A) = D$ with $u_j \to u$ and $Au_j \to v = \bar{A}u$. Because $\xi \leqslant c\alpha$, we have $|X_i| \leqslant c\alpha$, $i = 1, 2, \ldots, d$. Consequently, for all $i = 1, 2, \ldots, d$,

$$\|X_i(u_j - u_k)\| \leqslant c(\|A(u_j - u_k)\| + \|u_j - u_k\| \to 0,$$

as $j, k \to \infty$. Hence, u is in $D(\bar{X}_i)$. By the same argument, we obtain that if $C = \operatorname{ad}X_{i_k} \ldots \operatorname{ad}X_{i_1} A$, then u is in $D(\bar{C})$, i.e.,

$$D(\bar{A}) \subset D(\overline{\operatorname{ad}X_{i_k} \ldots \operatorname{ad}X_{i_1} A}). \tag{45}$$

We now show that

$$D(\bar{A}^n) \subset D(\bar{A}\bar{X}_{i_1} \ldots \bar{X}_{i_{n-1}}). \tag{46}$$

Notice that eq. (46) implies eq. (43) for the case $n = 1$ considered above. Suppose that relation (46) is satisfied for some n and let u be in $D(\bar{A}^{n+1})$. Because $\bar{A} = A^*$, it will suffice to show that $\bar{X}_{i_1} \ldots \bar{X}_{i_n} u$ is in $D(A^*)$, i.e., that

$$(Av, \bar{X}_{i_1} \ldots \bar{X}_{i_n} u) = (X_{i_n} \ldots X_{i_1} Av, u)$$

is a continuous linear functional of v in D. By virtue of eq. (20), we have $X_{i_n} \dots X_{i_1} A = A X_{i_n} \dots X_{i_1} + Q_n$. By the induction hypothesis, and lemma 2, $Q_n(v, u)$ is a linear continuous functional of v in D. Moreover, because u is in $D(\bar{A}^{n+1})$. $A^* u = (\bar{A} u)$ is in $D(\bar{A}^n)$; and, therefore, $(A X_{i_n} \dots X_{i_1} v, u) = (X_{i_n} \dots \dots X_{i_1} v, A^* u)$ is, by the induction hypothesis, a continuous linear functional of v. Hence, relation (46) is satisfied.

Furthermore, eq. (46) implies that the operator $\tilde{X}_1, \dots, \tilde{X}_d$ and $\tilde{A}$ leave $\tilde{D}$ invariant.

We now show that $\tilde{\xi} \leqslant c\tilde{\alpha}$. If u is in $\tilde{D}$, then u is in $D(\bar{A})$, and there is a sequence u_j in D with $u_j \to u$, $A u_j \to \bar{A} u$. According to the relation $\xi \leqslant c\alpha$, we have

$$\sum_{i=1}^{d} \|\tilde{X}_i u\| = \sum_{i=1}^{d} \lim_{j\to\infty} \|X_i u_j\| \leqslant \lim_{j\to\infty} c(\|A u_j\| + \|u_j\|) = c(\|\tilde{A} u\| + \|u\|).$$

Hence, $\tilde{\xi} \leqslant c\tilde{\alpha}$. Similarly, it can be shown using $(\operatorname{ad}\xi)^n \alpha \leqslant c_n \alpha$ that $(\operatorname{ad}\tilde{\xi})^n \tilde{\alpha} \leqslant c_n \tilde{\alpha}$.

Now let α analytically dominate ξ, so that the power series $\nu(s)$ and $\varkappa(s)$ of th. 4 have positive radii of convergence. Let $E(\lambda)$ be the resolution of the identity for the self-adjoint operator $\bar{A}$ and let B be the set of all vectors u, such that for the same bounded set Δ, $E(\Delta) u = u$. We know by corollary 2 of the spectral theorem (app. B.3) that B is dense in H. Moreover, $B \subset \tilde{D}$ and $\|\exp(\tilde{\alpha} t) u\| < \infty$, for all u in B and $0 \leqslant t \leqslant \infty$ by eq. (2). Taking s such that $\varkappa(s) < \infty$, we obtain, by th. 4, that $\|\exp(\tilde{\xi} s) u\| \leqslant \|\exp[c\tilde{\alpha}\varkappa(s)] u\| < \infty$ for all u in B. Any analytic vector for $\tilde{\xi}$ is an analytic vector for each $\tilde{X}_i$. Hence, each $\tilde{X}_i$ is essentially self-adjoint, by remark 1 to lemma 1. Because $X_i \subset \tilde{X}_i \subset \bar{X}_i$, we see that each $\bar{X}_i (= \bar{\tilde{X}}_i)$ is self-adjoint, i.e., X_i is essentially self-adjoint. ▼

Let L be a Lie algebra of skew-symmetric operators having a linear subspace $D \subset H$ as a common dense invariant domain and let E be the enveloping algebra of L. An element $Y \in E$ is said to be of order $\leqslant n$ if it is a real linear combination of operators of the form $Y_1 Y_2 \dots Y_k$ with $k \leqslant n$ and each Y_i in L. The set of all elements of E of order $\leqslant n$ will be denoted by $E^{(n)}$. If $X_1, \dots, X_d$, $d = \dim L$, are generators of L, then the operators

$$X_i \quad \text{and} \quad H_{ij} \equiv X_i X_j + X_j X_i \tag{47}$$

constitute a set of linear generators for $E^{(2)}$ (cf. proposition 9.2.1).

The next two lemmas and the remark show that the elliptic operator $\Delta = X_1^2 + \dots + X_d^2$ (cf. eq. 2. (21)) plays a special role in the representation theory of Lie algebras.

Lemma 6. *Let $\Delta = X_1^2 + \dots + X_d^2$. If A is an arbitrary operator in $E^{(2)}$, then for some $k < \infty$*

$$|A| \leqslant k|\Delta - I|. \tag{48}$$

Proof: For any $B = a_i X_i + a_{ij} H_{ij} \in E^{(2)}$, we have $|B| \leqslant |a_i||X_i| + |a_{ij}||H_{ij}|$, hence it suffices to prove lemma 6 for generators (47) of $E^{(2)}$. For the generators X_i, $i = 1, \ldots, d$, and all u in D, we get

$$\sum_{i=1}^{d} \|X_i u\|^2 = \sum_{i=1}^{d} (X_i u, X_i u) = (-\Delta u, u)$$

$$\leqslant \left(\left(\frac{1}{2}\Delta^2 - \Delta + \frac{1}{2}\right)u, u\right) = \left(\frac{1}{2}(\Delta - I)^2 u, u\right) = \frac{1}{2}\|(\Delta - I)u\|^2. \quad (49)$$

Consequently, we can write

$$\|X_i u\| \leqslant \left(\frac{d}{2}\right)^{1/2} \|(\Delta - I)u\|. \quad (50)$$

If $u \notin D$ both sides of eq. (50) are infinite. Thus

$$|X_i| \leqslant (d/2)^{1/2} |\Delta - I|. \quad (51)$$

We now prove eq. (48) for $A = H_{ij}$. Let B^+ denote the restriction of B^* to D if B is in E. Thus, $(X_{i_1} \ldots X_{i_n})^+ = (-1)^n X_{i_n} \ldots X_{i_1}$. Let P be the set of elements in E consisting of finite sums of the form $\sum_r Y_r^+ Y_r$. The operator $-\Delta$ is clearly in P. Moreover,

$$\begin{aligned} -\Delta + H_{ij} &= (X_i - X_j)^+ (X_i - X_j) + \sum_{k \neq i,j} X_k^+ X_k, \quad i \neq j, \\ -2\Delta + H_{ii} &= 2 \sum_{k \neq i} X_k^+ X_k, \\ -\Delta - H_{ij} &= (X_i + X_j)^+ (X_i + X_j) + \sum_{k \neq i,j} X_k^+ X_k. \end{aligned} \quad (52)$$

Hence, if $A = \sum a_{ij} H_{ij}$, a_{ij} real, then there is an $a \geqslant 0$ such that

$$-a\Delta + A \in P. \quad (53)$$

Consider now the operator $4\Delta^2 - H_{ij}^2$. We have

$$4\Delta^2 - H_{ij}^2 = (2\Delta - H_{ij})(2\Delta + H_{ij}) + A_1,$$

where $A_1 = 2[H_{ij}, \Delta]$ is in $E^{(3)}$ by eq. (21). The operator $-(2\Delta - H_{ij}) = \sum_k Y_k^+ Y_k$ with Y_k in L by eq. (52). Similarly $-(2\Delta + H_{ij}) = \sum_l Z_l^+ Z_l$ with Z_l in L. Therefore, using the commutation relations of the Lie algebra L, we obtain

$$4\Delta^2 - H_{ij}^2 = \sum_{k,l} (Y_k Z_l)^+ (Y_k Z_l) + A_2, \quad (54)$$

where A_2 is in $E^{(3)}$.

By proposition 9.2.1, $E^{(3)}$ is spanned by operators (47), and by those of the form

$$H_{ijk} = X_i X_j X_k + X_i X_k X_j + X_j X_i X_k + X_j X_k X_i + X_k X_i X_j + X_k X_j X_i. \quad (55)$$

Consequently, we may write $A_2 = \sum a_{ij}H_{ij}+S$, where S is a real linear combination of the X_i and H_{ijk}. Because the other terms in eq. (54) are symmetric, S must be symmetric. But the X_i and H_{ijk} are skew-symmetric so that S is also skew-symmetric. Consequently, $S = 0$. Therefore, by eq. (53) there is an $a \geqslant 0$ such that $-a\Delta+A_2$ is in P. Hence, by eq. (54), $4\Delta^2-H_{ij}^2-a\Delta$ is also in P. Using the fact that $(\Delta^2 u, u) \geqslant 0$, $(\Delta u, u) \leqslant 0$ and setting $k = \max(2, a/4)$, we obtain for all u in D

$$\begin{aligned}||H_{ij}u||^2 &\leqslant \big((4\Delta^2-a\Delta)u, u\big) \leqslant 4(\Delta^2 u, u)-a(\Delta u, u)+\frac{a^2}{16}(u, u)\\ &\leqslant k^2(\Delta^2 u, u)-2k^2(\Delta u, u)+k^2(u, u) = k^2||(\Delta-I)u||^2.\end{aligned}$$

Hence, $||H_{ij}u|| \leqslant k||(\Delta-I)u||$. If $u \notin D$, then both sides of this inequality are infinite. Hence, $|H_{ij}| \leqslant k|\Delta-I|$. ▼

LEMMA 7. *Let* $\xi = |X_1|+ \ldots +|X_d|$ *and let* $\alpha = |\Delta-I|$. *Then* α *analytically dominates* ξ. *In fact,* $\xi \leqslant \left(\frac{d}{2}\right)^{1/2}\alpha$ *and there is a* $c < \infty$ *such that for all* $n \geqslant 1$, $(\operatorname{ad}\xi)^n\alpha \leqslant c^n\alpha$. *Also,* $|\Delta|+|I|$ *analytically dominates* ξ.

PROOF: The inequality $\xi \leqslant \left(\frac{d}{2}\right)^{1/2}\alpha$ directly follows from eq. (49). To prove $(\operatorname{ad}\xi)^n\alpha \leqslant c^n\alpha$, we first introduce the norm $|||\cdot|||$ in $E^{(2)}$. Notice that $E^{(2)}$ is in fact a finite-dimensional vector space because it is spanned by elements (47). If A is in $E^{(2)}$, we define $|||A|||$ to be the smallest number, such that $|A| \leqslant k\alpha$. By lemma 6 this is always finite and provides a norm in $E^{(2)}$. Moreover, if $|||A||| = 0$, then $A = 0$. Hence, $E^{(2)}$ with this norm is a finite-dimensional Banach space. For any A in $E^{(2)}$ $(\operatorname{ad}X_i)A$ is in $E^{(2)}$ by eq. (21). Because $(\operatorname{ad}X_i)$ is a linear map in the finite-dimensional space $E^{(2)}$, it is continuous in the norm $|||\cdot|||$, hence there is a $c_i < \infty$ such that $|||(\operatorname{ad}X_i)A||| \leqslant c_i|||A|||$. Set $c = d\max c_i$. By eq. (17) $(\operatorname{ad}\xi)^n\alpha$ is the sum of d^n terms of the form $|\operatorname{ad}X_{i_n} \ldots \operatorname{ad}X_{i_1}\Delta|$ which is $\leqslant c_{i_n} \ldots c_{i_1}\alpha$. Therefore, $(\operatorname{ad}\xi)^n\alpha \leqslant c^n\alpha$. Because $c < \infty$ and the quantity $\nu(s)$ of th. 4 is finite ($\nu(s) = \exp(cs)$), α analytically dominates ξ. If we set $\alpha' = |\Delta|+|I|$, then $(\operatorname{ad}\xi)^n\alpha' < c^n\alpha \leqslant c^n\alpha'$, by eq. (17) and inequality $||(\Delta-I)u|| \leqslant ||\Delta u||+||u||$. Hence, $|\Delta|+|I|$ also analytically dominates ξ. ▼

Remark: Lemma 6 can be generalized. One can show that if B is in $E^{(2m)}$ ($m = 1, 2, \ldots$), then for some $k < \infty$

$$|B| \leqslant k\alpha^m,$$

where $\alpha^m = |(\Delta-I)^m|$. Moreover, if the operator coresponding to $\eta = |Y_1|+ \ldots + |Y_l|$ is in $E^{(2m)}$ and $\operatorname{ad}Y_j$, $j = 1, 2, \ldots, l$, maps $E^{(2m)}$ onto itself, then α^m analytically dominates η (cf. Nelson 1959, lemma 6.3). However, we shall not need these results.

§ 4. Analytic Vectors for Unitary Representations of Lie Groups

We have shown in § 1 that every representation T of a Lie group G gives rise in a natural manner to a representation of its Lie algebra L, defined on the Gårding subspace D_G. However, this correspondence as it stands is not very satisfactory. In fact, it may occur that a subspace $D \subset D_G$ or D_G itself which is invariant under the Lie algebra L is not invariant under G. For instance, if G is the one-parameter translation group represented in the Hilbert space $L^2(-\infty, +\infty)$ by the formula $T_x f(y) = f(x+y)$, then every subspace $C_0^\infty(0, n)$, $n = 1, 2, \ldots$, is invariant under the operator $X = \frac{\mathrm{d}}{\mathrm{d}x}$ of the Lie algebra. However, it is obviously not invariant under the group of translations.

In general, the problem stems from the fact that a Taylor series of regular functions does not necessarily converge to a regular function. Indeed on the Gårding subspace for a generator X_i in L, φ in $C_0^\infty(G)$ and $u(\varphi)$ in D_G, the vector

$$T(X_i)^n u(\varphi) = u(\tilde{X}_i^n \varphi) \tag{1}$$

is a regular vector for T of G, because $T_x u(\tilde{X}_i^n \varphi) = \int (\tilde{X}_i^n \varphi)(x^{-1}y) T_y u \mathrm{d}y$, but the expansion

$$\sum_{n=0}^{N} \frac{t^n}{n!} T(X_i)^n u(\varphi) = \int \sum_{n=0}^{N} \frac{t^n}{n!} \tilde{X}_i^n \varphi(l) T_x u \mathrm{d}x, \tag{2}$$

in general, does not even converge as $N \to \infty$. Those vectors u in H, for which the expansion $\sum \frac{t^n}{n!} T(X_i)^n u$, $i = 1, 2, \ldots, \dim L$, converges, are of special interest and, according to sec. 3, are called *analytic vectors* for the representatives $T(X_i)$. The properly chosen analytic vectors assure the satisfactory connection between representations of L and G in H. We shall show in particular that the invariant subspaces of analytic vectors relative to $T(L)$ are also invariant subspaces of analytic vectors relative to $T(G)$ and conversely.

This section is devoted to the analysis of properties of analytic vectors for unitary representations of Lie groups. We first establish the connection between analytic vectors for a group representation $x \to T_x$ and analytic vectors for operators $T(X)$, $X \in L$, in the sense of eq. 3 (1).

LEMMA 1. *Let $x \to T_x$ be a representation of a Lie group G in a Hilbert space H. Let $X_1, \ldots, X_d$, $d = \dim L$, be a basis for the Lie algebra L of G and let $\xi = |T(X_1)| + \ldots + |T(X_d)|$. Then, if $u \in H$ is an analytic vector for ξ, then u is an analytic vector for the representation T of G.*

PROOF: Notice that if $T_x u$ is analytic in a neighborhood of e in G, then $T_x u$ is analytic everywhere. Indeed,

$$T_x u = T_y T_{y^{-1}x} u$$

is analytic if x is near to a fixed element $y \in G$. We know, moreover, that the exponential map is an analytical isomorphism of a neighborhood of 0 in L with a neighborhood of e in G (cf. th. 3.10.1). Hence, it is sufficient to prove that if $u \in H$ is an analytic vector for ξ, then $X \to T_{\exp X} u$ is analytic in some neighborhood of 0 in L. Let $X = X_1 t_1 + \ldots + X_d t_d$. Then, $T_{\exp X} u$ is analytic in some neighborhood of 0 in L, if and only if

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha_1 + \ldots + \alpha_d = n} \|\psi_{\alpha_1 \ldots \alpha_d}\| t_1^{\alpha_1} \ldots t_d^{\alpha_d} < \infty \tag{3}$$

for $t_1, \ldots, t_d$ sufficiently small, where $\psi_{\alpha_1 \ldots \alpha_n}$ is the coefficient of $t_1^{\alpha_1} \ldots t_d^{\alpha_d}$ in the expansion of $\exp[T(X_1)t_1 + \ldots + T(X_d)t_d]u$. Using eq. 3 (15), we see that $\|\psi_{\alpha_1 \ldots \alpha_d}\|$ is the norm of the sum of several term which also occur in the expansion of $\|\exp(\xi s)u\|$, if we take $t_1 = \ldots = t_d = s$. Therefore, the left-hand side of eq. (3) is $\leqslant \|\exp(\xi s)u\|$, when $s = \max(|t_1|, \ldots, |t_d|)$ and consequently it is $< \infty$, by our assumption. ▼

Remark 1: The above result can be sharpened. One can show that a vector $u \in H$ is an analytic vector if and only if u is analytic for the representation T of G (cf. Nelson 1959, lemma 7.1). ▼

We now show two basic properties of the set of analytic vectors for a unitary representation T of a Lie group G.

THEOREM 2. *Let $x \to T_x$ be a unitary representation of a Lie group G on the Hilbert space H. Then the set A_T of analytic vectors for T is dense in H. If $X_1, \ldots, X_d$ is a basis for the Lie algebra L of G, and $\Delta = X_1^2 + \ldots + X_d^2$, then any analytic vector for $\overline{T(\Delta)}$ is an analytic vector for T of G, and the set $A_{T(\Delta)}$ of such vectors is dense in H.*

PROOF: Let $T(X_i)$, $i = 1, 2, \ldots, d$, denote the images of the generators X_i in L, and let $T(\Delta) = T(X_1)^2 + \ldots + T(X_d^2)$. By proposition 1.2, the $T(X_i)$ are skew-symmetric on the Gårding domain D_G, and by th. 2.3 $T(\Delta)$ is essentially self-adjoint.

Set $\tilde{D} = \bigcap_{n=1}^{\infty} D(\overline{T(\Delta)}^n)$. By eq. 3 (43), every vector in $\tilde{D}$ is in $D\big(\overline{T(X_{i_1})} \ldots \overline{T(X_{i_n})}\big)$ for all finite sequences $i_1, \ldots, i_n$. By Stone's theorem the domain $D\big(\overline{T(X)}\big)$ for X in L is the set of all vectors u in H, for which the limit (i.e., derivative) in eq. 1(11) exists. Therefore, if u is in $\tilde{D}$, then u is in $D\big(\overline{T(X)}\big)$ and, consequently, $T_x u$ has all partial derivatives at $x = e$. Now the inner automorphism $y \to xyx^{-1}$ in G implies the inner automorphism

$$X \to \mathrm{Ad}_x(X) = xXx^{-1} \tag{4}$$

in the Lie algebra L of G (cf. eq. 3.3 (28)). Hence, for images (relative to T), we have

$$\overline{T(X)}\, T_x u = T_x\big(T_{x^{-1}} \overline{T(X)}\, T_x\big) u = T_x \overline{T(Y)} u, \tag{5}$$

where $Y = \mathrm{Ad}_{x^{-1}}(X)$. For X', X'' in L, we have

$$\overline{T(X'')}\,\overline{T(X')}\,T_x u = T_x \overline{T(Y'')}\,\overline{T(Y')}u$$

and similar expression for arbitrary products $\overline{T(X_{i_1})} \dots \overline{T(X_{i_n})}$. Thus, $T_x u$ has partial derivatives of all orders for all x in G and u in $\tilde{D}$. Consequently, $\tilde{D}$ is contained in the space $\tilde{D}_G$ of all infinitely differentiable vectors. Clearly $\tilde{D}_G \supset D_G$. If u is in $\tilde{D}_G$, then u is in $D(\overline{T(\Delta)})^n$ for any n and, therefore, $u \in \tilde{D}$. Hence, $\tilde{D} \supset \tilde{D}_G$; consequently, $\tilde{D} = \tilde{D}_G$. Any analytic vector for $\overline{T(\Delta)}$ is in $\tilde{D}$ and, consequently, it is also an analytic vector for $T(\Delta)$. The operator $\overline{T(\Delta)}$ is self-adjoint and so it has a dense set of analytic vectors by lemma 3.1. Because, by lemma 3.7, $\alpha = |T(\Delta) - I|$ analytically dominates ξ, these are all analytic vectors for ξ and by lemma 1 they are analytic vectors for the representation T of G. ▼

EXAMPLE 1. Let G be the three-dimensional nilpotent group of example 2.1, and let

$$T_{[\alpha\beta\gamma]}u(x) = \exp[-i(\gamma + x\beta)]\,u(x+\alpha) \tag{6}$$

be the unitary representation of G in $L^2(-\infty, +\infty)$. Then the generators of one-parameter subgroups are

$$P = \frac{\mathrm{d}}{\mathrm{d}x}, \quad Q = -\mathrm{i}x, \; C = -\mathrm{i}I. \tag{7}$$

They satisfy the Heisenberg commutation relation $[P, Q] = -\mathrm{i}I$. We take as a common dense invariant domain D for the Lie algebra (7) the Schwartz's space S of C^∞-functions $u(x)$ with

$$\sup \left| x^\alpha \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\beta u(x) \right| < \infty, \qquad \alpha, \beta = 0, 1, \dots$$

Now we calculate the dense set of analytic vectors for the operator $T(\Delta) = P^2 + Q^2 + C^2$. The unit operator $-C^2$ in $T(\Delta)$ can be dropped since every analytic vector for $T(\Delta)$ is analytic for

$$T(\Delta') = P^2 + Q^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2. \tag{8}$$

The operator (8) has the same form as the Hamiltonian for one-dimensional harmonic oscillator. It is well known that eigenfunctions $u_n(x)$ normalized to one of $T(\Delta')$ can be expressed in terms of Hermite polynomials, i.e.,

$$u_n(x) = (\sqrt{(\pi)}\,2^n n!)^{-1/2} \exp(-x^2/2)\,H_n(x) \tag{9}$$

and correspond to the eigenvalues $\lambda_n = 2n+1$ (cf. e.g. Messiah 1961, vol. I, p. 492). These eigenfunctions are analytic vectors for $T(\Delta')$. Indeed, for $s < \infty$,

$$\sum_{k=0}^{\infty} \frac{1}{k!} \| T(\Delta')^k u_n \| s^k = \sum_{k=0}^{\infty} \frac{1}{k!} (\lambda_n s)^k = \exp(\lambda_n s) < \infty .$$

The set $\{u_n(x)\}$ forms a complete orthonormal set in $L^2(-\infty, +\infty)$. Hence by virtue of th. 2, the finite linear hull $A_{T(\Delta)}$ of $\{u_n(x)\}$ provides a dense set of analytic vectors for the representation (6) of G.

Every eigenfunction $u_n(x)$ lies in the Schwartz space S, which is the domain D for $T(L)$. Moreover, from elementary properties of Hermite polynomials, it follows that

$$\begin{aligned} Pu_n &= -\left(\frac{n+1}{2}\right)^{1/2} u_{n+1} + \left(\frac{n}{2}\right)^{1/2} u_{n-1}, \\ Qu_n &= -\mathrm{i}\left(\frac{n+1}{2}\right)^{1/2} u_{n+1} - \mathrm{i}\left(\frac{n}{2}\right)^{1/2} u_{n-1}. \end{aligned} \tag{10}$$

Hence, the dense subspace $A_{T(\Delta)}$ of analytic vectors of $T(\Delta)$ is the common invariant dense domain for the representation (7) of the Lie algebra of Heisenberg commutation relations. ▼

The proof of lemma 3.1 provides a method for the explicit construction of a dense set of analytic vectors for $\overline{T(\Delta)}$. Namely, let $E(\lambda)$ be the spectral resolution of the identity for $\overline{T(\Delta)}$ and $\delta = [a, b]$ run over bounded intervals of R^1, then the vectors of the form $E(\delta)u$, $u \in H$, constitute the dense set of analytic vectors for $\overline{T(\Delta)}$.

We shall see that with a proper choice, the dense set A_{T_G} of analytic vectors for a representation $x \to T_x$ of G forms a common dense invariant domain for a representation $X \to T(X)$ of the Lie algebra L of G and its enveloping algebra E and every element of A_{T_G} is in $A_{T(\Delta)}$ (cf. eq. 7(32)).

If we take as a common dense invariant domain for a representation $X \to T(X)$ of a Lie algebra L, the dense set A_{T_G} of analytic vectors for a representation $x \to T_x$ of G, then we have a satisfactory connection between invariant closed subspaces of the Lie algebra and those of the Lie group. Indeed, if $D \subset A_{\overline{T(\Delta)}}$ is an invariant closed subspace for Lie algebra, then for any $T(X)$ in L and u in D, $T(X)^n u$ is in D and, consequently,

$$T_{\exp(hX)}u = \exp[hT(x)]u = \sum_{k=0}^{\infty} \frac{h^n}{n!} T(x)^n u \tag{11}$$

converges and gives an element in D. Consequently, D is also invariant relative to the representation T of G. Conversely if D is a closed subspace invariant relative to G containing a dense invariant set A of analytic vectors then A is also invariant relative to the Lie algebra L. In fact, by eq. (11) for any $h > 0$ and u in A, we have

$$\frac{T_{\exp(hX)}u - u}{h} = T(X)u + hT(X)^2 u + \ldots \in D. \tag{12}$$

Setting $h \to 0$, we obtain that $T(X)u \in D$.

§ 5. Integrability of Representations of Lie Algebras

In quantum theory and in particle physics, we work in general directly with the representations of the Lie algebras. However, in many problems, the global group transformations themselves have a direct physical significance. For instance, if the group G contains the physical Poincaré group Π as a subgroup, then the (unitary) representatives T_x of $x \in \Pi$ will describe changes of the given physical system associated with the changes of the reference frames. Thus, in many cases, we are interested in those representations of the Lie algebra L, which can be integrated to a global (unitary) representation of the group G. There are many examples where the global representations bring new physical relationships showing that Nature makes use of the representations of groups rather than of Lie algebras.

In this section we give Nelson's fundamental theorem which establishes when a representation of a Lie algebra L in terms of skew-symmetric operators can be associated with a unitary representation of the simply connected Lie group G having L as its Lie algebra.

We first discuss the connection of this problem with analytic vectors.

LEMMA 1. *Let a Lie algebra L be represented by skew-symmetric operators on a Hilbert space H having a common, invariant, dense domain D. Let $X_1, \ldots, X_d$ be an operator basis for L, $\xi = |X_1| + \ldots + |X_d|$. If for some $s > 0$ the set of vectors u in D for which $\|\exp(\xi s)u\| < \infty$ is dense in H, then there is on H a unique unitary representation T of the simply connected Lie group G, having L as its Lie algebra such that for all X in L, $\overline{T(X)} = \overline{X}$.*

PROOF: The condition $\|\exp(\xi s)u\| < \infty$ for some $s > 0$ means that u is an analytic vector for ξ. Because any analytic vector for ξ is an analytic vector for any element $X \in L$, we conclude that any $X \in L$ has a dense set of analytic vectors. Consequently, by remark 1 to lemma 3.1, the operator $i\overline{X}$ is self-adjoint.

Let exp be the exponential mapping in the sense it is defined for Lie groups (cf. ch. 3.10.A), and let N be a neighborhood of e in G such that the exp is a one-to-one mapping from a neighborhood of 0 in L to N. For $x = \exp X$ in N, we define T_x to be the unitary operator $\exp\overline{X}$.

Let X, Y and Z be in L and suppose that $\exp X \exp Y = \exp Z$ in G. Then the two power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} Z^n \quad \text{and} \quad \sum_{k,l=0}^{\infty} \frac{1}{k!}\frac{1}{l!} X^k Y^l$$

are formally equal. Consequently, if u is a vector such that $\|\exp(|X|+|Y|)u\| < \infty$, $\|\exp(|Z|)u\| < \infty$, then $\exp(\overline{X})\exp(\overline{Y})u = \exp(\overline{Z})u$, i.e., $T_{\exp Z}u = T_{\exp X}T_{\exp Y}u$. Now, for X, Y, Z sufficiently close to 0 in L (such that the absolute values of coordinates are less than $\frac{1}{2}s$), there is by our supposition a dense set of vectors

$u \in D$, such that $\|\exp(|X|+|Y|)u\| < \infty$ and $\|\exp(|Z|)u\| < \infty$. Therefore, if x and y are sufficiently close to e in G, $T_x T_y = T_{xy}$. It means that in a neighborhood of e the map $x \to T_x$ defines a unitary homomorphism of the local group. This homomorphism is strongly continuous. Indeed, as $X \to 0$ in L, $\|\exp(X)u - u\| \to 0$ on the dense set, because $\|\exp(\xi s)u\| < \infty$ on the dense set by our supposition. Moreover, the set $\{T_x\}$ being uniformly bounded, $\|T_x\| = 1$, strong continuity on the dense set can be extended to strong continuity in H by continuity of all operators T_x. Consequently, the map $T:(x, u) \to T_x u$ of $N \times H$ into H defines a local unitary representation of G. Because G is simply connected, there is a unique extension of T to a unitary representation of G on H. ▼

This lemma, in fact, provides the criterion for the integrability of a skew-symmetric representation of a Lie algebra. However, its criterion may not be easily applicable in concrete cases. The next theorem provides a simpler criterion for integrability.

THEOREM 2. *Let L be a Lie algebra of skew-symmetric operators on a Hilbert space H which have a common invariant dense domain D. Let $X_1, \ldots, X_d$ be a operator, basis for L and $\Delta = X_1^2 + \ldots + X_d^2$. If Δ is essentially self-adjoint, then there is on H a unique unitary representation T of the simply connected Lie group G which has L as its Lie algebra such that for all X in L, $\overline{T(X)} = \bar{X}$.*

PROOF: Let $\xi = |X_1| + \ldots + |X_d|$ and $\beta = |\Delta| + |I|$. By lemma 3.7, β analytically dominates ξ, and by the last assertion of lemma 3.5 the set of vectors for which $\|\exp(\xi s)u\| < \infty$ for some $s > 0$ is dense in H. Hence, by lemma 1 we obtain the assertion of the theorem. ▼

Th. 2 implies the following corollary which provides a useful criterion for the *non*-integrability of a given skew-symmetric representation of a Lie algebra L.

COROLLARY 1. *Let L be a Lie algebra as in th. 2. If even a single element* iX *for X in L is not essentially self-adjoint, then no unitary representation $T(G)$ of the simply connected group G can be associated with L. Consequently, Δ cannot be essentially self-adjoint.*

PROOF: If Δ is e.s.a., then by th. 2, there is a unique unitary representation $T(G)$ of the simply-connected group G and $\overline{T(X)} = \bar{X}$ for all X in L. By corollary 4 to th. 2.3, $\overline{T(\mathrm{i}X)}$ is self-adjoint and consequently every iX must be essentially self-adjoint. Hence if some iX is not essentially self-adjoint, then there cannot exist a global unitary representation T of G such that $\overline{T(X)} = \bar{X}$. Consequently, Δ cannot be essentially self-adjoint. ▼

EXAMPLE 1. Let L be the Heisenberg Lie algebra, defined by the following commutation relations

$$[X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0. \tag{1}$$

Let $H = L^2(S)$ where S is the open interval $(0, 2\pi)$ and let

$$X = \frac{\mathrm{d}}{\mathrm{d}\varphi}, \quad Y = \mathrm{i}\varphi, \quad Z = \mathrm{i}I \tag{2}$$

be the skew-symmetric representation of L defined on $C_0^\infty(S)$ as the common invariant dense domain. For u in $C_0^\infty(S)$ and any v in $C^1(S)$ we have

$$(Xu, v) = \int_0^{2\pi} Xu(\varphi)v(\varphi)\mathrm{d}\varphi = (u, -Xv)+u(2\pi)v(2\pi)-u(0)v(0)$$
$$= (u, -Xv) = (u, X^*v). \tag{3}$$

Hence $D(X^*) \supset D(X)$, and $X^* = -\dfrac{\mathrm{d}}{\mathrm{d}\varphi}$ on $C^1(S)$. Now the operator X would be essentially self-adjoint if it would have the deficiency indices $(n_+, n_-) = (0, 0)$. From app. B.1(11) it follows that the deficiency indices n_+ and n_- of a symmetric operator iX are equal to the number of solutions of the equation

$$X^*v_\pm = \pm v_\pm. \tag{4}$$

For $X^* = -\dfrac{\mathrm{d}}{\mathrm{d}\varphi}$ on $C^1(S)$, we obtain $v_\pm = \exp(\mp\varphi)$. Hence deficiency indices n_+ and n_- of the operator X are (1,1); thus the operator X is not essentially self-adjoint. Consequently the representation (2) of the Heisenberg algebra is not integrable to a global representation of the corresponding nilpotent group, by virtue of corollary 1. ▼

The following useful corollary provides the main result of th. 2 but with a weaker assumption on the domain of the representation. Roughly speaking, th. 2 concerns the passage from a domain D of the type C^∞ to C^k, $k < \infty$, and the corollary below the passage from C^k to C^2.

COROLLARY 2. *Let L be a real Lie algebra and H a Hilbert space. For each X in L let $\varrho(X)$ be a skew-symmetric operator on H. Let D be a dense, linear subspace of H, such that for all X, Y in L, D is contained in the domain of $\varrho(X)\varrho(Y)$. Suppose that for all X, Y in L, u in D and real numbers a and b, we have*

$$\varrho(aX+bY)u = a\varrho(X)u+b\varrho(Y)u, \tag{5}$$
$$\varrho([X, Y])u = \big(\varrho(X)\varrho(Y)-\varrho(Y)\varrho(X)\big)u. \tag{6}$$

Let $X_1, \ldots, X_d$ be a basis for L. If the restriction A of $\varrho(X_1)^2+ \ldots +\varrho(X_d)^2$ to D is essentially self-adjoint, then there is on H a unique unitary representation T of the simply-connected Lie group G, which has L as its Lie algebra such that for all X in L, $\overline{T(X)} = \overline{\varrho(X)}$.

PROOF: In the present case, as in lemma 3.5, we have for each n

$$D(\bar{A}^n) \subset D(\overline{\varrho(X_{i_1})} \ldots \overline{\varrho(X_{i_n})}) \tag{7}$$

The proof of this relation is identical with the proof of eq. 3 (43). Only instead of considering ad$\varrho(X)A$, which might have only 0 in its domain, we consider $\varrho\big((\mathrm{ad}X)\Delta\big)$ where $\Delta = X_1^2+ \ldots +X_d^2$. Because $(\mathrm{ad}X)\Delta \in E^2$ by eq. 3 (21), the operator $\varrho\big((\mathrm{ad}X)\Delta\big)$ is well defined on D. Set $\tilde{D} = \bigcap_{n=1}^{\infty} D(\bar{A}^n)$, $\tilde{A}$ the restriction of $\bar{\tilde{A}}$ to $\tilde{D}$ and $\tilde{X}_i$ the restriction of $\overline{\varrho(X_i)}$ to $\tilde{D}$. By eq. (7), $\tilde{D}$ is invariant relative to $\tilde{X}_i$. Moreover, since $A \subset \tilde{A} \subset \bar{A}$ we find that $\bar{\tilde{A}}(=\bar{A})$ is self-adjoint, i.e., $\tilde{A}$ is essentially self-adjoint. Thus, all assumptions of th. 2 are satisfied and, consequently, the corollary follows. ▼

As a byproduct of th. 2 and corollary 2 we obtain a convenient criterion for the strong commutativity of unbounded operators. We recall that two unbounded self-adjoint operators are called *strongly commuting* if their spectral resolutions commute.

COROLLARY 3. *Let A and B be symmetric operators on a Hilbert space H and let D be a dense linear subspace of H, such that D is contained in the domain of A, B, A^2, AB, BA and B^2, and such that $ABu = BAu$ for all u in D. If the restriction of A^2+B^2 to D is essentially self-adjoint, then*

1° *A and B are essentially self-adjoint,*

2° *$\bar{A}$ and $\bar{B}$ strongly commute.*

PROOF: ad 1°. Let L be a two-dimensional abelian Lie algebra with a basis X, Y and let $\varrho\,(aX+bY) = \mathrm{i}aA+\mathrm{i}bB$. Then all assumptions of corollary 2 are satisfied. Consequently there exists on H a unique unitary representation T of a simply-connected abelian group (i.e. isomorphic to R^2) G, such that $\overline{T(Z)} = \overline{\varrho(Z)}$ for all Z in L. By corollary 4 to th. 2.3. $\mathrm{i}\overline{T(Z)}$ is self-adjoint. Consequently, $A = -\mathrm{i}T(X)$ and $B = -\mathrm{i}T(Y)$ are e.s.a.

ad 2°. Because G is abelian, the unitary operators $T_{\exp Z} = \exp[\overline{T(Z)}]$, $x = \exp Z \in G$, $Z \in L$ commute. Hence the self-adjoint operators $\mathrm{i}\overline{T(X)} = A$ and $\mathrm{i}\overline{T(Y)} = B$ are strongly commuting. ▼

We know by corollary 3 to th. 2.3 that symmetric elements L, M of the center Z of an enveloping algebra E are mapped onto essentially self-adjoint operators $T(L)$ and $T(M)$. However, if G is a physical symmetry group, we require that the corresponding self-adjoint operators $\overline{T(L)}$ and $\overline{T(M)}$, as well as T_x and $T(L)$, $x \in G$, $L = L^+ \in Z$ are strongly commuting. The following theorem shows that this is indeed the case.

THEOREM 3. *Let T be a unitary representation of a connected Lie group G in a Hilbert space H and let Z be the center of the left invariant enveloping algebra E of G. Then*

1° *For any symmetric L, M in Z the self-adjoint operators $\overline{T(L)}$ and $\overline{T(M)}$ are strongly commuting.*

2° *For any symmetric N in Z and x in G, the operators $\overline{T(N)}$ and T_x are strongly commuting.*

PROOF: ad 1°. For any

$$u(\varphi) = \int_G \varphi(x)\,T_x u\,\mathrm{d}x \in D_G$$

(D_G-Gårding domain), we have by formula 1(17)

$$T(L)T(M)u(\varphi) = u(\tilde{L}\tilde{M}\varphi) = u(\tilde{M}\tilde{L}\varphi) = T(M)T(L)u(\varphi).$$

Hence, taking $D = D_G$, we obtain that all assumptions of corollary 3 are satisfied. Consequently, the self-adjoint operators $\overline{T(L)}$ and $\overline{T(M)}$ are strongly commuting.

ad 2°. Let $u(\varphi) \in D_G$. Then using the fact that the left translations $L_y\varphi(x) = \varphi(y^{-1}x)$ commute with any element $N \in Z$ and by eqs. 1(17) and 1(13) we obtain for any $u(\varphi) \in D_G$ the equality

$$\begin{aligned} T_y T(N)u(\varphi) &= T_y u(\tilde{N}\varphi) = u(L_y\tilde{N}\varphi) = u(\tilde{N}L_y\varphi) = T(N)u(L_y\varphi) \\ &= T(N)T_y u(\varphi). \end{aligned}$$

Hence, the operators $T(N)$ and T_x for any N in Z and x in G commute on the Gårding domain. The operators $\overline{T(N)}$ and T_x also commute on $D(\overline{T(N)})$. Indeed, if $u \in D(\overline{T(N)})$, then from the definition of the closure $\overline{T(N)}$ of $T(N)$ it follows that there exists a sequence $D_G \ni u_n \to u$ such that $T(N)u_n \to v$ and $T(N)u_n \to \overline{T(N)}u = v$. Because every operator T_x, $x \in G$, is continuous, then for every $u \in D(\overline{T(N)})$ we have

$$\begin{aligned} T_x\overline{T(N)}u &= T_x \lim_{n\to\infty} T(N)u_n = \lim_{n\to\infty} T_x T(N)u_n \\ &= \lim_{n\to\infty} T(N)T_x u_n = \overline{T(N)}T_x \lim u_n = \overline{T(N)}T_x u. \end{aligned}$$

The assertion of item 2° follows now from the fact that every bounded operator commuting with a self-adjoint operator is strongly commuting. ▼

§ 6. FS^3-Theory of Integrability of Lie Algebra Representations

We shall present in this section a beautiful theory of integrability of Lie algebra representations elaborated by Flato, Simon, Snellman and Sternheimer ($\equiv FS^3$-theory). In contradistinction to Nelson's theory presented in sec. 5 it gives integrability criteria directly in terms of the properties of the the generators of the Lie algebras: therefore it is generally more effective in practical applications, especially for higher dimensional Lie algebras.

We begin with the observation that in the case of finite-dimensional real Lie algebras one may introduce several other definitions of analytic vectors, which are inequivalent (and weaker) than that used in secs. 3–4 (compare definition 3.1):

(i) A vector $u \in H$ is analytic for every element in the Lie algebra (in the sense of analyticity for a single operator as defined in sec. 3).

(ii) A vector $u \in H$ is analytic for every element X_i, $i = 1, 2, \ldots, \dim L$, in a given linear basis of the Lie algebra.

(iii) A vector $u \in H$ is analytic for every element X_k of Lie generators of the Lie algebra.*

It is evident that the notions of analyticity according (i)–(iii) are weaker than the notion introduced in sec. 4.

We shall now prove the basic result which says that it is enough to have a common invariant dense set of analytic vectors only for a basis of the Lie algebra (in the sense (ii)) to ensure the integrability of the *a priori* given Lie algebra representation. We begin with some preliminary results.

Let $x \to T(x)$ be a representation of a Lie algebra L on a complex Hilbert space H by skew-symmetric operators defined over a common dense invariant domain D. It is evident that such a representation is strongly continuous in H. In what follows we shall write $X = T(x)$, $Y = T(y)$, etc., and we shall denote

$$A(tX, Y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (\mathrm{ad} X)^n Y.$$

LEMMA 1. *For any $x, y \in L$ and $u \in D$, the series*

$$A(tX, Y)u = \sum_{n=0}^{\infty} \frac{t^n}{n!} \big((\mathrm{ad} X)^n Y\big) u \tag{1}$$

is convergent for all $t \in R$ and we have

$$T(e^x y e^{-x} y) u = A(X, Y) u. \tag{2}$$

PROOF: Eq. (2) follows from eq. 3.3(43) and from the above-mentioned strong continuity. Changing x into tx, we obtain the convergence of the series $A(tX; Y)u$ for all $t \in R$. ▼

One readily verifies that for any $x, y \in L$ and $u \in D$ we have

$$YX^m u = \sum_{p=0}^{m} \binom{m}{p} X^p \big((-\mathrm{ad} X)^{m-p} Y\big) u. \tag{3}$$

PROPOSITION 2. *Let T and T' be two representations of the real Lie algebra L by skew-symmetric operators over common invariant domains D and D' (respectively), dense in H, with $D \subset D'$, and such that for any $y \in L$, $Y = T(y)$ is the restriction to D of $Y' = T'(y)$. Then if D is a domain of analytic vectors for some $X = T(x)$ $\in T(L)$ we have, denoting by $\langle , \rangle$ the scalar product in H, for any $t \in R$, $u \in D$ and $v \in D'$:*

$$\langle -e^{t\bar{X}} Y u, v \rangle = \langle e^{t\bar{X}} u, A(tX', Y') v \rangle. \tag{4}$$

---

* We recall that a set $\{X_k\}_{k=1}^n$, $n \leq \dim L$ is called a set of Lie generators of L if L is generated by $\{X_k\}_{k=1}^n$ by linear combinations of repeated commutators.

PROOF: By virtue of remark 1 to lemma 3.1 the closure $\overline{X}$ of X (and of X') is skew-adjoint and therefore generates a unique one-parameter unitary group which we denote by $e^{t\overline{X}}$. For all $u \in D$, the function $t \to e^{t\overline{X}} u$ and $t \to e^{t\overline{X}} Yu$ are analytic in R. By lemma 1, the function $t \to A(tX', Y')v$, with $v \in D'$, is also analytic in R. The functions $a(t) = \langle -e^{t\overline{X}} Yu, v\rangle$ and $b(t) = \langle e^{t\overline{X}} u, A(tX', Y')v\rangle$ are therefore also analytic for all real t. Now we have

$$\frac{d^n a}{dt^n}(0) = \langle -X^n Yu, v\rangle, \tag{5}$$

$$\frac{d^n b}{dt^n}(0) = \sum_{p=0}^{n} \binom{n}{p} \langle X^p u, (\mathrm{ad}X')^{n-p} Y'v\rangle. \tag{6}$$

Since $Y \subset Y' \subset -Y'^*$ (and the same for X), we obtain from (3) that $\frac{d^n a}{dt^n}(0) = \frac{d^n b}{dt^n}(0)$, hence $a(t) = b(t)$ for all $t \in R$. ▼

COROLLARY 1. *Under the conditions of proposition* 1, $e^{-t\overline{X}}v$ *belongs to the domain* $D(Y^*)$ *of the adjoint* Y^* *of* Y *for all* $v \in D'$ *and* $t \in R$, *and*

$$T'(e^{tx} y e^{-tx})v = -e^{t\overline{X}} Y^* e^{-t\overline{X}} v. \tag{7}$$

Indeed, due to the continuity in u of the right-hand side of (4), $e^{-t\overline{X}}v \in D(Y^*)$, whence the formula (since D is dense in H).

HYPOTHESIS (A): *T is a representation of a Lie algebra L on a dense invariant domain D of vectors that are analytic for all skew-symmetric representatives $X_i = T(x_i)$ of a basis $x_1, \ldots, x_r$ of L.*

LEMMA 3. *Hypothesis* (A) *being satisfied, define H_∞ as the intersection of the domains of all monomials $\overline{X}_{i_1} \ldots \overline{X}_{i_n}$, for all $1 \leqslant i_1, \ldots, i_n \leqslant r$, $n \in N$. Let X_i' be the restriction of $\overline{X}_i$ to H_∞ and define, for all $y = \sum_{i=1}^{r} \lambda_i x_i \in L$,*

$$Y' \equiv T'(y) \equiv \sum_{i=1}^{r} \lambda_i X_i'$$

(with invariant domain H_∞). Then T' is a representation of L by skew-symmetric operators (on H_∞) and we have, for any two elements x_i and x_j in the basis and $v \in H_\infty$:

$$A(tX_i', X_j')v = e^{t\overline{X}_i} \overline{X}_j e^{-t\overline{X}_i} v. \tag{8}$$

PROOF: By definition, H_∞ contains D and is invariant under all

$$\overline{X}_i = \overline{X}_i' = -X_i^* = -X_i'^*,$$

hence under all Y' which, due to the hypothesis made, are skew-symmetric. By definition also, T' is linear. Now, if

$$[x_i, x_j] = \sum_{k=1}^{r} c_{ijk} x_k$$

we have, for all $v \in H_\infty$ and $u \in D$:

$$\begin{aligned}\langle (X_i' X' - X_j' X_i') v, u\rangle &= \langle v, (X_j X_i - X_i X_j) u\rangle \\ &= \langle v, -\sum_k c_{ijk} X_k u\rangle = \langle \sum_k c_{ijk} X_k' v, u\rangle.\end{aligned}$$

Therefore T' is a representation and we can apply formula (7) with $D' = H_\infty$ to $y = x_j$, whence (8). ▼

LEMMA 4. *Under hypothesis* (A), *the above-defined domain* H_∞ *is invariant under* $T'(L)$ *and under all one-parameter groups* $e^{t\bar{X}_i}$, *and if* $t_1, \ldots, t_r$ *are differentiable functions of some parameter* t, *then for all* $u \in H_\infty$ *the vector-valued function*

$$t \to e^{t_1 \bar{X}_1} \ldots e^{t_r \bar{X}_r} u$$

has a first derivative in t.

PROOF: From (8) and the invariance of H_∞ under $T'(L)$ we obtain

$$A(tX_i', X_{j_1}') \cdots A(tX_i', X_{j_n}') v = e^{t\bar{X}_i} \bar{X}_{j_1} \ldots \bar{X}_{j_n} e^{-t\bar{X}_i} v$$

for all base elements $x_i, x_{j_1}, \ldots, x_{j_n}$ and all $v \in H_\infty$, and $e^{-t\bar{X}_i} v$ belongs to the domain of all operators $\bar{X}_{j_1} \ldots \bar{X}_{j_n}$, whence the invariance of H_∞ under the $e^{-t\bar{X}_i}$.

The differentiability property follows by induction from the differentiability of the vector-valued function $t \to U(t)u(t)$ where $t \to u(t) \in H_\infty$ is strongly differentiable and $t \to U(t)$ is a unitary operator-valued function strongly differentiable on H_∞ (such that the map $(t, u) \to U(t)u$ is continuous $R \times H \to H$). ▼

We now give the main result.

THEOREM 5. *Let* T *be a Lie algebra representation in a complex Hilbert space satisfying hypothesis* (A). *Then* T *is integrable to a unique unitary group representation.*

PROOF: Let x, y be any elements of L close enough to 0 so that e^x, e^y, $e^x e^y$, and therefore $e^{tx} e^y$ for $0 \leqslant t \leqslant 1$, belong to the neighbourhood W of the identity of G introduced in 3.3.G. We shall write

$$\begin{aligned} e^{tx} e^y &= e^{\alpha_1 x_1} \ldots e^{\alpha_r x_r}, \\ e^{tx} &= e^{t_1 x_1} \ldots e^{t_r x_r}, \\ e^y &= e^{\beta_1 x_1} \ldots e^{\beta_r x_r}. \end{aligned} \tag{9}$$

For any $z \in L$ such that $e^z \in W$, we write (in a unique way, once the basis is chosen) $e^z = e^{z_1 x_1} \ldots e^{z_r x_r}$ and define

$$T(e^z) = e^{z_1 \bar{X}_1} \ldots e^{z_r \bar{X}_r}. \tag{10}$$

Since G is generated by finite products of elements of W, we only have to show that the group law holds in W, i.e. that for any $e^x, e^y \in W$ such that $e^x e^y \in W$, we have $T(e^x e^y) = T(e^x)T(e^y)$, and that $T(L)$ is on D the differential of $T(G)$. The uniqueness is obvious since relation (10) is a necessary condition.

From lemma 4, for $u \in H_\infty$, $T(e^{tx})u$ and $T(e^{tx}e^y)T(e^y)^{-1}u$ are differentiable functions of t. Since

$$\frac{d}{dt_i} e^{t_i \overline{X}_i} u = \overline{X}_i e^{t_i \overline{X}_i} u = e^{t_i \overline{X}_i} \overline{X}_i u,$$

we have by direct computation

$$\frac{d}{dt} T(e^{tx})u = \left(\frac{dt_1}{dt}\overline{X}_1 + \ldots + \frac{dt_r}{dt} e^{t_1\overline{X}_1} \ldots e^{t_{r-1}\overline{X}_{r-1}} \overline{X}_r e^{-t_{r-1}\overline{X}_{r-1}} \ldots e^{-t_1\overline{X}_1}\right) T(e^{tx})u,$$

and similarly

$$\frac{d}{dt} T(e^{tx})u = T(e^{tx})\left(\frac{dt_1}{dt} e^{-t_r\overline{X}_r} \ldots e^{-t_2\overline{X}_2}\overline{X}_1 e^{t_2\overline{X}_2} \ldots e^{t_r\overline{X}_r} + \ldots + \frac{dt_r}{dt}\overline{X}_r\right)u.$$

From relations 3.3(41), (2), (8), (9), and (10) we then get:

$$\frac{d}{dt} T(e^{tx})u = X'T(e^{tx})u = T(e^{tx})X'u. \tag{11}$$

On the other hand we have by direct computation, for all $u \in H_\infty$

$$\frac{d}{dt} T(e^{tx}e^y)T(e^y)^{-1}u$$

$$= \left(\frac{d\alpha_1}{dt}\overline{X}_1 + \ldots + \frac{d\alpha_r}{dt} e^{\alpha_1\overline{X}_1} \ldots e^{\alpha_{r-1}\overline{X}_{r-1}} \overline{X}_r e^{-\alpha_{r-1}\overline{X}_{r-1}} \ldots e^{-\alpha_1\overline{X}_1}\right) T(e^{tx}e^y)T(e^y)^{-1}u.$$

Hence, from relations 3.3(42), (2), (8), (9) and (10) we obtain that for all $u \in H_\infty$, $T(e^{tx}e^y)T(e^y)^{-1}u$, which belongs to H_∞, is also a differentiable solution of the vector-valued differential equation (with values in H_∞ and derivation in the H-topology)

$$\frac{d}{dt} u(t) = X'u(t), \qquad u(t) \in H_\infty. \tag{12}$$

Such an equation has a unique solution (cf. e.g. Kato 1966, p. 481). Indeed one checks easily that for any solution $u(s) \in H_\infty$, and $0 \leqslant s \leqslant t \leqslant 1$, $T(e^{(t-s)x})u(s)$ is differentiable in s and that

$$\frac{d}{ds}\left(T(e^{(t-s)x})u(s)\right) = -T(e^{(t-s)x})X'u(s) + T(e^{(t-s)x})X'u(s) = 0.$$

Therefore $T(e^{(t-s)x})u(s)$ does not depend on s. Equating its values for $s = 0$ and $s = t$ we obtain $u(t) = T(e^{tx})u(0)$, whence the uniqueness of the solution of (12) in H_∞ and the group law (which we can extend from H_∞ to H by continuity)

$$T(e^{tx}e^y) = T(e^{tx})T(e^y).$$

Moreover, relation (11) shows that $T(L)$ is the restriction to D of the differential of $T(G)$.▼

Let us note that while according to Nelson's result, an analytic vector for Δ was necessarily analytic for the whole algebra the situation in th. 5 is not similar: we only know from what was said up till now that the existence of a dense invariant set of analytic vectors for the basis implies integrability and therefore, by the global theorem, the existence of (another) dense invariant set of analytic vectors for the whole algebra (in the Nelson sense), therefore analytic for the group.

We now give a stronger version of th. 5 which utilizes the weakest concept of analyticity as defined by (iii).

THEOREM 6. *Let hypothesis* (A) *be satisfied. Let* $\{x_1, \ldots, x_n\}$ *be a set of Lie generators of the Lie algebra L, and let A denote a set of analytic vectors for* $T(x_1), \ldots, T(x_n)$ *separately. Then*

(i) *The set of all analytic vector for a given arbitrary element of L is invariant under* $T(L)$.

(ii) *There exists a unique unitary representation of the corresponding connected and simply-connected Lie group* (*having L as its Lie algebra*) *on the closure of the smallest set* A' *containing A and invariant under* $T(L)$, *the differential of which on* A' *is equal to* T.▼

(For the proof cf. Flato and Simon 1973.)

It is noteworthy that the invariance of the set of analytic vectors under $T(L)$ was obtained automatically.

The following theorem clarifies completely the connection between the weak analyticity, the strong analyticity of Nelson and the analyticity for group representation.

THEOREM 7. *Let G be a real finite-dimensional Lie group. Then there exists a basis* $\{x_1, \ldots, x_n\}$ *of the corresponding Lie algebra L such that given any representation of G on a Hilbert space, any vector analytic separately for the closures of the representatives of the basis* $\{x_1, \ldots, x_n\}$ *will be analytic for the group representation* (*which means analytic for the whole Lie algebra namely jointly analytic*).▼

(For the proof cf. Flato and Simon 1973.)

We now come to another question: Are analytic vectors really necessary in order to ensure integrability?

The first example showing the necessity was constructed by Nelson 1959. The following theorem well illustrates this problem.

THEOREM 8. *Every compact Lie algebra of dimension* $n > 1$, *has at least one representation in Hilbert space on an invariant domain such that every element of the algebra is represented by an essentially skew-adjoint operator on this domain, every element of some linear basis of the algebra is integrable to a one-parameter compact group, but the representation is not integrable.*

(For the proof cf. Flato, Simon and Sternheimer 1973.)

The FS^3-theory provides a very convenient framework for applications. In particular Niederle and Mickelsson 1973 and Niederle and Kotecký 1975 utilizing FS^3-integrability criteria have shown that the representations of su(p, q) and so(p, q) obtained by Gel'fand–Zetlin method (see sec. 8) are integrable.

The FS^3-theory has also found an interesting applications in Wightman quantum field theory. In particular Snellman 1972 has proved that the action of the polynomial ring in the field operators on the vacuum contains a dense set of analytic vectors for the Poincaré group. This analysis was extended later on by Nagel and Snellman 1974.

The FS^3-theory admits generalizations to other spaces from the Hilbert space. In fact, the theory of integrability of representations of Lie algebras in quasi-complete locally convex spaces was given by FS^3 (see Flato *et al.* 1972, sec. 5) The case of Banach space representations was treated in details by Kisyński 1973.

§ 7. The 'Heat Equation' on a Lie Group and Analytic Vectors

We describe in this section an interesting global method for the construction of analytic vectors for a representation T of a Lie group G. This is a generalization of the Gårding's method for the construction of regular vectors. We replace in fact the function $\varphi \in C_0^\infty(G)$ in the integral 1 (12), i.e.

$$u(\varphi) = \int_G \varphi(x) T_x u \, dx, \tag{1}$$

by a certain analytic function which decreases rapidly enough at infinity together with all its mixed partial derivatives. The class of all such functions is provided by the solutions of the heat equation on a Lie group. The case of nonunitary representations requires a slight extension of the proof of lemma 3 (below) only. Hence we treat here both the cases of unitary and nonunitary representations simultaneously. For simplicity of exposition we restrict ourselves to unimodular Lie groups.

Let X_i, $i = 1, 2, \ldots, d = \dim L$, be the generators of the right translations in $H = L^2(G)$ and let $\Delta = X_1^2 + \ldots + X_d^2$. Then by the heat equation on the Lie group G, we mean the equation of the form

$$\left(\frac{\partial}{\partial t} - \Delta\right)\varphi(t, x) = 0, \tag{2}$$

where t is a real parameter and $x \in G$.

If G is the translation group of the real line, then eq. (2) has the form

$$\left(\frac{\partial}{\partial t} - \frac{d^2}{dx^2}\right)\varphi(t, x) = 0, \tag{3}$$

i.e., it is the ordinary one-dimensional heat equation.

We know by th. 2.2 that the symmetric elliptic operator Δ in $L^2(G)$ is essentially self-adjoint. We denote by $\bar{\Delta}$ its self-adjoint extension in H and set

$$\varphi(t, x) = \big(\exp(t\bar{\Delta})f\big)(x), \tag{4}$$

where $f \in C_0^\infty(G)$ and $t > 0$. By the spectral theorem (app. B.3, th. 1) we have $\partial_t \varphi = \bar{\Delta}\varphi = \Delta\varphi$. Hence, φ is a solution of the heat equation (2). Moreover, as $t \to 0$, $\varphi(t, x) \to f(x)$.

EXAMPLE 1. Let G be the translation group of the real line. Then $\Delta = \mathrm{d}^2/\mathrm{d}x^2$ and the general solutions (4) of the corresponding heat equation (3) have the form

$$\varphi(t, x) = \exp\left(t\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)f(x) = \int_{-\infty}^{+\infty} \exp\left(t\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)\delta(x-y)f(y)\,\mathrm{d}y.$$

The kernel function $\exp(t\mathrm{d}^2/\mathrm{d}x^2)\,\delta(x-y)$ can be written in the form

$$\exp(t\mathrm{d}^2/\mathrm{d}x^2)\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp[\mathrm{i}p(x-y)]\,\mathrm{d}p = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-tp^2)\exp[\mathrm{i}p(x-y)]\,\mathrm{d}p$$

$$= \frac{\exp\left(-\dfrac{(x-y)^2}{4t}\right)}{\sqrt{4\pi t}}. \tag{5}$$

Hence

$$\varphi(t, x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4t}\right]f(y)\,\mathrm{d}y, \quad f \in C_0^\infty(G). \tag{6}$$

The kernel (5) is the well-known fundamental solution (Green's function) of the heat equation.

Note that if $t \to 0$,

$$\frac{1}{\sqrt{4\pi t}}\exp\left[-\frac{(x-y)^2}{4t}\right] \to \delta(x-y),$$

and consequently, $\varphi(t, x) \to f(x)$. ▼

We observe that the functions (5) and (6), as well as all their derivatives, decrease more rapidly at infinity than any exponential function. This turns out to be a general feature of the solution (4) of the heat equation (2) for an arbitrary Lie group. To state this property precisely, we make use of the concept of distance on an arbitrary Lie group (cf. ch. 3.9).

Let $\tau(x) \equiv \tau(x, e)$ be the distance between the unity e and x in a connected Lie group G. We shall show that the function $\varphi(t, x) = \exp(t\bar{\Delta})f$ with f in $C_0^\infty(G)$ decreases more rapidly than any exponent.

LEMMA 1. *Let* $\varphi(t, x) = \exp(t\bar{\Delta})f(x)$, $f \in C_0^\infty(G)$ *and let* s *be a real number. Then*

$$(\exp(s\tau)\varphi, \varphi) \leqslant \exp(\tfrac{1}{2}s^2 t)\,(\exp(s\tau)f, f), \tag{7}$$

where $(\cdot, \cdot)$ *denotes the scalar product in* $H = L^2(G)$. ▼

Remark: Clearly the right hand side of eq. (7) is finite because f has a compact support. We multiply both sides of eq. (7) by $\exp[-\frac{1}{2}s^2(t+\varepsilon)]$, where $\varepsilon > 0$,

$$(\exp[-\tfrac{1}{2}s^2(t+\varepsilon)+s\tau]\varphi, \varphi) \leqslant (\exp[-\tfrac{1}{2}\varepsilon s^2+s\tau]f, f), \tag{8}$$

then integrate both sides over s in the interval $(-\infty, +\infty)$ and get

$$(t+\varepsilon)^{-1/2}\left(\exp\left[-\frac{\tau^2}{2(t+\varepsilon)}\right]\varphi, \varphi\right) \leqslant \varepsilon^{-1/2}(\exp[-\tau^2/2\varepsilon]f, f). \tag{9}$$

Thus due to the factor $\exp\left[-\dfrac{\tau^2}{2(t+\varepsilon)}\right]$ on the left hand side of eq. (9), the decrease of the function φ at fixed t and for $x \to \infty$ must be at least as fast as

$$\exp\left(-\frac{\tau^2(x)}{4t}\right). \tag{10}$$

This is a direct generalization to an arbitrary connected Lie group of the result which we derived for the translation group on the straight line (cf. eq. (6)).

PROOF OF LEMMA 1: Let $X_1, \ldots, X_d$ be a basis in the left invariant Lie algebra L and let

$$\nabla u = \{X_1 u, \ldots, X_d u\} \tag{11}$$

be the gradient of a function u in H. The scalar product of ∇u and ∇v is

$$(\nabla u, \nabla v) = \sum_{i=1}^{d}(X_i u, X_i v). \tag{12}$$

Clearly,

$$(\nabla u, \nabla v) = (-\Delta u, v) = (u, -\Delta v),$$

where $\Delta = X_1^2 + \ldots + X_d^2$ is the left-invariant Laplacian on G. Let H^1 be the linear subspace of H consisting of elements u in H satisfying the condition

$$\|u\| + \|\nabla u\| < \infty. \tag{13}$$

For $v \in H^1$, differentiable relative to t and for a real function h which is bounded together with its derivatives, $h_t = \partial_t h$ and $X_i h$, the following identities are satisfied ($v_t \equiv \partial_t v$)

$$\begin{aligned}(v_t, h^2 v) + (h^2 v, v_t) &= \partial_t(hv, hv) - 2(h_t hv, v),\\ (\nabla v, \nabla(h^2 v)) + (\nabla(h^2 v), \nabla v) &= 2(\nabla(hv), \nabla(hv)) - 2(v\nabla h, v\nabla h).\end{aligned} \tag{14}$$

Summing up these identities and dividing by 2, we obtain

$$\operatorname{Re}\{(v_t, h^2v)+(\nabla v, \nabla(h^2v))\} = \tfrac{1}{2}\partial_t\|hv\|^2+\|\nabla(hv)\|^2-\|v\nabla h\|^2-(h_t hv, v). \quad (15)$$

Let $N > 0$. Set

$$\psi(x) = \psi_N(x) \equiv \min\{\exp[\tfrac{1}{2}s\tau(x)], N\}.$$

Then ψ is bounded. By virtue of exercise 3.11.9.1 we obtain

$$|\nabla \exp[\tfrac{1}{2}s\tau(x)]|^2 = \sum_{n=1}^{d} |X_n \exp[\tfrac{1}{2}s\tau(x)]|^2 = \frac{s^2}{4}\exp[s\tau(x)]\sum_{n=1}^{d}|X_n\tau(x)|^2$$

$$\leqslant \frac{s^2}{4}\psi^2(x)|\nabla\tau(e)|^2 = \frac{s^2}{4}\psi^2(x). \quad (16)$$

In eq. (15) we put $v(t, x) = \varphi(t, x) = \exp[t\bar{\Delta} f(x)]$ and $h(t, x) = \psi(x)$. Then because

$$(\varphi_t, \omega)+(\nabla\varphi, \nabla\omega) = (\varphi_t-\Delta\varphi, \omega) = 0,$$

for any ω in H^1, the left hand side of eq. (15) is zero. Hence

$$\partial_t\|\psi\varphi\|^2 \leqslant 2\|\varphi\nabla\psi\|^2. \quad (17)$$

From inequality (16) we obtain

$$\partial_t\|\psi\varphi\| \leqslant \tfrac{1}{2}s^2\|\psi\varphi\|, \qquad \text{i.e.} \qquad \partial_t\exp(-\tfrac{1}{2}s^2t)\|\psi\varphi\| \leqslant 0.$$

Consequently we can put

$$\|\psi\varphi\| \leqslant \exp(\tfrac{1}{2}s^2t)\|\psi f\|.$$

For $N\to\infty$ we obtain the inequality (7).▼

Next we derive asymptotic properties analogous to (7) and (10) for arbitrary finite mixed derivatives of $\varphi(t, x)$. Let $\alpha = \alpha_1, \ldots, \alpha_p$ represent a multi-index, where $\alpha_j = 1, 2, \ldots, d$, and let X_α be the product

$$X_\alpha = X_{\alpha_1}\ldots X_{\alpha_p}, \quad (18)$$

where X_{α_j} are left-invariant generators of L.

Set

$$P_\alpha = \partial_t^{(k)}X_\alpha, \qquad \tilde{P}_\alpha = \partial_t^{(k)}\tilde{X}_\alpha, \quad (19)$$

where $\tilde{X}_\alpha \equiv \tilde{X}_{\alpha_1}\ldots\tilde{X}_{\alpha_p}$ are the corresponding elements of the right-invariant enveloping algebra (of eq. 1(16)):

LEMMA 2. *Let*

$$\psi(t, x) = (P_\alpha\varphi)(t, x), \quad \text{or} \quad \psi(t, x) = (\tilde{P}_\alpha\varphi)(t, x), \quad (20)$$

and let $\chi(x)$ be a numerical continuous function on G, which does not increase more rapidly than some exponent. Then $(\chi\psi, \psi)$ is bounded for bounded t.

PROOF: Consider first $\psi = \tilde{P}_\alpha \varphi$. Then, because left- and right-invariant generators commute, we have

$$\tilde{P}_\alpha \varphi = \exp(t\bar{\Delta})(\Delta)^k \tilde{X}_\alpha f = \exp(t\bar{\Delta})\tilde{f} = \tilde{\varphi}, \tag{21}$$

where $\tilde{f} \in C_0^\infty(G)$. Consequently, $\tilde{\varphi}$ is the solution of the heat equation. Hence, in this case Lemma 2 is a corollary of lemma 1.

The case $\psi = P_\alpha \varphi$ can be reduced to the previous one if use is made of the relation

$$X_\alpha = \sum_\beta a_{\alpha\beta}(x)\tilde{X}_\beta$$

between the elements of the left- and the right-invariant enveloping algebra.

We have then

$$P_\alpha \varphi = \partial_t^k X_\alpha \varphi = \sum_\beta a_{\alpha\beta}(x)\,\partial_t^k \tilde{X}_\beta \varphi = \sum_\beta a_{\alpha\beta}(x)\tilde{P}_\beta \varphi.$$

By exercise 3.11.9.3 the coefficients $a_{\alpha\beta}(x)$ do not increase more rapidly than some exponential. Hence, the last sum and, consequently, $P_\alpha \varphi$ satisfies the assertion of the lemma.

We shall now construct a new set of vector solutions of the heat equation and show that this set is a dense set of analytic vectors for a representation T of G.

LEMMA 3. *Let $f \in C_0^\infty(G)$ and $\varphi(t, x) = \exp(t\bar{\Delta})f(x)$. Then*

$$\Phi(t, x) \equiv \int_G \varphi(t, y^{-1}x)T_y u\,dy, \qquad u \in H, \tag{22}$$

is also a solution of the heat equation (2), *regular for $t \geqslant 0$ with the initial condition*

$$\Phi(0, x) = \int f(y^{-1}x)T_y u\,dy. \tag{23}$$

Moreover, for $t > 0$ the function $\Phi(t, x)$ is analytic relative to x in G.

PROOF: Let L_y be a left translation on G, i.e., $L_y\varphi(x) = \varphi(y^{-1}x)$. Because Δ is left-invariant, we have $[L_y, \Delta] = 0$ for every y in G. Hence

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\Phi(t, x) &= \int_G \left(\frac{\partial}{\partial t} - \Delta\right)\varphi(t, y^{-1}x)T_y u\,dy \\ &= \int_G L_y\left(\frac{\partial}{\partial t} - \Delta\right)\varphi(t, x)T_y u\,dy = 0,\end{aligned} \tag{24}$$

i.e., $\Phi(t, x)$ is the solution of the heat equation (2). For $t \to 0$, $\varphi(t, y^{-1}x) \to f(y^{-1}x)$ and, consequently, $\Phi(t, x) \to \int f(y^{-1}x)T_y u\,dy$. Let ψ be the derivative (20) of the function φ. Consider the integral

$$\Psi(t, x) \equiv \int \psi(t, y^{-1}x)T_y u\,dy. \tag{25}$$

Setting $y \to xy^{-1}$, we obtain

$$\Psi(t, x) = \int \psi(t, y) H(x, y) \mathrm{d}y, \tag{26}$$

where

$$H(x, y) = T_{xy^{-1}} u.$$

We now show that the integral (26) is absolutely convergent. Indeed, by Schwartz inequality

$$\Big\{\int_G |\psi(t, y)| \|H(x, y)\| \mathrm{d}y\Big\}^2$$
$$= \Big\{\int |\psi(t, y)| \exp\Big[\frac{\lambda}{2}\tau(y)\Big] \|H(x, y)\| \exp\Big[-\frac{\lambda}{2}\tau(y)\Big] \mathrm{d}y\Big\}^2$$
$$\leqslant \int_G |\psi(t, y)|^2 \exp[\lambda\tau(y)] \|H(x, y)\|^2 \mathrm{d}y \int_G \exp[-\lambda\tau(y)] \mathrm{d}y, \tag{27}$$

where λ is real. Taking λ sufficiently large and using exercise 3.11.9.2, we obtain that the last integral in (27) is convergent: for the first integral we use lemma 2. Since

$$\|H(x, y)\| \leqslant \exp[c + c\tau(y)],$$

where c is finite for finite x, then the next to the last integral in (27) is also finite by lemma 2. Consequently, the integral (26) is absolutely convergent for bounded t and x.

Let $G_N = \{x \in G: \tau(x) \geqslant N\}$. Then if we replace G by G_N in eq. (27), the right-hand side of (27) provides a convenient majorant for

$$\Big\{\int_{G_N} |\psi(t, y)| \|H(x, y)\| \mathrm{d}y\Big\}^2. \tag{28}$$

Indeed due to the last integral in eq. (27) the expression (28) approaches zero as $N \to \infty$, uniformly for bounded t and x. Consequently, the function

$$\Psi(t, x) = \lim_{N\to\infty} \int_{G - G_N} \psi(t, y) H(x, y) \mathrm{d}y \tag{29}$$

is continuous because it is the uniform limit of continuous functions. Set now $\psi = P\varphi$ and take $\chi \in C_0^\infty(Z \times G)$, where Z is semi-straight line $t > 0$. Because P is left-invariant, we have

$$\int \varphi(t, y^{-1}x) P\chi(t, x) \mathrm{d}t \mathrm{d}x = \int L_y P\varphi(t, x) \chi(t, x) \mathrm{d}t \mathrm{d}x$$
$$= \int \psi(t, y^{-1}x) \chi(t, x) \mathrm{d}t \mathrm{d}x.$$

Multiplying both sides by $T_y u$ and integrating over y, we can, according to the Fubini theorem change the order of integration. From the definitions (22) and (25), we obtain

$$\int \Phi(t, x) P^* \chi(t, x) \, \mathrm{d}t \, \mathrm{d}x = \int \Psi(t, x) \chi(t, x) \, \mathrm{d}t \, \mathrm{d}x. \tag{30}$$

Because this equality is satisfied for an arbitrary $\chi \in C_0^\infty(Z\times G)$, all derivatives $P\Phi$ are continuous in the sense of distribution theory. Consequently, Φ is regular. Strictly speaking, this was proved only for numerical functions. However, the elementary derivation of this fact (Schwartz 1957) can be easily extended to functions with values in a Hilbert space. Now the generators

$$X_i = \sum_{k=1}^{d} a_{ki}(x)\frac{\partial}{\partial x_k}, \quad i = 1, 2, \ldots, d = \dim L,$$

are analytic vector fields on G. Hence, the coefficients $a_k(x)$ are analytic. Therefore, the operator Δ is an elliptic operator with analytic coefficients. It is well known that any solution of a parabolic equation with analytic coefficients is analytic (cf. Gårding 1960, lemma 10.1). Consequently, the function $\Phi(t, x)$ is analytic relative to x in G.

We now give the main theorem of this section.

THEOREM 4. *Let $x \to T_x$ be a representation of a connected Lie group G in a Hilbert space H. Let $f \in C_0^\infty(G)$ and $\varphi(t, x) = \exp(t\bar{\Delta})f(x)$ be a solution of the heat equation. Then for any fixed finite $t > 0$ (and all f in $C_0^\infty(G)$ and u in H), the set of all vectors*

$$u(t) = \int_G \varphi(t, y^{-1})T_y u\,\mathrm{d}y \tag{31}$$

forms a dense set of analytic vectors for T.

PROOF: The function

$$T_x u(t) = \int \varphi(t, y^{-1})T_{xy}u\,\mathrm{d}y = \int \varphi(t, y^{-1}x)T_y u\,\mathrm{d}y$$

satisfies all the assumptions of lemma 3. Consequently, the function $T_x u(t) \equiv \Phi(t, x)$ is regular for $t \geqslant 0$ and analytic relative to x in G for $t > 0$. For $t \to 0$ the vector $u(t)$ tends to the regular vector

$$v = u(0) = \int f(y^{-1})T_y u\,\mathrm{d}y.$$

The set of all such vectors is dense in H by th. 1.1. Consequently, due to the continuity relative to t, the set of analytic vectors (31) is also dense in H. ▼

The dense set of analytic vectors (31) for T of G forms the common dense invariant domain for the representation $T(X)$ of the Lie algebra L of G. In fact, for any $h > 0$ and X in L we have

$$\frac{1}{h}[T_{\exp(hX)}u(t) - u(t)] = \frac{1}{h}\int \varphi(t, y^{-1})(T_{\exp(hX)} - I)T_y u\,\mathrm{d}y$$

$$= \int\left[\frac{1}{h}(\varphi(t, y^{-1}\exp(hX)) - \varphi(t, y^{-1})\right]T_y u\,\mathrm{d}y.$$

The function in the bracket has the limit

$$\frac{1}{h}[\varphi(t, y^{-1}\exp(hX))-\varphi(t, y^{-1})] = \frac{\exp(t\bar{\Delta})}{h}[f(y^{-1}\exp(hX))-f(y^{-1})]$$
$$\underset{h\to 0}{\to} \exp(t\bar{\Delta})[(\tilde{X}f)(y^{-1})] = \exp(t\bar{\Delta})\tilde{f}(y^{-1}) \equiv \tilde{\varphi}(t, y^{-1}),$$

where

$$\tilde{f}(x) \equiv (\tilde{X}f)(x) = \sum_{i=1}^{d} a_i(x)\frac{\partial f}{\partial x_i}.$$

Because $\tilde{f}(y^{-1}) \in C_0^\infty(G)$, the function $\tilde{\varphi}(t, y^{-1})$ is a solution of the heat equation. Consequently, the vector

$$[T(X)u](t) \equiv \int \tilde{\varphi}(t, y^{-1})T_y u \, dy \tag{32}$$

is analytic by th. 4. Clearly, in the same manner, the action of any product of generators can be defined.

Consequently, the linear hull of all analytic vectors (31) provides a common dense invariant domain for the representation $X_{i_1} \dots X_{i_n} \to T(X_{i_1}) \dots T(X_{i_n})$ of the enveloping algebra E.

§ 8. Algebraic Construction of Irreducible Representations

In the previous sections we discussed the general representation theory of Lie algebras by unbounded operators. Here we present some alternative explicit construction of irreducible representations for non-compact Lie algebras based on diagrammatical technique. This technique works for any simple noncompact classical Lie algebra. We shall illustrate it in the case of $u(p, q)$-algebras, since $u(1, 1)$, $u(2, 1)$, $u(2, 2)$, $u(6, 6)$, etc. occur in particle physics.

We consider the algebraic description of irreducible self-adjoint representations of Lie algebras of the class $u(p, q)$. The approach is a direct generalization of the Gel'fand–Zetlin approach for compact Lie algebras $u(n)$ considered in ch. 10, § 1. The discrete series of irreducible representations of $u(p, q)$ which is constructed here can be considered as 'branches' of the discrete series of irreducible representations of the compact Lie algebra $u(p+q)$.

We recall that the pseudo-unitary group $U(p, q)$ is defined as the group of linear transformations of $(p+q)$-dimensional complex space C^{p+q}, which conserve the hermitian form

$$\bar{z}^1 z^1 + \dots + \bar{z}^p z^p - \bar{z}^{p+1} z^{p+1} - \dots - \bar{z}^{p+q} z^{p+q}. \tag{1}$$

Because $U(p, q)$ and $U(q, p)$ are isomorphic, we can restrict ourselves to the case of $U(p, q)$ with $p \geqslant q$.

The hermitian form (1) can be written as

$$z^* \sigma z, \tag{2}$$

where z represents the column consisting of z_k, $k = 1, 2, \dots, p+q$, $z^* = (\bar{z}^1, \dots, \bar{z}^{1+q}) = \bar{z}^T$ and the matrix σ is of the form

$$\sigma = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad \text{where } I_p = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}. \tag{3}$$

Using the definition of $U(p, q)$ we get

$$z^* u^* \sigma u z = z^* \sigma z, \quad u \in U(p, q).$$

Therefore due to the arbitrariness of z, we have

$$u^* \sigma u = \sigma, \quad \text{for} \quad u \in U(p, q). \tag{4}$$

We can now give an equivalent definition of $U(p, q)$; namely, $U(p, q)$ is the group of all linear transformations of C^{p+q} which obey the condition (4). Setting $u(t) = \exp(\mathrm{i}tM)$ in eq. (4), differentiating with respect to t and putting $t = 0$, we find that the generators M obey the condition

$$M^* = \sigma M \sigma, \tag{5}$$

with the commutation relations

$$[M, M'] = MM' - M'M. \tag{6}$$

Consider now the matrix A_{ij} given by

$$(A_{ij})_{lk} = \delta_{il}\delta_{jk}, \quad i, j, l, k = 1, 2, \dots, p+q. \tag{7}$$

These matrices obey the commutation relations of the Lie algebra $\mathrm{gl}(p+q, R)$

$$[A_{ij}, A_{kl}] = \delta_{kj}A_{il} - \delta_{il}A_{kj}. \tag{8}$$

Clearly every $n \times n$ complex matrix can be expressed as a linear combination of the A_{ij}-matrices. In particular, every generator $M \in u(p, q)$ obeying the condition (5) can be expressed in terms of A_{ij}. In fact, the $(p+q)^2$ independent matrices

$$M_{kk} = A_{kk}, \quad k = 1, 2, \dots, n,$$

$$M_{kl} = A_{kl} + A_{lk}, \quad \tilde{M}_{kl} = \mathrm{i}(A_{kl} - A_{lk}); \quad k < l \leqslant p, \text{ or } p < k < l,$$

$$N_{kl} = A_{kl} - A_{lk}, \quad \tilde{N}_{kl} = \mathrm{i}(A_{kl} + A_{lk}), \quad k \leqslant p < l, \tag{9}$$

obey the condition (5) and therefore represent the generators of $U(p, q)$. The commutation relations for the generators (9) of $U(p, q)$ can be obtained with the help of that of $\mathrm{gl}(p+q, R)$ given in (8). The hermiticity condition for generators M_{ik} and N_{ik}

$$M_{ik}^* = M_{ik}, \quad N_{ik}^* = N_{ik} \tag{10}$$

impose the following conditions on the generators A_{kl}

$$A_{kl}^* = \varepsilon_{kl} A_{lk}, \quad \varepsilon_{kl} = \begin{cases} +1, & k, l \leqslant p \text{ or } p < k, l, \\ -1, & k \leqslant p < l \text{ or } l \leqslant p < k. \end{cases} \tag{11}$$

Therefore, the problem of finding the irreducible hermitian representations of $u(p, q)$ is reduced to that of finding those irreducible representations of $\mathrm{gl}(p+q, R)$ whose generators obey the condition (11). From the practical point of view this is a considerable simplification of the problem, because the generators of $\mathrm{gl}(n, R)$, in contradistinction to those of $u(p, q)$, obey the simple and symmetric commutation relations of the form (8) (see also chs. 10.2 and 9.4).

The construction of hermitian irreducible representations of $u(p, q)$ can be carried out along the following steps:

(i) The construction of the carrier space.

(ii) The definition of the action of the generators A_{ij} obeying the commutation relations (8) and the 'hermiticity' condition (11).

(iii) The proof of irreducibility and inequivalence of the representations obtained.

In the case of the compact algebra $u(n)$ the carrier space H_{m_n} was specified by the highest weight $m_n = (m_{1n}, \ldots, m_{nn})$, which was placed in the top row of the Gel'fand–Zetlin pattern (see ch. 10, § 1). We specify the carrier space of the non-compact Lie algebra $u(p, q)$ also by the top row ('highest weight') $m_n = (m_{1n}, \ldots, m_{nn})$, $n = p+q$, of patterns and by the so-called '*type*' of representation. The type of the representation is determined by the decomposition of the number p into two non-negative integers α and β:

$$p = \alpha + \beta. \tag{12}$$

If the top row $m_n = (m_{1n}, \ldots, m_{nn})$, and the type (α, β) are given, then we define the generalized Gel'fand–Zetlin patterns by the following set of inequalities

$$\begin{gathered} m_{j,k+1} \geqslant m_{jk} \geqslant m_{j+1,k+1}, \quad k = 1, 2, \ldots, p-1, \\ m_{1k} \geqslant m_{1,k+1}+1 \geqslant m_{2k} \geqslant m_{2,k+1}+1 \geqslant \ldots \geqslant m_{\alpha k} \geqslant m_{\alpha,k+1}+1, \\ k = p, \ldots, n-1, \\ m_{j,k+1} \geqslant m_{jk} \geqslant m_{j+1,k+1}, \quad j = \alpha+1, \ldots, k-\beta, \; k = p, \ldots, n-1, \\ m_{k-\beta+2,k+1}-1 \geqslant m_{k-\beta+1,k} \geqslant m_{k-\beta+3,k+1}-1 \geqslant \ldots \geqslant m_{k+1,k+1}-1 \geqslant m_{kk}, \\ k = p, \ldots, n-1. \end{gathered} \tag{13}$$

In the present non-compact case the numbers m_{1k} run over the interval $[m_{1,k+1}+1, \infty)$, and the numbers m_{kk} over the interval $(-\infty, m_{k+1,k+1}-1]$. Therefore, the carrier space H_{m_n} is always infinite-dimensional.

The structure of a pattern m is defined as follows: place all numbers m_{jk} of the kth row between the numbers $m_{j,k+1}$ and $m_{j+1,k+1}$ of the $(k+1)$th row, as in the case of the compact $u(n)$ patterns. Then shift each of the first α elements

$m_{1,n-1}, \dots, m_{\alpha,n-1}$ of the $(n-1)$th row one place to the left. Similarly, shift each of the last β elements of the $(n-1)$th row one place to the right. Then shift the elements $m_{i,n-2}$ of the $(n-2)$nd row with respect to the $(n-1)$st row and so on, till the elements m_{ip} of the pth row, inclusive. The position of the elements of m_{ij} of the jth row, $j < p$ with respect to the neighbouring higher row, remains unchanged. In this manner the structure of the pattern reflects the properties of the inequalities (13) imposed on the numbers m_{ij}. It is assumed that the unit vectors corresponding to different patterns are orthonormal and they span the carrier space H_{m_n}.

EXAMPLE. *The Lie algebra* $u(2, 1)$. In this case we have three types:

$$(\alpha, \beta) \sim (2,0),\ (1,1) \text{ and } (0,2).$$

The corresponding patterns are:

$$\left|\begin{array}{cccc} & m_{13} & m_{23} & m_{33} \\ m_{12} & & m_{22} & \\ & m_{11} & & \end{array}\right|, \quad \left|\begin{array}{ccccc} & m_{13} & m_{23} & m_{33} & \\ m_{12} & & & & m_{22} \\ & & m_{11} & & \end{array}\right|, \quad \left|\begin{array}{cccc} m_{13} & m_{23} & m_{33} & \\ & m_{12} & & m_{22} \\ & & m_{11} & \end{array}\right|.$$

The numbers m_{ij} of the first pattern obey the following inequalities:

$$m_{12} \geqslant m_{13}+1, \quad m_{13} \geqslant m_{22} \geqslant m_{23}, \quad m_{12} \geqslant m_{11} \geqslant m_{22}$$

and of the second one the inequalities

$$m_{12} \geqslant m_{13}+1, \quad m_{33}-1 \geqslant m_{22}, \quad m_{12} \geqslant m_{11} \geqslant m_{22}.$$

Thus the infinite-dimensional carrier space H_{m_n} is determined by a set of integers $m_n = (m_{1n}, \dots, m_{nn})$ ('highest weight') which obey the inequalities

$$m_{1n} \geqslant m_{2n} \geqslant \dots \geqslant m_{nn}, \quad n = p+q,$$

and by a type (α, β). ▼

As in the case of $u(n)$, let the symbol $\hat{m}^j_{k-1}(m^j_{k-1})$ denote the pattern obtained from a pattern m by replacing the number $m_{j,k-1}$ in the $(k-1)$st row of m by the number $m_{j,k-1}-1$ $(m_{j,k-1}+1)$.

In order to obtain a hermitian representation of $u(p, q)$ in H_{m_n}, we shall define the action od A_{ij}, $i, j = 1, 2, \dots, n$, on the basis vectors of H_{m_n} in such a way that the commutation relations (8) and the 'hermiticity' conditions (11) will be fulfilled. It was shown in ch. 10, § 1 that any generator $A_{k-h,k}$, $h > 0$, can be expressed as the commutator of the generators $A_{k,k}$, $A_{k,k-1}$ and $A_{k-1,k}$ by

$$A_{k,k-h} = [A_{k,k-1}, A_{k-1,k-h}], \quad A_{k-h,k} = [A_{k-h,k-1}, A_{k-1,k}], \quad h > 1. \tag{14}$$

Therefore, in order to define the action of an arbitrary generator A_{ij}, $i, j = 1, 2, \dots, p+q$, in the carrier space H_{m_n} it is sufficient to define the action of the generators $A_{kk}, A_{k-1,k}$, and $A_{k,k-1}$, $k = 1, 2, \dots, p+q-1$. The action of these generators on the pattern $m \in H_{m_n}$ may be specified in a similar manner as in the case of the algebra $u(n)$. We denote by $m^j_{k-1}(\hat{m}^j_{k-1})$ the pattern obtained from m by replacement of $m_{j,k-1}$ by $m_{j,k-1}(m_{j,k-1}+1)$.

THEOREM 1. *Let H_{m_n} be a linear space spanned by patterns defined by eqs.* (12) *and* (13). *Let the action of operators A_{kk}, $A_{k-1,k}$ and $A_{k,k-1}$, $k = 1, 2, \ldots, p+q$, be defined by*

$$A_{kk}m = (r_k - r_{k-1})m, \tag{15}$$

$$A_{k,k-1}m = \sum_{j=1}^{k-1} a_{k-1}^{j}(m) m_{k-1}^{j}, \tag{16}$$

$$A_{k-1,k}m = \sum_{j=1}^{k-1} b_{k-1}^{j}(m) m_{k-1}^{\hat{j}}, \tag{17}$$

where

$$r_0 = 0, \quad r_k = \sum_{j=1}^{k} m_{jk}, \quad k = 1, 2, \ldots, n, \tag{18}$$

and

$$a_{k-1}^{j}(m) = \left[- \frac{\prod_{i=1}^{k} (l_{ik} - l_{j,k-1} + 1) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1})}{\prod_{i \neq k} (l_{i,k-1} - l_{j,k-1} + 1)(l_{i,k-1} - l_{j,k-1})} \right]^{1/2}, \tag{19}$$

$$b_{k-1}^{j}(m) = \left[- \frac{\prod_{i=1}^{k} (l_{ik} - l_{j,k-1}) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i \neq j} (l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} - 1)} \right]^{1/2}, \tag{20}$$

$$l_{ik} = m_{ik} - i, \tag{21}$$

$$\arg a_{k-1}^{j} = \arg b_{k-1}^{j} = \begin{cases} 0 & \text{for } k \neq p+1, \\ \dfrac{\pi}{2} & \text{for } k = p+1. \end{cases} \tag{22}$$

Then the action of an arbitrary operator A_{ki} is given by the following formula

$$A_{kl}m = \sum_{i_{k-1} \ldots i_l} a_{i_{k-1} \ldots i_l}(m) m_{i_{k-1} \ldots i_l}, \quad k > l \tag{23}$$

and

$$A_{kl}m = \sum_{i_k \ldots i_{l-1}} b_{i_k \ldots i_{l-1}}(m) m_{\hat{i}_k \ldots \hat{i}_{l-1}}, \quad k < l, \tag{24}$$

where

$$a_{i_{k-1} \ldots i_l}(m) = \prod_{s=l+1}^{k} a_{i_{s-1}}(m) \prod_{s=l+2}^{k} \frac{\varepsilon_{i_{s-1} i_{s-2}}(m)}{c_{i_{s-1} i_{s-2}}(m)}, \quad k > l^*, \tag{25}$$

---

* The product $\prod_{s=l+2}^{k}$ is, by definition, equal to 1 when $l = k-1$.

$$b_{l_k \dots l_{l-1}}(m) = \prod_{s=k+1}^{l} b_{l_{s-1}}(m) \prod_{s=k+2}^{l} \frac{\varepsilon_{l_{s-1} l_{s-2}}(m)}{c_{l_{s-1} l_{s-2}}(m)}, \quad k < l, \tag{26}$$

$$a_{l_s}(m) = a_s^{l_s}(m), \quad b_{l_s}(m) = b_s^{l_s}(m), \tag{27}$$

$$c_{l_s l_t}(m) = [(l_{l_s s} - l_{l_t t} + 1)(l_{l_s s} - l_{l_t t})]^{1/2} \geqslant 0, \tag{28}$$

$$\varepsilon_{l_s l_t}(m) = \operatorname{sign}(l_{l_s s} - l_{l_t t}), \tag{29}$$

and i_s runs over the set of values $1, 2, \dots, s$. *The patterns $m_{l_{k-1}, \dots, l_l}$ and $m_{\hat{l}_k, \dots, \hat{l}_{l-1}}$ are defiined inductively as follows:*

$$m_{l_{k-1} \dots l_l} = (m_{l_{k-1} \dots l_{l-1}})_l^{l_l}, \quad k > l, \quad m_{\hat{l}_k \dots \hat{l}_{l-1}} = (m_{\hat{l}_k \dots \hat{l}_{l-2}})_{l-1}^{\hat{l}_{l-1}}, \quad k < l, \tag{30}$$

where

$$m_{l_s} = m_s^{l_s}, \quad m_{\hat{l}_s} = m_s^{\hat{l}_s}. \tag{31}$$

The operators A_{kl} satisfy the hermiticity conditions (11) *and commutation relations* (8).

Every self-adjoint representation of $u(p, q)$, $p+q = n$, determined by the highest weight $m_n = (m_{1n}, \dots, m_{nn})$ and the type (α, β) is irreducible. Two representations are unitarily equivalent if and only if their highest weights and types coincide. ▼

The proof of this theorem involves algebraic manipulations only, and because it is very lengthy we omit it. (For details cf. Nikolov and Rerich 1966.)

The set of self-adjoint representations of $u(p, q)$ given by th. 1 is called the *discrete series*. It is evident from the formulas (15), (23) and (24) that a common, dense, linear invariant domain D in H_{m_n} for $u(p, q)$ consists of all finite linear combinations of basis elements m determined by the type (12) and inequalities (13). Clearly the space D is also the invariant domain for the enveloping algebra E of $u(p, q)$.

Decomposition with Respect to Subalgebras. The set of generators A_{ij}, $i, j = 1, 2, \dots, p+q-1$, determines a subalgebra $u(p, q-1)$ of $u(p, q)$. A subalgebra $u(p, k)$ can be selected in a similar manner from the algebra $u(p, k+1)$. Therefore, we finally obtain the decreasing chain of algebras

$$u(p, q) \supset u(p, q-1) \supset \dots u(p, 0) \supset \dots \supset u(2, 0) \supset u(1, 0). \tag{32}$$

We end the section with some comments:

(a) The above construction of representations follows the work of Gel'fand and Graev 1965. They simply guessed the form of patterns for $u(p, q)$ and the action of generators A_{kk}, $A_{k,k-1}$ and $A_{k-1,k}$ in $H_{m,n}^{(\alpha,\beta)}$ on the basis of the compact case $u(n)$.

There is so far no systematic derivation of the structure of admissible patterns of irreducible representations for $u(p, q)$, $q > 1$.

(b) *Representations of* $u(p, 1)$. The case of $u(p, 1)$ was treated in more detail by Gel'fand and Graev 1965 and Ottoson 1967. This case is considerably simpler than the general case with $q > 1$ because $u(p, 1)$ has the compact subalgebra $u(p)$ as a maximal subalgebra; hence the well-elaborated Gel'fand–Zetlin technique for $u(p)$ can be used for an analysis of irreducible representations of $u(p, 1)$. The very detailed derivation of the structure of patterns for $u(p, 1)$, the action of generators $A_{k,k}$, $A_{k-1,k}$ and $A_{k-1,k}$ as well as the classification of all irreducible representations of $u(p, 1)$ were given by Ottoson 1967. It is interesting that in the case of $u(p, 1)$ one obtains—besides a discrete series of representations—semidiscrete ones which are defined by $p-1$ discrete parameters and a complex number.

(c) *Degenerate Representations of* $u(p, q)$. It was remarked by Todorov 1966 that there exists a discrete series of so-called degenerate representations which are not contained in the set of discrete representations considered up to now. In fact, if we impose the restriction

$$m_{k,n} = m_{k+1,n} = \dots = m_{k+r,n}, \quad r > q \tag{33}$$

upon r neighboring components of a 'weight' m_n then we have

$$n-r \leqslant \alpha+\beta < p \tag{34}$$

and therefore we get the new discrete series of representations of $u(p, q)$. Moreover, the inequalities $m_{\alpha,n} \geqslant m_{\alpha+1,n}$ and $m_{n-\beta,n} \geqslant m_{n-\beta+1,n}$ might be violated. The resulting $(a_{jk})^2$ and $(b_{jk})^2$ which would have in general the wrong sign for such α and β here either vanish (due to the large number of inequalities between m_{ik}) or may be redefined in such a way as to give a hermitian representation of $u(p, q)$. There are two interesting cases. In the first one, we have,

$$m_{2,n} = \dots = m_{n,n} = 0, \quad \alpha = 1,\ \beta = 0. \tag{35}$$

The basis vectors for such a representation are of the form

$$\left|\begin{array}{ccccccccc} & & & m_{1n} & 0 & 0 & 0 & \dots\ 0 & 0 \\ & & m_{1,n-1} & 0 & 0 & & & & 0 \\ & \cdot & & & \cdot & & & \cdot & \\ m_{1p} & & & & 0 & \dots & 0 & & \\ & \cdot & & & & \cdot & & & \\ & & \cdot & m_{12} & & 0 & & & \\ & & & & m_{11} & & & & \end{array}\right| \tag{36}$$

These degenerate representations correspond to the so-called 'ladder' representations, which were used for the description of the properties of elementary particles multiplets (see, e.g., Anderson and Rączka 1967 a, b). The other discrete series of so-called *maximally degenerate self-conjugate representations* is obtained when the components m_{in} obey:

$$m_{1n} = -m_{n,n}, \quad m_{2n} = \ldots = m_{n-1,n} = 0, \quad \alpha = \beta = 1. \tag{37}$$

The corresponding patterns are

$$\left|\begin{array}{ccccccccc}
 & & m_{1n} & 0 & 0\ldots0 & 0 & -m_{1n} & & \\
 & m_{1,n-1} & & 0 & & 0 & & m_{1,n-1} & \\
 & \cdot^{\cdot} & & \vdots & & \vdots & & \ddots & \\
m_{p1} & & & & 0\ldots0 & & & & m_{p,p} \\
 & \ddots & & & & & & \cdot^{\cdot} & \\
 & & m_{13} & & & & m_{33} & & \\
 & & & m_{12} & 0 & m_{22} & & & \\
 & & & & m_{11} & & & &
\end{array}\right|. \tag{38}$$

These representations were also used for the description of the properties of elementary particle multiplets (Todorov 1966). The full classification of discrete degenerate series and semidiscrete series for the general $u(p, q)$-algebra has not yet been worked out.

Representations of $U(p, q)$. The problem of integrability of representations of discrete series was solved by Mickelsson and Niederle 1973 and Kotecký and Niederle 1975. Using J. Simon's integrability condition for real Lie algebras they proved that every representation of the discrete series of $u(p, q)$ constructed by Gel'fand–Graev method is differential of a unitary one-valued representation of group $U(p, q)$.

The global form of one-parameter subgroups of $u(p, q)$ was explicitly calculated by Gel'fand and Graev. Then using the representation of a group element $g \in U(p, q)$ in terms of one-parameter subgroups, one may calculate the global form of the representation. Gel'fand and Graev expressed the matrix elements for global representations in terms of generalized β-functions. They were not able, however, to obtain the final formulas in a closed form: an effective solution of this problem would be very useful for many applications.

Representations of so(p,q). The discrete series of irreducible hermitian representation of so(p, q) was constructed by Nikolov 1967. The very detailed analysis of all principal supplementary and exceptional series of representations of so$(n, 1)$ in the framework of Gel'fand patterns was given by Ottoson 1968.

§ 9. Comments and Supplements

A. *Comments*

The representation theory of Lie algebras discussed in sec. 1 is based on the work of Gårding 1947. It is interesting that the th. 1.1 giving a common dense invariant domain for the generators of a Lie algebra can be easily extended to semigroups and even to arbitrary differentiable manifolds.

The representation theory of enveloping algebras in sec. 2 is based on the work of Nelson and Stinespring 1959. Some scattered results were known before; in particular corollary 4 to th. 2.3 was proved previously by Segal 1951.

The theory of analytic vectors was originated by Harish–Chandra 1953. He showed, in particular, that for certain representations of semisimple Lie groups, the set of analytic vectors (well-behaved vectors in his terminology) is dense in H. Next Cartier and Dixmier 1958 showed that if T is either bounded or scalar-valued on a certain discrete central subgroup Z of G then the set of analytic vectors for T is dense.

The content of secs. 3, 4 and 5 on analytic vectors and their applications is based on the fundamental work of Nelson 1959. A more detailed proof of theorem of Nelson 8.5.2 and various extensions of the concept of analytic vectors were given by Goodman 1969. The most important th. 5.2 which gives a convenient criterion for integrability of a representation of a Lie algebra found many applications in particle physics and quantum field theory (cf. ch. 21).

The example 5.1 of nonintegrable representations has the shortcoming that the operator $X = \frac{\mathrm{d}}{\mathrm{d}\varphi}$ is not essentially self-adjoint. Nelson 1959 has found an example which was surprising to many physicists: namely, he has constructed two essentially self-adjoint operators A and B commuting on the common dense invariant domain and showed that the global transformations $\exp(\mathrm{i}tA)$ and $\exp(\mathrm{i}tB)$ do not commute. This demonstrates that an abelian Lie algebra might not be integrable to a global abelian group if the Nelson operator $\Delta = X_1^2 + \ldots + X_n^2$ is not essentially self-adjoint.

The th. 5.3 was derived by K. Maurin and L. Maurin 1964.

The theory of integrability of Lie algebras representation using the concept of weak analyticity was elaborated by Flato, Simon, Snellman and Sternheimer 1972. The simplified version of this theory, using the conditions for Lie generators of Lie algebra only was elaborated by Simon 1972 and Flato and Simon 1973.

The idea of using the solution of the heat equation on a Lie group G for the construction of analytic vectors for a representation T of G was first suggested by Nelson 1959, § 8. In sec. 6 we followed the simplified version of this theory elaborated by Gårding 1960.

One could ask whether it is not possible to develop a representation theory of Lie algebras dealing with skew-symmetric bounded operators only. One could then get rid of almost all difficulties encountered in this chapter. This interesting problem was considered by Doebner and Melsheimer who proved

THEOREM 1. *A nontrivial representation of a noncompact Lie algebra by skew-symmetric operators contains at least one unbounded operator.*

(For the proof cf. Doebner and Melsheimer 1967, th. 1.)

§ 10. Exercises

§ 1.1. Show that every skew-symmetric representation $X \to T(X)$ of a Lie algebra L in a complex Hilbert space having a common dense invariant domain D is strongly continuous on D.

Hint: Endow with the Euclidean topology and use the linearity of the representation.

§ 1.2. Let $G = \mathrm{SO}(3)$ and $H = L^2(S^2, \mu)$. Show that the self-adjoint generators of the left quasi-regular representation $T_x u(s) = u(x^{-1}s)$ have the form

$$
\begin{aligned}
L_x &= \mathrm{i}\left(\sin\varphi \frac{\partial}{\partial\vartheta} + \cot\vartheta\cos\varphi\frac{\partial}{\partial\varphi}\right),\\
L_y &= \mathrm{i}\left(\cos\varphi\frac{\partial}{\partial\vartheta} + \cot\vartheta\sin\varphi\frac{\partial}{\partial\varphi}\right),\\
L_z &= -\mathrm{i}\frac{\partial}{\partial\varphi}.
\end{aligned}
\tag{1}
$$

§ 1.3. Let $G = \mathrm{SO}(3)$, φ, ϑ, ψ the Euler angles for G and $H = L^2(G, \mu)$. Let T be the right regular representation of G in H. Show that the generators of the Lie algebra so (3) have the form

$$
\begin{aligned}
L_x &= -\cot\vartheta\sin\psi\frac{\partial}{\partial\psi} + \frac{\sin\psi}{\sin\vartheta}\frac{\partial}{\partial\varphi} + \cos\psi\frac{\partial}{\partial\vartheta},\\
L_y &= -\cot\vartheta\cos\psi\frac{\partial}{\partial\psi} + \frac{\cos\psi}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \sin\psi\frac{\partial}{\partial\vartheta},\\
L_z &= \frac{\partial}{\partial\psi}
\end{aligned}
\tag{2}
$$

Find the Gårding domain D_G in H. Find generators of the left regular representation T^L in H.

§ 1.4. Let $G = T^4 \circledS \mathrm{SO}(3, 1)$ and let $H = L^2(R^4)$. Let the T be the left quasi-regular representation of G in H. Find the form of generators of the Lie algebra $t^4 \circledS \mathrm{so}(3, 1)$ in H.

§ 1.5.* Let $E(R^{2n})$ be the Schwartz space of test functions of R^{2n}. Let $F(q, p)$ be a function on the phase space R^{2n} which is in $E(R^{2n})$. Show that the operator $\hat{F}$ associated with the function F by the formula

$$
F \to \hat{F} = F - \frac{1}{2}\sum_i\left(q_i\frac{\partial F}{\partial q_i} + p_i\frac{\partial F}{\partial p_i}\right) - \mathrm{i}\sum_i\left(\frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q_i}\right) \tag{3}
$$

provides in the space $E(R^{2n})$ a representation of the Lie algebra of classical Poisson brackets, i.e.

$$
\widehat{\{F, G\}} = -\mathrm{i}[\hat{F}, \hat{G}]. \tag{4}
$$

§ 1.6.* Let W be any function in $E(R^{2n})$. Show that the map

$$
\hat{F}' = \{W, F\} + \hat{F} \tag{5}
$$

still provides a representation of the Lie algebra of Poisson brackets in space $E(R^{2n})$.

§ 1.7.* Let σ be a canonical transformation in R^{2n}, $\sigma: (p, q) \to (p^\sigma, q^\sigma)$. Let for an arbitrary F in $E(R^{2n})$ the quantity $\hat{F}$ and $\hat{F}^\sigma$ denote a representation (3) in variables (p, q) and (p^σ, q^σ) respectively. Show that there exists a unitary transformation U_σ in $L^2(R^{2n})$ such that

$$U_\sigma \hat{F} U_\sigma^{-1} = \hat{F}^\sigma \quad \text{for all } F. \tag{6}$$

§ 1.8.** Let $\mathscr{F}$ be the space of initial conditions (φ, π) for a classical scalar field $\Phi(x)$ satisfying the nonlinear relativistic wave equation

$$(\Box + m^2)\Phi(x) = \lambda\Phi^3(x), \quad \lambda < 0, \tag{7}$$

$$\Phi(0, x) = \varphi(x), \quad (\partial_t \Phi)(0, x) = \Pi(0, x) = \pi(x). \tag{8}$$

Let $F(\varphi, \pi)$ be a smooth functional over the inital data. Show that the operator $\hat{F}$ associated with F by the formula

$$F \to \hat{F} = F - \frac{1}{2} DF[(\varphi, \pi)](\varphi, \pi) - \mathrm{i}\int \mathrm{d}^3 z \left(\frac{\delta F}{\delta\varphi(z)}\frac{\delta}{\delta\pi(z)} - \frac{\delta F}{\delta\pi(z)}\frac{\partial}{\partial\varphi(z)}\right) \tag{9}$$

where DF is the Frechet's differential in the point (φ, π) and $\dfrac{\delta}{\delta\varphi}$ and $\dfrac{\delta}{\delta\pi}$ are Frechet's derivatives, provides an algebraic representation of the classical Lie algebra of Poisson brackets.

§ 1.9.** Introduce a topology in the space $E(\mathscr{F})$ of smooth functionals over the space $\mathscr{F}$ of initial data and find a class of smooth functionals for which the map $F \to \hat{F}$ given by eq. (9) will provide an operator representation (cf. Bałaban, Jezuita and Rączka 1976 for a particular solution of this problem).

§ 1.10. Let $\Phi[x|\varphi, \pi]$ be a solution of eq. (7) defined by the initial conditions (φ, π). Show that the operators $\hat{\Phi}$ and $\hat{\Pi}$ associated with Φ and Π respectively by the formula (9) satisfy the following equal time commutation relations

$$[(\hat{\Phi}(t, x), \hat{\Pi}(t, y)] = i\delta^{(3)}(x-y), \tag{10}$$

$$[\hat{\Phi}(t, x), \hat{\Phi}(t, y)] = [\hat{\Pi}(t, x), \hat{\Pi}(t, y)] = 0. \tag{11}$$

§ 1.11. Let $\{h_l\}_1^\infty$ be an orthonormal basis in $L^2(R^3)$. Let $\varphi(x) = \Sigma q_l h_l(x)$ and $\pi(x) = \Sigma p_l h_l(x)$ be the expansion of the canonical variables (φ, π) in the basis $\{h_l\}$. Show that the quantization formula (9) written in terms of variables $\{q_l\}_1^\infty$ and $\{p_l\}_1^\infty$ coincides with the formula (3).

§ 2.1.* Let $G = \mathrm{SO}(3)$. Show that the angular momentum operator J^2 associated in the space $L^2(R^3)$ with the quasi-regular representation $T_x u(\gamma) = u(x^{-1}\gamma)$ has only non-negative integer eigenvalues, i.e.

$$J^2\Psi_\lambda(r) = \lambda\Psi_\lambda(r) \quad \text{with} \quad \lambda = J(J+1),\ J = 0, 1, 2, \ldots, \tag{12}$$

whereas in the space $L^2(\mathrm{SO}(3))$ it has integer and half-integer nonnegative eigenvalues.

§ 2.2. Let T be the representation of $T^4 \rtimes \mathrm{SO}(3, 1)$ as in exercise 1.4. Find the spectrum of the Casimir operator $M^2 = P_\mu P^\mu$. Show that the second Casimir operator $S^2 = W_\mu W^\mu$ is identically zero in H.

§ 4.1.* Let $x \to T_x$ be the left regular representation of the translation group R on $H = L^2(R)$. Show that the function

$$u(x) = \sum_{n=1}^{\infty} 2^{-n}[(x+n)^2+n^{-1}]^{-1}, \quad x \in R \tag{13}$$

is analytic in H but is not an analytic vector for T.

§ 4.2.* Show that $u(x)$ is the analytic vector for the representation of 4.1 if its Fourier transform $\hat{u}(p)$ satisfies the condition

$$\exp(\lambda|p|)\hat{u}(p) \in L^2(R) \tag{14}$$

for some $\lambda > 0$.

§ 5.1. Show that the operators

$$\begin{aligned} J_+ &= J_x + \mathrm{i}J_y = z^2\frac{\mathrm{d}}{\mathrm{d}z} - 2\lambda z, \\ J_- &= J_x - \mathrm{i}J_y = -\frac{\mathrm{d}}{\mathrm{d}z}, \\ J_3 &= z\frac{\mathrm{d}}{\mathrm{d}z} - \lambda, \end{aligned} \tag{15}$$

with $\lambda \in C$, form the infinite-dimensional representation of the Lie algebra so(3) in the space of analytic functions integrable with respect to the Gaussian measure. Show that the Lie algebra cannot be integrated to a global continuous representation of the group SO(3).

§ 5.2.*** Give the classification of non integrable representations of the canonical commutation relations

$$[q, p] = \mathrm{i}I.$$

§ 6.1. Show that the functions

$$\psi(t, \varphi, \vartheta, \psi) = \exp[-tJ(J+1)]D^J_{M\lambda}(\varphi, \vartheta, \psi) \tag{16}$$

are solutions of the heat equation on the rotation group SO (3)

§ 6.2.*** Find the solutions of the heat equation on the Poincaré group.

§ 7.1.*** Give the full classification of discrete degenerate and semidiscrete degenerate series of $u(p, q)$.

§ 7.2.*** Let X be a linear operator in a Hilbert space H. Find a necessary condition for the existence of a dense set of analytic vectors of X in H.

§ 7.3.*** Show that every irreducible essentially skew-adjoint representation of a real finite-dimensional Lie algebra defined on an invariant dense domain in a Hilbert space is integrable.

§ 7.4.*** Give a Lie algebraic formulation of axiomatic quantum field theory.

§ 7.5.*** Find integrability conditions for representations of infinite-dimensional (Hilbertian) Lie algebras in a Hilbert space.

§ 7.6. Show that three series of selt-adjoint irreducible representations of the Lie algebra su(1, 1) ~ so(2, 1) ~ sp(2R) ~ sl(2, R) can be constructed in terms of creation and annihilation operators as follows: Let the Lie algebra be represented by

$$X_1 = \frac{i}{2} a^* \sigma_1 a, \quad X_2 = \frac{i}{2} a^* \sigma_2 a, \quad X_3 = \frac{1}{2} a^* \sigma_3 a \quad \text{with } C_2 = X_3^2 - X_1^2 - X_2^2,$$

where σ_k, $k = 1, 2, 3$, are the Pauli matrices and $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $a^* = (a_1^*, a_2^*)$; $[a_i, a_j^*] = \delta_{ij}$; $i, j = 1, 2$. Consider the vectors

$$|\varphi, m\rangle = N_{\varphi m} a_1^{*(\varphi+m)} a_2^{*(\varphi-m)} |0\rangle, \quad a_i |0\rangle = 0, \ i = 1, 2.$$

Then $X_3|\varphi, m\rangle = m|\varphi, m\rangle$ and $C_2|\varphi, m\rangle = \varphi(\varphi+1)|\varphi, m\rangle$ and the unitary representations ($\varphi(\varphi+1)$ real, X_i hermitian) are

(i) discrete series $D^\pm$ for $\varphi < 0$, real. For D^+:

$$m = -\varphi, -\varphi+1, -\varphi+2, \ldots \quad \text{and for} \quad D^-: m = \varphi, \varphi-1, \varphi-2, \ldots;$$

(ii) supplementary series: U^{φ, E_0} for $-1+|E_0| < \varphi < -|E_0|$, $-1 < |E_0| < \frac{1}{2}$

$$m = E_0, E_0 \pm 1, E_0 \pm 2, \ldots;$$

(iii) principal series $D^{\varphi, \sigma}$ for $\varphi = -\frac{1}{2} + i\sigma$, σ real; for global forms of these representations see ch. 16.

§ 7.7. For the Lie algebra so(3, 1) ~ sl(2, C), with generators

$$J_k = \frac{1}{2} a^* \sigma_k a, \quad N_k = \frac{1}{2} i(a^* \sigma_k C a^* + a C \sigma_k a), \quad k = 1, 2, 3, \ C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

acting on the Hilbert space of states spanned by

$$|jm\rangle = [(j+m)!(j-m)!]^{-1/2} a_1^{*(j+m)} a_2^{*(j-m)} |0\rangle$$

show that $C_2 = J^2 - N^2 = 0$ and $C_2' = J \cdot N = -\frac{3}{4}$. If the lowest state is $|0\rangle$ or $a^*|0\rangle$, respectively, one obtains two self-adjoint representations with $j_0 = j_{\min} = 0$ or $\frac{1}{2}$, respectively. The algebra sl(2, C) has two series of unitary representations labelled by two numbers:

(i) principal series: $j_0 = \frac{1}{2}, 1, \frac{3}{2}, \ldots, -\infty < a < +\infty$,

(ii) supplementary series: $j_0 = 0$, $0 \leqslant ia \leqslant 1$, where

$$C_2 = j_0 a, \quad C_2' = 1 + a^2 - j_0^2.$$

Next use the two pairs of boson operators a_i, b_i, $i = 1, 2$, and

$$J_k = \tfrac{1}{2}(a^* \sigma_k a + b^* \sigma_k b), \quad N_k = -\tfrac{1}{2}(a^* \sigma_k C b^* - a C \sigma_k b),$$

construct other representations (cf. also ch. 19).

Chapter 12

Quantum Dynamical Applications of Lie Algebra Representations

In this chapter we discuss the direct applications of Lie algebra representations in solving kinematical and dynamical problems in quantum theory. It is designed to show how these representations occur and how they are used without first reading about all the subtleties of the formalism of quantum theory. Only the knowledge of the Schrödinger equation is assumed. The more detailed discussion of quantum mechanical framework, the concept of symmetry and group representations is relegated to ch. 13 and the relativistic problems to ch. 20 and 21.

§ 1. Symmetry Algebras in Hamiltonian Formulation

We begin with the theory of non-relativistic quantum systems described by the Schrödinger equation ($\hbar = 1$)

$$i\partial_t\psi(q, t) = H\psi(q, t), \tag{1}$$

where ψ is the wave function of the system, q the set of dynamical coordinates and H the Hamiltonian operator of the system, a differential operator, function of q and $\partial/\partial q$. Alternatively, we could consider, the Heisenberg algebra $[p_i, q_j] = -i\delta_{ij}I$ and H would be a function of p's and q's. The equivalence of Heisenberg and Schrödinger pictures is treated in ch. 20. As we noted at the beginning of ch. 11 most quantum mechanical operators are unbounded operators and for a rigorous treatment the theory developed in ch. 11 must be used.

For stationary solutions of eq. (1) of the form $\psi(t, q) = \exp(-iEt)u(q)$ we obtain the eigenvalue equation

$$Hu = Eu. \tag{2}$$

The symmetry of the Hamiltonian H of a quantum system is generated by those operators which commute with H and possess together with H a common dense invariant domain D in the carrier space. If

$$[H, X_1] = 0 \quad \text{and} \quad [H, X_2] = 0$$

holds on D, then

$$[H, [X_1, X_2]] = 0$$

also on D, by virtue of the fact that for linear operators the Jacobi identity holds. Hence the subset of operators satisfying the above condition forms a Lie algebra. Usually, the representations of the Lie algebras occurring in simple problem are integrable, and symmetry Lie algebras and symmetry Lie groups are used in physics literature generally indiscriminatly.

This notion of symmetry group in the narrow sense should be really called 'group of degeneracy of the energy', because an eigenspace of H for a fixed value E of the energy, is a representation space of the Lie algebra commuting with H. We first discuss in terms of examples this simpler notion of symmetry, in § 2 we shall generalize it considerably. It should be remarked that in non-relativistic quantum mechanics we are led directly to the representation of Lie algebras, the representation of the corresponding symmetry groups enter eventually, because certain important global concepts belong to the level of groups, not algebras.

The largest Lie group (or algebra) whose elements commute with H will be called the *maximal symmetry group*. In many cases it is much larger than the kinematical symmetry group.

EXAMPLE 1. The simple quantum rotator with the Hamiltonian $H = \frac{1}{2I}J^2$ where I is the moment of inertia and $J^2 = J_1^2+J_2^2+J_3^2$ is the Casimir operator of the Lie algebra of SO (3) provides a simple example of a quantum system with the symmetry group, in this case the group of rotations. In the carrier space of quantum mechanics $H = L^2(R^3)$ this Hamiltonian has eigenfunctions $u_\lambda(q)$ given by spherical harmonics $Y_M^J(\vartheta, \varphi)$ with eigenvalues $\lambda = (2I)^{-1}J(J+1)$, $J = 0, 1, 2, \ldots$ Hence the eigenspace $H^\lambda \subset H$ is the carrier space of $(2J+1)$-dimensional irreducible representation of SO (3).

According to the basic postulates of quantum mechanics, we must realize H and J as self-adjoint operators, hence by virtue of Nelson theorem we are restricted ourselves to integrable representations of the Lie algebra, i.e. to quantum mechanical representations of the rotation group SO(3). Without this restriction, the Lie algebra of SO (3) admits a large class of irreducible representations which are however not all integrable (see 11.10.5.1).

We also note that the Hamiltonian does not tell us the multiplicity of each representation of the symmetry group, i.e. how many times a given value J occur. This question can be answered, as we shall see, within the framework of larger groups, namely by the existence of other operators distinguishing states with the same J.

EXAMPLE 2. *Quantum mechanical rigid rotator.* Here we have two sets of commuting angular momentum operators J_s and J_b, namely angular momenta with respect to the space-fixed and body-fixed axes. For a completely symmetric rigid rotator the Hamiltonian is $H = \frac{1}{2I}(J_s^2+J_b^2)$. The Casimirs of the two algebras are however equal by definition of the rigid rotator $J_s^2 = J_b^2 = J^2$ and the Hamiltonian is again proportional to J^2.

Because we have two SO(3) algebras, the symmetry Lie algebra of H now is $SO(3)\times SO(3) \sim SO(4)$. The relation $J_s^2 = J_b^2$ indicates that only special representations of SO(4) of dimension $(2J+1)^2 = n^2$ occur. If we define the generators of SO(4), $L = J_s+J_b$, $K = J_s-J_b$, we have $K^2 = 0$, $K\cdot L = 0$ and for each J, L ranges from 0 to $2J$. Notice that in the present case the symmetry Lie algebra is larger than the geometrical symmetry SO(3).

EXAMPLE 3. *Three-dimensional harmonic oscillator.* The dynamical conjugate variables are the three position operators q_i and the three momenta p_i (represented by $\frac{\partial}{i\partial q_i}$ in the Schrödinger picture). The Hamiltonian is given by (in suitable units)

$$H = \tfrac{1}{2}(p^2+q^2). \tag{3}$$

We consider again the stationary states given by (2).

One way to exhibit the group of degeneracy of the Hamiltonian (3) is to introduce the creation and annihilation operators a_l^*, a_l, $l = 1, 2, 3$ (cf. eqs. 10.4 (1), (2)) by $a_l = \frac{1}{\sqrt{2}}(q_l+ip_l)$, $a_l^* = \frac{1}{\sqrt{2}}(q_l-ip_l)$ and write (3) as

$$H = \sum_{l=1}^{3} a_l^* a_l + \tfrac{3}{2}. \tag{4}$$

It is easy to see that the operators $X_{ij} = a_i^* a_j + \frac{1}{2}\delta_{ij}$ commute with H. By virtue of eq. 10.4(12) X_{ij} form the basis elements of the Lie algebra $u(n)$. In fact, $H = \sum_{i=1}^{3} X_{ii} = \mathrm{Tr}(X) = C_1$. Hence, the remaining elements of $u(3)$ form the Lie algebra su(3). The calculation of the invariant operators of su(3) shows that higher order invariant operators C_2, C_3, etc. are functions of the Hamiltonian. Hence only most degenerate representations of $u(3)$ of dimension 1, 6, 10, 15, ..., are realized in the present example.

An important feature of this example is the fact that the Hamiltonian (3) possesses a larger symmetry group than the immediate geometric symmetry of the problem, namely the rotational invariance. The generators of the rotation group $J_1 = \frac{1}{2}(a_1^* a_2 + a_2^* a_1)$, $J_2 = -\frac{i}{2}(a_1^* a_2 - a_2^* a_1)$, $J_3 = \frac{1}{2}(a_1^* a_1 - a_2^* a_2)$ (cf. 10.4 (16)) are among the elements of the symmetry algebra su(3), but the latter has additional elements. For this reason it is called a *dynamical symmetry of H* (in contrast to the geometric symmetry of H). In an irreducible representation of su(3), the representations of the rotation group occur more than once, but the multiplicity of the representations of the dynamical symmetry group is one. This is because the energy values and the quantum numbers of the representations of the maximal symmetry group constitute a complete set of labellings of all states.

EXAMPLE 4. *Non-relativistic Kepler problem.* Here the dynamical variables are as in example 3: p_i, q_i. The Hamiltonian is given by (again in suitable units)

$$H = \tfrac{1}{2}p^2 - \frac{\alpha}{r}, \qquad r \equiv |q|, \qquad \alpha > 0. \tag{5}$$

The obvious geometrical symmetry of the problem is still the rotational symmetry: $[H, \boldsymbol{L}] = 0$, $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$. However, it is easy to verify that one other vector operator

$$A = \frac{1}{\sqrt{-2E}}\left\{\tfrac{1}{2}[J\times p - p\times J] + \frac{\alpha r}{r}\right\} \tag{6}$$

commutes with H on the space of eigenfunctions of H corresponding to a fixed eigenvalue E. One readily verifies that the operators A (called the *Runge–Lenz vector* for historical reasons) and $\boldsymbol{L}$ form the basis element of the Lie algebra so(4). For $E < 0$, bound states, the dynamical symmetry group of (5) is SO(4). For the representation (6) one of the Casimir operators of SO(4) vanishes:

$$\boldsymbol{L}\cdot A = 0 \tag{7}$$

Hence only special representations of SO(4) are realized. With $J_1 = \frac{1}{2}(\boldsymbol{L}+A)$, $J_2 = \frac{1}{2}(\boldsymbol{L}-A)$, $J_1^2 = J_2^2 = J^2$, the degeneracy of the discrete levels is $(2J+1)^2 = n^2$, $J = 0, \frac{1}{2}, 1, \ldots$ (same as in example 2). The states can be labelled by $|n, J, M\rangle$ where n is equivalent to the energy label or to the remaining Casimir operator of SO(4). Hence the representations of the dynamical symmetry algebra have multiplicities one.

For $E > 0$, because of the occurrence of the factor i in eq. (6), we see that the real dynamical symmetry Lie algebra is now so(3, 1) and eq. (7) still holds. Physically these states correspond to the scattering states of the particle in the Kepler potential. The unitary representations of SO(3, 1) are infinite-dimensional: this implies that in a scattering experiment with a fixed energy, there are infinitely many partial waves of angular momentum, each with multiplicity one, equal to the multiplicity of irreducible representation T^L of SO(3) in the representation of SO(3,1).

Remark: The concept of symmetry can be applied to any other observable beside the energy, e.g. angular momentum, spin, etc. If A is an observable, and we consider the eigenvalue problem

$$Af = af$$

then the operators commuting with A generate a Lie algebra whose representations determine the degeneracy of the states with the same value a.

§ 2. Dynamical Lie Algebras

In § 1 we discussed the group of degeneracy of the Hamiltonians. The representations of the maximal symmetry group gives us the dimensionality of the eigenspace of H for a given energy E. In order to solve the quantum mechanical problem 1(2) completely, we have still to determine the spectrum of H. We shall solve this problem in the framework of the following general formalism.

Let W be a differential operator and consider the (wave) equation

$$W\psi = 0. \tag{$*$}$$

If there are operators L_i, $i = 1, \ldots, r$, forming a Lie algebra L and satisfying, on the space of solutions of $(*)$,

$$[W, L_i]\psi = 0, \quad i = 1, \ldots, r,$$

then all solutions of the wave equation $(*)$ span a representation space for the Lie algebra L. Clearly if ψ is a solution of $(*)$, so is $L_i\psi$ and we have $[W, L_i] = f(W)$ where f is an arbitrary polynomial with coefficients depending on coordinates and satisfying $f(0) = 0$. In particular, if $W = \mathrm{i}\partial_t - H$, or $W = (\partial_\mu - A_\mu)^2$, then the Lie algebra L is called the *dynamical Lie algebra* of the quantum system. It contains in general time-dependent operators $L(t)$ which on the space of solutions ψ of eq. $W\psi = 0$ satisfy the Heisenberg equation

$$[\mathrm{i}\partial_t, L_k(t)] = [H, L_k(t)].$$

The subalgebra $L' \subset L$ consisting of operators commuting with W is a more narrow definition of symmetry; both L' and L are represented on the same Hilbert space. Finally, the subalgebra $L'' \subset L'$ of time-independent operators satisfies $[H, L] = 0$, and is the *symmetry algebra of H* discussed in § 1.

The Heisenberg equation has the solution $L_k(t)$ given by the formula

$$L_k(t) = \exp[\mathrm{i}tH] L_k(0) \exp[-\mathrm{i}tH].$$

Because the energy operator H commutes with the evolution operator $\exp[\mathrm{i}tH]$ the time dependent dynamical Lie algebra $\{H, L_k(t)\}$ and the time independent dynamical Lie algebra $\{H, L_k(0)\}$ are unitarily equivalent. This allows us to restrict ourselves in concrete problems to the analysis of time-independent dynamical Lie algebras.

We now solve explicitly some important quantum-dynamical problems by the method of Lie algebra representations. We begin with a presentation of some supplementary results in the form of lemmas. The proof of these lemmas is straightforward and is left as an exercise for the reader.

LEMMA 1. *The following three operators* ($[p, q] = -\mathrm{i}$)

$$\begin{aligned} \Gamma_0 &= \tfrac{1}{4}(p^2+q^2), \\ T &= \tfrac{1}{4}(pq+qp), \\ \Gamma_4 &= \Gamma_0-\tfrac{1}{2}q^2 \end{aligned} \tag{1}$$

satisfy the commutation relation of the Lie algebra $o(2, 1)$ ($\mathrm{su}(1, 1)$)

$$[\Gamma_0, \Gamma_4] = \mathrm{i}T, \quad [\Gamma_4, T] = -\mathrm{i}\Gamma_0, \quad [T, \Gamma_0] = \mathrm{i}\Gamma_4. \tag{2}$$

The Casimir operator

$$C_2 = \Gamma_0^2-\Gamma_4^2-T^2 \tag{3}$$

is calculated to be

$$C_2 = -\tfrac{3}{16} = \varphi(\varphi+1), \quad \varphi = -\tfrac{3}{4}, -\tfrac{1}{4}. \blacktriangledown$$

Thus eq. (1) is a realization of a representation D^+ of $\mathrm{su}(1, 1)$ (see 11.10.7.6 (i)). Equations (1) and (3) immediately solve the dynamical equation for the linear oscillator with the Hamiltonian equation

$$Hu = \frac{\hbar\omega}{2}\left(p^2+\frac{\omega m}{\hbar}x^2\right)u = Eu, \tag{4}$$

for with the substitutions $q = \left(\frac{m\omega}{\hbar}\right)^{1/2} x$ and $2\Gamma_0 = \frac{1}{\hbar\omega} H$, eq. (4) can be written as

$$(2\Gamma_0-E/\hbar\omega)u = 0.$$

Thus, the eigenstates $|n\rangle$ of Γ_0 by 11.10.7.6(i) have discrete eigenvalues $n+\frac{1}{2}$, $n = 0, 1, 2, \ldots$ and provide the space of solutions

$$H|n\rangle = \hbar\omega(n+\tfrac{1}{2})|n\rangle.$$

As a second typical example, we use

LEMMA 2. *The operators*

$$\begin{aligned} \Gamma_0 &= \tfrac{1}{2}(rp^2+r), \\ \Gamma_4 &= \tfrac{1}{2}(rp^2-r), \\ T &= \boldsymbol{r}\cdot\boldsymbol{p}-\mathrm{i}, \end{aligned} \tag{5}$$

where $r = \sqrt{\boldsymbol{r}^2}$, $p = \sqrt{\boldsymbol{p}^2}$, *again satisfy the commutation relations* (2) *of the Lie algebra* $o(2, 1)$. *The Casimir operator* (3) *has the value*

$$C_2 = \boldsymbol{J}^2 = (\boldsymbol{r}\times\boldsymbol{p})^2 = j(j+1), \quad \textit{i.e.} \quad \varphi = -j-1 \text{ or } \varphi = j. \blacktriangledown$$

Consider now the Hamiltonian equation

$$Hu = \left(\frac{p^2}{2m}-\frac{\alpha}{r}\right)u = Eu \tag{6}$$

for the stationary solutions of the motion of a particle in a Coulomb field.

We introduce the related equation

$$\Theta\psi = [r(H-E)]\psi = 0. \tag{7}$$

The operator (6) can be expressed as a linear combination of the generators (4):

$$\Theta = \left(\frac{1}{2m} - E\right)\Gamma_0 + \left(\frac{1}{2m} + E\right)\Gamma_4 - \alpha. \tag{8}$$

To solve eq. (7), we first diagonalize Γ_0. Defining

$$\tilde{\psi} \equiv \exp(-\mathrm{i}\theta T)\psi, \tag{9}$$

where T is given in (5), and using the commutation relations (2), we have

$$\left[\left(-E + \frac{1}{2m}\right)(\Gamma_0 \cosh\theta + \Gamma_4 \sinh\theta) + \left(E + \frac{1}{2m}\right)(\Gamma_4 \cosh\theta + \Gamma_0 \sinh\theta) - \alpha\right]\tilde{\psi} = 0. \tag{10}$$

Hence if we choose

$$\tanh\theta = (E + 1/2m)/(E - 1/2m), \tag{11}$$

we obtain immediately from (10) the simple equation

$$[(-2E/m)^{1/2}\Gamma_0 - \alpha]\tilde{\psi} = 0. \tag{12}$$

The spectrum of Γ_0 in the discrete representation of $o(2, 1)$ determined by 11.10.7.6(i) are the values: $n = s+j+1$, $s = 0, 1, 2, \ldots$ Hence eq. (12) yields

$$E_n = -\frac{\alpha^2 m}{2n^2}, \tag{13}$$

which is the well-known H-atom spectrum. The solutions $\tilde{\psi}$ are now normalized as follows:

$$\int \bar{\tilde{\psi}}\tilde{\psi} r^{-1} \mathrm{d}^3 x = 1. \tag{14}$$

In order to determine the continuous spectrum of our problem, we choose in eq. (10) the parameter θ differently, namely,

$$\tanh\theta = (E - 1/2m)/(E + 1/2m) \tag{15}$$

and obtain, instead of (12),

$$[(2E/m)^{1/2}\Gamma_4 - \alpha]\psi = 0. \tag{16}$$

Denoting the generalized (non-normalizable) eigenvectors of Γ_4 by

$$\Gamma_4 |Q, \lambda\rangle = \lambda |Q, \lambda\rangle \tag{17}$$

we obtain

$$E_\lambda = \frac{\alpha^2 m}{2\lambda^2}, \qquad \lambda \in R. \tag{18}$$

The dynamical algebra (5) does not solve the complete degeneracy of the levels of the Hamiltonian (6), because we have not yet studied the angular momenta of the levels. The following lemma solves this problem.

LEMMA 3. *The operators* (5) *together with*

$$J = r \times p \tag{19}$$

and

$$\begin{aligned} A &= \tfrac{1}{2} r p^2 - p(r, p) - \tfrac{1}{2} r, \\ M &= \tfrac{1}{2} r p^2 - p(r, p) + \tfrac{1}{2} r, \\ \Gamma &= r p \end{aligned} \tag{20}$$

satisfy the commutation relations of the Lie algebra of so(4, 2). *The second, third and fourth order invariant operators have the values*

$$C_2 = -3, \quad C_3 = 0, \quad C_4 = -12, \tag{21}$$

where $C_2 = \frac{1}{2} L_{ab} L^{ab}$, $C_3 = \varepsilon_{abcdef} L^{ab} L^{cd} L^{ef}$, $C_4 = L_{ab} L^{bc} L_{cd} L^{de}$, $a, b = 0, 1, 2, 3, 4, 5$.

[Note: $L_{ij} = \varepsilon_{ijk} J_k$, $L_{i4} = A_i$, $L_{i0} = M_i$, $L_{i5} = \Gamma_i$, $L_{05} = \Gamma_0$, $L_{45} = \Gamma_4$, $L_{04} = T$.]

The next lemma gives the degeneracy of states of Hamiltonian (6).

LEMMA 4. *The representation given by* (5), (19), (20), *and* (21) *of* so(4, 2) *in the basis where* Γ_0, J^2 *and* J_3 *are diagonalized with eigenvalues* n, $j(j+1)$ *and* m, *respectively, has the weight diagram, i.e., the states* $|njm\rangle$, *given by Fig.* 1.

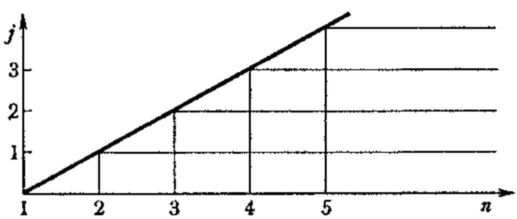

Fig. 1

PROOF: The elements Γ_0, Γ_4, T form the algebra su(1, 1). The Casimir operator $C_2(o(2, 1)) = \Gamma_0^2 - \Gamma_4^2 - T^2 = J^2$. Let the eigenvalues of the operator Γ_0 be denoted by n. Because J^2 has eigenvalues $j(j+1)$ for fixed j, the range of n is $j+1 \leqslant n < \infty$. This gives the horizontal lines in Fig. 1. ▼

For more general problems, we can use the following generalization of lemma 3:

LEMMA 5. *The operators*

$$\begin{aligned} J &= r \times \pi - \mu \hat{r}, \\ A &= \tfrac{1}{2} r \pi^2 - \pi(r \cdot \pi) + \frac{\mu}{r} J + \frac{\mu^2}{2r^2} r - \tfrac{1}{2} r, \\ M &= \tfrac{1}{2} r \pi^2 - \pi(r \cdot \pi) + \frac{\mu}{r} J + \frac{\mu^2}{2r^2} r + \tfrac{1}{2} r, \\ \Gamma &= r \pi, \end{aligned} \tag{22}$$

$$\Gamma_0 = \tfrac{1}{2}(r\pi^2+r+\mu^2/r),$$
$$\Gamma_4 = \tfrac{1}{2}(r\pi^2-r+\mu^2/r),$$
$$T = \boldsymbol{r}\cdot\boldsymbol{\pi}-\mathrm{i}$$

with

$$\hat{\boldsymbol{r}} = \boldsymbol{r}/r, \quad \boldsymbol{\pi} = \boldsymbol{p}-\mu\boldsymbol{D}(\boldsymbol{r}), \quad \boldsymbol{D}(\boldsymbol{r}) = \frac{\boldsymbol{r}\times\boldsymbol{n}(\boldsymbol{r}\cdot\boldsymbol{n})}{r[r^2-(\boldsymbol{r}\cdot\boldsymbol{n})^2]}$$

and $\boldsymbol{n}$ *an arbitrary constant unit vector also have the commutation relations of* so(4, 2) *where, instead of* (21), *we have now*

$$C_2 = -3(1-\mu^2), \quad C_3 = 0, \quad C_4 = 0. \tag{23}$$

For each value of $\mu = 0, \pm\frac{1}{2}, \pm1, \ldots$ *eq.* (22) *gives an irreducible representation of* so(4, 2) *in the discrete series. The representations* (22) *can be characterized by a so-called representation relation*

$$\{L_{AB}, L^A_C\} = -2ag_{BC}, \tag{24}$$

where L_{AB} *are the generators of* so(4, 2) *and* $a = 1-\mu^2$. ▼

A more general theory of symmetries in quantum mechanics is presented in chs. 13 and 21.

§ 3. Exercises

§ 2.1. Consider the angular momentum of a charge-monopole system given in eq. (22), i.e.

$$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{\pi}-\mu\hat{\boldsymbol{r}},$$
$$\boldsymbol{\pi} = \boldsymbol{p}-\mu\boldsymbol{D}(\boldsymbol{r}), \quad \boldsymbol{D}(\boldsymbol{r}) = \frac{\boldsymbol{r}\times\boldsymbol{n}(\boldsymbol{r}\cdot\boldsymbol{n})}{r\left(r^2-(\vec{\boldsymbol{r}}\cdot\boldsymbol{n})^2\right)}.$$

Show that

(a) For fixed $\boldsymbol{n}$, $\boldsymbol{D}$ is singular; $\boldsymbol{J}$ can be represented on the space of functions which rapidly go to zero along the singularity line $\hat{\boldsymbol{r}} = n$; it has a deficiency index (1, 1), hence can be extended to a self-adjoint operator. The integrability conditions on the representations lead to the so-called charge quantization condition, $\mu = 0, \pm\frac{1}{2}, \pm1, \ldots$

(b) If $\boldsymbol{n}$ is rotated together with $\boldsymbol{r}$, $\boldsymbol{D}$ is rotationally invariant and we have a configuration space $R^3\otimes S^2$. However, two different choices $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ can be connected by a gauge transformation provided we use the group space S^3 of SU(2). Hence we can represent $\boldsymbol{J}$ on $L^2(S^3)$, and consequently both integer and half integer values of spin are allowed.

(Cf. C. A. Hurst 1968 and A. O. Barut 1974.)

§ 2.2. Consider the differential equation

$$(H-W)u = \left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2}+q^2+\frac{K}{q^2}-W\right)u = 0, \quad 0 < q < \infty.$$

Let
$$\Gamma_0 = \frac{1}{4}\left(p^2+q^2+\frac{K}{q^2}\right), \quad T = \tfrac{1}{4}(pq+qp) \text{ and } \Gamma_4 = \tfrac{1}{4}(H-2q^2)$$

and calculate the spectrum of H by the representations of $o(2, 1)$ (cf. § 2. eq. (1)).

Hint: $C_2 = \Gamma_0^2 - \Gamma_4^2 - T^2 = \frac{1}{4}(K - \frac{3}{4})$.

§ 2.3.* Show that the radial Schrödinger wave equation for an N-dimensional oscillator, with an addition potential a/r^2, can be brought to the form given in the previous exercise.

§ 2.4. Show that the radial wave equation for the non-relativistic Kepler problem can also be brought to the form given in exercise 2.2 by suitable substitutions, and obtain the Balmer formula.

Hint: $\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2me^2}{\hbar r} - \frac{2mE}{\hbar^2}\right] u_l(r) = 0$. Set $\varepsilon = 2mE/\hbar^2$, $q = (2r\sqrt{-\varepsilon})^{1/2}$, $u_l(r) = (2r\sqrt{-\varepsilon})^{1/4}\tilde{u}_l(q)$, $K = \frac{3}{4} + 4l(l+1)$, $W = \frac{4me^2}{\sqrt{-2E}}$.

§ 2.5. Consider the equation

$$[(1+b)\Gamma_0 + (1-b)\Gamma_4 + c]\tilde{\psi} = 0,$$

where Γ_0, Γ_4 and T are generators of SO(2, 1) as before. Many equations of quantum theory can be written in this form. Give a complete classification of solutions of this equation as a function of b, c and the eigenvalues of the Casimir operator $C_2 = \Gamma_0^2 - \Gamma_4^2 - T = \varphi(\varphi+1)$ (cf. 11.10.7.6).

(a) Let $\tilde{\psi} = e^{i\theta T}\psi$, $\tanh\theta = \left(\frac{b-1}{b+1}\right)$ then $\left[b^{1/2}\Gamma_4 + \frac{c}{2}\right]\psi = 0$. Find the range of the spectrum for discrete, principal and supplementary series of representations of so(2, 1).

(b) Let $\tilde{\psi} = e^{i\theta T}\psi$, $\tanh\theta = \frac{b+1}{b-1}$, then $\left[(-b)^{1/2}\Gamma_4 + \frac{c}{2}\right]\psi = 0$. Find the nature and range of the spectra. (Note that Γ_0 has discrete but Γ_4 continuous spectrum.)

(Cf. Barut 1973.)

§ 2.6.** *Occurrence of principal and supplementary series of* $o(2,1)$ *and self-adjoint extension of Hamiltonians.* Consider the $o(2,1)$-algebra representation given by

$$\Gamma_0 = \frac{1}{2}\left(r\pi^2 - \frac{a}{r} + r\right), \quad \Gamma_4 = \frac{1}{2}\left(r\pi^2 - \frac{a}{r} - r\right), \quad T = r\cdot\pi - i$$

(π was defined in Lemma 15, § 2).

Show that in this representation the Casimir operator C_2 has the value $J^2 - \mu^2 - a$. Show that as in lemmas 3, 4 we can solve the Hamiltonian $H = \pi^2 + a/r^2 + b + c/r$. For large positive a, H is no longer self-adjoint (Kato 1966). Let $C_2 = \varphi(\varphi+1)$ and for simplicity, $\mu = 0$. Show that for $a < j(j+1)$, $C_2 > 0$ we have to use the discrete series of representation of $o(2, 1)$. For $j(j+1) < a < j(j+1) + \frac{1}{4}$, $-\frac{1}{4} < C_2 < 0$ and we must use the supplementary series of representations, and for $a > j(j+1) + \frac{1}{4}$, $C_2 < -\frac{1}{4}$, i.e. $\varphi = -\frac{1}{2} + i\lambda$

and we must use the principal series of unitary representations of $O(2, 1)$. It is remarkable that all *three* series of representations of $O(2, 1)$ occur in physical problems, and that the dynamical group provides a method of self-adjoint extension for a class of Hamiltonians which includes relativistic Dirac Hamiltonian for Coulomb problem (cf. Barut 1973).

2.7.* *Tensor method for* so(4, 2)*-algebra*; so(4), su(2), su(1, 1)*-subalgebras, Clebsch–Gordan coefficients.* Consider two su(2) algebras with generators

$$(J_1)_{ij} = \tfrac{1}{2}\varepsilon_{ijk}(a^*\sigma_k a), \qquad (J_2)_{ij} = \tfrac{1}{2}\varepsilon_{ijk}(b^*\sigma_k b) \tag{1}$$

where $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are two boson annihilation operators $[a, a^*] = 1$, $[b, b^*] = 1$, and σ_k are the Pauli matrices.

Show that in an su(2)⊗su(2)-basis with vectors

$$\begin{aligned} |j_1 m_1 j_2 m_2\rangle &= N^{j_1 j_2}_{m_1 m_2} a_1^{*j_1+m_1} a_2^{*j_1-m_1} b_1^{*j_2+m_2} b_2^{*j_2-m_2}|0\rangle, \\ (N^{j_1 j_2}_{m_1 m_2})^{-2} &= [(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!] \end{aligned} \tag{2}$$

the basis elements of the SU(2, 2)-algebra are represented by

$$\begin{aligned}
L_{ij} &= \frac{1}{2}\varepsilon_{ijk}[a^*\sigma_k a + b^*\sigma_k b] = (J_1)_{ij} + (J_2)_{ij} \equiv J_{ij}, \\
L_{i4} &= -\frac{1}{2}(a^*\sigma_i a - b^*\sigma_i b) = A_i, \\
L_{i5} &= -\frac{1}{2}(a^*\sigma_i C b^* - aC\sigma_i b) = M_i, \\
L_{i6} &= \frac{1}{2i}(a^*\sigma_i C b^* + aC\sigma_i b) = \Gamma_i, \\
L_{46} &= \frac{1}{2}(a^*Cb^* + aCb) = T, \\
L_{45} &= \frac{1}{2i}(a^*Cb^* - aCb) = \Gamma_4, \\
L_{56} &= \frac{1}{2}(a^*a + b^*b + 2) = \Gamma_0, \qquad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\end{aligned} \tag{3}$$

Prove that the most degenerate discrete self-adjoint irreducible representations are labelled by the eigenvalue $\mu = (j_1 - j_2)$ of

$$K = \frac{1}{2}(a^*a - b^*b), \qquad [K, L_{ab}] = 0 \quad \text{for all } a, b, \tag{4}$$

and that this corresponds precisely to the representation given in eq. 2(22) by differential operators with the Casimir operators $C_2 = -3(1-\mu^2)$, $C_3 = 0$, $C_4 = 0$. The basis $|\mu, m, n, \alpha\rangle$ where the labels are the eigenvalues of K, L_{12}, L_{56}

and L_{34}, respectively, we call parabolic states. Find by direct calculation

$$\mu = j_1 - j_2, \quad m = m_1 + m_2, \quad n = j_1 + j_2 + 1, \quad \alpha = m_2 - m_1,$$

and show that the series of representations satisfy the representation relation

$$\{L_{AB}, L^A_C\} = L_{AB}L^A_C + L^A_C L_{AB} = 2(\mu^2 - 1)g_{BC}, \quad A, B, C = 1, 2, \ldots, 6.$$

Consider the reduction ($J_k \equiv \varepsilon_{klm} J_{lm}$)

$$\mathrm{su}(2) \otimes \mathrm{su}(2) \supset \mathrm{su}(2) \supset u(1), \quad \text{with } \boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2.$$

We define the spherical basis $|\mu, m; n; j(j+1)\rangle$, eigenvectors of $(K, L_{12}, L_{56}, \boldsymbol{J}^2)$ by

$$|(j_1 j_2) jm\rangle = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | (j_1 j_2) jm\rangle | j_1 m_1 j_2 m_2\rangle \delta_{m_1+m_2, m}. \tag{5}$$

This expansion can be viewed as the expression of the spherical states in terms of the parabolic states.

Next consider the subgroup $O(2, 1) \times O(2, 1) \subset O(4, 2)$ generated by

$$N_1^{(r)} = \frac{1}{2}\left(L_{46} + (3-2r)L_{35}\right),$$

$$N_2^{(r)} = \frac{1}{2}\left(L_{45} - (3-2r)L_{36}\right),$$

$$N_3^{(r)} = \frac{1}{2}\left(L_{56} + (3-2r)L_{34}\right), \quad r = 1, 2,$$

and the states

$$\begin{aligned} |\varphi_1 n_1 \varphi_2 n_2\rangle &= N b_1^{*n_1+\varphi_1-1} a_2^{*n_1-\varphi_1} a_1^{*n_2+\varphi_2-1} b_2^{*n_2-\varphi_2} |0\rangle, \\ N^{-2} &= [(n_1+\varphi_1-1)!(n_1-\varphi_1)!(n_2+\varphi_2-1)!(n_2-\varphi_2)!], \end{aligned} \tag{7}$$

where

$$\begin{aligned} N_3^{(r)} |\varphi_1 n_1 \varphi_2 n_2\rangle &= n_r |\varphi_1 n_1 \varphi_2 n_2\rangle, \quad r = 1, 2, \\ N^{(r)2} |\varphi_1 n_1 \varphi_2 n_2\rangle &= \varphi_r(\varphi_r - 1) |\varphi_1 n_1 \varphi_2 n_2\rangle, \quad r = 1, 2, \\ N^{(r)2} &= N_3^{(r)2} - N_1^{(r)2} - N_2^{(r)2}, \quad r = 1, 2. \end{aligned} \tag{8}$$

Now in the reduction

$$\mathrm{su}(1, 1) \otimes \mathrm{su}(1, 1) \supset \mathrm{su}(1, 1) \supset u(1)$$

with $\boldsymbol{N} = \boldsymbol{N}^{(1)} + \boldsymbol{N}^{(2)}$, show that

$$|(\varphi_1 \varphi_2)\varphi n\rangle = \sum_{n_1 n_2} \langle \varphi_1 n_1 \varphi n_2 | (\varphi_1 \varphi_2) \varphi n\rangle | \varphi_1 n_1 \varphi_2 n_2\rangle \delta_{n_1+n_2, n}. \tag{9}$$

Prove that a property of the discrete, degenerate series of representations of $O(4, 2)$ is that

$$\boldsymbol{J}^2 = \boldsymbol{N}^2 \tag{10}$$

Hence in the basis $|(\varphi_1\varphi_2)\varphi n\rangle$ the set K, J^2, L_{56}, L_{12} is again diagonal with $\mu = \varphi_2-\varphi_1$, $m = \varphi_1+\varphi_2-1$, $n = n_1+n_2$. On the other hand in the states $|\varphi_1 n_1 \varphi_2 n_2\rangle$ the set $K, L_{12}, L_{56}, L_{34}$ is diagonal with $\alpha = n_1-n_2$.

Hence (5) and (9) represent expansions of the $O(4,2)$-states into the same states, consequently with proper identification of indices show that the Clebsch-Gordan coefficients of SU(2) are identical with those of SU(1, 1) for coupling of discrete series of representations, namely

$$\langle j_1 m_1 j_2 m_2 | (j_1 j_2) jm\rangle = \langle \varphi_1 n_1 \varphi_2 n_2 | (\varphi_1\varphi_2)\varphi n\rangle$$

with

$$j_1 = \frac{1}{2}(n_1+n_2+\varphi_2-\varphi_1-1),$$

$$j_2 = \frac{1}{2}(n_1+n_2+\varphi_1-\varphi_2-1),$$

$$m_1 = \frac{1}{2}(n_2-n_1+\varphi_2+\varphi_1-1),$$

$$m_2 = \frac{1}{2}(n_1-n_2+\varphi_2+\varphi_1-1),$$

$$j = \varphi-1, \quad m = \varphi_1+\varphi_2-1$$

and

$$\mu = j_1-j_2 = \varphi_2-\varphi_1.$$

§ 2.8. *Coherent States for* SU(1, 1). For the Heisenberg algebra $\{1, a, a^*\}$ with

$$[a, a^*] = 1$$

the coherent states $|z\rangle$ are the eigenstates of a:

$$a|z\rangle = z|z\rangle$$

and can be written as

$$|z\rangle = e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle = e^{|z|^2/2}e^{za^*}|0\rangle$$

in terms of the eigenstates $|n\rangle$ of the number operator a^*a. They satisfy $\langle z|z\rangle = 1$, $\langle z'|z\rangle = \exp\left[-\frac{1}{2}|z|^2-\frac{1}{2}|z'|^2+\bar{z}'z\right]$,

$$\frac{1}{\pi}\int |z\rangle\langle z| d^2z = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1.$$

For the su(1, 1)-algebra $\{L^+, L^-, L_3\}$ with

$$[L^+, L^-] = -L_3, \quad [L_3, L^\pm] = \pm L^\pm$$

define similar coherent states by

$$L^-|z\rangle = z|z\rangle.$$

Show that for the discrete-representation $D^+(\varphi)$:

$$|z\rangle = [\Gamma(-2\varphi)]^{1/2} \sum_{n=0}^{\infty} \frac{(\sqrt{2}z)^n}{[n!\Gamma(-2\varphi+n)]^{1/2}} |\varphi, n\rangle,$$

and the resolution of identity becomes

$$\frac{4}{\pi\Gamma(-2\varphi)} \int r\mathrm{d}r\mathrm{d}\theta (\sqrt{2r})^{-2\varphi-1} K_{1/2+\varphi}(2\sqrt{2r})|z\rangle\langle z| = 1, \quad z = re^{i\theta}$$

and we obtain a family of Hilbert spaces of entire functions of growth (1, 1) (cf. A. O. Barut and L. Girardello 1971).

Chapter 13

Group Theory and Group Representations in Quantum Theory

§ 1. Group Representations in Physics

Discrete and continuous groups occur in classical physics as transformation groups and express generally the symmetry of dynamical equations of particles or fields: crystal symmetry, the Galilei, Poincaré or Einstein group of transformations of space-time, the group of canonical transformations acting on the phase-space, the group of gauge transformation, or conformal transformations of the electromagnetic potentials, symplectic transformation of thermodynamic functions, etc. We gave even a Lie algebra structure in the Poisson-brackets of classical mechanics and classical field theory, for which the canonical transformations act as the group of automorphisms. However, in all these only a defining representation of the groups occurs. The theory of group representations in linear spaces which is the subject matter of this book, is really the domain of quantum physics. The particular adaptability of the theory of group representations to quantum physics stems from the following basic kinematical difference between classical and quantum theories: In both classical and quantum theory a physical system can be described by the notion of 'state', and in both cases the system has continuously infinitely many states. In quantum theory, due to the linearity of the equations of motions, the states can be represented as linear combinations of a selected orthogonal sets of states, often a set of countable many elements; i.e., the states form a linear vector space. In contrast, in the classical case the dynamical equations in terms of coordinates and momenta are non-linear, hence there is no such set of basis states.

The group-structure of physical theories may be studied along the following two different approaches:

A. *Theories Based on Dynamical Equations*

Most physical theories postulate a certain set of dynamical equations, e.g., equations of particle mechanics, fluid dynamics, electrodynamics, statistical and quantum mechanics. These equations govern the behavior of functions ψ of particle or field coordinates, and may be written symbolically as

$$L\psi = 0. \tag{1}$$

These equations may be linear or non-linear, differential or integral equations, or more general operator equations. The role of group theory in this line of approach is to facilitate the solutions of eq. (1) and to better recognize the structure of the underlying dynamics. If we start from this point of view, we are led to study the space Φ of solutions of eq. (1). In order to characterize the space we may search first for set of the operators $\{X_i\}$ for which

$$X_i\psi \in \Phi, \quad \text{if} \quad \psi \in \Omega \subset \Phi,\ i = 1, 2, \ldots$$

For a class of operators X_i we have then the property that $[L, X_i]\psi = 0$ and $[L, [X_i, X_j]]\psi = 0$: we may then give the set a Lie algebra structure. Note that the X's are general operators transforming in general both the arguments of ψ and the form of ψ itself. More generally, we may have $[L, X_i]\psi = \lambda L\psi$, i.e., a multiple of the eq. (1), which leads again to a Lie algebra structure on the space of solutions.

If such a formulation is possible, then the set X_i contains not only the usual symmetry operations, but also the more general dynamical symmetry transformations.

In particular let Y be an operator satisfying the eigenvalue equation

$$Y\psi_n = y_n\psi_n, \quad \psi_n \in \Omega.$$

For a maximal subset C of operators such that

$$[C_i, Y] = 0, \quad C_i \in C,$$

we have

$$Y(C_i\psi_n) = y_n(C_i\psi_n),$$

i.e., the 'value' of Y is 'conserved' under the operation C_i. The set C is a symmetry algebra with respect to the quantity Y. In particular, if Y is the Hamiltonian of the system in classical or quantum mechanics, C is called the *symmetry algebra* or the *algebra of degeneracy of the energy*.

This approach is formally the same for classical and quantum theories, and leads directly to Lie algebras rather than Lie groups, provided we can define the product of operators. This discussion can also be based on a variational principle instead of eq. (1).

In contrast, in the next approach the concept of groups plays a fundamental role.

B. *Theories Based on Prescribed Symmetries*

If the dynamical equations of the system are unknown, we can be guided by general symmetry principles to discover or guess the equations, or we can investigate general properties of the system which results from prescribed symmetries. The general covariance principle of Einstein in general relativity, or the properties of the S-matrix of the interaction of elementary particles are typical examples.

C. *The Idea of Relativity*

In Euclidean geometry points are all indistinguishable and have an objective existence whereas coordinates are man-made. All Cartesian coordinates are equally admissible. Thus the objective properties of points must be independent of the choice of coordinate frame. The coordinates of a point in different frames are related by a group of transformations; or a transformation of the group maps one point into another (passive and active views, respectively). Hence we have a group of automorphisms of our geometry. Conversely, any group of transformations may serve as the group of automorphism of some geometry. The characterization of the geometry by its group of authomorphisms is the basic idea in Klein's Erlangen Program (1872). This idea of the relativity of coordinate frames extends from points, to lines, areas,... and to physical quantities, like forces. velocities, fields... More generally, physical processes taking place at some time at some place are independent of the observers (frames). All observers in a class are equally admissible. Observers are related to each other by a group of transformations, or a transformation takes one observer into another. This group of transformations is the group of authomorphisms of a physical theory, or of a physical law. Physicists call the class of observers related by this group of transformations also as inertial frames. This idea is basic to much of the kinematical applications of group theory to physics.

We begin with the general kinematical framework of quantum theory which is essential for the formulation of symmetry principles and for the use of symmetry and dynamical group.

§ 2. Kinematical Postulates of Quantum Theory

Superposition Principle and Probability Interpretation

Quantum theory originated from the wave properties of matter. The most important property of wave phenomena is the property of superposition or interference: linear combinations of solutions are again solutions of the wave equation because the equation is linear. Furthermore, the matter waves describe probability amplitudes rather than the amplitudes for certain densities of particles or fields. These two fundamental physical ideas put together give us a general framework in which physical states ψ, φ, ... are described by elements of a linear space, and the observed positive definite conditional probabilities are represented by the square of the sesquilinear forms $|(\psi, \varphi)|^2$. Historically, in simple cases, the space of solutions of the wave equations (e.g., Schrödinger equation) can be imbedded in a Hilbert space, and hence the theory has been formulated traditionally within a Hilbert space framework since its axiomatization by von Neumann. However, the two basic physical ideas allow more general spaces of physical states. The requirement that limiting points of sequences of states ψ_n should be included gives a general topological vector space V and its dual V' so that the sesquilinear form (probability amplitudes) can be taken to be

$$(\psi, \varphi), \quad \varphi \in V \text{ and } \psi \in V'.$$

Even within the Hilbert space formalism it is often convenient to take a more general formalism and use a Gel'fand triplet

$$\Phi \subset H \subset \Phi',$$

where Φ is a dense nuclear subspace in the Hilbert space H and Φ' is dual to Φ, in order to accommodate operators with continuous spectra and their generalized eigenvectors in Φ' which are not normalizable (cf. app. B).

Besides these mathematical generalizations of the framework of quantum theory in the use of more general spaces, physical generalizations of the theory may also be considered in the future, for example, one may weaken the universal validity of the linearity of states, i.e., the superposition principle. Keeping these points in mind, we describe in this section the standard form of quantum postulates and the role of group representations.

States and Rays

The basic framework of the quantum theoretical description of a physical system is a linear space H whose unit rays are in one-to-one correspondence with the states of the system which are called *pure states*. A *unit ray* Ψ is the set of vectors $\{\lambda\psi\}$, $||\psi|| = 1$, $\lambda = \exp(\mathrm{i}\alpha)$, $\psi \in H$. The reason for the introduction of rays rather than vectors themselves lies a) in the use of a space over the complex numbers, and b) in the basic probability interpretation of quantum theory. The quantities related to observable effects are the absolute values of a sesquilinear form $|(\psi, \varphi)|^2$ which are independent of the parameters λ, λ' characterizing a ray. Consequently the space of rays is the quotient space $H = H/S^1$, i.e. the projective space of one dimensional subspaces of H.

This basic correspondence between physical states and the elements of H incorporates the *superposition principle* of quantum theory—namely, that there is a set of basis states out of which arbitrary states can be constructed by linear superpositions. Thus if the rays $\{\lambda\psi_n\}$, $n = 1, 2, \ldots$, describe physical states, then $\psi'_\alpha = \sum_n \alpha_n \psi_n$ is another vector of H, so that the ray $\{\lambda\psi'_\alpha\}$ corresponds to another possible state of the system. Note that $\psi'_\alpha = \sum_n \alpha_n \psi_n$ and $\lambda\psi'_\alpha$ represent the same state, but $\sum_n \alpha_n(\lambda_n \psi_n)$ is in general a different state, although ψ_n and $(\lambda_n\psi_n)$ represent the same state. Herein lies the problem of relative phases of quantum theory.

States that can be obtained from each other by linear superpositions are called pure 'coherent' states.

Superselection Rules

In general, there are physical limitations on the validity of the superposition principle. One cannot realize pure states out of superposition of certain states; for example, one cannot form a pure state consisting of a positively and a negatively charged particle, or a pure state consisting of a fermion and a boson. This does not mean that two such states cannot interact; it only means that their formal linear combination is not a physically realizable pure state (superselection rule). The existence of superselection rules is connected with the measurability of the relative phase of such a superposition and depends on further properties of the system, like charge, baryon number, etc. The superselection rule on fermions (i.e., separation of states of integral and half integral fermions) follows from the rotational invariance (see below). In all such cases we divide the linear space H into subsets, such that the superposition principle holds within each subset. These subsets are called *coherent subspaces*. In each subspace, $\sum \alpha_n \psi_n$ and $\lambda \sum \alpha_n \psi_n$ correspond to the same state, but $\sum \alpha_n \psi_n$ and $\sum \alpha_n' \psi_n$, in general, correspond to distinct states. We shall come back to this problem at the end of this section.

Probability Interpretation

Physical experiments consist in preparing definite states, in letting them interact and in observing the rate of occurrence of other well-defined states. The transition probability between two states ψ and φ is defined by the square of a sesquilinear form $|(\psi, \varphi)|^2$. We can also say the transition probability between two rays Ψ and Φ because this quantity is the same for all vectors of the rays; overall phases are unimportant. However, if ψ and φ are themselves linear combinations of some basis vectors, then the transition probability depends on the relative phases of their components. The quantity $|(\psi, \varphi)|^2$ can be related, by multiplying it with certain kinematical factors, to the experimentally observed quantities like cross sections of reactions, and lifetimes of unstable states.

The Dynamical Problem

Now in order to evaluate quantities like $|(\psi, \varphi)|^2$ we must have a definite realization of the linear space H, and must obtain a *number* that can be compared with experiment. Thus we need a definite labelling of the states $\psi, \varphi, \ldots$ and a definite expression for the sesquilinear product. We shall refer to this realization as the *concrete linear space* (CLS). This is the more important and the more difficult part of the theory. Although all Hilbert spaces of the same dimension are isomorphic and one can transform one realization into another, some definite explicit realization with a physical correspondence is necessary.

If the linear or the Hilbert space framework provides the *kinematical principle* of quantum theory, the explicit calculation of states $\psi, \varphi, \ldots$ or, of sesquilinear (or scalar) products (ψ, φ), is the *dynamical* part of quantum theory.

In simple cases, the dynamical problems is solved by postulating a differential equation for the states $\psi, \varphi, \ldots$ represented, for example, as elements of $H = L^2(R^3)$, and identifying all solutions of the equation with all the states of the physical system. This is the case in Schrödinger theory. For more complicated systems, or for unknown new systems, this is not possible. Even if we know all the states of an isolated system, measurements on the system are carried out by additional external interactions which change the system.

Short of the complete calculation of the sesquilinear products (ψ, φ) some very general principles allow one to derive a number of important symmetry properties of these quantities. It is along these lines that the traditional use of group representations in quantum theory has been developed. More recently the quantum theoretical Hilbert space has been identified with the explicit carrier space of the representations of more general groups and algebras. In this second sense the group representations solve also the dynamical problem. We shall elaborate both of these aspects. In order to be specific, we consider from now on the space of states to be a Hilbert space.

Equivalent Description or Symmetry Operations

As in any correspondence, one should first consider equivalent mappings between the physical states and the rays in the Hilbert space. For the knowledge of physically equivalent descriptions of a system reflects already, as we shall see, important properties of the system itself.

If the same physical system can be described in two different ways in the same coherent subspace of the Hilbert space H, once by rays $\Psi_1, \Phi_1, \ldots$ and once by rays $\Psi_2, \Phi_2, \ldots$ (for example, by two different observers), such that the same physical state is once described by Ψ_1, in the other case by Ψ_2—equivalently we can speak of symmetry operation of the system—then the transition probabilities must be the same by the definition of physical equivalence. We have then a norm preserving mapping $\hat{T}$ between the rays Ψ_1 and Ψ_2. Mathematically, it is more convenient to find out the corresponding map $H \to H$ between the vectors $\psi, \varphi, \ldots$ in the Hilbert space. Because only the absolute values are invariant, the transformation in the Hilbert space can be unitary or anti-unitary. In fact, one can prove that given two descriptions of a system in the space of rays, *one can choose* unit vectors, $\psi_1, \varphi_1, \ldots$ from the rays $\Psi_1, \Phi_1, \ldots$ in the first description, and unit vectors $\psi_2, \varphi_2, \ldots$ from the rays $\Psi_2, \Phi_2, \ldots$ in the second description, such that the correspondence $\psi_1 \leftrightarrow \psi_2$, $\varphi_1 \leftrightarrow \varphi_2, \ldots$ is either *unitary* or *anti-unitary.* That is, one can *construct* a unitary or anti-unitary correspondence $H \to H$. More precisely, we have

THEOREM 1 (Wigner). *Let $\Psi_2 = \hat{T}\Psi_1$ be a mapping of the rays of a Hilbert space H which preserves the inner product of rays, then there exists a mapping $\psi_2 = T\psi_1$ of all vectors of H such that $T\psi$ belongs to the ray $\hat{T}\Psi$* if *ψ belongs to the ray Ψ and, in addition $1°T(\psi+\varphi) = T\psi + T\varphi$, $2°T(\lambda\psi) = \chi(\lambda)T(\psi)$, $3°(T\psi, T\varphi) = \chi[(\psi, \varphi)]$, where either $\chi(\lambda) = \lambda$ (unitary case), or $\chi(\lambda) = \bar{\lambda}$ (anti-unitary case) for all λ.*

PROOF: Let $\psi_1, \varphi_1, \ldots$ and $\psi_2, \varphi_2, \ldots$ be two sets of orthonormal bases chosen from the first and second set of rays, respectively. We are given a transformation which preserves the absolute values. The problem is to *construct* the corresponding transformation T on the vectors by suitable choices of the phases. It could be *a priori* that $T\psi_1 = c_1\psi_2, T\varphi_1 = c_2\varphi_2, \ldots$ and that no relations between the c's can be established, in which case T is not even a linear operator. We want, in fact, to show that T can be so defined that it is a unitary or an anti-unitary operator.

We single out the unit vector ψ_1 and choose ψ_2, and define $T\psi_1 = \psi_2$. This is the only arbitrary choice, and shall show that all other phases are uniquely determined. Thus T is determined up to an overall phase factor.

Next consider the vector $\psi_1 + \varphi_1$, where φ_1 is orthogonal to ψ_1. It is easy to show that a representative vector of the corresponding ray in the second description is $a\psi_2 + b\varphi_2$. We have then

$$T(\psi_1+\varphi_1) = c(a\psi_2+b\varphi_2) = \psi_2+b'\varphi_2,$$

where we must have $c = 1/a$ by the previous choice, and put $cb = b' = b/a$. We now *define* $T\varphi_1$ by $T(\psi_1+\varphi_1) - \psi_2$ or simply by $b'\varphi_2$. Hence we can set

$$T(\psi_1+\varphi_1) = T\psi_1 + T\varphi_1.$$

Similarly for a general $f_1 = a_\psi\psi_1 + a_\varphi\varphi_1 + \ldots$ we choose a representative $f_2 = \hat{a}_\psi\psi_2 + \hat{a}_\varphi\varphi_2 + \ldots$, write $Tf_1 = cf_2$ with $c\hat{a}_\psi = a_\psi$, so that

$$\begin{aligned} T(a_\psi\psi_1 + a_\varphi\varphi_1 + \ldots) &= a_\psi\psi_2 + c\hat{a}_\varphi\varphi_2 + \ldots \\ &= a_\psi T\psi_1 + a'_\varphi T\varphi_1 + \ldots \end{aligned}$$

Now we form the absolute values of the scalar products

$$|(\psi_1+\varphi_1, f_1)| = |a_\psi + a_\varphi|$$

and

$$|(T\psi_1+T\varphi_1, Tf_1)| = |a_\psi + a'_\varphi|.$$

These two numbers must be equal. This plus the fact that $|a'_\varphi| = |a_\varphi|$ allows us to calculate a'_φ in terms of a_φ and a_ψ. One obtains two solutions:

$$a'_\varphi = a_\varphi \quad \text{and} \quad a'_\varphi = \bar{a}_\varphi \frac{a_\psi}{\bar{a}_\psi}.$$

Clearly for the first solution T is linear *and* unitary. For the second solution we find $Tf_1 = (a_\psi/\bar{a}_\psi)[\bar{a}_\psi T\psi_1 + \bar{a}_\varphi T\varphi_1 + \ldots]$. An overall phase factor is unimportant and by a new normalization of T—which by the way does not change the choice $T\psi_1 = \psi_2$—we obtain an *anti-unitary* operator. ▼

Remarks: 1. The two possibilities in th. 1 come from the fact that the complex field has two (and only two) automorphisms that preserve the absolute values: the identity automorphism and the complex conjugation. In the case of the Hilbert space over a real field Wigner's theorem yields only unitary transformations (up to a phase), because the only automorphism of the real field is the identity automorphism. In fact, Wigner's theorem is closely related to the fundamental theorem of projective geometry.

2. One or the other case occurs for a given situation. Whether the transformation is unitary or anti-unitary depends on further properties of the two equivalent descriptions of the system. It does *not* depend, however, on the choice of vectors $\psi, \varphi, \ldots$ from the rays; if the transformation is, for example, unitary for a choice $\psi_1, \varphi_1, \ldots$ there is no other choice $\lambda\psi_1, \lambda'\varphi_1, \ldots$ such that it becomes anti-unitary and vice versa. Furthermore, once a vector ψ_2 is chosen, the others, $\varphi_2, \chi_2, \ldots$ are uniquely determined from the requirement that the correspondence is unitary (or anti-unitary).

Symmetry Transformations

The description of the *symmetry properties* of the system in the standard sense belongs to the situation characterized by the above theorem. For, if under a symmetry transformation measured probabilities are unchanged, we obtain automatically two equivalent descriptions in H, one corresponding to the original and the other to the transformed frame; and these two descriptions must be related to each other by unitary (or anti-unitary) transformations. Conversely, and this is more important from our point of view, *the Hilbert space of states must be isomorphic to the carrier space of unitary (or anti-unitary) representations of the symmetry transformations* (they may form a group or an algebra, etc.). Note that we wish to obtain a concrete Hilbert space to calculate transition probabilities. Thus, if we know the symmetry transformations of the system we can start from an arbitrary *collection* of irreducible unitary (or anti-unitary) representation spaces of the symmetry transformations to build up the Hilbert space H. This solves the problem partly, but not completely because we do not know *what* collection of irreducible representations we have. The explicit forms of the symmetry operations are discussed in sec. 3.

Uniqueness of Operators

We have said that representative vectors from the rays of two equivalent descriptions can be so chosen that the mapping of vectors $\psi_1 \leftrightarrow \psi_2$ is either unitary, or anti-unitary. There is one other important phase problem in quantum theory and this concerns the *uniqueness* of the unitary (or anti-unitary) correspondence $\psi_1 \leftrightarrow \psi_2$. It follows from the proof of Wigner's theorem that this correspondence is unique *up to an overall phase factor.*

Ray Representations or Projective Representations

If there are two equivalent descriptions with rays $\Psi_1, \Phi_1, \ldots$ and $\Psi_2, \Phi_2, \ldots$ respectively, corresponding to the same physical states as seen by the two different observers (passive view), or with rays Ψ_1, $\Phi_1, \ldots$ corresponding to states $\{s\}$ in the first description and to the transformed states $\{gs\}$ in the second description (active view), then we know that we can choose vectors $\psi_1 \in \Psi_1, \psi_2 \in \Psi_2, \ldots$ such that

$$\psi_2 = T_g \psi_1, \qquad \varphi_2 = T_g \varphi_1, \ldots \tag{1}$$

That is, if ψ_1 is a vector associated with Ψ_1, then $T_g \psi_1$ is a vector associated with the ray Ψ_2. Now if there are two operators T_g and $T_{g'}$ with the property (1), they can differ only in a constant factor of modulus 1. This result has an implication on the group law of transformations. For the product of two transformations $T_g T_{g'}$ gives the same results as the transformation $T_{gg'}$. Consequently,

$$T_{gg'} = \omega(g, g') T_g T_{g'}, \tag{2}$$

where $\omega(g, g')$ is a phase factor. Because T_g is a representation of the symmetry group, the group law for the representations is more general than the group law itself $g(g's) = (gg')s$. Representation of the type (2) are called 'ray representations' or 'representations up to a factor', or 'projective representations'. This is again the result of the fact that we have a correspondence between physical states and the rays in Hilbert space, not vectors.

DEFINITION. A *ray* (or projective) *representation* T *of* a topological group G is a continuos homomorphism $T: G \to L(\hat{H})$, the set of linear operators in the projective space $\hat{H}$ with the quotient topology relative to map $H \to \hat{H}$, i.e., $\psi \to \Psi$. ▼

Although the representation T is determined up to an arbitrary factor, the phase $\omega(g, g')$ in eq. (2) is not arbitrary. First of all, two phase systems $\omega(g, g')$ and $\omega'(g, g')$ may be defined to be equivalent if

$$\omega'(g, g') = \omega(g, g') \frac{c(gg')}{c(g) c(g')}, \qquad g, g' \in G, \tag{2a}$$

where $c(g)$ is an arbitrary continuous function, because then the corresponding T_g and $T'_g = c(g) T_g$ have the same phase $\omega'(g, g')$. Furthermore, the associativity law of the group multiplication puts another restriction on the phase system $\omega(g, g')$, namely,

$$\omega(g, g')\omega(gg', g'') = \omega(g', g'')\omega(g, g'g''). \tag{2b}$$

Note that $\omega'(g, g')$ defined in (2a) satisfies (2b) if $\omega(g, g')$ does.

It is easy to see by taking very simple examples that eq. (2b), even up to equivalence given by (2a), does not uniquely determine the phases $\omega(g, g')$, so that we have, in general, a number of new inequivalent ray representations for a given group G, besides the usual representations with $\omega = 1$.

Let $g \to T_g$ be a projective representation of G, $g \in G$, in H. Let v_i be the components of some $v \in H$ in some basis. A ray can be represented by the quantities, $\bar{v}_i \equiv v_i/v_1$ where v_1 is any component of v. For clearly all vectors in the ray $\{\lambda v\}$ induce the same $\bar{v}$. By choosing a special vector v with $v_1 = 1$ we see that the transformations induced on $\bar{v}$ by T are

$$\bar{v}'_1 = v_1,$$

$$\bar{v}'_i = \Big[\sum_{k=2}^{\infty} D_{ik}(g)\bar{v}_k + D_{i1}(g)\Big] \Big/ \Big(\sum_{k=2}^{\infty} D_{1k}(g)\bar{v}_k + D_{11}(g)\Big),$$

$$i = 2, 3, \ldots$$

These transformations are nonlinear and are called *projective transformations*. It is easy to check that representations T_g and $c(g)T_g$ as well as inequivalent ray representations $\omega(g, g') \neq 1$ induce the same projective representation. The phase ambiguity has completely disappeared in this formulation, but it is only hidden, because the inverse problem of finding all inequivalent projective representations is equivalent to finding all inequivalent phases.

Projective Representations and the Central Extension

The remaining phase ambiguity precludes the application of the mathematical theory of ordinary representations when $\omega(g, g') \neq 1$. In this case we can try to construct a larger (or extended) group $\mathscr{E}$ whose ordinary representations give *all* the inequivalent ray representations (2) of G. This is the problem of *lifting* the projective representations of G into the ordinary representations of $\mathscr{E}$ and can be done simply as follows: Let K be the abelian group generated by multiplying the inequivalent phases $\omega(x, y)$ satisfying (2b). Consider the pairs (ω, x), $\omega \in K$, $x \in G$. In particular, $K = \{(\omega, e)\}$ and $G = \{(e, x)\}$. The pairs (ω, x) form a group with the multiplication law of a semidirect product

$$(\omega_1, x_1)(\omega_2, x_2) = \big(\omega_1 \omega(x_1, x_2)\omega_2, x_1 x_2\big).$$

In the present case we can think of (ω, x) as ωx. The group $\mathscr{E} = \{(\omega, x)\}$ is called a *central extension* of G by K (cf. 21, § 4) and we see that the vector representations of $\mathscr{E}$ contain all ray representations of G. Thus the extended group $\mathscr{E}$ may be considered as the proper *quantum mechanical group*. The theory and applications of group extensions will be discussed later in ch. 21. Here we give only

PROPOSITION 2. *Finite-dimensional projective representations of simply connected continuous groups are equivalent to ordinary representations.*

PROOF: First, quite generally, we take the determinant of eq. (2): $\det T(x)\det T(y) = \omega^n(x, y)\det T(x, y)$, where n is the dimension of the representations. The new representation $T'(x) = T(x)/[\det T(x)]^{1/n}$ formally satisfies $T'(x)T'(y) = T'(xy)$: There are different values of $[\det T]^{1/n}$ and we can pass to an equivalent phase system such that $T'(x)T'(y) = \omega'(x, y)T'(xy)$ with $\omega'^n = 1$. Now if the group space is simply connected $[x']^{1/n}$ can be uniquely defined and is the same for all x by continuity. Hence we arrive at an equivalent ordinary representation. ▼

Similarly the ray representations of the one-parameter subgroups of Lie groups are always equivalent to ordinary representations (cf. ch. 21, § 4).

In many physical examples, such as the cases of rotation, Lorentz and Poincaré groups, the projective representations of the group can be reduced to true unitary representations of its universal covering group (Bargmann 1954). A notable exception is the Galilei group, where the 'quantum mechanical group' is truly an eleven-parameter group, a central extension of the universal covering group of the Galilei group. The cases where we do not need a central extension are covered by

LIFTING CRITERION. *Let G be a connected and simply connected Lie group with Lie algebra L. Assume that for each skew-symmetric real valued bilinear form $\theta(x, y)$ on L satisfying*

$$\theta([x, y]z)+\theta([y, z]x)+\theta([z, x], y) = 0$$

there exists a linear form f on L such that

$$\theta(x, y) = f([x, y])$$

for all x, y in L. Then each strongly continuous projective representation of G is induced by a strongly continuous unitary representation on the corresponding Hilbert space. ▼

(For proof see Bargmann 1954, or Simms 1971.)

Remark: This condition is often expressed by the statement that the second cohomology group $H^2(G, R)$ is trivial (cf. ch. 21, § 4).

Summarizing, we have: A symmetry group G of the physical system *induces* a representation T of invertible mappings of H onto itself, which is unitary or anti-unitary and is a representation of a central extension $\mathscr{E}$ of G or of the covering group of G. In the unitary case we proved that H is a certain direct integral of irreducible carrier spaces of T by virtue of Mautner theorem (5.6.1).

Continuity

Mathematical theory of representations of topological groups requires the assumption of continuity of representations. Physically this means the continuity of the probabilities $|(\varphi, T_g\psi)|^2$ as a function of g, i.e., when we compare the probabilities of finding two states $T_g\psi$ and $T_{g'}\psi$ in a fixed state φ.

Unitary and Anti-Unitary Operators

The group property of the transformations, eq. (2), and the continuity allow us to determine the unitary or the anti-unitary character of the representation T of the symmetry group G.

If for every element g of G we have

$$g = h^2, \tag{3}$$

where h is also a group element, we have

$$T_g = \omega(g) T_h^2, \tag{4}$$

where $\omega(g)$ is a phase factor.

The square of an anti-unitary or unitary operator is unitary. Thus, T is unitary. For the identity component of any Lie group G, eq. (3) is satisfied. Indeed let $g(t)$ be a one parameter subgroup of G such that $g(t_0) = g$. Then for $h = g(t_0/2)$, eq. (3) holds. Consequently connected Lie symmetry groups will be represented by unitary operators. For the anti-unitary case, eq. (3), must break down. If eq. (3) does not hold, as for the extended Poincaré group with space and time reflections, further physical considerations are necessary to decide the unitary or anti-unitary character of T.

One can also see that the invariance of a state, which is a superposition of two stationary states with different energies, at time t under a symmetry transformation also eliminates the anti-unitary representation. For then the operator corresponding to the second solution in Wigner's theorem denoted by A would give

$$A\big(\psi_1 \exp(-iE_1 t) + \varphi_1 \exp(-iE_2 t)\big) = \exp(iE_1 t) A\psi_1 + \exp(iE_2 t) A\varphi_1,$$

whereas the correct evolution of the state obtained from the Schrödinger equation is

$$\exp(-iE_1 t) A\psi_1 + \exp(-iE_2 t) A\varphi_1.$$

PROPOSITION 3. *The symmetry corresponding to the 'reversal of the direction of motion' (time reversal) must be represented by anti-unitary operators A.*

PROOF: An arbitrary state $\psi(t)$ can be represented as a superposition of stationary states ψ_n. The state $\psi(t)$ and the time reversed state $(A\psi)(t)$ evolve as

$$\psi(t) = \sum_n \exp(-iE_n t)\psi_n, \qquad (A\psi)(t) = \sum_n \exp(-iE_n t) A\psi_n.$$

This last state must be, by time reversal invariance, also the transform of the state $\psi(-t)$, i.e.,

$$A\psi(-t) = A \sum_n \exp(iE_n t)\psi_n.$$

Thus, A must be anti-unitary in order for both states to be the same. ▼

In the unitary case, one can define a normalized operator T_g such that $T_{g^{-1}}$

$= T_g^{-1}$. Then $T_{gg^{-1}} = \omega(g, g^{-1})I$ by eq. (2). For two commuting transformations we have from (2)

$$T_g T_{g'} = c(g, g')T_{g'}T_g, \quad c(g, g') = \frac{\omega(g', g)}{\omega(g, g')}$$

and we find $c(g, g') = +1$ only if T_g and $T_{g'}$ also commute. In general, if the *commutator* $T_g T_{g'} T_g^{-1} T_{g'}^{-1}$ (which is independent of the normalizations of T_g and $T_{g'}$ and which is uniquely determined from T_g and $T_{g'}$) is a multiple of I, i.e., $C = gg'g^{-1}g'^{-1} = I$, then $T_C = c(g, g')I$. The factor $c(g, g')$ is a characteristic of the coherent subspace only, i.e., it has a unique value in each coherent subspace. If in particular $T_{g'}$ and T_g are members of the same one-parameter subgroup, then $c(g, g') = 1$.

Superselection Rules and Symmetry

In section 1.2, we have seen that the vectors $\sum_n \alpha_n \psi_n$ and $\sum_n \alpha_n(\lambda_n \psi_n)$ belong to different rays (states) although ψ_n and $\lambda_n\psi_n$ belong to the same ray. Now if a physical symmetry transformation of the system changes ψ_n into $\lambda_n\psi_n$ then, because the state of the system has not changed, a superposition of the form $\sum_n \alpha_n\psi_n$ is not possible, unless $\lambda_n = 1$. The relative phase λ_n between vectors in different coherent sectors is not an observable because the physics has not changed under the symmetry transformation. If this is the case, no physical measurement can distinguish the state $\sum_n \alpha_n\psi_n$ from the state $\sum_n \alpha_n(\lambda_n\psi_n)$. Thus, to show the existence of a superselection rule, we need a symmetry transformation (a physical postulate) and the existence of vectors ψ_n which go into $\lambda_n\psi_n$ under this transformation, and which represent eigenstates of a measurable physical quantity, e.g. charge.

EXAMPLE 1. *Rotational Invariance and Fermion Superselection Rule.* Consider, for concreteness, a state $\psi_1 = \left|\frac{j}{2}, m\right\rangle$ belonging to the representation $D^{j/2}$ of the rotation group SO(3) and a state $\psi_2 = |j, m'\rangle$ belonging to the representation $D^j, j = \text{integer}$. Consider a rotation $\hat{n}\omega$ by an angle ω in a direction $\hat{n}$. The states transform by the formula

$$|JM\rangle' = \mathrm{D}^J_{M',M}(\hat{n}\omega)|JM'\rangle.$$

It follows from exercise 5.8.3.1 that matrices D^J satisfy the condition

$$D^J(\hat{n}\omega + 2\pi n) = (-1)^{2nJ}D^J(\hat{n}\omega).$$

Thus for the rotation $\hat{n}_y 2\pi$, $\hat{n}_y = (0, 1, 0)$ we have in particular

$$|JM\rangle' = D^J_{M',M}(\hat{n}_y 2\pi)|JM'\rangle = (-1)^{2J}|JM\rangle.$$

Thus, we get an extra relative phase of $(-1)^{2J}$ in the linear combination of the two states ψ_1 and ψ_2; hence, if rotational invariance holds, according to our previous discussion, there is a superselection rule between the states with integer J and those with half-odd integer J-values; they cannot mix in physically realizable states.

EXAMPLE 2. SU(2) *Group for Isospin and Superselection Rules.* We give here an example for an approximate symmetry group. It was mentioned in ch. 7, § 4.C that particles of same spin and parity and of roughly equal masses with strong interactions may be grouped into multiplets according to irreducible representations of the group SU(2) (just like spin). The corresponding new quantum numbers are I and I_3, called *isospin*.

If the group $\mathrm{SU}(2)_I$ describing the isotopic spin multiplets of particles were an exact symmetry group of nature in the same way as the group SU(2) for spin, then by the result of example 1 there would be a superselection rule between the integer and half-odd integer I-spin states. Now for strong interactions which are independent of the electric charge, $\mathrm{SU}(2)_I$ is a good symmetry group. This means that there are no *pure* states of the form $|I=1\rangle+|I=1/2\rangle$, $\{$e.g., $|\Sigma\rangle+|\Lambda\rangle\}$.* There are, however, pure states like $|I=1/2\rangle+|I=1/2\rangle$, for example $|n\rangle+|p\rangle$ for strong interactions alone. These superpositions violate, however, the superselection rule on charge (see example 3 below); consequently, there is no superselection rule for charge *for strong interactions alone.* In the presence of electromagnetic and weak interactions, $\mathrm{SU}(2)_I$ is not a symmetry group, but then charge superselection rule holds exactly; a pure state $|\Sigma^0\rangle+|\Lambda^0\rangle$ now exists, but not a state $|n\rangle+|p\rangle$. In fact, an SU(2) -rotation taking n into p does not leave the system unchanged but corresponds to the weak interaction process: $n \to p+e+\bar{\nu}$.

Similarly, if a hypothetical 'superweak interaction' would violate the rotational invariance, then we could have pure states of the form $|j=1/2\rangle+|j=0\rangle$, e.g., $|N\rangle+|\pi\rangle$.

EXAMPLE 3. *Superselection Rules for Gauge Groups.* Two equivalent descriptions obtained from each other by a commutative one-parameter continuous group (not obviously related to space-time transformations) implies the existence of an additive quantum number a, and the eigenstates transform as

$$|q\rangle' = \exp(\mathrm{i}\lambda q)|q\rangle .$$

---

* $|n\rangle$, $|p\rangle$, $|\Lambda\rangle$, $|\Sigma\rangle$, $|\pi\rangle$, ... denote particle states with definite values of isotopic spin I, i.e. the neutron, the proton, the Λ-particle, the sigma particle, the pion, etc.

For two-states with different values of q, e.g. $+1$ and -1, we obtain two different phases $\exp(i\lambda)$ and $\exp(-i\lambda)$, hence a superselection rule for q. The basic *physical* assumption underlying all such superselection rules, such as electric charge, baryon number, lepton number, we repeat, is the requirement that the multiplication of all states by $\exp(i\lambda q)$ produces no observable change in the system, hence equivalent descriptions and gauge groups.

One can form, instead of pure states, *mixed* states out of vectors from different coherent subspace. But this will not interest us here any further.

Implications of the Superselection Rules on Parity and Other Group Extensions

Within a coherent subspace the parity of each state (relative to one of them) is well determined. In fact, we use the ray representations of the full orthogonal group $O(3)$, or the full Lorentz group, including reflections. In this case the parity is defined either in the same representation space as SO(3) (or proper homogeneous Lorentz group) or in a doubled Hilbert space. Thus relative parities are well determined, e.g., for the levels of H-atom and for particle-antiparticle pair in Dirac theory. However, for states in different coherent subspaces, the relative parity is not determined because we cannot take a linear combination of two such states and see how it transforms under parity.

Very similar considerations apply to other group extensions, e.g., by charge conjugation.

The extension of the isotopic spin group SU(2) by a reflection operator implies a doubling of $I = n/2$ states, but not necessarily of $I = n$, $n =$ integer, states. Now this extension is carried out by $C =$ charge conjugation, or by isospin parity $G = Ce^{i\pi}$ also called G-parity. (The use of C or G, respectively, corresponds in the rotation group for spin to the use of reflection operator Σ or the parity P; G commutes with all isospin rotations as P commutes with all space rotations.) G tells us whether we have polar or axial vectors in I-space (e.g., π meson is a polar vector). Therefore, the doubling with G takes us into antiparticles. Consequently, we have among others the result that $I = n/2$ boson-multiplets cannot contain antiparticles; they must lie in the other half of the doubled space. In the limit of an exact $SU(2)_I$ the relative G-parity (isospin-parity) between $I = n/2$ and $I = n$ multiplets is not defined; nor is it defined between states with different charges or baryon numbers. It is defined, however, between, e.g., $I = 1$ multiplet (π) and two $I = \frac{1}{2}$-multiplets with $N = 0$ (e.g., $N\bar{N}$) (cf. ch. 21.4).

See § 4 for another example of the superselection rule, the mass superselection rule in non-relativistic quantum mechanics.

§ 3. Symmetries of Physical Systems

The concept of symmetry is associated with the following statements, which are all different expressions of the same fundamental phenomenon:

(i) It is impossible to know or measure certain quantities e.g., the absolute positions, directions, the absolute left or right, ...

(ii) It is impossible to distinguish between a class of situations, e.g., two identical particles.

(iii) The physical equations (or laws) are independent of some coordinates, e.g., the equations may contain only relative coordinates, and are independent of absolute coordinates.

(iv) The invariance of the equations of physics under a certain group of transformations, e.g., rotational invariance of Newton equations for the Kepler problem.

(v) The existence of certain permanencies pertaining to a system in spite of the constant change of its motion, or state,

(vi) Equivalent descriptions of the same physical system by two observers which are in different states.

If we have an established theory, we can explicitly study its symmetry properties. On the other hand, because symmetries and permanencies are easier to recognize, we can use these properties as requirements in establishing new theories of physical phenomena. How do the group representations enter into the discussion of the symmetry of physical systems?

The symmetry operation takes one state into another possible state which are in the same equivalence class with respect to the symmetry group G. Having recognized an equivalence relation between the phenomena we can classify them into equivalence classes. Thus the dual object $\hat{G}$ to G, i.e. the set of all irreducible representations of G, really enumerates distinct physical objects. For example, from the point of view of relativistic invariance distinct objects are the possible mass and spin values.

A. *Geometric Symmetry Principles*

The geometric invariant principles refer to the description of physical phenomena in space and time.

An *event* is a point P in space-time manifold M with a scale of units λ at P, (M, λ), i.e. we make a *measurement* at space-point x at time t and with a scale $\lambda(t, x)$, e.g. a position measurement relative to an origin with some choice of scale.

The manifold can be given the structure of the Minkowski space, in relativistic theories, or the structure of $R^1(t) \times R^3(x)$ in non-relativistic theories, for a fixed scale.

An *observer* is a local coordinate system (a *chart*) on M. A space-time *interval* is the dimensionless distance between two events measured in some units $dI = (dx, dx)^{1/2}\lambda$. In the case of the Minkowski structure, the distance is given by the indefinite scalar product $(x, x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$, relative to an observer.

An equivalent description by another observer corresponds by the principle of relativity to a map of the space-time-scale manifold into itself which preserves the interval dI. For fixed scale, in the Minkowski space, the group of transformations which preserve the distance $d(x, y) = (x-y, x-y)^{1/2}$ is isomorphic with the semidirect product $T^{3,1} \circledS O(3, 1)$ of the group of translations and full homogeneous Lorentz transformations on the Minkowski space M relative to the natural action of $O(3, 1)$ on M.

A partial ordering of events can be introduced by the relation

$$x > y, \text{ if and only if } x^0 > y^0 \text{ and } (x-y, x-y) > 0, \tag{1}$$

i.e., the event x is 'later' than the event y, and the relative vector $(x-y)$ is time-like. A transformation on the space-time manifold $\varphi : M \to M$ for which eq. (1) implies

$$\varphi(x) > \varphi(y) \tag{2}$$

and vice versa, is called a causal *automorphism* of space-time with respect to a local coordinate system. The causal automorphisms form a group (*causality group*). We have then

THEOREM 1 (Zeeman). *For fixed scale of units, the complete group of causal automorphisms of the Minkowski space is the semidirect product $T^{3,1} \circledS (\Lambda^{\uparrow} \times D)$, where $\Lambda^{\uparrow}$ is the group of orthochronous Lorentz transformations, and D the group of dilatations $x \to \varrho x$, $x \in M$, $\varrho \in$ the multiplicative group of non-zero real numbers.* ▼

We indicate the main steps of the proof. Given a causal automorphism $\varphi: M \to M$ which keeps the origin in M fixed (without loss of generality), we choose four linearly independent light-like vectors l_i, $i = 1, 2, \ldots, 4$, as a basis in M, i.e. $x = x^i l_i$ for all $x \in M$. Let g be the *linear* map in M given by $gx = x^i \varphi(l_i)$. Then it can be proved that φ is linear by showing that $\varphi = g$ by induction on the subspaces M_i spanned by vector l_j, $1 \leqslant j \leqslant i$, for each i. Then because φ preserves the light cone, it follows then that φ belongs to $T^{3,1} \circledS (\Lambda^{\uparrow} \otimes D)$. ▼

Remark: The group of causal automorphisms of M is thus isomorphic to the group of automophisms of the Lie algebra of the Poincaré group. The latter is $T^{3,1} \circledS (\Lambda^{\uparrow} \times D)$ (cf. exercise 1.10.1.11). Another proof of Zeeman's theorem can be given by this isomorphism.

The complete relativistic invariance implies the *full* inhomogeneous Lorentz group, thus includes the discrete transformations of space-reflection, as well as time reversal. The connected component of the identity in homogeneous Lorentz group is the restricted Lorentz group $\Lambda^{\uparrow}_{+}$, where + refers to the condition $\det \Lambda = +1$.

According to sec. 1, we are interested in the projective representations of the group of relativity, or in the representations of the extended groups. The simply connected covering group of $T^{3,1} \circledS SO(3, 1)$ is the group $T^{3,1} \circledS SL(2, C)$, also called the *Poincaré group*.

In non-relativistic theories we replace the full inhomogeneous Lorentz group by the full *Galilei group*. The latter is a *contraction* of the Poincaré group as we showed in example 3.4.2.

If we allow the change of units in measuring the interval of events, and further change the units from point to point in space-time, we arrive at the *group of conformal transformations* of the Minkowski space, which thus contains in addition to the inhomogeneous Lorentz transformations, the dilatations

$$D_1: x'^{\mu} = \varrho x^{\mu} \tag{3}$$

and the special conformal transformations denoted by

$$\begin{aligned} C_4\colon\ x'^{\mu} &= (x^{\mu}+c^{\mu}x^2)/\sigma(x), \\ \sigma(x) &= 1+2c^{\nu}x_{\nu}+c^2x^2. \end{aligned} \tag{4}$$

Equations (4) are *nonlinear* transformations. The 15-parameter conformal group is realized in a nonlinear manner as a group of transformations in the Minkowski space although the Poincaré subgroup is linear. It is possible to introduce a six-dimensional space, and a linear realization in it, because of the isomorphism of the conformal group with the group O (4, 2). It is reasonable to use the six-dimensional manifold as the basic space-time-scale to describe physical systems, the additional two coordinates may be viewed as representing the scale and the change of scale from point to point. As such, this space is the largest possible geometric framework, and its group of motion will be the inhomogeneous conformal group with a covering group

$$T^{4,2} \circledS \mathrm{SU}(2, 2). \tag{5}$$

The group (5) is perhaps the largest *kinematical* group (excluding the curved space of general relativity). For $m = 0$ the usual wave equations are exactly invariant under (5) (cf. § 4). We shall see, however, that much larger groups occur in quantum theory, but not to represent geometric symmetries, but to represent dynamics.

Representations of the Geometric Symmetry Groups

According to the results of sec. 1, we need the unitary representations of the covering groups of geometric symmetry groups, also unitary representations of most of the discrete symmetry groups, except those which contain the time reversal operation T. The discrete operations are best introduced, after the representations of the connected part of the symmetry Lie groups (e.g. Poincaré group) have been determined (cf. ch. 21, § 2).

Because the geometric symmetry groups are in general non-compact, the space of states quantum theory, even for a single particle, is infinite-dimensional; in general, we will have a direct integral of infinite-dimensional representations.

We have noted that the elements of the Lie algebra of the symmetry group have a direct physical meaning, in the sense that we prepare quantum systems initially to be in the eigenstates of a complete set of commuting operators, including the elements of the Cartan subalgebra and those from the enveloping algebra; and these operators have physical names, as linear and angular momentum, spin, helicity, etc. In fact, we begin, in quantum theory, with the representations of the Lie algebra. However, many results show that physics is really using the global representations of the group; i.e., those representations of the Lie algebra, which are integrable.

We know that some of the generators of non-compact symmetry groups are unbounded, and have continuous spectra. Because we do want to label the physical states by these continuous eigenvalues (rather than use always wave packets) and because such states are not in Hilbert space, we see that it is more convenient to base quantum theory on a more general fremework than that given by a Hilbert space. The use of such generalized eigenvectors, their normalization with Dirac δ-functions, etc. is now made mathematically rigorous by the use of distributions and nuclear spectral theory (cf. app. B, § 3).

The representations of the discrete geometric symmetry operations, like parity and time reversal, lead to the concept of central group extensions and the representations of the extended groups (cf. ch. 21, § 4).

Conservation laws or the constants of motion are initimately related with the notion of symmetry. Physically important quantities like energy, momentum, angular momentum, ... are the generators of symmetry transformations. In quantum theory their eigenvalues label the representations of the symmetry groups, hence the physical states.

B. *Symmetry Transformations Which are not of Geometric Origin*

As in geometrical symmetry transformations, the non-geometric symmetry transformations are associated with the impossibility of making certain absolute measurements:

(a) The impossibility of knowing the absolute difference between identical particles leads to symmetry properties of the total wave function of a system of N identical particles. The symmetry group here is the permutation group S_N and the physical requirement is that two states differing in their description by the exchange of identical particles represent the same physics and hence belong to the carrier space of a representation of S_N. As a result of this symmetry, one deduces that, for example, two electrons cannot be simultaneously in the same quantum state. Results following from the permutation symmetry have the quality of dynamical laws. For if we did not know the postulate of indistinguishability of identical particles, we would have been forced to invent fictitious forces between the electrons to prevent them from occupying the same state. This is an instructive example in which certain dynamics is best expressed in terms of a symmetry.

(b) The impossibility of knowing the absolute *sign* of the electric charge in the interactions of a number of charged particles. The physical phenomena depend only on the relative signs of the charges; they do not change if we replace all charges by their negatives. This symmetry is called *charge conjugation.* In the interactions of elementary particles, we have to interpret this symmetry more generally as *particle-antiparticle conjugation*, everywhere each particle is replaced by its antiparticle which has opposite values of *all* additive quantum numbers.

(c) The impossibility of knowing the relative phases between certain states. We know from sec. 1, that absolute value of phases of a state $\psi \in H$ is immaterial. Here we say that even the *relative* phase is sometimes not measurable. We already discussed this symmetry in detail under the name of superselection rules in section 1. As noted there, the superselection rules are associated with the absolute additive quantum numbers, like charge Q, baryon number B, lepton numbers L (and L—the muonic lepton number). These in turn can be represented as generators of one-parameter groups of transformations (abelian gauge groups).

Table I shows the list of symmetry principles in physics. The approximate symmetry groups and dynamical groups are discussed in the next chapter.

Table I
Physical Symmetries and Dynamical Groups

	Conserved Generators or Physical Implications
A. Geometrical Symmetry Groups	
Translations	P_μ
Rotations	J_l
Lorentz or Galileo Transformations	N_l
Parity P	P
Time Reversal T	anti-unitary
B. Group of Scale Transformations	
Dilatations	D
Special Conformal Transformations	K_μ
C. Nongeometrical Symmetries	
Identical Particles	Symmetry types of the wave functions
Gauge Groups (Non-Measurability of Relative Phases)	Charge Q Baryon number B Lepton numbers L
Particle-Antiparticle Conjugation	C
D. General Covariance	Equivalence principle Equations of motion of sources and fields
E. Approximate Dynamical Symmetries, e.g., US(2), SU(3), SU(6), $O(4)$	Multiplets
F. Dynamical Groups	
$O(3, 1)$, $O(4, 2)$	Infinite multiplets
G. Groups of Diffeomorphisms	
Infinite Parameter Groups	Geometrization of dynamics

§ 4. Dynamical Symmetries of Relativistic and Non-Relativistic Systems

According to our discussion in § 2, the dynamical problem in quantum theory amounts to an explicit construction of the concrete linear space (CLS), to identify the states, and to label them with physical observables in order to calculate the transition probabilities. The kinematical symmetries such as the Poincaré or the conformal invariance allow us to label these states by global quantum numbers, e.g. total linear and angular momenta. Consequntly to assure relativistic invariance on the one hand, and to partly determine the concrete linear space on the other hand, it is always appropriate to determine which class of representations of the symmetry group are realized for a given quantum system. For non-relativistic problems the Poincaré invariance goes over into the Galilean invariance (cf. ch. 1, § 8 for the contraction of the Poincaré Lie algebra to the Galilean Lie algebra). It is more convenient however, even for non-relativistic problems to start from the conformal invariance, as we shall see.

We start from the 15 parameter conformal group of the Minkowski space isomorphic to SO(4, 2), introduced in § 3.

PROPOSITION 1. *The Lie algebra basis elements of the conformal group on the space of scalar functions over the Minkowski space are represented by the following differential operators*

$$\begin{aligned} M_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu, \\ P_\mu &= \partial_\mu, \\ K_\mu &= 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu, \\ D &= x^\nu \partial_\nu, \end{aligned} \qquad \mu, \nu = 0, 1, 2, 3 \tag{1}$$

where $M_{\mu\nu}$, P_μ are the basis elements of the Poincaré subalgebra, K_μ the generators of the so-called special conformal transformations and D generates the dilatations.

The proof follows from the definition of the conformal group given by eqs. 3(3), 3(4).

Modifications in the case of spinor or vector valued functions can easily be found (see exercise 4.1).

Remark: Strictly speaking the conformal group (4) is not well defined globally in the Minkowski space. A proper definition needs a compactified form of the Minkowski space. For the properties of the Lie algebra however the Minkowski space can be used without difficulties. ▼

The commutation relations of the conformal Lie algebra are easily derived from eq. (1) to be those of the Poincaré Lie algebra plus the following additional ones:

$$[M_{\mu\nu}, K_\lambda] = g_{\nu\lambda} K_\mu - g_{\mu\lambda} K_\nu,$$

$$[M_{\mu\nu}, D] = 0,$$

$$\begin{aligned} [P_\mu, K_\nu] &= 2(g_{\mu\nu}D - M_{\mu\nu}), \\ [P_\mu, D] &= P_\mu, \\ [K_\mu, K_\nu] &= 0, \\ [K_\mu, D] &= -K_\mu. \end{aligned} \tag{2}$$

The important property of the conformal group from the physical point of view is embedded in the following observation:

PROPOSITION 2. *Relativistic wave equations for massless particles of spin 0 and spin $\frac{1}{2}$ are invariant under the conformal group of the Minkowski space.*

The statement means that if

$$W\varphi = 0 \tag{3}$$

is the wave equation, one can realize the generators L_i of the conformal Lie algebra on the space of solutions such that in general

$$[W, L_i] = \lambda W. \tag{4}$$

Hence L_i acting on the space of solutions of the wave equation takes one solution into another.

PROOF: Consider first the equation for a spinless and massless particle, i.e. the ordinary wave equation

$$\Box\varphi \equiv (\partial_t^2 - \partial_i\partial_i)\varphi = 0. \tag{5}$$

It is easy to verify that K_μ and D given by eq. (1) result in

$$\begin{aligned} [\Box, K_\mu] &= 4x_\mu\Box, \\ [\Box, D] &= 2\Box. \end{aligned} \tag{6}$$

The wave equation for spin $\frac{1}{2}$ can be treated in a similar fashion (cf. exercise 6.4.2).

PROPOSITION 3. *Wave equations for massive particles, e.g.*

$$(\Box - m^2c^2/\hbar^2)\varphi = 0, \tag{7}$$

are formally invariant under the conformal group provided the mass m is transformed as follows:

under dilatations:

$$m^2 \to e^{2\alpha}m^2, \tag{8}$$

under special conformal transformations:

$$m^2 \to \sigma(x)^2 m^2, \qquad \sigma(x) = 1 + 2c^\mu x_\mu + c^2x^2,$$

or, alternatively, m^2 is viewed as an operator having the same commutation relations as $\Box$ in eq. (6). ▼

The proof follows by direct calculation and is straightforward.

Remark: The invariance in proposition 3 is not a symmetry, in the usual sense, for a definite single particle of mass m, because it connects states of particles of mass m with those of another particle of mass m'. However, the transformations (8) have some important applications in physics. Furthermore, if the conformal symmetry is interpreted as a change of scale (cf. § 3), eq. (8) can be interpreted as the change due to the dimension of mass.

We shall now discuss the contraction of the wave equation (7) to the nonrelativistic limit. In this limit the energy is measured after the rest mass is taken out. Hence we set

$$\partial_0 \to mc + \frac{1}{c}\partial_t. \tag{9}$$

Then the generators of the conformal group become

$$\begin{aligned}
P_0 &= mc + \frac{1}{c}\partial_t,\\
P_i &= \partial_i,\\
M_{ij} &= (x_i\partial_j - x_j\partial_i),\\
M_{0i} &= c(t\partial_i - mx_i) - \frac{1}{c}x_i\partial_t,\\
D &= c^2mt + (t\partial_t - x_k\partial_k),\\
K_0 &= c^3mt^2 + c(t^2\partial_t - 2tx_k\partial_k + mx^2) + \frac{1}{c}x^2\partial_t,\\
K_i &= c^2(2x_imt - t^2\partial_i) + (2x_it\partial_t - 2x_ix_k\partial_k + x^2\partial_i).
\end{aligned} \tag{10}$$

At the same time the wave operator becomes

$$\Box - m^2c^2 \to (2m\partial_t - \partial_i\partial_i) + \frac{1}{c^2}\partial_t\partial_t, \tag{11}$$

i.e. the Schrödinger operator plus an additional term of order $1/c^2$.

PROPOSITION 4. *The operators*

$$\begin{aligned}
\bar{P}_0 &= \partial_t,\\
P_i &= \partial_i,\\
M_{ij} &= x_i\partial_j - x_j\partial_i,\\
\bar{M}_{0i} &= (t\partial_i - mx_i)
\end{aligned} \tag{12}$$

commute with the Schrödinger operator $S = (2m\partial_t - \partial_i\partial_i)$ *and generate the Lie algebra of the Galilean group* (cf. ch. 1.8), *except for the following commutator*:

$$[P_i, \bar{M}_{0j}] = -m\delta_{ij} \tag{13}$$

for which the right-hand side was zero in the purely geometric definition of the Galilei group. ▼

The proof of this and the following propositions is again by direct computation and we leave it to the reader.

The difference noted in proposition 4 is due to the replacement (9) and expresses the fact that the mass m is really an operator commuting with all the ten generators of the Galilei group. The solutions of the Schrödinger equation thus realizes a slightly different representation of the Galilei group, namely the projective representation given by eq. (12). Equivalently we can say that eq. (12) is an extension of the Galilean Lie algebra, or its quantum mechanical representation (cf. ch. 21, § 4).

In fact, when we go over to the global form of the representation (12) we obtain an example of a representation up to a factor discussed in § 2 and an example of the superselection rule, namely the

MASS SUPERSELECTION RULE (Bargmann). *For two elements of the Galilei group*

$$g = (b, \boldsymbol{a}, \boldsymbol{v}, R), \quad g' = (b', \boldsymbol{a}', \boldsymbol{v}', R')$$

we have the global representation

$$U(g')\,U(g) = \omega(g', g)\,U(g'g), \tag{14}$$

where the phase $\omega(g', g)$ is given by

$$\omega(g', g) = \exp\left[\mathrm{i}\frac{m}{2}(\boldsymbol{a}'\cdot R'\boldsymbol{v} - \boldsymbol{v}'\cdot R\boldsymbol{a} + b\boldsymbol{v}'\cdot R'\boldsymbol{v})\right]. \blacktriangledown \tag{14'}$$

The superselection rule arises because the following *identity* group element

$$(0, 0, -\boldsymbol{v}, 1)(0, -\boldsymbol{a}, 0, 1)(0, 0, \boldsymbol{v}, 1)(0, \boldsymbol{a}, 0, I) = (0, 0, 0, I)$$

is not represented by $U = I$, but by the phase factor

$$\exp(-\mathrm{i}m\boldsymbol{a}\cdot\boldsymbol{v}). \tag{15}$$

Hence the superposition of two states with different masses m_1 and m_2 transforms into

$$\psi(m_1)+\psi(m_2) \to \exp(-\mathrm{i}m_1\boldsymbol{a}\cdot\boldsymbol{v})\psi(m_1)+\exp(-\mathrm{i}m_2\boldsymbol{a}\cdot\boldsymbol{v})\psi(m_2). \tag{16}$$

This implies that the relative phase is not observable, hence according to our general discussion, the superposition $|m_1\rangle + |m_2\rangle$ is not a realizable state.

PROPOSITION 5. *The Schrödinger operator $S = (2m\partial_t - \partial_i\partial_i)$ is also invariant under a modified dilatation operator $\tilde{D}$ and a modified special conformal transformation with generator $\tilde{K}_0$* (17) *which together with the Galilei Lie algebra form a 12-parameter Lie algebra, called the Schrödinger Lie algebra.* $\blacktriangledown$

The modification comes because, as we have seen, the operators D and K_0 as invariant operators of the wave operator $(\Box - m^2c^2)$ also transform the mass m. Now in the Schrödinger operator $(2m\partial_t - \partial_i\partial_i)$ the mass m occurs as a factor of ∂_t. Hence if we wish to determine the invariance of the Schrödinger operator for fixed m, we can transfer the transformation property of m to ∂_t or to $\partial_i\partial_i$.

This process gives immediately the following expressions for the modified generators

$$\begin{aligned} \tilde{D} &= 2t\,\partial_t + x_k\,\partial_k + 3/2, \\ \tilde{K}_0 &= t^2\,\partial_t + tx_k\,\partial_k + \frac{3}{2}t - \frac{m}{2}x^2. \end{aligned} \tag{17}$$

PROPOSITION 6. *The Schrödinger group generators are represented in the momentum space (in Schrödinger picture of quantum mechanics) by*

$$\begin{aligned} H_0 &= \frac{1}{2m}p^2, \\ \boldsymbol{P} &= \boldsymbol{p}, \\ \boldsymbol{J} &= \boldsymbol{q}\times\boldsymbol{p}, \\ \boldsymbol{M} &= -t\boldsymbol{p}+m\boldsymbol{q}, \\ \tilde{D} &= \frac{t}{m}p^2 + \frac{1}{2}(\boldsymbol{p}\cdot\boldsymbol{q}+\boldsymbol{q}\cdot\boldsymbol{p}), \\ \tilde{K}_0 &= -\frac{t^2}{2m}p^2 + \frac{1}{2}t(\boldsymbol{p}\cdot\boldsymbol{q}+\boldsymbol{q}\cdot\boldsymbol{p}) - \frac{m}{2}q^2. \end{aligned} \tag{18}$$

PROPOSITION 7. *All the generators in the previous proposition including the time dependent ones, satisfy*

$$[H_0, L_A] + \frac{\partial L_A}{\partial t} \equiv \dot{L}_A = 0 \tag{19}$$

and their explicit time dependence is given by

$$L_A(t) = \exp(-\mathrm{i}tH_0)L_A(0)\exp(\mathrm{i}tH_0). \;\blacktriangledown \tag{20}$$

In particular, the time independent generators commute with the Hamiltonian, and generate, what is commonly called a degeneracy group of the Hamiltonian. All operators satisfying (19) generate a symmetry group in the large sense, not of Hamiltonian, but of the time dependent operator $(\mathrm{i}\partial_t - H)$, hence of the quantum mechanical system *per se*.

PROPOSITION 8. *The Schrödinger group has as a subgroup, a dynamical group* SU (1, 1), *generated by* $\tilde{D}$, $\tilde{K}_0$, H, *or*

$$\begin{aligned} L_1(t) &= \frac{1}{2}(\tilde{K}_0 + H_0), \\ L_2(t) &= -\frac{1}{2}D, \\ L_3(t) &= -\frac{1}{2}(\tilde{K}_0 - H_0). \end{aligned} \tag{21}$$

Here L_3 is the compact generator of SU(1, 1) *with a discrete spectrum.*

In particular, the algebra at $t = 0$ is

$$L_1(0) = \frac{1}{2}\left(\frac{p^2}{2m} - \frac{m}{2}q^2\right),$$

$$L_2(0) = -\frac{1}{2}(p\cdot q+q\cdot p), \tag{22}$$

$$L_3(0) = -\frac{1}{2}\left(\frac{p^2}{2m} + \frac{m}{2}q^2\right). \blacktriangledown$$

PROPOSITION 9. *The Lie algebra* (22) *of proposition* 8 *solves the dynamical problem for the* 3*-dimensional quantum oscillator with the Hamiltonian*

$$H = \frac{p^2}{2m} + \lambda q^2 \tag{23}$$

and the free particle with Hamiltonian $H = L_1 - L_3 = \dfrac{p^2}{2m}$. ▼

The proof consist in the observation that in suitable units, the Hamiltonian (23) is identical with the compact generator of the su(1, 1) algebra (22). The Casimir operator

$$C_2 = L_3^2 - L_1^2 - L_2^2$$

can be evaluated in the representation (22):

$$C_2 = \frac{1}{4}(q\times p) - \frac{3}{16}.$$

Thus we know the representation of the Lie algebra, hence the spectrum of H, as well as the states of the oscillator.

Remark: Note that a free particle and the quantum mechanical oscillator are realized on the same representation space of the group SU(1,1). The difference is that the energy eigenstates are obtained by diagonalizing the compact generator L_3 in the oscillator case, but the non-compact generator $(L_1 - L_3)$ in the case of the free particle. The latter operator has of course a continuous spectrum. But even for a free particle there exists an operator with a discrete spectrum. This fact illustrates an important point in quantum theory: The physical identification of the Lie algebra elements is essential in quantum theory; we do not use abstract groups rather groups with definite identifications.

§ 5. Comments and Supplements

Historical Remarks

Quantum theory, developed by Heisenberg, Schrödinger, Dirac, Pauli, Born, Jordan, and others, found its first mathematical formulation by von Neumann, who gave an axiomatic Hilbert space formulation and proved the uniqueness and equivalence of the Heisenberg and Schrödinger formulations. This equivalence was also proved by Pauli and Lanczos. The applications of group representations to quantum theory were originated by Wigner. The formulation of relativistic invariance in quantum theory is due to Dirac and Wigner; the latter gave the first complete discussion of the representations of the Poincaré group (1939). The group theoretical discussion of wave equations is due to Bargmann and Wigner 1948 (cf. chs. 16-21).

The concept of superselection rules was introduced by Wick, Wightman and Wigner 1952. The representations of symmetry groups by unitary or anti-unitary operators in Hilbert space (Wigner theorem) has been elaborated by Wigner, Bargmann 1964 and, more generally, by Emch and Piron 1963 and Uhlhorn 1963.

The invariance of classical Maxwell's equations under the conformal group goes back to Bateman 1910 and Cunningham 1910. The theory of conformally invariant wave equations goes back to Dirac 1936. Dynamical groups were introduced by Barut 1964.

§ 6. Exercises

§ 3.1. Show that the dilatations and the non-linear special conformal transformations eqs. 3 (3), 3 (4) can be written as linear transformations in the six-dimensional space of the form

$$\begin{aligned} D_1: \quad & \eta'^{\mu} = \eta^{\mu}, \ k' = \varrho^{-1}k, \ \lambda' = \varrho\lambda, \\ C_4: \quad & \eta'^{\mu} = \eta^{\mu} + c^{\mu}\lambda, \\ & k' = -2c_\nu\eta^\nu + k + c^2\lambda, \\ & \lambda' = \lambda, \end{aligned}$$

where $\eta^\mu = kx^\mu$, k and $\lambda = kx^2$ are taken to be the six new coordinates.

§ 3.2. Show that the Maxwell equations

$$\operatorname{div}\boldsymbol{E}(x) = \varrho(x),$$

$$\operatorname{div}\boldsymbol{H}(x) = 0,$$

$$\operatorname{curl}\boldsymbol{H}(x) - \frac{1}{c}\frac{\partial \boldsymbol{E}(x)}{\partial t} = \frac{1}{c}\varrho(x)\boldsymbol{u}(x) = \frac{1}{c}\boldsymbol{j}(x),$$

$$\operatorname{curl}\boldsymbol{E}(x) + \frac{1}{c}\frac{\partial \boldsymbol{H}(x)}{\partial t} = 0,$$

where $x = (t, \boldsymbol{x})$, are not invariant under the Galilei transformations

$$t' = t,$$

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t.$$

§ 3.3. Show that in the Minkowski space M^2 (with one space and one time dimensions) the causality group is much larger than $T^{1,1} \rtimes (\mathrm{SO}(1,1) \otimes D)$. In particular non-linear transformations which map space and time lines into curved lines would be allowed.

§ 3.4. On the Minkowski space M^4 define the relation xLy, if $(x-y)$ is an oriented light-like vector: $x^0 > y^0$, $(x-y)^2 = 0$. Let $\varphi : M \to M$ be a one-to-one mapping. Show that φ preserve the partial ordering $x > y$ if and only if it preserves the relation xLy. Note that the relation xLy is not a partial ordering because it is not transitive. Show also that a causal automorphism maps light rays into light rays.

§ 4.1. Show that for a Dirac particle the generators of the conformal group (corresponding to eq. (4.1)) are

$$M_{\mu\nu} = \mathring{M}_{\mu\nu} + \frac{\mathrm{i}}{2}\gamma_\mu \gamma_\nu, \quad \mu < \nu,$$

$$P_\mu = \mathring{P}_\mu - \frac{1}{2}\gamma_\mu (1+\gamma_5),$$

$$K_\mu = \mathring{K}_\mu - \frac{1}{2}\gamma_\mu (1-\gamma_5),$$

$$D = \mathring{D} - \frac{1}{2}\gamma_5,$$

where $\mathring{M}_{\mu\nu}$, $\mathring{P}$, ... are given in eq. (4.1).

§ 4.2. The wave operator for a massless Dirac particle is $\gamma^\mu \partial_\mu$. Show that the wave equation $\gamma^\mu \partial_\mu \psi = 0$ is invariant under conformal transformations in the sense of eq. (4.4), using the representation given in the previous exercise.

§ 4.3. Discuss the conformal group in M^2, two-dimensional space-time.

§ 4.4.* Show the conformal invariance of massless spin 1 wave equations in fact of all massless wave equations.

§ 4.5.* Let M^n, $n = 2, 3, \ldots$ be the Minkowski space. Show that the conformal invariance restricts the interaction term $F(\varphi)$ of the nonlinear relativistic equation

$$\Box \varphi = F(\varphi)$$

to the form $F(\varphi) = \lambda \varphi^{(n+2)/(n-2)}$. Note that the resulting interaction is, on the second quantized level, renormalizable, but not superrenormalizable.

§ 4.6.*** Derive the similar result for the nonlinear Dirac equation

$$\partial_\mu \gamma^\mu \psi = F(\overline{\psi}, \psi).$$

Hint: Use only invariance under dilatation $x \to \varrho x$, $\varrho > 0$.

Chapter 14

Harmonic Analysis on Lie Groups. Special Functions and Group Representations

Besides the fundamental role of the theory of group representations in formulating the basic equations of physics, we must also mention the important method of *harmonic analysis* in the solution of dynamical problems.

In many physical problems we deal with functions over the homogeneous or symmetric spaces, in particular, on group spaces. For example, functions over the so-called mass-hyperboloid in momentum space: $\varphi(p_\mu), p_\mu^2 = m^2$. The argument in this case is an element of the homogeneous space SO(3, 1)/SO(3). These functions can be decomposed over the set of eigenfunctions of Casimir operators. Such decompositions are extremely powerful and have a physical interpretation. They also form a basis for approximations when in suitable cases only few terms of the expansion are important. The expansions in terms of the special functions of mathematical physics can be reformulated in terms of harmonic analysis on homogeneous spaces. These problems will be treated in detail in chs. 14 and 15.

Let G be a unimodular Lie group with a Haar measure μ and let $H = L^2(G, \mu)$. We restrict our analysis to type I groups only. The main purpose of harmonic analysis in H is to solve the following problems:

(i) To determine a basis* $\{e_k(\lambda, g)\}$ in H and a dense subspace $\Phi \subset H$ such that if the generalized Fourier transform of $\varphi \in \Phi$ is given by the formula

$$\hat{\varphi}_k(\lambda) = (\varphi, e_k(\lambda)), \tag{1}$$

then the spectral synthesis formula for φ is given by

$$\varphi(g) = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_k \hat{\varphi}_k(\lambda) e_k(\lambda, g). \tag{2}$$

* We assume that the index λ corresponds to the set of eigenvalues of invariant operators of G and k corresponds to the set of eigenvalues of the remaining operators, which together with the invariant operators form a maximal set of commuting operators in H. For convenience we use this notation as though $\{\lambda\}$ would be a continuous set and $\{k\}$ would be a discrete one. However in general both sets might be discrete, continuous or mixed.

(ii) To establish the Plancherel equality*

$$(\varphi, \psi) = \int_{\Lambda} d\varrho(\lambda) \sum_{k} \hat{\varphi}_k(\lambda)\overline{\hat{\psi}_k(\lambda)}. \tag{3}$$

(iii) To construct explicitly the measure $d\varrho(\lambda)$ (Plancherel measure).

The main difficulty in the harmonic analysis on Lie groups is associated with the fact that in most cases a maximal set of commuting operators in H, which determines the basis $e_k(\lambda, g)$ contains unbounded operators with continuous spectra and therefore the eigenvectors $e_k(\lambda, g)$ are distributions. Consequently, in order to give a proper interpretation to the functions $e_k(\lambda, g)$ and to the eigenfunction expansion (2), we have to deal with the so-called *Gel'fand triplet* $\Phi \subset H \subset \Phi'$ rather than with a simple Hilbert space H. In this triplet Φ is a certain nuclear space of smooth functions dense in H, and Φ' is the dual space to Φ. It is evident, therefore, that a natural framework for harmonic analysis on Lie groups is via the nuclear spectral theory. This theory allows a clear and elegant formulation of harmonic analysis on groups. At the same time, the theory provides a generalization of the classical Fourier analysis, and is useful for applications in quantum physics, where the concepts of eigenfunctions and eigenfunction expansions play the central role.

§ 1. Harmonic Analysis on Abelian and Compact Lie Groups

We shall first give the extension of ordinary Fourier analysis on R^n to an arbitrary abelian Lie group.

THEOREM 1. *Let G be an arbitrary abelian Lie group and let $H = L^2(G, \mu)$, where μ is the Haar measure on G. Let $g \to T_g$ be the regular representation of G in H given by*

$$T_g u(\tilde{g}) = u(\tilde{g}+g). \tag{1}$$

Then

(i) *There exists a generalized Fourier transform F such that*

$$\begin{aligned} F\colon H \to FH \equiv \hat{H} &= \int_{\Lambda} \hat{H}(\lambda) d\varrho(\lambda), \\ F\colon T_g \to FT_gF^{-1} \equiv \hat{T}_g &= \int_{\Lambda} \hat{T}_g(\lambda) d\varrho(\lambda), \end{aligned} \tag{2}$$

where $\hat{H}(\lambda)$ and $\hat{T}_g(\lambda)$ are ϱ-a.a. irreducible and $\dim \hat{H}(\lambda) = 1$. *The spectrum Λ coincides with the character group $\hat{G}$ of G.*

(ii) *There exists a Gel'fand triplet $\Phi \subset H \subset \Phi'$ and a basis $e(\lambda, g)$ in $H(\lambda)$ such that for every element X in the enveloping algebra E for ϱ-a.a. λ, we have***

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* We follow the convention adopted in mathematical literature and take the scalar product in H which is linear with respect to the first factor and antilinear with respect to the second one.

** For the definition of the Gel'fand triplet, the formulation of nuclear spectral theorem and notation see app. B, § 3.

$$\langle \overline{T(X)}\varphi, e(\lambda)\rangle = \hat{X}(\lambda)\langle \varphi, e(\lambda)\rangle, \tag{3}$$

where $\hat{X}(\lambda)$ *is a real number. The basis elements* $e(\lambda, g)$ *are regular functions on* G.

(iii) *The spectral synthesis formula has the form*

$$\varphi(g) = \int d\varrho(\lambda)\hat{\varphi}(\lambda)e(\lambda g), \quad \varphi \in \Phi, \tag{4}$$

where

$$\hat{\varphi}(\lambda) = \int \varphi(g)\overline{e(\lambda, g)}d\mu(g), \quad \hat{\varphi}(\lambda) \in \hat{H}(\lambda). \tag{5}$$

(iv) *For* $\varphi, \psi \in \Phi$ *the Plancherel equality has the form*

$$\int_G \varphi(g)\overline{\psi(g)}d\mu(g) = \int_\Lambda \hat{\varphi}(\lambda)\overline{\hat{\psi}}(\lambda)d\varrho(\lambda). \tag{6}$$

PROOF: ad (i). An abelian Lie group is of type I: hence th. 5.6.3 implies the decomposition (2). By virtue of proposition 6.1.1 every irreducible component $\hat{T}_g(\lambda)$ is one-dimensional.

ad (ii)–(iv). Let $D_G \subset H$ be the Gårding domain for the enveloping algebra E of G. Because the elliptic operator $T(\Delta) = \sum_{i=1}^{\dim G} T(X_i)^2$ commutes with all the elements $T(X), X \in E$ of G, by virtue of th. 11.2.3 the closure $\overline{T(X)}$ of a symmetric element $X^+ = X \in E$ of $T(X)$ is a self-adjoint operator. By virtue of th. 11.5.3, all operators $\overline{T(X)}$, $X \in E$, are mutually commuting and commute also with all T_g, $g \in G$. Let $\{X_i\}_{i=1}^{\dim G}$ be a basis in the Lie algebra L of G. Then the self-adjoint operators $\overline{T(X_i)}$ provide a maximal set of commuting operators in H and the elliptic Nelson operator $\overline{T(\Delta)}$ is also diagonal. Consequently all assertions ad (ii)–ad(iv) follow from the nuclear spectral theorem. ▼

The measure $d\varrho(\lambda)$ on the spetral set $\Lambda = G$ in eq. (2), is called the *Plancherel measure*.

Remark 1: By virtue of eq. app. B.3(27), eq. (3) may be written in the form

$$\overline{T(X)}' e(\lambda, g) = \hat{X}(\lambda)e(\lambda, g), X \in L, \tag{7}$$

where $\overline{T(X)}'$ is the extension of $\overline{T(X)}$ obtained by extending the domain $D(\overline{T(X)})$ by those elements φ' in Φ' for which the equality

$$\langle \overline{T(X)}\varphi, \varphi'\rangle = \langle \varphi, \overline{T(X)}'\varphi'\rangle, \quad \varphi \in \Phi, \ \varphi' \in \Phi', \tag{8}$$

is satisfied.

Remark 2: Since the Plancherel measure on the spectral set $\Lambda = \hat{G}$ is absolutely continuous relative to the Lebesgue measure $d\lambda$ (i.e., $d\varrho(\lambda) = \varrho(\lambda)d\lambda$, $\varrho(\lambda)$ continuous on Λ), by virtue of eq. app. B.3(29) we obtain the following orthogonality relation for generalized eigenvectors

$$\int_G e(\lambda, g)\overline{e(\lambda', g)}d\mu(g) = \varrho^{-1}(\lambda)\delta(\lambda - \lambda'). \tag{9}$$

This is a generalization of the well-known orthogonality relation in ordinary Fourier analysis on R:

$$\int_{R^n} \exp(i\lambda x)\overline{\exp(i\lambda' x)}\,\mathrm{d}^n x = (2\pi)^n \delta^{(n)}(\lambda-\lambda'). \tag{10}$$

In both cases these integrals are understood as weak integrals of the regular distributions $e(\lambda, g)\,\overline{e(\lambda', g)}$ on G.

We see therefore that the nuclear spectral theory provides a direct extension of harmonic analysis on R^n to arbitrary abelian Lie groups.

In the case of compact groups the harmonic analysis in the Hilbert space $H = L^2(G, \mu)$, μ – normalized Haar measure on G, is essentially given by the Peter–Weyl theorem 7.2.1, which states that an arbitrary function $u(g) \in H$ may be represented in the form

$$u(g) = \sum_{\lambda, p, q} \hat{u}_{pq}(\lambda) D^{\lambda}_{pq}(g), \tag{11}$$

where $\Lambda = \{\lambda\}$ is the dual object $\hat{G}$ of G and $D^{\lambda}_{pq}(g)$ are the matrix elements of the irreducible representation T^{λ} of G. The generalized Fourier transform $\hat{u}_{pq}(\lambda)$ of $u \in H$ is given by the formula 7.2(6)

$$\hat{u}_{pq}(\lambda) = d^{\lambda} \int_G u(g)\overline{D^{\lambda}_{pq}(g)}\,\mathrm{d}\mu(g), \tag{12}$$

where d^{λ} = dimension of the representation T^{λ} of G. The matrix elements $D^{\lambda}_{pq}(g)$ satisfy the following orthogonality and the completeness relations:

$$\int_G D^{\lambda}_{pq}(g)\overline{D^{\lambda'}_{p'q'}(g)}\,\mathrm{d}\mu(g) = \frac{1}{d^{\lambda}}\delta^{\lambda\lambda'}\delta_{pp'}\delta_{qq'}, \tag{13}$$

$$\sum_{\lambda, p, q} d^{\lambda} D^{\lambda}_{pq}(g)\overline{D^{\lambda}_{pq}(g')} = \delta(g - gg'^{-1}) \tag{14}$$

(cf. eq. 7.1 (9) and 7.2 (20)).

§ 2. Harmonic Analysis on Unimodular Lie Groups

The simplicity of harmonic analysis on compact groups was associated with the fact that the commutant T' of an arbitrary representation T of G was generated by a compact self-adjoint operator K_u given by eq. 7.1(4). Because every compact operator has only discrete spectrum the decomposition of an arbitrary function was given in the form of a discrete sum 1(11). In addition, the basic functions $D^{\lambda}_{pq}(g)$ which provide an expansion of an arbitrary function $u \in L^2(G, \mu)$ were matrix elements of irreducible representations T^{λ} of G and satisfied the orthogonality and completeness relations (1(13) and 1(14)).

In the case of an arbitrary Lie group the commutant T' of the regular representation T of G might contain operators with continuous spectra. Hence, in general, one will obtain a direct integral decomposition of both the representations in $H = L^2(G, \mu)$ and functions $u \in H$. This is a typical feature of non-compact groups. In addition, the orthogonality and the completeness relations 1(13) and 1(14) will hold only in special cases and will need an additional interpretation as products of distributions. Because the eigenfunctions of operators with continuous spectra are not elements of $L^2(G, \mu)$, but only linear functionals over a dense space $\Phi \subset H$ of smooth functions, we have to deal with the triplet $\Phi \subset H \subset \Phi'$ rather than with a single space $H = L^2(G, \mu)$. The elegant and effective formalism for dealing with continuous spectra of self-adjoint operators is provided by the nuclear spectral theory presented in app. B, § 3. This theory provides a satisfactory framework for an extension of harmonic analysis from abelian and compact groups to the case of noncompact Lie groups.

Let G be a unimodular Lie group. We first give a description of invariant operators which generate the center of the commutant T' of the regular representation T in the Hilbert space $H = L^2(G, \mu)$.

Let $g \to T_g^L$ and $g \to T_g^R$ be the left and the right regular representations of G in H, i.e.,

$$T_g^L u(\tilde{g}) = u(g^{-1}\tilde{g}), \quad T_g^R u(\tilde{g}) = u(\tilde{g}g), \quad u \in H. \tag{1}$$

Denote by $\mathscr{R}_L$ (or $\mathscr{R}_R$) the closure, in the weak operator topology of $L(H)$, of the set of all linear combinations of the T_g^L (or T_g^R). Then by virtue of Segal's theorem 9.6.3 we have

$$\mathscr{R}_L' = \mathscr{R}_R, \quad \mathscr{R}_R' = \mathscr{R}_L \tag{2}$$

and

$$(\mathscr{R}_L \cup \mathscr{R}_R)' = \mathscr{R}_L' \cap \mathscr{R}_R', \tag{3}$$

i.e. the commutant of the algebra $\mathscr{R}_L \cup \mathscr{R}_R$ is the intersection of the center of the algebra $\mathscr{R}_L'$ and the center of $\mathscr{R}_R'$. Segal's theorem shows the important fact that in the space $L^2(G, \mu)$ we have no other invariant operators besides those associated with the algebra of spectral resolutions of two-sided invariant operators.

In order to determine the generalized Fourier expansion for non-compact groups, in analogy to the compact case, we introduce an additional set of non-invariant operators in the carrier space H. To carry out this, we first construct the Gårding domain D_G for the elements of the enveloping algebras E^L and E^R of G.

Let $\{g_1, g_2\} \to T_{\{g_1, g_2\}}$ be a unitary representation of $G \times G$ in $H = L^2(G, \mu)$ given by

$$T_{\{g_1, g_2\}} u(g) = u(g_1^{-1} g g_2). \tag{4}$$

It is evident that $T_g^L = T_{\{g, e\}}$ and $T_g^R = T_{\{e, g\}}$. (5)

The Gårding subspace D_G associated with the representation $T_{\{g_1, g_2\}}$ consists of the elements $u(\varphi)$, $\varphi \in C_0^\infty(G\times G)$, of the form

$$u(\varphi) = \int\limits_{G\times G} \varphi(g_1, g_2) T_{\{g_1, g_2\}} u \,\mathrm{d}\mu(g_1)\mathrm{d}\mu(g_2),\ u \in H. \tag{6}$$

It provides a common, dense, linear, invariant domain for all operators in the enveloping algebra of $G\times G$. Thus by virtue of eq. (4) it provides also a common, dense, linear invariant domain for the left- and right-invariant enveloping algebras E^L and E^R of G respectively.

Let $\{A_j\}_{j=1}^m$ be a maximal set of self-adjoint mutually commuting operators in the right-invariant enveloping algebra E^L of G. Clearly all operators A_j commute with two-sided invariant operators C_i, $i = 1, 2, \ldots, n$. Let $\{B_k\}_{k=1}^m$ be the corresponding maximal set of self-adjoint mutually commuting operators in the left-invariant enveloping algebra E^R of G. The operators B_k commute with all C_i as well as with all A_j. Clearly we can select operators B_k in such a manner that the algebraic form of B_k is the same as that of A_k, $k = 1, 2, \ldots, m$. Then B_k and A_k are related by a unitary transformation. In fact, let J denote the involution operator in H defined by

$$(Ju)(g) \equiv u(g^{-1}). \tag{7}$$

This operator is unitary by virtue of the invariance of the Haar measure. Moreover,

$$JT_g^L J^{-1} = T_g^R. \tag{8}$$

Equation (8) implies

$$JX_i J^{-1} = Y_i, \tag{9}$$

where X_i and Y_i are generators of one-parameter subgroups $T^L_{g(t_i)}$ and $T^R_{g(t_i)}$, respectively. Next, if

$$M^R \equiv \sum a_{\alpha_1\ldots\alpha_k} X_{\alpha_1} \ldots X_{\alpha_k} \in E^R$$

and

$$M^L \equiv \sum a_{\alpha_1\ldots\alpha_k} Y_{\alpha_1} \ldots Y_{\alpha_k} \in E^L,$$

then we have also

$$JM^L J^{-1} = M^R. \tag{10}$$

This equation implies that the spectrum p_i of an operator A_i and the spectrum q_i of the corresponding operator B_i coincide. Consequently, the range of the set $p = \{p_1, \ldots, p_{l(s)}\}$ and $q = \{q_1, \ldots, q_{l(s)}\}$ is the same. In addition, if operators A_j are related with compact subgroups of G, then all indices p and q are discrete.

The sets $\{A_j\}_1^m$ and $\{B_k\}_1^m$ may often be associated with a specific sequence of successive maximal subgroups of G. For instance, if $G = \mathrm{SO}(p, 1)$, then the sets $\{A_j\}_1^m$ and $\{B_k\}_1^m$ may be associated with the set of Casimir operators of successive maximal subgroups $\mathrm{SO}(p) \supset \mathrm{SO}(p-1) \supset \ldots \supset \mathrm{SO}(2)$. If we take

$C_2(\mathrm{SO}(p,1))$ and $C_2(\mathrm{SO}(p))$ to our system of commuting operators, then by virtue of corollary 2 to th. 11.2.3, the closures of all operators $\{C_i\}_1^{[p/2]}$, $\{A_j\}_1^m$ and $\{B_k\}_1^m$ are self-adjoint.

Because we have two sets $\{A_j\}_1^m$ and $\{B_k\}_1^m$ of operators, in addition to the set $\{C_i\}_1^n$ of two-sided invariant operators, we shall denote the eigenvectors by the symbol $e_{pq}(\lambda, g)$, where the multiindex $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ is associated with the spectrum of two-sided-invariant operators, the multiindex $p = \{p_1, \ldots, p_m\}$ corresponds to the set of eigenvalues of operators $A_1, \ldots, A_m$, and $q = \{q_1, \ldots, q_m\}$ is the multiindex associated with a set of eigenvalues of operators $B_1, \ldots, B_m$. For instance, in the case of $G = \mathrm{SL}(2, C)$ one could take $G_1 = \mathrm{SU}(2)$ $G_2 = U(1)$; in this case $\lambda = (\varrho, m)$ are eigenvalues of two-sided-invariant operators C_2 and C_2' of $\mathrm{SL}(2, C)$, $p = \{J, M\}$ and $q = \{J', M'\}$.

We use in the following for the sake of simplicity a continuous direct sum notation for direct integral decomposition associated with the invariant operators $\bar{C}_i$ and the direct sum notation for decompositions associated with operators A_i and B_i, $i = 1, 2, \ldots, m$ although in general these operators might have continuous, discrete or mixed spectra.

The following theorem represents a generalization of the Peter–Weyl theorem for non-compact groups.

THEOREM 1. *Let G be a unimodular Lie group, $H = L^2(G, \mu)$, and let $g \to T_g$ be the regular representation of G in H given by eq. (4). Let $\{C_i\}_1^n$ be the maximal set of algebraically independent '+'-symmetric two-sided-invariant operators in the center Z of the enveloping algebra E, and $\{A_i\}_1^m$ and $\{B_i\}_1^m$ be the maximal set of right- and left-invariant, respectively, self-adjoint commuting operators in $E^L \oplus E^R$. Then*

(i) *There exists a direct integral decomposition*

$$H \to \hat{H} = \int_\Lambda \hat{H}(\lambda)\, d\varrho(\lambda), \quad T_g \to \hat{T}_g = \int_\Lambda \hat{T}_g(\lambda)\, d\varrho(\lambda) \tag{11}$$

of H and T_g such that $(\hat{H}(\lambda), \hat{T}_g(\lambda))$ are ϱ-a.a. irreducible.

(ii) *There exists a Gel'fand triplet $\Phi \subset H \subset \Phi'$ and a basis $e_{pq}(\lambda, g)$ in $H(\lambda)$ such that for $\varphi \in \Phi$ we have*

$$\langle C_i\varphi, e_{pq}(\lambda)\rangle = \hat{C}_i(\lambda_i)\langle \varphi, e_{pq}(\lambda)\rangle, \tag{12}$$

$$\langle A_j\varphi, e_{pq}(\lambda)\rangle = \hat{A}_j(p_j)\langle \varphi, e_{pq}(\lambda)\rangle, \tag{13}$$

$$\langle B_k\varphi, e_{pq}(\lambda)\rangle = \hat{B}_k(q_k)\langle \varphi, e_{pq}(\lambda)\rangle. \tag{14}$$

(iii) *For $\varphi \in \Phi$ the spectral synthesis formula has the form*

$$\varphi(g) = \int_\Lambda d\varrho(\lambda) \sum_{p,q=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda) e_{pq}(\lambda, g), \tag{15}$$

where

$$\hat{\varphi}_{pq}(\lambda) = \langle \varphi, e_{pq}(\lambda)\rangle. \tag{16}$$

(iv) *For $\varphi, \psi \in \Phi$ the Plancherel equality has the form*

$$\int_G \varphi(g)\overline{\psi(g)}\mathrm{d}\mu(g) = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{p,q=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda)\overline{\hat{\psi}_{pq}(\lambda)}. \tag{17}$$

PROOF: Let $D_G \subset H$ be the Gårding domain for operators C_i, A_j and B_k, whose elements are given by eq. (6).

By virtue of th. 11.2.3 we conclude that the closures $\overline{C}_i$ of C_i are self-adjoint operators on D_G. Using then th. 11.5.3, we conclude that all C_i, $i = 1, 2, \ldots, n$, are mutually strongly commuting and also commute with all T_g, $g \in G$.

Now let A be a maximal '*'-algebra in the commutant T' of T which contains all spectral resolutions of $\overline{C}_i$, $i = 1, 2, \ldots, n$. Then using the Mautner theorem 5.6.1 we obtain the decomposition (11).

The assertions (ii)–(iv) follow from the nuclear spectral theorem. ▼

The measure $\mathrm{d}\varrho(\lambda)$ on the spectral set Λ in eq. (11) is called the *Plancherel measure* and is determined by the spectral measure of the Casimir operators C_i.

Remark 1: By virtue of eq. app. B.3(27) eqs. (12)–(14) may be written in the form

$$\overline{C}_i' e_{pq}(\lambda, g) = \hat{C}_i(\lambda_i) e_{pq}(\lambda, g), \tag{18}$$

$$A_j' e_{pq}(\lambda, g) = \hat{A}_j(p_j) e_{pq}(\lambda, g), \tag{19}$$

$$B_k' e_{pq}(\lambda, g) = \hat{B}_k(q_k) e_{pq}(\lambda, g), \tag{20}$$

where, e.g., $\overline{C}_i'$ is the extension of $\overline{C}_i$ obtained by extending the domain $D(\overline{C}_i)$ by those elements φ' in Φ' for which the equality

$$\langle \overline{C}_i\varphi, \varphi'\rangle = \langle \varphi, \overline{C}_i'\varphi'\rangle, \qquad \varphi \in \Phi$$

is satisfied.

Remark 2: In many cases the set Λ coincides with R^n or is a regular subset of it on which Lebesgue measure is well defined.

If the spectral measure $\mathrm{d}\varrho(\lambda)$ in the decomposition (11) is absolutely continuous relative to the Lebesgue measure $\mathrm{d}\lambda$ (i.e., $\mathrm{d}\varrho(\lambda) = \varrho(\lambda)\mathrm{d}\lambda$, $\varrho(\lambda)$ continuous on Λ), then eq. app. B.3(29) provides the following orthogonality relation for the generalized eigenvectors $e_{pq}(\lambda, g)$

$$\int_G e_{pq}(\lambda, g)\overline{e_{p'q'}(\lambda', g)}\mathrm{d}\mu(g) = \varrho^{-1}(\lambda)\,\delta(\lambda - \lambda')\,\delta_{pp'}\,\delta_{qq'}. \tag{21}$$

These formulas represent an alternative way of writing the Plancherel equality (17) and are understood as weak integrals of the distributions

$$e_{pq}(\lambda, g)\overline{e_{p'q'}(\lambda', g)} \quad \text{defined on } G.$$

Notice that a generalized Fourier component $\hat{\varphi}_{pq}(\lambda)$ of an element $\varphi \in \Phi$ given by eq. (16) cannot be represented in general as an integral of the form 1(5). However, we have

PROPOSITION 2. *Let the set $\{C_i\}_1^n, \{A_j\}_1^m, \{B_k\}_1^m$ of differential operators contain an elliptic operator. Then all eigenvectors are regular functions on G. For $\varphi \in \Phi$ we have*

$$\hat{\varphi}_{pq}(\lambda) = \int_G \varphi(g)\overline{e_{pq}(\lambda, g)}\,\mathrm{d}\mu(g). \tag{22}$$

We have also the following completeness relation

$$\int \Big(\sum_{pq} e_{pq}(\lambda, g)\overline{e_{pq}(\lambda, g')}\Big)\,\mathrm{d}\mu(\lambda) = \delta(gg'^{-1}). \tag{23}$$

PROOF: Equation (22) and eq. (23) follow from assertion 3(22) and 3(24) of app. B respectively, of nuclear spectral theorem. The integral (23) is understood in the sense of the weak integral of the regular distributions.

Remark 1: If G is a semisimple Lie group which contains among its maximal subgroups a maximal compact subgroup K (like, e.g., $\mathrm{SO}(n, 1)$), $\mathrm{SU}(n, 1)$, $\mathrm{Sp}(n, 1)$), then the assumption of proposition 2 can be easily satisfied. Indeed, it is sufficient to take in this case the sets $\{A_j\}_1^m$ and $\{B_k\}_1^m$ to be the operators associated with Casimir operators of the successive maximal subgroups $G_1 \supset G_2 \supset \ldots \supset G_s$, where $G_1 \doteq K$. Then, because the elliptic Nelson operator Δ satisfies the equality

$$\Delta(G) = \sum_{i=1}^{\dim G} X_i^2 = C_2(G) + 2C_2(K),$$

it is simultaneously diagonalized together with operators $C_2(G)$ and $C_2(K)$ which enter into the maximal set of commuting operators in $L^2(G, \mu)$. Hence for this class of groups the Fourier transform $\hat{\varphi}_{pq}(\lambda)$ and completeness relation have the explicit forms (22) and (23), respectively.▼

We assume in the following that the eigenfunctions $e_{pq}(\lambda, g)$ are regular functions (cf. proposition 2) and we normalize them in the following manner:

$$e_{pq}(\lambda, e) = \delta_{pq}, \tag{24}$$

where e is the unity of G. The following proposition shows that the eigenfunctions $e_{pq}(\lambda, g)$ are in fact matrix elements of irreducible representations. Indeed, we have

PROPOSITION 3. *The eigenfunctions $e_{pq}(\lambda, g)$ satisfy the following relations:*

$$e_{pq}(\lambda, g^{-1}) = \overline{e_{qp}(\lambda, g)}, \tag{25}$$

$$e_{pq}(\lambda, g_1 g_2) = e_{pr}(\lambda, g_1)e_{rq}(\lambda, g_2). \tag{26}$$

PROOF: Let us perform a 'rotation' in the space $H \doteq L^2(G, \mu)$ by means of the operator T_g^R. Then, because the operators $\overline{T(C_l)}$ are two-sided invariants and $\overline{T(A_j)}$ are right-invariant, we have

$$T_g^R \overline{C}_i T_{g^{-1}}^R = \overline{C}_i, \quad T_g^R A_j T_{g^{-1}}^R = A_j,$$
$$T_g^R B_k T_{g^{-1}}^R = \check{B}_k. \tag{27}$$

The new eigenfunctions in the 'rotated' system are

$$e_{pq}^{(g)}(\lambda, \tilde{g}) \equiv (T_g^R e)_{pq}(\lambda, \tilde{g}) = e_{pq}(\lambda, \tilde{g}g). \tag{28}$$

On the other hand, because the operators A_j are unchanged, we have

$$(T_g^R e)_{pq}(\lambda, \tilde{g}) = D_{q'q}^\lambda(g) e_{pq'}(\lambda, \tilde{g}), \tag{29}$$

where $D_{q'q}^\lambda(g)$ are the matrix elements of the operator T_g^R in the subspace $\hat{H}(\lambda)$. Clearly because $T_{g_1 g_2}^R = T_{g_1}^R T_{g_2}^R$ and $T_{g^{-1}}^R = (T_g^R)^*$, the functions $D_{q'q}^\lambda(g)$ satisfy the conditions

$$D_{q'q}^\lambda(g_1 g_2) = D_{q'r}^\lambda(g_1) D_{rq}^\lambda(g_2), \tag{30}$$

and

$$D_{q'q}^\lambda(g^{-1}) = \overline{D_{qq'}^\lambda(g)}. \tag{31}$$

Equations (28) and (29) imply

$$e_{pq}(\lambda, \tilde{g}g) = D_{q'q}^\lambda(g) e_{pq'}(\lambda, \tilde{g}). \tag{32}$$

Setting $\tilde{g} = e$ and utilizing the normalization condition (24), one obtains

$$e_{pq}(\lambda, g) = D_{pq}^\lambda(g). \tag{33}$$

The assertion of proposition 3 follows now from eqs. (30) and (31).▼

Following the current convention we shall use the symbol $D_{pq}^\lambda(g)$ for the generalized eigenvectors $e_{pq}(\lambda, q)$. If the set $\{\overline{T}(C_i)\}$, $\{\overline{T}(A_j)\}$ and $\{\overline{T}(B_k)\}$ of commuting operators in $H = L^2(G, \mu)$ satisfies the assumptions of remark 2 and proposition 2, then we have

$$\int_G D_{pq}^\lambda(g) D_{p'q'}^{\lambda'}(g) \mathrm{d}\mu(g) = \varrho^{-1}(\lambda)\,\delta(\lambda - \lambda')\,\delta_{pp'}\delta_{qq'}, \tag{34}$$

$$\int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{pq} D_{pq}^\lambda(g) \overline{D_{pq}^\lambda(g')} = \delta(gg'^{-1}) \tag{35}$$

and

$$\varphi(g) = \int_\Lambda \sum_{pq} \hat{\varphi}_{pq}(\lambda) D_{pq}^\lambda(g) \mathrm{d}\varrho(\lambda), \tag{36}$$

where

$$\hat{\varphi}_{pq}(\lambda) = \int_G \varphi(g) \overline{D_{pq}^\lambda}(g) \mathrm{d}\mu(g). \tag{37}$$

Formulas (34)–(37) provide a generalization of the corresponding formulas for compact groups given in § 1, to the case of the unimodular Lie groups (satisfying the assumption of remark 2 and proposition 2).

It is interesting and very useful in applications that the spectral synthesis (15) and the Plancherel equality (17) can be put in an operator form. In fact, let $\varphi \in \Phi(G)$ and set

$$F(\lambda) \equiv \int_G \varphi(g) T_{g^{-1}}(\lambda) \mathrm{d}\mu(g), \tag{38}$$

where $\hat{T}_g(\lambda)$ is an irreducible component of $\hat{T}^R_g$ on the carrier space $\hat{H}(\lambda)$. Since $\hat{T}^R$ is a factor representation in $\hat{H}(\lambda)$ all irreducible components of it are equivalent. We have

Proposition 4. *Let $D^\lambda_{pq}(g)$ be the eigenfunctions in $L^2(G)$, which satisfy assertions of proposition* 2. *Then we have*

(i) *The operator $F(\lambda)$ is a Hilbert–Schmidt operator in $\hat{H}(\lambda)$ for $\varrho-aa\lambda$.*

(ii) *The spectral synthesis formula* (15) *has the form*

$$\varphi(g) = \int_\Lambda \mathrm{d}\varrho(\lambda) \mathrm{Tr}[F(\lambda) T_g(\lambda)]. \tag{39}$$

(iii) *The Plancherel equality* (17) *has the form*

$$(\varphi, \psi)_H = \int_\Lambda \mathrm{d}\varrho(\lambda) \mathrm{Tr}\big(F(\lambda) G^*(\lambda)\big), \quad \varphi, \psi \in \Phi,$$

where $G(\lambda)$ is the $\hat{T}(\lambda)$-transform of ψ given by eq. (38).

Proof: Let $\{\hat{e}_s(\lambda)\}$ be an orthonormal basis in $\hat{H}(\lambda)$; then by eq. (38), we have

$$\begin{aligned} F(\lambda)\hat{e}_s(\lambda) &= \int_G \varphi(g) D^\lambda_{qs}(g^{-1}) \hat{e}_q(\lambda) \mathrm{d}\mu(g) \\ &= \int_G \varphi(g) \overline{D}^\lambda_{sq}(g) \hat{e}_q(\lambda) \mathrm{d}\mu(g) = \hat{\varphi}_{sq}(\lambda)\hat{e}_q(\lambda). \end{aligned}$$

Hence, the square of the Hilbert–Schmidt norm $|F(\lambda)|$ of the operator $F(\lambda)$ is

$$\begin{aligned} |F(\lambda)|^2 &= \sum_{p=1}^{\dim \hat{H}(\lambda)} \|F(\lambda)\hat{e}_p(\lambda)\|^2_{\hat{H}(\lambda)} \\ &= \sum_{p,q,q'=1}^{\dim H(\lambda)} \hat{\varphi}_{pq}(\lambda)\overline{\hat{\varphi}_{pq'}(\lambda)}\big(\hat{e}_q(\lambda), \hat{e}_{q'}(\lambda)\big)_{\hat{H}(\lambda)} \\ &= \sum_{p,q=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda)\overline{\hat{\varphi}_{pq}(\lambda)} = \|\hat{\varphi}(\lambda)\|^2_{\hat{H}(\lambda)}. \end{aligned}$$

Since $\{\hat{\varphi}(\lambda)\} \in L^2(\Lambda)$, the Hilbert–Schmidt norm $|F(\lambda)|$ is ϱ-almost everywhere bounded and consequently $F(\lambda)$ is a ϱ-almost everywhere Hilbert–Schmidt operator.

The matrix elements of the operator $F(\lambda)$ are

$$F_{pq}(\lambda) = \big((F(\lambda)\hat{e}_q(\lambda), \hat{e}_p(\lambda)\big)_{\hat{H}(\lambda)}$$

$$= \int_G \varphi(g)\big((T_g(\lambda))^{-1}\hat{e}_q(\lambda), \hat{e}_p(\lambda)\big)_{\hat{H}(\lambda)} \mathrm{d}\mu(g)$$

$$= \int_G \varphi(g)\overline{\big(T_{g(\lambda)}\hat{e}_p(\lambda), \hat{e}_q(\lambda)\big)}_{\hat{H}(\lambda)} \mathrm{d}\mu(g)$$

$$= \int_G \varphi(g)\overline{D^\lambda_{pq}(g)}\,\mathrm{d}\mu(g) \tag{40}$$

Hence, $F_{pq}(\lambda) = \hat{\varphi}_{qp}(\lambda)$ and thus, the spectral synthesis (15) of φ in $\Phi(G)$ can be written in the form

$$\varphi(g) = \int_\Lambda \sum_{pq=1}^{\dim H(\lambda)} F_{qp}(\lambda) D^\lambda_{pq}(g)\,\mathrm{d}\varrho(\lambda)$$

$$= \int_\Lambda \mathrm{Tr}\big(F(\lambda)T_g(\lambda)\big)\mathrm{d}\varrho(\lambda). \tag{41}$$

Here, the integration runs over the set of unitary irreducible representations on which the Plancherel measure $\varrho(\cdot)$ does not vanish.

The Plancherel equality (17) can now be put in the form:

$$(\varphi, \psi)_H = (\hat{\varphi}, \hat{\psi})_{\hat{H}} = \int_\Lambda \sum_{pq} F_{qp}(\lambda)\overline{G_{qp}(\lambda)}\,\mathrm{d}\varrho(\lambda)$$

$$= \int_\Lambda \mathrm{Tr}\{F(\lambda)G^*(\lambda)\}\,\mathrm{d}\varrho(\lambda). \blacktriangledown \tag{42}$$

§ 3. Harmonic Analysis on Semidirect Product of Groups

The general theory of harmonic analysis on unimodular Lie groups embraces also the case of unimodular semidirect products $G = N \circledS G_A$, where G_A is the group of automorphisms of N. We restrict ourselves in this section to the presentations of general theory for the two most important semidirect products, namely the Euclidean groups $E_n = T^n \circledS \mathrm{SO}(n)$ and the generalized Poincaré groups $\Pi_n = T^n \circledS \mathrm{SO}(n-1, 1)$, $n = 2, 3, ..$ One readily verifies using th. 3.10.5 that all groups E_n and Π_n are unimodular.

We shall first describe explicitly the maximal set of commuting differential operators in the space $H = L^2(G, \mu)$. We know that the set $\{C_l\}$ of algebraically independent invariant operators of E_n or Π_n consists of $\left\{\frac{n}{2}\right\}$ operators (see 9.7.3.1). We shall now determine the additional operators which together with the set $\{C_l\}$ will provide a maximal set of algebraically independent commuting operators.

Let

$$\mathrm{SO}(n) \supset \mathrm{SO}(n-1) \supset \ldots \supset \mathrm{SO}(2), \tag{1}$$

$$\mathrm{SO}(n-1, 1) \supset \mathrm{SO}(n-1) \supset \ldots \supset \mathrm{SO}(2) \tag{2}$$

be a sequence of successive maximal subgroups of SO(n) and SO($n-1,1$), respectively. Let $\{A_j\}_1^m$ be the maximal set of '+'-symmetric, algebraically independent Casimir operators in the enveloping algebra E^L of G, associated with successive subgroups in the sequence (1) or (2). Let $\{B_k\}_1^m$ be the corresponding maximal set of '+'-symmetric, algebraically independent operators in the enveloping algebra E^R of G.

One readily verifies that $m = \frac{1}{2}\left(\left[\frac{n}{2}\right] + \dim G_A\right)$. Because $\left[\frac{n+1}{2}\right] + 2m = \dim G$, the sets $\{C_i\}_1^{\left[\frac{n+1}{2}\right]}$, $\{A_j\}_1^m$ and $\{B_k\}_1^m$ provide the maximal set of independent commuting operators in the carrier space $H = L^2(G, \mu)$.

The basic features of harmonic analysis on E_n or Π_n are described by the following theorem.

THEOREM 1. *Let G_n be the group E_n or Π_n, $n = 2, 3, \ldots$, and let $g \to T_g$ be the regular representation of G_n in the Hilbert space $H = L^2(G, \mu)$, given by eq.* 2(4).

Let $\{C_t\}_1^{[(n+1)/2]}$, be the sequence of two-sided invariant operators of G_n in H and let $\{A_j\}_1^m$ and $\{B_k\}_1^m$ be the maximal sets of independent '+'-symmetric Casimir operators associated with the sequence of subgroups (1) *or* (2), *respectively. Then*

(i) *There exists a direct integral decomposition*

$$H \to \hat{H} = \int_\Lambda \hat{H}(\lambda)\,\mathrm{d}\varrho(\lambda), \qquad T_g \to \hat{T}_g = \int \hat{T}_g(\lambda)\,\mathrm{d}\varrho(\lambda) \tag{3}$$

of H and T_g such that $\left(\hat{H}(\lambda), \hat{T}_g(\lambda)\right)$ are ϱ-a.a. irreducible.

(ii) *There exists a Gel'fand triplet $\Phi \subset H \subset \Phi'$ and a basis $e_{pq}(\lambda, g)$ such that the relations* 2(12)–2(14) *hold.*

(iii) *For $\varphi \in \Phi$ the spectral synthesis formula has the form*

$$\varphi(g) = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{p,q=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_{pq}(\lambda) e_{pq}(\lambda, g), \tag{4}$$

where

$$\hat{\varphi}_{pq}(\lambda) = \int_G \varphi(g)\overline{e_{pq}(\lambda, g)}\,\mathrm{d}\mu(g). \tag{5}$$

(iv) *For $\varphi, \psi \in \Phi$ the Plancherel equality has the form* 2(17).

PROOF: The proofs of all assertions of Theorem 1, except eq. (5), are parallel to those of 2.1, and we omit them. The relation (5) is implied by the fact that, e.g., for E_n, together with the operators $C_2(\mathrm{SO}(n))$ and $M^2 = P_\mu P^\mu$, the Nelson operator $\Delta = P_\mu P_\mu + C_2(\mathrm{SO}(n))$ is also diagonalized. Because Δ is elliptic, the formula (5) follows from proposition 2.2.▼

Clearly propositions 2.3 and 2.4 are also valid for the groups E_n and Π_n.

We shall now consider an example which clearly illustrates the main features of harmonic analysis on non-compact, non-commutative Lie groups.

EXAMPLE 1. Let $G = E_2 = T^2 \circledS SO(2)$. If $x = (x_1, x_2) \in T^2$ and $\alpha \in SO(2)$, $0 \leqslant \alpha < 2\pi$ then the composition law in G is given by the formula

$$(x, \alpha)(x', \alpha') = (x+x'_\alpha, \alpha+\alpha'), \tag{6}$$

where

$$x'_\alpha = (x'_1 \cos\alpha - x'_2 \sin\alpha, x'_1 \sin\alpha + x'_2 \cos\alpha).$$

The composition law (6) implies

$$(x, \alpha)^{-1} = (-x_{-\alpha}, 2\pi-\alpha). \tag{7}$$

One readily verifies that the invariant measure on G has the form

$$d\mu[(x, \alpha)] = dx_1 dx_2 d\alpha. \tag{8}$$

Let $H = L^2(G, \mu)$. The right and left regular representations T^R and T^L of G in H are given by the formulas

$$(T^R_{(x', \alpha')} u)[(x, \alpha)] = u[(x, \alpha)(x', \alpha')] = u[(x+x'_\alpha, \alpha+\alpha')] \tag{9}$$

and

$$(T^L_{(x', \alpha')} u)[(x, \alpha)] = u[(x', \alpha')^{-1}(x, \alpha)] = u[(-x'_{-\alpha'} + x_{2\pi-\alpha'}, 2\pi-\alpha'+\alpha)]. \tag{10}$$

The generators X_i, $i = 1, 2, 3$, of the one-parameter subgroups $g(t_i)$ associated with the left regular representation (i.e., belonging to the right-invariant Lie algebra) are given by the formula

$$X_i u = \lim_{t \to 0} \left(\frac{T^L_{g(t_i)} - I}{t_i} u \right), \tag{11}$$

where u is an element of the Gårding domain. Using eqs. (10) and (11), one obtains

$$X_1 = -\frac{\partial}{\partial x_1}, \quad X_2 = -\frac{\partial}{\partial x_2}, \quad X_3 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - \frac{\partial}{\partial \alpha}. \tag{12}$$

Similarly, using the formula

$$Y_i u = \lim_{t_i \to 0} \left(\frac{T^R_{g(t_i)} - I}{t} u \right) \tag{13}$$

and eq. (9), one obtains the following expressions for the generators of the left-invariant Lie algebra of G

$$\begin{aligned} Y_1 &= \cos\alpha \frac{\partial}{\partial x_1} + \sin\alpha \frac{\partial}{\partial x_2}, \\ Y_2 &= -\sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_2}, \\ Y_3 &= \frac{\partial}{\partial \alpha}. \end{aligned} \tag{14}$$

Clearly, each operator X_i commutes with all operators Y_k, $i, k = 1, 2, 3$. The left- and the right-invariant Lie algebras satisfy the commutation relations of the form

$$[Z_1, Z_2] = 0, \quad [Z_2, Z_3] = Z_1, \quad [Z_3, Z_1] = Z_2. \tag{15}$$

One readily verifies that the invariant operator has the form

$$C_1 = Z_1^2 + Z_2^2. \tag{16}$$

Hence,

$$C_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \tag{17}$$

or, in the spherical coordinates,

$$\begin{gathered} x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi, \\ C_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}. \end{gathered} \tag{18}$$

Using the general method prescribed in th. 1 for the selection of the maximal set of commuting operators, one then obtains

$$C_1, \quad A_1 = X_3 = -\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial\alpha}, \quad B_1 = Y_3 = \frac{\partial}{\partial\alpha}. \tag{19}$$

This is the maximal set of independent, commuting, differential operators, because the dimension of G equals three.

Now, we shall find an explicit form of the eigenfunctions $e_{pq}(\lambda, g) \equiv D^\lambda_{pq}(g)$. The expressions (18) and (19) suggest that we should look for common eigenfunctions of the operators C_1, A_1 and B_1 of the form

$$D^\lambda_{pq}(\varphi, r, \alpha) = \exp(\mathrm{i}p\varphi)\mathrm{d}^\lambda_{pq}(r)\exp[-\mathrm{i}q(\alpha+\varphi)]. \tag{20}$$

These are the eigenfunctions of the operators X_3 and Y_3, while $C_1 D^\lambda_{pq} = \hat{C}_1(\lambda) D^\lambda_{pq}$ provides the following equation for $d^\lambda_{pq}(r)$:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(p-q)^2}{r^2}\right) d^\lambda_{pq}(r) = \hat{C}_1(\lambda) d^\lambda_{pq}(r). \tag{21}$$

This is the classical form of the Bessel equation, whose (regular) solution is

$$d^\lambda_{pq}(r) = i^{p-q} J_{p-q}\big(-i\sqrt{\hat{C}_1(\lambda)}r\big). \tag{22}$$

The operator C_1 is skew-adjoint (by the Nelson–Stinespring theorem) and negative definite by virtue of eq. (17). Hence, the eigenvalues $\hat{C}_1(\lambda)$ of C_1 are negative. Setting $\hat{C}_1(\lambda) = -\lambda^2$, $\lambda \in R$, one obtains by eqs. (22) and (20)

$$D^\lambda_{pq}(\varphi, r, \alpha) = i^{p-q}\exp[\mathrm{i}(p-q)\varphi] J_{p-q}(\lambda r)\exp[\mathrm{i}q\alpha]. \tag{23}$$

It is well known that the spectral measures $\mathrm{d}\varrho(\lambda)$ for the Bessel equation has the form $d\varrho(\lambda) = \lambda d\lambda$. Hence, the orthogonality and completeness relations for the functions $D^\lambda_{pq}(\varphi, r, \alpha)$ take the form

$$\int D^\lambda_{pq}(\varphi, r, \alpha)\overline{D^{\lambda'}_{p'q'}(\varphi, r, \alpha)}\, r\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\alpha = \delta_{p'p}\delta_{q'q}\frac{\delta(\lambda-\lambda')}{\lambda}, \tag{24}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda \sum_{pq} D^\lambda_{pq}(\varphi r\alpha)\overline{D^\lambda_{pq}(\varphi' r' \alpha')} = \delta(\varphi-\varphi')\frac{1}{r}\delta(r-r')\delta(\alpha-\alpha'). \tag{25}$$

Because $J_{p-q}(0) = \delta_{pq}$, the functions $D^\lambda_{pq}(e)$ satisfy the normalization condition 2 (24), i.e., $D_{pq}(e) = \delta_{pq}$, where $e = (0, 0, 0)$ is the unity of G. Therefore, the functions $D^\lambda_{pq}(g)$ satisfy the unitarity condition

$$D^\lambda_{pq}(g^{-1}) = \overline{D^\lambda_{qp}(g)} \tag{26}$$

and the composition law

$$\sum_s D^\lambda_{ps}(g_1)D^\lambda_{sq}(g_2) = D^\lambda_{pq}(g_1 g_2). \tag{27}$$

Notice that the last formula allows us to derive various composition laws for Bessel functions. One readily verifies that if $g_1 = (0, r_1, 0)$ and $g_2 = (\varphi_2, r_2, 0)$, then, $g_1 \cdot g_2 = (\varphi, r, \alpha)$ is given by the relations

$$r = [r_1^2+r_2^2+2r_1r_2\cos\varphi_2]^{1/2}, \quad \exp(\mathrm{i}\varphi) = \frac{r_1+r_2\exp(\mathrm{i}\varphi_2)}{r}, \quad \alpha = 0. \tag{28}$$

Then, for $\lambda = 1$, eq. (27) gives

$$\sum_{k=-\infty}^{\infty} \exp(\mathrm{i}k\varphi_2)J_{n-k}(r_1)J_k(r_2) = \exp(\mathrm{i}n\varphi)J_n(r). \tag{29}$$

In particular, for $\varphi_2 = 0$, we have $r = r_1+r_2$ and eq. (29) gives

$$\sum_{k=-\infty}^{\infty} J_{n-k}(r_1)J_k(r_2) = J_n(r_1+r_2) \tag{30}$$

and, for $\varphi_2 = \pi$, $r_1 \geqslant r_2$ we have $r = r_1-r_2$, and $\varphi = 0$. Hence eq. (29) gives

$$\sum_{k=-\infty}^{\infty} (-1)^k J_{n-k}(r_1)J_k(r_2) = J_n(r_1-r_2). \tag{31}$$

§ 4. Comments and Supplements

A. *Plancherel Measure*

The first significant general result of harmonic analysis on separable unimodular groups was obtained by Segal 1950. With each $\varphi \in H = L^2(G, \mu)$, he associated an operator

$$T(\varphi) = \int \varphi(g) T_g \mathrm{d}\mu(g), \tag{1}$$

where T_g is a unitary representation of G.

The Mautner theorem implies that $T(\varphi)$ has the following direct integral decomposition

$$T(\varphi) \to \hat{T}(\varphi) = \int \hat{F}(\lambda) \mathrm{d}\varrho(\lambda).$$

Segal calls $\{\hat{F}(\lambda)\}$ a Fourier transform of φ and proves the following Plancherel Theorem

$$\int_G |\varphi(g)|^2 \mathrm{d}\mu(g) = \int_\Lambda \langle \hat{F}(\lambda), \hat{F}(\lambda)\rangle_\lambda a(\lambda) \mathrm{d}\varrho(\lambda), \tag{2}$$

where $\langle \cdot, \cdot\rangle_\lambda$ denotes the norm square in the Banach space of bounded operators in the space $\hat{H}(\lambda)$ and $a(\lambda)$ is a positive, ϱ-measurable function.

(For the proof see Segal 1950, th. 3.)

It is crucial to know for applications the explicit form of the Plancherel measure. This problem was solved for classical complex Lie groups by Gel'fand and Naimark 1950 (see also Gel'fand 1963 for simplified derivation). For instance in the case of $\mathrm{SL}(n, C)$ group the irreducible unitary representations $\hat{T}(\lambda)$ are labelled by the multiindex $\lambda = (m_2, \ldots, m_n, \varrho_2, \ldots, \varrho_n)$ where m_i are integers and ϱ_i—real numbers. (cf. ch. 19, § 3).

The explicit form of Plancherel measure in these variables is

$$\mathrm{d}\varrho(\lambda) = c \prod_{1 \leqslant p \leqslant q \leqslant n} [(\varrho_p - \varrho_q)^2 + (m_p - m_q)^2], \quad \varrho_1 = m_1 = 0, \tag{3}$$

where $c = 2^{n/2(n-1)} [n!(2\pi)^{(n-1)(n+2)}]^{-1}$ and m_i, ϱ_i, $i = 1, 2, \ldots, n$, are invariant numbers which characterize an irreducible unitary representation of $\mathrm{SL}(n, C)$ The Plancherel equality is

$$\int \varphi(g) \overline{\psi}(g) \mathrm{d}\mu(g) = \int \mathrm{Tr}\left(\hat{F}(\lambda) \hat{G}^*(\lambda)\right) \mathrm{d}\varrho(\lambda),$$

where

$$\hat{F}(\lambda) = \int \varphi(g) \hat{T}_g(\lambda) \mathrm{d}\mu(g).$$

The Plancherel measure is zero on the supplementary series of $\mathrm{SL}(n, C)$. The generalization of Gel'fand–Naimark results to arbitrary connected semisimple Lie groups was given by Harish-Chandra in a series of papers (cf. 1954, and 1970).

The explicit form of the Plancherel measure for some real groups was found recently. In particular, Hirai found the explicit form of the Plancherel measure for Lorentz type groups $\mathrm{SO}(n, 1)$ 1966 and for $\mathrm{SU}(p, q)$ groups 1970. Romm has found Plancherel measure for group $\mathrm{SL}(n, R)$ 1965. Recently Leznov and Savelev gave an explicit form of the Plancherel measure for all real forms of complex classical groups 1970 for principal nondegenerate series. (See also 1971 for $\mathrm{SU}(p, q)$ groups). However, their derivation contains some assumptions whose validity was not verified.

B. *Comments*

The harmonic analysis on unimodular groups presented in secs. 2 and 3 is based on lecture notes given by Rączka 1969. This presentation based on the nuclear spectral theory provides the most natural extension to non-compact noncommutative groups of the classical Fourier theory for abelian groups and of Peter–Weyl theory for compact groups. It provides also a convenient framework for applications in quantum theory, where the concept of generalized eigenfunctions plays a central role. A series of fundamental problems of harmonic analysis for semisimple Lie groups were solved by Harish–Chandra 1954, 1965, 1966 and review article 1970. The monumental works of Harish–Chandra are presented in two-volume monograph elaborated by Warner 1972. An interesting treatment of various problems of harmonic analysis on semisimple complex Lie groups is presented in the recent monograph of Želobenko 1974. It contains in particular an excellent reviev of achievements of Russian mathematicians in this domain.

Example 3.1 demonstrates the usefulness of harmonic analysis on groups for the analysis of properties of special functions. This subject is extensively treated in monographs of Vilenkin 1968, Miller 1968 and lecture notes of Wigner elaborated by Talman 1968.

The harmonic analysis on the Lorentz group $SL(2, C)$ is extensively treated in the monograph of Gel'fand and Vilenkin, v. 5, 1966, and in the recent book by Rühl 1970 which also contains interesting applications in particle physics. The harmonic analysis on Poincaré group was treated by Rideau 1966 and by Nghiem Zuan Hai in the doctoral dissertation (Orsay 1969).

A very elegant and general treatment of harmonic analysis on locally compact groups based on the C^*-algebra theory is presented in the monograph of Dixmier 1969 (see also Naimark 1970).

Chapter 15

Harmonic Analysis on Homogeneous Spaces

The harmonic analysis on homogeneous spaces is another one of the most important but difficult parts of group representation theory. The degree of difficulty is well illustrated by the classical treatise *Abstract Harmonic Analysis* by Hewitt and Ross, where the concept of harmonic analysis appears only after some 1065 pages of 'introductory material'.

We shall first state the basic problems in harmonic analysis.

Let X be a homogeneous space, G a locally compact transformation group on X, and K the stability subgroup of G. Let $\mathrm{d}\mu(x)$ be a quasi-invariant measure on X provided by the Mackey Theorem (ch. 4.3.1) and let $H = L^2(X, \mu)$. The map $g \to T_g$ given by

$$T_g u(x) = \sqrt{\frac{\mathrm{d}\mu(xg)}{\mathrm{d}\mu(x)}}\, u(xg) \tag{1}$$

provides a unitary representation T of G in H.

The two basic problems of harmonic analysis are the following:

(i) *Spectral analysis*: The decomposition of the representation (1) and of the carrier space H onto the direct integrals

$$T_g \to \hat{T}_g = \int_{\Lambda} \hat{T}_g(\lambda)\mathrm{d}\varrho(\lambda), \qquad H \to \hat{H} = \int_{\Lambda} \hat{H}(\lambda)\mathrm{d}\varrho(\lambda) \tag{2}$$

of irreducible representations $\hat{T}_g(\lambda)$ of G in $\hat{H}(\lambda)$ and the determination of the spectrum.

(ii) *Spectral synthesis*: The determination of a dense subspace $\Phi \subset H$ such that for every $\varphi \in \Phi$ we have:

$$\varphi(x) = \int_{\Lambda} \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda) e_k(\lambda, x)\mathrm{d}\varrho(\lambda), \tag{3}$$

where $\{e_k(\lambda, x\}_{k=1}^{\dim H(\lambda)}$ is a basis in $H'(\lambda)$ and

$$\hat{\varphi}_k(\lambda) = \langle \varphi, e_k(\lambda)\rangle \tag{4}$$

is a component of $\varphi \in H$ in $\hat{H}(\lambda)$.

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* Note that $\hat{H}(\lambda)$ and $H'(\lambda) \subset \Phi'$ are isomorphic but different; cf. app. B.3(30).

§ 1. Invariant Operators on Homogeneous Spaces

Associated with the problem of spectral analysis is the problem of finding the maximal set $\{C_i\}_1^n$ of independent invariant commuting operators. Contrary to a common belief among physicists, the set $\{C_i\}_1^n$ might contain more invariant operators than those obtained from the center Z of the enveloping algebra E of G. In fact, let $N(K)$ be a nontrivial normalizer in G of the stability subgroup K of X, i.e., the set of all $n \in G$ such that $nKn^{-1} \subset K$. Then using the correspondence $x_g \to Kg$ between elements of the space X and the cosets Kg we obtain

$$nx_g \sim nKg = Kng = x_{ng}.$$

This implies that the left translations of X by elements of N and the right translations of X by elements of G commute. Hence

$$(T_n^L T_g u)(x) = (T_g T_n^L u)(x). \tag{5}$$

Consequently, if the quotient group $N(K)/K$ is nontrivial, then the maximal set of operators associated with the group $N(K)/K$ provides an additional set of invariant operators besides those from the center Z of the enveloping algebra E of G. If $N(K)/K$ is non-commutative, the additional set of invariant differential operators associated with $N(K)/K$ is also non-commutative.

Let us note that a Lie group might have invariant operators which are not elements of the enveloping algebra nor even differential operators; for instance, in the case of the Poincaré group in addition to the mass square operator $P_\mu P^\mu$ ($\sim m^2 = p_\mu p^\mu$) and the square of the spin operator $W_\mu W^\mu$ ($\sim m^2 J(J+1)$) which are differential operators in the center $Z(E)$, we have the invariant operator $Q = \operatorname{sign} p_0$, where p_0 is the eigenvalue of the generator P_0 in the carrier space. The operator Q is neither an element of $Z(E)$ nor it is a differential operator.

EXAMPLE 1. Let G be the Poincaré group $T^4 \circledS \mathrm{SL}(2, C)$ and let

$$K = T^4 \circledS Z, \tag{6}$$

where Z is the two-dimensional nilpotent group consisting of all complex matrices of the form

$$z = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}, \quad z \in C^1. \tag{7}$$

Hence $X = G/K$ is a four-dimensional homogeneous space. The subgroup $S = ZD$, where

$$D = \left\{ \begin{bmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{bmatrix}, \ \delta \in C^1 \right\},$$

has Z as its normal subgroup by virtue of Gauss decomposition 3.6(3) for $\mathrm{SL}(2, C)$. Hence the normalizer $N(K)$ for K is the group

$$N(K) = T^4 \circledS S. \tag{8}$$

Thus

$$N(K)/K = D. \tag{9}$$

Consequently, in the space $H = L^2(X, \mu)$, $X = G/K$, there will be two additional invariant operators associated with the generators of D, in addition to mass square $P_\mu P^\mu$ and spin-square $W_\mu W^\mu$ operators which are generators of the center Z of the enveloping algebra E of the Poincaré group. ▼

In the case of arbitrary Lie groups G and $K \subset G$ the problem of finding the normalizer $N(K)$ of K in G is unsolved. Hence, we do not have a general characterization of the set $\{C_i\}_1^n$ of independent invariant operators in the space $H = L^2(X, \mu)$, $X = G/K$. However, for more special groups G and K, there is a more specific characterization of the set $\{C_i\}_1^n$. For intance, if G is a semisimple connected Lie group and K is the maximal compact subgroup of G then it follows from the Cartan decomposition of G: $G = KP$ that there is no subgroup G_0 between G and K, of dimension greater than the dimension of K. Hence $N(K)/K$ is at most discrete. This implies that there are no additional invariant differential operators in $\{C_i\}_1^n$ which come from $N(K)/K$.

One should also remark the following connection between the properties of homogeneous spaces and the properties of the set $\{C_i\}_1^n$ of invariant operators: if the stability group K of X is small, then the number of invariant operators in $\{C_i\}$ is large and may contain invariant operators even from outside of the enveloping algebra; conversely, if the stability subgroup K becomes bigger, there are more constraints in X and the number of invariant operators decreases; in this case even those invariant operators which were algebraically independent in the center Z of E become dependent as the differential operators on the space $H = L^2(X, \mu)$. The following example will illustrate this point.

EXAMPLE 2. Let G be the Euclidean group $T^n \circledS SO(n)$. In general G has $[(n+1)/2]$ algebraically independent operators in the center Z of E and $X = G/K \sim R^n$. We shall show that the center Z of E in $L^2(X, dx)$ is generated by a single operator. Indeed, let $C_i(x_k, \partial/\partial x_l\}$ be differential operators in Z. Because C_i must be invariant under all translations $t \in T^n$, the differential operators $C_i(x_k, \partial/\partial x_l)$ have constant coefficients. Consequently, $C_i = P_i(\partial_1, \ldots, \partial_n)$, where P_i is a polynomial. Now the only rotational invariant quantity in R^n is the radius $r = |x| = \sqrt{\sum x_i^2}$. Hence the polynomials P_i must be functions of $\partial_1^2 + \ldots + \partial_n^2 \equiv \Delta$. Hence, an arbitrary invariant differential operator in $L^2(X, dx)$ from the enveloping algebra E of G has the form

$$C = \sum_{l=1}^{s} C_l \Delta^l. \tag{10}$$

Clearly, if $G = T^n \circledS SO(p, q)$, $p+q = n$, then an arbitrary invariant operator in $L^2(X, dx)$ from the enveloping algebra E has also the form (10) with

$$\Delta = \partial_1^2 + \ldots + \partial_p^2 - \partial_{p+1}^2 - \ldots - \partial_{p+q}^2 . ▼ \tag{11}$$

The following theorem gives the description of the set of invariant operators for the symmetric space $X = G/K$.

THEOREM 1. *Let $X = G/K$ be a symmetric space of rank l. Then the algebra of all G-invariant differential operators in the space $H = L^2(X, \mu)$ is a commutative algebra with l algebraically independent generators.*

If X is of rank one, then every invariant differential operator C is a polynomial in the second order Casimir operator of G which in a proper coordinate system on X is equal to the Laplace–Beltrami operator

$$\Delta = \bar{g}^{-1/2}\,\partial_\alpha\, g^{\alpha\beta}\sqrt{\bar{g}}\,\partial_\beta \tag{12}$$

where $g^{\alpha\beta}(x)$ is the invariant metric tensor on the space X and $\bar{g} \equiv |\det g|$.

(For the proof cf. Helgason 1962, ch. 10).

§ 2. Harmonic Analysis on Homogeneous Spaces

We shall now elaborate on the harmonic analysis on general homogeneous spaces $X = G/K$ where G is a connected Lie group and K a closed subgroup of G. The following theorem provides a general solution of the basic problems of harmonic analysis on homogeneous spaces.

THEOREM 1. *Let $H = L^2(X, \mu)$ where μ is a quasi-invariant measure on X and let $g \to T_g$ be a unitary representation of G in $H = L^2(X, \mu)$ given by eq.* 1(1).

Let $\{T(C_i)\}_1^n$ be the maximal set of independent G-invariant operators in the representation $T(E)$ of the enveloping algebra E of G. Let $Q = \{A_j\}_1^m$ be the maximal set of commuting self-adjoint (non-invariant) operators in $T(E)$. Then

(i) *There exists a direct integral decomposition*

$$H \to \hat{H} = \int_\Lambda \hat{H}(\lambda)\,\mathrm{d}\varrho(\lambda), \qquad T_g \to \hat{T}_g = \int \hat{T}_g(\lambda)\,\mathrm{d}\varrho(\lambda) \tag{1}$$

of H and T_g such that $\hat{H}(\lambda)$ and $\hat{T}_g(\lambda)$ are ϱ-a.a. irreducible.

(ii) *There exists a Gel'fand triplet $\Phi \subset H \subset \Phi'$ and a basis $\{e_k(\lambda, x)\}$ in $H(\lambda)$ such that*

$$\langle \overline{T(C_i)}\varphi, e_k(\lambda)\rangle = \hat{C}_i(\lambda)\langle\varphi, e_k(\lambda)\rangle, \quad i = 1, \ldots, n, \tag{2}$$

$$\langle A_j\varphi, e_k(\lambda)\rangle = \hat{A}_j(k)\langle\varphi, e_k(\lambda)\rangle, \quad j = 1, \ldots, m. \tag{3}$$

(iii) *For $\varphi \in \Phi(X)$ the spectral synthesis formula is*

$$\varphi(x) = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda) e_k(\lambda, x), \tag{4}$$

where

$$\hat{\varphi}_k(\lambda) = \langle\varphi, e_k(\lambda)\rangle.$$

(iv) *For $\varphi, \psi \in \Phi(X)$ the Parseval equality is given by*

$$\int \varphi(x)\overline{\psi}(x)\mathrm{d}\mu(x) = \int_{A} \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda)\overline{\hat{\psi}_k(\lambda)}. \tag{6}$$

PROOF: We shall first construct the Gel'fand triplet $\Phi(X) \subset H \subset \Phi'(X)$. Let $u_1 \in H$ and let $A_{u_1} \equiv \{\psi \in D(G): T(\psi)u_1 = 0\}$, where $T(\psi) = \int_G \psi(g) T_g \mathrm{d}g$. If $\psi_n \to \psi$ in $D(G)$, then $T(\psi_n)u_1 \to T(\psi)u_1$. Thus, if $\psi_n \in A_{u_1}$, then $\lim \psi_n = \psi \in A_{u_1}$, i.e., A_{u_1} is closed in $D(G)$. Hence the quotient space $\tilde{D}(G) \equiv D(G)/A_{u_1}$ is nuclear. Take $u_1 \in H$ and set $H_{u_1} \equiv \{T(\psi)u_1, \psi \in D(G)\}$ and equip H_{u_1} with the nuclear topology of the space $\tilde{D}(G)$. If the space H_{u_1} is not dense in H, we take $u_2 \perp H_{u_1}$ and form the topological direct sum $H_{u_1} \oplus H_{u_2}$, and so on until we reach a dense space $\oplus_k H_{u_k}$ in H. Because H is separable, this is always possible. The space $\Phi \equiv \oplus_k H_{u_k}$ is a countable sum of nuclear spaces and it is therefore itself nuclear. The natural embedding H_{u_k} in H is continuous; in fact,

$$\|T(\psi)u_k\| = \|\int \psi(g) T_g u_k \mathrm{d}g\| \leqslant |[\psi]_p| \|u_k\|,$$

where $|[\psi]_p|$ is a seminorm in $\tilde{D}(G)$ induced by the Schwarz norm p in $D(G)$. Consequently, the imbedding $I: \Phi \to H$ is also continuous.

For an element M in the enveloping algebra E of G, we obtain by virtue of eq. 11.1 (17), $T(M)\Phi \subset \Phi$. By the same equation, for every $M \in E$, $T(M)$ is a continuous operator of Φ.

We shall now prove the direct integral decomposition (1). Let C_i, $i = 1, 2, \ldots, n$, be the set of algebraically independent symmetric elements in the center $Z(E)$ of E (i.e., $C_i^+ = C_i$). By virtue of corollary 3 to th. 11.2.3, we conclude that the closures $\overline{T(C_i)}$ of $T(C_i)$ are self-adjoint operators. Using then th. 11.5.3, we conclude that all $\overline{T(C_g)}$, $i = 1, 2, \ldots, n$, are mutually strongly commuting and also commute with all $T_g, g \in G$.

Let A be a maximal '*'-algebra in the commutant T' of T, which contains all spectral resolutions of $\overline{T(C_i)}$, $i = 1, 2, \ldots, n$. Then using the Mautner Theorem 5.6.1 we obtain the decomposition (1). The spectral set Λ contains as a subset the spectra Λ_{C_i} of the self-adjoint operators $\overline{T(C_i)}$, $i = 1, 2, \ldots, n$.

Let $\{e_k(\lambda, x\}$ be the common generalized eigenvectors of the operators $\overline{T(C_i)}$ and $\{A_j\}_1^m$ provided by the nuclear spectral theorem. Then eq. (2) and (3), spectral synthesis formula (4), and Parseval equality (5) follow from assertions 3(23), 3(25) and 3(24) of app. B.▼

Remark 1. By virtue of eq. app. B3(27), eqs. (2) and (3) can be written in the form

$$\overline{T(C_i)}' e_k(\lambda, x) = \hat{C}_i(\lambda) e_k(\lambda, x) \tag{7}$$

and

$$A_j' e_k(\lambda, x) = \hat{A}_j(k) e_k(\lambda, x), \tag{8}$$

where, e.g., A_j' denotes the extension of A_j obtained by extending the domain $D(A_j)$ by those elements $\varphi' \in \Phi'$ for which the equality

$$\langle A_j \varphi, \varphi' \rangle = \langle \varphi, A_j' \varphi' \rangle, \quad \varphi \in \Phi, \ \varphi' \in \Phi'$$

is satisfied.▼

For the sake of simplicity we shall use a notation as though all Casimir operators would have purely continuous spectrum and all noninvariant operators A_j would have purely discrete spectrum.

PROPOSITION 2. *Let the set $\{T(C_i)\}_1^n$ or $\{A_j\}_1^m$ contain an elliptic differential operator. Then all eigenvectors $e_k(\lambda, x)$ are regular functions on X. For $\varphi \in \Phi$ we have*

$$\hat{\varphi}_k(\lambda) = \int_X \varphi(x) \overline{e_k(\lambda, x)} \mathrm{d}\mu(x). \tag{9}$$

We have also the following completeness relation

$$\int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} e_k(\lambda, x) \overline{e_k(\lambda, x')} = \delta(x-x'). \tag{10}$$

The proof of proposition 2 is similar to that of proposition 14.2.2 and we omit it. The integral (10) is understood as weak integral of regular distributions $e_k(\lambda, x) e_k(\lambda, x')$ on $X \times X$ and is essentially equivalent to Parseval equality.

Finally, if, in addition, all operators in $\{T(C_i)\}_1^n$ have spectra which are absolutely continuous with respect to the Lebesgue measure (i.e. $\mathrm{d}\varrho(\lambda) = \varrho(\lambda)\mathrm{d}\lambda$), then we have the following orthogonality relation

$$\int_X e_k(\lambda, x) \overline{e_{k'}(\lambda', x)} \mathrm{d}\mu(x) = \varrho(\lambda)^{-1} \delta(\lambda - \lambda') \delta_{kk'}. \tag{11}$$

The condition of proposition 2 that the maximal set of commuting operators in $L^2(X, \mu)$ contains an elliptic operator, and the condition that all operators in $\{T(C_i)\}_1^n$ and $\{A_j\}_1^m$ have absolutely continuous spectra and purely discrete spectra, respectively, are often satisfied in applications. In particular, these conditions are satisfied in the case considered in the next section (sec. 3), where X is a symmetric space of rank one, with respect to group $\mathrm{SO}(p, q)$.

It is interesting that the eigenvectors $e_k(\lambda, x)$ are linear combinations of matrix elements $D^\lambda_{pq}(g)$ of the representation $\hat{T}_g(\lambda)$. In fact, let $\{e_k(\lambda, x)\}$ be a basis in $H(\lambda)$, whose elements satisfy assertions of proposition 2: then

$$\hat{T}_g(\lambda) e_k(\lambda, x) = e_k(\lambda, xg) = D^\lambda_{lk}(g) e_l(\lambda, x). \tag{11'}$$

Now by virtue of the homogeneity of X, any point in X can be written in the form og_x, where o is the origin of X (o corresponds to the coset K). Hence, formula (11) implies

$$e_k(\lambda, x) = e_k(\lambda, og_x) = D^\lambda_{lk}(g_x)e_l(\lambda, o), \tag{12}$$

where g_x is an element of the Borel set $S \subset G$ determined by the Mackey decomposition $G = KS$ of G. Formula (12) reveals the important fact that the eigenvectors $e_k(\lambda, x)$ in the space $L^2(X, \mu)$, $X = K^{\cdot} G$, may be obtained by the reduction of the matrix elements $D^\lambda_{pq}(g)$ on G to the Borel set S. This procedure is well known to physicists in the case of the group SO(3) where the eigenvectors $Y^J_M(\vartheta, \varphi)$ on the sphere S^2 may be obtained from the matrix elements $D^J_{MN}(\varphi, \vartheta, \psi)$ by the formula $Y^J_M(\vartheta, \varphi) = (-1)^M \left(\dfrac{4\pi}{(2J+1)}\right)^{1/2} D^J_{-M,0}(\varphi, \vartheta, 0)$.

EXAMPLE 1. Let G be the Lorentz group SO(3, 1) and let

$$X = \mathrm{SO}(3, 1)/\mathrm{SO}(3).$$

According to ch. 4, table 1, X is a Cartan symmetric space of rank one. It can be realized as the three-dimensional hyperboloid

$$(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2 = 1. \tag{13}$$

Because SO(3, 1) and SO(3) are unimodular groups, it follows from corollary 1 to th. 4.3.1 that there exists an invariant measure $d\mu(x)$ on X. This measure has the form $d\mu(x) = d^3x/x^0$ (cf. example 4.3.2). We shall find the explicit form of harmonic functions on X which provide the spectral synthesis formula (4), completeness (10) and orthogonality relation (11), as well as the direct integral decomposition (1) of the space $H = L^2(X, d\mu)$ and of the representation T_g: $u(x) \to u(g^{-1}x)$ onto irreducible components.

We know by th. 1.1 that in the space $L^2(X, \mu)$ the ring of invariant differential operators is generated by the Laplace-Beltrami operator Δ, 1(12). We shall find first the explicit form of Δ. Introducing the polar coordinates

$$\begin{aligned} x^1 &= \sin\vartheta\cos\varphi\,\mathrm{sh}\,\theta, & \varphi &\in [0, 2\pi),\\ x^2 &= \sin\vartheta\sin\varphi\,\mathrm{sh}\,\theta, & \vartheta &\in [0, \pi/2],\\ x^3 &= \cos\vartheta\,\mathrm{sh}\theta, & &\\ x^0 &= \mathrm{ch}\theta, & \theta &\in [0, \infty) \end{aligned} \tag{14}$$

and utilizing expression 1(12) for the Laplace–Beltrami operator, one finds

$$\Delta(X) = -\frac{1}{\mathrm{sh}^2\theta}\frac{\partial}{\partial\theta}\,\mathrm{sh}^2\theta\frac{\partial}{\partial\theta}+\frac{\boldsymbol{J}^2}{\mathrm{sh}^2\theta}, \tag{15}$$

where $\boldsymbol{J}^2$ is the invariant operator of SO(3). Setting

$$e_k(\lambda, x) = V^\lambda_J(\theta)\, Y^J_M(\vartheta, \varphi), \tag{16}$$

where $Y^J_M(\vartheta, \varphi)$ are the harmonic functions on the sphere S^2, one reduces the eigenvalue equation

$$\Delta(X)e_k(\lambda, x) = \hat{\Delta}(\lambda)e_k(\lambda, x) \tag{17}$$

to the solution of a second-order ordinary differential equation of the form

$$\left[-\frac{1}{\sinh^2\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sinh^2\theta\frac{\mathrm{d}}{\mathrm{d}\theta}+\frac{J(J+1)}{\sinh^2\theta}\right]V^\lambda_J(\theta) = \hat{\Delta}(\lambda)V^\lambda_J(\theta). \tag{18}$$

The spectrum Λ of this operator and the spectral measure $\mathrm{d}\varrho(\lambda)$ may be found either by casting this equation into a Schrödinger equation, or by using the standard Titchmarsh–Kodaira technique. One finds that

$$\hat{\Delta}(\lambda) = -\Lambda^2-1, \quad \lambda\in\Lambda = [0, \infty), \tag{19}$$

and the spectral measure $\mathrm{d}\varrho(\lambda) = \mathrm{d}\lambda$ (cf. Limić, Niederle and Rączka 1967). Equation (17) is a hypergeometric equation whose solution, regular at $\xi = 0$, is

$$V^\lambda_J(\theta) = N^{-1/2}\tanh^J\theta\cosh^{[i\lambda-1]}\theta\, {}_2F_1\left\{\frac{1}{2}(J-i\lambda+1),\right.$$
$$\left.\frac{1}{2}\left(J-i\lambda+2;\; J+\frac{3}{2},\; \tanh\theta\right)\right\}, \tag{20}$$

where

$$N = \left|\frac{(2\pi)^{1/2}\Gamma(i\lambda)\Gamma(J+{}^3/_2)}{\Gamma[\frac{1}{2}(i\lambda+1+J)]\Gamma(\frac{1}{2}(i\lambda+2+J)]}\right|^2. \tag{21}$$

Consequently, the direct integral 2(1) for $H = L^2(X, \mu)$ implied by Δ has the form

$$\hat{H} = \int_0^\infty \hat{H}(\lambda)\mathrm{d}\lambda.$$

Because the second-order Casimir operators of SO(3, 1) and of the maximal compact subgroup $K = \mathrm{SO}(3)$ are diagonal, the elliptic Nelson operator

$$\Delta_N = \sum_1^{\dim G} X_i^2 = C_2(\mathrm{SO}(3,1))+2C_2(\mathrm{SO}(3))$$

is also diagonal. This implies by virtue of proposition 2 that we have:

(i) *Spectral synthesis*: The nuclear space $\Phi(X)$ can be taken to be the Schwartz space $S(X)$. For $\varphi\in S(X)$, the spectral synthesis (3) is

$$\varphi(\theta, \vartheta, \varphi) = \int_0^\infty \mathrm{d}\lambda\sum_{J=0}^\infty\sum_{M=-J}^J \hat{\varphi}_{JM}(\lambda)V^\lambda_J(\theta)Y^J_M(\vartheta, \varphi), \tag{22}$$

where

$$\hat{\varphi}_{JM}(\lambda) = \int_X \mathrm{d}\mu(x)\varphi(\theta, \vartheta, \varphi)V^\lambda_J(\theta)Y^J_M(\vartheta, \varphi). \tag{23}$$

(ii) Parseval equality

$$\int_X \varphi(x)\overline{\psi}(x)\mathrm{d}\mu(x) = \int_0^\infty \mathrm{d}\lambda \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \hat{\varphi}_{JM}(\lambda)\overline{\hat{\psi}_{JM}(\lambda)}. \tag{24}$$

In eqs. (23) and (24):

$$\mathrm{d}\mu(x) = \cosh^2\theta \mathrm{d}\theta \sin\vartheta \mathrm{d}\vartheta \mathrm{d}\varphi. \tag{25}$$

(iii) Completeness relation (8)

$$\int_0^\infty \mathrm{d}\lambda \sum_{J=0}^{\infty} \sum_{M=-J}^{J} V_J^\lambda(\theta)\overline{Y_M^J(\vartheta,\varphi)}V_J^\lambda(\theta')\overline{Y_M^J(\vartheta',\varphi')}$$
$$= \mathrm{ch}^{-2}\theta\,\delta(\theta-\theta')\,\delta(\cos\vartheta-\cos\vartheta')\,\delta(\varphi-\varphi'). \tag{26}$$

(iv) The orthogonality relation (9) takes the form

$$\int_X V_{J'}^{\lambda'}(\theta)\,Y_{M'}^{J'}(\vartheta,\varphi)\,V_J^\lambda(\theta)\,Y_M^J(\vartheta,\varphi)\,\mathrm{d}\mu(x) = \delta(\lambda-\lambda')\,\delta_{JJ'}\,\delta_{MM'}. \tag{27}$$

Every generator Y of SO(3, 1) commutes with the operator Δ. Hence, each Hilbert space $H(\lambda)$ is invariant relative to the action of the representation T_g of SO(3, 1). Using the Bruhat criterion given by th. 19.1.2 one may verify that almost every representation $\hat{T}_g(\lambda)$ in $\hat{H}(\lambda)$ obtained by the restriction of $\hat{T}_g$ to $\hat{H}(\lambda)$ is irreducible. Hence the decomposition (1) of H implies the decomposition

$$T_g \to \hat{T}_g = \int_0^\infty \hat{T}_g(\lambda)\,\mathrm{d}\lambda \tag{28}$$

of T_g onto irreducible components.▼

§ 3. Harmonic Analysis on Symmetric Spaces Associated with Pseudo-Orthogonal Groups SO(p, q)

The pseudo-orthogonal groups play an important role in theoretical physics. The most important one is the Lorentz group SO(3, 1) of special theory of relativity.

The group SO(4, 1), known as the de Sitter group, appears in general relativity, as the dynamical group in the theory of hydrogen atom in the periodic table of elements, as well as in hadron models. The conformal group SO(4, 2) is the symmetry group of Maxwell equations and also appears as a symmetry or dynamical group in the theory of elementary particles. Other groups like SO(2, 1) or SO(4, 3) also often appear in applications. Hence it is appropriate to present a detailed study of the properties of harmonic analysis on symmetric spaces associated with the groups SO(p, q) and the associated representations of SO(p, q) on symmetric spaces.

The full classification of symmetric spaces associated with the groups SO(p, q) was given in ch. 4, tables I and II. The most important for applications are the following symmetric spaces:

(i) Cartan symmetric spaces

$$X = \mathrm{SO}_0(p,q)/\mathrm{SO}(p)\otimes\mathrm{SO}(q). \tag{1}$$

The rank k of these spaces is $k = \min(p,q)$ and the dimension equals pq.

(ii) The symmetric spaces with noncompact semisimple stability groups

$$X_{r,s} = \mathrm{SO}_0(p,q)/\mathrm{SO}_0(r,s)\otimes\mathrm{SO}_0(p-r,q-s), \quad 0\leqslant r\leqslant p, \quad 0\leqslant s\leqslant q. \tag{2}$$

(iii) The symmetric spaces with noncompact nonsemisimple stability group

$$X_0 = \mathrm{SO}_0(p,q)/T^{p-1,q-1} \rtimes \mathrm{SO}_0(p-1,q-1). \tag{3}$$

Here $T^{n,m}$ is the group of translations of the Minkowski space $M^{n,m}$.

We shall consider in this section the harmonic analysis on symmetric spaces (1)–(3) of rank one; these are

$$\begin{aligned} X_+^{p+q-1} &\equiv \mathrm{SO}_0(p,q)/\mathrm{SO}_0(p-1,q), \\ X_-^{p+q-1} &\equiv \mathrm{SO}_0(p,q)/\mathrm{SO}_0(p,q-1) \end{aligned} \tag{4}$$

and X_0 given in (3).

The groups $\mathrm{SO}_0(p,q)$ and the stability subgroups of the spaces X_+, X_- and X_0 are all unimodular. Hence by corollary 1 to th. 4.3.1, there exists on these spaces X_+, X_- and X_0 an invariant measure $\mathrm{d}\mu(x)$. Hence a unitary representation $g\to T_g$ of $\mathrm{SO}_0(p,q)$ is given in the space $H = L^2(X,\mu)$ by the formula

$$T_g u(x) = u(g^{-1}x). \tag{5}$$

We know by virtue of th. 1.1 that in the spaces $H(X_+)$, $H(X_-)$ and $H(X_0)$ the ring of invariant differential operators is generated by the second order Casimir operator C_2, which on the spaces $H(X_+)$ and $H(X_-)$ is equal to the Laplace–Beltrami operator.

We may always select a coordinate system on X_+, X_- or X_0 such that the second-order Casimir operators of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ will be diagonal. Hence the elliptic Nelson operator

$$\Delta_N = \sum_{i=1}^{\dim G} X_i^2 = C_2(\mathrm{SO}(p,q)) - 2C_2(\mathrm{SO}(p)) - 2C_2(\mathrm{SO}(q))$$

is also diagonal. Consequently, all assumptions of th. 2.1 are satisfied and we obtain:

(i) The space H and the representation T_g given by eq. (5) can be represented as a direct integral

$$H\to\hat{H} = \int_\Lambda \hat{H}(\lambda)\,\mathrm{d}\mu(\lambda), \qquad T_g\to\hat{T}_g = \int_\Lambda T_g(\lambda)\,\mathrm{d}\mu(\lambda)$$

of components $\hat{H}(\lambda)$ and $\hat{T}_g(\lambda)$, respectively.

(ii) There exists a Gel'fand-triplet $\Phi \subset H \subset \Phi'$ such that all elements $e(\lambda)$ in $H'(\lambda)$ satisfy the eigenvalue eq. 2(7).

(iii) The spectral synthesis 2(4) for functions $\varphi(x) \in \Phi$ and Parseval equality 2(6) hold.

To carry out the decomposition of H and T_g onto irreducible components explicitly and to get the spectral synthesis for the functions $\varphi(x) \in \Phi$ in explicit form, we have to find the maximal abelian algebra A in the commutant T' of the representation T and find its spectrum Λ. We shall solve these problems along the following steps:

(i) Construct a convenient coordinate system on X for which the metric tensor $g_{\alpha\beta}(x)$ is diagonal.

(ii) Solve the eigenvalue problem for the Laplace–Beltrami operator

$$\Delta(X)e(\lambda, x) = \hat{\Delta}(\lambda)e(\lambda, x).$$

(Because $[\Delta, Y] = 0$ for all Y in the Lie algebra L of G, the space $H'(\lambda)$ for fixed λ, spanned by all $e(\lambda, x)$ is invariant.)

(iii) Find additional invariant operators which decompose the space $\hat{H}(\lambda)$ onto irreducible subspaces.

We stress that according to th. 1.1 the ring of invariant differential operators is generated by the Laplace–Beltrami operator. It, however, says nothing about the other invariant operators in H. We shall find that in the case of symmetric spaces of rank one these additional invariant operators are certain reflection operators in $H(X)$.

A. *Harmonic Analysis on Symmetric Spaces Associated with Groups* $SO_0(p, q)$, $p \geqslant q > 2$

To choose a suitable coordinate system, we have to introduce some convenient model for the space $X_{\pm}^{p+q-1}$ in (4). This means that we have to introduce a manifold, with the same dimension and the same stability group as $X_{\pm}^{p+q-1}$ on which the group $SO_0(p, q)$ acts transitively.

It can be shown, by means of considerations similar as in the case of the group $SO(p)$ (cf. 4.2, example 1) that a model for the space X_{+}^{p+q-1} can be realized by the hyperboloid $H^{p,q}$ determined by the equation

$$(x^1)^2 + \ldots + (x^p)^2 - (x^{p+1})^2 - \ldots - (x^{p+q})^2 = 1. \tag{6}$$

And as a model appropriate for the space X_{-}^{p+q-1}, we take the hyperboloid $H^{q,p}$ defined by the equation

$$(x^1)^2 + \ldots + (x^q)^2 - (x^{q+1})^2 - \ldots - (x^{p+q})^2 = 1. \tag{7}$$

If we introduce internal coordinates $\Omega = \{\eta^1, \ldots, \eta^{p+q-1}\}$ on the space $H^{p,q}$ (which is imbedded in the flat Minkowski space $M^{p,q}$), then the metric tensor

$g_{\alpha\beta}(H^{p,q})$ on the hyperboloid $H^{p,q}$ induced by the metric tensor $g_{ab}(M^{p,q})$ on the Minkowski space $M^{p,q}$, is given by

$$g_{\alpha\beta}(H^{p,q}) = g_{ab}(M^{p,q})\,\partial_\alpha x^a(\Omega)\,\partial_\beta x^b(\Omega), \tag{8}$$

where $a, b = 1, 2, \ldots, p+q$, $\alpha, \beta = 1, 2, \ldots, p+q-1$ and x^a are Cartesian coordinates of the hyperboloid Ω.

Generally, we may choose a large number of different coordinate systems on the hyperboloid $H^{p,q}$ such that the Laplace–Beltrami operator admits separation of variables. However, the most convenient coordinate system is the biharmonic one, because in this system the generators of the maximal abelian compact subgroup of the group $SO_0(p, q)$ are automatically contained in the maximal set of commuting operators.

The biharmonic coordinate system on the sphere S^n was introduced in eq.10.3(5a).

The biharmonic coordinate system on the hyperboloid $H^{p,q}$, (6), is constructed as follows:

$$\begin{aligned} x^k &= x'^k\cosh\theta, & k &= 1, 2, \ldots, p, \\ x^{p+l} &= \tilde{x}^l\sinh\theta, & l &= 1, 2, \ldots, q, \end{aligned} \qquad \theta \in [0, \infty), \tag{9}$$

where the form of the x'^k and $\tilde{x}^l$ depends on whether p and q are even or odd. We must distinguish four cases:

$$\begin{aligned} &\text{(i) } p = 2r, & q &= 2s, \\ &\text{(ii) } p = 2r, & q &= 2s+1, \\ &\text{(iii) } p = 2r+1, & q &= 2s, \\ &\text{(iv) } p = 2r+1, & q &= 2s+1, \end{aligned} \qquad r, s = 1, 2, \ldots \tag{10}$$

If p is even ($p = 2r$), the corresponding x'^k ($k = 1, 2, \ldots, 2r$) are given by the recursion formulas

$$\begin{aligned} &\text{for } r = 1 & x'^1 &= \cos\varphi^1, \\ & & x'^2 &= \sin\varphi^1, & &\varphi^1 \in [0, 2\pi), \\ &\text{for } r > 1 & x'^i &= x^{*i}\sin\vartheta^r, & &i = 1, 2, \ldots, 2r-2, \\ & & x'^{2r-1} &= \cos\varphi^r\cos\vartheta^r, & &\varphi^j \in [0, 2\pi),\ j = 1, 2, \ldots, r, \\ & & x'^{2r} &= \sin\varphi^r\cos\vartheta^r, & &\vartheta^k \in [0, \tfrac{1}{2}\pi],\ k = 2, 3, \ldots, r, \end{aligned} \tag{11}$$

where x^{*i} are the coordinates for $p = 2(r-1)$.

If p is odd ($p = 2r+1$), we first construct the x^{*i}, $i = 1, 2, \ldots, 2r$, by using the above-mentioned method for $p = 2r$; we then obtain the corresponding x'^k, $k = 1, 2, \ldots, 2r+1$, as

$$\begin{aligned} x'^i &= x^{*i}\sin\vartheta^{r+1}, & i &= 1, 2, \ldots, 2r, \\ x'^{2r+1} &= \cos\vartheta^{r+1}, & \vartheta^{r+1} &\in [0, \pi). \end{aligned} \tag{12}$$

The recursion formulas for $\tilde{x}^l$, q even or odd, are the same as those for x'^k, p even or odd, respectively, except that angles φ^i, ϑ^j in x'^k are replaced by $\tilde{\varphi}^i$, $\tilde{\vartheta}^j$.

Choosing the parametrization $\Omega \equiv \{\omega, \tilde{\omega}, \theta\}$ on the hyperboloid $H^{p,q}$ in the form

$$\begin{aligned} \omega &\equiv \{\varphi^1, \dots, \varphi^{[p/2]}, \vartheta^2, \dots, \vartheta^{\{p/2\}}\}, \\ \tilde{\omega} &\equiv \{\tilde{\varphi}^1, \dots, \tilde{\varphi}^{[q/2]}, \tilde{\vartheta}^2, \dots, \tilde{\vartheta}^{\{q/2\}}\}, \end{aligned} \tag{13}$$

and denoting

$$\{\partial_\gamma\} \equiv \left\{\frac{\partial}{\partial\varphi^1}, \frac{\partial}{\partial\vartheta^2}, \dots; \frac{\partial}{\partial\varphi^{[p/2]}}, \frac{\partial}{\partial\varphi^{\{p/2\}}}, \frac{\partial}{\partial\tilde{\varphi}^1}, \frac{\partial}{\partial\tilde{\vartheta}^2}, \dots, \frac{\partial}{\partial\tilde{\varphi}^{[q/2]}}, \frac{\partial}{\partial\tilde{\vartheta}^{\{q/2\}}}, \frac{\partial}{\partial\theta}\right\}, \gamma = 1, 2, \dots, p+q-1, \tag{14}$$

we can calculate the metric tensor $g_{\alpha\beta}(H^{p,q})$ as well as the Laplace–Beltrami operator $\Delta(H^{p,q})$.

Since in all four cases shown in (10) the variables in the Laplace–Beltrami operator are separated in the same way due to the properties of the metric tensor (8), we can write the operator $\Delta(H^{pq})$ in the form

$$\Delta(H^{pq}) = -(\cosh^{p-1}\theta\sinh^{q-1}\theta)^{-1}\frac{\partial}{\partial\theta}\cosh^{p-1}\theta\sinh^{q-1}\theta\frac{\partial}{\partial\theta} + \frac{\Delta(S^{p-1})}{\cosh^2\theta} - \frac{\Delta(S^{q-1})}{\sinh^2\theta}, \tag{15}$$

where $\Delta(S^{p-1})[\Delta S^{q-1})]$ is the Laplace–Beltrami operator of the rotation group $SO(p)$ $[SO(q)]$. (See ch. 10, sec. 3.) If we represent the eigenfunctions of $\Delta(H^{p,q})$ as a product of the eigenfunctions of $\Delta(S^{p-1})$, $\Delta(S^{q-1})$, and a function $\psi^\lambda_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)$ we obtain the following equation:

$$\left[-(\cosh^{p-1}\theta\sinh^{q-1}\theta)^{-1}\frac{\mathrm{d}}{\mathrm{d}\theta}\cosh^{p-1}\theta\sinh^{q-1}\theta\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{l_{\{p/2\}}(l_{\{q/2\}}+p-2)}{\cosh^2\theta} + \frac{\tilde{l}_{\{q/2\}}(\tilde{l}_{\{q/2\}}+q-2)}{\sinh^2\theta} - \hat{\Delta}(\lambda)\right]\cdot\psi^\lambda_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta) = 0, \tag{16}$$

where $l_{\{p/2\}}(l_{\{p/2\}}+p-2)$, $[\tilde{l}_{\{q/2\}}(\tilde{l}_{\{q/2\}}+q-2)]$ are eigenvalues of the operator $\Delta(S^{p-1}), [\Delta(S^{q-1})]$ with $l_{\{p/2\}}[l_{\{q/2\}}]$ certain non-negative integers for $p > 2$ ($q > 2$]).

A discrete series of representations exists if there exist solutions of (16), which are square integrable functions $\psi^\lambda_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)$, $\theta \in [0, \infty)$, with respect to the measure

$$\mathrm{d}\mu(\theta) = \cosh^{p-1}\theta\sinh^{q-1}\theta\,\mathrm{d}\theta, \tag{17}$$

which is induced by the measure $\mathrm{d}\mu(x)$ on the hyperboloid $H^{p,q}$, which in the biharmonic coordinates has the form*

$$\mathrm{d}\mu(\Omega) = \bar{g}(H^{p,q})^{1/2}\mathrm{d}\Omega = \mathrm{d}\mu(\omega)\mathrm{d}\mu(\tilde{\omega})\cosh^{p-1}\theta\sinh^{q-1}\theta\,\mathrm{d}\theta. \tag{18}$$

* The measure $\mathrm{d}\mu(x) = [\bar{g}(H^{p,q})]^{1/2}\mathrm{d}\Omega$ is the Riemannian measure, which is left-invariant under the action of $SO_0(p, q)$. See ch. 4, § 3.

The left-invariant measure $d\mu(\omega)$ (with respect to SO(p)) is defined in eq. 10.3 (14). Since the differential eq. (16) has meromorphic coefficients regular in the interval $(0, \infty)$, any two linearly independent solutions are also regular analytic in this interval (Ince, 1956). Since at the origin and at infinity the coefficients are singular, the solutions are not generally regular there, and we can easily find two essentially distinct behaviours of the solutions at the origin:

$$\psi_1^0 \sim \theta^{\tilde{l}_{\{q/2\}}}, \qquad \psi_2^0 \sim \theta^{-\tilde{l}_{\{q/2\}}-q+2},$$

and at infinity:

$$\psi_{1,2}^{\infty} \sim \exp\left\{-\frac{1}{2}(p+q-2)\pm\left[\frac{1}{2}(p+q-2)^2-\hat{\Delta}(\lambda)\right]^{1/2}\right\}\theta.$$

The only satisfactory solution, i.e., the solution which is square-integrable with respect to our measure $d\mu(\theta)$ (17), is one that behaves like $\psi_1^0(\theta)$ at the origin and like $\psi_2^\infty(\theta)$ at infinity. We obtain the solution of (16) with these properties by converting (16) into the hypergeometric equation whose solution is

$$\psi^{\lambda}_{l_{\{p/2\}}\tilde{l}_{\{q/2\}}}(\theta) = \tanh^{\tilde{l}_{\{q/2\}}}\theta\cdot\cosh^{-\{(p+q-2)/2+[(p+q-2)^2/2-\hat{\Delta}(\lambda)]^{1/2}\}}\theta\times$$
$$\times\,{}_2F_1\left(-n+l_{\{p/2\}}+\frac{p-2}{2},-n;\ \tilde{l}_{\{q/2\}}+\frac{q}{2};\ \tanh^2\theta\right),$$

where the non-negative integer n is connected with $l_{\{p/2\}}$, $l_{\{q/2\}}$ and $\hat{\Delta}(\lambda)$ by the condition that ${}_2F_1$ be a polynomial, i.e.,

$$l_{\{p/2\}}-\tilde{l}_{\{q/2\}}-2n = \tfrac{1}{2}(p+q-2)+\{[\tfrac{1}{2}(p+q-2)]^2-\Delta(\lambda)\}^{1/2}-p+2, \qquad n = 0,1,2,\ldots \tag{19}$$

From this restrictive condition we can find that the discrete spectrum of the operator $\Delta(H^{p,q})$ is of the form

$$\hat{\Delta}(\lambda) = -L(L+p+q-2), \qquad L = -\{\tfrac{1}{2}(p+q-4)\},-\{\tfrac{1}{2}(p+q-4)\}+1,\ldots \tag{20}$$

and

$$L = l_{\{p/2\}}-\tilde{l}_{\{q/2\}}-q-2n. \tag{21}$$

Thus the orthonormal eigenfunctions of the invariant operator $\Delta(H^{p,q})$ are:

$$Y^{L,l_2,\ldots,l_{\{p/2\}}\tilde{l}_2,\ldots,\tilde{l}_{\{q/2\}}}_{m_1,\ldots,m_{[p/2]},\tilde{m}_1,\ldots,\tilde{m}_{[q/2]}}(\omega,\tilde{\omega},\theta)$$
$$= Y^{l_2,\ldots,l_{\{p/2\}}}_{m_1,\ldots,m_{[p/2]}}(\omega)\cdot Y^{\tilde{l}_2,\ldots,\tilde{l}_{\{q/2\}}}_{\tilde{m}_1,\ldots,\tilde{m}_{[q/2]}}(\tilde{\omega})\cdot V^{L}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta), \tag{22}$$

where

$$Y^{l_2,\ldots,l_{\{p/2\}}}_{m_1,\ldots,m_{[p/2]}}(\omega)$$
$$\equiv\begin{cases} Y^{l_2,\ldots,l_r}_{m_1,\ldots,m_r}(\omega) = (N_r^{-1/2})\prod_{k=2}^{r}\sin^{2-k}(\vartheta^k)\cdot d^{J_k}_{M_k,M_k'}(2\vartheta^k)\cdot\prod_{k=1}^{r}\exp im_k\varphi^k, \text{ if } p=2r,\\ Y^{l_2,\ldots,l_{r+1}}_{m_2,\ldots,m_r}(\omega) = (N_{r+1}^{-1/2})\sin^{1-r}(\vartheta^{r+1})\cdot d^{J_{r+1}}_{M_{r+1},0}(\vartheta^{r+1})\times\\ \quad\times\prod_{k=2}^{r}\sin^{2-k}(\vartheta^k)\cdot d^{J_k}_{M_k,M_k'}(2\vartheta^k)\cdot\prod_{k=1}^{r}\exp im_k\varphi^k, \quad \text{if } p=2r+1,\end{cases} \tag{23}$$

are eigenfunctions of $\Delta(S^{p-1})$ derived in ch. 10, § 3, eq. (19), eq. (20); and $Y^{\tilde{l}_2,\ldots,\tilde{l}_{[q/2]}}_{\tilde{m}_1,\ldots,\tilde{m}_{[q/2]}}(\tilde{\omega})$ are eigenfunctions of $\Delta(S^{q-1})$ expressed as the product of the usual d-functions of angular momenta and exponential functions exactly as in (23), but of the variables $\tilde{\varphi}^i$, $\tilde{\vartheta}^j$ and $\tilde{l}_k$, $\tilde{m}_l$ instead of φ^i, ϑ^j and l_k, m_l. The function $V^L{}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)$ is the solution of (16) given by

$$V^L_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta) = (N^{-1/2})\tanh^{\tilde{l}_{\{2/q\}}}(\theta)\cdot\cosh^{-(L+p+q-2)}(\theta)\times$$

$$\times {}_2F_1[\tfrac{1}{2}(p+q-2+l_{\{p/2\}}+\tilde{l}_{\{q/2\}}+L), \tfrac{1}{2}(L+q+\tilde{l}_{\{q/2\}}-l_{\{p/2\}});\ \tilde{l}_{\{q/2\}}+\tfrac{1}{2}q;\ \tanh^2\theta], \quad (24)$$

where, for a definite representation, L is fixed and $l_{\{p/2\}}$, $\tilde{l}_{\{q/2\}}$ are restricted by the condition that ${}_2F_1$ be a polynomial, i.e.,

$$l_{\{p/2\}}-\tilde{l}_{\{q/2\}} = L+q+2n, \quad n = 0, 1, 2, \ldots \quad (25)$$

In eq. (24) N is a normalization factor given by

$$N = \frac{\Gamma[\frac{1}{2}(l_{\{p/2\}}-\tilde{l}_{\{q/2\}}-L-q+2)]\Gamma(\tilde{l}_{\{q/2\}}+q/2)\Gamma[\frac{1}{2}(L-\tilde{l}_{\{q/2\}}+l_{\{p/2\}}+p)]}{2[L+\frac{1}{2}(p+q-2)]\Gamma[\frac{1}{2}(l_{\{p/2\}}+\tilde{l}_{\{q/2\}}+L+p+q-2)]\Gamma[\frac{1}{2}(l_{\{p/2\}}+\tilde{l}_{\{q/2\}}-L)]}. \quad (26)$$

Let $\hat{H}(L)$, L fixed, denote the subspace of $H = L^2(H^{p,q}, \mu)$ spanned by the harmonic functions (23). Because

$$[\Delta(H^{p,q}), Z_{ij}] = 0, \quad i, j = 1, 2, \ldots, p+q$$

for any generator $Z_{ij} \in \mathrm{so}(p, q)$, the space $\hat{H}(L)$ is an invariant space for the quasi-regular representation (1.1). We denote the unitary representation by (1.1) restricted to $\hat{H}(L)$ by $\hat{T}(L)$.

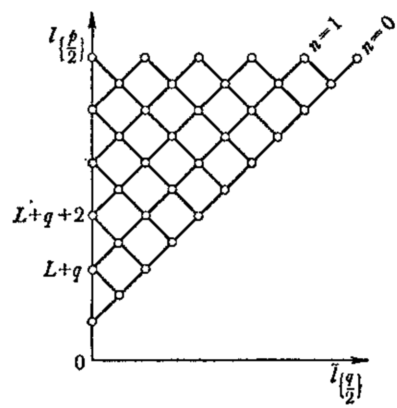

Fig. 1. Representations $\hat{T}(L)$ in $L^2(H^{p,q}, \mu)$ for $p > q > 2$.

The structure of the representation space $\hat{H}(L)$ can be illustrated graphically by a net in the plane $(l_{\{p/2\}}\, l_{\{q/2\}})$. Namely, utilizing relation (25), we get the diagram shown in Fig. 1. Every node of the net in the figure represents a finite-dimensional subspace $\hat{H}^{(L)}_{l_{\{p/2\}},\, l_{\{q/2\}}}$ of an irreducible representation of the maximal compact subgroup $SO(p)\times SO(q)$ which is determined by the pair of integers $l_{\{p/2\}}$ and $\tilde{l}_{\{q/2\}}$. The generators $L_{ij}\in SO(p)\times SO(q)$ act inside $\hat{H}^{(L)}_{l\{p/2\},\tilde{l}_{\{q/2\}}}$. On the other hand, the generators B_{ij} of the non-compact type map the subspace $\hat{H}^{(L)}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}$ into four neighboring subspaces $\hat{H}^{(L)}_{l_{\{p/2\}}\pm 1,\tilde{l}_{\{q/2\}}\pm 1}$. It is interesting that the structure of the whole representation space is determined by the lowest value of $l_{\{p/2\}}$.

The set of all representations

$$\hat{T}(L),\quad L=-\{\tfrac{1}{2}(p+q)-4\},-\{\tfrac{1}{2}(p+q)-4\}+1,\ldots,$$

constitutes the discrete series of most degenerate irreducible unitary representations of the group $SO_0(p,q)$ which are realized in the Hilbert spaces $L^2(H^{p,q},\mu)$. Let $\tilde{H}=L^2(H^{p,q},\mu)$. There also exists a discrete series of representations $\tilde{T}(L)$ on the Hilbert space $\tilde{H}(L)\subset\tilde{H}$ spanned by the harmonic functions obtained by exchanging p, $l_{\{p/2\}}$ with q, $\tilde{l}_{\{q/2\}}$, respectively, in eq. (22). The representations $\hat{\tilde{T}}(L)$ of $SO_0(p,q)$ realized $\hat{\tilde{H}}(L)$ are not unitarily equivalent to $\hat{T}(L)$ on $\hat{H}(L)$ except in the case $p=q$, which is the case when both Hilbert spaces coincide. The structure of the representation space $\hat{\tilde{T}}(L)$ is illustrated graphically in Fig. 2.

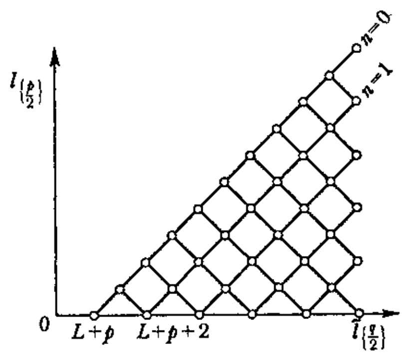

Fig. 2. Representation $\hat{\tilde{T}}(L)$ in $L^2(H^{p,q},\mu)$ for $p\geqslant q>2$.

The differential operator (16) and therefore the Laplace–Beltrami operator (1), has, in the case of the group $SO_0(p,q)$, a discrete spectrum given by eq. (20) as well as a continuous spectrum. The latter is of the form (see Limić, Niederle and Rączka 1967)

$$\hat{\Delta}(\lambda)=+\lambda^2+\left(\frac{p+q-2}{2}\right)^2,\quad \lambda\in[0,\infty). \tag{27}$$

For the continuous spectrum the solution of eq. (16), regular at the origin, is given by the function

$$V^{\lambda}_{l_{\{p/2\}},\tilde{l}_{\{\tilde{q}/2\}}}(\theta)=(N^{-1/2})\cdot\tanh^{|\tilde{l}_{\{q/2\}}|}\theta\cdot\cosh^{[-(p+q-2)/2+i\lambda]}\theta\times$$
$$\times{}_2F_1\{\tfrac{1}{2}[|\tilde{l}_{\{q/2\}}|+|l_{\{p/2\}}|-i\lambda+\tfrac{1}{2}(p+q-2)],$$

$$\cdot\tfrac{1}{2}[|\tilde{l}_{\{q/2\}}|-|l_{\{p/2\}}|-i\lambda+\tfrac{1}{2}(q-p+2)];\ |\tilde{l}_{\{q/2\}}|+\tfrac{1}{2}q;\ \tanh^2\theta\},\tag{28}$$

with

$$N=$$

$$=\left|\frac{\sqrt{2\pi}\,\Gamma(|\tilde{l}_{\{q/2\}}+\tfrac{1}{2}q|)\cdot\Gamma(i\lambda)}{\Gamma\{\tfrac{1}{2}[i\lambda+|l_{\{p/2\}}|+|\tilde{l}_{\{q/2\}}|+\tfrac{1}{2}(p+q-2)]\}\cdot\Gamma\{\tfrac{1}{2}[i\lambda+|\tilde{l}_{\{q/2\}}|-|l_{\{p/2\}}|+\tfrac{1}{2}(q-p+2)]\}}\right|^2.$$

These functions obey the following orthogonality condition

$$\int V^{\lambda}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)\overline{V}^{\lambda'}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)\,d\mu(\theta)=\delta(\lambda-\lambda'),\tag{29}$$

where $d\mu(\theta)$ is given by formula (17).

The orthogonal eigenfunctions of the Laplace–Beltrami operator $\Delta(H^{p,q})$ are then harmonic functions of the form

$$Y^{\lambda,l_2,\dots,l_{\{p/2\}},\tilde{l}_2,\dots,\tilde{l}_{\{q/2\}}}_{m_1,\dots,m_{[p/2]},\tilde{m}_1,\dots,\tilde{m}_{[q/2]}}(\theta,\omega,\tilde{\omega})$$

$$=V^{\lambda}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)\cdot Y^{l_2,\dots,l_{\{p/2\}}}_{m_1,\dots,m_{[p/2]}}(\omega)\cdot Y^{\tilde{l}_2,\dots,\tilde{l}_{\{q/2\}}}_{\tilde{m}_1,\dots,\tilde{m}_{[q/2]}}\tag{30}$$

where $V^{\lambda}_{l_{\{p/2\}},\tilde{l}_{\{q/2\}}}(\theta)$ is given in (28), and

$$Y^{l_2,\dots,l_{\{p/2\}}}_{m_1,\dots,m_{[p/2]}}(\omega)\quad\text{and}\quad Y^{\tilde{l}_2,\dots,\tilde{l}_{\{q/2\}}}_{\tilde{m}_1,\dots,\tilde{m}_{[q/2]}}(\tilde{\omega})$$

are the eigenfunctions of $\Delta(S^{p-1})$ and $\Delta(S^{q-1})$ respectively. Because of the orthogonality property (29), the set of eigenfunctions (30) associated with a definite point λ of the continuous part of the spectrum does not span a Hilbert subspace* $\hat{H}(\lambda)$ of $\hat{H}$, but an isomorphic space $H'(\lambda)\subset\Phi'(H^{p,q})$.

It turns out that the set of invariant operators of the so(p,q) algebra in H contains an invariant operator which is outside the enveloping algebra of so(p,q). Namely, the reflection operator R defined by

$$Rf(x)=f(-x),\qquad f(x)\in H\tag{31}$$

commutes with all the generators of so(p,q) and therefore represents an invariant operator.** In the case of the discrete series of representations $\hat{T}^L$ the eigenvalue p of the reflection operator R is determined by the invariant number L, i.e.,***

$$RY^{L,l,\tilde{l}}_{m,\tilde{m}}=(-1)^{L+q}Y^{L,l,\tilde{l}}_{m,\tilde{m}}\qquad\text{for }Y^{L,l,\tilde{l}}_{m,\tilde{m}}\in H(H^{p,q},\mu)$$

and (32)

$$RY^{L,l,\tilde{l}}_{m,\tilde{m}}=(-1)^{L+p}Y^{L,l,\tilde{l}}_{m,\tilde{m}}\qquad\text{for }Y^{L,l,\tilde{l}}_{m,\tilde{m}}\in H(H^{p,q},\mu).$$

* $\hat{H}(\lambda)$ represents in our case l^2—Hilbert space of sequences. The vectors $\hat{e}_k(\lambda)$ are the sequences of type $(0,0,\dots,0,1,0,\dots)$ with a unit on the k-th place and zero elsewhere.

** The existence of the invariant operator R is not in contradiction with Helgason's theorem which states that the ring of invariant operators in the enveloping algebra of so(p,q) for a symmetric space of rank one is generated by the Laplace–Beltrami operator.

*** In what follows we shall use for the harmonic function (22) and (30) the abbreviations

$$Y^{L,l,\tilde{l}}_{m,\tilde{m}}\quad\text{and}\quad Y^{\lambda,l,\tilde{l}}_{m,\tilde{m}}.$$

However, in the case of the continuous series of representations, the eigenfunctions (30) of $\Delta(H^{p,q})$ associated with a definite eigenvalue λ are eigenvectors of R with eigenvalues

$$r = (-1)^{(l_{\{p/2\}}+\tilde{l}_{\{q/2\}})}. \tag{33}$$

Therefore, the set of functions $\{Y^{\lambda,l,\tilde{l}}_{m,\tilde{m}}\}$ will be split into two subsets of functions

$$\{Y^{\lambda,+,l,\tilde{l}}_{m,\tilde{m}}\} \quad \text{and} \quad \{Y^{\lambda,-,l,\tilde{l}}_{m,\tilde{m}}\}$$

spanning invariant linear subspaces of so(p, q).

We shall now give the general form of harmonic analysis on homogeneous spaces for the present special case (cf. th. 2.1). In our case the decomposition 2(1) of $H = L^2(H^{p,q}, \mu)$ into direct integral $\hat{H}$ is determined by the set of invariant operators $\{\Delta, R\}$ and has the form

$$H \xrightarrow{F} \hat{H} = \sum_{L=-\{\frac{1}{2}(p+q-4)\}}^{\infty} \hat{H}(L, (-1)^{L+q}) + \sum_{\pm} \int_0^{\infty} \hat{H}(\lambda, \pm)\,\mathrm{d}\lambda. \tag{34}$$

As the nuclear space $\Phi \subset H$ we can take the Schwartz S-space on X. The space Φ is a dense invariant domain of the invariant operators $\Delta[\mathrm{SO}_0(p, q)]$ and R, as well as of all the generators of both compact and non-compact types.

The isomorphism F in (34) is given by means of the generalized Fourier transform with respect to the eigenfunctions (22) and (30) associated with the discrete and continuous part of the spectrum of the invariant operators $\Delta[\mathrm{SO}_0(p, q)]$ and R

$$H \supset \Phi \ni \psi(x) \xrightarrow{F} \{(F\psi)(\lambda, r)\} = \{\hat{\psi}(\lambda, r)\}, \tag{35}$$

where $\lambda = L$, or λ and $r = \pm 1$. For a definite λ the vector $\hat{\psi}(\lambda, r)$ is an element of l^2-Hilbert space, whose components $\psi^{\lambda,r,l,\tilde{l}}_{m,\tilde{m}}$ are given by the formula

$$\hat{\psi}^{\lambda,r,l,\tilde{l}}_{m,\tilde{m}} = \langle \psi, Y^{\lambda,r,l,\tilde{l}}_{m,\tilde{m}} \rangle = \int_{H^{p,q}} \psi(\theta, \omega, \tilde{\omega}) \overline{Y^{\lambda,r,l,\tilde{l}}_{m,\tilde{m}}}\, \mathrm{d}\mu(\theta, \omega, \tilde{\omega}). \tag{36}$$

Using eqs. 2(4)–2(6) one readily writes in the present case the spectral synthesis formula and the Parseval equality.

The action of any generator $Z_{ij} \in$ so(p, q) in the Hilbert space $H(\lambda, r)$ is given by

$$Z_{ij}\hat{\psi}^{\lambda,r} \equiv \{\langle Z_{ij}\psi, Y^{\lambda,r}_{m,\tilde{m}} \rangle\}, \qquad \psi \in \Phi. \tag{37}$$

The proof of irreducibility of the space $H(L, \pm)$ and $\hat{H}(\lambda, \pm)$ with respect to the action (38) of the Lie algebra so(p, q) is straightforward and we omit it. (See Limić, Niederle and Rączka 1966, for details.)

The harmonic functions on the hyperboloid $H^{q,p}$ can be obtained by exchanging p, $l_{\{p/2\}}$ with q, $\tilde{l}_{\{q/2\}}$, respectively, in eqs. (22) and (30).

The continuous series $\hat{T}(\lambda)$ of the irreducible unitary representations of SO(p, q), $p \geqslant q > 1$, on $H = L^2(H^{q,p}, \mu)$ can be constructed by the same procedure as described above.

B. *Decomposition with Respect to the Maximal Compact and Maximal Noncompact Subgroups.*

The decomposition of an irreducible representation of the group $SO_0(p, q)$ with respect to the irreducible representations of the maximal compact subgroup $SO(p)\times SO(q)$ can be easily obtained from Figures 1 and 2. For example, for the discrete representations $\hat{T}(L)$ the carrier space $\hat{H}(L)$ can be represented as the direct sum

$$\hat{H}(L) = \sum_{l_{\{p/2\}}, \tilde{l}_{\{q/2\}}} \oplus \hat{H}^{(L)}_{l_{\{p/2\}}, \tilde{l}_{\{q/2\}}}, \tag{38}$$

where the sum runs over all $l_{\{p/2\}}$ and $\tilde{l}_{\{q/2\}}$ obeying the condition

$$l_{\{p/2\}} - \tilde{l}_{\{q/2\}} = L+2n+q, \quad n = 0, 1, \ldots,$$

and every finite-dimensional space $H^{(L)}_{l_{\{p/2\}}, \tilde{l}_{\{q/2\}}}$ enters with multiplicity one. Thus for $g \in SO(p)\times SO(q)$ we have

$$\hat{T}_g(L) = \sum_{l_{\{p/2\}}, \tilde{l}_{\{q/2\}}} \oplus \hat{T}^{\,l_{\{p/2\}}} \otimes \hat{T}^{\tilde{l}_{\{q/2\}}}, \tag{39}$$

where $T_g^{\,l_{\{p/2\}}}$ $(T_g^{\,\tilde{l}_{\{q/2\}}}$ are symmetric finite-dimensional representations of $SO(p)$ $[SO(q)]$ determined by the highest weight m of the type (see eq. 10.3(29))

$$m = (l_{\{p/2\}}, 0, 0, \ldots, 0). \tag{40}$$

Every representation $\hat{T}_g^{\,l_{\{p/2\}}} \otimes \hat{T}_g^{\,\tilde{l}_{\{q/2\}}}$ enters in the decomposition (40) with multiplicity one.

The decomposition of a representation $\hat{T}(\Lambda, +)$ and $\hat{T}(\Lambda, -)$ of the continuous series has the same form as that of the discrete series $\hat{T}(L)$; only the summation in (39) and (40) is taken over all $l_{\{p/2\}} + l_{\{q/2\}}$, even or odd, respectively.

In physical problems for which a noncompact higher symmetry group exists, we are also interested in the decomposition of a given irreducible representation of the noncompact group with respect to irreducible representations of a maximal noncompact subgroup. We can do this in the case of the group $SO_0(p, q)$ if, on the manifold $H^{p,q}$ we introduce the biharmonic coordinate system in which the Laplace–Beltrami operator $\Delta[SO(p, q-1)]$ will be diagonal. Such a biharmonic coordinate system is given by

$$\begin{aligned} x^i &= x'^i \cosh\eta, \quad i = 1, 2, \ldots, p+q-1, \quad p \geqslant q > 2, \\ x^{p+q} &= \sinh\eta, \quad \eta \in (-\infty, \infty), \end{aligned} \tag{41}$$

where the x'^i are the biharmonic coordinates on the hyperboloid $H^{p,q-1}$ given by eq. (9). Using the formula 1(12), we find that in the coordinate system (41) the Laplace–Beltrami operator has the following form

$$\Delta(H^{p,q}) = \frac{-1}{\cosh^{p+q-2}\eta} \frac{\partial}{\partial\eta} \cosh^{p+q-2}\eta \frac{\partial}{\partial\eta} + \frac{\Delta(H^{p,q-1})}{\cosh^2\eta}. \tag{42}$$

Here $\Delta(H^{p,q-1})$ is the Laplace–Beltrami operator of the group $SO_0(p, q-1)$ related to the hyperboloid $H^{p,q-1}$. The discrete and continuous parts of the spectra of $\Delta(H^{p,q})$ and $\Delta(H^{p,q-1})$ are clearly the same as those found in subsec. A. If we represent the eigenfunctions of $\Delta(H^{p,q})$ as a product of an eigenfunction of

$\Delta(H^{p,q-1})$ and a function $V_{\tilde{L}}^{L}(\eta)$, we obtain the following differential equation for the latter function

$$\left[\frac{-1}{\cosh^{p+q-2}\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\cosh^{p+q-2}\eta\frac{\mathrm{d}}{\mathrm{d}\eta}-\frac{\tilde{L}(\tilde{L}+p+q-3)}{\cosh^2\eta}+L(L+p+q-2)\right]V_{\tilde{L}}^{L}(\eta)=0. \tag{43}$$

Using the transformation $V_{\tilde{L}}^{L}(\eta)=\cosh^{(2-p+q)/2}\eta\cdot\psi_{\tilde{L}}^{L}(\eta)$, we obtain for $\psi_{\tilde{L}}^{L}(\eta)$ the differential equation of the type which has been treated by Titchmarsch (1962, part I, § 4, 19). Therefore, we immediately know that both independent solutions ${}_{(\alpha)}V_{\tilde{L}}^{L}(\eta)$, $\alpha=1,2$, enter into the eigenfunction expansion associated with the differential operator (43). The final solution of eq. (42) for the case when spectra of $\Delta(H^{p,q})$ and $\Delta(H^{p,q-1})$ are discrete can be written in the form

$${}_{(\alpha)}Y_{m\tilde{m}\Omega}^{L\tilde{L}l\tilde{l}}={}_{\alpha}V_{\tilde{L}}^{L}(\eta)\,Y_{m\tilde{m}}^{\tilde{L}l\tilde{l}}(\theta,\omega,\tilde{\omega}),\qquad \alpha=1,2, \tag{44}$$

with

$$\begin{aligned}\tilde{L}&=L+(2n+3-\alpha),\qquad n=0,1,2,\dots,\\ L&=-\{\tfrac{1}{2}(p+q-4)\}+k,\qquad k=1,2,\dots\end{aligned} \tag{45}$$

The functions ${}_{\alpha}V_{\tilde{L}}^{L}(\eta)$ can be expressed in terms of Gegenbauer polynomials

$${}_{\alpha}V_{\tilde{L}}^{L}(\eta)=\frac{(-1)^{(\tilde{L}-L+1-\alpha)}}{\sqrt{{}_{\alpha}M}}\cosh^{-(L+p-1)}\eta\;C_{\tilde{L}-L-1}^{L+p/2}(\eta),\qquad \alpha=1,2, \tag{46}$$

where

$${}_{(1)}M=\frac{{}_{(1)}N\cdot\Gamma^2[\tfrac{1}{2}(L+\tilde{L}+p)]}{\Gamma^2(L+\tfrac{1}{2}p)\Gamma^2[\tfrac{1}{2}p(\tilde{L}-L)]},$$

$${}_{(\alpha)}N=\frac{2\pi\Gamma[\tfrac{1}{2}(\tilde{L}-L+\alpha-1)]\Gamma[\tfrac{1}{2}(\tilde{L}+L+p+q+\alpha-3)]}{(2L+p+q-2)\Gamma[\tfrac{1}{2}(\tilde{L}+L+p+q-\alpha)]\Gamma[\tfrac{1}{2}(\tilde{L}-L-\alpha+2)]}$$

and

$${}_{(2)}M=\frac{{}_{(2)}N\cdot 4\cdot\Gamma^2[\tfrac{1}{2}(L+\tilde{L}+p+1)]}{(L+\tilde{L}+p-1)^2\cdot\Gamma^2(L+\tfrac{1}{2}p)\cdot\Gamma^2[\tfrac{1}{2}(\tilde{L}-L+1)]}.$$

The solutions corresponding to the continuous part of the spectra of the invariant operators $\Delta(H^{p,q})$ and $\Delta(H^{p,q-1})$ can be found in a similar manner (Niederle and Limić 1968).

The form (44) of the harmonic functions for the group $SO_0(p,q)$ implies that the carrier space $\hat{H}(L)$ has the following structure

$$H(L)=\sum_{\tilde{L}=L+1}^{\infty}\oplus H(L,\tilde{L}), \tag{47}$$

where $\hat{H}(L,\tilde{L})$ is the infinite dimensional space on which the irreducible representation $\hat{T}(L,\tilde{L})$ of $SO(p,q-1)$ is realized. The space $\hat{H}(L,\tilde{L})$ is spanned by the harmonic functions (22) with L and $\tilde{L}$ fixed. The decomposition of the representation $\hat{T}(L)$ is

$$g\in SO_0(p,q-1),\qquad \hat{T}_g(L)=\sum_{\tilde{L}=L+1}^{\infty}\oplus\hat{T}_g(L,\tilde{L}).$$

C. *Maximal Set of Commuting Operators and Their Spectra*

For applications to physical problems of the group $SO_0(p, q)$ the discrete and continuous series of most degenerate representations $\hat{T}(L)$, $\tilde{T}(L)$ and $\hat{T}(\Lambda, \pm)$, $\tilde{T}(\Lambda, \pm)$ are especially convenient due to the following facts:

(i) The maximal set of commuting operators is maximally reduced in these representations of groups $SO_0(p, q)$. That is, for the discrete most degenerate representations of $SO_0(p, q)$, the maximal set of independent commuting operators in the enveloping algebra consists of

$$\Delta[SO(p, q)],\ R,$$

$$C_p \equiv \begin{Bmatrix} \Delta[SO(p)], \Delta[SO(p-2)], \dots, \Delta[SO(4)] & \text{for } p \text{ even} \\ \Delta[SO(p)], \Delta[SO(p-1)], \Delta[SO(p-3)], \dots, \Delta[SO(4)] & \text{for } p \text{ odd} \end{Bmatrix}, \tag{48}$$

$$\tilde{C}_q \equiv \begin{Bmatrix} \Delta[SO(q)], \Delta[SO(q-2)], \dots, \Delta[SO(4)] & \text{for } p \text{ even} \\ \Delta[SO(q)], \Delta[SO(q-1)], \Delta[SO(q-3)], \dots, \Delta[SO(4)] & \text{for } p \text{ odd} \end{Bmatrix},$$

$$\dot{H} \equiv \left\{ -\frac{\partial}{\partial\varphi^k}, -\frac{\partial}{\partial\varphi^l}, \begin{matrix} k = 1, 2, \dots, [\frac{1}{2}p] \\ l = 1, 2, \dots, [\frac{1}{2}q] \end{matrix} \right\},$$

where $\Delta[SO(p, q)]$ represents the second-order Casimir operator of $SO(p, q)$, and C_p and C_ϱ the sequence of corresponding Casimir operators of the maximal compact subgroup $SO(p) \times SO(q)$. The set H contains operators of the Cartan subalgebra, except when p and q are odd, in which case H represents the maximal abelian compact subgroup of $SO_0(p, q)$. It should be noted that the reflection operator R, which is outside the enveloping algebra of $SO_0(p, q)$ is necessary for the characterization of an irreducible representation of the continuous series.

The number of operators contained in the maximal set of commuting operators in the enveloping algebra for the discrete most degenerate representatons of $SO_0(p, q)$ is equal to

$$N = p+q-1, \tag{49}$$

while the corresponding number for principal non-degenerate representations is

$$N' = \tfrac{1}{2}(r+l) = \tfrac{1}{4}[N(N+1)+2l],$$

where r and l are the dimension and the rank of $SO_0(p, q)$, respectively.

(ii) The additive quantum numbers may be related to the eigenvalues of the set H. It turns out that the set H is largest in the biharmonic coordinate system, which we have used.

(iii) The eigenfunctions of the maximal commuting set of operators are given in explicit form by formulas (22) and (30) ; the range of the numbers $L, l_2, \dots$, $\dots, l_{[p/2]}\, \tilde{l}_2, \dots, \tilde{l}_{[q/2]}, m_1, \dots, m_{[p/2]}, \tilde{m}_1, \dots, m_{[q/2]}$ which may play the role of quantum numbers, is determined by (19), (21) and 10.3(22), respectively.

The main achievement of the present method consists in recasting some of the difficult problems of the representation theory of locally compact Lie groups into the language of the relatively simple theory of second-order differential equations. This method may also be applied to the explicit construction of the less degenerate representations which are determined by two, three, ..., k ($k \leqslant n$— rank of $SO_0(p, q)$) invariant numbers.

§ 4. Generalized Projection Operators

We developed in 7.3 the formalism of projection operators P^{λ}_{pq} which we used for an effective solution of various problems in the representation theory of compact groups and in particle physics. The operators P^{λ}_{pq} were defined by the formula

$$P^{\lambda}_{pq} = d_{\lambda} \int_G \overline{D^{\lambda}_{pq}}(x)\, T_x \mathrm{d}\mu(x). \tag{1}$$

If one tries to extend the formula (1) for noncompact groups one encounters the following difficulties:

(i) the matrix elements $D^{\lambda}_{pq}(x)$ are distributions from $\Phi'(G)$,

(ii) the volume $\int_G \mathrm{d}x$ is infinite.

Hence, the proper meaning of the integral (1) should be clarified. Consider first as an illustration the case of an abelian vector group G. In this case $D^{\lambda}_{pq}(X)$ reduces to $\exp(ipx)$ and the integral (1) takes the form

$$P^{\lambda} = (2\pi)^{-n/2} \int_G \exp(-\mathrm{i}\lambda x)\, T_x \mathrm{d}x, \tag{2}$$

where $x \to T_x$ is the regular representation of G. For $\varphi(x) \in \Phi(G)$ we have

$$\begin{aligned}(P^{\lambda}\varphi)(x) &= (2\pi)^{-n/2} \int_G \exp(-\mathrm{i}\lambda x')\varphi(x+x')\mathrm{d}x' \\ &= (2\pi)^{-n/2} \exp(\mathrm{i}\lambda x) \int_G \exp(-i\lambda y)\varphi(y)\mathrm{d}y \\ &= \exp(\mathrm{i}\lambda x)\hat{\varphi}(\lambda) \in H(\lambda) \subset \Phi'.\end{aligned}$$

Thus P^{λ} represents a map from $\Phi(G)$ into $\Phi'(G)$, i.e. it is an operator-valued distribution. To be precise, one should consider first the quantity

$$P^{\lambda}_{N} = (2\pi)^{-n/2} \int_{G_N} \exp(-\mathrm{i}\lambda x)\, T_x \mathrm{d}x, \tag{3}$$

where G_N is a compact subset of G and $\lim_{N\to\infty} G_N = G$. Since $\exp(-\mathrm{i}\lambda x)\, T_x$ is a continuous function on G and G_N has a finite Haar measure, the integral (3) is well defined. We have, moreover,

$$(P^{\lambda}\varphi)(x) \stackrel{\text{def}}{=} \lim_{N\to\infty} (P^{\lambda}_{N}\varphi)(x) = \exp(\mathrm{i}\lambda x)\hat{\varphi}(\lambda),$$

where the limit is taken in the topology of $\Phi'(G)$. We see therefore that the quantity P^λ given by eq. (2) is well defined for noncompact abelian vector groups as a weak limit of the operators given by (3), i.e.,

$$P^\lambda = \lim_{N\to\infty} (2\pi)^{-n/2} \int_{G_N} \exp(-i\lambda x)\, T_x \mathrm{d}x.$$

If G is a vector group, then, the space Λ of indices λ which determines the irreducible representations of G is also a vector group. Consider the Schwartz's space $D(\Lambda)$ of functions with support in Λ. Then for $f \in D(\Lambda)$, one can define a 'smeared out' operator $P(f)$ of the form

$$P(f) = \int_\Lambda f(\lambda) P^\lambda \mathrm{d}\lambda. \tag{4}$$

For $u \in \Phi(G)$ we have

$$||P(f)u|| \leqslant \max_{\lambda \in \hat{G}} |f(\lambda)|\, ||u||_H.$$

This means that $P(f)$ is a densely defined bounded operator in $H(G)$.

One readily verifies that

$$P(f_1)P(f_2) = P(f_1 f_2).$$

This equality can be written in the form of product of operator-valued distributions

$$P^\lambda P^{\lambda'} = \delta(\lambda-\lambda')P^\lambda.$$

This is the generalization for noncompact groups of the orthogonality relations 7.3(3).

We see therefore that for a proper description of the projection operators for noncompact groups we have to use the technique of operator-valued distributions. Hence we begin with a review of the basic notions concerning operator-valued distributions in a Hilbert space.

Let $H = L^2(\Lambda)$, where Λ is a subset of an n-dimensional Euclidean or Minkowski space R^n. Let $K(\Lambda)$ be a space of test functions (e.g. Schwartz $D(\Lambda)$ or $S(\Lambda)$ space).

DEFINITION 1. An *operator-valued distribution* P is a mapping $f \to P(f)$, $f \in K(\Lambda)$, with values in a set of linear operators in H satisfying

1° The operators $P(f)$ and $P(f)^*$, $f \in K(\Lambda)$, have a common dense domain D, which is a linear subset of H satisfying

$$P(f)D \subset D, \quad P(f)^*D \subset D. \tag{5}$$

2° On the domain D, $P(f)$ fulfils the conditions

$$P(\alpha f) = \alpha P(f),$$
$$P(f_1+f_2) = P(f_1)+P(f_2),$$

where $\alpha \in C^1$ and $f_1, f_2 \in K(\Lambda)$.

3° $P(f)$ is weakly continuous in $K(\Lambda)$, i.e. if $u, v \in D$ and $f \to 0$ then

$$(P(f)u, v) \to 0. \tag{6}$$

The operator-valued distribution $f \to P(f)$ can sometimes be written in the form of an integral

$$P(f) = \int_\Lambda f(\lambda)P(\lambda)\mathrm{d}\lambda. \tag{7}$$

The symbol $P(\lambda)$ has often a direct meaning and is currently also called an *operator-valued distribution.*

DEFINITION 2. The *adjoint* $P^*(\lambda)$ *of an operator distribution* $P(\lambda)$, is an operator distribution such that to a test function $f(\lambda)$ in $K(\Lambda)$ it assigns the operator $[P(\bar{f})]^*$, i.e.,

$$P^*(f) \equiv \int_\Lambda f(\lambda)P^*(\lambda)\mathrm{d}\lambda \equiv [P(\bar{f})]^* = \Big[\int_\Lambda \overline{f(\lambda)}P(\lambda)\mathrm{d}\lambda\Big]^*. \tag{8}$$

An operator-valued distribution is said to be *real* if $P(\lambda)^* = P(\lambda)$.

Let G be a locally compact Lie group, for which the assertions of th. 14.2.1 are valid, e.g. G is a semisimple Lie group, a group of motions of Euclidean or Minkowski space, etc. Let $C_1, \ldots, C_n$ be '+'-symmetric generators in the center $Z(E)$ of the enveloping algebra E of G and let $\bar{C}_1, \ldots, \bar{C}_n$ be the corresponding self-adjoint invariant operators. Let $H = L^2(G)$ and let $\Phi(G) \subset H \subset \Phi'(G)$ be a Gel'fand triplet, such that all $\bar{C}_i$ map Φ continuously into Φ. Let $\{D^\lambda_{pq}(x)\}$ be a set of generalized eigenfunctions of the operators $\bar{C}_1, \ldots, \bar{C}_n$ provided by th. 14.2.1 which satisfy the completeness 14.2(35) and orthogonality 14.2(34) relations and give the decomposition 14.2(36) of any element φ in $\Phi(G)$.

Let us now introduce the quantity

$$P^\lambda_{pq} = \varrho(\lambda)\int_G \overline{D^\lambda_{pq}(x)}\,T_x\mathrm{d}x, \tag{9}$$

where T_x is the right regular representation, i.e.,

$$(T_x\varphi)(y) = \varphi(yx), \tag{10}$$

and $\varrho(\lambda)$ is a spectral measure associated with the invariant operators $\bar{C}_1, \ldots, \bar{C}_n$ of G. We have

PROPOSITION 1. *The quantity* P^λ_{pq} *given by* eq. (9) *represents an operator-valued distribution in the space* $H = L^2(G)$ *and maps* $\Phi(G)$ *into* $\Phi'(G)$.

PROOF: Set

$$P^\lambda_{pqN} = \varrho(\lambda)\int_{G_N} \overline{D^\lambda_{pq}(x)}\,T_x\mathrm{d}x, \tag{11}$$

where G_N is a compact subset of G and $\lim_{N\to\infty} G_N = G$. Since $\overline{D^\lambda_{pq}(x)}\,T_x$ is continuous on G, the integral (11) gives a well defined operator in H. For $\varphi \in \Phi(G)$ we have

$$(P^{\lambda}_{pqN}\varphi)(y) = \varrho(\lambda) \int_{G_N} dx \overline{D^{\lambda}_{pq}(x)} (T_x\varphi)(y) = \varrho(\lambda) \int_{yG_N} dz \overline{D^{\lambda}_{pq}(y^{-1}z)} \varphi(z)$$
$$= \varrho(\lambda) \sum_r \overline{D^{\lambda}_{pr}(y^{-1})} \int_{yG_N} dz \overline{D^{\lambda}_{rq}(z)} \varphi(z)$$
$$\xrightarrow[N\to\infty]{} \varrho(\lambda) \sum_r D^{\lambda}_{rp}(y) \hat{\varphi}_{rq}(\lambda). \tag{12}$$

The interchange of integration and summation in eq. (12) is justified by Fubini–Tonelli theorem if $\varphi \in L^1(G)$. Indeed,

$$\Big|\sum_r \overline{D^{\lambda}_{pr}(y^{-1})}\, \overline{D^{\lambda}_{rq}(z)}\, \varphi(z)\Big|^2 \leqslant |\varphi(z)|^2 \sum_r |\overline{D^{\lambda}_{pr}(y^{-1})}|^2 \sum_{r'} |\overline{D^{\lambda}_{r'q}(z)}|^2 = |\varphi(z)|^2,$$

since e.g.

$$\sum_r |\overline{D^{\lambda}_{pr}(z)}|^2 = \sum_r \langle e_p, T^{\lambda}_z e_r \rangle_{\hat{H}(\lambda)} \langle T^{\lambda}_z e_r, e_p \rangle_{\hat{H}(\lambda)} = 1.$$

We see that, as for abelian vector groups, we have

$$P^{\lambda}_{pq} = \lim_{N\to\infty} \varrho(\lambda) \int_{G_N} dx \overline{D^{\lambda}_{pq}(x)}\, T_x, \tag{13}$$

in the sense of the weak limit of operators $(P^{\lambda}_{pq})_N$. The expression (4.12) shows that for any $\varphi(y)$ in $\Phi(G)$ the quantity $(P^{\lambda}_{pq}\varphi)(y)$ is an element of $\varphi(\lambda, y) \in H(\lambda) \subset \Phi'(G)$. Therefore, by eq. (3) of app. B, $P^{\lambda}_{pq}\varphi$ is a generalized eigenvector of the invariant operators $C_1, \ldots, C_n$. Thus, the domain $D(P^{\lambda}_{pq})$ of P^{λ}_{pq} in $H = L^2(G)$ consists of the null-vector only and consequently, P^{λ}_{pq} cannot be considered as an operator in H. However, if $f(\lambda)$ is an element of the space $C_0(\Lambda)$ of continuous functions on Λ then, the smeared-out operator

$$P_{pq}(f) = \int_{\Lambda} f(\Lambda) P^{\lambda}_{pq} d\lambda, \tag{14}$$

represents a bounded linear operator. Indeed, for $\varphi \in \Phi(G)$, one obtains from eq. (12)

$$P_{pq}(f)\, \varphi(y) = \int_{\Lambda} d\lambda f(\lambda) \varrho(\lambda) \sum_r D^{\lambda}_{rp}(y) \hat{\varphi}_{rq}(\lambda).$$

Using the Plancherel equality 14.2(17) we have

$$||P_{pq}(f)\varphi||^2 = \int_{\Lambda} |f(\lambda)|^2 \sum_r \hat{\varphi}_{rq}(\lambda) \overline{\hat{\varphi}_{rq}(\lambda)} \varrho(\lambda) d\lambda \leqslant \max_{\lambda\in\Lambda} |f(\lambda)|^2 ||\varphi||^2,$$

i.e.

$$||P_{pq}(f)\varphi|| \leqslant \max_{\lambda\in\Lambda} |f(\lambda)|\, ||\varphi||. \tag{15}$$

Since the operators $P_{pq}(f)$ are bounded for any $f \in C_0(\Lambda)$ one can take the whole space H as a common dense domain D in def. 1. It is then evident that $P_{pq}(f)$ satisfies the conditions 1° and 2° of def. 1. Condition 3° follows from eq. (15). In fact,

$$|(P_{pq}(f)\varphi, \psi)|^2 \leqslant \|P(f)\varphi\|^2\|\psi\|^2 \leqslant \max_{\lambda\in\Lambda}|f(\lambda)|^2\|\varphi\|^2\|\psi\|^2.$$

Hence, $(P_{pq}(f)\varphi, \psi) \to 0$ whenever $f \to 0$ in $C_0(\Lambda)$. We see, therefore, that the mappings $P^\lambda_{pq}: \Phi(G) \to H(\lambda) \subset \Phi'(G)$ given by eq. (13) represent operator-valued distributions in the Hilbert space $H = L^2(G)$. ▼

Remark: If T_x represents the operator of a left translation in $L^2(G)$ then the operator distribution P^λ_{pq} has to be taken in the form

$$P^\lambda_{pq} = \varrho(\lambda)\int \overline{D^\lambda_{pq}(x)}\, T_{x^{-1}}\mathrm{d}x. \tag{16}$$

It is only with such a definition that $(P^\lambda_{pq}\varphi)(y)$ represents an element $\varphi(\lambda, y)$ in $H(\lambda)$ (cf. eq. (12)).

The operator-valued distribution P^λ_{pq} satisfies certain hermicity and orthogonality relations which are very useful in applications. In fact, we have:

PROPOSITION 2. *Let P^λ_{pq} be an operator-valued distribution given by formula* (9). *Then,*

$$(P^\lambda_{pq})^* = P^\lambda_{qp} \tag{17}$$

and

$$P^\lambda_{pq} P^{\lambda'}_{p'q'} = \delta(\lambda-\lambda')\,\delta_{qp'}P^\lambda_{pq'}. \tag{18}$$

PROOF: The operator $P^\lambda_{pq}(f)$ is bounded. Hence, for any φ, ψ in $\Phi(G)$, by eqs. (7), (8) and (12) and the Plancherel equality, 14.2(17) one obtains

$$(\varphi, P^{*\lambda}_{pq}(f)\psi) = (P^\lambda_{pq}(\bar{f})\varphi, \psi) = \left(\int d\lambda \overline{f(\lambda)} P^\lambda_{pq}\varphi, \psi\right) = \int \mathrm{d}\lambda \overline{f(\lambda)}\varrho(\lambda)\hat{\varphi}_{rq}(\lambda)\overline{\hat{\psi}_{rp}(\lambda)}.$$

Using the same formulae one verifies that the last expression is equal to $(\varphi, P^\lambda_{pq}(f)\psi)$. Hence,

$$P^{*\lambda}_{pq}(f) = P^\lambda_{qp}(f), \tag{19}$$

and eq. (17) follows by def. 1 and eq. (7). We now prove eq. (18). In fact, for any f, g in $C_0(\Lambda)$ and φ, ψ in $L^2(G)$, eqs. (19) and (12) and the Plancherel equality 14.2 (17) yield

$$\begin{aligned}(P^\lambda_{pq}(f)P^\lambda_{p'q'}(g)\varphi, \psi) &= (P^\lambda_{p'q'}(g)\varphi, P^\lambda_{qp}(\bar{f})\psi) \\ &= \delta_{p'q}\int \mathrm{d}\lambda f(\lambda)g(\lambda)\varrho(\lambda)\hat{\varphi}_{rq'}(\lambda)\overline{\hat{\psi}_{rp}(\lambda)}.\end{aligned} \tag{20}$$

One readily verifies that the last expression can be written in the form

$$\left(\left\{\int_\Lambda f(\lambda')d\lambda' \int_\Lambda g(\lambda)d\lambda\,\delta(\lambda-\lambda')\,\delta_{p'q}P^\lambda_{pq'}\right\}\varphi, \psi\right).$$

Comparing this with the first expression of eq. (20), one obtains eq. (18) by proposition 1 and eq. (7). ▼

The operator-valued distributions P^{λ}_{pq} have simple transformation properties with respect to the action of the group G. Indeed, we have

PROPOSITION 3. *Let P^{λ}_{pq} be an operator-valued distribution given by formula* (9) *and let $x \in G$. Then*

$$T_x P^{\lambda}_{pq} = \sum_r D^{\lambda}_{rp}(x) P^{\lambda}_{rq}, \tag{21}$$

$$P^{\lambda}_{pq} T_x = \sum_r D^{\lambda}_{rq}(x) P^{\lambda}_{pr}. \tag{22}$$

PROOF: Since T_x is continuous, the product $T_x P^{\lambda}_{pq}$ is an operator-valued distribution and $T_x P_{pq}(f)$ is a bounded operator in H by proposition 1. Utilizing formula (9) and the Plancherel equality one obtains

$$\begin{aligned}(T_x P^{\lambda}_{pq}(f)\varphi, \psi) &= \int_{G} dy (P^{\lambda}_{pq}(f)\varphi)(yx)\overline{\psi(y)} \\ &= \int_{\Lambda} d\lambda f(\lambda)\varrho(\lambda) D^{\lambda}_{q'p}(x)\hat{\varphi}_{p'q}(\lambda)\overline{\hat{\psi}_{p'q'}(\lambda)} \\ &= \Big(\int_{\Lambda} d\lambda f(\lambda) D^{\lambda}_{q'p}(x') P^{\lambda}_{q'q}\varphi, \psi\Big).\end{aligned}$$

Comparing the first and last terms of this equality and utilizing def. 1 and eq. (7), one obtains eq. (21). Similarly, one proves eq. (22). ▼

By eqs. (21) and (22) one also obtains

$$T_x P^{\lambda}_{pq} T_x^{-1} = D^{\lambda}_{rp}(x)\overline{D^{\lambda}_{sq}(x)} P^{\lambda}_{rs}. \tag{23}$$

Formula (23) means that P^{λ}_{pq} transforms as a tensor operator corresponding to the tensor product of a basis vector $e_p(\lambda)$ and an adjoint vector to $e_q(\lambda)$ (i.e., as the product $|\lambda{:}p\rangle\langle\lambda{:}q|$ in Dirac's notation).

In some cases, like e.g. in the case of semisimple groups or the Poincaré group, the character $\chi^{\lambda}(x) = \operatorname{Tr} T_x(\lambda)$ is a well defined distribution on G. In this case one may define the following operator-valued distributions in $H = L^2(G)$

$$P^{\lambda} = \varrho(\lambda)\int_G dx \overline{\chi^{\lambda}(x)} T_x. \tag{24}$$

One verifies similarly as in propositions 2 and 3 that

$$(P^{\lambda})^* = P^{\lambda}, \tag{25}$$

$$P^{\lambda}P^{\lambda'} = \delta(\lambda-\lambda')P^{\lambda}, \tag{26}$$

$$T_x P^{\lambda} = P^{\lambda} T_x. \tag{27}$$

The operator-valued distributions P^{λ} are useful in applications. If $H(X)$ is a carrier Hilbert space of a unitary representation T of G and $\Phi \subset H \subset \Phi'$ is the Gel'fand triplet, then P^{λ} projects Φ onto the generalized eigenspace $H(\lambda) \subset \Phi'$ The space $P^\lambda\Phi$ isomorphic with $H(\lambda)$ is invariant under T and it is isomorphic to the Hilbert space $\hat{H}(\lambda)$ by formula 3(30) of app. B.

We have considered so far the formalism of operator-valued distributions in the Hilbert space $H = L^2(G)$. However, all of our results can be extended to the space $H = L^2(X)$, where $X = G/G_0$ is the homogeneous space of right G-cosets $\{G_0 g\}$, G is a connected Lie group and G_0 is a closed subgroup of G. The group G acts on the elements $\varphi \in \Phi(x) \subset L^2(x)$ by means of the right translation

$$(T_g\varphi)(x) = \varphi(xg),$$

i.e., the map $g \to T_g$ gives the unitary quasi-regular representation of G in $L^2(X)$.

Suppose that all assumptions of th. 2.1 are satisfied. Then the generalized Fourier expansion of an element $\varphi \in \Phi(X) \subset L^2(X)$ is given by the formula 1(4). Using formula (9) for P^λ_{pq} and eqs. 1(4) and 2(12) one obtains

$$\begin{aligned}(P^\lambda_{pq}\varphi)(x) &= \varrho(\lambda)\int \mathrm{d}g\,\overline{D^\lambda_{pq}(g)}\,\varphi(xg)\\ &= \varrho(\lambda)\int_G \mathrm{d}g\,\overline{D^\lambda_{pq}(g)}\int_{\tilde{\Lambda}}\sum_{r,s} D^{\lambda'}_{s,r}(g)\hat{\varphi}_r(\lambda')e_s(\lambda' x)\,\mathrm{d}\tilde{\varrho}(\lambda')\\ &= \tilde{\varrho}(\lambda)\hat{\varphi}_g(\lambda)e_p(\lambda, x) \in H(\lambda) \subset \Phi'(X),\end{aligned} \tag{28}$$

provided $\hat{\varphi}_r(\lambda)$ and $e_s(\lambda, x)$ are continuous functions of λ; this condition is satisfied in most cases of practical interest. Hence, as in case of $L^2(G)$, the quantity P^λ_{pq} represents a map from $\Phi(X)$ into $H(\lambda) \subset \Phi'(x)$, i.e., it defines an operator-valued distribution. The proof of propositions 2 and 3 for operator-valued distributions P^λ_{pq} in $L^2(X)$ runs similarly and we omit them.

As we have shown, the spaces $H(\lambda)$ spanned by the generalized eigenvectors $e_k(\lambda)$ play an important role in applications. The operator-valued distributions P^λ_{pq} and P^λ provide a natural tool for the separation of these spaces from the space $H(G)$ or $H(X)$.

Operator-valued distributions P^λ_{pq} provide a convenient method for the solution of various practical problems encountered in group representation theory and quantum physics. For instance, one can derive a general formula for the Clebsch–Gordan coefficients for non-compact Lie groups. Indeed, let $\tilde{H} = H(\lambda_1) \otimes H(\lambda_2)$ be the carrier space of the tensor product $T = T^{\lambda_1} \otimes T^{\lambda_2}$ of irreducible representations T^{λ_1} and T^{λ_2} of G and let $\{e_k(\lambda_i)\}_{k=1}^{\infty}$ be a basis in $H(\lambda_i)$, $i = 1, 2$. Then, for constant q, r and s the element

$$e_p(\lambda) = P^\lambda_{pq}e_r(\lambda_1)e_s(\lambda_2), \tag{29}$$

by virtue of eq. (21), has the following transformation properties

$$T_x e_p(\lambda) = \sum_r D^\lambda_{rp}(x)e_r(\lambda),$$

i.e., it transforms according to an irreducible unitary representation T^λ of G. Consequently the expression

$$\begin{aligned}\langle\lambda, p|\lambda_1, p_1; \lambda_2 p_2\rangle &\equiv N^{-1}\langle e_p(\lambda), e_{p_1}(\lambda_1)e_{p_2}(\lambda_2)\rangle \\ &\equiv N^{-1}\langle P^{\lambda}_{pq}e_r(\lambda_1)e_s(\lambda_2), e_{p_1}(\lambda_1)e_{p_2}(\lambda_2)\rangle \qquad (30)\end{aligned}$$

is the projection of a basis vector $e_p(\lambda)$ upon the basis vector $e_{p1}(\lambda_1)e_{p2}(\lambda_2)$ and represents the Clebsch–Gordan coefficient. The constant N represents a normalization constant of the vector (29). We see that the Clebsch–Gordan coefficient (30) is, in fact, the matrix element of the operator-valued distribution P^{λ}_{pq} in the tensor product basis of the space $\tilde{H}$. Using eq. (9) for P^{λ}_{pq} and the relation

$$\begin{aligned}T_x e_r(\lambda_1)e_s(\lambda_2) &= T^{\lambda_1}_x e_r(\lambda_1)\cdot T^{\lambda_2}_x e_s(\lambda_2) \\ &= \sum_{r_1} D^{\lambda}_{r_1 r}(x)e_{r_1}(\lambda_1)\sum_{s_2} D^{\lambda_2}_{s_2 s}e_{s_2}(\lambda_2),\end{aligned}$$

one obtains

$$\langle\lambda, p|\lambda_1, p_1; \lambda_2 p_2\rangle = N^{-1}d^{\lambda}\int_G \mathrm{d}x\,\overline{D^{\lambda}_{pq}(x)}\,D^{\lambda_1}_{p_1 r}(x)D^{\lambda_2}_{p_2 s}(x). \qquad (31)$$

§ 5. Comments and Supplements

A. *Harish–Chandra and Helgason Theory*

We shall now describe a very interesting approach to harmonic analysis on symmetric spaces G/K based on geometric ideas. This theory was originated by Gel'fand and Harish–Chandra and finally completed by Helgason 1967, 1972, 1973, 1974.

In the case of ordinary Fourier analysis, we have

$$\hat{\varphi}(p) = \int_{R^n} \varphi(x)\exp[\mathrm{i}(x,p)]\,\mathrm{d}x,$$

where $(x,p) \equiv x_\mu p^\mu$, and

$$\varphi(x) = (2\pi)^{-n}\int_{R^n} \hat{\varphi}(p)\exp[\mathrm{i}(x,p)]\,\mathrm{d}p,$$

or, in polar coordinates $p = \lambda\omega$, $\lambda \geqslant 0$ and ω a unit vector,

$$\hat{\varphi}(\lambda\omega) = \int_{R^n} \varphi(x)\exp[\mathrm{i}\lambda(x,\omega)]\,\mathrm{d}x, \qquad (1)$$

$$\varphi(x) = (2\pi)^{-n}\int_{R^+}\int_{S^{n-1}} \hat{\varphi}(\lambda\omega)\exp[\mathrm{i}\lambda(x,\omega)]\,\lambda^{n-1}\mathrm{d}\lambda\,\mathrm{d}\omega, \qquad (2)$$

where $R^+ = \{\lambda \in R: \lambda \geqslant 0\}$ and $\mathrm{d}\omega$ is the volume element on the unit sphere S^{n-1}. The function e_p: $x \to \exp[\mathrm{i}(x,p)]$ has the following properties;

(i) e_p is an eigenfunction of the Laplace operator on R^n.

(ii) e_p is constant on each hyperplane perpendicular to p (i.e., e_p is a plane wave with the normal p).

In order to extend the harmonic analysis from R^n to the symmetric spaces $X = G/K$, one needs a generalized 'plane wave' e_p on X which would satisfy properties (i) and (ii) and provide an expansion of functions from $L^2(X, \mu)$.

The simplest nontrivial case where this extension can be done is the symmetric space $X = \mathrm{SU}(1, 1)/U(1)$, which is isomorphic to a disc $D = \{z \in C: |z| < 1\}$. (See ch. 4, exercise 5.2.4.) We shall now generalize the geometric properties of plane waves to the curved space D. Let B be the boundary of D, i.e., $B = \{z \in C: |z| = 1\}$. The parallel geodesics in D are by definition geodesics originating from the same point b on the boundary B of D (see Fig. 1).

A horocycle with normal $b \in B$ is by definition an orthogonal trajectory to the family of all parallel geodesics corresponding to b (see Fig. 1.) Hence a horo-

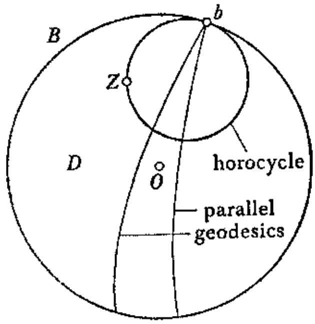

Fig. 1

cycle in D is the non-euclidean analog of a hyperplane in R^n. Now the inner product (x, ω) in eq. (1) is the distance from the origin to the hyperplane with normal ω passing through x. By analogy, we define $\langle z, b\rangle$ for $z \in D$, $b \in B$ to be the Riemannian distance from 0 to the horocycle $\xi(z, b)$ with normal b, passing through z. Now the function

$$e_{p,b}: z \to \exp(p\langle z, b\rangle), \quad p \in C,\ b \in B,\ z \in D \tag{3}$$

has the properties of plane waves on R^n. Indeed,

(i) $e_{p,b}$ is the eigenfunction of the Laplace–Beltrami operator Δ on D ($\Delta = [1-(x^2+y^2)]^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$, see exercise 4.5.2.5).

(ii) $e_{p,b}$ is constant on each horocycle $\xi(z, b)$ with normal b.

The following theorem shows that the eigenfunctions $e_{p,b}(z)$ on D are, in fact, the natural extension of the plane waves e_p on R^n.

THEOREM 1. *Let $\varphi \in C_0(D)$. Set*

$$\hat{\varphi}(p, b) = \int_D \varphi(z)\exp[(-ip+1)\langle z, b\rangle]\mathrm{d}z, \quad p \in R,\ b \in B, \tag{4}$$

where $\mathrm{d}z = (1-|z|^2)^{-2}\mathrm{d}x\mathrm{d}y$ *is the volume element on* D. *Then*

$$\varphi(z) = (2\pi)^{-2} \int_R \int_B \hat{\varphi}(p, b) \exp[(\mathrm{i}p+1)\langle z, b\rangle] p \tanh(\tfrac{1}{2}\pi p) \mathrm{d}p\, \mathrm{d}b, \tag{5}$$

where $\mathrm{d}b$ *is the ordinary angular measure on* B $(= S^1)$. ▼

(For the proof see Helgason 1967, th. 3.1.)

To extend this theorem to an arbitrary symmetric space, we need a generalization of the concepts of the boundary B, the horocycle ξ and the complex distance $\langle z, b\rangle$.

Let K be the maximal compact subgroup of a semisimple Lie group G and let L be the Lie algebra of G. Let $L = K \oplus H_p \oplus N_0$ be the Iwasawa decomposition of L. Denote by M the centralizer of H_p in K; that is,

$$M = \{k \in K : (\mathrm{Ad}k) Y = Y, \text{ for all } Y \in H_p\}. \tag{6}$$

It turns out that a generalization of the boundary B for an arbitrary symmetric space is given by the coset space $B = K/M$.

The group-theoretical analysis of the horocycle on the disc D reveals that horocycles are the orbits in D of all groups of the form $g\mathcal{N}g^{-1}$, where $\mathcal{N}$ is the nilpotent group associated with the algebra $\mathcal{N}_0$ of SU(1,1). This suggests that for an arbitrary symmetric space $X = G/K$ we may define a horocycle to be an orbit in X of a subgroup of G of the form $g\mathcal{N}g^{-1}$, where $\mathcal{N}$ is the nilpotent group in the Iwasawa decomposition $G = K\mathcal{A}_p\mathcal{N}$. One readily verifies the following properties of horocycles:

LEMMA 2.

(i) *The group* G *permutes the horocycles transitively.*

(ii) *Each horocycle* ξ *can be written in the form*

$$\xi = ka\xi_0, \tag{7}$$

where a *is the unique element of the subgroup* $\mathcal{A}_p$.

(iii) *Given* $x \in X$, $b \in B$, *there exists exactly one horocycle passing through* x *with normal* b. ▼

(For the proof see Helgason 1967.)

The element $a \in \mathcal{A}_p$ in eq. (7) is called the *complex distance* from the coset $K = 0$ to ξ. We denote the complex distance $a \in \mathcal{A}_p$ from 0 to the horocycle determined by lemma 2, (iii), by the symbol $\exp A(x, b)$, $A(x, b) \in H_p$.

Now we give a generalization of plane waves on arbitrary symmetric spaces. Let $b \in B$ and let p be a complex linear functional on H_p and set

$$e_{p,b}: x \to \exp[p(A(x, b))], \ x \in X.$$

It is evident that $\exp[p(A(x, b))]$ is constant on horocycles. We have

THEOREM 3. *Let* $X = G/K$ *be a Cartan symmetric space. Then*

(i) *The functions* $e_{p,b}$ *are eigenfunctions of all the invariant operators* C *from* *the center Z of the enveloping algebra E of G, represented by differential operators in $L^2(X, \mu)$.*

(ii) *Define for $\varphi \in C_0^\infty(X)$, a generalized Fourier transform by the formula*

$$\hat{\varphi}(p, b) = \int_X \varphi(x) \exp[(-ip+\varrho)A(x, b)] \, d\mu(x), \tag{8}$$

where p is an element of the real dual H_p^* of H_p, $b \in B$ and $\varrho = \sum_{\alpha>0} \alpha$. Then the spectral synthesis formula for $\varphi(x)$ has the form

$$\varphi(x) = \int_{H_p^*} \int_B \hat{\varphi}(p, b) \exp[(ip+\varrho)A(x, b)] |c(p)|^{-2} dp \, db, \tag{9}$$

if the Euclidean measure dp *on H_p^* is suitably normalized. The spectral density $|c(p)|^{-2}$ is defined by the formula*

$$c(p) = \int_{\bar{\mathcal{N}}} \exp[(-ip-\varrho)Y(\bar{n})] d\bar{n}, \tag{10}$$

where $\bar{\mathcal{N}}$ is the analytic subgroup associated with the Lie algebra $\bar{N} = \sum_{\alpha<0} L_\alpha$ and $Y(\bar{n})$, $\bar{n} \in \bar{\mathcal{N}}$ is determined by the Iwasawa decomposition $\bar{n} = k(\bar{n}) \exp[Y(\bar{n})] n(\bar{n})$.

Furthermore, we have the Parseval formula

$$\int |\varphi(x)|^2 d\mu(x) = \int_{H_p^*} \int_B \hat{\varphi}(p, b) |c(p)|^{-2} dp \, db, \tag{11}$$

and the direct integral representation of the space $H = L^2(X, \mu)$ and of the representation T_g: $\varphi(x) \to \varphi(g^{-1}x)$ are given by

$$H \to \hat{H} = \int \hat{H}(p) |c(p)|^{-2} dp, \quad \hat{T}_g = \int \hat{T}_g(p) |c(p)|^{-2} dp, \tag{12}$$

where p runs over H_p^* modulo the Weyl groups. All the functions $u_p(x) \in H(p)$ given by

$$u_p(x) = \int_B \exp[i(p+\varrho)A(x, b)] u(b) db, \quad u(b) \in L^2(B, db), \tag{13}$$

are eigenfunctions of all invariant differential operators of the center Z of the enveloping algebra E of G. ▼

(For the proof see Helgason 1967.)

Th. 3 represents one of the most remarkable results in the theory of harmonic analysis on homogeneous spaces. We emphasize that the proof of the theorem is essentially geometric and does not use the spectral analysis of self-adjoint operators and all the machinery of functional analysis, yet it still provides the spectral measure in explicit form.

B. *Comments*

(i) The first work dealing with harmonic analysis on homogeneous spaces was done by Hecke in 1918, where finite-dimensional spaces of continuous functions on the sphere S^2, invariant under rotations were classified. Later on E. Cartan in 1929 extended Peter–Weyl harmonic analysis on compact groups to harmonic analysis on compact Riemannian spaces with transitive compact Lie group of isometries. However, the full scale activity in this field of research began after 1950. The most important contributions were done in the works of Gel'fand 1950, Godement 1952, Berezin and Gel'fand 1956, Gel'fand and Graev 1959, Harish-Chandra 1958, I and II, Berezin 1957, Gindikin and Karpelevic 1962, Helgason 1959, 1962, 1965, 1967, 1970, 1972, 1973, 1974, and Vilenkin 1956, 1963 and 1968.

The harmonic analysis on arbitrary homogeneous spaces, presented in sec. 2 was elaborated by K. Maurin and L. Maurin 1964 (see also K. Maurin 1969, ch. VII). It provides an elegant solution of basic problems of harmonic analysis on homogeneous spaces, based on general nuclear spectral theorem. The harmonic analysis on symmetric spaces of rank one with pseudoorthogonal transformation group was elaborated by Limić, Niederle and Rączka 1966 a and b and 1967. The harmonic analysis on homogeneous spaces of rank one of $SO_0(p, q)$ groups with noncompact stability group was elaboratet by Niederle 1967 and Limić and Niederle 1968. The extension of this theory for symmetric spaces associated with pseudo-unitary groups $U(p, q)$ and symplectic groups $Sp(n)$ was elaborated by Fischer and Rączka 1966, 1967 and Pajas and Rączka 1969, respectively.

The geometric approach to harmonic analysis on symmetric spaces presented in sec. 4.A was originated by Harish-Chandra 1958, I and II and completed by Helgason 1967, 1970, 1972, 1973, 1974.

(ii) We considered in sec. 3 the harmonic analysis on symmetric spaces of rank one associated with pseudo-orthogonal groups $SO_0(p, q)$, $p \geqslant q > 2$. The same analysis can be done for conformal type groups $SO_0(p, 2)$ and Lorentz type groups $SO_0(p, 1)$. The detailed analysis may be found in a series of papers by Limić, Niederle and Rączka 1966 a, b, 1967. In the papers a further case where the stability group is not simple is also treated, e.g., for symmetric spaces X_0 given by eq. 3(3).

(iii) The generalized projection operators P^{λ}_{pq} for noncompact groups were introduced by Rączka 1969. The application of these operators for the explicit calculation of Clebsch–Gordan coefficients for Lorentz group was given by Anderson, Rączka, Rashid and Winternitz 1970 a, b.

§ 6. Exercises

§ 1.1. Show that the Laplace–Beltrami operator does not exist on the cone

$$(x^1)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^{p+q})^2 = 0. \tag{1}$$

Hint: Show that the metric tensor 3(8) is singular.

§ 1.2. Show that on symmetric spaces $X_{+}^{p,q} = U(p,q)/U(p-1,q)$ given by

$$|z^1|^2 + \dots + |z^p|^2 - |z^{p+1}|^2 - \dots - |z^{p+q}|^2 = 1 \tag{2}$$

the ring of invariant operators is generated by the Laplace-Beltrami operator and the operator $C_1 = \sum_{i=1}^{p+q} X_i$, where X_i are generators of $U(p,q)$.

§ 2.1. Find the matrix elements of irreducible most degenerate representations of the group $\mathrm{Sp}(n)$.

Hint: Represent every element x of $\mathrm{Sp}(n)$ as a product of one-parameter subgroups and use the method of Pajas and Rączka 1968 for the explicit construction of the carrier space.

§ 3.1. Let $G = \mathrm{SO}(2,2)$ and let $H = L^2(X, \mu)$, where $X = \mathrm{SO}(2,2)/\mathrm{SO}(1,2)$, is realized by the hyperboloid

$$(x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 = 1. \tag{3}$$

Show that there exists in H the discrete series of representations of G characterized by the eigenvalues

$$\lambda = -L(L+2), \quad L = 0, 1, 2, \dots \tag{4}$$

of the Laplace–Beltrami operator and by the eigenvalues of the reflection operator $Ru(x) = u(-x)$.

§ 3.2. Show that the carrier spaces $H_{\pm}(L)$ of irreducible representations of the discrete series of the previous exercise have the structure given on p. 472, Fig. 3, where m and $\tilde{m}$ are invariant numbers characterizing the representation of $\mathrm{SO}(2)\times\mathrm{SO}(2)$ subgroup, satisfying the condition $|m|-|\tilde{m}| = \pm(L+2+2n)$, $n = 0, 1, 2, \dots$

The representations obtained after changing m and $\tilde{m}$ are equivalent to a pair of previous representations (and are denoted by the dotted lines in the figure).

Hint: Use the reduction of the Laplace–Beltrami operator for $\mathrm{SO}(2,2)$ with respect to $\mathrm{SO}(2)\times\mathrm{SO}(2)$-subgroup.

§ 3.3.* Let $G = U(2,2)$ and let $H = L^2(X, \mu)$, where $X = U(2,2)/U(1,2)$ is represented by the following hypersurface in C^4

$$|z^1|^2 + |z^2|^2 - |z^3|^2 - |z^4|^2 = 1. \tag{5}$$

Show that there exists in H the discrete series of representations of G, which is characterized by the eigenvalues

$$\lambda = -L(L+6), \quad L = -2, -1, 0, 1, 2, \dots,$$

of the Laplace–Beltrami operator and by the eigenvalues of the operator $C_1 = \sum_{i=1}^{4} X_i$, where X_i are generators of the Cartan subgroup of G.

Hint: Introduce the biharmonic coordinate system on X and reduce $\Delta(X)$ to a one-dimensional Schrödinger operator.

§ 3.4.*** Let $G = \mathrm{Sp}(p, q)$. Construct the degenerate series of irreducible representations of G in the spaces $H = L^2(X^+, \mu)$, where $X^+ = \mathrm{Sp}(p, q)/\mathrm{Sp}(p-1, q)$ and $X^- = \mathrm{Sp}(p, q)/\mathrm{Sp}(p, q-1)$.

Hint: Use the method of Pajas and Rączka 1968 which they applied for the construction of degenerate series of irreducible representations of $\mathrm{Sp}(n)$.

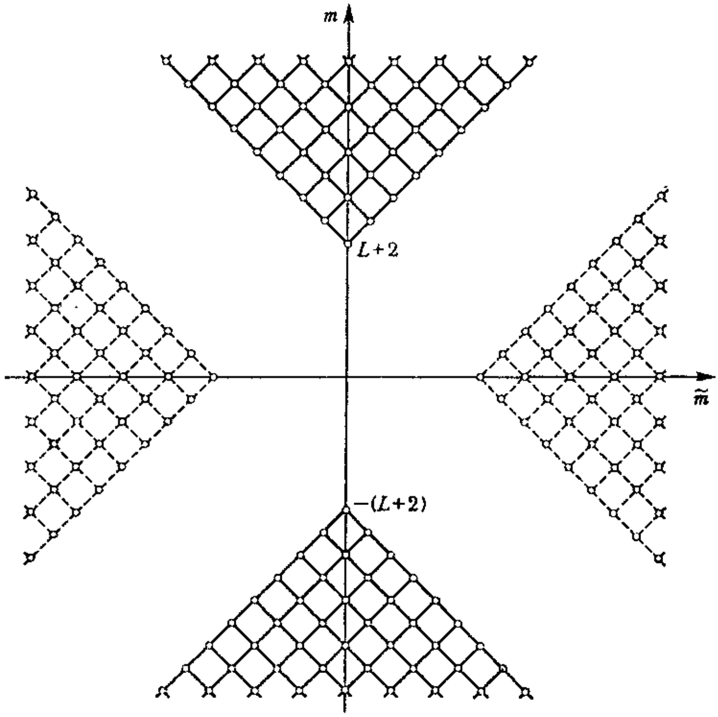

Fig. 3

Chapter 16

Induced Representations

§ 1. The Concept of Induced Representations

We constructed in ch. 7 all irreducible finite-dimensional representations of an arbitrary connected Lie group, using the technique of induced representations. In this chapter we give a method of construction of induced unitary representations of *any* topological, locally compact, separable group G. We begin with the construction of the carrier space, which is a generalization of the construction given in ch. 8, § 2, for the finite-dimensional case.

A. *The Construction*

Let K be a closed subgroup of G and let $k \to L_k$ be a unitary representation of K in a separable Hilbert space H. Let μ be any quasi-invariant measure in the homogeneous space $X = K\backslash G = \{Kg, g \in G\}$ of the right K-cosets. Consider the set H^L of all functions u with domain in G and range in H, satisfying the following conditions:

1° $\big(u(g), v\big)$ is measurable (relative to dg) for all $v \in H$.

2° $u(kg) = L_k u(g)$, for all $k \in K$ and all $g \in G$. (1)

3° $\int_X \|u(g)\|^2 \mathrm{d}\mu(\dot{g}) < \infty$, $\dot{g} = Kg$, where $\|u(g)\|$ is the norm in the space H.

Condition 3° requires some further explanation. We first note that

$$\|u(kg)\| = \|L_k u(g)\| = \|u(g)\|,$$

because L_k is unitary. Hence, $\|u(g)\|$ defines a function on the right K-cosets (i.e., on the space $X = K\backslash G$).

Remark: Each point $x = Kg$ of $K\backslash G$ remains fixed under the action of the subgroup $g^{-1}Kg \cong K$ from right. Hence the subgroup K is called the *stability* (stationary, isotropy, little) *group* of the space X (cf. ch. 4, § 1).

LEMMA 1. *The space H^L defined by eq.* (1) *is isomorphic to the Hilbert space $L^2(X, \mu, H)$ of square integrable vector functions with domain in $X = K\backslash G$ and values in the Hilbert space H. This isomorphism is given by the formula*

$$u(g) = L_{k_g}\tilde{u}(\dot{g}),$$

where k_g is the factor of g in the Mackey decomposition $g = k_g s_g$ (2.4(1)).

PROOF: The vector $u(g)$ plays, by condition 1°(1), the role of a linear, continuous functional in H. The norm $||u(g)||$ of this functional is, by definition, given by the formula

$$||u(g)|| = \sup_{||v||\leqslant 1} |(u(g), v)|.$$

Because H is separable, there exists a sequence v_n of elements in H with $||v_n|| \leqslant 1$ and dense in the unit ball in H. Hence, $||u(g)|| = \sup_n |(u(g), v_n)|$. Because, by condition 1°(1) all of the functions $(u(g), v_n)$ are measurable, and because the upper limit of a sequence of measurable functions is itself measurable, it follows that $||u(g)||$ is a measurable function of g, and consequently also of $\dot{g}$ (cf. app. A.5). Hence, the integral (3° of eq. (1)) is meaningful. If $u \in H^L$, $\lambda \in C^1$ then $\lambda u \in H^L$. Moreover if u_1 and u_2 satisfy conditions 1°, 2° and 3° of eq. (1), then u_1+u_2 evidently satisfy 1° and 2°. Due to the inequality

$$||u_1(g)+u_2(g)||^2 \leqslant 2\big(||u_1(g)||^2+||u_2(g)||^2\big), \tag{2}$$

u_1+u_2 also satisfies the condition 3°. Thus the functions, which satisfy conditions 1°–3° form a vector-space. Using the identity

$$\begin{aligned} 4\,(u_1(g), u_2(g)) = {} & ||u_1(g)+u_2(g)||^2-||u_1(g)-u_2(g)||^2+ \\ & +\mathrm{i}||u_1(g)+\mathrm{i}u_2(g)||^2-\mathrm{i}||u_1(g)-\mathrm{i}u_2(g)||, \end{aligned} \tag{3}$$

one deduces that the function $g \to (u_1(g), u_2(g))$ is measurable. From condition 2° it follows that $(u_1(kg), u_2(kg)) = (L_k u_1(g), L_k u_2(g)) = (u_1(g), u_2(g))$. Hence, the function $(u_1(g), u_2(g))$ is a measurable function of $\dot{g}$. This function is also summable relative to the measure $\mu(\cdot)$ in X. In fact, by Schwarz's inequality, we have

$$|(u_1(g), u_2(g))| \leqslant ||u_1(g)||\,||u_2(g)||.$$

By condition 3° the functions $||u_i(g)||^2$, $i = 1, 2$, are summable. Furthermore, by Hölder's inequality the product $||u_1(g)||\,||u_2(g)||$ is also summable.* Consequently, the function $(u_1(g), u_2(g))$ is summable with respect to $\mathrm{d}\mu(\dot{g})$.

The above properties of the function $g \to (u_1(g), u_2(g))$ allows us to introduce a positive, hermitian form in the space H^L by the formula

$$[(u_1, u_2)]_{H^L} \equiv \int_X (u_1(g), u_2(g))_H \mathrm{d}\mu(\dot{g}). \tag{4}$$

Identifying in H^L two functions which are equal almost everywhere, and utilizing the last equation for the scalar product in H^L, we can convert the space H^L into

* If $f(x) \in L^p(\Omega)$ and $g(x) \in L^q(\Omega)$, where

$$1 < p < \infty, \quad 1 < q < \infty, \quad 1/p+1/q = 1,$$

then the product $f(x)g(x)$ is integrable and

$$\left|\int f(x)g(x)\mathrm{d}x\right| \leqslant ||f||_p\,||g||_q.$$

a scalar product space. In order to show the isomorphy of H^L to $L^2(X, \mu; H)$ let $\tilde{u}(g) \in L^2(X, \mu; H)$ and set

$$u(g) = L_{k_g}\tilde{u}(\dot{g}).* \tag{5a}$$

The functions (5a) are in the space H^L. In fact, the measurability of the map $g \to kg$ and continuity of L_k imply that for any $v \in H$, the function $g \to (u(g), v)$ is measurable. Because $kg = kk_g s_g$ implies $k_{kg} = kk_g$, we have $u(kg) = L_{kk_g}u(g) = L_k u(g)$. Finally, the unitartiy of L_{k_g} implies

$$\int_X \|u(g)\|^2 \mathrm{d}\mu(\dot{g}) = \int_X \|\tilde{u}(\dot{g})\|^2 \mathrm{d}\mu(\dot{g}) < \infty .$$

The map $\tilde{u}(\dot{g}) \to u(g)$ given by eq. (5a) is thus the isometry of $L^2(X, \mu, H)$ into H^L. Conversely, if $u(g) \in H^L$, then the function

$$\tilde{u}(\dot{g}) = L_{k_g}^{-1} u(g) \tag{5b}$$

satisfies $\tilde{u}(k\dot{g}) = L_{kk_g}^{-1} u(kg) = L_{kk_g}^{-1} L_k u(g) = \tilde{u}(\dot{g})$ and belongs to $L^2(X, \mu; H)$. Consequently, the map (5a) represents the isomorphism of $L^2(X, \mu; H)$ onto H^L. ▼

Next we construct a unitary representation of G in the Hilbert space H^L.

LEMMA 2. *The map* $g_0 \to U_{g_0}^L$ *given by*

$$U_{g_0}^L u(g) \equiv (\varrho_{g_0}(g))^{1/2} u(gg_0), \tag{6}$$

where $\varrho_{g_0}(g) = \mathrm{d}\mu(\dot{g}g_0)/\mathrm{d}\mu(\dot{g})$ *is the Radon–Nikodym derivative of the quasi-invariant measure* $\mathrm{d}\mu$ *in X, defines a unitary representation of G in* H^L.

PROOF: The function $v(g) = [\varrho_{g_0}(g)]^{1/2} u(gg_0)$ clearly satisfies the condition 1°(1). Furthermore, because

$$\int_X \|v(g)\|^2 \mathrm{d}\mu(\dot{g}) = \int_X \varrho_{g_0}(g) \|u(gg_0)\|^2 \mathrm{d}\mu(\dot{g}) = \int_X \|u(g)\|^2 \mathrm{d}\mu(\dot{g}) < \infty, \tag{7}$$

the function $v(g) \in H^L$. The operator U_g^L is isometric and possesses an inverse $(U_g^L)^{-1}$. Consequently, the map $g \to U_g^L$ is unitary. Using eqs. (6) and the composition law for $\varrho_{g_0}(g)$ we get

$$\begin{aligned}[U_{g_1}^L U_{g_2}^L u](g) &= (\varrho_{g_1}(g))^{1/2} (\varrho_{g_2}(gg_1))^{1/2} u(gg_1 g_2) \\ &= (\varrho_{g_1 g_2}(g))^{1/2} u(gg_1 g_2) = U_{g_1 g_2}^L u(g).\end{aligned}$$

Consequently,

$$U_{g_1}^L U_{g_2}^L = U_{g_1 g_2}^L . \tag{8}$$

It remains to show that the map $g \to U_g^L$ is strongly continuous. To see this, note that in the formula

$$(U_g^L u, v) = \int_X (\varrho_g(g_1))^{1/2} (u(g_1 g), v(g_1)) \mathrm{d}\mu(\dot{g}_1) \tag{9}$$

the integrand is a measurable function of both variables by th. 4.3.1 and eq. (3). Hence, $((U_g^L u, v))$ is a measurable function of g. By proposition 5.7.2.a (weakly)

* Here and in some next formulas we omit, for the sake of simplicity, the standard verification that the equality is independent on the choice of element u from the equivalence class.

measurable unitary representation is strongly continuous. Thus, the map $g \to U_g^L$, given by eq. (6), defines a strongly continuous unitary representation of G in the Hilbert space H^L, called the *representation of G induced by L*, or simply, the *induced representation.* ▼

If L is a one-dimensional representation of the closed subgroup $K \subset G$, U^L is called a *monomial representation*. If L is the one-dimensional identity representation of K, then U^L is called the *quasi-regular representation* of G. In this case $H^L = L^2(X, \mu)$. If K is the identity subgroup, then U^L is the right regular representation of G. The monomial induced representations play a fundamental role in representation theory of complex, classical Lie groups (cf. ch. 19) and of nilpotent groups.

EXAMPLE 1. Let $G = N \rtimes M$ be the two-dimensional Poincaré group. The action of G in the two-dimensional space-time is given by the formula

$$\begin{bmatrix} x \\ t \end{bmatrix} \to [n, \Lambda] \begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} n_x \\ n_t \end{bmatrix}, \qquad -\infty < \alpha < +\infty.$$

This implies the following composition law

$$(n, \Lambda)(n_0, \Lambda_0) = (n + \Lambda n_0, \Lambda\Lambda_0).$$

We consider representations of G induced by the one-dimensional characters $n \to L_n = \langle n, \hat{n} \rangle = \exp[i(n, \hat{n})]$ of N where $(n, \hat{n}) = n_\mu \hat{n}^\mu$ Because N is normal in G, the space $X = N \backslash G$ is isomorphic with the group space of the subgroup M. The Haar measure $\mathrm{d}\mu$ on M considered as the measure on X is G-invariant. In fact, from the composition law for G it follows that

$$xg \simeq (0, \Lambda_x)(n, \Lambda) = (\Lambda_x n, \Lambda_x \Lambda) \simeq (0, \Lambda_x \Lambda) \in X.$$

Hence $\mathrm{d}\mu(xg) = \mathrm{d}\mu(\Lambda_x \Lambda) = \mathrm{d}\mu(x)$. This implies

$$\mathrm{d}\mu(xg)/\mathrm{d}\mu(x) \equiv 1.$$

The carrier space H^L of U^L consists of functions $u(g)$ on G satisfying the condition 2°(1), i.e.,

$$u\big((n', I)(n, \Lambda)\big) = \langle n', \hat{n} \rangle u\big((n, \Lambda)\big).$$

The action of $U_{g_0}^L$ in H^L is given by the formula (6) with $\varrho \equiv 1$:

$$U_{g_0}^L u(g) = u(gg_0).$$

By lemma 1 we can represent U^L in the carrier space $L^2(X, \mathrm{d}\mu(x))$. Eq.(5b), mapping the space H^L unitarily onto $L^2\big(X, \mathrm{d}\mu(x)\big)$ becomes the resctriction operator

$$H^L \ni u \to u|_M M \in L^2(M, \mathrm{d}\mu) \cong L^2(X, \mathrm{d}\mu).$$

The action of G in the last space is now given by ($g_0 = (n_0, \Lambda_0)$) (cf. eq. (15))

$$U_{g_0}^L u(\Lambda) = \langle \Lambda n_0, \hat{n} \rangle u(\Lambda\Lambda_0). \text{▼}$$

We show in ch. 17, § 1, that every such induced representation is irreducible.

There arises the following natural questions:

(i) Are there functions, not identically zero, which satisfy the conditions 1°, 2° and 3° of eq. (1)?

(ii) How are the representations ${}^{\mu}U^L$ and ${}^{\nu}U^L$ of G corresponding to two different quasi-invariant measures μ and ν on $K\backslash G$ related to each other?

(iii) Can one define a representation $g \to U_g^L$ induced by a representation L of K directly on the space $L^2(X, \mu, H)$, where $X = K\backslash G$?

The answer to the first question, as well as a clear description of the structure of the space H^L is given by the following

PROPOSITION 3. *Let $w(g)$ be an arbitrary, continuous function with domain in G, range in H and with a compact support. Set*

$$\hat{w}(g) \equiv \int_K L_k^{-1} w(kg)\mathrm{d}k, \tag{10}$$

where dk *is right-invariant Haar measure in K. Then,*

1° *$\hat{w}(g)$ is a continuous function on G with compact support* on $K\backslash G$.*

2° *$\hat{w}(g)$ belongs to H^L.*

3° *The set $C_0^L = \{\hat{w}(g);\ w(g) = \lambda(g)v,\ \lambda(g) \in C_0(G), v \in H\}$ forms a dense set in H^L.*

PROOF: Ad 1°. Every continuous function $w(g)$ on G with a compact support D is uniformly continuous (cf. proposition 2.2.4) i.e., for any $\varepsilon > 0$ there exists a compact neighborhood V of the identity, such that

$$g_1^{-1} g_2 \in V \Rightarrow \|w(g_1) - w(g_2)\| < \varepsilon.$$

Let $g_0 \in G$ and let $g \in g_0 V$. Then,

$$\begin{aligned}\hat{w}(g) - \hat{w}(g_0) &= \int_K \left(L_k^{-1} w(kg) - L_k^{-1} w(kg_0)\right)\mathrm{d}k \\ &= \int_{K \cap DV^{-1}g_0^{-1}} L_k^{-1}\left(w(kg) - w(kg_0)\right)\mathrm{d}k.\end{aligned}$$

Hence,

$$\|\hat{w}(g) - \hat{w}(g_0)\| \leqslant \int_{K \cap DV^{-1}g_0^{-1}} \|w(kg) - w(kg_0)\|\mathrm{d}k < \varepsilon \operatorname*{mes}_K (K \cap DV^{-1}g_0^{-1}),$$

i.e., $\hat{w}(g)$ is a continuous function on G.

Ad 2°. For any $k_0 \in K$ we have

$$\hat{w}(k_0 g) = \int_K L_k^{-1} w(kk_0 g)\mathrm{d}k = \int_K L_{\tilde{k}k_0^{-1}}^{-1} w(\tilde{k}g)\mathrm{d}\tilde{k} = L_{k_0}\hat{w}(g).$$

---

* I.e., $\pi(\operatorname{supp} \hat{w})$ is compact in $X = K\backslash G$, where π is the canonical projection $\pi\colon G \to X$.

Furthermore, it follows from eq. (10) that $\hat{w}(g) = 0$ if $g \notin KD$. Thus, $\|w(\dot{g})\|$ is a continuous function of $\dot{g}$ with a support $\pi(D)$. Therefore, all conditions 1°–3°(1) are satisfied and consequently $\hat{w} \in H^L$.

Ad 3°. Let u be an element of H^L such that $(u, \hat{w})_{H^L} = 0$ for an arbitrary continuous function w with domain in G, range in H and a compact support. By virtue of 2°(1) we have

$$\big(u(g), \hat{w}(g)\big)_H = \int_K \big(u(g), L_k^{-1} w(kg)\big)_H \,\mathrm{d}k = \int_K \big(u(kg), w(kg)\big)_H \,\mathrm{d}k.$$

Set now $w(g) = \lambda(g)v$, where $\lambda \in C_0(G)$. Then,

$$\big(u(g), \hat{w}(g)\big)_H = \int_K \lambda(kg)\big(u(kg), v\big)_H \mathrm{d}k,$$

and by eq. (4)

$$(u, \hat{w})_{H^L} = \int_{K\backslash G} \Big[\int_K \lambda(kg)\big(u(kg), v\big)_H \mathrm{d}k\Big] \mathrm{d}\mu(\dot{g}).$$

Then, using the equality 4.3(8) one obtains

$$(u, \hat{w})_{H^L} = \int_G \lambda(g)\big(u(g), v\big)\varrho(g)\mathrm{d}g = 0.$$

This equation implies, because λ is arbitrary, $\big(u(g), v\big) = 0$ up to a set $N_v \subset G$ of measure zero. Let now $\{v_n\}$ be an everywhere dense sequence in H and let $N = \bigcup_n N_{v_n}$. Because N is of measure zero, for $g \notin N$ we have $\big(u(g), v_n\big)_H = 0$ for every n. Consequently $u(g) = 0$. ▼

We shall now solve the problem (ii).

Let μ and ν be two quasi-invariant measures in $X = K\backslash G$. Denote by ${}^{\mu}H^L$ and ${}^{\nu}H^L$ the corresponding carrier spaces and by ${}^{\mu}U^L$ and ${}^{\nu}U^L$ unitary representations of G in ${}^{\mu}H^L$ and ${}^{\nu}H^L$, respectively. Then one has

PROPOSITION 4. *There exists a unitary transformation V from ${}^{\mu}H^L$ onto ${}^{\nu}H^L$ such that*

$$V\big({}^{\mu}U_g^L\big)V^{-1} = {}^{\nu}U_g^L \tag{11}$$

for all g in G.

PROOF: The measures μ and ν in X are equivalent by th. 4.3.1. Let ψ denote a Radon–Nikodym derivative of μ with respect to ν. It is a measurable function by th. 4.3.1. Let π denote the canonical projection of G onto $K\backslash G$ given by $\pi: g \to Kg$. Then, for each u in ${}^{\mu}H^L$ the function $\sqrt{(\psi \circ \pi)}u$ is in ${}^{\nu}H^L$ and the norm of u in ${}^{\mu}H^L$ is equal to that of $\sqrt{(\psi \circ \pi)}u$ in ${}^{\nu}H^L$. It is evident that every v in ${}^{\nu}H^L$ is of the form $\sqrt{(\psi \circ \pi)}u$ for some u in ${}^{\mu}H^L$. Denote by V the operator of the multiplication by $\sqrt{\psi \circ \pi}$. Then V defines a unitary map of ${}^{\mu}H^L$ onto ${}^{\nu}H^L$. In fact, let $v = \sqrt{(\psi \circ \pi)}u \in {}^{\nu}H^L$. Then, because $\mathrm{d}\mu = \psi \mathrm{d}\nu$ one obtains

$$\varrho^{\mu}_{g_0}(g) \equiv \frac{\mathrm{d}\mu(\dot{g}g_0)}{\mathrm{d}\mu(\dot{g})} = \frac{\mathrm{d}\nu(\dot{g}g_0)}{\mathrm{d}\nu(\dot{g})}\frac{\psi(\dot{g}g_0)}{\psi(\dot{g})} = \varrho^{\nu}_{g_0}(g)\frac{\psi(\dot{g}g_0)}{\psi(\dot{g})},$$

and consequently, by eq. (6),

$$\begin{aligned} V^{\mu}U^L_{g_0}V^{-1}(u) &= V\big(\varrho^{\mu}_{g_0}(g)\big)^{1/2}u(gg_0) = (\psi\circ\pi)^{1/2}(g)\,\big(\varrho^{\mu}_{g_0}(g)\big)^{1/2}u(gg_0) \\ &= (\psi\circ\pi)^{1/2}(gg_0)\,\big(\varrho^{\nu}_{g_0}(g)\big)^{1/2}u(gg_0) = {}^{\nu}U^L_g u(g), \end{aligned}$$

i.e. eq. (11) follows. ▼

We give now the solution of problem (iii). The knowledge of the representation $g \to U^L_g$ directly on the space $L^2(X, \mu; H)$ is essential in many applications; for instance, in particle physics one wants to know properties of representations of the Poincaré group Π in the space of functions with domain on the mass hyperboloid $p_0^2-p_1^2-p_2^2-p_3^2 = m^2 > 0$ which is the quotient space Π/K, where K is the semi-direct product $T^4 \circledS \mathrm{SO}(3)$.

Let $g \to B_g$ be an operator function from G into the set of unitary operators in H which satisfies

1° $B_{kg} = L_k B_g$, for all $k \in K$ and all $g \in G$. (12)

2° The map $g \to B_g$ is weakly measurable.

The last condition means that for each pair u, v in H the function

$$g \to (B_g u, v) \tag{13}$$

is dg-measurable.

Denote by s_g the unique element of G characterizing the coset $Kg = Kk_g s_g = Ks_g$ and by $k_{s_g g_0} \in K$ the factor of the Mackey decomposition 2.4 (1) of the element $s_g g_0$. We have:

PROPOSITION 5. *Let K be a closed subgroup of G and let $k \mapsto L_k$ be a unitary representation of K in H. If $g \to B_g$ is an operator function satisfying the conditions* (12), *then the map $g_0 \to U^L_{g_0}$ given by**

$$U^L_{g_0}u(\dot{g}) = \varrho^{1/2}_{g_0}(g)B_g^{-1}B_{gg_0}u(\dot{g}g_0) \tag{14}$$

provides a unitary representation of G in $L^2(X, \mu, H)$. If we put $B_g = L_{k_g}$ and $\dot{g} \equiv x$ then

$$U^L_{g_0}u(x) = [\mathrm{d}\mu(xg_0)/\mathrm{d}\mu(x)]^{1/2}L_{k_{s_g g_0}}u(xg_0). \tag{15}$$

PROOF: Let $\tilde{u}(\dot{g}) \in L^2(X, \mu; H)$ and set

$$u(g) = B_g\tilde{u}(\dot{g}). \tag{16a}$$

The functions (16a) are in the space H^L. In fact, by condition 2°(12), for any $v \in H$, the function $g \to \big(u(g), v\big)$ is measurable. By condition 1° (12) and $\dot{\widetilde{kg}} = Kkg = Kg = \dot{g}$ we have $u(kg) = L_k u(g)$. Finally, the unitarity of B_g implies

$$\int_X \|u(g)\|^2\mathrm{d}\mu(\dot{g}) = \int_X \|\tilde{u}(\dot{g})\|^2\mathrm{d}\mu(\dot{g}) < \infty.$$

---

* We use for simplicity the same symbol U^L_g for representations (6) and (15), and we omit the tilda over $u(\dot{g})$.

The map $u(g) \to \tilde{u}(\dot{g})$ given by eq. (16a) is thus the isometry of $L^2(\mu, X, H)$ into H^L. Conversely, if $u \in H^L$ then the function

$$\tilde{u}(\dot{g}) = B_g^{-1} u(g) \tag{16b}$$

satisfies $\tilde{u}(k\dot{g}) = B_{k_g}^{-1} L_k u(g) = B_g^{-1} u(g) = \tilde{u}(\dot{g})$ and belongs to $L^2(X, \mu, H)$. Consequently, the map (16a) represents the isomorphism of $L^2(X, \mu; H)$ onto H^L.

The following diagram illustrates the transformations of the operators $\tilde{U}_g^L$ by isomorphisms (16)

$$\begin{array}{ccc}
\tilde{u}(\dot{g}) & \xrightarrow{\quad B_g \quad} & u(g) = B_g \tilde{u}(\dot{g}) \in H^L \\
\Big\downarrow \tilde{U}_{g_0}^L & & \Big\downarrow U_{g_0}^L \\
\varrho_{g_0}^{1/2}(g) B_g^{-1} B_{gg_0} \tilde{u}(\dot{g}g_0) & \xleftarrow{\quad B_g^{-1} \quad} & \varrho_{g_0}^{1/2}(g) B_{gg_0} \tilde{u}(\dot{g}g_0)
\end{array}$$

which proves the assertion (14).

In order to complete the solution of problem (iii), one should show that there always exists an operator function $g \to B_g$ which satisfies conditions (12). This can be done by means of th. 2.4.1. In fact, let S be a Borel set such that any $g \in G$ can be uniquely written in the form $g = k_g s_g$, $k_g \in K$ and $s_g \in S$. Then, setting

$$B_g = L_{k_g}, \tag{17}$$

where $k \to L_k$ is a unitary representation of the subgroup K, one obtains an operator function $g \to B_g$ which satisfies conditions 1°(12) and 2°(12).

Because $gg_0 = k_g s_g g_0 = k_g k_{s_g g_0} s_{s_g g_0}$ we obtain $k_{gg_0} = k_g k_{s_g g_0}$. Hence

$$B_g^{-1} B_{gg_0} = L_{k_{s_g g_0}} \tag{18}$$

which proves the formula (15). ▼

In the following we denote the space $L^2(X, \mu; H)$ also by the symbol H^L, whenever their distinction is unimportant.

EXAMPLE 2. Let G be the set of all 2×2, real, unimodular matrices

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha\delta - \beta\gamma = 1,$$

i.e., $G = \mathrm{SL}(2, R)$.

Take as the subgroup K the set of all elements k in G of the form

$$k = \begin{bmatrix} \lambda & \nu \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \neq 0. \tag{19}$$

In fact $K = R \otimes\!\!\!\!\!\!\;\!\!\!\!\!\!\;\; \Lambda$, where

$$R = \left\{ \begin{bmatrix} 1 & \nu \\ 0 & 1 \end{bmatrix} \right\}$$

is the invariant subgroup in K, and

$$\Lambda = \left\{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}\right\}.$$

We want to find unitary representations U^L of G induced by the one-dimensional representations (characters) of K which are of the form

$$k \to L_k \equiv L_{(\nu,\lambda)} = I \cdot L_\lambda = |\lambda|^{i\sigma}\left(\frac{\lambda}{|\lambda|}\right)^{\varepsilon} \tag{20}$$

where $\sigma \in (-\infty, \infty)$ and $\varepsilon = 0$ or 1. Because the carrier space H of L is C^1, the carrier space of U^L is $H^L = L^2(X, \mu)$, where $X = K\backslash G$ and μ is a quasi-invariant measure on X. By virtue of eq. (15) the action of $U^L_{g_0}$ in H^L is given by formula

$$U^L_{g_0} u(x) = (\mathrm{d}\mu(xg_0)/\mathrm{d}\mu(x))^{1/2} L_{k_{s_g}g_0} u(xg_0). \tag{21}$$

In order to obtain the explicit form of the operator $U^L_{g_0}$ we have to find the explicit realization of

(i) the homogeneous space X and the action of G on X,

(ii) the measure $\mathrm{d}\mu(x)$, and

(iii) the function $L_{k_{s_g}g_0}$.

We now give the solutions of these problems.

(i) To find an explicit realization of the space X we use the Mackey decomposition 2.4(1), i.e., $g = k_g s_g$. Notice first that every g for which $\delta \neq 0$ can be represented in the form

$$g = \begin{bmatrix} \delta^{-1} & \beta \\ 0 & \delta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \gamma/\delta & 1 \end{bmatrix}. \tag{22}$$

The remaining elements in G of the form $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, $\gamma = -\beta^{-1}$, can be represented as

$$g = \begin{bmatrix} \beta & -\alpha \\ 0 & \beta^{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{23}$$

Thus any element $g \in G$ can be represented as

$$g = k_g s_g, \tag{24}$$

where $k_g \in K$, and

$$s_g = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, \quad x \in (-\infty, \infty), \quad \text{or} \quad s_g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \equiv S_0, \tag{25}$$

respectively.

The set S of all points (25) provides the Mackey set of the th. 2.4(1). Since

KS_0 is of measure zero in $X = K\backslash G$ we can disregard it in the following considerations. Then

$$X \in x \equiv \dot{g} = Kg = Kk_g s_g = Ks_g, \tag{26}$$

thus every element of the homogeneous space $X = K\backslash G$ can be uniquely determined by the parameter x of s_g; hence by virtue of eq. (25) we have a one-to-one correspondence between the ponts of X and the points of the real line R^1.

To find the point x corresponding to a coset Kg we decompose g by the formula (22) and set $x = \gamma/\delta$.

We obtain the action of $g_0 \in G$ in X on a characteristic element $s_g = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$ of a given coset Kg as follows:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & \delta_0 \end{bmatrix} &= \begin{bmatrix} \alpha_0 & \beta_0 \\ \alpha_0 x + \gamma_0 & \beta_0 x + \delta_0 \end{bmatrix} \\ &= \begin{bmatrix} (\beta_0 x + \delta_0)^{-1} & \beta_0 \\ 0 & \beta_0 x + \delta_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \dfrac{\alpha_0 x + \gamma_0}{\beta_0 x + \delta_0} & 1 \end{bmatrix}. \end{aligned} \tag{27}$$

This implies

$$x \to x g_0 = \frac{\alpha_0 x + \gamma_0}{\beta_0 x + \delta_0}, \tag{28}$$

i.e., the action of G in X is given by the fractional transformation (28).

(ii) We now find the quasi-invariant measure $d\mu(x)$ on X.

The differential of eq. (28), gives

$$d(xg_0) = (\beta_0 x + \delta_0)^{-2} dx. \tag{29}$$

This shows that one can take the Lebesgue measure dx as a quasi-invariant measure on X. By virtue of (29), the Radon–Nikodym derivative $\varrho_{g_0}(g)$ is

$$\varrho_{g_0}(g) = \frac{d(xg_0)}{dx} = (\beta_0 x + \delta_0)^{-2}. \tag{30}$$

(iii) From eq. (27), we have immediately

$$k_{s_g g_0} = \begin{bmatrix} (\beta_0 x + \delta_0)^{-1} & \beta_0 \\ 0 & \beta_0 x + \delta_0 \end{bmatrix}. \tag{31}$$

Thus by virtue of eq. (20)

$$L_{k_{s_g g_0}} = |\beta_0 x + \delta_0|^{-i\sigma} \left(\frac{\beta_0 x + \delta_0}{|\beta_0 x + \delta_0|} \right)^{\varepsilon} \tag{32}$$

Using eqs. (21), (30) and (32), we obtain the explicit form of induced representations

$$(U_g^L u)(x) = |\beta x+\delta|^{-i\sigma-1}\left(\frac{\beta x+\delta}{|\beta x+\delta|}\right)^{\varepsilon} u\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right). \tag{33}$$

It will be proven in ch. 19 that all these representations are irreducible except the one corresponding to $\varepsilon = 1$, $\sigma = 0$.

Remark on the Choice of the Quasi-Invariant Measure

One can use for the construction of the representation U^L of G the carrier space $H^L = L^2(X, \nu; H)$ with any quasi-invariant measure $\nu(\cdot)$ on X. In general, the requirement that the measure is quasi-invariant implies only the condition $\mathrm{d}\nu(x) = \varphi(x)\mathrm{d}x$, $\varphi \geqslant 0$. The function $\varphi(x)$ can be uniquely determined, for instance, by the requirement that $\nu(\cdot)$ is invariant under the subgroup of rotations $\begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}$, for which

$$x \to \frac{x\cos\vartheta - \sin\vartheta}{x\sin\vartheta + \cos\vartheta}. \tag{34}$$

This condition implies $\varphi(x) = (1+x^2)^{-1}$. It is evident that the measure so obtained, $\mathrm{d}\nu(x) = \varphi(x)\mathrm{d}x$, is also quasi-invariant relative to the remaining subgroup consisting of elements of the form $g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, or $g = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$. In the present case the Radon–Nikodym derivative has the form

$$\tilde{\varrho}_{g_0}(g) = \frac{\mathrm{d}\nu(xg_0)}{\mathrm{d}\nu(x)} = \frac{1+x^2}{(\alpha_0 x+\gamma_0)^2+(\beta_0 x+\delta_0)^2}. \tag{35}$$

The carrier space $H^L = L^2\,(X, \nu)$ consists now of all functions satisfying

$$\int_X |\tilde{u}(x)|^2 \frac{\mathrm{d}x}{1+x^2} < \infty. \tag{36}$$

The representation U^L induced by the one-dimensional representation (20) of the subgroup K has now the form

$$\begin{aligned}(U_g^L \tilde{u})(x) = {} & (1+x^2)^{1/2}[(\alpha x+\beta)^2+(\beta x+\delta)^2]^{-1/2} \times \\ & \times |\beta x+\delta|^{-i\sigma}\cdot\left(\frac{\beta x+\delta}{|\beta x+\delta|}\right)^{\varepsilon} \tilde{u}\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right).\end{aligned} \tag{37}$$

This representation, by proposition 4, is unitarily equivalent to the representation (33). The operator V which realizes the equivalence has the form

$$\tilde{u} \xrightarrow{V} u(x) = \tilde{u}(x)/(1+x^2)^{1/2}.$$

Although the representations with different measures are mathematically equivalent, they may not be equivalent in physical applications.

B. *Induced and Multiplier Representations*

Historically most of the irreducible representations of various groups were first derived in the form of the so-called multiplier representations defined below. The proposition following the definition describes the one-to-one correspondence between induced and multiplier representations.

DEFINITION 1. The representation $g \to U_g$ of G in $L^2(X, \mu, H)$ defined by

$$U_g u(x) = [\mathrm{d}\mu(xg)/\mathrm{d}\mu(x)]^{1/2}\sigma(x, g)u(xg), \tag{38}$$

where $\sigma(x, g)$ is a measurable operator function on $X\times G$ and satisfies the following functional equation

$$\sigma(x, g_0 g_1) = \sigma(x, g_0)\sigma(xg_0, g_1) \tag{39}$$

is called a *multiplier representation*. The function $\sigma(x, g)$ is called the *multiplier*.

PROPOSITION 6. *The representation $g \to U_g^L$ of G induced by the representation $k \to L_k$ of K provides a multiplier* (18), *i.e.*,

$$\sigma(\dot{g}, g_0) = B_g^{-1}B_{gg_0}, \quad \dot{g} \equiv gK. \tag{40}$$

Conversely, for every multiplier $\sigma(x, g_0)$ with values in the set of unitary operators in the Hilbert space H, the corresponding multiplier representation (38) *is a unitary representation in $L^2(X, \mu, H)$ of G which is equivalent to a representation U^L induced by the representation $k \to L_k$ of K defined by the formula*

$$K \in k \to L_k = \sigma(K, k). \tag{41}$$

PROOF: The operator-valued function

$$G\times G \ni (g, g_0) \to B_g^{-1}B_{gg_0} \in L(H)$$

is left K-invariant with respect to the first variable:

$$(kg, g_0) \to B_{kg}^{-1}B_{kgg_0} = (L_k B_g)^{-1}L_k B_{gg_0} = B_g^{-1}B_{gg_0}.$$

It follows that the map on $X\times G$:

$$(\dot{g}, g_0) \to \sigma(\dot{g}, g_0) \equiv B_g^{-1}B_{gg_0}$$

is well defined.

The map σ satisfies the functional equation (39) (also be referred to as multiplier equation). Indeed,

$$\sigma(\dot{g}, g_0 g_1) = B_g^{-1}B_{gg_0g_1} = B_g^{-1}B_{gg_0}B_{gg_0}^{-1}B_{gg_0g_1} = \sigma(\dot{g}, g_0)\sigma(\dot{g}g_0, g_1).$$

It is easily seen that the function $\varrho_{g_0}^{1/2}(g)$ also satisfies this equation.

We have then obtained the correspondence between induced representations U^L given by equation (14) and the multipliers. It is our aim to prove that this correspondence is, in fact, one-to-one. Indeed, it follows from the multiplier equation that $k \to L_k \equiv \sigma(K, k)$ defines the unitary representation of the subgroup K. The function

$$B_g = \sigma(K, g)$$

satisfies condition (12); hence the corresponding multiplier representation (38) is unitarily equivalent to the representation U^L defined by the formula (14). Indeed,

$$B_g^{-1} B_{gg_0} = \sigma(K, g)^{-1} \sigma(K, gg_0) = \sigma(\dot{g}, g_0)$$

and the proof is complete. ▼

C. *Realization of Induced Representations in Terms of Left Cosets*

So far we have discussed the properties of induced representations realized in a space $H^L = L^2(X, \mu; H)$, where X was a quotient space $K \backslash G$ of right cosets Kg, $g \in G$. However, in many applications the space $\hat{X} = G/K$ of left cosets gK, $g \in G$ is customary. Hence, we give an alternative formulation of the results in the space of left cosets.

The carrier space $\hat{H}^L$ of induced representations $\hat{U}^L$ is now defined by the conditions 1° (1), 3° (1), but instead of 2° (1), we have

$$u(gk) = L_k^{-1} u(g). \tag{42}$$

The action of $\hat{U}^L$ in the carrier space $\hat{H}^L$ is now given by the formula (cf. eq. (6))

$$\hat{U}^L_{g_0} u(g) = \hat{\varrho}^{1/2}_{g_0}(g) u(g_0^{-1} g)$$

where, in the present case, $\hat{\varrho}_{g_0}(g) = \mathrm{d}\mu(g_0^{-1}\dot{g})/\mathrm{d}\mu(\dot{g})$. We introduce, as previously, an operator function $g \to \hat{B}_g$ from G into a set of unitary operators in H which satisfies the following conditions (cf. eq. (12)):

$$\left.\begin{array}{l} 1^\circ\ \hat{B}_{gk} = L_{k^{-1}} \hat{B}_g. \\ 2^\circ\ \text{The map } g \to \hat{B}_g \text{ is weakly measurable.} \end{array}\right\} \tag{43}$$

We readily verify as previously that the map

$$u(g) = \hat{B}_g u(\dot{g}) \tag{44}$$

defines a one-to-one isomorphism of the spaces $\hat{H}^L$ and $L^2(\hat{X}, \mu; H)$ (cf. eqs. (16a) and (16b)). This implies that the action of $\hat{U}^L_{g_0}$ in the space $L^2(\hat{X}, \mu; H)$ is given by the formula (cf. proposition 5)

$$\hat{U}^L_{g_0} u(\dot{g}) = \hat{\varrho}^{1/2}_{g_0}(g) \hat{B}_g^{-1} \hat{B}_{g_0^{-1} g} u(g_0^{-1} \dot{g}). \tag{45}$$

Setting

$$\hat{B}_g = L_{k_g}^{-1}, \tag{46}$$

one verifies that both of the conditions of eq. (43) are satisfied.

The element g in the formula (45) can be taken to be any element of the left coset $\dot{g} = gK = s_g k_g K = s_g K$; hence we can set s_g instead of g in eq. (45). Because $\hat{B}_{s_g} = I$, the formula (45) can be written in the form

$$\hat{U}^L_{g_0} u(x) = \left(\mathrm{d}\mu(g_0^{-1} x)/\mathrm{d}\mu(x)\right)^{1/2} L^{-1}_{k_{g_0^{-1} s_g}} u(g_0^{-1} x), \tag{47}$$

where $g_0, g \in G$, $x \equiv \dot{g} = gK$, $s_g \in S$ (i.e. $g = s_g k_g$) and $\hat{\varrho}_{g_0}(g)$ is the Radon–Nikodym derivative $\mathrm{d}\mu(g_0^{-1}x)/\mathrm{d}x$.

This completes the construction of induced, unitary representations of a group G associated with a space $\hat{X}$ of left cosets $\dot{g} = gK$. ▼

PROPOSITION 7. *If G is unimodular, then the representations U^L and $\hat{U}^L$ of G given by* (15) *and* (47), *respectively, both induced by the same unitary representation L of K are unitarily equivalent, i.e.,*

$$J\hat{U}^L J^{-1} = U^L, \tag{48}$$

where J denotes the involution given by the formula

$$(Ju)(g) = u(g^{-1}). \tag{49}$$

PROOF: If $u(g) \in H^L$, (i.e., $u(kg) = L_k u(g)$) then

$$Ju(gk) = u(k^{-1}g^{-1}) = L_{k^{-1}}u(g^{-1}) = L_k^{-1}(Ju)(g).$$

Hence, $Ju(g) \in H^{\hat{L}}$.

Given the quasi-invariant measure $\mathrm{d}\mu$ on $K\backslash G$, let us define a measure $\mathrm{d}\hat{\mu}$ on G/K by the formula:

$$\int_{G/K} f(\dot{g})\mathrm{d}\hat{\mu}(\dot{g}) = \int_{K\backslash G} I(f\circ\pi)(\dot{g})\mathrm{d}\mu(\dot{g}).$$

Because $J(f\circ\pi)$ is the left K-invariant function on G, the right hand side of this formula is meaningful.

The measure $\mathrm{d}\hat{\mu}$ is quasi-invariant on G, and by a simple computation we find

$$\hat{\varrho}_{g_0}(g) \equiv \frac{\mathrm{d}\hat{\mu}(g_0^{-1}\dot{g})}{\mathrm{d}\hat{\mu}(\dot{g})} = \frac{\mathrm{d}\hat{\mu}(g_0^{-1}gK)}{\mathrm{d}\hat{\mu}(gK)} = \frac{\mathrm{d}\mu(Kg^{-1}g_0)}{\mathrm{d}\mu(Kg^{-1})} = \varrho_{g_0}(g^{-1}).$$

Then

$$J\hat{U}^L_{g_0}u(g) = (\hat{U}^L_{g_0}u)(g^{-1}) = \hat{\varrho}_{g_0}(g^{-1})u(g_0^{-1}g^{-1}) = \varrho_{g_0}(g)\,(Ju)(gg_0) = (U^L_{g_0}Ju)(g).$$

Because $J = J^{-1}$ one obtains

$$J\hat{U}^L J^{-1} = U^L$$

which is precisely eq. (48). ▼

The following example shows the beauty and the effectiveness of the theory of induced representations.

EXAMPLE 3. Let G be the Poincaré group $T^4 \rtimes SO(3,1)$ and $K = T^4 \rtimes SO(3)$. Let $k \to L_k$, $k = (a, r)$, $a \in T^4$, $r \in SO(3)$ be given by the formula

$$L_k = L_{(a,r)} = \exp[\mathrm{i}(\overset{0}{p}, a)]D^J(r), \tag{50}$$

where $\overset{0}{p} = (m, 0, 0, 0)$ and $D^J(r)$ is an irreducible representation of SO(3). We shall construct the representation U^L of G using eq. (47). We first find the explicit realization of the space $X = G/K$. By virtue of the Cartan decomposition of SO(3, 1) we have

$$\mathrm{SO}(3,1) = \mathscr{P}\,\mathrm{SO}(3) \tag{51}$$

where $\mathscr{P}$ is the set of pure Lorentz transformations, which may be parametrized by velocities $v = \{v_\mu\}$, $v_\mu v^\mu = 1$, or momenta $p = \{p_\mu\}$, $p_\mu = mv_\mu$. Hence the space X

$$X = T^4 \rtimes \mathrm{SO}(3,1)/T^4 \rtimes \mathrm{SO}(3) \sim \mathscr{P}$$

and may be realized as the mass hyperboloid

$$\{p_\mu\colon p_\mu p^\mu = m^2\}.$$

Now, by virtue of example 4.3.2 the measure $d\mu$ on the mass hyperboloid has the form $d\mu(p) = d^3p/p_0$; this measure is invariant under the action of G on X, hence the Radon–Nikodym derivative in (47) equals to one.

To complete the construction of U^L we have to find the operator $L^{-1}_{k_{g_0^{-1}s_g}}$. Note first that by virtue of (51) an element g in SO(3, 1) may be written in the form: $g = \Lambda_p r$, where $\Lambda_p \in \mathscr{P}$ and $r \in \mathrm{SO}(3)$. Observing that $\Lambda_p = s_g$ we obtain

$$\begin{aligned} g_0^{-1}g &= (a_0, \Lambda_0)^{-1}(0, \Lambda_p) = (-\Lambda_0^{-1}a_0, \Lambda_0^{-1})(0, \Lambda_p) \\ &= (-\Lambda_0^{-1}a_0, \Lambda_0^{-1}\Lambda_p) = (0, \Lambda_{\Lambda_0^{-1}p})(-\Lambda^{-1}_{\Lambda_0^{-1}p}\Lambda_0^{-1}a_0, \Lambda^{-1}_{\Lambda_0^{-1}p}\Lambda_0^{-1}\Lambda_p). \end{aligned} \tag{52}$$

Hence

$$k_{g_0^{-1}s_g} = (-\Lambda^{-1}_{\Lambda_0^{-1}p}\Lambda_0^{-1}a_0, \Lambda^{-1}_{\Lambda_0^{-1}p}\Lambda_0^{-1}\Lambda_p). \tag{53}$$

Taking into account that $\Lambda_p\overset{0}{p} = p$ and $(\Lambda p, \Lambda a) = (p, a)$ we obtain by eq. (50)

$$\begin{aligned} \bar{L}^{-1}_{k_{g_0^{-1}s_g}} &= \exp[i(\overset{0}{p}, \Lambda_p^{-1}a)][D^J(\Lambda^{-1}_{\Lambda_p^{-1}}\Lambda_0^{-1}\Lambda_p)]^{-1} \\ &= \exp[i(p, a)]D^J(\Lambda_p^{-1}\Lambda_0\Lambda_{\Lambda_p^{-1}}). \end{aligned}$$

Setting $r_\Lambda \equiv \Lambda_p^{-1}\Lambda\Lambda_{\Lambda_p^{-1}}$ we obtain: $((p, a) \equiv pa)$

$$U^L_{(a,\Lambda)}u(p) = \exp[ipa]D^J(r_\Lambda)u(\Lambda^{-1}p). \tag{54}$$

We show in ch. 17 that for an arbitrary $m > 0$ and J the representations (54) are irreducible. ▼

§ 2. Basic Properties of Induced Representation

We shall now derive a series of important properties of induced representations.

A. *Conjugate Representations**

We first show the equivalence of a representation $\overline{U^L}$ conjugate to U^L and a representation $U^{\bar{L}}$ induced by a representation $\bar{L}$ that can be written in the form

$$\bar{L}_k = CL_kC,$$

* Note that for unitary representations the conjugate and contragradient representations coincide (cf. eq. 5.1(14)).

where C is a conjugation (5.1 (14)) in the carrier space H of L. If we form a representation $U^{\bar{L}}$ induced by $\bar{L}$, then every vector function $\bar{u}(g)$ in $H^{\bar{L}}$ satisfies the condition

$$\bar{u}(kg) = \bar{L}_k\bar{u}(g) = CL_kCu(g). \tag{1}$$

By virtue of proposition 3.1.3°, the conjugation C can be lifted to the space $H^{\bar{L}}$ ie. $\bar{u}(g) = Cu(g)$. Thus eq. (1) and 1(6) implies

$$U^{\bar{L}}_{g_0}\bar{u}(g) = (\varrho_{g_0}(g))^{1/2}\bar{u}(gg_0) = CU^{L}_{g_0}u(g) = \bar{U}^{L}_{g_0}\bar{u}(g),$$

i.e.,

$$U^{\bar{L}}_g = \bar{U}^L. \tag{2}$$

B. *Representations Induced by the Direct Sum of Representations*

Let $\overset{1}{L}$ and $\overset{2}{L}$ be two unitary representations of a closed subgroup K of a group G in Hilbert spaces $\overset{1}{H}$ and $\overset{2}{H}$, respectively. The direct sum $\overset{1}{L}\oplus\overset{2}{L}$ of these representations is given in the Hilbert space $\overset{1}{H}\oplus\overset{2}{H}$ by the formula

$$\Big(\bigoplus_{i=1}^{2} \overset{i}{L}_g\Big)\{\overset{1}{u}, \overset{2}{u}\} = \{\overset{1}{L}_g\overset{1}{u}, \overset{2}{L}_g\overset{2}{u}\}$$

(cf. def. 5.3.3). The carrier space $H^{\overset{1}{L}\oplus\overset{2}{L}}$ of induced representation $U^{\overset{1}{L}\oplus\overset{2}{L}}$ will consist of all vector functions $\{\overset{1}{u}(g), \overset{2}{u}(g)\}$, with values in $\overset{1}{H}\oplus\overset{2}{H}$, each $\overset{i}{u}(g)$ satisfying conditions 1°, 2° and 3° of eq. 1(1).

This implies

$$H^{\overset{1}{L}\oplus\overset{2}{L}} = H^{\overset{1}{L}}\oplus H^{\overset{2}{L}} \tag{3}$$

and consequently

$$U^{\overset{1}{L}\oplus\overset{2}{L}} \simeq U^{\overset{1}{L}}\oplus U^{\overset{2}{L}}.$$

Hence the operations of inducing and taking direct sum are interchangable.

This result can be extended for any discrete sum $\oplus\overset{i}{L}$. In general we have

THEOREM 1. *Let K be a closed subgroup of a locally compact separable group G. Let L be a unitary representation of K, which decomposes into a direct integral of unitary representations $\overset{s}{L}$ of K, i.e.,*

$$L = \int L^s \mathrm{d}\mu(s). \tag{4}$$

Let H be the carrier space of L

$$H = \int \overset{s}{H}\mathrm{d}\mu(s)$$

where every $\overset{s}{H}$ is a separable Hilbert space.

Then, the representation U^L of G induced by L is unitarily equivalent to $\int U^{\overset{s}{L}} d\mu(s)$.

SKETCH OF THE PROOF: The representation U^L of G is realized in the space H^L of vector functions on G with values in H. The assertion of the theorem follows essentially from the fact that, as in the case of the finite sum (3), to the decomposition $H = \int \overset{s}{H} d\mu(s)$ there corresponds a decomposition

$$H^L = \int H^{\overset{s}{L}} d\mu(s),$$

where $H^{\overset{s}{L}}$ is a Hilbert space of vector functions on G with values in $\overset{s}{H}$. For measure-theoretical details of the proof cf. Mackey 1952, th. 10.1. ▼

COROLLARY. *If the representation U^L of G, induced by a representation L of a subgroup K is irreducible, then the representation L is also irreducible.*

PROOF: In fact, if L is reducible it could be written as a direct sum $L = \overset{1}{L} \oplus \overset{2}{L}$. This would imply by th. 1 that $U^L = U^{\overset{1}{L}} \oplus U^{\overset{2}{L}}$, which contradicts its irreducibility. ▼

Remark: The converse of this statement is false: L might be irreducible, while U^L is reducible. For instance, let G be any group, $K = \{e\}$ and let L be the one-dimensional identity representation of K in $H = C^1$. Then U^L is the regular representation which is reducible.

C. 'Inducing in Stages'

Let N and K be two closed subgroups of G, such that $N \subset K$ and let L be a representation of N. One may form an induced representation of G either directly, by forming ${}_GU^L$, or in stages, i.e., by forming first the induced representation ${}_KU^L \equiv V$, and then, constructing ${}_GU^V$. The following theorem states that both methods lead to an equivalent result.

THEOREM 2. *Let $N, K, N \subset K$, be closed subgroups of the separable, locally compact group G. Let L be a representation of N and let $V \equiv {}_KU^L$. Then, ${}_GU^L$ and ${}_GU^V$ are unitarily equivalent representations of G.* ▼

The proof of this important theorem is long and difficult (cf. Mackey 1952, th. 4.1). In order to give, however, the idea of the proof we consider a special case, when homogeneous spaces $N\backslash K$ and $K\backslash G$ both possess invariant measures.

PROOF: We start with the construction of carrier spaces ${}_GH^L$ and ${}_GH^V$ for induced representations ${}_GU^L$ and ${}_GU^V$, respectively. Let H be the carrier space of L. Then, the representation $V = {}_KU^L$ of K is realized in the Hilbert space ${}_KH^L$ of functions on K with values in H, satisfying the condition (cf. eq. 1(1))

$$u(nk) = L_n u(k), \quad n \in N,\ k \in K. \tag{5}$$

The representation $k \to {}_KU^L = V_k$ of K in ${}_KH^L$ is given by the formula

$$V_{k_0} u(k) = u(kk_0). \tag{6}$$

On the other hand, the space ${}_GH^V$ in which the representation ${}_GU^V$ is realized consists of functions on G with values in ${}_KH^L$ satisfying the condition

$$v(kg) = V_k v(g), \quad k \in K,\ g \in G. \tag{7}$$

Because values of the function $v(g) \in {}_GH^V$ belong to ${}_KH^L$, one can consider elements of ${}_GH^V$ as vector functions $F(g, k)$ on $G \times K$ with values in H. Equations (5) and (7) with this convention take the form

$$F(g, nk) = L_n F(g,k), \tag{5'}$$

$$F(k_0 g, k) = V_{k_0} F(g, k) = F(g, kk_0). \tag{7'}$$

Setting in eq. (7′) $k = e$, one obtains

$$F(g, k_0) = F(k_0 g, e) \equiv \Phi(k_0 g). \tag{8}$$

We now show that the map $S: F \to \Phi$ provides an isomorphism between the spaces ${}_GH^L$ and ${}_GH^V$ and an isomorphism of the operators ${}_GU_g^V$ and ${}_GU_g^L$. In fact, eq. (5′) implies

$$\Phi(ng) = L_n \Phi(g), \quad n \in N, \tag{5''}$$

i.e., $\Phi(g) \in {}_GH^L$. Conversely, to each function $\Phi \in {}_GH^L$ there corresponds, by eq. (8), the function $F(g, k)$ satisfying (5′) and (7′). Moreover, ${}_GU_{g_0}^V F(g, k) = F(gg_0, k)$ implies $\Phi(gg_0) = {}_GU_{g_0}^L \Phi(g)$. Consequently,

$$S_G U^V S^{-1} = {}_GU^L.$$

Thus, to conclude the theorem it is now sufficient to show that the map $S: F \to \Phi$ is unitary. Denote by X, Y and Z the homogeneous spaces $N\backslash G$, $K\backslash G$ and $N\backslash K$ respectively. The norm of a function Φ in ${}_GH^L$ is given by the formula

$$\|\Phi\|^2_{{}_GH^L} = \int_X \|\Phi(g_x)\|^2_H \mathrm{d}\mu(x), \tag{9}$$

where g_x is any element of the right coset Ng, corresponding to a point $x \in X$ and $\mathrm{d}\mu(x)$ is an invariant measure on X. Similarly

$$\|F\|^2_{{}_GH^V} = \int_Y \|F(g_y, k)\|^2_{{}_KH^L} \mathrm{d}\sigma(y), \tag{10}$$

$$\|v\|^2_{{}_KH^L} = \int_Z \|v(k_z)\|^2_H \mathrm{d}\varrho(z). \tag{11}$$

In eq. (10), g_y is any element of the right coset Kg corresponding to a point y in Y (cf. th. 2.4.1) and $\mathrm{d}\sigma(y)$ is an invariant measure in Y. The notation in eq. (11) is analogous. Notice that after a selection of g_y and k_z corresponding to a point $y \in Y$ and $z \in Z$, an element g_x corresponding to $x \in X$ can be taken as $k_z g_y$. Hence substituting eq. (11) into eq. (10), one obtains

$$\|F\|^2_{{}_GH^V} = \int_Y \left(\int_Z (\|F(g_y, k_z)\|^2_H \mathrm{d}\varrho(z) \right) \mathrm{d}\sigma(y)$$

$$= \int_{Y \times Z} \|\Phi(k_z g_y)\|^2_H \mathrm{d}\varrho(z) \mathrm{d}\sigma(y) = \int_X \|\Phi(g_x)\|^2_H \mathrm{d}\tilde{\mu}(x), \tag{12}$$

where $d\tilde{\mu}(x) = d\sigma(y)d\varrho(z)$. We now show that $d\tilde{\mu}(x)$ is an invariant measure on X equal to $d\mu(x)$. In fact, if $g_x = k_z g_y$ then, by th. 2.4.1, for any $g \in G$, one obtains

$$g_x g = k_z g_y g = k_z k_{(y,g)} g_{yg} = n_{(z,y,g)} k_{zk(y,g)} g_{yg}, \tag{13}$$

where $n_{(z,y,g)} \in N$.

This implies, because we assumed $d\varrho$ and $d\sigma$ to be invariant measures,

$$d\tilde{\mu}(xg) = d\varrho(zk_{(y,g)})d\sigma(yg) = d\varrho(z)d\sigma(y) = d\tilde{\mu}(x). \tag{14}$$

Because the invariant measures $d\mu$, $d\sigma$ and $d\varrho$ are defined up to a constant factor one can normalize them in such a manner that $d\mu = d\varrho d\sigma$. Thus the invertible and isometric map $S: F \to \Phi$ is unitary. ▼

D. *Representations Induced by the Tensor Product of Representations*

Let $\overset{1}{T}_{g_1}$ and $\overset{2}{T}_{g_2}$ be representations of G_1 and G_2 in Hilbert spaces H_1 and H_2, respectively. According to def. 5.5.2 the *outer tensor product representation* $\overset{1}{T}_{g_1} \otimes \overset{2}{T}_{g_2}$ of the direct product $G_1 \times G_2$ in the tensor product space $H_1 \otimes H_2$ is given by means of the formula:

$$\begin{aligned}(\overset{1}{T} \otimes \overset{2}{T})_{(g_1,g_2)}(u_1 \otimes u_2) &= (\overset{1}{T}_{g_1} \otimes \overset{2}{T}_{g_2})(u_1 \otimes u_2) \\ &= \overset{1}{T}_{g_1} u_1 \otimes \overset{2}{T}_{g_2} u_2.\end{aligned} \tag{15}$$

The next theorem describes the connection between a representation of $G_1 \times G_2$ induced by an outer tensor product of the representation $L_1 \otimes L_2$ of closed subgroups K_1 and K_2 and outer tensor product of induced representations.

THEOREM 3. *Let L_1 and L_2 be unitary representations of the closed subgroups K_1 and K_2 of the separable, locally compact groups G_1 and G_2, respectively. Then, the two representations*

$$_{G_1 \times G_2}U^{L_1 \otimes L_2} \quad \textit{and} \quad {_{G_1}U^{L_1}} \otimes {_{G_2}U^{L_2}}$$

of $G_1 \times G_2$ are unitarily equivalent.

PROOF: Let H_1 (respectively H_2) denote the carrier space of the representation $L_1(L_2)$ of $K_1(K_2)$. The proof consists in showing the existence of a unitary map S of $H_1^{L_1} \otimes H_2^{L_2}$ onto $(H_1 \otimes H_2)^{L_1 \otimes L_2}$ such that

$$S({_{G_1}U_{g_1}^{L_1}} \otimes {_{G_2}U_{g_2}^{L_2}}) = {_{(G_1 \times G_2)}U_{(g_1,g_2)}^{L_1 \otimes L_2}} S. \tag{16}$$

In order to show this, let us associate with each pair

$$\{u_1(g_1), u_2(g_2)\}; \quad u_1(g_1) \in H_1^{L_1}, \quad u_2(g_2) \in H_2^{L_2},$$

a function

$$G_1 \times G_2 \ni (g_1, g_2) \to u_1(g_1) \otimes u_2(g_2). \tag{17}$$

One readily verifies that in this manner a linear map S of $H_1^{L_1} \otimes H_2^{L_2}$ into $(H_1 \otimes H_2)^{L_1 \otimes L_2}$ is defined. In fact, the conditions 1° and 2° of eq. 1(1) are satisfied. For instance, if $(k_1, k_2) \in K_1 \times K_2$, then,

$$\begin{aligned} u_1(k_1 g_1) \otimes u_2(k_2 g_2) &= L_{1k_1} \otimes L_{2k_2}(u_1(g_1) \otimes u_2(g_2)) \\ &= (L_1 \otimes L_2)_{(k_1, k_2)}(u_1(g_1) \otimes u_2(g_2)). \end{aligned}$$

The condition 3°1(1) is also satisfied. In fact, if $d\mu_1$ (resp. $d\mu_2$) is a quasi-invariant measure on $X_1 = K_1 \backslash G_1$ ($X_2 = K_2 \backslash G_2$) and $\varrho_{g_1}(x_1)\,[\varrho_{g_2}(x_2)]$ is the corresponding Radon–Nikodym derivative, then $d\mu_1 d\mu_2$ is a quasi-invariant measure on $X_1 \times X_2 \cong K_1 \times K_2 \backslash G_1 \times G_2$ with $\varrho_{g_1}(x_1)\varrho_{g_2}(x_2)$ being the corresponding Radon–Nikodym derivative. We have then

$$\begin{aligned} &\int_{X_1 \times X_2} \|u_1(g_1) \otimes u_2(g_2)\|^2_{H_1 \otimes H_2} d\mu_1(\dot{g}_1) d\mu_2(\dot{g}_2) \\ &= \int_{X_1 \times X_2} \|u_1(g_1)\|^2_{H_1} \|u_2(g_2)\|^2_{H_2} d\mu_1(\dot{g}_1) d\mu_2(\dot{g}_2) \\ &= \|u_1\|^2_{H_1 L_1} \|u_2\|^2_{H} {}_{L} < \infty . \end{aligned}$$

Consequently, $u_1(g_1) \otimes u_2(g_2) \in (H_1 \otimes H_2)^{L_1 \times L_2}$.

Thus, there exists a densely defined linear map S of the tensor product space $H_1^{L_1} \otimes H_2^{L_1}$ into $(H_1 \otimes H_2)^{L_1 \otimes L_2}$. This map is isometric. In fact, if

$$w = \sum_{i=1}^{n} u_i(g_1) \otimes v_i(g_2),$$

where

$$u_i(g_1) \in H_1^{L_1}, \quad v_i(g_2) \in H_2^{L_2},$$

then

$$\begin{aligned} \|S(w)\|^2_{(H_1 \otimes H_2)^{L_1 \times L_2}} &= \int_{X_1 \times X_2} \Big\|\sum_{i=1}^{n} u_i(g_1) \otimes v_i(g_2)\Big\|^2_{H_1 \otimes H_2} d\mu_1(\dot{g}_1) d\mu_2(\dot{g}_2) \\ &= \int_{X_1 \times X_2} \sum_{i,j=1}^{n} (u_i(g_1), u_j(g_1))_{H_1} (v_i(g_2), v_j(g_2))_{H_2} d\mu_1(\dot{g}_1) d\mu_2(\dot{g}_2) \\ &= \sum_{i,j=1}^{n} (u_i, u_j)_{H_1 L_1} (v_i, v_j)_{H_2 L_2} \\ &= \Big\|\sum_{i=1}^{n} u_i \otimes v_i\Big\|^2_{H_1^{L_1} \otimes H_2^{L_2}} \\ &= \|w\|^2_{H_1^{L_1} \otimes H_2^{L_2}} . \end{aligned}$$

Therefore, the map S is extendible to a unitary operator of the whole space $H_1^{L_1} \otimes H_2^{L_2}$ into $(H_1 \times H_2)^{L_1 \otimes L_2}$. In order to conclude the proof, it will be sufficient to show that the image of S forms a total set* in $(H_1 \otimes H_2)^{L_1 \otimes L_2}$.

In fact, by proposition 1.3, elements $\hat{\Phi}(g_1, g_2)$ given by eq. 1(10), where

$$\Phi(g_1, g_2) = \varphi(g_1, g_2)(u \otimes v),$$

$\varphi \in C_0(G_1 \times G_2)$, $u \in H_1$ and $v \in H_2$, form a total set in the space $(H_1 \otimes H_2)^{L_1 \times L_2}$. Because the functions

$$\varphi(g_1, g_2) = \psi(g_1)\eta(g_2), \quad \psi \in C_0(G_1),\ \eta \in C_0(G_2),$$

form a total set in $C_0(G_1 \times G_2)$, the images $\hat{\Phi}$ of the functions $\Phi = \psi(g_1)u \otimes \eta(g_2)v = u_\psi \otimes v_\eta$ still form a total set in $(H_1 \otimes H_2)^{L_1 \times L_2}$. Using formula 1(10), one readily verifies that

$$\hat{\Phi} = S(\hat{u}_\psi \otimes \hat{u}_\eta).$$

Consequently, the assertion of th. 3 follows. ▼

§ 3. Systems of Imprimitivity

A. *The Imprimitivity Theorem*

Let K be a closed subgroup of a locally compact, separable group G, L a unitary representation of K in H, and U^L the induced representation of G in H^L. Let Z be a Borel subset of $X = K\backslash G$ and χ_Z the characteristic function of the set Z. For any $u \in H^L$ let

$$\big(E(Z)u\big)(g) \equiv \chi_Z(\dot{g})u(g), \quad \dot{g} = Kg. \tag{1}$$

This function is weakly measurable (cf. eq. 1.1°(1)). Moreover, for $k \in K$, we have

$$E(Z)u(kg) = \chi_Z(\dot{g})u(kg) = L_k\big(\chi_Z(\dot{g})u(g)\big) = L_k E(Z)u(g).$$

We have also

$$\int_X \|\chi_Z(\dot{g})u(g)\|^2 d\mu(\dot{g}) = \int_X \chi_Z(\dot{g})\|u(g)\|^2 d\mu(\dot{g})$$
$$= \int_Z \|u(g)\|^2 d\mu(\dot{g}) < \infty.$$

Hence, the function (1) satisfies the conditions 1°, 2° and 3° of eq. 1(1) and, consequently, it belongs to H^L. It follows, moreover, from eq. (1) that the operator function $Z \to E(Z)$ has the properties

$$\begin{gathered} E(X) = I, \quad E(\emptyset) = 0, \\ E(Z_1 \cap Z_2) = E(Z_1)E(Z_2), \\ E^*(Z) = E(Z), \end{gathered} \tag{2}$$

---

* A subset Q in Hilbert space H is called a *total set* if the linear envelope of elements of Q is a dense set in H.

and $E(\cdot)$ is countably additive in the strong operator topology of $\mathscr{L}(H^L)$. Hence the map

$$X \supset Z \to E(Z) \in \mathscr{L}(H^L)$$

defines on the space X a spectral measure (cf. app. B. 3). This measure has the definite transformation properties under the representations U^L of G. In fact,

$$\begin{aligned}(U^L_{g_0} E(Z)\, U^L_{g_0^{-1}} u)(g) &= [\varrho_{g_0}(g)]^{1/2} \big(E(Z)\, U^L_{g_0^{-1}} u\big)(gg_0) \\ &= [\varrho_{g_0}(g)]^{1/2} \chi_Z(\dot{\widehat{gg_0}})\, (U^L_{g_0^{-1}} u)\, (gg_0) \\ &= [\varrho_{g_0}(g)]^{1/2} [\varrho_{g_0^{-1}}(gg_0)]^{1/2} \chi_Z(\dot{\widehat{gg_0}}) u(g) = E(Z_{g_0^{-1}}) u(g). \qquad (3)\end{aligned}$$

The last step follows from the obvious equality $\chi_Z(\dot{\widehat{gg_0}}) = \chi_{Z_{g_0^{-1}}}(\dot{g})$ and the composition law for Radon–Nikodym derivatives. Thus

$$U^L_g E(Z)\, U^L_{g^{-1}} = E(Zg^{-1}). \qquad (4)$$

We see, therefore, that with every induced representation U^L of G, one can associate a spectral measure $E(Z)$ having the transformation property (4).

In general, let X be a G-space and let U be a unitary representation of G in a Hilbert space H. If $E(Z)$, $Z \subset X$, is a spectral measure with values in $\mathscr{L}(H)$, which transforms under U_g according to (4), then $E(Z)$ is called a *system of imprimitivity* for U based on X. If the base X is transitive under G, $E(Z)$ is called a *transitive system of imprimitivity.* A representation which has at least one system of imprimitivity is said to be *imprimitive.* The system of imprimitivity given by a spectral function (1) is called the *canonical system of imprimitivity.*

EXAMPLE 1. Let P_μ be the momentum operator of a single relativistic particle and let $\Lambda \to U_\Lambda$ be a unitary representation of the Lorentz group in the Hilbert space H of wave functions. The spectral decomposition of P_μ has the form (cf. example 6.2.1)

$$P_\mu = \int_{p^2=m^2} p_\mu \mathrm{d}E(p),$$

where $E(p)$ is the spectral measure associated with momenta P_μ. Because P_μ is a tensor operator, we have (cf. 9.1(11))

$$U_\Lambda^{-1} P_\mu U_\Lambda = \Lambda_\mu^{-1\nu} P_\nu.$$

Thus

$$U_\Lambda^{-1} P_\mu U_\Lambda = \int_{p^2=m^2} \Lambda_\mu^{-1\nu} p_\nu \mathrm{d}E(p) = \int_{p'^2=m^2} p'_\mu \mathrm{d}E(\Lambda p').$$

On the other hand, we have

$$U_\Lambda^{-1} P_\mu U_\Lambda = \int p_\mu \mathrm{d}\big(U_\Lambda^{-1} E(p)\, U_\Lambda\big).$$

Consequently, for a Borel subset Z on the mass hyperboloid we have*

$$U_\Lambda^{-1} E(Z) U_\Lambda = E(\Lambda Z).$$

Thus, the spectral measure of the momentum operator is a system of imprimitivity for U based on the momentum space. Because the action of the translations in momentum space is trivial, $E(Z)$ is also a system of imprimitivity for the Poincaré group.

In a similar manner one may show that every self-adjoint tensor operator acting in a carrier space of physical states provides an imprimitivity system for the representation U based on momentum space. ▼

We denote $E(\psi) = \int \psi(z) \mathrm{d}E(z)$.

Equations (1)–(4) show that with every induced representation U^L one can associate the transitive canonical system of imprimitivity. It turns out that conversely a unitary representation possessing a transitive system of imprimitivity is unitarily equivalent to an induced representation. This fundamental result is described by the following theorem.

THEOREM. *Let U be a unitary continuous representation of G in a Hilbert space $H, X = K\backslash G$ and E: $C_0(X) \to L(H)$ a $*$-holomorphism with $E[C_0(X)]H$ dense in H and*

$$U_g E(\psi) U_{g^{-1}} = E(T_g^R \psi), \quad g \in G, \ \psi \in C_0(X). \tag{5}$$

Then there exists a unique (up to unitary equivalence) continuous unitary representation L of K in a Hilbert space H^L such that the pair (U, E) is unitarily equivalent to the pair (U^L, E^L), i.e. there exists a unitary operator V: $H \to H^L$ such that

$$VU_g = U_g^L V, \quad g \in G \tag{6}$$

and

$$VE(\psi) = E^L(\psi) V, \quad \psi \in C_0(X). \tag{7}$$

PROOF: Since the kernel of E is a translation invariant ideal we see that $\|E(\psi)\| = \|\psi\|_\infty$, the supremum norm of ψ; now consider the Gårding domain D_G

$$D_G = \operatorname{span}\{u(\varphi) = \int \varphi(g) U_g u \mathrm{d}g, \ u \in H, \ \varphi \in C_0(G)\} \tag{8}$$

and the Radon measure $\varphi \to (E(\tau\varphi)u, v)$, $\tau\varphi(x) = \int_K \varphi(kx)\mathrm{d}k$ for $u, v \in H$, denoted $\mathrm{d}\mu_{u,v}$, so

$$(E(\tau\varphi)u, v) = \int_G \varphi(g) \mathrm{d}\mu_{u,v} g, \quad \varphi \in C_0(G).$$

We claim that for $u, v \in D_G$, $\mathrm{d}\mu_{u,v}$ is a continuous function. To see this, let $u, v \in H$ and $\psi_1, \varphi \in C_0(G)$ then

$$\begin{aligned} |(E(\tau\varphi)u(\psi_1), v)| &\leqslant \|\tau\varphi\|_\infty \|\psi_1\|_\infty \operatorname{vol}(\operatorname{supp}\psi_1) \|u\| \|v\| \\ &\leqslant C \operatorname{vol}(\operatorname{supp}\psi_1) \|u\| \|v\| \|\varphi\|_\infty \|\psi_1\|_\infty \end{aligned}$$

* Here G acts on the left in momentum space, because in physics we realize momentum space by means of left cosets, $p \sim \Lambda_p \mathrm{SU}(2)$.

where C is a constant depending on the support of φ, so $(E(\tau,\varphi)u(\psi_1),v)$ defines a Radon measure $\mathrm{d}\lambda(g_1,g_2)$ on $G\times G$.

By Fubini theorem we have:

$$\begin{aligned}\int_G \varphi(g)\,\mathrm{d}\mu_{u(\psi_1),v(\psi_2)}(g) &= (E(\tau\varphi)u(\psi_1),v(\psi_2))\\ &= \int_G \bar\psi_2(g)\big(E(\tau(\varphi))u(\psi_1),U_g v\big)\,\mathrm{d}g = \int_G \bar\psi_2(g)\big(E\tau(T^R_{g^{-1}}\varphi)u(T^R_{g^{-1}}\psi_1),v\big)\mathrm{d}g\\ &= \int_G \bar\psi_2(g)\iint_{G\times G}\varphi(g_1g^{-1})\psi_1(g_2g^{-1})\,\mathrm{d}\lambda(g_1,g_2)\,\mathrm{d}g\\ &= \int_G \varphi(g)\iint_{G\times G}\bar\psi_2(g_1^{-1}g^{-1})\psi_1(g_2g_1^{-1}g^{-1})\,\mathrm{d}\lambda(g_1,g_2)\,\mathrm{d}g,\end{aligned}$$

where we have made a change of variables in the g-integration. Hence

$$\mathrm{d}\mu_{u(\psi_1),v(\psi_2)}(g) = h_{u(\psi_1),v(\psi_2)}(g)$$

where

$$h_{u(\psi_1),v(\psi_2)}(g) = \iint_{G\times G}\bar\psi_2(g_1^{-1}g^{-1})\psi_1(g_2g_1^{-1}g^{-1})\,\mathrm{d}\lambda(g_1,g_2)$$

is a continuous function on G by standard arguments. Note also that $h_{U_g u(\psi_1),v(\psi_2)}(g_1)$ is a continuous function in (g,g_1) and more generally for $u,v\in D_G$ that $h_{U_{g_1}u,U_{g_2}v}$ is a continuous function in (g_1,g_2,g).

We define a sesquilinear form on $D_G\times D_G$ by $\beta(u,v) = h_{u,v}(e)$ and check that

(i) $\beta(u,u)\geqslant 0,\ u\in D_G$,

(ii) $\beta(U_k u, U_k v) = \beta(u,v)$,

(iii) $(E(\tau\varphi)u,v) = \int_G \varphi(g)\beta(U_g u, U_g v)\,\mathrm{d}g$.

Now the rest of the proof is standard: we let

$$H^L = \{[u\in D_G|\beta(u,u)<\infty]/[u\in D_G|\beta(u,u)=0]\}$$

(Hilbert space completion) and $L_k[u] = [U_k u]$ (the class defined by $U_k u$). Then $(L_k[u],[v]) = \beta(U_k u, v)$ is continuous and L is a unitary continuous representation of K in H^L. For $u\in D_G$, $w_u(g) = [U_g u]$ is a continuous H^L-valued function on G satisfying $w_u(kg) = L_k w_u(g)$, $k\in K$, $g\in G$; in fact $V\colon u\to w_u$ extends to an isometry from H onto H^L intertwining (U,E) and (U^L,E^L) as required. The uniqueness of L follows from $(U^L,E^L)\simeq(U^{L'},E^{L'})$ (unitary equivalence) if and only if $L=L'$. ▼

This fundamental theorem is a starting point for many applications in physics; in particular, the classification of irreducible representations of Euclidean, Galilei and Poincaré groups, the construction of relativistic position operators as well as the proof of the equivalence of Schrödinger and Heisenberg formulation of quantum mechanics can be achieved with the help of the Imprimitivity Theorem (cf. ch. 17.2 and 20.1 and 2).

B. *Irreducibility and Equivalence of Induced Representations*

The remaining part of this section is devoted to an elaboration of convenient criteria for the irreducibility of an induced representation U^L and for the equivalence of two induced representations in terms of a system of imprimitivity.

Let $g \to U_g^L$ be an induced unitary representation of a topological locally compact group G in a Hilbert space H^L. A linear hull of the set of vectors of the form

$$\int_G \varphi(g)\, U_g^L v\, dg \quad \text{for all } \varphi \in C_0(G) \text{ for all } v \in H^L,$$

is called the *Gårding space* D_G of the representation U^L. One readily verifies, similarly as in the case of Lie groups, that D_G is a linear and dense subset of H^L invariant with respect to U_g (cf. th. 11.1.1).

A vector $v \in H^L$ is said to be *continuous* if it can be represented as a continuous vector function on G.

LEMMA 1. *Let $\mu(\cdot)$ be a quasi-invariant measure on $K\backslash G$ and choose its continuous Radon–Nikodym derivative. Then every vector $v \in D_G$ is a continuous vector function on G.*

We leave the proof as an exercise for the reader.

In the sequel we shall represent elements $v \in D_G$ by continuous vector functions on G. We now construct a dense set of elements in the space H.

LEMMA 2. *The set*

$$\{v(e) \colon v \in D_G\} \tag{9}$$

is dense in the carrier space H of a representation L of the subgroup K.

The proof follows almost directly from proposition 16.1.3.

Let T and T' be representations of G in $H(T)$ and $H(T')$, respectively. Let $R(T, T')$ be the set of all intertwining operators (cf. ch. 5, § 2). $R(T, T')$ is a vector space. In the case $T = T'$, $R(T, T)$ is a closed subalgebra of $\mathscr{L}(H)$. In fact, if for a sequence $R_n \in R(T, T)$ and $\lim_{n\to\infty} R_n = R \in \mathscr{L}(H(T))$, then for an arbitrary u in $H(T)$

$$\begin{aligned} T_g R u &= T_g \lim_n R_n u = \lim_n T_g R_n u \\ &= \lim_n R_n T_g u = R T_g u. \end{aligned} \tag{10}$$

The algebra $R(T, T)$ is called a *commuting algebra* of the representation T. If $R(T, T')$ contains a unitary operator $\tilde{V}$, then $\tilde{V} T_g \tilde{V}^{-1} = T'_g$ for all $g \in G$, and consequently T and T' are unitarily equivalent.

THEOREM 3. *Let U^L and $U^{L'}$ be representations of G in Hilbert spaces H^L and $H^{L'}$, respectively, induced by the representations L and L' of the closed subgroup $K \subset G$. Let $E(Z)$ (resp. $E'(Z)$) denote the corresponding canonical system of im-*

primitivity, where Z is a Borel set in $K\backslash G$. Then, the set $R(L, L')$ is isomorphic with the set S of all operators $V \in \mathscr{L}(H^L \to H^{L'})$ such that

$1°$ $U_g^{L'} V = VU_g^L$ *for all g in G, i.e. $V \in R(U^L, U^{L'})$,*

$2°$ $E'(Z)V = VE(Z)$ *for all Borel sets $Z \subset K\backslash G$.* (11)

PROOF: The proof consists in constructing the isomorphism φ of the vector space $R(L, L')$ onto the vector space S of operators which satisfy conditions (11). Let $R \in R(L, L')$ and $v \in H^L$. Set

$$(\tilde{R}v)(g) \underset{g \in G}{\equiv} Rv(g). \tag{12}$$

We first show that $\tilde{R}v$ is in $H^{L'}$. In fact, it is evident that $\tilde{R}v(g)$ is weakly measurable (cf. 1(1)1°). Moreover, for any $k \in K$

$$\begin{aligned}(\tilde{R}v)(kg) &= Rv(kg) = RL_k v(g) \\ &= L'_k Rv(g) = L'_k(\tilde{R}v)(g).\end{aligned}$$

Finally, from the inequality

$$\begin{aligned}\int_X (\tilde{R}v(g), \tilde{R}v(g))\mathrm{d}\mu(\dot{g}) &= \int_X (Rv(g), Rv(g))\mathrm{d}\mu(\dot{g}) \\ &\leqslant \|R\|^2 \int_X (v(g), v(g))\mathrm{d}\mu(g) = \|R\|^2(v, v)\end{aligned} \tag{13}$$

it follows that $\tilde{R}v$ satisfies condition 1(1)3°. Hence $\tilde{R}v \in H^{L'}$. It is evident from the definition (12) that the operator $\tilde{R}$ is linear. Moreover, eq. (13) implies

$$\|\tilde{R}v\| \leqslant \|R\| \, \|v\|, \quad \text{i.e.} \quad \|\tilde{R}\| \leqslant \|R\|.$$

Thus $\tilde{R} \in \mathscr{L}(H^L \to H^{L'})$. We now show that $\tilde{R} \in S$. In fact,

$$\begin{aligned}(U_{g_0}^{L'} \tilde{R}v)(g) &= \varrho_{g_0}^{1/2}(g)\,(\tilde{R}v)\,(gg_0) = \varrho_{g_0}^{1/2}(g) Rv\,(gg_0) \\ &= R(U_{g_0}^L v)(g) = (\tilde{R}U_{g_0}^L v)(g).\end{aligned}$$

Moreover, for any Borel set Z in $K\backslash G$, one obtains

$$\begin{aligned}\big(E'(Z)\tilde{R}v\big)(g) &= \chi_Z(\dot{g})(\tilde{R}v)(g) = \chi_Z(\dot{g}) Rv(g) \\ &= R\big(E(Z)v\big)(g) = \big(\tilde{R}E(Z)v\big)(g).\end{aligned}$$

Denote by φ the map $R(L, L') \ni R \to \tilde{R} \in S$ given by eq. (12). In order to conclude the proof of th. 3, it is now sufficient to show that φ is the map of $R(L, L')$ onto S, i.e., that for every V in S there exists R in $R(L, L')$ such that $\varphi(R) = V$.

Let D_G and D'_G be the Gårding subspaces of U^L and $U^{L'}$, respectively, and let $V \in S$. Then, by virtue of (11)1° $VD_G \subset D'_G$. Let $v \in D_G$ and let Z be an arbitrary Borel subset of X. Then, utilizing (11)2°, one obtains

$$\begin{aligned}\int_Z \big((Vv)(g), Vv(g)\big)\,\mathrm{d}\mu(\dot{g}) &= \|E'(Z)Vv\|^2 \\ &= \|VE(Z)v\|^2 \leqslant \|V\|^2 \|E(Z)v\|^2 = \|V\|^2 \int \big(v(g), v(\dot{g})\big)\mathrm{d}\mu(\dot{g}).\end{aligned}$$

Because the functions under the integral are continuous by lemma 1, then

$$((Vv)(g), Vv(g)) \leqslant \|V\|^2 (v(g), v(g))$$

for all g in G. In particular

$$\|(Vv)(e)\| \leqslant \|V\| \, \|v(e)\| .$$

It follows therefore from lemma 2 that there exists a linear map $\tilde{R} \in \mathscr{L}(H \to H')$ such that

$$Vv(e) = \tilde{R}v(e).$$

We now show that $\tilde{R} \in R(L, L')$. In fact, for any $g \in G$ we have

$$\begin{aligned}(Vv)(g) &= \varrho_g^{-1/2}(k)[U_g^{L'} Vv](e) = \varrho_g^{-1/2}(k)[VU_g^L v](e) \\ &= \varrho_g^{-1/2}(k)\tilde{R}[U_g^L v](e) = \tilde{R}v(g).\end{aligned} \tag{14}$$

Now, let $k \in K$. Because v (resp. Vv) satisfies the condition 1(1)2° relative to the representation L_k (resp. L'_k), then, by eq. (14),

$$L'_k \tilde{R}v(e) = L'_k(Vv)(e) = (Vv)(k) = \tilde{R}v(k) = \tilde{R}L_k v(e).$$

Thus, by lemma 1,

$$L'_k \tilde{R} = \tilde{R} L_k \quad \text{for all } k \in K,$$

i.e., $\tilde{R} \in R(L, L')$. Consequently, by eq. (14) and the definition of the map φ for an arbitrary v in D_G, one obtains

$$Vv = \varphi(\tilde{R})v.$$

Because D_G is dense in H^L, then, $V = \varphi(\tilde{R})$. ▼

Remark 1: The irreducibility of L does not guarantee, in general, the irreducibility of U^L (cf. the remark and the counter-example after corollary to th. 2.1). However, in many important cases, for instance in the case of semi-direct products, the irreducibility of L implies the irreducibility of U^L (cf. th. 17.1.5).

We now prove a theorem, which provides in many important cases a convenient criterion for the irreducibility of an induced representation U^L of G.

Let U^L be an induced representation in H^L and let $E(Z)$ be the corresponding canonical system of imprimivity. We call a pair (U^L, E) *irreducible*, if for an arbitrary $V \in \mathscr{L}(H^L)$, $g \in G$, and a Borel subset $Z \subset X$,

$$\begin{pmatrix} VU_g^L = U_g^L V \\ VE(Z) = E(Z)V \end{pmatrix} \to (V = \lambda I). \tag{15}$$

The following theorem gives a simple criterion for the irreducibility of the pair (U^L, E).

THEOREM 4. *Let U^L be a representation of a locally compact, separable group G induced by the representation L of the closed subgroup $K \subset G$. Then*

$$\begin{pmatrix} \textit{A pair } (U^L, E) \textit{ is} \\ \textit{irreducible} \end{pmatrix} \leftrightarrow \begin{pmatrix} \textit{The representation } L \\ \textit{is irreducible} \end{pmatrix}.$$

PROOF: We apply th. 3 in the special case in which $L = L'$. It then follows from the definition of irreducibility, that

$$\begin{pmatrix}\text{A pair } (U^L, E) \text{ is} \\ \text{irreducible}\end{pmatrix} \leftrightarrow (S = \{\lambda I, \lambda \in C^1\}),$$

$$\begin{pmatrix}\text{A representation } L \\ \text{is irreducible}\end{pmatrix} \leftrightarrow \big(R(L, L) = \{\lambda I, \lambda \in C^1\}\big).$$

The assertion of the theorem follows from the fact that the map φ defined by eq. (12) establishes one-to-one correspondence of $R(L, L)$ and S. ▼

Now let U^L (resp. $U^{L'}$) be a representation of G in H^L (resp. $H^{L'}$) induced by a representation L (resp. L') of the closed subgroup $K \subset G$ and let $E(Z)$ (resp. $E'(Z)$) be the corresponding system of imprimitivity. A pair (U^L, E) is said to be *equivalent* to a pair $(U^{L'}, E')$ (which we write as $(U^L, E) \simeq (U^{L'}, E')$), if there exists a unitary operator $V: H^L \to H^{L'}$ such that

$$VU_g^L V^{-1} = U_g^{L'} \quad \text{for all } g \text{ in } G,$$
$$VE(Z)V^{-1} = E'(Z) \quad \text{for all Borel subsets } Z \subset X. \tag{16}$$

The following theorem provides a criterion of equivalence of the pairs (U^L, E) and $(U^{L'}, E')$

THEOREM 5.

$$[(U^L, E) \simeq (U^{L'}, E')] \Leftrightarrow (L \simeq L').$$

PROOF: It follows from the definition of equivalence that $(U^L, E) \simeq (U^{L'}, E')$ if and only if there exists a unitary operator V satisfying conditions (16). On the other hand, $L \simeq L'$ if and only if there exists a unitary, intertwining operator $R \in R(L, L')$. By virtue of eq. (13), R is unitary if and only of the corresponding $V = \varphi(R)$ is unitary. Thus the assertion of the theorem follows from th. 3 which, in fact, states that the map φ is one to one. ▼

Remark 2: It follows from th. 5 and eq. (16) that $L \simeq L'$ implies that the induced representation U^L is equivalent to the induced representation $U^{L'}$.

Theorems 3–5 do not provide in general a direct criterion for the irreducibility of a representation U^L of G induced by an irreducible representation L of a subgroup $K \subset G$. However, in cases in which the system of imprimitivity $E(Z)$ is associated with a spectral measure of a representation of an invariant commutative subgroup N of G (as it is in case of a semidirect product $G = N \circledS M$) then th. 4 provides the irreducibility of U^L (cf. th. 17.1.5).

The direct criteria of irreducibility for induced representations of semisimple Lie groups are given in ch. 19, sec. 1.

§ 4. Comments and Supplements

The theory of induced representations was originated by Frobenius in 1898. He gave the basic construction of inducing presented in § 1 in the case of finite groups. It is interesting that this simple method which could be extended immediately to many groups was used in the case of continuous groups only forty years later. This was done by Wigner in 1939 in his classic paper on the classification of irreducible unitary representations of the Poincaré group. Later on this method was applied by Bargmann 1947 and Gel'fand and Naimark 1947 for the construction of representations of the Lorentz group. Soon Gel'fand and Naimark recognized the generality and the power of the technique of induced representations and in their fundamental paper in 1950 gave the construction of 'almost all' irreducible unitary representations of all complex classical simple Lie groups.

The systematic analysis of the general properties of induced representations was done by Mackey. He gave a general construction of induced representations for an arbitrary locally compact topological group, proved with the generality the Imprimitivity Theorem, the Induction in Stages Theorem, the Frobenius Reciprocity Theorem, the Tensor Product Theorem and others. The work of Mackey and Gel'fand and Naimark stimulated the development of group representation theory and its applications in quantum physics. In particular the method of induced representations was applied for various concrete groups like the groups of motion of the n-dimensional Minkowski or Euclidean space, $SL(n, R)$, $SU(p, q)$ and other groups. The systematic analysis of the properties of induced representations of real semisimple Lie groups was carried out by Bruhat (1956). In particular he derived the important criteria for irreducibility of induced representations.

The generalization of the technique of induced representation to the construction of the so-called holomorphic and partially holomorphic induced representations as well as the simplified derivation of some of Mackey's results was done by Blattner 1961a, b.

The idea of the proof of Imprimitivity Theorem given in sec. 3 is due to N. S. Poulsen (unpublished) and in the present form was communicated to us by B. Ørsted.

§ 5. Exercises

§ 1.1. Let G be the three-dimensional real nilpotent group with the composition law

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+x_1y_2), \qquad x, y, z \in R^1.$$

The group G is the semidirect product $G = N \circledS S$ of the abelian normal subgroup $N = \{(0, y, z)\}$ and $S = \{(x, 0, 0)\}$. Let $(0, y, z) \to L_{(0, y, z)} = \exp(\mathrm{i}\hat{z}z)$, $\hat{z} \in R$ be the one-dimensional representation of N. Show that the representation U_g^L has the form

$$U^L_{(x, y, z)}u(\xi) = \exp \mathrm{i}[\hat{z}(z+\xi y)]u(\xi+x)]. \tag{1}$$

§ 1.2. Let $G = SO(3)$. Show that any representation of G induced by an irreducible representation of any subgroup K of G is reducible.

§ 1.3. Let G be the Euclidean group $T^n \circledS SO(n)$ and let $K = T^n \circledS SO(n-1)$. Take

$$k \to L_k = L_{(a,r)} = \exp(i\overset{0}{p}a)D^m(r), \tag{2}$$

where $\overset{0}{p} = (M, 0, 0, \ldots, 0)$ and $D^m(r)$ is an irreducible representation of $SO(n-1)$ characterized by highest weight m. Show that

1° The space $X = G/K$ is isomorphic with the sphere $p_\mu p_\mu = M^2$.

2° The action of the induced representation U^L of G in $H = L^2(X, \mu)$ where $d\mu(p)$ is the invariant measure on the sphere is given by the formula

$$(U^L_{(a,R)}u)(p) = \exp(ipa)D^J(r_R)u(R^{-1}p), \tag{3}$$

where $r_R = R_p^{-1}RR_{R^{-1}p}$ and R_p is the rotation defined by the formula

$$p = R_p\overset{0}{p}.$$

Hint: Use the formula 1 (47) and the method of example 1.3.

§ 2.1. Let $k \to L_k$ be an indecomposable representation of a closed subgroup K of a topological group G. Show that the induced representation U^L of G is also indecomposable.

§ 1.4.* Let $G = SO(3)$ and $K = SO(2)$. Take irreducible representation L of K (given by a character) and find the induced representation U^L of G in $H = L^2(X, \mu)$, $\mu = K/G \simeq S^2$. Show that the obtained representation is reducible. Show that every induced representation of G from any subgroup K is reducible.

§ 1.5.* Show that the group $SL(2, R)$ has also a supplementary series of representations given by the formula

$$T_g u(x) = |\beta x+\delta|^{\varrho-1}u\left(\frac{\alpha x+\gamma}{\beta x+\delta}\right), \quad g = \alpha\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix} \tag{4}$$

which is realized in a Hilbert space with $(\cdot, \cdot)$ given by

$$(u, v) = \Gamma^{-1}(-\varrho)\int |x_1-x_2|^{-1-\varrho}u(x_1)\bar{u}(x_2)dx_1 dx_2$$

with $-1 < \varrho < 1$, $\varrho \neq 0$. Find the little group for the representation (4).

Chapter 17

Induced Representations of Semidirect Products

The method of induced representations is most effective in the case of regular semidirect products $G = N \rtimes S$ where the subgroup N is abelian. We show that in this case *every* irreducible unitary representation of G is induced from a nontrivial subgroup of G. Further we give the complete classification of all induced irreducible unitary representations of G. The classification is given in terms of the parameters which characterize an orbit $\hat{O}$ of S in the dual $\hat{N}$ of the group $\overline{N}$, and additional parameters which characterize the representation of the stability subgroup $K_{\hat{O}} \subset S$ of the orbit $\hat{O}$. If G is a physical symmetry group, these parameters have a direct physical meaning.

Once the general theory is established, the construction of all irreducible unitary representations of any specific semidirect product becomes an easy exercise. We present the details of the derivation for the Euclidean and the Poincaré group. In the case of the Poincaré group the parameter which characterizes an orbit $\hat{O}$ has the meaning of mass and the parameter which characterizes the representation of the stability group has the meaning of the spin of the particle. The remaining quantum numbers for a particle, like the components of momentum and the projection of the spin also appear in a natural manner. These examples clearly show the power, elegance and usefulness in physics of the method of induced representations.

§ 1. Representation Theory of Semidirect Products

Let G be a semidirect product $N \rtimes S$ of separable, locally compact groups N and S, and let N be abelian. We recall that the composition law in G is given by the formula (cf. ch. 3, § 4, def. 2)

$$(n_1, s_1)(n_2, s_2) = \big(n_1\, s_1(n_2),\, s_1 s_2\big), \qquad s_1(n_2) \in N. \tag{1}$$

Because N is abelian one can also write $n_1 s_1(n_2) = n_1 + s_1(n_2)$. Let o and e be the identity elements in N and S, respectively. The sets

$$\{(n, e),\ n \in N\} \quad \text{and} \quad \{(o, s), s \in S\} \tag{2}$$

are closed subgroups of G isomorphic, in a natural manner, to the subgroups N and S, respectively. We shall identify N and S with these subgroups. The subgroup S acts as a group of automorphisms of N by the formula

$$sns^{-1} = (o, s)(n, e)(o, s^{-1}) = \big(s(n), e\big).$$

We denote this automorphism by the symbol α_s, i.e.,

$$\alpha_s(n) \equiv s(n). \tag{3}$$

Let T be a unitary representation of G and let U and V be its restriction to N and S, respectively, i.e.

$$U_n \equiv T_{(n, e)}, \quad V_s \equiv T_{(o, s)}.$$

Because $(n, s) = (n, e)\,(o, s)$, for all $n \in N$ and $s \in S$, then

$$T_{(n, s)} = U_n V_s. \tag{4}$$

Thus a representation T of G is completely determined by its restriction U and V to the subgroups N and S, respectively. The representations U of N and V of S cannot be chosen arbitrarily. In fact, the composition law in G

$$(n_1, s_1)(n_2, s_2) = (n_1 + s_1(n_2), s_1 s_2) \tag{5}$$

implies that the representations U and V must satisfy the following operator equation

$$U_{n_1} V_{s_1} U_{n_2} V_{s_2} = U_{n_1} U_{\alpha_{s_1}(n_2)} V_{s_1} V_{s_2} \tag{6}$$

which reduces to the following equation

$$V_s U_n V_s^{-1} = U_{\alpha_s(n)} = U_{s(n)}. \tag{7}$$

We now show that if T is a unitary representation then this equation is equivalent to a system of imprimitivity for T. In fact, let $\hat{N}$ be the dual space (of characters) to N (cf. ch. 6). If $\hat{n}(n) = \langle n, \hat{n}\rangle$ is a character and α is an automorphism of N, then $\hat{n}\,(\alpha(n))$ is also a character, which we denote $\hat{n}\alpha$. Using the relation 6.1(6′)

$$\langle n, \hat{n}_1\rangle\langle n, \hat{n}_2\rangle = \langle n, \hat{n}_1\hat{n}_2\rangle \tag{8}$$

one readily verifies that the map $\hat{n} \to \hat{n}\alpha$ is an automorphism of the character group $\hat{N}$, i.e.,

$$(\hat{n}_1\hat{n}_2)\alpha = (\hat{n}_1\alpha)(\hat{n}_2\alpha), \qquad \hat{n}_i\alpha \in \hat{N}.$$

In particular, the automorphism (3) of N implies the automorphism $\hat{n} \to \hat{n}s$ of $\hat{N}$.

Now, using the SNAG theorem (cf. ch. 6, § 2) and the equality

$$\langle s(n), \hat{n}\rangle = \langle n, \hat{n}s\rangle$$

one obtains

$$U_n = \int_{\hat{N}} \langle n, \hat{n}\rangle \mathrm{d}E(\hat{n})$$

and, by (7),

$$V_s U_n V_{s^{-1}} = U_{s(n)} = \int_{\hat{N}} \langle s(n), \hat{n} \rangle \mathrm{d}E(\hat{n})$$

$$= \int_{\hat{N}} \langle n, \hat{n} \rangle \mathrm{d}E(\hat{n}s^{-1}). \tag{9}$$

On the other hand, the left-hand side of eq. (7) is

$$\int_{\hat{N}} \langle n, \hat{n} \rangle \mathrm{d}(V_s E(\hat{n}) V_{s^{-1}}). \tag{9'}$$

Because the characters separate the points of G, equations (9) and (9′) imply

$$V_s E(Z) V_s^{-1} = E(Zs^{-1}), \tag{10}$$

for any Borel set $Z \subset \hat{N}$. Thus, $E(Z)$ is a system of imprimitivity for a representation V based on the dual space $\hat{N}$. Because,

$$U_n E(Z) U_n^{-1} = E(Z),$$

the projection $E(Z)$ form also a system of imprimitivity for a representation T of G. Consequently, every unitary representation of a semidirect product is imprimitive.

The set of all $\hat{n}s$ for a given $\hat{n} \in N$ and all s in S is called the *orbit of the character* $\hat{n}$, and is denoted by the symbol $\hat{O}_{\hat{n}}$. We assume that topology on the orbit $\hat{O}_{\hat{n}}$ was chosen in such a manner that it is a locally compact space. It is evident that two orbits $\hat{O}_{\hat{n}_1}$ and $\hat{O}_{\hat{n}_2}$ either coincide or are disjoint. Consequently, the dual space $\hat{N}$ decomposes onto nonintersecting sets $\hat{O}_{\hat{n}}$.

Any two points $\hat{n}s_1$ and $\hat{n}s_2$ of $\hat{O}_{\hat{n}}$ may be connected by the transformation $s_1^{-1}s_2 \in S$. Hence an orbit $\hat{O}_{\hat{n}}$ is a homogeneous G-space. By virtue of th. 4.1.1, this space is homeomorphic with the homogeneous space $X = K_{\hat{n}} \backslash S$, where $K_{\hat{n}}$ is the closed subgroup consisting of all elements s in S for which $\hat{n}s = \hat{n}$.

By th. 4.3.1, one knows that there exists a measure $\mathrm{d}\mu(x)$ in the homogeneous space $X = K_{\hat{n}} \backslash S$, quasi-invariant with respect to S. It might occur, however, that there exist quasi-invariant measures in $\hat{N}$ which are not concentrated on any orbit $\hat{O}_{\hat{n}}$ of $\hat{N}$. To see this, consider the following example:

EXAMPLE 1. Let G be the semidirect product consisting of all pairs (z, m), $z \in C^1$, m an integer, in which the composition law is given by the following formula

$$(z_1, m_1)(z_2, m_2) = (z_1 + \exp(\mathrm{i}m_1 \pi \alpha) z_2, m_1 + m_2), \tag{11}$$

where α is an irrational number.

Here, the abelian, invariant subgroup N consists of all elements of the form $u = (z, 0)$ and $S = \{(0, m)\}$. Because N is a non-compact vector group, the dual group $\hat{N}$ is isomorphic to N (cf. example 6.1.1). The group S acts on $\hat{N}$ by the formula

$$\hat{N} \ni \hat{n} \to (\hat{n})s = \exp(\mathrm{i}m\pi\alpha)\hat{n}, \tag{12}$$

where $s = (0, m) \in S$.

Thus every orbit $\hat{O}_{\hat{n}}$ consists of a countable number of points lying on the circle of radius $r = |\hat{n}|$.

For every $r > 0$, let $\mu_r(Z)$ denote a linear Lebesgue's measure of the intersection of a Borel set $Z \subset N$ and the circle $|\hat{n}| = r$. Clearly, every $\mu_r(Z)$ is invariant because the action of S, by eq. (12), corresponds to a rotation. However, since every orbit $\hat{O}_{\hat{n}}$ is a countable set, $\mu_r(\hat{O}_{\hat{n}}) = 0$ for every $\hat{O}_{\hat{n}} \subset \hat{N}$. Thus none of the measures $\mu_r(\cdot)$ is concentrated on an orbit $\hat{O}_{\hat{n}}$. ▼

In order to avoid such pathological cases we impose some regularity conditions on the semidirect product $G = N \rtimes S$. We say that G is a *regular semidirect product* of N and S if $\hat{N}$ contains a countable family $Z_1, Z_2, \ldots$ of Borel subsets, each a union of G orbits, such that every orbit in $\hat{N}$ is the intersection of the members of a subfamily $Z_{n_1}, Z_{n_2}, \ldots$ containing that orbit. Without loss of generality, we can suppose that the intersection of a finite number of Z_i is an element of the family $\{Z_i\}_1^\infty$. This is equivalent to the assumption that any orbit is the limit of a decreasing sequence $\{Z_{n_i}^{\hat{n}}\} \subset \{Z_i\}_1^\infty$ i.e.: $Z_{n_i}^{\hat{n}} \searrow O_{\hat{n}}$.

We shall see that most of the interesting semidirect products occurring in physical applications are regular. One readily verifies however that the semidirect product of example 1 is not regular.

PROPOSITION 1. *Let T be a unitary representation of a regular semidirect product $G = N \rtimes S$, and let $E(\cdot)$ be the projection-valued measure associated with the restriction U of the representation T to N. Then, if T is irreducible there exists an orbit $\hat{O}_{\hat{n}}$ such that $E(\hat{O}_{\hat{n}}) = 1$ and $E(\hat{N} - \hat{O}_{\hat{n}}) = 0$.*

PROOF: Because every set Z_i is a union of orbits then, $Z_i g = Z_i$, for all $g \in G$. The irreducibility of T implies then, by eq. (10), and by Schur's Lemma, that $E(Z_i) = 0$ or 1. Similarly, $E(\hat{O}_{\hat{n}}) = 0$ or 1 for every orbit $\hat{O}_{\hat{n}} \subset \hat{N}$. Suppose that $E(\hat{O}_{\hat{n}}) = 0$ for all orbits $\hat{O}_{\hat{n}}$. Because the orbit is the limit

$$Z_{n_i}^{\hat{n}} \searrow \hat{O}_{\hat{n}}$$

and because the countably additive measure $E(\cdot)$ satisfies

$$E\Big(\bigcap_i Z_{n_i}^{\hat{n}}\Big) = \prod_i E(Z_{n_i}^{\hat{n}}) = \lim_{i\to\infty} E(Z_{n_i}^{\hat{n}})$$

we obtain

$$\lim_{i\to\infty} E(Z_{n_i}^{\hat{n}}) = E(\hat{O}_{\hat{n}}) = 0.$$

Thus for any orbit $O_{\hat{n}}$ there exists an element $Z_{n_i}^{\hat{n}} \supset \hat{O}_{\hat{n}}$ of measure zero. Then because the set of such $Z_{n_i}^{\hat{n}}$ covers $\hat{N}$, we obtain $E(\hat{N}) = 0$ which is a contradiction.

Hence, there exists at least one orbit $\hat{O}_{\hat{n}}$ such that $E(\hat{O}_{\hat{n}}) = 1$. If there were two orbits $\hat{O}_{\hat{n}_1}$ and $\hat{O}_{\hat{n}_2}$ satisfying the condition $E(\hat{O}_{\hat{n}_i}) = I$, $i = 1, 2$, then

$$E(\hat{O}_{\hat{n}_1} \cup \hat{O}_{\hat{n}_2}) = E(\hat{O}_{\hat{n}_1}) + E(\hat{O}_{\hat{n}_2}) = 2I.$$

But $E(\hat{N}) = I$. Hence the spectral measure $E(\cdot)$ is concentrated only on one orbit $\hat{O}_{\hat{n}}$. ▼

Let $\hat{O}$ denote an orbit, which is the support of the spectral measure $E(\cdot)$. We showed that the orbit $\hat{O}$ is homeomorphic with a transitive space $S_{\hat{O}} \backslash S$, where $S_{\hat{O}}$ is the stability subgroup of a point $\hat{n}_0$ of the orbit $\hat{O}$. Because for every $n \in N$ and $\hat{n} \in \hat{O}$, $\hat{n}n = \hat{n}$, the orbit $\hat{O}$ can also be considered as homeomorphic to a transitive manifold $N \otimes S_{\hat{O}} \backslash G$. Then, eq. (10) implies

$$T_{(n,s)} E(Z) T^{-1}_{(n,s)} = E\big(Z(n, s)^{-1}\big), \tag{13}$$

where Z is a Borel subset of $\hat{N}$.

Thus, by the Imprimitivity Theorem (ch. 16, § 3), every irreducible representation T of a regular, semidirect product is unitarily equivalent to a representation $U^{\hat{L}}$ of G induced by a representation $\hat{L}$ of the subgroup $N \otimes S_{\hat{O}}$. By the corollary to th. 16.2.1, the representation $\hat{L}$ is irreducible. The representation $U^{\hat{L}}$ is realized in a Hilbert space $H^{\hat{L}} = L^2(\hat{O}, \mu\colon H)$, where μ is a quasi-invariant measure in $\hat{O}$ and H is the carrier space of the representation $\hat{L}$ of the subgroup $N \otimes S_{\hat{O}}$.

We now prove the important property of the representation $\hat{L}$ of the stability group $N \otimes S_{\hat{O}}$ of the orbit $\hat{O}$:

LEMMA 2. *The restriction $\hat{L}_n$ of an irreducible inducing representation $\hat{L}$ of $N \otimes S_{\hat{O}}$ to the subgroup N is a one-dimensional representation, i.e.,*

$$\hat{L}_n u = \langle n, \hat{n}_0 \rangle u, \tag{14}$$

where $u \in H$ and $\hat{n}_0$ is that element of $\hat{O}$ which has the stability subgroup $N \otimes S_{\hat{O}}$.

PROOF: As we have just proved, the irreducible representation U of $N \otimes S$ is unitarily equivalent to an induced representation $U^{\hat{L}}$. By virtue of eq. 16.1(15) the action of $U^{\hat{L}}_g|_N$ is given in $H^{\hat{L}} = L^2(\hat{O}, \mu; H)$ by the formula

$$U^{\hat{L}}_n u(\hat{n}) = \hat{L}_{sns^{-1}} u(\hat{n}), \tag{15}$$

where $\hat{n}_0 s = \hat{n}$ and $\hat{n}_0$ is the element of $\hat{O}$ having the stability subgroup $N \otimes S_{\hat{O}}$. On the other hand, by virtue of SNAG's Theorem for $u, v \in H^{\hat{L}}$ we have

$$(U^{\hat{L}}_n u, v) = \int_{\hat{N}} \langle n, \hat{n} \rangle \mathrm{d}\mu_{u,v}(\hat{n}) = \int_{\hat{O}} \langle n, \hat{n} \rangle \big(u(\hat{n}), v(\hat{n})\big)_H \mathrm{d}\mu(\hat{n}). \tag{16}$$

By proposition 16.1.3 we can set $u(\hat{n}) = \alpha(\hat{n})u$, $v(\hat{n}) = \beta(\hat{n})v$, $\alpha, \beta \in L^2(\hat{O}; \mu)$, $u, v \in H$. Then eqs. (15) and (16) imply

$$\int_{\hat{O}} \langle n, \hat{n} \rangle \alpha(\hat{n}) \bar{\beta}(\hat{n}) (u, v)_H \mathrm{d}\mu(\hat{n}) = \int_{\hat{O}} \alpha(\hat{n}) \bar{\beta}(\hat{n}) (\hat{L}_{sns^{-1}} u, v)_H \mathrm{d}\mu(\hat{n}). \tag{17}$$

Consequently

$$\hat{L}_{sns^{-1}}u = \langle n, \hat{n}\rangle u \tag{18}$$

for all $n \in N$, $u \in H$ and almost all $\hat{n} \in \hat{O}$. If $\hat{n} = \hat{n}_0 s$ is such that the last formula is true and recalling that $\langle n, \hat{n}_0 s\rangle = \langle sns^{-1}, \hat{n}_0\rangle$ we obtain

$$\hat{L}_n u = \langle n, \hat{n}_0\rangle u. \blacktriangledown$$

The next lemma provides a convenient characterization of irreducible representations $\hat{L}$ of the stability group $N \rtimes S_{\hat{O}}$ in terms of irreducible representations L of $S_{\hat{O}}$.

LEMMA 3. *Every irreducible unitary representation L of $N \rtimes S_{\hat{O}}$ is determined and determines an irreducible unitary representation L of $S_{\hat{O}}$.*

PROOF: Let L be a unitary representation of $S_{\hat{O}}$. For

$$g_{\hat{O}} = (n, s_{\hat{O}}), \quad g_{\hat{O}} \in N \rtimes S_{\hat{O}}, \quad n \in N, \; s_{\hat{O}} \in S_{\hat{O}}$$

set

$$\hat{L}_{(n, s_{\hat{O}})} = \langle n, \hat{n}_0\rangle L_{s_{\hat{O}}}, \tag{19}$$

where

$$\hat{n}_0 s_{\hat{O}} = \hat{n}_0 \quad \text{for all } s_{\hat{O}} \in S_{\hat{O}}.$$

The map $(n, s_{\hat{O}}) \to \hat{L}_{(n, s_{\hat{O}})}$ defines a unitary representation of $N \rtimes S_{\hat{O}}$. Indeed,

$$\begin{aligned}\hat{L}_{(n,s_{\hat{O}})(n',s'_{\hat{O}})} = \hat{L}_{(n(s_{\hat{O}}n'),\, s_{\hat{O}}s'_{\hat{O}})} &= \langle n(s_{\hat{O}}n'), \hat{n}_0\rangle L_{s_{\hat{O}}s'_{\hat{O}}} = \langle n, \hat{n}_0\rangle \langle s_{\hat{O}}n', \hat{n}_0\rangle L_{s_{\hat{O}}} L_{s'_{\hat{O}}} \\ &= \hat{L}_{(n,s_{\hat{O}})} \hat{L}_{(n',s'_{\hat{O}})}\end{aligned}$$

and

$$\hat{L}^*_{(n, s_{\hat{O}})} = \overline{\langle n, \hat{n}_0\rangle} L^*_{s_{\hat{O}}} = \langle n^{-1}, \hat{n}_0\rangle L^{-1}_{s_{\hat{O}}} = \hat{L}^{-1}_{(n, s_{\hat{O}})}.$$

If L is irreducible, then $\hat{L}$ is also irreducible. Conversely, by eqs. (4) and (19), every irreducible representation $\hat{L}$ of $N \rtimes S_{\hat{O}}$ defines an irreducible unitary representation of $S_{\hat{O}}$. $\blacktriangledown$

In the following we shall denote the representation $\hat{L}$ of $N \rtimes S_{\hat{O}}$ described in lemma 3 by the symbol $\hat{n}L$ where $\hat{n} \in \hat{O}$ and L is a representation of $S_{\hat{O}}$. The following theorem gives a convenient characterization of irreducible, unitary representations of semidirect products.

THEOREM 4. *Let G be a regular, semidirect product $N \rtimes S$ of separable, locally compact groups N and S, and let N be abelian. Let T be an irreducible unitary representation of G. Then*

1° *One can associate with T an orbit $\hat{O}$ in $\hat{N}$.*

2° *The representation T is unitarily equivalent to an induced representation $U^{\hat{n}L}$,* *where L is an irreducible unitary representation of $S_{\hat{O}}$ in a Hilbert space H. The representation $U^{\hat{n}L}$ is realized in the Hilbert space $H^{\hat{n}L} = L^2(\hat{O}, \mu; H)$.*

PROOF: By proposition 1 the spectral measure $E(\cdot)$ associated with the restriction of T to N is concentrated on an orbit $\hat{O}$. Further, by eq. (13) and the Imprimitivity Theorem 16.3.1, T is unitarily equivalent to a representation $U^{\hat{L}}$ induced by a representation $\hat{L}$ of the stability subgroup $N \rtimes S_{\hat{O}}$. Finally, by lemmas 2 and 3, $\hat{L}$ is of the form $\hat{n}L$, where L is an irreducible, unitary representation of the subgroup $S_{\hat{O}}$. ▼

The next theorem states that, conversely, a representation $U^{\hat{L}}$ induced by an irreducible unitary representation $\hat{L}$ of the stability subgroup $N \rtimes S_{\hat{O}}$ of an orbit $\hat{O}$ is irreducible. In fact, we have

THEOREM 5. *Let G be as in th. 4. Then:*

1° *With each orbit $\hat{O}$ in $\hat{N}$ and with each irreducible unitary representation $\hat{n}L$ of the stability subgroup $N \rtimes S_{\hat{O}}$ one can associate the induced representation $U^{\hat{n}L}$ which is irreducible.*

2° *The spectral measure $E(\cdot)$, which is defined by the restriction of $U^{\hat{n}L}$ to N, is concentrated on the orbit $\hat{O}$.*

3° *The representation $U^{\hat{n}L}$ is realized in the Hilbert space $H^{\hat{n}L} = L^2(\hat{O}, \mu; H)$, where H is the carrier space of the representation L and μ is a quasi-invariant measure in $\hat{O}$. We have*

$$U^{\hat{n}L}_{(n,s)} u(\hat{n}) = \langle n, \hat{n} \rangle_S U^L_s u(\hat{n}), \tag{20}$$

where $_SU^L$ is a representation of S given by eq. 16.1(15) *which is induced by the representation L of the stability subgroup $S_{\hat{O}} \subset S$.*

PROOF: Let $\hat{O}$ be an orbit in $\hat{N}$ and let $S_{\hat{O}} \subset S$ be the stability subgroup of a point $\hat{n}_0 \in \hat{O}$. Because N acts in $\hat{N}$ as the identity, we have:

$$\hat{O} \cong S_{\hat{O}} \backslash S \cong N \rtimes S_{\hat{O}} \backslash N \rtimes S = N \rtimes S_{\hat{O}} \backslash G. \tag{21}$$

Moreover a measure μ on $\hat{O}$, quasi-invariant relative to S, remains quasi-invariant for G.

We now find an explicit form of the induced representation $U^{\hat{n}L}$. Let ϱ denote the Radon–Nikodym derivative of a measure μ on $\hat{O}$ and B_g an operator function on S satisfying conditions 16.1(12) relative to $S_{\hat{O}}$. Let L be an irreducible unitary representation of the subgroup $S_{\hat{O}}$ and let H be the carrier space of L. Then, the representation $_SU^L$ of the group S induced by the representation L has the form

$$_SU^L_s u(\hat{n}) = \varrho_s(\gamma) B_\gamma^{-1} B_{\gamma s} u(\hat{n}s),$$

where $u \in L^2(\hat{O}, \mu, H)$ and $\gamma \in S$ is such that $\hat{n} = n_0\gamma$.

Let us now find functions $\tilde{\varrho}$, $\tilde{B}$, which allow to form the representation $_GU^{\hat{n}L}$ induced by the representation $\hat{n}L$ of the subgroup $N \rtimes S_{\hat{O}}$. Because N acts as the identity in $\hat{N}$, we have $(g = (n, s))$

$$\tilde{\varrho}_g(\gamma) = \frac{\mathrm{d}\mu(\hat{n}g)}{\mathrm{d}\mu(\hat{n})} = \frac{\mathrm{d}\mu(\hat{n}s)}{\mathrm{d}\mu(\hat{n})} = \varrho_s(\gamma), \quad \hat{n} = \hat{n}_0\gamma.$$

Further, setting

$$\hat{B}_g = \langle n, \hat{n}_0 \rangle B_s \tag{22}$$

we readily verify that both conditions of eq. 16.1(12) relative to the subgroup $N \circledS S_{\hat{O}}$ are satisfied. Hence, by eqs. 16.1(15) and (22) we obtain

$$\begin{aligned} {}_G U_g^L u(\hat{n}) &= \tilde{\varrho}_g^{1/2}(\gamma)\tilde{B}_\gamma^{-1}\tilde{B}_{\gamma g}u(\hat{n}g) \\ &= \varrho_s^{1/2}(\gamma)\tilde{B}_\gamma^{-1}\tilde{B}_{\gamma n \gamma^{-1} s}u(\hat{n}(n, s)) \\ &= \varrho_s^{1/2}(\gamma)B_\gamma^{-1}\langle \gamma n \gamma^{-1}, \hat{n}_0\rangle B_{\gamma s}u(\hat{n}s) \\ &= \langle n, \hat{n}\rangle {}_S U_s^L u(\hat{n}). \end{aligned}$$

The restriction of $U^{\hat{n}L}$ to the subgroup N by virtue of eq. (18) gives

$${}_G U_n^{\hat{n}L} u(\hat{n}) = \langle n, \hat{n}\rangle u(\hat{n}).$$

Thus,

$$\begin{aligned} ({}_G U_n^{\hat{n}L} u, v) &= \int_{\hat{O}} \langle n, \hat{n}\rangle \big(u(\hat{n}), v(\hat{n})\big)\mathrm{d}\mu(\hat{n}) \\ &= \int_{\hat{O}} \langle n, \hat{n}\rangle \mathrm{d}\mu_{u,v}(\hat{n}). \end{aligned}$$

Comparing this with the expression resulting from SNAG's theorem:

$$({}_G U_n^{\hat{n}L} u, v) = \int_{\hat{N}} \langle n, \hat{n}\rangle \mathrm{d}\mu_{u,v}(\hat{n}),$$

one sees that the spectral measure $E(\cdot)$ associated with the unitary operators $n \to U_n^{\hat{n}L}$ is concentrated on the orbit $\hat{O}$.

If an operator A commutes with all operators ${}_G U_g^{\hat{n}L}$ (i.e., $A \in R(U^{\hat{n}L}, U^{\hat{n}L})$) then it also commutes with all ${}_G U_n^{\hat{n}L}$, $n \in N$, and consequently with the spectral measure $E(\cdot)$. Thus, by th. 16.3.3, we conclude that the dimension of algebra of operators commuting with all ${}_G U_g^{\hat{n}L}$ is equal to $\dim R(\hat{n}L, \hat{n}L)$. If L is irreducible, then $\hat{n}L$ is also irreducible by proposition 3; consequently $\dim R(\hat{n}L, \hat{n}L) = 1$. Hence, ${}_G U^{\hat{n}L}$ is irreducible. ▼

According to ths. 4 and 5, the classification of all irreducible unitary representations of a regular, semidirect product $N \circledS S$ can be performed along the following steps:

1° Determine the set $\hat{N}$ of all characters of N.

2° Classify all orbits $\hat{O}$ in $\hat{N}$ under the subgroup S.

3° Select an element $\hat{n}_0$ in a given orbit $\hat{O}$ and determine the stability subgroup $S_{\hat{O}} \subset S$.

4° Take an irreducible representation L of $S_{\hat{O}}$ and form the induced irreducible unitary representation ${}_SU^L$ and finally form ${}_GU^{\hat{n}L}$ by formula (20).

EXAMPLE 2. Let G be the semidirect product $N \circledS S$ of the translation group N in the Euclidean space R^3 and the rotation group $S = \mathrm{SO}(3)$. Because N is a noncompact vector group, the dual space $\hat{N}$ is isomorphic to N, i.e., $\hat{N} \simeq R^3$. If $n = (n_1, n_2, n_3,) \in N$ and $\hat{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3) \in \hat{N}$, then an arbitrary unitary character has the form

$$\langle n, \hat{n} \rangle = \exp i(n_1\hat{n}_1 + n_2\hat{n}_2 + n_3\hat{n}_3).$$

The set of all $\hat{n}_0 s$ for a given $\hat{n}_0 \in \hat{N}$ and all $s \in \mathrm{SO}(3)$ forms an orbit $\hat{O}$ associated with a character $\hat{n}_0$. It is evident that in the present case the orbits are spheres with the center at $(0, 0, 0) \in R^3$ and the radius $r \geqslant 0$. We now verify that the Euclidean group $R^3 \circledS \mathrm{SO}(3)$ is a regular semidirect product. In fact, let Z be a countable family of Borel subsets of the dual space $\hat{N}$ consisting of the following sets

(i) Z_{00} = the orbit $r = 0$,

(ii) $Z_{r_1 r_2}$ = the union of all orbits with $r > 0$ such that $r_1 < r < r_2$, where $r_1, r_2, r_1 < r_2$, are any two positive rational numbers.

We see that $Z_{r_1 r_2} g = Z_{r_1 r_2}$ for all $Z_{r_1 r_2} \in Z$ and all g in $R^3 \circledS \mathrm{SO}(3)$. Moreover, each orbit is the intersection of the members of a subfamily of Z which contain the orbit. Thus, $R^3 \circledS \mathrm{SO}(3)$ is a regular semidirect product. Consequently, according to ths. 4 and 5, every irreducible, unitary representation of this group is a representation induced by an irreducible unitary representation of the stability subgroup associated with orbits $r = 0$ or $r > 0$. We shall consider the cases $r > 0$ and $r = 0$ separately.

1° $r > 0$. Take $\hat{n}_0 = (0, 0, r)$. The stability subgroup $S_{\hat{O}} \subset S$ of the point $\hat{n}_0$ is isomorphic to SO(2). The irreducible representations L of $S_{\hat{O}}$ have the form

$$\varphi \to \exp(il\varphi), \qquad \varphi \in [0, 2\pi], \qquad l = 0, \pm 1, \pm 2, \ldots$$

The carrier space H of L is C^1. The measure μ on the orbit $\hat{O} = S_{\hat{O}} \backslash S = \mathrm{SO}(2) \backslash \mathrm{SO}(3) \cong S^2$ is the ordinary invariant measure relative to rotations on the sphere S^2. Hence, the Hilbert space $H^{\hat{n}L}(\hat{O}, \mu; H)$ consists now of all complex functions on the sphere, square integrable relative to the measure $\mu(\cdot)$. Every irreducible representation L gives rise to an irreducible representation $U^{\hat{n}L}$ of G. According to eq. (20), the action of $U_g^{\hat{n}L}$ in the space $H^{\hat{n}L}(\hat{O}, \mu, H)$ is

$$U^{\hat{n}L}_{(n,s)} u(\hat{n}) = \exp[i(n_1\hat{n}_1 + n_2\hat{n}_2 + n_3\hat{n}_3)]_S U_s^L u(\hat{n}), \tag{23}$$
$$n \in N, \; s \in S, \; \hat{n} \in S^2.$$

Here, ${}_SU^L$ is the representation of SO(3) induced by the representation L of SO(2). They are constructed in exercise 16.4.1.4.

2° $r = 0$. Taking $\hat{n}_0 = (0, 0, 0)$ we see that the stability subgroup $S_{\hat{O}} = S = \mathrm{SO}(3)$. The irreducible induced representations of G associated with this orbit are all finite-dimensional, irreducible, unitary representations of SO(3) 'lifted' to the group G. By virtue of ths. 4 and 5, these are all irreducible unitary representations of $R^3 \rtimes SO(3)$. ▼

Th. 5 gives the method of construction of irreducible representations $U^{\hat{n}L}$ of $N \rtimes S$ using the special properties of the semidirect products. One may also construct the unitary representation $U^{\hat{n}L}$ induced by the representation $\hat{n}L$ of the subgroup $N \rtimes S_{\hat{O}}$ directly, using the general method described in ch. 16, § 1. We recall that in that method the representation $U^{\hat{n}L}$ was constructed in the Hilbert space of functions $u(g)$ on G satisfying the condition

$$u(kg) = \hat{L}_k u(g), \quad k \in K \equiv N \rtimes S_{\hat{O}}.$$

In our case this condition has the form

$$u\big((n, s_{\hat{O}})g\big) = (\hat{n}L)_{(n, s_{\hat{O}})} u(g) = \langle n, \hat{n}_0 \rangle L_{s_{\hat{O}}} u(g). \tag{24}$$

The set of functions $u(g)$ on G satisfying eq. (24) can easily be found. In fact, let $u(s)$ be the functions on S satisfying, for all $s_{\hat{O}} \in S_{\hat{O}}$, the condition $u(s_{\hat{O}} s) = L_{s_{\hat{O}}} u(s)$, i.e., $u(s) \in H^L$ which is the carrier space of the representation ${}_S U^L$. Then setting

$$u(g) = \langle n, \hat{n}_0 \rangle u(s), \quad g = (n, s), \quad n \in N, \ s \in S \tag{25}$$

and using eqs. (1), (25), (8) and (24), one obtains

$$\begin{aligned} u\big((n', s_{\hat{O}})g\big) &= u\big((n' s_{\hat{O}} n, s_{\hat{O}} s)\big) = \langle n' s_{\hat{O}} n, \hat{n}_0 \rangle u(s_{\hat{O}} s) \\ &= \langle n', \hat{n}_0 \rangle \langle s_{\hat{O}} n, \hat{n}_0 \rangle L_{s_{\hat{O}}} u(s) = \langle n', \hat{n}_0 \rangle L_{s_{\hat{O}}} u(g) \\ &= \hat{n}L_{(n', s_{\hat{O}})} u(g). \end{aligned}$$

Using eq. (25) one verifies that the representation $U^{\hat{n}L}$ of G induced by the representation $\hat{n}L$ of $N \rtimes S_{\hat{O}}$ has the form

$$U^{\hat{n}L}_{g'} u(g) = \varrho^{1/2}_{g'}(g) u\big(g(n's')\big) = \varrho^{1/2}_{g'}(g) \langle n, \hat{n}_0 \rangle \langle n', \hat{n}_0 s \rangle u(ss'). \tag{26}$$

Furthermore, if we use the equality

$$\varrho_{g'}(g) = \frac{\mathrm{d}\mu(\hat{n}g')}{\mathrm{d}\mu(\hat{n})} = \frac{\mathrm{d}\mu(\hat{n}s')}{\mathrm{d}\mu(\hat{n})} = \varrho_{s'}(s)$$

and cancel on both sides of eq. (26) the factor $\langle n, \hat{n}_0 \rangle$ we obtain

$$U^{\hat{n}L}_{(n', s')} u(s) = \varrho^{1/2}_{s'}(s) \langle n', \hat{n}_0 s \rangle u(ss') = \langle n', \hat{n} \rangle {}_S U^L_s u(s). \tag{27}$$

Thus we see that the representation $U^{\hat{n}L}$ of G is realized in the carrier space of functions over the group S of the representations ${}_S U^L$, induced by the representation L of $S_{\hat{O}}$. This method of construction of the induced representations $U^{\hat{n}L}$ of $N \rtimes S$ is often convenient in applications. The representation ${}_S U^L_{s'}$ in (27) is an application of eq. 16.1(6). If we realize it on the space $H^L(\hat{O}, \mu; H)$ using eq. 16.1(15), we obtain precisely eq. (20).

§ 2. Induced Unitary Representations of the Poincaré Group

A. *The Lorentz and the Poincaré Groups*

In this section we give a complete classification of irreducible unitary representations of the Poincaré group, using the general formalism of induced representations developed in sec. 1.

We start with a discussion of some of the properties of the Lorentz and the Poincaré groups. Let $g^{\mu\nu} = g_{\mu\nu}$ be the diagonal metric tensor in the four-dimensional Minkowski space M, with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. For

$$x = \{x^0, x^1, x^2, x^3\} = \{x^\mu\}, \quad \text{and} \quad y = \{y^\mu\}$$

let

$$x \cdot y = x^0 y^0 - \boldsymbol{x} \cdot \boldsymbol{y} = x^\mu y_\mu, \quad y_\mu = g_{\mu\nu} y^\nu \tag{1}$$

be the (indefinite) scalar product in M.

The Lorentz group is the set of all linear transformations L of M into M, which preserve the scalar product (1), i.e.,

$$(Lx) \cdot (Ly) = xy. \tag{2}$$

From eq. (2) one obtains

$$L^\alpha{}_\mu L_{\alpha\nu} = g_{\mu\nu}, \text{ or } L^T g L = g, \tag{3}$$

where

$$L_{\alpha\nu} = g_{\alpha\beta} L^\beta{}_\nu = (gL)_{\alpha\nu} \quad \text{and} \quad (L^T)_\mu{}^\alpha = L^\alpha{}_\mu. \tag{4}$$

Clearly from eq. (3) we have

$$L^T = gL^{-1}g \quad \text{or} \quad L^T{}_\mu{}^\alpha = g_{\mu\tau}(L^{-1})^\tau{}_\varrho g^{\varrho\alpha} = L^{-1}{}_\mu{}^\alpha. \tag{5}$$

Equation (3) also implies $\det L = \pm 1$. Furthermore, from eq. (3) for $\mu = 0$, $\nu = 0$ we have

$$(L^0{}_0)^2 - \sum_{k=1}^{3} (L^k{}_0)^2 = 1.$$

Thus, $|L^0{}_0| \geqslant 1$; consequently $\det L$ and $\operatorname{sign} L^0{}_0$ are both continuous functions of the variables $L^\mu{}_\nu$, and therefore they must be constant on every component of the Lorentz group. Thus every Lorentz transformation falls into one of the four pieces:

$$\begin{aligned}
&\text{I. } L_+^\uparrow\text{: } \det L = +1,\ \operatorname{sign} L^0{}_0 = +1,\\
&\text{II. } L_-^\uparrow\text{: } \det L = -1,\ \operatorname{sign} L^0{}_0 = +1,\\
&\text{III. } L_+^\downarrow\text{: } \det L = +1,\ \operatorname{sign} L^0{}_0 = -1,\\
&\text{IV. } L_-^\downarrow\text{: } \det L = -1,\ \operatorname{sign} L^0{}_0 = -1.
\end{aligned} \tag{6}$$

The transformations $L \in L_+^\uparrow$ form a subgroup, which is called the *proper orthochronous Lorentz* group. It is the connected component of the identity (i.e., it consists of all Lorentz transformations, which can be reached from the identity in a continuous manner). The remaining components do not form subgroups of the Lorentz group.

We now establish a connection between $L_+^\uparrow$ and the group SL(2, C) of all 2×2 complex matrices of determinant one. Let $\sigma = \{\sigma^\mu\}$ be the set of four hermitian matrices of the form:

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{7}$$

With each four-vector $x = (x^\mu) \in M$ one associates a 2×2-hermitian matrix X by the formula

$$X = \sum_{\mu=0}^{3} x^\mu \sigma_\mu = \begin{bmatrix} x^0+x^3 & x^1-\mathrm{i}x^2 \\ x_1+\mathrm{i}x^2 & x^0-x^3 \end{bmatrix}. \tag{8}$$

The map $x \to X$ is linear and one-to-one. In fact, using the formula ($\tilde{\sigma}^\mu \equiv \sigma_\mu$)

$$\mathrm{Tr}(\tilde{\sigma}^\mu \sigma_\nu) = 2g^\mu{}_\nu$$

one can associate to every hermitian matrix X a real four-vector

$$x^\mu = \tfrac{1}{2}\mathrm{Tr}\,(X\sigma^\mu)$$

such that $X = x^\mu \sigma_\mu$.

Using eq. (8) we obtain

$$\det X = x^\mu x_\mu, \quad \tfrac{1}{2}[\det(X+Y) - \det X - \det Y] = x^\mu y_\mu. \tag{9}$$

Set now

$$\hat{X} = \Lambda X \Lambda^*, \qquad \Lambda \in SL(2, C). \tag{10}$$

The matrix $\hat{X}$ is hermitian. Therefore the corresponding vector $\hat{x}$ belongs to M. Consequently, eq. (10) defines a real linear map $\Lambda \to L_\Lambda$ of M into itself. By eq. (10), we have $\Lambda_1\Lambda_2 \to L_{\Lambda_1\Lambda_2} = L_{\Lambda_1}L_{\Lambda_2}$. Moreover, because $\det\Lambda = 1$, we obtain by eqs. (8) and (10)

$$\hat{x}_\mu \hat{x}_\mu = \det\hat{X} = \det X = x^\mu x_\mu.$$

Hence, by eq. (9), the transformations L_Λ conserve the scalar product (1) and, consequently, represent Lorentz transformations in M. By virtue of eq. (3), $\det L_\Lambda$ is either $+1$ or -1. If there would exist elements L_Λ with determinant $+1$ as well as -1, then, the set of all elements L_Λ would be disconnected. This is, however, impossible because SL(2, C) is connected and the map $\Lambda \to L_\Lambda$ is continuous. Consequently, the map λ: $\Lambda \to L_\Lambda$ is a homomorphism of SL(2C) into $L_+^\uparrow$.

We now show that the homomorphism λ is 'two-to-one'. To see this, we find the kernel Z of the homomorphism λ. This is the set of all Λ in SL(2, C), which for any hermitian matrix X satisfy the equality

$$X = \Lambda X \Lambda^*. \tag{11}$$

Taking, in particular, $X = e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ we see that the elements $\Lambda \in Z$ satisfy the condition $\Lambda\Lambda^* = e$, i.e., $\Lambda = \Lambda^{-1}$. Thus, eq. (11) reduces to

$$X\Lambda - \Lambda X = 0$$

which must be satisfied for any hermitian X. This implies $\Lambda = \lambda I$. By virtue of the condition $\det\Lambda = 1$, we obtain $\Lambda = \pm I$. Consequently, $L_{\Lambda_1} = L_{\Lambda_2}$ if and only if $\Lambda_1 = \pm\Lambda_2$.

The group SL(2, C) is simply connected (th. 3.7.1). Hence, it is the universal covering group of the proper Lorentz group $L_+^\uparrow$. Denoting the invariant subgroup of SL(2, C) consisting of elements I and $-I$ by Z, we have

$$L_+^\uparrow = \mathrm{SL}(2, C)/Z. \tag{12}$$

There exists two automorphisms of SL(2, C), which are important in applications:

$$\Lambda \to (\Lambda^T)^{-1} \quad \text{and} \quad \Lambda = \bar{\Lambda}. \tag{13}$$

It is interesting that the first automorphism has an explicit realization. In fact, if $\Lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ then the matrix $\sigma_2\Lambda(\sigma_2)^{-1} = \begin{bmatrix} \delta & -\gamma \\ -\beta & \alpha \end{bmatrix}$, is inverse to $\Lambda^T = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$. Hence,

$$(\Lambda^T)^{-1} = \sigma_2\Lambda(\sigma_2)^{-1}. \tag{14}$$

The Poincaré Group

Notice that the Lorentz group leaves the interval $(x-y)^2$ in the Minkowski space invariant. On the other hand, the translations $x^\mu \to x^\mu + a^\mu$ where a^μ is a constant four-vector also leave the length $(x-y)^2$ invariant. This leads to the definition of the Poincaré group Π as the group of all real transformations in the Minkowski space M,

$$x^\mu \to L^\mu{}_\nu x^\nu + a^\mu \tag{15}$$

leaving the length $(x-y)^2$ invariant.

Definition (15) gives the following composition law for the elements of the Poincaré group

$$\{n_1, L_1\}\{n_2, L_2\} = \{n_1 + L_1 n_2, L_1 L_2\}. \tag{16}$$

Thus Π is the semidirect product $N \circledS L$ of the translation group N and the Lorentz group L. Similarly, as for the Lorentz group, Π has four pieces distinguished by det L and sign $L^0{}_0$, namely $\Pi_+^\uparrow, \Pi_-^\uparrow, \Pi_+^\downarrow, \Pi_-^\downarrow$.

In the following we shall consider the inhomogeneous group $\tilde{\Pi}$ corresponding to SL(2, C) group. It is the semidirect product $N \circledS \mathrm{SL}(2, C)$ defined by the following composition law

$$\{n_1, \Lambda_1\}\{n_2, \Lambda_2\} = \{n_1 + L_{\Lambda_1} n_2, \Lambda_1 \Lambda_2\}. \tag{17}$$

We recall that in a semidirect product $N \circledS S$, the topology is defined by the topology of the Cartesian product $N \times S$ of the group spaces N and S (cf. ch. 3, § 4). Thus, because N and SL(2, C) are simply connected, the semidirect product $\tilde{\Pi} = N \circledS \mathrm{SL}(2, C)$ is also simply connected. By virtue of the connection between SL(2, C) and the proper orthochronous Lorentz group $L_+^\uparrow$ we see that $\tilde{\Pi}$ is the universal covering group of the group $\Pi_+^\uparrow$. Moreover, because N and SL(2,C) act in N by unimodular transformations, the group $\tilde{\Pi}$ is also unimodular by th. 3.10.5. In addition, by virtue of eq. 3.10(16) the product of the invariant measures in N and SL(2, C) provides an invariant measure $\mu(\cdot)$ on $\tilde{\Pi}$, i.e.,

$$\begin{aligned} \mathrm{d}\mu(\{n, \Lambda\}) &= \mathrm{d}\sigma(n)\mathrm{d}\nu(\Lambda) \\ &= \mathrm{d}^4 n \frac{\mathrm{d}\beta \mathrm{d}\gamma \mathrm{d}\delta \mathrm{d}\bar{\beta} \mathrm{d}\bar{\gamma} \mathrm{d}\bar{\delta}}{|\delta|^2}, \qquad n \in N,\ \Lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL(2, C) \end{aligned} \tag{18}$$

(cf. eq. 2.3 (9)).

B. *Classification of Orbits*

The translation subgroup $N = \{(n, I)\}$ is a non-compact vector group. Hence, the corresponding dual group $\hat{N}$ can be identified with N. To every $\hat{n} = (\hat{n}_0, \hat{n}_1, \hat{n}_2, \hat{n}_3) \in \hat{N}$ there correspond a character given by the formula:

$$\langle n, \hat{n} \rangle = \exp[\mathrm{i}(n_0 \hat{n}_0 - n_1 \hat{n}_1 - n_2 \hat{n}_2 - n_3 \hat{n}_3)] = \exp(\mathrm{i} n^\mu \hat{n}_\mu). \tag{19}$$

The action of SL(2, C) in $\hat{N}$ by eqs. (17) and (5) follows from the equality

$$\langle L_\Lambda n, \hat{n} \rangle = \exp(\mathrm{i} L_\Lambda{}^\mu{}_\nu n^\nu \hat{n}_\mu) = \exp[\mathrm{i} n^\nu (L_\Lambda^T)_\nu{}^\mu \hat{n}_\mu] = \langle n, L_\Lambda^{-1} \hat{n} \rangle, \tag{20}$$

i.e.,

$$\hat{n} \to L_\Lambda^{-1} \hat{n}, \tag{21}$$

where $\Lambda \in \mathrm{SL}(2, C)$ and $L_\Lambda^{-1} \in L_+^\uparrow$. Thus, the group SL(2, C) acts in the dual space $\hat{N}$ in the same manner as in N. Consequently, the set of all $L_\Lambda \hat{n}_0$ for a given $\hat{n}_0 \in \hat{N}$ and all $L_\Lambda \in L_+^\uparrow$ forms an orbit $\hat{O}$ associated with the character $\hat{n}_0$. This implies that every orbit is contained in one of the hyperboloides

$$\hat{n}_0^2 - \hat{n}_1^2 - \hat{n}_2^2 - \hat{n}_3^2 = m^2, \tag{22}$$

where m^2 is any real number.

If $m^2 > 0$, then eq. (22) describes a two-sheeted hyperboloid (see Fig. 1a). The upper sheet $\hat{O}_m^+$ and the lower sheet $\hat{O}_m^-$ represent, separately, orbits relative to $L_+^\uparrow$. If $m^2 < 0$, eq. (22) defines a one-sheeted hyperboloid in $\hat{N}$ (see Fig. 1b).

Finally, if $m^2 = 0$, eq. (22) describes a cone, which consists of three orbits: $\hat{O}_0^+$—the upper cone, $\hat{O}_0^0$ consisting of the point $(0, 0, 0, 0)$ only, and $\hat{O}_0^-$—the lower cone (see Fig. 1c).

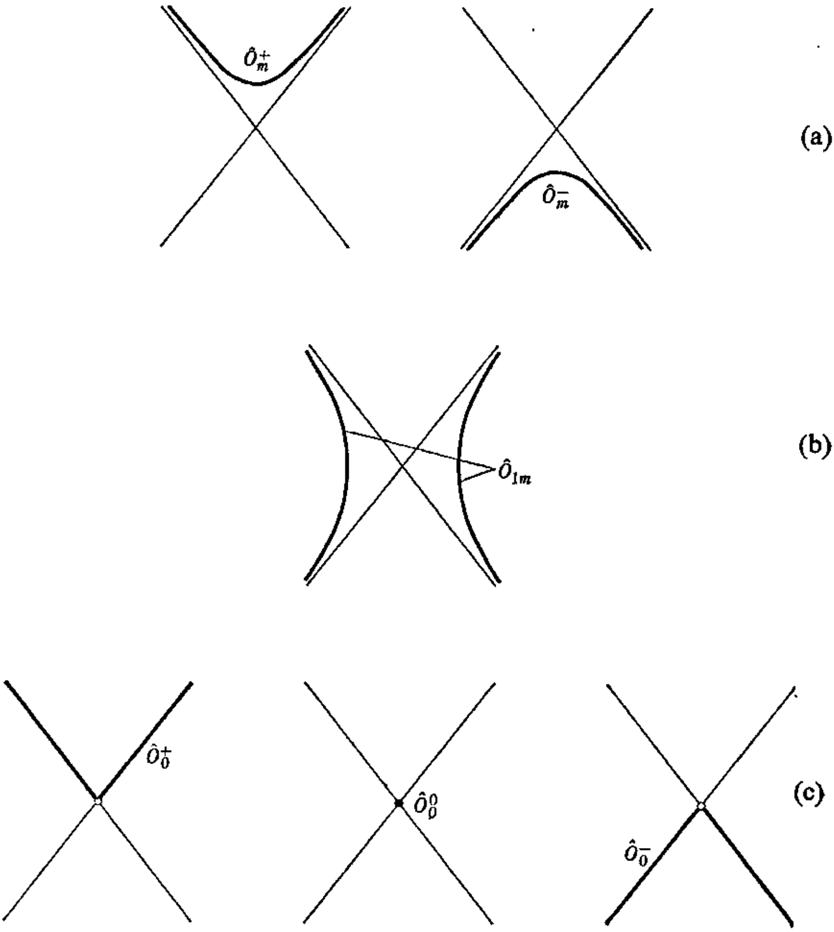

Fig. 1

Summarizing, we have six types of orbits:

1° $\hat{O}_m^+$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = m^2$, $m > 0$, $\hat{n}_0 > 0$,

2° $\hat{O}_m^-$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = m^2$, $m > 0$, $\hat{n}_0 < 0$,

3° $\hat{O}_{lm}$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = -m^2$, $m > 0$,

4° $\hat{O}_0^+$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = 0$, $m = 0$, $\hat{n}_0 > 0$,

5° $\hat{O}_0^-$: $\hat{n}_0^2-\hat{n}_1^2-\hat{n}_2^2-\hat{n}_3^2 = 0$, $m = 0$, $\hat{n}_0 < 0$,

6° $\hat{O}_0^0$: the point $0 = (0, 0, 0, 0)$, $m = 0$.

C. *The Classification of Irreducible Unitary Representations*

We verify firstly that the Poincaré group $\tilde{\Pi} = N \rtimes SL(2, C)$ is indeed a regular semidirect product as defined earlier, i.e. there exists a countable family Z of

Borel sets $Z_1, Z_2, \ldots$ of $\hat{N}$, each a union of orbits, such that every orbit in $\hat{N}$ is the intersection of a subfamily $Z_{n_1}, Z_{n_2}, \ldots$ of sets containing the orbit. Using the classification of orbits given in subsec. B, we can easily construct the family Z. Indeed, consider the family of subsets of the dual space $\hat{N}$ consisting of the following sets

(i) $\hat{O}_0^0$, $\hat{O}_0^+$ and $\hat{O}_0^-$;

(ii) $Z_m^+(r_1, r_2)$ = the union of all orbits $\hat{O}_m^+$ with $r_1 < m < r_2$, where r_1, r_2, $r_1 < r_2$, are any two positive rational numbers;

(iii) The sets $Z_m^-(r_1, r_2)$ and $Z_{im}(r_1, r_2)$ defined in the same way as $Z_m^+(r_1, r_2)$.

Then, the countable family of sets (i), (ii) and (iii) satisfies all the conditions imposed on the family Z. Thus, the Poincaré group $\tilde{\Pi}$ is a regular semidirect product. Consequently, we can directly apply for the classification of irreducible unitary representations of $\tilde{\Pi}$ the general formalism of induced representations developed in § 1 for the regular semidirect products.

Next we enumerate the classes of irreducible unitary representations of the Poincaré group $\tilde{\Pi}$ associated with each type of orbit:

1° $\hat{O}_m^+$: A representative of this orbit is the character $\hat{n}_0 = (m, 0, 0, 0)$, $m > 0$. The stability subgroup $S_{\hat{O}_m^+}$ of the point $\hat{n}_0$ is the unitary group SU(2). The group SU(2) has irreducible unitary representations $L^j \equiv D^j$ of dimension $2j+1$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ Hence, the corresponding representations $U^{\hat{n}L^j}$ of $\tilde{\Pi}$ induced by these irreducible representations D^j of SU(2) will be labelled by two parameters: m and j. These parameters can be identified in particle physics with the total mass and the total spin of a stable free system (cf. subsec. D). We denote the irreducible induced representations $U^{\hat{n}L}$ by the symbol $U^{m,+,j}$

2° $\hat{O}_m^-$: Take $\hat{n}_0 = (-m, 0, 0, 0)$. Then the stability subgroup of $\hat{n}_0$ is again SU(2). Thus we obtain again a series of induced irreducible unitary representations $U^{m,-,j}$ of $\tilde{\Pi}$ labelled by the mass m and the spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ In the physical identification $U^{m,\pm,j}$ the sign $\pm$ refers to the sign of the energy.

3° $\hat{O}_{im}$: We can choose the character $\hat{n}_0 = (0, m, 0, 0)$. The (2×2)-hermitian matrix (8) corresponding to $\hat{n}_0$ has the form $\hat{n}_0 = m\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = mi\sigma_2$. Thus the stability subgroup $S_{\hat{O}}$ consists of all matrices $g \in \mathrm{SL}(2, C)$ satisfying the condition

$$\sigma_2 = \Lambda\sigma_2\Lambda^*, \quad \text{or} \quad (\sigma_2)^{-1}\Lambda\sigma_2 = (\Lambda^*)^{-1}. \tag{23}$$

Because $(\sigma_2)^{-1} = \sigma_2$, eqs. (14) and (23) imply that $(\Lambda^T)^{-1} = (\Lambda^*)^{-1}$, i.e., $\Lambda = \bar{\Lambda}$. Consequently, the stability group of any orbit $\hat{O}_{im}$ is the group of (2×2)-real unimodular matrices, i.e., SL(2, R).

The group SL(2, R) has three series of unitary irreducible representations.

(i) The principal series $D^{i\sigma,\varepsilon}$, σ real, $\varepsilon = 0$ or 1. We have explicitly constructed these series of irreducible representations in example 16.1.2. The representations $D^{i\sigma,\varepsilon}$ and $D^{-i\sigma,\varepsilon}$ are equivalent.

(ii) The discrete series D^n, $n = 0, 1, 2, \ldots$ These representations can be realized in the Hilbert space H^n of complex functions with the domain in the upper half-plane, $\operatorname{Im} z > 0$. The scalar product in H^n is given by the formula*

$$(u, v) = \frac{\mathrm{i}}{2\pi\Gamma(n)} \int_{\operatorname{Im} z>0} u(z)\overline{v(z)}\,(\operatorname{Im} z)^{n-1} \mathrm{d}z\,\mathrm{d}\bar{z}, \qquad n = 0, 1, 2, \ldots \tag{24}$$

If $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2, R)$, then,

$$(D_g^n u)(z) = (\beta z + \delta)^{-n-1} u\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{25}$$

These discrete series are of two types $D^{n,+}$ and $D^{n,-}$ in which the spectrum of the compact generator is bounded below and above respectively (cf. exercise 11.10.7.6).

(iii) The supplementary series of representations D^ϱ, $-1 < \varrho < 1$, $\varrho \neq 0$. These can be realized in the Hilbert space H^ϱ of functions with domain on the real line, and with the scalar product

$$(u, v) = \frac{1}{\Gamma(-\varrho)} \int_{R^1 \times R^1} |x_1 - x_2|^{-1-\varrho} u(x_1)\overline{v(x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{26}$$

The action of D^ϱ in H^ϱ is given by the formula

$$(D_g^\varrho u)(x) = |\beta x + \delta|^{\varrho-1} u\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right). \tag{27}$$

(cf. exercise 16.4.1.5).

Corresponding to each series of these irreducible representations $D^{i\sigma,\varepsilon}$, $D^{n,\pm}$ and L^ϱ of SL$(2, R)$ and to a given orbit $\hat{O}_{im}$ we have the irreducible, unitary representations $U^{im,i\sigma,\varepsilon}$, $U^{im,n,\pm}$ and $U^{im,\varrho}$ of the Poincaré group $\tilde{\Pi}$. Isolated free quantum systems with imaginary masses are not known, but two particle states with a total imaginary mass can be constructed if one of the particles has a space-like momentum. These representations have found some applications also in the harmonic analysis of scattering amplitudes (cf. ch. 21.6).

4° $\hat{O}_0^+$. We can choose the character $\hat{n}_0 = (\frac{1}{2}, 0, 0, \frac{1}{2})$. Then, the corresponding 2×2-matrix is $\hat{n}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The condition $\Lambda \hat{n}_0 \Lambda^* = \hat{n}_0$ implies

$$\begin{bmatrix} \alpha\bar{\alpha} & \alpha\bar{\gamma} \\ \bar{\alpha}\gamma & \gamma\bar{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

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* For $n = 0$, we have $\lim_{s\to 0} y_+^{s-1}/\Gamma(s) = \delta(y)$. Hence the scalar product (24) takes the form

$$(u, v) = \int_{-\infty}^{\infty} u(x)\overline{v(x)}\,\mathrm{d}x.$$

Note that D^0 is reducible.

which gives $\alpha = \exp(i\theta)$, $\gamma = 0$ and β an arbitrary complex number. Thus, the stability subgroup $S_{\hat{O}_0}$ of the point $\hat{n}_0$ consists of all 2×2 matrices of the form

$$\begin{bmatrix} \exp(i\theta) & z \\ 0 & \exp(-i\theta) \end{bmatrix}, \qquad \theta \in [0,2\pi],\ z \in C^1. \tag{28}$$

The group $S_{\hat{O}_0}^+$ is actually isomorphic to a semidirect product. To see this we write an arbitrary element $k \in S_{\hat{O}_0}^+$ in the form

$$k = \begin{bmatrix} \exp\left(i\frac{\theta}{2}\right) & \exp\left(-\frac{i\theta}{2}\right)z \\ 0 & \exp\left(-\frac{i\theta}{2}\right) \end{bmatrix}, \qquad \theta \in [0, 4\pi]. \tag{29}$$

Then the product $k_1 \cdot k_2$, $k_1, k_2 \in S_{\hat{O}_0}^+$ is given by the formula

$$k_1 k_2 = \begin{bmatrix} \exp\left[\frac{i(\theta_1+\theta_2)}{2}\right] & \exp\left[-\frac{i(\theta_1+\theta_2)}{2}\right](z_1+\exp(i\theta_1)z_2) \\ 0 & \exp\left[-\frac{i(\theta_1+\theta_2)}{2}\right] \end{bmatrix}.$$

If we set $k = (z, \theta)$, then we have the following composition law

$$(z_1, \theta_1)(z_2, \theta_2) = (z_1+\exp(i\theta_1)z_2, \theta_1+\theta_2). \tag{30}$$

This shows that $S_{\hat{O}}^+$ itself is indeed the semidirect product $T^2 \rtimes S^1$ of the two-dimensional translation group $T^2 \doteq \{(z, 0)\}$ and the rotation group $S^1 = \{(0, \theta)\}$. Notice that the composition law (30) in the semidirect product $T^2 \rtimes S^1$ is the same as that of the group of the motion of the two-dimensional Euclidean space, i.e., $T^2 \rtimes O(2)$. However, in the present case $(0, 2\pi) \neq (0, 0)$, but only $(0, 4\pi) = 0$ (i.e., $T^2 \rtimes S^1$ covers twofold the Euclidean group).

Because T^2 is a noncompact vector group, the dual group $\hat{T}^2$ can be identified with T^2. Every character $\hat{z} \in \hat{T}^2$ is given by the formula

$$\langle z, \hat{z}\rangle = \exp[i(x\hat{x}+y\hat{y})], \quad z = (x, y),\ z = (\hat{x}, \hat{y}). \tag{31}$$

The action of S^1 in T^2 is

$$(0, \theta)(z, 0)(0, -\theta) = (\exp(i\theta)z, 0),$$

i.e., it produces a rotation in T^2 by an angle θ. Hence, by eq. (31) the action of S^1 in the dual space is of the form $\hat{z} \to \exp(i\theta)\hat{z}$. The orbits are circles in the complex plane $\hat{T}^2$, with the center at $0 = (0, 0)$ and radius $r \geqslant 0$. Consequently, we can distinguish two kinds of orbits:

(i) $r = 0$: In this case the stability subgroup is S^1. The irreducible representations of S^1 are one-dimensional: $\theta \to \exp(ij\theta)$ $j = 0, \pm1/2, \pm1, \ldots$ The representations $\theta \to \exp(ij\theta)$ and $\theta \to \exp(-ij\theta)$ are equivalent but not unitarily equivalent. The equivalence is given by the anti-unitarity transformation V: $u \to \bar{u}$ of the carrier space $H = C^1$. We denote the irreducible representations of $S_{\hat{O}_0}^+$

$= T^2 \circledS S^1$ induced by these one-dimensional representations of S^1 by the symbol L^j or L^{-j}, respectively. The representations L^j, $j = 0, \pm\frac{1}{2}, \pm 1, \ldots$ give rise to a series of irreducible unitary induced representations of $\tilde{\Pi}$. We denote these by the symbol $U^{0,+;j}$.

(ii) $r > 0$: Choose $\hat{z}_0 = (r, 0)$. The stability subgroup of the point $\hat{z}_0$ is the center $Z = \{I, -I\}$ of $SL(2, C)$. This group has two irreducible, unitary representations. Hence, we obtain two series $L^{r,\varepsilon}$, $r > 0$, $\varepsilon = 0$ or 1, of irreducible, unitary representations of $S_{\hat{O}_0^+}$ induced by one-dimensional irreducible representations of Z. Consequently, we obtain two series of irreducible representations of $\tilde{\Pi}$ induced by the representations $L^{r,\varepsilon}$ of $S_{\hat{O}_0^+}$. We denote them by the symbol $U^{0,+;r,\varepsilon}$, $r > 0$, $\varepsilon = 0, 1$.

5° $\hat{O}_0^-$: The classification of irreducible, unitary representations of $\tilde{\Pi}$ associated with this orbit runs parallel to the case 4°. We obtain three new series of irreducible, unitary representations of $\tilde{\Pi}$, which we denote by the symbols $U^{0,-;j}$, $U^{0,-;r,\varepsilon}$ $r > 0$, $\varepsilon = 0, 1$, respectively.

6° $\hat{O}_0^0$: In this case the stability subgroup of the orbit $\hat{O}_0^0 = (0, 0, 0, 0)$ is the whole group $SL(2, C)$ itself. The irreducible unitary representations of $\tilde{\Pi}$ associated with this orbit, are all the irreducible unitary representations of $SL(2, C)$, '*lifted*' to the group $\tilde{\Pi}$. The full classification of these representations is given in ch. 19, § 1 and 2. The group $SL(2, C)$ has a principal series and a supplementary series of irreducible unitary representations. Representations of the principal series are labelled by two numbers $i\varrho$ and j, $\varrho \geqslant 0$, $j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots$. The representations of the supplementary series are labelled by a single real number ϱ, $-1 < \varrho < 1$, $\varrho \neq 0$. The corresponding representations of $\tilde{\Pi}$ induced by the irreducible representations of the principal and the supplementary series are denoted by the symbols $U^{0,i\varrho,j}$ and $U^{0,\varrho}$, respectively.

To summarize, we have the following series of irreducible, unitary representations of the Poincaré group $\tilde{\Pi}$, associated with orbits 1°–6°; the first upper index characterizes the orbit $\hat{O}$ (i.e., mass and the sign of energy), and the second one characterizes the irreducible, unitary representations of the stability subgroup of $\hat{O}$.

1° $U^{m,+;j}$, $m > 0$, $\hat{n}_0 > 0$, $j = 0, \frac{1}{2}, 1 \ldots$,
2° $U^{m,-;j}$, $m > 0$, $\hat{n}_0 < 0$, $j = 0, \frac{1}{2}, 1, \ldots$,
3° $U^{im;i\sigma,\varepsilon}$, $m > 0$, $\sigma \geqslant 0$, $\varepsilon = 0, 1$,
$U^{im;n,\pm}$, $m > 0$, $n = 0, 1, 2, \ldots$,
$U^{im;\varrho}$, $m > 0$, $-1 < \varrho < 1$, $\varrho \neq 0$,
4° $U^{0,+;j}$, $m = 0$, $\hat{n}_0 > 0$, $j = 0, \pm\frac{1}{2}, \pm 1 \ldots$,
$U^{0,+;r,\varepsilon}$, $m = 0$, $\hat{n}_0 > 0$, $r > 0$, $\varepsilon = 0, 1$,
5° $U^{0,-;j}$, $m = 0$, $\hat{n}_0 < 0$, $j = 0, \pm\frac{1}{2}, \pm 1, \ldots$,
$U^{0,-;r,\varepsilon}$, $m = 0$, $\hat{n}_0 < 0$, $r > 0$, $\varepsilon = 0, 1$,
6° $U^{0,0;i\varrho,j}$, $m = 0$, $\hat{n}_0 = 0$, $\varrho \geqslant 0$, $j = 0, \frac{1}{2}, 1 \ldots$,
$U^{0,0;\varrho}$, $m = 0$, $\hat{n}_0 = 0$, $-1 < \varrho < 1$, $\varrho \neq 0$.

D. *The Explicit Realization of Irreducible Representations* ($m > 0$)

We now derive explicit formulas for the unitary operators $U_g^{m,+;j}$, $g \in \tilde{\Pi}$, and discuss the physical identification of these representations.

The stability subgroup of the orbit $\hat{O}_m^+$ is the group $K = T^4 \circledS \mathrm{SU}(2)$. By lemmas 1.2 and 3, the irreducible, unitary representations of $T^4 \circledS \mathrm{SU}(2)$ are of the form

$$k = (a, r) \to L_k^j = L_{(a,r)}^j = \exp(i\overset{0}{p}a)D^j(r), \tag{31}$$

where $\overset{0}{p} = (m, 0, 0, 0)$ and D^j is a unitary irreducible representation of SU(2) derived in exercise 5.8.1. The action of $U_g^{m,+j}$ in the space $H^{m,+;j}$ is given by eq. 16.1(47). Hence to complete the construction of $U_g^{m,+;j}$ we have to find the operator $L_{kg_0^{-1}s_g}^{-1}$. Note first that by virtue of the Cartan decomposition 3.6 (17) every element of SL(2, C) has the following decomposition

$$\Lambda = \Lambda_p r, \tag{32}$$

where

$$\Lambda_p = \begin{bmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in R,\ z \in C \text{ and } r \in \mathrm{SU}(2) \tag{33}$$

(cf. exercise 3). By virtue of eq. (10) we have

$$\hat{p} = \begin{bmatrix} p_0 - p_3 & p_2 + ip_1 \\ p_2 - ip_1 & p_0 + p_3 \end{bmatrix} = \Lambda\hat{\overset{\circ}{p}}\Lambda^* = \Lambda_p\hat{\overset{\circ}{p}}\Lambda_p^*. \tag{34}$$

Hence, Λ_p has the meaning of a pure Lorentz transformations which transform $\overset{\circ}{p}$ onto p. Using the map $\Lambda \to L_\Lambda$ of SL(2, C) onto the Lorentz group SO(3,1) we can write eq. (34) in the form

$$p = L_\Lambda \overset{\circ}{p} = L_{\Lambda_p} L_r \overset{\circ}{p} = L_{\Lambda_p} \overset{\circ}{p}. \tag{35}$$

Using eqs. (34) and (33) we find that the, explicit correspondence between the elements Λ_p and p is given by the formula

$$\begin{aligned} p_0 &= \frac{m}{2}(\lambda^{-2} + \lambda^2 + |z|^2), \\ p_3 &= \frac{m}{2}(\lambda^{-2} - \lambda^2 - |z|^2), \\ p_2 - ip_1 &= \frac{m}{2}\lambda^{-1}\bar{z}. \end{aligned} \tag{36}$$

These equations give the explicit correspondence between the elements of the set S of Mackey decomposition $G = SK$ and the homogeneous space O_m^+. Observing that $\Lambda_p = s_g$ we obtain

$$\begin{aligned} g_0^{-1} s_g &= (a_0, \Lambda_0)^{-1}(0, \Lambda_p) = (-L_{\Lambda_0}^{-1} a_0, \Lambda_0^{-1})(0, \Lambda_p) \\ &= (-L_{\Lambda_0}^{-1} a_0, \Lambda_0^{-1}\Lambda_p) = (0, \Lambda_{L_{\Lambda_0 p}^{-1}})(-L_{L_{\Lambda_0 p}^{-1}}^{-1} L_{\Lambda_0}^{-1} a_0, \Lambda_{L_{\Lambda_0 p}^{-1}}^{-1}\Lambda_0^{-1}\Lambda_p). \end{aligned}$$

Hence

$$k_{g_0^{-1}s_g} = (-L^{-1}_{L^{-1}_{\Lambda_0 p}} L^{-1}_{\Lambda_0} a_0,\ \Lambda^{-1}_{L^{-1}_{\Lambda_0 p}} \Lambda_0^{-1} \Lambda_p). \tag{37}$$

Taking into account that $L_{\Lambda_p}\mathring{p} = p$ and $(L_\Lambda p,\ L_\Lambda a) = (p, a)$ we obtain for the representation (31)

$$L^{-1}_{k_{g_0^{-1}s_g}} = \exp[ipa] D^j(\Lambda_p^{-1}\Lambda_0\Lambda_{L^{-1}_{\Lambda_0 p}}). \tag{38}$$

Setting $r_{\Lambda_0} = \Lambda_p^{-1}\Lambda_0\Lambda_{L^{-1}_{\Lambda_0 p}}$ and noticing that the Radon–Nikodym derivative equals to one we finally obtain by virtue of eq. 16.1(47), the formula

$$U^{m,+;j}_{(a,\Lambda)} u(p) = \exp[ipa] D^j(r_\Lambda) u(L_\Lambda^{-1} p). \tag{39}$$

We recall that $u(p)$ is a vector-valued function from $\hat{O}_m^+$ into the $(2j+1)$-dimensional vector space H of the representation D^j of SU(2): $u(p) = \{u_n(p)\}$, $n = -j, -j+1, \ldots, j-1, j$. In terms of the components, eq. (39) can be written as

$$(U^{m,+;j}_{(a,\Lambda)} u)_n(p) = \exp(ipa) D^j_{nn'}(\Lambda_p^{-1}\Lambda\Lambda_{L_\Lambda^{-1}p}) u_{n'}(L_\Lambda^{-1}p). \tag{40}$$

The representation D^j of SU(2) may be extended to $D^{(j,0)}$ of SL(2, C). Utilizing the multiplicative properties of $D^{(j,0)}$ matrices and introducing the so-called spinor basis $v_l(p) \equiv D^{(j,0)}_{ll'}(\Lambda_p) u_{l'}(p)$ one may also write eq. (40) in the simpler form:

$$(U^{m,+;j}_{(a,\Lambda)} v)_l(p) = \exp(ipa) D^{(j,0)}_{ll'}(\Lambda) v_{l'}(L_\Lambda^{-1}p). \tag{41}$$

The set of functions $\{u_n(p)\}_{n=-j}^{j} \in H^{m,+;j}$ may be identified with the wave functions of a free physical system with spin j and mass m. Indeed, in the rest system $(p = \overset{0}{p})$, and under rotations, $(0, r) \in \mathrm{SU}(2)$, the formula (40) gives

$$(U^{m,+;j}_{(0,r)} u_n)(\mathring{p}) = D^j_{nn'}(r) u_{n'}(\mathring{p}), \tag{42}$$

i.e., the set $\{u_n(\overset{0}{p})\}$ transforms in the rest frame according to the spinor representations D^j. This is, of course, the property of a free physical system which has a total spin j. The number n, $n = -j, -j+1, \ldots, -1, j$, in the rest system represents the projection of the spin on a given axis of quantization.

Next we consider the generators of the translations by taking the one-parameter subgroups of the form $a_\mu(t) = (a_\mu(t), I)$, $\mu = 0, 1, 2, 3$ and by using eq. (40). We find

$$(P_\mu u)_n(p) = p_\mu u_n(p). \tag{43}$$

Thus, for the mass operator $M = \sqrt{P_\mu P^\mu}$ we obtain

$$M u_n(p) = \sqrt{(p_0^2 - p^2)} u_n(p) = m u_n(p). \tag{44}$$

Equations (42) and (44) show that the set $\{u_n(p)\}$ describes a physical system with the rest mass m, in addition to the spin j.

If j is an integer, the representation $U^{m,+;j}$ of $\tilde{\Pi}$ are also representations of the proper Poincaré group $\Pi_+^\uparrow$. However, if j is half-odd-integer, then the representation (40) becomes a two-valued representation of $\Pi_+^\uparrow$. This follows from the fact that for half-odd-integer j the representation D^j of SO(3) becomes two-valued. (Cf. group extension by parity, ch. 21, § 4.)

We note, for completeness, that the proposition 16.1.3 provides a full description of the structure of elements $u(p) = \{u_n(p)\}$ of the carrier space $H^{m,+;j}$. In fact, the space $H^{m,+;j}$ is spanned by vectors of the form

$$u(p) = \sum_{n=-j}^{j} u_n(p)\, Y_n^j, \qquad u_n(p) \in C_0(\hat{O}_m^+), \tag{45}$$

where Y_n^j are the basis vectors of the carrier space H^j of the representation D^j of the stability subgroup SU(2). We have shown that Y_n^j can be represented as homogeneous polynomials of order $2j$ in the form

$$Y_n^j(\xi_1, \xi_2) = \frac{\xi_1^{j+n}\xi_2^{j-n}}{\sqrt{[(j+n)!\,(j-n)!]}}, \tag{46}$$

where $\xi_1, \xi_2 \in C^1$ (cf. exercise 8.9.2.1). The functions $\{Y_n^j\}, n = -j, -j+1, \dots, j-1, j$, are called spinors of order $2j$. The action of an irreducible representation D^j of SU(2) in the space H^j is given by the formula

$$\big(D^j(r)\, Y^j\big)_n(\xi_1, \xi_2) = D^j_{n'n}(r)\, Y^j_{n'}(\xi_1, \xi_2). \tag{47}$$

Elementary Systems

We consider the largest symmetry groups associated with the geometrical transformations of space-time (with fixed scales): the Galilei group or the Poincaré group. All the transformations of these groups have a physical, geometrical interpretation:

a) space and time translations (displacements of the coordinate frame);

b) rotations and reflections;

c) transformations which give to the system a velocity ('boost' transformations).

An isolated system must allow equivalent descriptions under the Poincaré group. Consequently we can define *elementary systems* whose concrete Hilbert space (CHS) is the carrier space of a single irreducible representation of the full Poincaré group Π. An elementary system is characterized by the invariants of Π, mass (m^2) and spin $\big(j(j+1)\big)$, or helicity. The eigenstates of the displacements are labelled by $|p_\mu;\, \sigma\rangle$, $\sigma \equiv \{j, n\}$; the rest frame states $|p_0, \boldsymbol{p} = 0,\, \sigma\rangle$ are rotational invariant and a velocity imparting is given by the transformation

$$|p_\mu;\, \sigma\rangle = \exp(\mathrm{i}\boldsymbol{\xi}\boldsymbol{M})|p_0, \boldsymbol{p} = 0,\, \sigma\rangle$$

for every fixed $\boldsymbol{\xi}$. Here $\boldsymbol{M}$ are the generators of pure Lorentz transformations for the rest states and

$$\boldsymbol{\xi} = \hat{\boldsymbol{p}}\cosh^{-1}\frac{p_0}{m} = \hat{\boldsymbol{p}}\sinh^{-1}\frac{p}{m}, \quad p = \sqrt{\boldsymbol{p}^2}, \quad \hat{\boldsymbol{p}} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$$

for massive particles.

An elementary system may reveal under external probing a more complex internal structure. We can give an operational definition of an elementary particle. To do this we first say that two elementary systems are *connected* if physical interactions can connect the Hilbert spaces $H^{m_1,+;j_1}$ and $H^{m_2,+;j_2}$ of the two systems. For example, the connection of 1s and 2p states of the H-atom by photo-absorption or the connection of the neutron and proton states by β-decay. We have then:

DEFINITION. An *elementary particle* (EP) is an elementary system whose states in no way can be physically connected to the states of other systems. Its Hilbert space is isolated, i.e., the states of one elementary particle $|1\rangle$ do not form a linear space with those of other systems $|2\rangle$, that is, the superposition $|1\rangle+|2\rangle$ is physically meaningless. The only effect that outside interactions can have on an EP is to change the state *within* the irreducible representation, i.e., to change its momentum and spin projection. ▼

It follows that an EP can have only those internal quantum numbers for which there are absolute superselection rules.

This operational definition of an elementary particle reflects the dependence of the concept of elementarity on the nature of interactions, as it should be. Clearly, in the kinetic theory of gases, for example, the molecules are elementary particles for, under the processes considered, the internal structure of the molecule is not excited and there is no connection to other parts of the Hilbert space. Similarly, nuclei are elementary particles in atomic phenomena, and so on.

As we did in the previous section, a 'measurement' on an elementary particle will be described by an interaction vertex with a coupling constant λ

$$M = \lambda\langle p', \sigma'|T|p, \sigma\rangle = \lambda\langle \overset{0}{p}\sigma'|\exp(-\mathrm{i}\xi'\cdot M)T\exp(\mathrm{i}\xi\cdot M)|\overset{0}{p}\sigma\rangle.$$

Here m and j are fixed on both sides, and T is a general tensor operator and represents the external agent.

The use of the Poincaré group in the definition of elementary systems presupposes a flat Minkowski space as the physical space. If we change the topology of the space, or its metric, or both, the group of motions of the space will change too. For example, the asymptotically flat space of general relativity leads to the Bondi–Meltzner–Sachs group which is the semidirect product of an infinite abelian group with SL(2, C) and one could base the concept of elementary particles on the representations of this group.

§ 3. Representation of the Extended Poincaré Group

We shall now analyse the properties of representations of the Poincaré group including space and time reflections. Let I denote the space or the time inversion operators P. T in the carrier Hilbert space H of a unitary representation U of the

Poincaré group and let $\hat{I}$, $\hat{P}$ and $\hat{T}$ denote the corresponding operators in space-time or momentum space. By definition we have

$$\hat{P}a \equiv a_{\hat{P}} = (a_0, -a), \quad \hat{T}a \equiv a_{\hat{T}} = (-a_0, a). \tag{1}$$

We assume that the action of the transformation P and T on $U_{(a,\Lambda)}$ is given by the formula

$$I^{-1}U_{(a,\Lambda)}I = U_{(a_{\hat{I}},\Lambda_{\hat{I}})}. \tag{2}$$

The momentum vector $\hat{p}$ must transform like $\hat{a}$ under Lorentz transformations and must be linear in p: hence it must be of the form

$$p_{\hat{I}} = \lambda(p_0, -p),$$

where $\lambda \in C^1$. Because $\hat{I}\hat{I}p = p$ we have $\lambda^2 = 1$, i.e., $\lambda = \pm 1$. In order to determine the sign of λ we impose an additional condition, namely the definiteness of the energy. This gives $\lambda = +1$ and

$$p_{\hat{I}} = (p_0, -p).$$

Consequently

$$(\hat{P}a, \hat{P}p) = (a, p), \quad (\hat{T}a, \hat{T}p) = -(a, p). \tag{3}$$

By virtue of eq. (2) we have

$$(I^{-1}U_{(a,e)}I)\psi(p) = U_{(\hat{I}a,e)}\psi(p) = \exp[\mathrm{i}(\hat{I}ap)]\psi(p). \tag{4}$$

This implies by virtue of eq. (3) that P must be linear and T antilinear transformation in the carrier Hilbert space H i.e.

$$P\psi(p) = \eta\psi(p), \quad T\psi(p) = C\psi^*(p), \tag{5}$$

where η and C are matrices.

We now show that

$$\Lambda_{\hat{I}} = \hat{I}^{-1}\Lambda\hat{I} = \Lambda^{*-1}. \tag{6}$$

Indeed it follows, e.g. from the form of generators of the Lorentz group ($M_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$), that space and time reflections commute with the rotation group and anticommute with pure Lorentz transformations. Because by virtue of eq. 3.11.6.8 we have

$$\Lambda = u_1\varepsilon u_2, \quad u_1, u_2 \in \mathrm{SU}(2), \tag{7}$$

where $\varepsilon = \left[\begin{smallmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{smallmatrix}\right]$ is a pure Lorentz transformation we obtain

$$\Lambda_{\hat{I}} = u_1\varepsilon^{-1}u_2 = \Lambda^{*-1}. \tag{8}$$

LEMMA 1. *The matrix η in eq. (5) must satisfy the condition*

$$D^*(\Lambda)\eta D(\Lambda) = \eta, \tag{9}$$

whereas the matrix C is given by the formula

$$C = \lambda D(\mathrm{i}\sigma_2), \quad |\lambda| = 1, \quad \sigma_2 = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}. \tag{10}$$

If η exists, then it represents the parity operator for the representation $D(\Lambda)$.

PROOF: Because

$$U_{(0,\Lambda)}\psi(p) = D(\Lambda)\psi(L_\Lambda^{-1}p) \tag{11}$$

by virtue of eq. (5) we obtain

$$D(\Lambda_{\hat{P}}) = \eta D(\Lambda)\eta^{-1}, \qquad D(\Lambda_{\hat{T}}) = C\bar{D}(\Lambda)C^{-1}. \tag{12}$$

Now, eq. (8) implies

$$D(\Lambda_{\hat{T}}) = D(\Lambda^{*-1}) = D^{*-1}(\Lambda). \tag{13}$$

Using eqs. 2 (14) we see that the condition (12) for T can always be satisfied if $C = \lambda D(\sigma_2)$, $|\lambda| = 1$. To prove the last assertion we note that every representation $D(\Lambda)$ of SL(2, C) is the direct sum of irreducible representations $D^{(j_1,j_2)}$ by th. 8.1.4 Now in the SU(2)×SU(2) basis of representation $D^{(j_1,j_2)}$ all generators of SU(2) are hermitian whereas all generators of pure Lorentz transformation are antihermitian. Hence for $r \in$ SU(2), by virtue of eq. (9), we have

$$D^*(r)\eta \cdot D(r) = D^{-1}(r)\eta D(r) = \eta \tag{14}$$

and for pure Lorentz transformations Λ_p we have

$$D^*(\Lambda_P)\eta D(\Lambda_P) = \exp[\mathrm{i}\vartheta_k N_k]\eta\exp[\mathrm{i}\vartheta_k N_k] = \eta. \tag{15}$$

Eqs. (14) and (15) show that η must commute with J_k and anticommute with N_k, i.e. η is the parity operator for $D(\Lambda)$.▼

We now determine which representations of SL(2, C) satisfy the condition (9). It follows from eq. (1) that P commutes with the generators J of SL(2, C) and anticommutes with generators $N = (M_{01}, M_{02}, M_{03})$ of pure Lorentz transformations. The irreducible representation $D^{(j_1,j_2)}$ of SL(2, C) refer to SU(2)×SU(2) basis associated with generators $J_1 = \frac{1}{2}(J+\mathrm{i}N)$ and $J_2 = \frac{1}{2}\mathrm{i}(J-\mathrm{i}N)$ Consequently j_1 and j_2 are interchanged under parity. Hence among the irreducible representations of SL(2, C) only $D^{(j_1,j_2)}$ admit the definition of parity in the carrier space. If the wave function $\psi(p)$ transform according to the representation $D^{(j_1,j_2)}$, $j_1 \neq j_2$ of SL(2, C), then we have at least to double the carrier space by considering the representation

$$D = D^{(j_1,j_2)} \oplus D^{(j_2,j_1)}$$

in order to be able to define the parity operator P. The specific examples of this procedure as well as the general theory of the extentions of the Poincaré group are discussed in ch. 21.

§ 4. Indecomposable Representations of Poincaré Group

We present in this section the construction of indecomposable representations of the Poincaré group induced from the indecomposable representations of the stability subgroup $T^4 \otimes \mathrm{SU}(2)$. We give also an application of these representations to a description of unstable particles of arbitrary spin.

A. *SLS Representations of Topological Groups*

A *sesquilinear system*, SLS, is a pair $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle \equiv \Phi$ of complex linear spaces $\overset{1}{\Phi}$ and $\overset{2}{\Phi}$ together with a sesquilinear (linear-antilinear) form $(\cdot\,,\cdot)$ on $\overset{1}{\Phi} \times \overset{2}{\Phi}$, i.e.

$$(\alpha_i u_i, \beta_k w_k) = \alpha_i \bar{\beta}_k (u_i, w_k)$$

and

$$\begin{cases} (u, \overset{2}{\Phi}) = 0 & \text{iff } u = 0, \\ (\overset{1}{\Phi}, w) = 0 & \text{iff } w = 0. \end{cases}$$

An isomorphism F between two SLS, Φ and $\tilde{\Phi}$ is a pair $\langle F_1, F_2 \rangle$, where F_i is a linear isomorphism of $\overset{i}{\Phi}$ onto $\overset{i}{\tilde{\Phi}}$, $i = 1, 2$ and $(F_1 u, F_2 w) = (u, w)$ for all $u \in \overset{1}{\Phi}$, $w \in \overset{2}{\Phi}$.

Using the sesquilinear form $(\cdot,\ \cdot)$ on $\overset{1}{\Phi} \times \overset{2}{\Phi}$ one can define a locally convex topology $\tau(\overset{1}{\Phi})$ on $\overset{1}{\Phi}$, generated by functionals $u \to (u, w)$, $u \in \overset{1}{\Phi}$, where w runs over $\overset{2}{\Phi}$: one may define similarly a topology on $\overset{2}{\Phi}$.

A SLS representation T of a locally compact group G on a SLS, $\Phi(T) = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ is a pair $\langle \overset{1}{T}, \overset{2}{T} \rangle$, where

1. $\overset{1}{T}$ (resp. $\overset{2}{T}$) is a homomorphism of G into the group of invertible linear endomorphisms of $\overset{1}{\Phi}$ (resp. $\overset{2}{\Phi}$).

2. $(\overset{1}{T}_g u, \overset{2}{T}_g w) = (u, w)$ for all $g \in G$, $u \in \overset{1}{\Phi}$, $w \in \overset{2}{\Phi}$.

3. The map $g \to (\overset{1}{T}_g u, w)$ is continuous on G for each $u \in \overset{1}{\Phi}$, and $w \in \overset{2}{\Phi}$.

If X is a bounded linear operator in $\overset{1}{\Phi}$ then the adjoint X^* is defined by the equality

$$(X^* u, w) = (u, Xw).$$

The condition 2 means that

$$\overset{1}{T}_g = (\overset{2}{T}{}_g^{-1})^*,$$

i.e. a representation $g \to \overset{1}{T}_g$ in $\overset{1}{\Phi}$ is contragradient to $g \to \overset{2}{T}_g$. Clearly $\overset{1}{T} = \overset{2}{T}$ if $\overset{2}{T}$ is unitary and sesquilinear form is the scalar product.

A representation T is (topologically) irreducible if $\overset{1}{\Phi}$ has no non-trivial $\tau(\overset{1}{\Phi})$-closed $\overset{1}{T}$-stable subspaces. (Clearly this implies that $\overset{2}{\Phi}$ has no non-trivial $\tau(\overset{2}{\Phi})$-closed $\overset{2}{T}$-stable subspaces.)

B. *SLS Induced Representations of the Poincaré Group*

We shall now construct a class of nonunitary representations of the Poincaré group which might correspond to unstable particles. Let G be the Poincaré group $G = \Pi = T^4 \otimes \mathrm{SL}(2, C)$, and let K be the closed subgroup of P such that G/K has an invariant measure. Let $k \to L_k = \langle \overset{1}{L}_k, \overset{2}{L}_k \rangle$ be a finite-dimensional SLS representation of K in the vector space $\tilde{\Phi} = \langle \overset{1}{\tilde{\Phi}}, \overset{2}{\tilde{\Phi}} \rangle$.

If $\langle u, w \rangle \in \tilde{\Phi}$ then the sesquilinear form $(\cdot, \cdot)$ can be written as

$$(u, w) = u_s \bar{w}_s, \qquad s = 1, 2, \ldots, \dim \overset{1}{L} \tag{1}$$

and the SLS representation L_k, by definition, satisfies

$$(\overset{1}{L}_k u, \overset{2}{L}_k w) = (u, w) \quad \text{for all } \langle u, w \rangle \in \tilde{\Phi} \text{ and } k \in K. \tag{2}$$

Now let $D(G)$ be the vector space of functions $\langle u(g), w(g) \rangle$ on Π with values in $\tilde{\Phi}$ such that each component $u_i(g)$ or $w_s(g)$, $i, s = 1, 2, \ldots, \dim \overset{1}{L}$, is an element of the Schwartz space of infinitely differentiable functions with compact support. Denote by $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ the vector space of all functions $\langle u(g), w(g) \rangle \in D(G)$ such that

$$w(gk) = \overset{2}{L}{}_k^{-1} w(g) \tag{3}$$

and

$$u(gk) = (\overset{1}{L}{}_k^{-1}) u(g). \tag{4}$$

The symbol 'C' denotes the operation of taking the contragradient (i.e. for a bounded operator X in $\overset{2}{\Phi}$: $X^C \equiv (X^*)^{-1}$). The vector space of functions satisfying the conditions (3) and (4) can be easily constructed. Indeed, if $\langle u(g), w(g) \rangle \in D(G)$ then

$$\hat{w}(g) = \int_K L_k w(gk) \mathrm{d}k \tag{5}$$

and

$$\hat{u}(g) = \int_K L_k^C u(gk) \mathrm{d}k \tag{6}$$

satisfy the conditions (3) and (4), respectively. It is evident from eq. (5) (resp. (6)) that $\hat{w}(g) = 0$ (resp. $\hat{u}(g) = 0$) if $g \notin SK$ where S is the compact support of the function $w(g)$ (resp. $u(g)$). Hence if

$$\langle u(g), w(g) \rangle \in D(G) \quad \text{then} \quad \langle \hat{u}(g), \hat{w}(g) \rangle$$

has a compact support on G/K. Consequently the sesquilinear form

$$(\hat{u}, \hat{w}) \equiv \int_{G/K} \bar{\hat{u}}_s(\dot{g})\hat{w}_s(\dot{g})\mathrm{d}\mu(\dot{g}), \qquad \dot{g} \equiv gK, \tag{7}$$

is well defined.

The action of the SLS representation $T^L = \langle \overset{1}{T^L}, \overset{2}{T^L} \rangle$ of Π in the space $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ is given by the left translation

$$\overset{2}{T}{}^L_{g_0} w(g) = w(g_0^{-1}g), \tag{8}$$

$$\overset{1}{T}{}^L_{g_0} u(g) = u(g_0^{-1}g). \tag{9}$$

The sesquilinear form (7) is conserved by the representation $g \to T^L_g$. Indeed using Mackey decomposition $g = s_g k$, where s_g belongs to the Borel set $S \subset G (S \sim G/K)$ and $k \in K$ one obtains ($\dot{g} \equiv x_g$, $G/K \equiv X$):

$$\begin{aligned}
(\overset{1}{T}_{g_0} u, \overset{2}{T}_{g_0} w) &= \int_X u_s(g_0^{-1}g)\overline{w}_s(g_0^{-1}g)\mathrm{d}\mu(\dot{g}) \\
&= \int_X u_s(s_{g_0^{-1}g}k)\overline{w}_s(s_{g_0^{-1}g}k)\mathrm{d}\mu(x_g) \text{ by virtue of (3) and (4)} \\
&= \int_X u_s(g_0^{-1}x_g)\overline{w}_s(g_0^{-1}x_g)\mathrm{d}\mu(x_g) \\
&= \int_X u(x')\overline{w}_s(x')\mathrm{d}\mu(x') = (u, w).
\end{aligned} \tag{10}$$

Because $u(g), w(g) \in C_0(G/K)$ the map $g \to (u, T^L_g w)$ is continuous. Consequently the map $g \to T^L_g = \langle \overset{1}{T}{}^L_g, \overset{2}{T}{}^L_g \rangle$ is an SLS representation of Π in the space $\Phi = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ induced by the SLS representation $k \to L_k = \langle \overset{2}{L}_k, \overset{2}{L}_k \rangle$.

The formulas (8) and (9) give the action of induced representation $\langle \overset{1}{T}, \overset{2}{T} \rangle$ in the space $\langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ of functions defined on the group manifold. In many applications it is more convenient to have a realization directly on the function space on the homogeneous space $X = G/K$. This can be easily calculated; in fact, using Mackey decomposition $g = s_g k_g$ and the condition (3) one obtains a map $w(g) = L_{k_g} w(g)$ from the space of functions defined on the group manifold to the space of functions defined on the coset space $X = G/K$. The transformed function $(\overset{2}{T}{}^L_{g_0} w)(g)$ is mapped onto

$$L_{k_g}(\overset{2}{T}_{g_0} w)(g) = L_{k_g} L^{-1}_{k_{g_0^{-1}g}} w(x_{g_0^{-1}g}) = L^{-1}_{k_{g^{-1}}s_g} w(g_0^{-1}x_g).$$

Hence,

$$\overset{2}{T}{}^L_{g_0} w(x_g) = L^{-1}_{k_{g_0^{-1}s_g}} w(g_0^{-1}x_g). \tag{11}$$

Consequently, selecting a definite stability subgroup K of G and its arbitrary representation $k \to L_k$ one obtains an explicit realization of the induced representation $\overset{2}{T}{}^L$ of G by formula (11). Similarly we have

$$\overset{1}{T}{}^L_{g_0} u(x_g) = (L^{-1}_{k_{g_0^{-1}g}}) u(g_0^{-1}x), \qquad u \in \overset{1}{\Phi}(X). \tag{12}$$

Consequently, an SLS representation $g \to T^L_g = \langle \overset{1}{T}{}^L_g, \overset{2}{T}{}^L_g \rangle$ of G induced by a representation $k \to L_k$ of a closed subgroup K of Π is realized in the space $\Phi(X) = \langle \overset{1}{\Phi}(X), \overset{2}{\Phi}(X) \rangle$ by formula (11) and (12). The sesquilinear form $(\cdot, \cdot)$ on $\overset{1}{\Phi}(X) \times \overset{2}{\Phi}(X)$ is given by the formula

$$(u, w) = \int_X u_s(x) \overline{w}_s(x) \mathrm{d}\mu(x), \qquad X = \Pi/K, \tag{13}$$

where $u \in \overset{1}{\Phi}(X)$, $w \in \overset{2}{\Phi}(X)$, and $\mathrm{d}\mu(x)$ is an invariant measure on $X = \Pi/K$.

The whole formalism can be directly applied to an arbitrary locally compact topological group G. In the general case it is only necessary to put in front of formulas (11) and (12) the factor $\left(\frac{\mathrm{d}\mu(g^{-1}_0 x)}{\mathrm{d}\mu(x)} \right)^{1/2}$ representing the square root of the Radon–Nikodym derivative of $\mathrm{d}\mu$ on X.

C. *Applications of SLS Indecomposable Representations of the Poincaré Group for a Description of Unstable Particles*

There is so far no completely satisfactory definition of an unstable particle. Hence it seems most reasonable to use as a guide a phenomenological description. An unstable particle is experimentally determined as an object with the following properties:

(i) It has a definite spin J and a definite space parity P.

(ii) It has a mass distribution or equivalently it has a definite decay law.

The decay law is for most particles exponential, i.e. $p(t) \sim \mathrm{e}^{-\Gamma t}$. However, it was suggested in some cases, as for instance for the A_2 meson, that a decay law might be an algebraic-exponential of the form $p(t) = (a+bt+ct^2)\mathrm{e}^{-\Gamma t}$. In what follows we mean by an isolated unstable particle one which is under the influence of forces causing the decay only.

We shall construct in this section a class of indecomposable representations of the Poincaré group Π by means of which we can reproduce all properties possessed by the phenomenological unstable particle.

We begin with the determination of the stability subgroup K of the Poincaré group Π.

It is generally accepted that an unstable particle has a complex mass M. A complex mass determines a complex orbit $\mathcal{O}$ in the space of complex momenta $p = k - \mathrm{i}q$ for which $p^2 = M^2$. The stability subgroup G_p of a vector $p \in \mathcal{O}$ is the subgroup $T^4 \otimes G_k \cap G_q$. Putting k in the rest system and setting $q = (q_0, 0, 0, q_3)$

by a proper rotation, we conclude that in general $G_p = T^4 \otimes U(1)$. Since we want to have a definite spin J as a quantum number characterizing an unstable particle we must have $G_k \cap G_q = \mathrm{SU}(2)$: this is only possible if $q = \lambda k$. Hence $p = \lambda q + \mathrm{i}q$. It is convenient to write $p = Mv$ where $M = M_0 - \mathrm{i}(\Gamma/2)$ and $v = (v_0, \boldsymbol{v})$ is the relativistic four-velocity ($v_\mu v^\mu = 1$).

We usually consider in particle physics the irreducible representations of Π. It seems however that for the description of an unstable particle or a composite system a reducible representation of Π is more appropriate. Hence we now give a general construction of nonunitary representations T^L of Π induced by an arbitrary nonunitary reducible representation L of $K = T^4 \otimes \mathrm{SU}(2)$.

Let $k \to L_k = \langle \overset{1}{L}_k, \overset{2}{L}_k \rangle$ be an SLS representation of K in $\tilde{\Phi} = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$:

$$k = (a, r) \to \overset{2}{L}_k = N_a D^J(r), \quad \overset{1}{L} = \overset{2}{L}{}^c, \; a \in T^4, \; r \in \mathrm{SU}(2) \tag{14}$$

where $a \to N_a$ is a reducible representation of the translation group T^4 and $r \to D^J(r)$ is an irreducible representation of SU(2), characterized by an integer or half-integer number J. The composition law in K:

$$(a, r)(a', r') = (a + ra', rr') \tag{15}$$

implies that N_a must be of the form $N_{(a, \mathring{v})}$ where $\mathring{v} = (v_0, 0, 0, 0)$ is a timelike vector and $(a, \mathring{v})$ is the Minkowski scalar product. The sesquilinear form $(u, w)_L$ in $\tilde{\Phi} = \langle \overset{1}{\Phi}, \overset{2}{\Phi} \rangle$ has now the form

$$(u, w)_L = u_{i\mu} \overline{w}_{i\mu} \tag{16}$$

where $i = 1, 2, \dots, \dim N_{(a, \mathring{v})}$, and $\mu = -J, -J, -J+1, \dots, J-1, J$, is the spin index.

In this case the indecomposable representations $a \to N_{(av)}$ of T^4 play an important role. The simplest example of such a representation is given by the formula

$$T^4 \ni a \to N_{(a,v)} = \mathrm{e}^{-\mathrm{i}M(a,v)} \begin{bmatrix} 1 & \gamma(a, v) \\ 0 & 1 \end{bmatrix}, \quad \gamma \in C^1. \tag{17}$$

Using the induction method one may find that an n-dimensional indecomposable representation of T^4 may be taken to be in the form

$$T^4 \ni a \to N_{(a,v)}$$

$$= \mathrm{e}^{-\mathrm{i}M(a,v)} \begin{bmatrix} 1 & \gamma_{n-1}(a, v) & \gamma_{n-2}\gamma_{n-1}(a, v)^2/2 \dots & \dfrac{\gamma_1 \cdots \gamma_{n-1}}{(n-1)!}(a, v)^{n-1} \\ 0 & 1 & \gamma_{n-2}(a, v) \quad \dots & \dfrac{\gamma_1 \cdots \gamma_{n-2}}{(n-2)!}(a, v)^{n-2} \\ & \ddots & \cdots\cdots\cdots\cdots\cdots\cdots & \\ & & 1 \quad \gamma_2(a, v) & \dfrac{\gamma_1 \gamma_2}{2}(a, v)^2 \\ & 0 & 1 & \gamma_1(a, v) \\ & & & 0 \quad 1 \end{bmatrix}. \tag{18}$$

One may construct also other classes of indecomposable representations of T^4. However, the representations (18) are most important for us, since they provide an algebraic-exponential decay law (cf. eq. (32)).

We now give the explicit form of the representation T_g^L of Π induced by a generally reducible representation $k \to L_k$ of the subgroup $K = T^4 \circledS \mathrm{SU}(2)$.

PROPOSITION 1. *Let $k \to L_k$ be a representation of the subgroup $K = T^4 \circledS \mathrm{SU}(2)$ given by* eq. (14) *and let $\overset{i}{\Phi}(X)$, $i = 1, 2$, be the Schwartz space $D(X)$, where $X = \Pi/K$. Then the SLS representation*

$$g \to T_g^L = \langle \overset{1}{T}{}_g^L, \overset{2}{T}{}_g^L \rangle$$

of the Poincaré group Π is given in the space

$$\Phi(X) = \langle \overset{1}{\Phi}(X), \overset{2}{\Phi}(X) \rangle$$

by the formulas

$$\overset{2}{T}{}^L_{\{a,\Lambda\}} w(v) = N_{(a,v)} D^J(r_\Lambda) w(L_\Lambda^{-1} v), \quad w \in \overset{2}{\Phi}(X), \tag{19}$$

and

$$\overset{1}{T}{}^L_{\{a,\Lambda\}} u(v) = N^C_{(a,v)} (D^J)^C(r_\Lambda) u(L_\Lambda^{-1} v), \quad u \in \overset{1}{\Phi}(X), \tag{20}$$

where $X = \Pi/K$ is the velocity hyperboloid ($X \ni x \sim \{v_\mu\}$, $v_\mu v^\mu = 1$, $v_0 > 0$), and $r_\Lambda = \Lambda_v \Lambda \Lambda_{L_\Lambda^{-1} v}$ is the Wigner rotation, Λ_v is the Lorentz transformation implied by the Mackey decomposition

$$\Lambda = \Lambda_v r, \quad \Lambda_v = \begin{bmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in R^1,\ z \in C^1,\ r \in \mathrm{SU}(2) \tag{21}$$

of SL(2, *C*) *and $L_v \in \mathrm{SO}(3, 1)$ is the pure Lorentz transformation in T^4 implied by the element $\Lambda_v \in \mathrm{SL}(2, C)$.*

The sesquilinear form $(\cdot, \cdot)$ in $\Phi(X)$ is given now by the formula

$$(u, w) = \int \frac{d_3 v}{v_0} u_{i\mu}(v) \overline{w}_{i\mu}(v) \tag{22}$$

where $i = 1, 2, \ldots, \dim N_{(a,v)}$ and $\mu = -J, -J+1, \ldots, J-1, J$.

PROOF: Using eq. 2 (36) we find that the homogeneous space $X = \Pi/K$, can be realized as the velocity hyperboloid. The correspondence $\Lambda_v \to v$ is given by the formula

$$\Lambda_v = \begin{bmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{bmatrix} \to v = \begin{bmatrix} v_0 - v_3 & v_2 + iv_1 \\ v_2 - iv_1 & v_0 + v_3 \end{bmatrix} = \Lambda_v \hat{\hat{v}} \Lambda_v^*, \quad \hat{\hat{v}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{23}$$

The explicit action of $\overset{2}{T}{}_g^L$ in $\overset{2}{\Phi}(X)$ can be calculated in the following manner:

$$\begin{aligned} (\overset{2}{T}_{\{a\Lambda\}} w)(0, \Lambda_v) &= w((a, \Lambda)^{-1}(0, \Lambda_v)) \\ &= w((-L_\Lambda^{-1} a, \Lambda^{-1}))(0, \Lambda_v) = w((-L_\Lambda^{-1} a, \Lambda^{-1}\Lambda_v)) \\ &= w((0, \Lambda_{L_\Lambda^{-1} v})(-L^{-1}_{\Lambda_{L_\Lambda^{-1} v}} L_\Lambda^{-1} a, \Lambda^{-1}_{L^{-1}_{\Lambda} v} \Lambda^{-1}\Lambda_v)) \\ &= N_{(a,v)} D^J(\Lambda_v^{-1} \Lambda \Lambda_{L_\Lambda^{-1} v}) w(0, L_\Lambda^{-1} v). \end{aligned} \tag{24}$$

The element $\Lambda_v^{-1}\Lambda\Lambda_{L_{\Lambda}^{-1}v}$ transforms $\mathring{v}$ into $\mathring{v}$: consequently it represents a rotation $r_\Lambda \in \mathrm{SU}(2)$ (Wigner's rotation). If we use the correspondence $\Lambda_v \sim v$ given by eq. (23), then the formula (24) can be written in the form

$$(T_{\{a,\Lambda\}}w)(v) = N_{(a,v)}D^J(r_\Lambda)w(L_\Lambda^{-1}v).$$

Similarly one obtains formula (20). ▼

Let us find now a physical interpretation for the pair of spaces $\langle \overset{1}{\Phi}, \overset{2}{\Phi}\rangle$. According to the basic concept of quantum mechanics a measurement is an operation which prescribes a number to every wave function w; consequently a measurement is in a fact a functional on the space of wave functions. This suggests to consider in our case the space $\overset{2}{\Phi}$ as the space of wave functions and the space $\overset{1}{\Phi}$ as the space of measuring devices. The probability amplitude in the measurement of a state $w \in \overset{2}{\Phi}$ by a measuring device being in the state $u \in \overset{1}{\Phi}$ is then given by the sesquilinear form (22). Clearly by virtue of eq. (10) this probability amplitude is invariant with respect to simultaneous transformations of the state w and the measuring device u.

Consider now various special cases:

C_1. *Scalar Unstable Particle*

Consider first the case of a one-dimensional nonunitary representation of the translation group

$$a \to N_{(a,v)} = \mathrm{e}^{-\mathrm{i}M(a,v)} \text{ where } M = M_0 - \frac{\mathrm{i}\Gamma}{2}, \; v = (v_0\vec{v}).$$

Let $u(v) \in \overset{1}{\Phi}$ and $w(v) \in \overset{2}{\Phi}$ be the states of the measuring device and of the unstable particle, respectively, at $t = 0$. The time evolution of the wave function is given by the formula (19), i.e. $w(t, v) = N_{tv_0}w(v)$. By virtue of eq. (22) the probability of measuring the state $w(t, v)$ by a measuring device in a state u is given by the formula

$$p(t) = |(u(t=0), w(t))|^2. \tag{25}$$

This is the probability that an unstable particle has not decayed at time t. To obtain an expression for $p(t)$ in the rest frame of the unstable particle we assume a measuring device in the state $u(t = 0, v)$ in the form $u(t = 0; v) = \delta_\varepsilon(v)$, where $\delta_\varepsilon(v)$ is an ε-model with compact support of the Dirac δ-function. By virtue of eq. (22) we obtain the following formula for $p(t)$:

$$p(t) = |(u(t=0), w(t))|^2 = \left|\int \frac{\mathrm{d}_3 v}{v_0}\,\delta_\varepsilon(v)\,w(v)\,\mathrm{e}^{-\mathrm{i}Mv_0t}\right|^2 \underset{\varepsilon\to 0}{\to} |w(0)|^2\mathrm{e}^{-\Gamma t}. \tag{26}$$

This formula agrees with the conventional expression for the time-dependence of probability obtained in the Weisskopf–Wigner formalism.

Consider now the case of a scalar particle ($J = 0$) whose wave function $w_i(v)$, $i = 1, 2, \ldots, \dim N_{(a,\mathring{v})}$, transforms according to a reducible representation of the translation group $a \to N_{(a,v)}$, $\dim N_{(a,\mathring{v})} > 1$. Assuming now the state of the measuring device to be in the form $u_i(v) = \delta_\varepsilon(v)\alpha_i$ where $\{\alpha_i\}$ is a vector in $\overset{1}{\tilde{\Phi}}$ one obtains the following expression for the probability amplitude

$$\big(u(t = 0), w(t)\big) = \int \frac{d_3 v}{v_0}\, \delta_\varepsilon(v)\alpha_i \overline{(N_{tv_0})_{ik} w_k(v)}. \tag{27}$$

In the limit $\varepsilon \to 0$ this gives

$$p(t) = |\big(u(t = 0), w(t)\big)|^2 \to |(\alpha, N_{tv_0}\beta)|^2, \quad \beta \equiv w \quad (t = 0; v = 0). \tag{28}$$

The explicit form of the time-dependence of $p(t)$ depends now on the form of the representation of the translation group. In particular if we take the two-dimensional representation (17) then one obtains the following expression:

$$p(t) = (a + bt + ct^2)e^{-\Gamma t} \tag{29}$$

where

$$\begin{aligned} a &= |(\alpha, \beta)|^2, \\ b &= 2\mathrm{Re}[(\alpha, \beta)\gamma\alpha_1\overline{\beta_2}], \\ c &= |\gamma\overline{\alpha}_1\beta_2|^2. \end{aligned} \tag{30}$$

It is interesting that this type of time-dependence was sometimes suggested on the basis of experimental results for the decay of A_2 meson.

Notice that the states of the form

$$w(v) = \begin{bmatrix} w_1(v) \\ 0 \end{bmatrix} \tag{31}$$

form an invariant subspace $\check{\Phi}$ in $\overset{2}{\Phi}$ for the representation $\overset{2}{T^L}$; for $w \in \check{\Phi}$ by virtue of eq. (30) the probability $p(t)$ has the form $p(t) = ae^{-\Gamma t}$. This provides an illustration of a phenomenon that the decay law $p(t)$ depends on production and on detection arrangements (cf. eqs. (28) and (30)).

In general taking an n-dimensional representation $a \to N_{(a,v)}$ of the translation group T^4 given by eq. (18) one obtains the decay law $p(t)$ in the form

$$p(t) = e^{-\Gamma t} \sum_{k=0}^{2(n-1)} a_k t^k \tag{32}$$

where a_k, $k = 0, 1, \ldots, 2(n-1)$, in formula (32) depend on detection and production arrangements (cf. eq. (28)).

C_2. *Unstable Particle with Spin*

The wave function of an unstable particle with spin J is a vector function $w_{i\mu}(v)$, $i = 1, 2, \ldots, \dim N_{(a\mathring{v})}$, $\mu = -J, -J+1, \ldots, J-1, J$, on the velocity hyperboloid. Using the same arguments as above we obtain the following expression for the probability amplitude

$$\big(u(t=0), N_{tv_0}w(t=0)\big) = \int \frac{d_3v}{v_0}\,\delta_\varepsilon(v)\alpha_{i\mu}\overline{(N_{tv_0})_{ik}w_{k\mu}(v)}. \tag{33}$$

In the limit $\varepsilon \to 0$ this gives

$$p(t) = |\big(u, w(t)\big)|^2 = |\alpha_\mu \overline{N_{tv_0}\beta_\mu}|^2, \quad \beta_{i\mu} \equiv w_{i\mu} \quad (t=0; v=0). \tag{34}$$

This shows that the decay law for an unstable particle with an arbitrary spin J is determined in fact by the representation $a \to N_{(a,v)}$ of the translation group T^4.

For physical interpretation of the present theory and further results cf. Rączka 1973.

§ 5. Comments and Supplements

A. *Irregular Semidirect Products*

The theory presented in sec. 1 provides a complete description of all irreducible unitary representations in the case of regular semidirect products $N \otimes S$. If $N \otimes S$ is *not* regular the formalism of sec. 1 can still be applied and one obtains an extensive class of irreducible induced unitary representations. The difference is that we are not able to prove, in this case, that every irreducible unitary representation is induced. Hence the class of irreducible unitary representations so obtained might not be complete.

B. *Semidirect Products of Type I*

It is important to know, in applications, whether a given semidirect product is a group of type I. The following theorem provides a convenient criterion for the solution of this problem.

THEOREM 1 (Mackey). *A regular, semidirect product $N \otimes S$ is a group of type I, if and only if, for every $\hat{n} \in \hat{N}$ the stability subgroup $K_{\hat{o}_{\hat{n}}}$ is of type I.*

The Euclidean and the Poincaré groups are of type I. Indeed, let $G = E^3 \otimes SO(3)$ be the Euclidean group. We have shown in example 1.2 that the stability subgroups are isomorphic to SO(2) in the case $r > 0$ and to SO(3) in the case $r = 0$. Because every compact group is of type I, th. 1 implies that the Euclidean group is also of type I.

For Poincaré group $\tilde{\Pi}$, stability subgroups of any character $\hat{n} \in \hat{N}$ are either simple Lie groups (i.e., SU(2), SL(2, R), SL(2, C)), or the semidirect product $T^2 \otimes S^1$. Simple groups are of type I. The semidirect product $T^2 \otimes S^1$ has in turn only compact stability subgroups. Hence, it is also of type I. Consequently, $\tilde{\Pi}$ has stability subgroups all of type I, and, therefore, by virtue of th. 1, is itself of type I.

C. *Comments*

The representations $U^{m,j}$, $m \geqslant 0$, $j = 0, \frac{1}{2}, 1, \ldots$ are usually used for the description of elementary relativistic free particles with mass m and spin j. The representations with imaginary mass ($m^2 < 0$) have no direct interpretation. They appear however in the description of interactions of relativistic two-particle systems: in particular these representations were extensively used in the harmonic analysis of scattering amplitudes (cf. ch. 21.6).

§ 6. Exercises

§ 1.1. Let G be a topological group and $g \to T_g$ a representation of G in a topological vector space Φ. Show that the set $\{\Phi, G\}$ forms a semidirect product $\Phi \circledS G$ with the composition law

$$(\varphi, g)(\varphi', g') = (\varphi + T_g \varphi', gg'). \tag{1}$$

§ 1.2. Show that $G = T^n \circledS \mathrm{SO}(n)$ has only two distinct classes of irreducible representations.

§ 1.3. Give the classification of irreducible representations for Lorentz type groups $T^{n+1} \circledS \mathrm{SO}_0(n, 1)$.

§ 1.4. Let G be the affine group of the real line

$$x \to ax+b, \quad a > 0, \; b \in R,$$

i.e. $G = N \circledS K$ where $N = \{(b, 1)\}$. Show that G has only three distinct irreducible representations U^+, U^- and U^s given by the formula

$$U^{\pm}_{(a,b)} u(x) = \exp(\pm \mathrm{i}\, e^* b) u(x + \log a), \quad u \in L^2(R) \tag{2}$$

and U^s being the character given by the formula

$$U^s_{(a,b)} = \exp(\mathrm{i} s \log a) I, \quad s \in R. \tag{3}$$

§ 1.5. Show that the group K(2) consisting of all upper triangular matrices of the form

$$\begin{bmatrix} \delta & \zeta \\ 0 & \delta^{-1} \end{bmatrix}, \quad \delta, \zeta \in C,$$

is the semidirect product $\mathfrak{Z} \circledS D$ of two Abelian subgroups $\mathfrak{Z}$ and D. Show that K(2) has two distinct irreducible infinite-dimensional representations.

§ 2.1. Let $g \to U_g$ be a unitary representation of the Poincaré group in a carrier space H. Show that the following operator

$$Z = P_0 (P_0^2)^{-1/2}, \tag{4}$$

where P_0 is the generator of time translation, belongs to the intertwining algebra $R(U, U)$.

Is Z an element of the enveloping field of the Lie algebra of the Poincaré group?

§ 2.2. Show that the operators for two-particle system

$$\lambda_i,\ M,\ S_\mu S^\mu,\ P_\mu \quad (i = 1, 2), \quad P_\mu = P_{(1)\mu} + P_{(2)\mu}, \tag{5}$$

where

$$M = (P_\mu P^\mu)^{1/2}, \quad S_\mu = \varepsilon_{\mu\nu\varrho\lambda} J^{\nu\varrho} P^\lambda, \quad J^{\nu\varrho} = J^{\nu\varrho}_{(1)} + J^{\nu\varrho}_{(2)},$$
$$\lambda_i = [(P^\mu_i P_\mu)^2 - m_i^2 M^2]^{-1/2} S^\mu_i P_\mu, \tag{6}$$

form a maximal set of commuting operators in the tensor product space $H = H^{m_1 J_1} \otimes H^{m_2 J_2}$.

§ 2.3. Let $|p_i \lambda_i [m_i J_i]\rangle$ be the basis vectors in the spaces H^{m_i, J_i}, $i = 1, 2$, normalized in the following manner

$$\langle p_i \lambda_i [m_i J_i] | p_i' \lambda_i [m_i J_i]\rangle = \varepsilon_i \delta^{(3)}(p_i - p_i'), \quad \varepsilon_i = \sqrt{p_i^2 + m_i^2}.$$

Set $p = \frac{1}{2}(p_1 - p_2)$ and $P = p_1 + p_2$. Because $P^2 = (4M^2)^{-1}(M^4 + m_1^4 + m_2^4 - 2M^2 m_1^2 - 2M^2 m_2^2 - 2m_1^2 m_2^2)$, we can represent P by means of M, and the angles φ and ϑ. Show that the vectors

$$|P, \Lambda [M, J, \lambda_{(1)}, \lambda_{(2)}]\rangle = \left(\frac{2J+1}{4\pi}\right)^{1/2} \int \mathrm{d}(\cos\vartheta) \mathrm{d}\varphi \bar{D}^J_{\Lambda\lambda}(\varphi, \vartheta, 0) |PM\varphi\vartheta \lambda_{(1)} \lambda_{(2)}\rangle \tag{7}$$

where $\lambda = \lambda_1 - \lambda_2$, are the common eigenvectors of operators (6) and are normalized in the following manner

$$\langle P' \Lambda' [M' J' \lambda_1' \lambda_2'] | P\Lambda [M J \lambda_1 \lambda_2]\rangle = \delta^{(4)}(p - p') \delta_{JJ'} \delta_{\lambda_1 \lambda_1'}, \delta_{\lambda_2 \lambda_2'}. \tag{8}$$

§ 2.4. Let $g \to U_g^{(m, j)}$ be an irreducible representation of the Poincaré group characterized by a positive mass m and a spin j. Find the set of analytic vectors for $U^{(m, j)}$.

§ 2.5. Find the form and spectrum of invariant operators of the Poincaré group in n-dimensional space-time.

Hint: Use the fact that irreducible representations are characterized by an orbit and the invariant numbers of an irreducible representation of the stability group.

§ 2.6. Give the explicit realizations of the representations $U^{im, n, \pm}$, $U^{im, \rho}$, $U^{0, \pm, J}$, $U^{0, \pm, r, \varepsilon}$ of the Poincaré group in the same way as that of $U^{m, J}$ given in the text.

§ 2.7. *Representations of the Poincaré Group in Different Basis.* In some physical applications it is convenient not to use the explicit form of the induced representations but realizations in which other operators are diagonalized. Discuss: (i) Total angular momentum basis: $P_0, \boldsymbol{P}^2, \boldsymbol{P} \cdot \boldsymbol{J}, J_3$, (ii) Lorentz subgroup basis: $\boldsymbol{J}^2 - \boldsymbol{N}^2, \boldsymbol{J} - \boldsymbol{N}, \boldsymbol{J}^2, J_3$. This realization also solves the problem of decomposition of the representations of the Poincaré group with respect to Lorentz group.

§ 3.1. Let $G = \mathrm{SL}(2, C)$ and $\Lambda \to D(\Lambda)$ be a finite-dimensional representation of G. Show that the parity conjugate of the representation $D(\Lambda_p)$ is $D^{*-1}(\Lambda_p)$.

§ 4.1. Let $G = T^2 \circledS \mathrm{SO}(2)$. Show that the representation U^L induced by a character χ of T^2 is irreducible.

Hint: Use th. 3.4.

§ 4.2. Discuss and classify the induced representations of inhomogeneous conformal group $T^6 \circledS \mathrm{SO}_0(4, 2)$.

Chapter 18

Fundamental Theorems on Induced Representations

Let U^L be the representation of G induced by a representation L of a subgroup K. Let U^L_N denote the restriction of U^L to another subgroup N of G. We derive in sec. 1 the so-called Induction-Reduction Theorem (I-R Theorem) which provides an elegant and effective method of the decomposition of U^L_N into irreducible representations of N. The solution of this problem is crucial for many applications of group representation theory in particle physics.

We derive in sec. 2 the Tensor-Product Theorem which is, in fact, a direct consequence of I-R Theorem. The Tensor-Product Theorem allows us to decompose effectively the tensor product $U^{L_1} \otimes U^{L_2}$ of two arbitrary induced representations of G. In particular, we give the special form of both theorems in the case of semidirect products, for which we have a complete solution of the problem of the decomposition into irreducible components.

In sec. 3 we present a 'continuous' version of the classical Frobenius-Reciprocity Theorem. This theorem has many applications especially in representation theory of complex classical Lie groups.

§ 1. The Induction-Reduction Theorem

Let G be a locally compact group and K and N any two closed subgroups of G. Let U^L be a representation of G induced by a unitary representation L of K, and let U^L_N be its restriction to a subgroup N. It is crucial to know, in many applications, the multiplicity of an irreducible representation M of N in the representation U^L_N. We show that the theory of induced representations provides a satisfactory solution to this problem.

Let $X = K\backslash G$ be the space of right K-cosets. The group G acts transitively in X. The subgroup N, however, does not in general act transitively in X. Let X_1 and X_2 be any two N-invariant Borel subsets of X such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = 0$. Let $H(X_1)$ and $H(X_2)$ be the closed subspaces of the carrier space H^L of U^L consisting of functions $u(x) \in H^L$, which vanish outside X_1 and X_2, respectively. The spaces $H(X_1)$ and $H(X_2)$ are invariant under the action of representation U^L_N and are orthogonal complements of each other. Thus, we have

$$H^L = H(X_1) \oplus H(X_2) \quad \text{and} \quad U^L_N(H^L) = U^1(H(X_1)) \oplus U^2(H(X_2)). \tag{1}$$

The space X_1 (or X_2) is invariant relative to N, but in general not transitive. Thus, one can split them further until one obtains orbits relative to N. If X_i, $i = 1, 2, \ldots$, is a countable set of orbits of X relative to N, then one obtains, by previous procedure, the following decompositions

$$H^L = \sum_i \oplus H(X_i), \qquad U_N^L(H^L) = \sum_i \oplus U_N^i(H(X_i)). \tag{2}$$

Notice that each orbit X_i of X relative to N represents a double coset of G, i.e., the set of elements of the form Kg_iN, $g_i \in G$. Clearly, two double cosets are either disjoint, or they coincide. Thus, the summation in eqs. (2) extends over different double cosets K: N of G relative to subgroups K and N. This is, in fact, the essence of Induction-Reduction Theorem. The only problem, which is left, is to determine what representations U^i of N are associated with each double coset X_i. We show that it is again a certain induced representation of N.

We shall formulate the Induction-Reduction Theorem in the most general case, which includes the case, when all the orbits are sets of measure zero. In such a case, the direct sum (2) becomes a direct integral over double cosets.

The measure ν_D on the set $\mathscr{D}$ of double cosets is defined in the following way: let $s(g)$, $g \in G$, denote the double coset KgN associated with g, and let $\tilde{\nu}$ be any finite measure in G with the same sets of measure zero as the Haar measure. Then, if $E \subset \mathscr{D}$ we define $\nu = \tilde{\nu}(s^{-1}(E))$. Such a measure is said to be an *admissible measure*. Finally, we assume, as in the case of semidirect products, some regularity conditions for the subgroups K and N: Namely, we say that K and N act regularly in G (by $g(k, n) = k^{-1}gn$) if there exists a sequence of Borel sets Z_i in G such that

(i) $\tilde{\nu}(Z_0) = 0$, $Z_i(k, n) = Z_i$, for each $(k, n) \in K \times N$ and all i.

(ii) Every orbit $\hat{O}$ not contained in Z_0 relative to the action of $K \otimes N$, is an intersection of sets Z_i containing the orbit $\hat{O}$.

We are now in the position to formulate the Induction-Reduction Theorem.

THE INDUCTION-REDUCTION THEOREM. *Let G be a separable, locally compact group and let K and N be closed subgroups of G acting regularly in G. Let U^L be a representation of G induced by the representation L of K and let U_N^L denote its restriction to the subgroup N. Then,*

1° $$U_N^L \simeq \int_{\mathscr{D}} U_N(D)\,\mathrm{d}\nu(D), \tag{3}$$

where $\mathscr{D}$ is the set of double cosets $K\backslash G/N$, $U_N(D)$ is a unitary representation of N and $\nu(\cdot)$ is any admissible measure on $\mathscr{D}$.

2° *A representation $N \ni n \to U_n(D)$ in the decomposition* (3) *is determined to within equivalence by a double coset D. For every $g \in D$ the subgroup $N \cap g^{-1}Kg$ depends on the double coset D only: representations of it defined as $\gamma \to L_{g\gamma g^{-1}}$,* *are equivalent for all $g \in D$: hence representations of N induced by them are also equivalent, and $U_N(D)$ can be taken to be any one of them.* ▼

PROOF: Ad 1°. Let ν be an admissible measure in $\mathscr{D}$ and ν the corresponding finite measure in G. We define the finite quasi-invariant measure μ in $X = K\backslash G$ by the formula $\mu(E) = \tilde{\nu}(\pi^{-1}(E))$, where π is the canonical projection from G onto X. Let r be the equivalence relation in X defined by the formula $x_1 \approx x_2$ if and only if $x_1 = x_2 n$, $n \in N$. Then, because K and N are acting regularly in G, r is a regular equivalence relation. Hence, we can use the measure disintegration th. 4.3.2 for the measure μ on X. Thus, for every $D \ni K\backslash G/N \equiv \mathscr{D}$ we obtain the measure μ_D in X, such that $\mathrm{d}\mu(x) = \mathrm{d}\nu(D)\mathrm{d}\mu_D(x)$. Each measure μ_D in X is concentrated on the orbit of X relative to N, i.e., $\mu_D(X-r^{-1}(D)) = 0$ and this measure is quasi-invariant with respect to the action of N in X, for almost all D relative to measure $\mathrm{d}\nu(D)$.

The representation U^L of G induced by a unitary representation L of K is realized in the Hilbert space $H^L = L^2(X, \mu, H(L))$ by the standard formula (cf. eq. 16.1(14))

$$U^L_{g_0} u(\dot{g}) = \varrho^{1/2}_{g_0}(g) B_g^{-1} B_{gg_0} u(\dot{g}g_0), \tag{4}$$

where by virtue of eq. 4.3 (9)

$$\varrho_{g_0}(g) = \varrho^{-1}(g)\varrho(gg_0), \tag{5}$$

$\dot{g} = x = x_0 g$, $x_0 = \dot{e} = K$, and $u(\dot{g})$ is a function in X with values in the carrier space $H(L)$ of the representation L of the stability subgroup K.

Let $H(\mathscr{D})$ be the Hilbert space $L^2(X, \mu, H(L))$. Then, the disintegration of the measure $\mathrm{d}\mu(x) = \mathrm{d}\nu(D)\mathrm{d}\mu_D(x)$ and lemma app. B.3.3 imply that the Hilbert space $H^L = L^2(X, \mu, H(L))$ can be written in the form

$$H^L = \int_{\mathscr{D}} H(D)\mathrm{d}\nu(D). \tag{6}$$

This implies in particular

$$\|u\|_{H^L} = \int_{\mathscr{D}} \mathrm{d}\nu(D) \int_X \mathrm{d}\mu_D(x) \|u(x)\|_{H(L)} = \int_{\mathscr{D}} \mathrm{d}\nu(D) \|u\|_{H(\mathscr{D})}. \tag{7}$$

The inspection of formula (4) shows that U^L_N is decomposable with respect to the decomposition (6). Hence

$$U^L_N = \int_{\mathscr{D}} U_N(D)\mathrm{d}\nu(D). \tag{8}$$

Ad 2°. We now determine the representations $U(D)$ occurring in the decomposition (8).

Let $x_D \in D$. Then, $x_D = x_0 g_D$, where $g_D \in G$ and $x_0 = e = K$. The stabilizer of x_D in N is the subgroup $N \cap g_D^{-1} K g_D$. We know, by virtue of the Measure Disintegration Theorem, that if the measure μ is quasi-invariant relative to the group G, then the measure μ_D is also quasi-invariant relative to N. More precisely, if $n, n' \in N$, and $x = x_D n = x_0 g_D n$, we have

$$\mathrm{d}\mu_D(xn') = \frac{\varrho(g_D n n')}{\varrho(g_D n)} \mathrm{d}\mu_D(x). \tag{9}$$

Consequently, the function $\varrho_D(\cdot)$ corresponding to the measure μ_D is, for $\mathrm{d}\nu$—almost all D

$$\varrho_D(\dot{n}) = \varrho(g_D n).$$

Set $B_n(D) \equiv B_{g_D \dot{n}}$. Then, eqs. (4) and (5) imply ($g = g_D n$, $x = x_D n$)

$$U_{n'}(D)u(x) = \varrho_D^{-1/2}(n)\varrho_D(\dot{n}n')B_n^{-1}(D)B_{nn'}(D)u(xn').$$

Now, if $\dot{n} \in N$, and $y \in N \cap g_D^{-1} K g_D$ then,

$$B_{yn}(D) \equiv B_{g_D yn} = B_{g_D y g_D^{-1} g_D n}.$$

Because $g_D y g_D^{-1} \in K$ and $B_{kg} = L_k B_g$ for $k \in K$ and $g \in G$ by virtue of eq. 16.1(12) one obtains

$$B_{yn}(D) = L_{g_D y g_D^{-1}} B_{g_D n} = L_{g_D y g_D^{-1}} B_n(D). \tag{10}$$

This shows that a representation $n \to U_n(D)$ in the Hilbert space $H(D)$ is induced by a representation $y \to L_{g_D y g_D^{-1}}$ of the subgroup $N \cap g_D^{-1} K g_D$. ▼

EXAMPLE 1. Let G be the Poincaré group (i.e., $G = T^4 \rtimes SO(3, 1)$) and let $K = T^4 \rtimes SO(3)$. Let L be an irreducible representation of K defined by the formula $k = (a, R) \to L_k = \exp(ipa)I$, where $p^2 = m^2 > 0$. The representation U^L of G induced by this representation L of K is irreducible (cf. ch. 17, sec. 2.C) and corresponds to a particle with a positive mass m and a spin zero.

Consider first the restriction of U^L to the subgroup $N = T^4 \rtimes \{e\}$. The space $X = K/G \cong SO(3)/SO(3, 1)$ is isomorphic to the hyperboloid $H^{(1,3)}$ given by the equation $p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2$. The subgroup N, by virtue of eq. 17.1.20, leaves each point of the hyperboloid invariant. Hence, each point of the hyperboloid represents an orbit for N, and, consequently, the space $\mathscr{D}$ of double cosets $K \backslash G/N$ is isomorphic with the set of orbits of the hyperboloid $H^{(1,3)}$. Applying now the Induction-Reduction Theorem, one obtains

$$U^L_{T^4} = \int_{H^{(1,3)}} U_{T^4}(p)\mathrm{d}\nu(p), \qquad H^{(1,3)} = \{p\colon p^2 = m^2\}. \tag{11}$$

The representations $U(p)$ of T^4 occurring in (11) are induced by a representation $y \to L_{gyg^{-1}}$ of the subgroup $T^4 \cap g_{-1}(T^4 \rtimes SO(3))g = T^4$. Hence, $T^4 \ni a \to U_a(p) = \exp(ipa)I$, $p^2 = m^2$. Consequently, $U^L_{T^4}$ is a direct integral of irreducible, one-dimensional representations $U(p)$ of T^4.

Take next $N = SO(3, 1)$. This subgroup acts transitively on the hyperboloid $H^{(1,3)}$ isomorphic to $X = K \backslash G$. Consequently, we have in this case only one double coset $K:N$. Thus, by virtue of the Induction-Reduction Theorem, we obtain

$$U^{L}_{SO(3,1)} = U(D) = U(H^{(1,3)}). \tag{12}$$

The representation $U(H^{(1,3)})$ occurring in (12) is a representation of SO(3, 1) induced by the representation $y \to L_{gyg^{-1}}$ of the subgroup SO(3, 1) $\cap g_{-1} T^4 \circledS \circledS \mathrm{SO}(3) g \cong \mathrm{SO}(3)$. Setting $g = e$, one obtains $\mathrm{SO}(3) \ni y \to L_y = I$. Thus $U(H^{(1,3)})$ is the representation of SO(3, 1) induced by the identity representation, of SO(3). It may be realized as the quasi-regular representation

$$U_n u(p) = u(pn), \quad p \in H^{(1,3)}, \quad n \in \mathrm{SO}(3,1). \tag{13}$$

We showed (cf. ch. 15, sec. 3(28)) that $U_N(H^{(1,3)})$ has the following decomposition in terms of the so-called degenerate representations U^{Λ}_N, $\Lambda \in [0, \infty)$:

$$U_N(H^{(1,3)}) = \int_0^{\infty} U^{\Lambda}_N(H^{(1,3)})\mathrm{d}\Lambda, \tag{14}$$

where $U^{\Lambda}(H^{(1,3)})$ is irreducible.

Consequently, the irreducible representation U^L of the Poincaré group restricted to the Lorentz group is the direct integral (14) of irreducible representations U^{Λ}. ▼

We now derive a special case of the I-R Theorem for the case when G is a semidirect product $N \circledS M$, where N and M are separable and locally compact and N is commutative. This class of groups contains the Euclidean, the Galilean and the Poincaré groups, which play a fundamental role in physics. We show that in these cases one can obtain an effective decomposition of the representations $U(D)$, in eq. (3), associated with a double coset D into its irreducible components.

Let $G = N \circledS M$ and let $W = N_0 \circledS M_0$, where N_0 and M_0 are any closed subgroups of N and M, respectively. Let $\hat{n}$ be any member of the dual group $\hat{N}$ (of characters) of N, and let $M_{\hat{n}}$ be the subgroup of all $m \in M$ such that $\hat{n}m = \hat{n}$. Let L be any irreducible representation of $M_{\hat{n}}$ and let $U^{\hat{n}L}$ denote the (irreducible) representation of G induced by the representation $(n, m) \to \hat{n}(n)L_m$ of the subgroup $K = N \circledS M_{\hat{n}}$. We have

THEOREM 2. *Let $G = N \circledS M$ be a semidirect product and let M_0 and $M_{\hat{n}}$ be regularly related subgroups of M. Then,*

1°
$$U^{\hat{n}L}_W \cong \int_{\mathscr{D}} U_W(D)\mathrm{d}\nu(D), \tag{15}$$

where $\mathscr{D}$ is the space of double cosets K: W and $U_W(D)$ is a unitary representation of W.

2° *The integrand $U(D)$ corresponding to the double coset D containing an element $m \in M$ can be computed as follows: Let $L^{(m)}$ denote the representation of $M_{\hat{n}m} \equiv m^{-1}M_{\hat{n}}m$ which takes $m^{-1}ym$ into Ly. Restrict $L^{(m)}$ to $M_{\hat{n}m} \cap M_0$ and then induce to $M_{0,\chi}$, where χ is the restriction of $\hat{n}m$ to N_0 and $M_{0,\chi}$ is the subgroup* *of all $m \in M_0$ with $\chi m = \chi$. Let $\int L^\lambda \mathrm{d}\varrho(\lambda)$ denote the decomposition of this induced representation into irreducible components. Then*

$$U_W(D) \cong \int U_W^{\chi L^\lambda} \mathrm{d}\varrho(\lambda), \tag{16}$$

where $U^{\chi L^\lambda}$ are irreducible representations of W induced by the representation $(n, m) \to \chi(n) L_m^\lambda$ of $N_0 \rtimes M_{0,\chi}$.

PROOF: 1° and the first part of 2° follow directly from the I-R Theorem and we leave elaboration of details for the reader. Eq. (16) follows from the theorem about the decomposition of a representation induced by a direct integral of representations (cf. th. 16.2.1). Irreducibility of $U_W^{\chi L^\lambda}$ results from th. 17.1.5. ▼

Notice that the th. 2 does not provide an explicit solution of the problem of decomposition of the representation $U^{\hat{n}L}$ into irreducible components. It reduces, however, this problem to a problem of the decomposition of a representation of the subgroup $M_{0,\chi} \subset M_0$. In all known cases this is sufficient for an effective solution of the problem of decomposition of the representation $U_W^{\hat{n}L}$ into irreducible components.

EXAMPLE 2. Let $G = N \rtimes M$ be the Euclidean group in E^3, i.e., $N = T^3$, $M = SO(3)$. Let $U^{\hat{n}L}$, $\hat{n} = (1, 0, 0)$, be an (irreducible) representation of G induced by the representation $(n, m) \to \hat{n}(n) L_m$ of the subgroup $K = T^3 \rtimes SO(2)_z$, where $SO(2)_z$ is the rotation group around z-axis. We find the restriction of $U^{\hat{n}L}$ to the subgroup $W = T^2 \rtimes SO(2)_x$. In the present case, $K \backslash G$ is the two-dimensional sphere S^2. Thus, the space of double cosets $K \backslash G/W$ coincides with the set of one-dimensional circles on S^2. Let D be a double coset containing an element $m \in SO(3)$; $m \notin SO(2)_z$. Then, by th. 2, the character $\chi \neq 0$ and consequently $M_{0,\chi} = \{e\}$. Thus, the induced representation of $M_{0,\chi}$ is the identity representation. Consequently, by eq. (16), we have

$$U_W(D) \cong U^{\chi L}, \tag{17}$$

where $U^{\chi L}$ is the representation of W induced by the representation $n \to \chi(n) I$ of $T^2 \rtimes M_{0,\chi}$.

If $m \in SO(2)_z$, then,

$$D = KmW = KW$$

has zero Haar measure (relative to G). Hence, it does not give a contribution to the decomposition (15). Consequently, the representation $U^{\hat{n}L}$ of the Euclidean group $T^3 \rtimes SO(3)$ restricted to the subgroup $T^2 \rtimes SO(2)_x$ is the direct integral (15) over the irreducible representations (17). ▼

The Induction-Reduction Theorem is very useful in an explicit solution of various problems encountered in group representation theory. We shall use it extensively in the next sections of this chapter and in the representation theory of classical Lie groups.

§ 2. Tensor-Product Theorem

Let U^1 and U^2 be irreducible unitary representations of a group G. One of the central problems in the theory of group representations and in the applications is the problem of the reduction of the tensor product $U^1 \otimes U^2$ into its irreducible constituents. We show in this section that the Induction-Reduction Theorem provides an effective method for the decomposition of the tensor product of any two induced representations of a separable, locally compact group G. In fact, let K_1 and K_2 be two closed subgroups of G and let L and M be representations of K_1 and K_2, respectively. Let, further, U^L and U^M be representations of G induced by the representations L of K_1 and M of K_2, respectively. Let

$$\mathscr{G} \equiv G \times G = \{(g_1, g_2); g_1, g_2 \in G\}, \qquad \tilde{G} \equiv \{(g, g); g \in G\}$$

and $K = K_1 \times K_2$. Clearly, G is isomorphic with $\tilde{G}$. Let $U^L \otimes U^M$ be a representation of $\mathscr{G}$ given by the outer tensor product. This representation is equivalent to the representation $U^{L\otimes M}$ of $\mathscr{G}$ by th. 16.2.3. On the other hand, $U^L \otimes U^M$ restricted to $\tilde{G}$ is equivalent to the (inner) tensor product $U^L \otimes U^M$. We have, therefore,

$$U^L \otimes U^M \cong U^{L\otimes M}_{\tilde{G}}. \tag{1}$$

Thus, the problem of the reduction of the (inner) tensor product $U^L \otimes U^M$ of G is, in fact, the problem of the decomposition of induced representations $U^{L\otimes M}$ of $G \otimes G$ restricted to the subgroup $\tilde{G} \simeq G$. This problem, in turn, is solved by the Induction-Reduction Theorem. We have

TENSOR-PRODUCT THEOREM. *Let G be a separable, locally compact group, and let K_1 and K_2 be two regularly related, closed subgroups of G. Let L and M be representations of K_1 and K_2, respectively, and let U^L and U^M be representations of G induced by the representations L and M, respectively. Then,*

1° $$U^L \otimes U^M \cong \int_{\mathscr{D}} U(D)\,\mathrm{d}\nu(D), \tag{2}$$

where D is the set of double cosets $K_1 \backslash G / K_2$, $U(D)$ is a unitary representation of G und ν is any admissible measure in $\mathscr{D}$.

2° *A representation $G \ni g \to U_g(D)$ in the decomposition (2) is determined to within an equivalence by the double coset D. If L: $y \to \tilde{L}_{gyg^{-1}}$ and $M: y \to \tilde{M}_{\gamma y \gamma^{-1}}$, $g, \gamma \in G$, $g\gamma^{-1} \in D$ are representations of the subgroup $g^{-1}K_1 g \cap \gamma^{-1}K_2\gamma$ and $\tilde{L} \otimes \tilde{M}$ denote their tensor product, then $U(D)$ is unitarily equivalent to the representation $U^{\tilde{L}\otimes\tilde{M}}$.*

PROOF: The proof is reduced by eq. (1) to the I-R Theorem. We first determine the set $K\backslash\mathscr{G}/\tilde{G}$ of double cosets.

Two elements (g, γ) and (g_1, γ_1) of $\mathscr{G}$ belong to the same double coset in $K:\tilde{G}$ if and only if the formula

$$(k_1, k_2)(g, \gamma)(\tilde{g}, \tilde{g}) = (g_1, \gamma_1)$$

holds for some $k_i \in K_i$ and $\tilde{g} \in G$. This condition is equivalent to

$$k_1 g\gamma^{-1} k_2^{-1} = g_1 \gamma_1^{-1}$$

for some $k_i \in K_i$, that is, to

$$\pi(g\gamma^{-1}) = \pi(g_1 \gamma_1^{-1}),$$

where π denotes the canonical projection of G into $K_1 \backslash G/K_2$. Hence the map $\theta(g, \gamma) \equiv \pi(g\gamma^{-1})$ defines a one-to-one mapping between the (Borel) spaces $K\backslash\mathscr{G}/\tilde{G}$ and $K_1\backslash G/K_2$ in which $K(g, \gamma)\tilde{G}$ corresponds to $K_1 g\gamma^{-1} K_2$. One verifies, using the fact that K_1 and K_2 are regularly related, that θ is a Borel isomorphism. Thus, K and $\tilde{G}$ are regularly related in $G \times G$ and all hypotheses of the I-R Theorem are satisfied. This theorem states that $U^{L\otimes M}$ restricted to $\tilde{G}$ is a direct integral over the set of double cosets $\mathscr{D} = K(g, \gamma)\tilde{G}$ of the form

$$U_{\tilde{G}}^{L\times M} \simeq \int_{\mathscr{D}} U_{\tilde{G}}(D)\,\mathrm{d}\nu(D).$$

Each term $U_{\tilde{G}}(D)$ in the integrand is a representation of $\tilde{G}$ induced by the representation $(y, y) \to (L\times M)_{(g,\gamma)(y,y)(g,\gamma)^{-1}}$ of the subgroup

$$\tilde{G} \cap (g, \gamma)^{-1}(K_1 \times K_2)(g, \gamma).$$

But the subgroup $\tilde{G} \cap (g, \gamma)^{-1}(K_1 \times K_2)(g, \gamma)$ lifted to G by the isomorphism $(g, g) \to g$ is the subgroup $g^{-1}K_1 g \cap \gamma K_2 \gamma^{-1}$. Finally, the representation $(y, y) \to (L\otimes M)_{(g,\gamma)(y,y)(g,\gamma)^{-1}}$ becomes the representation $\tilde{L} \otimes \tilde{M}$, where $L: y \to L_{gyg^{-1}}$ and $M: y \to M_{\gamma y\gamma^{-1}}$ are representations of the subgroup $g^{-1}K_1 g \cap \gamma^{-1}K_2\gamma$. ▼

The Tensor-Product Theorem provides an elegant method of the reduction of tensor products of unitary representations of various physical symmetry groups, such as the Lorentz group, the Euclidean group, the Galilean group or the Poincaré group.

The last three groups are of the form of a semidirect product $N \rtimes M$, where N and M are separable and locally compact and N is commutative. Hence, we now derive a special case of the T-P Theorem for this class of groups.

Let $\hat{n}_1$ and $\hat{n}_2$ be two characters of N. Let $M_{\hat{n}_i}$ be the closed subgroup of M consisting of all $m \in M$ with $\hat{n}_i m = \hat{n}_i$, $i = 1, 2$. Let L^i be an irreducible representation of $M_{\hat{n}_i}$ and let $\hat{n}_i L_i$ be the representation $(n, m) \to \hat{n}_i(n) L^i_m$ of $N \rtimes M_{\hat{n}_i}$, $i = 1, 2$. Then, by virtue of th. 17.1.5, the representations $U^{\hat{n}_i L_i}$ induced by $\hat{n}_i L_i$, $i = 1, 2$, are irreducible. We have

THEOREM 2. *Let $G = N \rtimes M$ be a semidirect product of separable, locally compact groups N and M, N commutative, and let $M_{\hat{n}_1}$ and $M_{\hat{n}_2}$ be regularly related subgroups of M. Then,*

$$1^\circ \qquad U^{\hat{n}_1 L_1} \otimes U^{\hat{n}_2 L_2} \cong \int_{\mathscr{D}} U_G(D)\,\mathrm{d}\nu(D), \tag{3}$$

where $\mathscr{D}$ is the space of double cosets $M_{\hat{n}_1}\backslash M/M_{\hat{n}_2}$ in M and $U(D)$ is a unitary representation of G.

2° The representations $U(D)$ of G occurring in (3) corresponding to the double coset D containing an element $m \in M$ can be computed as follows: let $\chi_1 = \hat{n}_1 m$ and let $\chi = \hat{n}_2\chi_1$. Let $\tilde{L}^i$ be the restriction of L^i to $M_{\chi_1} \cap M_{\hat{n}_2} \subseteq M_\chi$. Form the inner tensor product $\tilde{L}^1 \otimes \tilde{L}^2$, then form the representation $U^{\tilde{L}^1\otimes\tilde{L}^2}$ of M_χ induced by $\tilde{L}^1 \otimes \tilde{L}^2$. Let $\int L^\lambda d\varrho(\lambda)$ be the decomposition of $U^{\tilde{L}^1\otimes\tilde{L}^2}$ as a direct integral of irreducible representations. Then,

$$U_G(D) \simeq \int U_G^{\chi L^\lambda} d\varrho(\lambda), \tag{4}$$

where $U^{\chi L^\lambda}$ are irreducible representations of $N \circledS M$ induced by the representation $(n, m) \to \chi(n) L_m^\lambda$ of $N \circledS M_\chi$.

Proof: 1° and the first part of 2° follow from th. 1.2. It remains only to observe the obvious one-to-one correspondence of the space $N \circledS M_{\hat{n}_1}\backslash \backslash G/N \circledS M_{\hat{n}_2}$ to the space $D = M_{\hat{n}_1}\backslash M/M_{\hat{n}_2}$, because N is normal in G: we leave an elaboration of details for the reader. Eq. (4) results from th. 16.2.1. The irreducibility of $U^{\chi L^\lambda}$ follows from th. 17.1.5. ▼

Th. 2 provides then a method of the decomposition of the tensor product of irreducible representations into irreducible constituents. We illustrate now the power of th. 2 in the case of the reduction of the tensor product for the Poincaré group in two-dimensions.

Example 1. Let $G = N \circledS M$ be the two-dimensional Poincaré group. The action of G in the two-dimensional space-time is given by the formula

$$\begin{bmatrix} x \\ t \end{bmatrix} \to \begin{bmatrix} \operatorname{ch}\alpha, & \operatorname{sh}\alpha \\ \operatorname{sh}\alpha, & \operatorname{ch}\alpha \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} n_x \\ n_t \end{bmatrix}. \tag{5}$$

The dual $\hat{N}$ of N consists of 'momentum' vectors $\hat{n} = (\hat{n}_x, \hat{n}_t) = p$, with 'mass' square $p^2 = \hat{n}_t^2 - \hat{n}_x^2$. Every non-zero $\hat{n} \in \hat{N}$ has the stability group $M_{\hat{n}} = \{e\}$. For $\hat{n} = (0, 0)$, $M_{\hat{o}} = M$. Hence, every irreducible representation of G is the representation $U^{\hat{n}I}$ induced by a representation $n \to \hat{n}(n)\cdot I$ of the stability subgroup $N \circledS \{e\}$ for $\hat{n} \neq \hat{o}$, or the representation of the Lorentz group M lifted to G for $\hat{n} = \hat{o}$. We want to decompose the tensor product $U^{\hat{n}_1 I} \otimes U^{\hat{n}_2 I}$ into irreducible components. Because the stability group $M_{\hat{n}}$ of every non-zero character $\hat{n}$ is the identity, one obtains $M_\chi = \{e\}$. Consequently, $U^{\tilde{L}^1\otimes\tilde{L}^2}$ is the identity representation. Thus,

$$\int U^{\chi L^\lambda} d\varrho(\lambda) = U^{\chi I} \tag{6}$$

is the contribution to $U^{\hat{n}_1 I} \otimes U^{\hat{n}_2 I}$ of the double coset containing $m \in M$ (i.e., $U(D) = U^{\chi I}$). The decomposition (3) of $U^{\hat{n}_1 I} \otimes U^{\hat{n}_2 I}$ will be the direct integral of representations (6) over admissible characters χ. Because by th. 2.2°

$$\chi(n) = \hat{n}_2(n)\chi_1(n) = \langle n, \hat{n}_2 + \chi_1 \rangle = \exp[i(\hat{n}_2 + \chi_1)n],$$

where $\chi_1 = \hat{n}_1 m$, we obtain

$$\chi^2 = \chi_1^2+2\chi_1 \hat{n}_2+\hat{n}_2^2 = p_1^2+2|p_1||p_2|\mathrm{ch}\,\alpha+p_2^2, \qquad |p_i| = \sqrt{p_i^2}\,, \tag{7}$$

i.e.,

$$|p_1|+|p_2| \leqslant |\chi| < \infty .$$

This implies

$$U^{\hat{n}_1 I} \otimes U^{\hat{n}_2 I} \cong \int\limits_{|p_1|+|p_2|}^{\infty} U^{\chi I} \mathrm{d}|\chi|. \tag{8}$$

The formula (8) shows that the abstract decomposition (3) into double cosets has a nice physical interpretation. In fact, eq. (8) represents the decomposition of a two-particle system 'with masses' $|p_1|$ and $|p_2|$ into subsystems with invariant mass $|p| = |\chi|$. ▼

Th. 2 can equally be easily applied in the case of the Poincaré group in four dimensions. In this case, the subgroup M_χ is nontrivial and one obtains additional invariant numbers λ_1, λ_2 which remove the degeneracy of the representation $U(D)$. These additional invariant numbers correspond to the helicities of particles one and two. This again shows that the abstract decomposition (3) applied to physical problems provides the results, which have an interesting physical interpretation (cf. exercises 5.2.2*).

§ 3. The Frobenius Reciprocity Theorem

First, we consider the case of finite groups. The classical Frobenius Reciprocity Theorem states:

THEOREM 1. *Let G be a finite group and K a subgroup of G. Let U^{L^i} be a representation of G induced by an irreducible representation L^i of K. Then, the multiplicity of an irreducible representation U^j of G in U^{L^i} is equal to the multiplicity of the representation L^i in the restriction of U^j to K.*

This theorem plays an important role in the representation theory of finite groups and in their applications. We now give Mackey's generalization of the Reciprocity Theorem to induced representations of locally compact, topological groups. We begin with a reformulation of th. 1.

Let us consider the array

$$\begin{bmatrix} n(1,1) & \dots & n(1,s) \\ n(2,1) & \dots & n(2,s) \\ \dots & \dots & \dots \\ n(r,1) & \dots & n(r,s) \end{bmatrix},$$

where rows are indexed by a number 'i' labelling the irreducible representations L^i of K and columns are indexed by a number 'j' labelling irreducible representations U^j of G and in the position (i,j), we put the multiplicity $n(i,j)$ of

U^j in U^{L^i}. The multiplicity $n(i,j)$ represents a function from the group $\hat{K}\times\hat{G}$ dual to the non-negative integers. The classical Frobenius Reciprocity Theorem can now be restated in the following form.

THEOREM 1′. *There exists a function $n(\cdot,\ \cdot)$ from the group $\hat{K}\times\hat{G}$ dual to non-negative integers such that:*

$$U_G^{L^i} = \sum_{j\in\hat{G}} n(i,j)\,U^j, \quad \textit{and} \quad U_K^j = \sum_{h\in\hat{K}} n(h,j)\,L^h. \blacktriangledown \tag{1}$$

It turns out that the Reciprocity Theorem in this formulation can be generalized for the case when both G and K are not necessarily compact.

We first discuss some properties of measures on $\hat{K}\times\hat{G}$. Let Z_1 and Z_2 be Borel spaces and let α be a finite measure on $Z_1\times Z_2$. Let α_1 and α_2 be the projections of α on Z_1 and Z_2, respectively, i.e., for Borel sets $E_1\subset Z_1$ and $E_2\subset Z_2$, we have

$$\alpha_1(E_1) = \alpha(E_1\times Z_2), \quad \alpha_2(E_2) = \alpha(Z_1\times E_2). \tag{2}$$

The Measure Disintegration Theorem (cf. th. 4.3.2) assures that there exists a finite Borel measure β_x in Z_2 such that

$$\alpha = \int_{Z_1} \beta_x \mathrm{d}\alpha_1(x).$$

This means that for all Borel sets $E\subset Z_1\times Z_2$ we have

$$\alpha(E) = \int_{Z_1} \beta_x\{y:(x,y)\in E\}\,\mathrm{d}\alpha_1(x).$$

The measure β_x in Z_2 is said to be *x-slice* of the measure α. Similarly, one introduces a finite Borel measure γ_y on Z_1 such that

$$\alpha = \int_{Z_2} \gamma_y \mathrm{d}\alpha_2(y).$$

The measure γ_y is called *y-slice* of the measure α. Now we are in a position to state Mackey's generalization of the Frobenius Reciprocity Theorem.

THEOREM 2. *Let K be a closed subgroup of a separable, locally compact group G. Let both G and K be of Type I and have smooth duals $\hat{G}$ and $\hat{K}$, respectively. Let U^{L^x} be a representation of G induced by a unitary, irreducible representation L^x of K. Then, there exists a finite Borel measure α in $\hat{K}\times\hat{G}$ and a measurable function $n(\cdot,\ \cdot)$ from $\hat{K}\times\hat{G}$ to the non-negative integers including $+\infty$ such that*

1° *The projections α_1 and α_2 of α onto $\hat{K}$ and $\hat{G}$ are equivalent to measures defined by the regular representations of K and G, respectively.*

2° *For α_1-almost all x in $\hat{K}$*

$$U^{L^x} \simeq \int_{\hat{G}} n(x,y)\,U^y \mathrm{d}\beta_x(y), \tag{3}$$

where U^y are irreducible representations of G and β_x is the x-slice of α.

3° *For α_2-almost all y in $\hat{G}$*

$$U_K^y \simeq \int_{\hat{K}} n(x, y) L^x \mathrm{d}\gamma_y(x), \tag{4}$$

where γ_y is y-slice of α. ▼

The proof is given in Mackey's paper, 1952, th. 5.1. A simplified proof is presented in Mackey's Chicago Lecture Notes, 1955.

Remark 1: Mackey's Theorem provides a two-fold generalization of the Reciprocity Theorem. In fact, it gives a single function $n(x, y)$, from which the multiplicities of both U^y in U^{L^x} and L^x in U_K^y, respectively, are obtained as well as a single measure α from which the families of measures $\beta_x(\cdot)$ and $\gamma_y(\cdot)$ are obtained. ▼

If G is compact, then the direct integrals (3) and (4) reduce to direct sums and we have

$$U_G^{L^i} \cong \sum_{j\in\hat{G}} n(i,j)\, U_G^j \quad \text{and} \quad U_K^j = \sum_{h\in\hat{K}} n(h,j)\, L_K^h, \tag{5}$$

where $n(i, j)$ is a function from $\hat{K}\times\hat{G}$ to the non-negative integers. This result is an extension to compact groups of th. 1′ for finite groups.

We now give two interesting applications of the Reciprocity Theorem.

EXAMPLE 1. Let G be a compact group and let $g \to U_g$ be the regular representation of G in the Hilbert space $H = L^2(G)$. This representation can be considered as the representation U^L of G induced by the identity representation $L = I$ of the subgroup $K = \{e\}$, where e is the unit element of G. Because the multiplicity of L in an irreducible representation U^j of G restricted to K is equal to $\dim U^j$, then, by virtue of th. 2, we have

$$\text{multiplicity } n_j \text{ of } U^j \text{ in } U^L = \dim U^j.$$

Let now G be a noncompact, simple Lie group. We know that every non-trivial, unitary representation of G is infinite-dimensional. Hence, we conclude by the same arguments that every non-trivial unitary, irreducible representation of G which enters in the regular representation is contained an infinite number of times. ▼

Next we consider an example, when K is noncompact.

EXAMPLE 2. Let $G = T^4 \rtimes SO(3, 1)$ be the Poincaré group and let $K = T^4 \rtimes SO(3)$. We ask for the multiplicity of the representation $\hat{n}^1L^1$: $(a, r) \to \hat{n}^1(a)\, L_r^1$ of K in the representation $U^{\hat{n}L}$ of G induced by the representation $\hat{n}L$ of K. We know that $U^{\hat{n}L}$ is irreducible (cf. th. 17.1.5). Hence, the representation $U^{\hat{n}L}$ restricted to K contains $\hat{n}^1L^1$ at most once. ▼

§ 4. Comments and Supplements

A. The assumption in th. 2 that G and K are both of Type I is essential if K is noncompact. Indeed, Mackey has shown that if G is the discrete group of transformations of the real line

$$x \to ax+b,$$

$a > 0$, a, b rationals, and K corresponds, e.g., to the noncompact subgroup $(\{1, b\})$, then, Frobenius Reciprocity Theorem does not hold (cf. Mackey 1951, p. 216).* However, Mautner has proved that if K is compact, then the Reciprocity Theorem does hold even for groups that are not of Type I. (Cf. also Mackey 1962, § 6.)

B. It is interesting that the classical Frobenius Reciprocity Theorem can be reformulated still in another more symmetric form. In fact, let K_1 and K_2 be subgroups of the finite group G and let U^{L^i} and U^{M^j} be representations of G induced by irreducible representations L^i and M^j of K_1 and K_2, respectively. Then, the Generalized Reciprocity Theorem can be stated in the following form (cf. Osima 1952).

THEOREM 3. *There exists a function* $n(\cdot\,,\,\cdot)$ *from the dual group* $\hat{K}_1 \times \hat{K}_2$ *to non-negative integers such that*

$$U^{L^i}_{K_2} = \sum_{l \in \hat{K}_2} n(i, l) M^l$$

and

$$U^{M^j}_{K_1} = \sum_{h \in \hat{K}_1} n(h, j) L^h . \blacktriangledown \tag{6}$$

Notice that, if $K_2 = G$, this theorem coincides with th. 1′. Similarly, th. 2 can also be stated for non-compact groups in the form analogous to th. 3 (cf. Mackey 1952, § 7).

C. The technique of induced representations provides the complete classification of irreducible unitary representations of regular semidirect products (see ch. 17). It was also shown by Dixmier 1957 that every irreducible unitary representation of a connected nilpotent group G is induced by a one-dimensional representation of some subgroup of G. These two examples illustrate the power of the theory of induced representations.

Kirillov presented a certain variant of the theory of induced representations for nilpotent groups based on the methods of orbits in the dual space to the vector space of the Lie algebra (1962). This method was extended to other classes of groups by Bernat 1965, Kostant 1965, Pukansky 1968 and others. In particular Auslander and Moore 1966 gave the classiffication of induced representations of certain solvable Lie groups.

There is an interesting connection of the method of orbits with the quantiza-

* Let us note that this is the classical example of von Neumann of a group whose factor representations are of type II (cf. e.g. Naimark 1968, p. 558).

tion problem of classical mechanics. This problem was analyzed by Kostant 1965, 1970, Kirillov 1970 and Simms 1973.

D. The generalization of the theory of induced representations to group extensions was also elaborated by Mackey 1958.

§ 5. Exercises

§ 1.1.*** Find the formulation of induction-reduction theorem for nonunitary, e.g. indecomposable induced representations.

§ 2.1. Let $G = T^n \rtimes SO(3,1)$ and $K = T^n \rtimes SO(2)$. Show that the representation U^L induced by the irreducible representation L_k of K of the form

$$k = (a, \varphi) \to L_k = \exp[i\mathring{p}a]\exp[iM\varphi], \tag{1}$$

where $\mathring{p} = (m, 0, 0, 0)$ and $M = 0, \pm 1, \pm 2$ is irreducible and has the following decomposition

$$U^L = \sum_{J \geqslant |M|}^{\infty} \oplus U^{m,J}. \tag{2}$$

§ 2.2.* Let U^{M_1,J_1} and U^{M_2,J_2}, $M_1, M_2 \in (0, \infty)$, $J_1, J_2 = 0, \frac{1}{2}, 1, \frac{1}{2}, \ldots$ be two irreducible representations of the Poincaré group. Show that

(a) for $M_1, M_2 > 0$ we have

$$U^{M_1,J_1} \otimes U^{M_2,J_2} \cong \int_{M_1+M_2}^{\infty} dM \sum_{l=0}^{\infty} \sum_{s=|J_1-J_2|}^{J_1+J_2} \sum_{J=|s-l|}^{s+l} \oplus U^{M,J}, \tag{3}$$

(b)
$$U^{M_1,J_1} \otimes U^{0,J} \cong \int_{M_1}^{\infty} dM \sum_{l=|J_2|}^{\infty} \sum_{J=|l-J_1|}^{l+J_1} \oplus U^{M,J}, \tag{4}$$

(c)
$$U^{0,J_1} \otimes U^{0,J_2} \cong \int_{0}^{\infty} dM \sum_{J=|J_1-J_2|}^{\infty} \oplus U^{M,J}. \tag{5}$$

Hint: Use the th. 2.2.

§ 2.3. Let $U^{(J_1,J_2)}$ be the finite-dimensional representation of the Poincaré group obtained by lifting the representation $D^{(J_1,J_2)}$ of SL(2, C) to the Poincaré group. Is the tensor product

$$U^{(J_1,J_2)} \otimes U^{M,J}$$

reducible?

§ 2.4. Let $H = H^{M_1,J_1} \otimes H^{M_2,J_2}$ be the carrier space of the tensor product $U^{M_1,J_1} \otimes U^{M_2,J_2}$. Show that the operators

$$\lambda_i = [(P_i^\mu P_\mu)^2 - m_i^2 M^2]^{-1/2} \zeta_i^\mu P_\mu, \qquad i = 1, 2,$$

where

$$P^\mu = P_1^\mu + P_2^\mu, \quad M^2 = P_\mu P^\mu, \quad m_i^2 = P_i^\mu P_{i\mu}$$

and

$$\zeta_i^\mu = \tfrac{1}{2}\varepsilon^{\mu\nu\varrho\lambda} M_{i\nu\varrho} P_{i\lambda}$$

are invariant operators in H. Show that in the center of mass system we have

$$\lambda_i = \boldsymbol{P}_i \boldsymbol{J}_i / |\boldsymbol{P}_i|, \qquad i = 1, 2,$$

i.e., operators λ_i are helicity operators.

Chapter 19

Induced Representations of Semisimple Lie Groups

We shall now construct several series of unitary irreducible representations of semisimple classical Lie groups using the method of induced representations. The set of unitary representations of semisimple noncompact groups, in contrast to the set of finite-dimensional representations, is very rich. We usually distinguish four series of unitary representations: the principal nondegenerate, principal degenerate, supplementary nondegenerate and supplementary degenerate series. The corresponding series are determined by various sets of invariant numbers: for instance, in the case of the group $SL(n, C)$ the principal nondegenerate series is determined by $2n-2$ invariant numbers and successive degenerate ones by $2n-2k$, $k = 2, 3, \ldots, n-1$, invariant numbers.

In sec. 1 we present a general theory of induced representations of semisimple Lie groups. The construction of induced representations is based on the induction of representation from irreducible representations of a minimal parabolic subgroup of G. We analyze also problems of irreducibility and nonequivalence of the resulting induced representations. Next we analyze in details the construction of induced representations of principal, supplementary and degenerate series for the groups $SL(n, C)$ and $GL(n, C)$, which are frequently considered as symmetry groups of various physical systems. Finally in sec. 6 we give a brief discussion of properties of induced representations of other classical Lie groups.

It is remarkable that the same technique of induced representations which we used in ch. 8 for construction of all irreducible finite-dimensional representations of classical Lie groups will allow us in the present case the construction of irreducible infinite-dimensional unitary representations of classical groups. The corresponding formulas which give the realization of finite and infinite-dimensional representations are almost identical (cf. eq. 8.3(4) and 3(16)).

§ 1. Induced Representations of Semisimple Lie Groups.

In this section we present the construction of induced representations for semisimple classical Lie groups and include some recent results concerning irreducibility.

Let G be a connected semisimple Lie group with a finite centre, $G = KAN$

the Iwasawa decomposition for G, $P = MAN$ the corresponding minimal parabolic subgroup of G (cf. 3.6.D for definitions). Let L be a finite-dimensional representation of P. The following lemma describes the structure of L.

LEMMA 1. *A finite-dimensional continuous irreducible representation L of P in a space H has the form*

$$L_{man} = \chi(a)L_m, \quad m \in M,\ a \in A,\ n \in N, \tag{1}$$

where χ is a character of A and $m \to L_m$ is a continuous irreducible representation of M in H.

PROOF: By Iwasawa Theorem AN is connected and solvable: consequently by Lie's Theorem there exists a non-zero vector $u_0 \in H$ such that $L_{an}u_0 = \chi(an)u_0$, where χ is a one-dimensional continuous representation of AN: Because N is the derived group of AN $\chi(n) = 1$ for all $n \in N$. Since $L_{an}L_m u_0 = \chi(a)L_m u_0$ the span $L_m u_0$ is L-stable; hence irreducibility of L implies that this span must coincide with H. Therefore $L_{an}u = \chi(a)u$ for all $a \in A$, $n \in N$ and $u \in H$. ▼

Let χL be a finite-dimensional irreducible unitary representation of P, $X = G/P$ or $X = P\backslash G$ and $\mu(\cdot)$ a quasi-invariant measure on X. The action of the induced representation $U^{\chi L}$ of G in the space $H^{\chi L} = L^2(X, \mu)$ is given by the formula 16.1(15) if we use the right translation $x \to xg$, or by the formula 16.1(47) if we use the left translation $x \to g^{-1}x$.

The unitary representations $U^{\chi L}$ of G induced by finite-dimensional irreducible representations χL of P are called the *principal P-series* of unitary representations of G.

Notice that the above construction of induced representation may be used for complex as well as real semisimple Lie groups. If G is complex then, by 3.6.C, K is the compact form of G, M is a maximal torus in K and MA is a Cartan subgroup of G: thus the members of the principal P-series of G are unitary representations of G induced from a Cartan subgroup of G: consequently the invariant numbers which characterize the representations of the principal P-series of G are pairs consisting of an integer and real which label characters of MA.

EXAMPLE 1. Let $G = \mathrm{SL}(2, R)$. The subgroups K, A, N and M were given in example 3.6.4. Since $M = \{e, -e\}$ it has only two irreducible nonequivalent representations given by

$$L_m^+ = 1,\ m \in M, \quad \text{and} \quad L_m^- = m_1,\ m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in M. \tag{2}$$

Consequently, by lemma 1 the irreducible finite-dimensional unitary representations of $P = MAN$ are one-dimensional and have the form:

$$man = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \to \chi(a)L_m^{\pm}, \tag{3}$$

where $\chi(a) = |a|^{ir}$, $r \in R$. Therefore the principal P-series of unitary representations $U^{\chi L}$ of SL(2, R) consist of two series $U^{r,+}$ and $U^{r,-}$. The explicit realization of these representations was given in example 16.1.2.▼

One of the basic problems of representation theory is the determination of irreducibility and equivalence of representations $U^{\chi L}$ of principal P-series. To formulate the corresponding theorems we have to introduce the action of the Weyl group on representations χL of MA and the concept of extendible representations. Let $\mathfrak{g}$ and $\mathfrak{a}$ be Lie algebras of G and A respectively and let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$; let χL be an irreducible finite-dimensional representation of MA and $w \in W$; then $w\chi L$ denote the representation of MA given by

$$ma \to \chi(m_w^{*-1} a m_w^*) L_{m_w^{*-1} m m_w^*}, \qquad m \in M, \ a \in A, \tag{4}$$

where m_w^* is any element of the normalizer M^* of A in K associated with w (cf. sec. 3.6.D for properties of M^*).

Let us assume that the nilpotent subgroup N in the Iwasawa decomposition corresponds to the positive restricted root spaces of the Lie algebra $\mathfrak{a}$ and let N^- be the nilpotent subgroup of G corresponding to the negative restricted root spaces. Let L be a unitary irreducible representation of M in a Hilbert space H. We say that the pair (L, H) is extendible if there is an irreducible finite-dimensional complex G module, V, so that the M-module,

$$V^{N^-} = \{u \in V;\ nu = u \text{ for all } n \in N^-\} \tag{5}$$

is equivalent with (L, H). We call V an extension of L.

THEOREM 2. *Let χL be a finite-dimensional representation of P. Then*

(i) *If $\chi L \nsim w\chi L$ for every $w \neq I$ of W then $U^{\chi L}$ is irreducible.*

(ii) *$U^{\chi' L'} \sim U^{\chi'' L''}$ if and only if there exists a $w \in W$ such that $\chi' L' \sim w\chi'' L''$*

(iii) *If $U^{\chi L}$ is reducible then*

$$U^{\chi L} = \sum_{i=1}^{r} U^i$$

where U^i is irreducible unitary subrepresentation of $U^{\chi L}$, $U^i \nsim U^{\chi' L'}$ for any representation $\chi' L'$ of P and $r \leqslant m$—the multiplicity of L in V as an M-module.

(The proof of these results was essentially given by Bruhat 1956. The upper bound on r given in (iii) was derived by Wallach 1971.)

Th. 2 provides an effective tool in the verification of irreducibility of unitary representations of the principal P-series. We shall illustrate its power for two classes of groups.

THEOREM 3. *Let G be a connected complex semisimple Lie group. Then every member of the principal P-series is irreducible.*

PROOF: In this case MA is a Cartan subgroup of G and W acts on MA as the Weyl group of G relative to MA. Let χ be a character of A and L—a character of M. Let V be the finite-dimensional irreducible (induced) representation of G with the lowest integral weight L constructed in ch. 8, § 2. The subgroup

N^- can be identified with the subgroup Z of the Gauss decomposition. Hence the M-module V^{N^-} defined by eq. (5), by virtue of corollary 1 to th. 8.2.2 is one-dimensional and by virtue of eq. 8.2(18) is equivalent with (L, H). Hence V is an extension of L. Now the corollary 1 to th. 8.2.2 implies that the multiplicity of L in V as an M-module is one. Consequently by assertion (iii) of th. 2 the representation $U^{\chi L}$ is irreducible. ▼

The second application concerns $\mathrm{SL}(n, R)$ groups. In this case $K = \mathrm{SO}(n)$ and A can be taken as the set of all diagonal matrices of G with positive entries. N is the group of all upper triangular matrices with ones on the diagonal. The centralizer M is the group of all diagonal elements of G with entries ± 1 on the diagonal. Let $m = \mathrm{diag}(m_1, \ldots, m_n)$ be an element of M: set $\varepsilon_0(m) = 1$ and $\varepsilon_i(m) = m_i$, $i = 1, \ldots, n-1$. Then every nontrivial unitary character of M is of the form $\varepsilon_{i_1} \ldots \varepsilon_{i_r}$, $1 \leqslant i_1 < \ldots < i_r \leqslant n-1$. Set $\varepsilon_n = \varepsilon_1 \ldots \varepsilon_{n-1}$. Then we have:

THEOREM 4. *Let* $G = \mathrm{SL}(n, R)$. (i) *If n is odd, every element of the principal P-series is irreducible.* (ii) *If n is even, χ—the character of A and*

$$L = \varepsilon_{i_1} \ldots \varepsilon_{i_j}, \qquad 1 \leqslant i_1 < \ldots < n-1 \text{ and } j \neq \frac{n}{2}, \tag{6}$$

then the representation $U^{\chi L}$ is irreducible. If $j = n/2$ and if $U^{\chi L}$ is reducible then

$$U^{\chi L} = U^1 \oplus U^2$$

is the direct sum of irreducible unitary representations. ▼

PROOF: The Weyl group W acts on M by permuting the entries along the diagonal: hence W also permutes $\varepsilon_1, \ldots, \varepsilon_n$. Consider now the representations V^i, $i = 0, \ldots, n-1$, where V^0 is the trivial representation of G, V^1 is the standard (matrix) action of G on C^n and $V^i \doteq V^1 \wedge V^1 \wedge \ldots \wedge V^1$ (i times) is the polyvector representation of G which is determined by the highest weight $(1, \ldots, 1, 0, \ldots, 0)$ (cf. ch. 8.3). Let $e_1, \ldots, e_n$ be the standard basis of C^n. Then if $m \in M$, $me_i = \varepsilon_i(m) e_i$; hence in general as an M-module V^k splits into a direct sum

$$V^0 = 1, \quad V^k = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n-1} \oplus\, \varepsilon_{i_1} \ldots \varepsilon_{i_k} + \sum_{1 \leqslant j_1 < \ldots < j_{n-k} \leqslant n-1} \oplus\, \varepsilon_{j_1} \ldots \varepsilon_{j_{n-k}}$$
$$\text{for } n-1 \geqslant k > 0. \tag{7}$$

Let χ be a character of A and L be the character of M which can be taken in the form $L = \varepsilon_0 \varepsilon_1 \ldots \varepsilon_r$, $r = 0, \ldots, n-1$. If $r = 0$ then ε_0 is the action of M on $(V^0)^{N^-}$: hence V^0 is the extension of L. Similarly, if $L = \varepsilon_0 \varepsilon_1 \ldots \varepsilon_r$ then V^{n-r} is the extension of L; indeed the M-module $(V^{n-r})^{N^-}$ by virtue of corollary 1 to th. 8.2, is one-dimensional with the lowest integral weight L. Now if n is odd by virtue of eq. (7) every representation of M appears exactly once: by virtue of th. 2(iii) this implies assertion (i). Similarly using (7) and th. 2 one verifies the assertion (ii).

EXAMPLE 2. Let $G = \mathrm{SL}(2, R)$. The Weyl group $W = M^*/M$ for SL(2, R) was calculated in example 3.6.4 and $W = \left\{w_1 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, w_2 = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\right\}$.

Since the action of W on the subgroup A has the form

$$w_1^{-1}aw_1 = a, \quad w_2^{-1}aw_2 = a^{-1}, \tag{8}$$

and the action of W on M is trivial the representations $L^{r,\pm}$ and $L^{-r,\pm}$ *of* MA satisfy

$$w_2 L^{r,\pm} = L^{-r,\pm}.$$

Consequently, by th. 2(ii) the induced representations $U^{r_1,\pm}$ and $U^{r_2,\pm}$ are equivalent if $r_2 = -r_1$. Now if $r = 0$ the representation $L^- = \varepsilon_1$ of M is contained twice in the representation V^1 of SL(2, R) considered as an M-module. Consequently, by virtue of th. 2(iii) the representation $U^{0,-}$ is a direct sum of two irreducible representations. Similarly using ths. 2 and 3 one verifies that $U^{r,+}$, for $r \neq 0$ and $U^{r,-}$ are irreducible.▼

§ 2. Properties of the Group SL(n, C) and Its Subgroups

In the case $G = \mathrm{SL}(n, C)$, K is a compact form of G, M is a maximal torus in K and MA is the Cartan subgroup D of G consisting of all unimodular diagonal $n \times n$ matrices. The parabolic subgroup $P = MAN$ consists of all upper triangular unimodular matrices. Comparison with the Gauss decomposition of SL(n, C)

$$\mathrm{SL}(n, C) = \overline{\mathfrak{Z}DZ}$$

given in 3.6.A shows that the parabolic subgroup P coincides with the subgroup $\mathfrak{Z}D$ of SL(n, C). This fact implies that the Gauss decomposition plays an important role in the explicit construction of unitary representations induced from the subgroup P.

We now recall the basic properties of the Gauss decomposition for SL(n, C). The subgroup $P = \mathfrak{Z}D$ consists of upper triangular matrices whose diagonal elements satisfy the condition

$$\prod_{i=1}^{n} k_{ii} = 1 \tag{1}$$

and is solvable. The commutative (Cartan) subgroup D consists of diagonal matrices also satisfying the condition (1). The subgroup $\mathfrak{Z}$ consists of upper triangular matrices with diagonal elements equal to unity. The subgroup Z consists of lower triangular matrices with diagonal elements equal to unity and is also nilpotent. The subgroup $\mathfrak{Z}$ is the normal subgroup of P, and P is the semidirect product $\mathfrak{Z} \circledS D$.

We recall that the Gauss decomposition $\mathrm{SL}(n, C) = \overline{\mathfrak{Z}DZ}$ means that almost every element $g \in \mathrm{SL}(n, C)$ can be uniquely decomposed in the form

$$g = \zeta\delta z, \quad \zeta \in \mathfrak{Z}, \ \delta \in D, \ z \in Z, \tag{2a}$$

or

$$g = kz, \tag{2b}$$

where

$$k = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ 0 & k_{22} & \dots & k_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & k_{nn} \end{bmatrix} \in P = \mathfrak{Z}D \tag{3}$$

and

$$z = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ z_{21} & 1 & 0 & \dots & 0 \\ z_{31} & z_{32} & 1 & \dots & 0 \\ \vdots & & & & \\ \dots & \dots & \dots & \dots & \dots \\ z_{n1} & z_{n2} & \dots & z_{n,n-1} & 1 \end{bmatrix} \in Z. \tag{4}$$

The matrix elements z_{pq}, $n \geqslant p > q > 0$ of a group element $z \in Z$ are arbitrary complex numbers. Hence, we can represent them in the form $z_{pq} = x_{pq} + iy_{pq}$. We know, due to the results of ch. 3, § 11, eq. (38) that the group of triangular matrices with diagonal elements one has invariant measure which can be taken to be of the form

$$d\mu(z) = \prod_{p,q=1} \mathrm{d}x_{pq} \mathrm{d}y_{pq}, \quad n \geqslant p > q > 0. \tag{5}$$

In the following we shall utilize this measure on the group Z for the construction of the space $L^2(Z, \mu)$ in which the irreducible, unitary representations of $SL(n, C)$ are realized.

§ 3. The Principal Nondegenerate Series of Unitary Representations of SL(n, C)

We now apply the general formalism of induced representations in order to construct a class of irreducible unitary representations of SL(n, C). Let us recall the basic steps of construction of induced representations. Let P be a closed subgroup of G, $X = P \backslash G$, and μ a quasi-invariant measure on X. Let $k \to L_k$ be a unitary representation of P in the Hilbert space H. Then, the carrier space H^L of the representation U^L induced by the representation L of P consists of functions satisfying

$$u(kg) = L_k u(g), \quad k \in P \tag{1}$$

and

$$\int_X \|u(g)\|_H^2 \mathrm{d}\mu(x) < \infty. \tag{2}$$

In the realization of the representation U^L in $H^L = L^2(X, \mu)$ the action of $U^L_{g_0}$ is given by the formula 16.1 (15), i.e.,

$$U^L_{g_0} u(x) = \sqrt{\left(\frac{d\mu(xg_0)}{d\mu(x)}\right)} L_{k_{x_g g_0}} u(xg_0), \tag{3}$$

where $\dfrac{d\mu(xg_0)}{d\mu(x)}$ is the Radon–Nikodym derivative, $x = \dot{g} = Pg = Pk_g x_g = Px_g$ and $k_{x_g g_0}$ is determined by the Mackey decomposition (2.4.1) of $x_g g_0$, i.e. $x_g g_0 = k_{x_g g_0} x'$ with $x' \in S$, $k_{x_g g_0} \in P$.

In the case of SL(n, C) we have $P = \mathfrak{Z}D$. Since D is commutative it has only one-dimensional irreducible representations, and because $\mathfrak{Z}$ is normal in P; this representation extends to P, by virtue of (3). Thus the construction of induced representations of SL(n, C) is reduced to a simple computation of the Radon–Nikodym derivative $d\mu(xg)/d\mu(x)$ and of the element $k_{x_g g_0} \in P$ corresponding to $x_g g_0 \in G$.

A. *The Determination of the Factor* $k_{x_g g_0}$

Let D be the diagonal (Cartan) subgroup of SL(n, C) and let

$$\delta \to L_\delta = \chi(\delta) = \prod_{s=2}^{n} |\delta_{ss}|^{m_s + i\varrho_s} \delta_{ss}{}^{-m_s}, \tag{4}$$

where m_s are integers and ϱ_s are real numbers, $s = 2, 3, \ldots, n$, be the one-dimensional representation of D determined by the character χ. The subgroup P is the semidirect product $\mathfrak{Z} \otimes D$ and $\mathfrak{Z}$ is normal in P. Hence, the map L: $(\zeta, \delta) \to I \cdot \chi(\delta)$ is the most general unitary one-dimensional representation L of P:

$$k = (\zeta, \delta) \to L_k = I\chi(\delta) = \prod_{s=2}^{n} |k_{ss}|^{m_s + i\varrho_s} k_{ss}{}^{-m_s}, \qquad k_{ss} = \delta_{ss}. \tag{5}$$

We shall use L to construct the induced representation U^L of SL(n, C).

A function $u(x)$ in the carrier space H^L is defined in the space $X = P \backslash G$. It is, however, more convenient to consider it as a function over the group space of the subgroup Z. This is possible due to the following lemma:

LEMMA 1. *The set $X = P \backslash G$ coincides with group space of the subgroup Z up to a subset of measure zero with respect to any quasi-invariant measure μ on X.*

PROOF: By the Gauss decomposition if follows that the set $G - KZ$ is a set of measure zero with respect to the Haar measure dg on G. The canonical projection π: $G \to P \backslash G$ maps dg-null sets into $d\mu$-null sets in X. Hence $X - Z$ is $d\mu$-null set and the lemma is proved. ▼

This lemma allows us to associate uniquely with almost every coset $Pg = x \in X$ the element $z_g \in Z$ defined by the equality $g = k_g z_g$. Comparing the decomposition $g = k_g z_g$ with the Màckey decomposition 2.4 (1), $g = k_g x_g$, we conclude that elements $z_g \in Z$ play the role of elements $x_g \in S$.

Let now the decomposition of the element $z_g g_0$ be defined by

$$\tilde{g} \equiv z_g g_0 = k_{\tilde{g}} z_{\tilde{g}} . \tag{6}$$

For the sake of simplicity of notation we set $z_{\tilde{g}} = \tilde{z}$, $k_{\tilde{g}} = \tilde{k}$. The element $\tilde{z}$ corresponds to the transformed point xg_0 in eq. (3). The explicit formulas for the matrix elements $\tilde{k}$ and $\tilde{z}$ can be obtained with the help of the Gauss decomposition. In fact, applying formula 3.11(18) for the group element

$$\tilde{g} = z_g g_0 = k\tilde{\tilde{z}} \qquad \left(\text{i.e., } \tilde{g}_{pq} = \sum_{s=1}^{p-1} z_{ps} g_{sq} + g_{pq}\right)$$

we obtain

$$\begin{aligned}
(\tilde{z})_{pq} &= \frac{\begin{vmatrix} \tilde{g}_{pq} & \tilde{g}_{p,p+1} & \cdots & \tilde{g}_{p,n} \\ \tilde{g}_{p+1,q} & \tilde{g}_{p+1,p+1} & \cdots & \tilde{g}_{p+1,n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{nq} & \tilde{g}_{n,p+1} & \cdots & \tilde{g}_{nn} \end{vmatrix}}{\begin{vmatrix} \tilde{g}_{pp} & \tilde{g}_{p,p+1} & \cdots & \tilde{g}_{pn} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{np} & \tilde{g}_{n,p+1} & \cdots & \tilde{g}_{nn} \end{vmatrix}}, \\
\tilde{k}_{pp} &= \frac{\begin{vmatrix} \tilde{g}_{pp} & \tilde{g}_{p,p+1} & \cdots & \tilde{g}_{pn} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{np} & \tilde{g}_{n,p+1} & \cdots & \tilde{g}_{nn} \end{vmatrix}}{\begin{vmatrix} \tilde{g}_{p+1,p+1} & \tilde{g}_{p+1,p+2} & \cdots & \tilde{g}_{pn} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{g}_{n,p+1} & \tilde{g}_{n,p+2} & \cdots & \tilde{g}_{nn} \end{vmatrix}} .
\end{aligned} \tag{7}$$

In particular, in the case of SL(2, C), we obtain

$$(\tilde{z})_{21} = \frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}, \qquad z \in G, \tag{8}$$

$$\tilde{k}_{11} = (g_{12}z + g_{22})^{-1}, \quad \tilde{k}_{12} = g_{12}, \quad \tilde{k}_{22} = g_{12}z + g_{22}. \tag{9}$$

We see that the action of SL(2, C) on Z, which in this case is isomorphic to the additive group C of complex numbers, is given by a fractional linear transformation (8). In the case of SL(n, C), $n > 2$, the mapping $z \to \tilde{z}$ is a natural generalization of the fractional linear transformation (8).

B. *Determination of the Radon–Nikodym Derivative* $\dfrac{d\mu(\tilde{z})}{d\mu(z)}$

The invariant measure $d\mu(z)$ on the nilpotent subgroup Z is given by formula 2(5). Consequently, the Radon–Nikodym derivative $d\mu(\tilde{z})/d\mu(z)$ represents, in fact, the Jacobian of the transformation $z \to \tilde{z}$. It compensates the factor, which results from the non-invariance of the measure $d\mu(z)$ on Z with respect to the action of the group $SL(n, C)$ on Z. We have

LEMMA 2. *The Radon–Nikodym derivative is given by the formula*

$$\frac{d\mu(\tilde{z})}{d\mu(z)} = \prod_{s=2}^{n} |\tilde{k}_{ss}|^{-4(s-1)}, \tag{10}$$

where $\tilde{k}_{ss}$ is given by equation (7).

PROOF: Let $(\tilde{z})_{pq}$ and z_{pq} be the matrix elements of $\tilde{z}$ and z, respectively, $(\tilde{g} = zg_0 = \tilde{k}\tilde{z})$. We introduce for simplicity some ordering in the variables $(\tilde{z})_{pq}$ and $(z)_{pq}$, $n \geqslant p > q \geqslant 1$, by setting

$$(\tilde{z})_{pq} = w_l = u_l + iv_l, \tag{11}$$

$$l = 1, 2, \ldots, \frac{n(n-1)}{2} = N.$$

$$(z)_{pq} = z_l = x_l + iy_l, \tag{12}$$

Note that by eq. (7) the variables (11) are analytic functions of the variables (12).

Eq. 2(5) implies that the Radon–Nikodym derivative is given by the following Jacobian

$$\frac{D(u_1, v_1, \ldots, u_n, v_n)}{D(x_1, y_1, \ldots, x_n, y_n)}.$$

In the evaluation of this Jacobian we use the following well-known fact: if $w_l = u_l + iv_l$, $l = 1, 2, \ldots, r$, are analytic functions of variables $z_k = x_k + iy_k$, $k = 1, 2, \ldots, r$, then

$$\frac{D(u_1, v_1, \ldots, u_r, v_r)}{D(x_1, y_1, \ldots, x_r, y_r)} = \left|\frac{D(w_1, w_2, \ldots, w_r)}{D(z_1, z_2, \ldots, z_r)}\right|^2. \tag{13}$$

We prove this by the method of induction. For $r = 1$, using Cauchy–Riemann equations and eqs. (8) and (9), we obtain

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left|\frac{d\tilde{z}}{dz}\right|^2.$$

Assuming that eq. (13) is true for $s = m-1$, we find for $s = m$:

$$\frac{D(u_1, v_1, u_2, v_2, \ldots, u_m, v_m}{D(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)} = \frac{D(u_1, v_1, u_2, v_2, \ldots, u_m, v_m)}{D(x_1, y_1, u_2, v_2, \ldots, u_m, v_m)} \times$$

$$\times \frac{D(x_1, y_1, u_2, v_2, \ldots, u_m, v_m)}{D(x_1, y_1, x_2, y_2, \ldots, x_m, y_m)} = \frac{D(u_1, v_1)}{D(x_1, y_1)} \cdot \frac{D(u_2, v_2, \ldots, u_m, v_m)}{D(x_2, y_2, \ldots, x_m, y_m)}$$

$$= \left|\frac{D(w_1)}{D(z_1)}\right|^2 \cdot \left|\frac{D(w_2, \ldots, w_m)}{D(z_2, \ldots, z_m)}\right|^2 = \left|\frac{D(w_1, w_2, \ldots, w_m)}{D(z_1, z_2, \ldots, z_m)}\right|^2$$

Using now eqs. (11) and (7) we obtain

$$\frac{D(w_1, w_2, \ldots, w_n)}{D(z_1, z_2, \ldots, z_n)} = \prod_{s=2}^{n} (\tilde{k}_{ss})^{-2(s-1)}, \tag{14}$$

where $\tilde{k} \in K$ is defined by the decomposition $\tilde{g} = zg_0 = \tilde{k}\tilde{z}$ and the explicit expressions for the matrix elements $\tilde{k}_{ss}$ are given in eq. (7). ▼

C. *Principal Series of Representations*

By virtue of eqs. (1)–(3), the carrier space H^L of the representation U^L induced by the one-dimensional representation L of K consists of all measurable functions $\varphi(z) = \varphi(\ldots, z_{pq}, \ldots)$, $0 < q < p \leqslant n$, satisfying the condition

$$\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} |\varphi(\ldots, z_{pq}, \ldots)|^2 \mathrm{d}x_{pq} \mathrm{d}y_{pq} < \infty, \qquad z_{pq} = x_{pq} + \mathrm{i}y_{pq}. \tag{15}$$

The operator U_g^L is, by virtue of eqs. (3), (5) and (10), given by

$$U_{g_0}^L \varphi(z) = \sqrt{\frac{\mathrm{d}\mu(\tilde{z})}{\mathrm{d}\mu(z)}} L_{\tilde{k}} \varphi(\tilde{z}) = \prod_{s=2}^{n} |\tilde{k}_{ss}|^{m_s + \mathrm{i}\varrho - 2(s-1)} \tilde{k}_{ss}^{-m_s} \varphi(\tilde{z}), \tag{16}$$

where

$$\tilde{g} = zg_0 = \tilde{k}\tilde{z}$$

and the parameters $\tilde{k}_{ss}$ and the components $(\tilde{z})_{pq}$ of $\tilde{z}$ are given by eq. (7).

This series of representations is called the *principal non-degenerate series.* It is characterized by two sets of invariant numbers: the integers $m_2, m_3, \ldots, m_n$, and the real numbers $\varrho_2, \varrho_3, \ldots, \varrho_n$, determining the characters 2(4) of D.

Example 1. Let $G = \mathrm{SL}(2, C)$, i.e. G is the universal covering group of the Lorentz group SO(3, 1). The subgroup Z, by example 3.6.1, consists of the matrices z of the form

$$z = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}, \qquad z = x + \mathrm{i}y \text{ and } x, y \in R^1, \tag{17}$$

where we denoted the matrix element z_{21} by the same letter z as the matrix $z \in Z$.

The subgroup P consists of matrices of the form

$$\begin{bmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{bmatrix}, \quad \text{with} \quad k_{11}k_{22} = 1. \tag{18}$$

The carrier space $H^L \doteqdot L^2(Z, \mu)$ consists of the equivalence classes of measurable functions for which

$$\int |\varphi(z)|^2 \mathrm{d}\mu(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\varphi(z)|^2 \mathrm{d}x\mathrm{d}y < \infty. \tag{19}$$

Let

$$\mathrm{SL}(2, C) \ni g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha\delta - \beta\gamma = 1.$$

Then, by eqs. (5), (8), (9) and (10), one obtains

$$L_{\tilde{k}} = |\tilde{k}_{22}|^{m_2 + i\varrho_2} \tilde{k}_{22}^{-m_2}, \quad \tilde{z} = \frac{\alpha z + \gamma}{\beta z + \delta}, \quad \tilde{k}_{22} = \beta z + \delta \tag{20}$$

and

$$\frac{d\mu(\tilde{z})}{d\mu(z)} = |\tilde{k}_{22}|^{-4}.$$

Hence, the representation U^L given in eq. (16) takes the form

$$U_g^L \varphi(z) = \sqrt{\frac{d\mu(\tilde{z})}{d\mu(z)}} \cdot L_{\tilde{k}} \varphi(\tilde{z}) = |\beta z + \delta|^{m_2 + i\varrho_2 - 2} (\beta z + \delta)^{-m_2} \varphi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{21}$$

Note that by analytic continuation of the multiplier, eq. (21) can be made to coincide with the finite-dimensional induced representations of SL(2, C), 8.2(24), (cf. exercise 8.2.1).

Th. 1.3 shows that every representation U^L of the principal series is irreducible.

In order to state the equivalence properties of irreducible representations of SL(n, C), it is convenient to introduce another normalization of the invariant numbers m_i and ϱ_i, $i = 1, 2, \ldots, n$. Because $\prod_{s=1}^{n} k_{ss} = 1$ we can either set $m_1 = \varrho_1 = 0$ (as we did), or we can keep the element k_{11} in

$$\chi(\delta) = \prod_{s=1}^{n} (k_{ss})^{m_s + i\varrho_s} k_{ss}^{-m_s}$$

and introduce a more 'symmetric' normalization

$$\begin{aligned} m_1 + m_2 + \ldots + m_n &= 0, \\ \varrho_1 + \varrho_2 + \ldots + \varrho_n &= 0. \end{aligned} \tag{22}$$

The problem of the equivalence of irreducible representations is solved by the following.

THEOREM 3. *Two irreducible representations of the principal non-degenerate series are equivalent if and only if their collection of pairs of invariant numbers satisfying* (22)

$$(m_1, \varrho_1),\ (m_2, \varrho_2),\ \ldots,\ (m_n, \varrho_n)$$

and

$$(m_1', \varrho_1'),\ (m_2', \varrho_2),\ \ldots,\ (m_n, \varrho_n'),$$

which determine these irreducible representations can be obtained from each other by a permutation. ▼

PROOF: The Weyl group W permute characters; hence theorem follows from th. 1.2(iii).

D. *Reduction of the Principal Series of Representations to the Subgroup* SU(n)

We now consider the problem of determining the irreducible representations of SU(n) which occur in the restriction of a representation U^L of the principal series of SL(n, C) to the subgroup SU(n). This problem appears in many applications of group theory in particle physics. The following theorem gives the complete answer to this question. The proof of the theorem demonstrates again the power of the I-R Theorem.

THEOREM 4. *Let U^L be an irreducible representation of* SL(n, C) *induced by a one-dimensional representation L of the minimal parabolic subgroup P, and let T be an irreducible representation of* SU(n). *Let M be the subgroup $M = \mathrm{SU}(n) \cap \cap P$. Then, the multiplicity of T in $U^L_{\mathrm{SU}(n)}$ is equal to the multiplicity of the one-dimensional representation* L_M in T_M.*

PROOF: The theorem represents a special case of the I-R Theorem. Because every element of $g \in \mathrm{SL}(n, C)$ can be written in the form $g = ku$ (cf. eq. 2(2a)), there is only one double coset $P{:}\ \mathrm{SU}(n)$. Applying the I-R Theorem, we find that U^L restricted to SU(n) is the representation of SU(n) induced by the representation L restricted to the subgroup $\mathrm{SU}(n) \cap P = M$. Applying now the Frobenius Reciprocity Theorem we obtain the assertion of the th. 4. ▼

Remark 1: The representation L of P restricted to M has the form

$$M \ni \gamma = \begin{bmatrix} \exp(i\varphi_1) & \cdots & 0 \\ & \ddots & \vdots \\ 0 & & \exp(i\varphi_n) \end{bmatrix} \to L_\gamma = \exp[i(m_2\varphi_2 + \ldots + m_n\varphi_n)] . \quad (23)$$

Every irreducible representation T of SU(n) is uniquely determinined by its highest weight m and the weight diagram is associated with the highest weight m. The assertion of th. 4 states, in fact, that the multiplicity of T in the representation $U^L_{\mathrm{SU}(n)}$ is equal to the multiplicity of the weight $m_L = (m_2, m_3, \ldots, m_n)$ in the weight diagram associated with the highest weight m.

Remark 2: The restriction of the representation L of P to the subgroup M does not depend on the invariant numbers $\varrho_2, \ldots, \varrho_n$ because L is, in fact, the character of D (cf. eq. (4)). Consequently, the non-equivalent members of the principal

* L_M and T_M denote the restriction of the representation L of P and T of SU(n) to the subgroup M, respectively.

series with the same invariant numbers $m_2, \ldots, m_n$, but with different $\varrho_2, \ldots, \varrho_n$ have the same content with respect to SU(n). ▼

We can derive from th. 4 the following useful corollary:

COROLLARY. *The irreducible representation T of* SU(n) *corresponding to the lowest possible highest weight is contained in $U^L_{SU(n)}$ only once. This lowest highest weight is equal to the weight m_L.*

PROOF: Indeed, the representation of T, whose highest weight m is equal m_L satisfies the condition of th. 5 and therefore is contained in $U^L_{SU(n)}$. It is contained only once because the highest weight in any irreducible representation is nondegenerate. Any other representation T', in which m_L is not a highest weight, is determined by a highest weight m', which is higher than m_L. ▼

The following example illustrates the content of th. 4.

EXAMPLE 2. Let $G = \mathrm{SL}(2, C)$ and let U^L be a representation of the principal series induced by a representation L of P. The representation L is defined by the character $\chi(\delta) = |\delta|^{m_2+i\varrho_2}\delta^{-m_2}$, $\delta = \begin{bmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{bmatrix} \in D$. An irreducible representation T^J of SU(2) is determined by J, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ Let Y^J_m, $m = -J, -J+1, \ldots, J-1, J$, be a basis of the carrier space of T^J. It is well known that an element $\gamma \in M$,

$$\gamma = \begin{bmatrix} \exp(-i\varphi) & 0 \\ 0 & \exp(i\varphi) \end{bmatrix}$$

corresponds to the rotation around z-axis by an angle 2φ. Hence, $T^J_\gamma Y^J_m = \exp(2im\varphi) Y^J_m$, and every representation $\varphi \to \exp(2im\varphi)$ of M appears with multiplicity one. Consequently, T^J restricted to M (i.e., T^J_M) contains L_M if and only if $m_2/2$ is one of the numbers $J, J-1, \ldots, -J$. Thus if $J \geqslant m_2/2$, the representation T^J enters in $U^L_{SU(2)}$ with the multiplicity one, i.e.,

$$U^L_{SU(2)} = \sum_{J=\left|\frac{m_2}{2}\right|}^{\infty} \oplus T^J(\mathrm{SU}(2)). \tag{24}$$

Note that two non-equivalent representations U^L and $U^{L'}$, for which $\varrho_2 \neq \varrho_2'$ but $m_2 = m_2'$ have the same decomposition (24).

§ 4. Principal Degenerate Series of SL(n, C).

We give now a description of so-called *principal degenerate series*. These series have various degrees of degeneracy and are described by $2n-2k$, $k = 2, 3, \ldots, n-1$, invariant numbers respectively.

Let

$$n = n_1 + n_2 + \ldots + n_r, \quad r \geqslant 2, \; r \neq n, \tag{1}$$

be a partition of the integer n into positive integers and let

$$g = \begin{bmatrix} g_{11} & \cdots & g_{1r} \\ \cdots & \cdots & \cdots \\ g_{r1} & \cdots & g_{rr} \end{bmatrix} \tag{2}$$

be a decomposition of $g \in \mathrm{SL}(n, C)$ into matrices g_{pq}, $p, q = 1, 2, \ldots, r$, with n_p rows and n_q columns. We choose the matrix blocks g_{pq} in such a way that when inserted into the eq. (2) they give exactly the matrix $g \in \mathrm{SL}(n, C)$. We introduce, moreover, the matrices k and z, $k, z \in \mathrm{SL}(n, C)$, of the form

$$k = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1r} \\ 0 & k_{22} & \cdots & k_{2r} \\ & & & \vdots \\ 0 & \cdots & \cdots & k_{rr} \end{bmatrix}, \quad z = \begin{bmatrix} I_{n_1} & 0 & \cdots\cdots & 0 \\ z_{21} & I_{n_2} & 0 \;\cdots & 0 \\ \vdots & & & \\ z_{r1}, & z_{r2} & \cdots\cdots & I_{n_r} \end{bmatrix}, \tag{3}$$

where k_{pq} and z_{pq} are arbitrary matrices of dimension $n_p \times n_q$, and I_{n_k}, $k = 1, 2, \ldots, r$, are the square unit matrices of order n_k. The set of all matrices k and z given by eq. (3) are subgroups of $\mathrm{SL}(n, C)$, which we denote by $P_{n_1 n_2, \ldots, n_r}$ and $Z_{n_1 n_2, \ldots, n_r}$, respectively.

The following lemma provides a decomposition of $g \in \mathrm{SL}(n, C)$, which is analogous to the decomposition 1(2b).

LEMMA 1. *Almost every element $g \in \mathrm{SL}(n, C)$ can be uniquely represented in the form*

$$g = k_g z_g, \tag{4}$$

where $k_g \in P_{n_1 n_2, \ldots, n_r}$ and $z_g \in Z_{n_1 n_2, \ldots, n_r}$.

The proof is straightforward and we omit it.

The unitary degenerate representations of $\mathrm{SL}(n, C)$ are constructed in a similar manner as the nondegenerate ones. Therefore, we restrict ourselves to a discussion of the main steps only.

We shall construct unitary representations U^L of $\mathrm{SL}(n, C)$ induced by the one-dimensional representations $k \to L_k$ of the subgroup $P_{n_1 n_2, \ldots, n_r}$. Let $\Delta_j = \det k_{jj}$, where k_{jj} are $n_j \times n_j$ matrices given in eq. (3). Then the map

$$L: k \to \chi(k) = \prod_{s=2}^{r} |\Delta_s|^{m_s + i\varrho_s} \Delta_s^{-m_s}, \tag{5}$$

where $\varrho_2, \ldots, \varrho_r$ are arbitrary real numbers and $m_2, \ldots, m_r$ are arbitrary integers, gives the one-dimensional representation of the subgroup $D_{n_1, \ldots, n_r}$ consisting of all matrices of the form $\begin{bmatrix} k_{11} & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & k_{rr} \end{bmatrix}$. The subgroup $\mathfrak{Z}_{n_1 n_2, \ldots, n_r}$ consisting of all matrices of the form

$$\begin{vmatrix} I_{n_1} & k_{12} & \cdots\cdots & k_{1r} \\ & I_{n_2} & k_{23} \ \cdots & k_{2r} \\ 0 & & \ddots & \vdots \\ & & & I_{n_k} \end{vmatrix} \tag{6}$$

is a normal subgroup in $P_{n_1,\ldots,n_r}$, and $P_{n_1,\ldots,n_r} = \mathfrak{Z}_{n_1,\ldots,n_r} \circledS D_{n_1,\ldots,n_r}$. Hence, the representation (5) of $D_{n_1,\ldots,n_r}$ can be lifted to the one-dimensional representation L of the subgroup $P_{n_1,\ldots,n_r}$. The representation U^L of SL(n, C) induced by the representation L of $P_{n_1,\ldots,n_r}$ is realized in the Hilbert space $L^2(X, \mu)$, where $X = P_{n_1,\ldots,n_r}\backslash G$ and μ is a quasi-invariant measure in X. However, using the decomposition (4), one can show that the group space of $Z_{n_1,\ldots,n_r}$ coincides with X up to a subset of a smaller dimension in X (cf. lemma 3.1). Consequently, we can take the measure μ as the invariant measure on the subgroup $Z_{n_1,\ldots,n_r}$. This measure is induced by the measure 2(5) on Z and is given by the formula

$$d\tilde{\mu}(z) = \Pi\, dx_{pq}dy_{pq}, \tag{7}$$

where only those factors $dx_{pq}dy_{pq}$ occur, which correspond to the matrix elements z_{ij}, $i > j$, of the matrix $z \in Z_{n_1,\ldots,n_r}$.

A carrier space $L^2(Z_{n_1,\ldots,n_r}, \tilde{\mu})$ of a degenerate representation U^L consists of all functions $\varphi(z)$ measurable in $Z_{n_1,\ldots,n_r}$ which satisfy the condition

$$\int |\varphi(z)|^2 d\tilde{\mu}(z) < \infty. \tag{8}$$

The representation U^L of SL(n, C) is given explicitly in the space $L^2(Z_{n_1,\ldots,n_r}, \tilde{\mu})$ by the formula 3(3), i.e.,

$$U_g^L\varphi(z) = \sqrt{\left(\frac{d\tilde{\mu}(\tilde{z})}{d\tilde{\mu}(z)}\right)} L_{\tilde{k}}\, \varphi(\tilde{z}), \tag{9}$$

where $\tilde{k}$ and $\tilde{z}$ are factors of the decomposition (4) of an element $\tilde{g} \equiv zg$, i.e., $zg = \tilde{k}\tilde{z}$. To complete the construction of U^L it is still necessary to compute the Radon–Nikodym derivative $d\tilde{\mu}(\tilde{z})/d\tilde{\mu}(z)$. The computations analogous to that in Lemma 3.2 give

$$\frac{d\tilde{\mu}(\tilde{z})}{d\tilde{\mu}(z)} = |\Lambda_2|^{-2(n_1+n_2)}|\Lambda_3|^{-2(n_1+2n_2+n_3)}\ldots|\Lambda_r|^{-2(n_1+2n_2+\ldots+2n_{r-1}+2n_r)} \tag{10}$$

where Λ_s, $s = 2, \ldots, r$, denote the determinants of the corresponding block-diagonal elements of the element $\tilde{k}$.

Notice that the number of the invariant labels ϱ_s, m_s depends on the partition of n into n_i given by eq. (1). In the case $r = 2$ we obtain the *so-called most degenerate series* of SL(n, C).

EXAMPLE 1. Consider the most degenerate representations of SL(n, C), which are defined by the following partition of n

$$n \equiv n_1 + n_2 = (n-1)+1. \tag{11}$$

In this case the subgroups $Z_{n-1,1}$ and $P_{n-1,1}$ consists of matrices of the form

$$z = \begin{bmatrix} I_{n_1} & 0 \\ z & 1 \end{bmatrix}, \quad k = \begin{bmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{bmatrix} \tag{12}$$

Here, I_{n_1} is the $(n-1)\times(n-1)$ unit matrix,

$$z = (z_{n,1}, z_{n,2}, \ldots, z_{n,n-1}) \equiv (z_1, z_2, \ldots, z_{n-1})$$

is the $1\times(n-1)$ matrix, k_{11} is the $(n-1)\times(n-1)$ matrix, k_{12} is the $(n-1)\times 1$ matrix, and k_{22} is a complex number. Note that in the present case, $\Lambda_2 = \det k_{22} = k_{22}$.

Hence, by eqs. (10) and (5), we have

$$\sqrt{\frac{\mathrm{d}\tilde{\mu}(\tilde{z})}{\mathrm{d}\tilde{\mu}(z)}} = |\Lambda_2|^{m_2+i\varrho_2-(n_1+n_2)}\Lambda_2^{-m_2}$$
$$= |\tilde{k}_{22}|^{m_2+i\varrho_2-n}\tilde{k}_{22}^{-m_2}.$$

Therefore, in order to define explicitly the action of an operator U^L_g, we have to find the form of $\tilde{k}_{22}$ and $\tilde{z}$. Comparing the matrix element of the matrix $\tilde{g} = zg$ with those of the product $\tilde{k}\tilde{z}$, we obtain

$$(\tilde{z})_{np} \equiv (\tilde{z})_p = \Big(\sum_{j=1}^{n-1} g_{jp}z_j + g_{np}\Big)\Big/\Big(\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big) \tag{13}$$

and

$$\tilde{k}_{22} = \Big(\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big) \tag{14}$$

Therefore, the explicit form of the action of an operator U^L_g in the space $L^2(Z_{n-1,1}, \tilde{\mu})$ is given by the formula

$$U^L_g\varphi(\ldots, z_p, \ldots) = \Big|\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big|^{m_2+i\varrho_2-n} \times$$
$$\times \Big(\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big)^{-m_2} \varphi\Big(\ldots, \Big(\sum_{j=1}^{n-1} g_{jp}z_j + g_{np}\Big)\Big/\Big(\sum_{j=1}^{n-1} g_{jn}z_j + g_{nn}\Big), \ldots\Big). \tag{15}$$

Eq. (13) shows that SL(n, C) acts on the manifold $Z_{n-1,1}$ as a group of projective transformations, i.e., $Z_{n-1,1}$ is an $(n-1)$-dimensional projective space.

One can readily show, applying the I-R theorem, that every representation of the principal degenerate series is also irreducible. One can also derive the analogs of ths. 3.3 and 3.4 for degenerate series (cf. Gel'fand and Naimark 1950, ch. 3.4.)

§ 5. Supplementary Nondegenerate and Degenerate Series

We considered so far unitary representations U^L of a group G induced by *unitary* representations $k \to L_k$ of a subgroup $P \subset G$. These representations were realized by functions on G satisfying the condition

$$u(kg) = L_k u(g). \tag{1}$$

The action of U_g^L in the carrier space H^L was given by formula 3 (3) and the scalar product was defined by 3(2).

One could also try to obtain a unitary induced representation U^L of G from a *nonunitary* representation L of P. We now show the explicit construction of such representations. The new class of representations of G induced by nonunitary representations of P is the class of supplementary series of representations.

Clearly, the scalar product 3(2) for a nonunitary representation L of P cannot be invariant under U^L. It turns out, however, that it is sufficient to replace 3(2) by the scalar product $(\cdot, \cdot)$ of the form

$$(\varphi, \psi)_{H^L} = \int_{X\times X} K(x_1, x_2)\left(\varphi(x_1), \psi(x_2)\right)_H \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2), \tag{2}$$

$$x \in X = P\backslash G.$$

The kernel $K(x_1, x_2)$ is selected in such a manner that it compensates the additional factor resulting from the nonunitarity of the representation L of P.

A. *Supplementary Series for* SL(2, *C*)

First, we shall construct the supplementary series of representations for SL(2, C) (cf. example 3.1). In this case the subgroup P consists of matrices of the form

$$\begin{bmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{bmatrix} \quad \text{with } k_{11}k_{22} = 1.$$

We find unitary representations U^L of SL(2, C) induced by one-dimensional nonunitary representations of P given by formula

$$k \to L_k = |k_{22}|^{m+i\varrho} k_{22}^{-m}, \tag{3}$$

where now ϱ is not real.

Using 3(21) one obtains

$$U_g^L \varphi(z) = \sqrt{\frac{\mathrm{d}\mu(\tilde{z}}{\mathrm{d}\mu(z)}} L_{\tilde{k}}\, \varphi(\tilde{z}) = |\beta z + \delta|^{m+i\varrho-2}(\beta z+\delta)^{-m} \varphi\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right), \tag{4}$$

i.e., the action of U^L in the carrier space H^L is in fact the same as in case of the principal series. The scalar product in H^L will, however, be different. We find it using the invariance and positive definiteness requirements

$$(U_g^L\varphi, U_g^L\psi) = (\varphi, \psi) \equiv \int K(z_1', z_2')\varphi(z_1')\overline{\psi}(z_2')\mathrm{d}z_1'\,\mathrm{d}z_2' \tag{5}$$

where we set $\mathrm{d}\mu(z) = \mathrm{d}z \equiv \mathrm{d}x\mathrm{d}y$.

LEMMA 1. *The kernel* $K(z_1, z_2)$ *has the form*

$$K(z_1, z_2) = |z_1 - z_2|^{-2+\sigma}, \tag{6}$$

where $\sigma = -i\varrho$ *and* $0 < \sigma < 2$.

PROOF: Set $z_1' = \tilde{z}_1 = \dfrac{\alpha z_1+\gamma}{\beta z_1+\delta}$ and $z_2' = \tilde{z}_2$ in the right-hand side of eq. (5). Then, using the formula $\mathrm{d}\tilde{z}/\mathrm{d}z = |\beta z+\delta|^{-4}$ one obtains

$$(\varphi, \psi) = \int K(\tilde{z}_1, \tilde{z}_2)\varphi(\tilde{z}_1)\overline{\psi(\tilde{z}_2)}|\beta z_1+\delta|^{-4}|\beta z_2+\delta|^{-4}\mathrm{d}z_1\,\mathrm{d}z_2.$$

Further, putting expression (4) in the left-hand side of eq. (5), and by virtue of the arbitrariness of $\varphi(\tilde{z}_1)$ and $\psi(\tilde{z}_2)$, one obtains

$$K(z_1, z_2)|\beta z_1+\delta|^{m+i\varrho-2}(\beta z_1+\delta)^{-m}|\beta z_2+\delta|^{m-i\bar{\varrho}-2}\overline{(\beta z_2+\delta)}^{-m} = K\left(\frac{\alpha z_1+\gamma}{\beta z_1+\delta}, \frac{\alpha z_2+\gamma}{\beta z_2+\delta}\right)|\beta z_1+\delta|^{-4}|\beta z_2+\delta|^{-4}.$$

Consequently,

$$K\left(\frac{\alpha z_1+\gamma}{\beta z_1+\delta}, \frac{\alpha z_2+\gamma}{\beta z_2+\delta}\right) = K(z_1, z_2)|\beta z_1+\delta|^{m+i\varrho+2}(\beta z_1+\delta)^{-m}|\beta z_2+\delta|^{m-i\bar{\varrho}+2}\overline{(\beta z_2+\delta)}^{-m}. \tag{7}$$

For the particular value $g = z_0 = \begin{bmatrix}1 & 0\\ z_0 & 1\end{bmatrix}$ of g we obtain

$$K(z_1+z_0, z_2+z_0) = K(z_1, z_2).$$

And for $z_0 = -z_2$

$$K(z_1, z_2) = K(z_1-z_2, 0) \equiv K_1(z_1-z_2). \tag{8}$$

Using eqs. (8) and (7), one obtains

$$K_1\left(\frac{z_1-z_2}{(\beta z_1+\delta)(\beta z_2+\delta)}\right) = K_1(z_1-z_2)|\beta z_1+\delta|^{m+i\varrho+2}\times \\ \times(\beta z_1+\delta)^{-m}|\beta z_2+\delta|^{m-i\bar{\varrho}+2}\overline{(\beta z_2+\delta)}^{-m}. \tag{9}$$

Setting here $z_2 = 0$ and $\beta = \dfrac{1-\delta}{z_1}$, we have

$$K_1\left(\frac{z_1}{\delta}\right) = K_1(z_1)|\delta|^{m-i\bar{\varrho}+2}\bar{\delta}^{-m}. \tag{10}$$

Now, putting in eq. (9), $z_1 = 0$ and $\beta = \dfrac{1-\delta}{z_2}$, we obtain

$$K_1\left(-\frac{z_2}{\delta}\right) = K_1(-z_1)|\delta|^{m+i\varrho+2}\delta^{m}. \tag{11}$$

Eqs. (10) and (11), by virtue of arbitrariness of z_1 and z_2, imply

$$|\delta|^{-i\bar{\varrho}}\bar{\delta}^{-m} = |\delta|^{i\varrho}\delta^{-m}. \tag{12}$$

Setting here $\delta = \exp(i\theta)$, θ real, we obtain

$$\exp(im\theta) = \exp(-im\theta), \quad \text{i.e.,} \quad m = 0. \tag{13}$$

Therefore, by eq. (12), we have $\varrho = -\bar{\varrho}$, i.e., ϱ is a purely imaginary number, $\varrho = \mathrm{i}\sigma$, σ real. Putting in eq. (10), $\delta = z_1 \equiv z$ we have

$$K_1(z) = C(z)^{-2+\sigma}$$

and

$$K(z_1, z_2) = C(z_1 - z_2)^{-2+\sigma},$$

where $C = K_1(1)$ is an arbitrary constant.

The application of the standard Fourier analysis of functions of one complex variable shows that the scalar product (5) with the kernel (6) is positive definite only for $0 < \sigma < 2$ (cf. Naimark 1964, ch. III, § 12). ▼

Equations (13) and (4) imply the following expression for U_g^L

$$U_g^L \varphi(z) = |\beta z + \delta|^{-2-\sigma} \varphi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{14}$$

This formula defines unitary representations of SL(2, C), when $0 < \sigma < 2$. It is instructive to investigate, what representations we obtain, when $\sigma \geqslant 2$. In the case $\sigma = 2$ the scalar product (5) takes the form

$$(\varphi, \psi) = \iint \varphi(z_1) \bar{\psi}(z_2) \mathrm{d}z_1 \mathrm{d}z_2 . \tag{15}$$

In particular

$$(\varphi, \varphi) = \left|\int \varphi(z) \mathrm{d}z\right|^2 \geqslant 0$$

and

$$(\varphi, \varphi) = 0, \quad \text{if} \quad \int \varphi(z) \mathrm{d}z = 0. \tag{16}$$

It is natural to consider the set of functions $\varphi(z)$ on the manifold $Z = C^1$ as elements of a Hilbert space H', obtained by identifying functions with the same value of integral $\int \varphi(z) \mathrm{d}z$. The scalar product in H_1 is induced by the form (15); in fact, H' is one-dimensional. Indeed, if $\int \varphi_2(z) \mathrm{d}z \neq 0$, then setting

$$c = \frac{\int \varphi_1(z) \mathrm{d}z}{\int \varphi_2(z) \mathrm{d}z}, \qquad \varphi = \varphi_1 - c\varphi_2,$$

we obtain

$$\int \varphi(z) \mathrm{d}z = 0 \Rightarrow \varphi_1 = c\varphi_2,$$

i.e., any two elements of $H'(z)$ are linearly dependent. Using eq. (14) and setting $\sigma = 2$, we find

$$\int U_g^L \varphi(z) \mathrm{d}z = \int |\beta z + \delta|^{-4} \varphi\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right) \mathrm{d}z.$$

Passing to the variable $\tilde{z} = \dfrac{\alpha z+\gamma}{\beta z+\delta}$ and utilizing the Jacobian of the transformation $z \to \tilde{z}$ (eq. 2(20) and below), we obtain

$$\int U_g^L \varphi(z)\mathrm{d}z = \int \varphi(z)\mathrm{d}\tilde{z} = \int \varphi(z)\mathrm{d}z,$$

i.e.,

$$U_g^L \varphi(z) = |\beta z+\delta|^{-4}\varphi\left(\frac{\alpha z+\gamma}{\beta z+\delta}\right) = \varphi(z) \quad \text{for every } \varphi \in H'(Z) \tag{17}$$

or

$$U_g^L = 1.$$

It should be noted that this is the only representation of the Lorentz group which is unitary and finite-dimensional.

When we set $\sigma > 2$, we obtain a Hilbert space with an indefinite metric, which contains a finite number of negative squares. These are the so-called *Pontryagin spaces*. The representation theory in Pontryagin spaces was originated by Gel'fand and Naimark 1947 and recently systematically developed by Naimark and collaborators (cf. excellent review by Naimark and Ismagilov 1968).

B. *Supplementary Series for* SL(n, C)

The derivation of the explicit form of representations of supplementary series for SL(n, C) is similar to the one of SL$(2, C)$. We again start with the class of functions on SL(n, C) satisfying the condition $\varphi(kg) = L_k\varphi(g)$, where the representation $k \to L_k$ of K is now not unitary. We write an element $k \in K$ in the form

$$k = \begin{bmatrix} k_{11} & k_{12} & \cdots & & & & & & & k_{1n} \\ & k_{22} & & & & & & & & k_{2n} \\ & & \ddots & & & & & & & \vdots \\ & & & k_{n-2\tau,n-2\tau} & & & & & & \vdots \\ & & & & \lambda_1 & & & & & \vdots \\ & & & & & \mu_1 & & & & \vdots \\ & & & & & & \lambda_2 & & & \vdots \\ & 0 & & & & & & \mu_2 & & \vdots \\ & & & & & & & & \ddots & \vdots \\ & & & & & & & & \lambda_\tau & k_{n-1,n} \\ & & & & & & & & & \mu_\tau \end{bmatrix} \in P, \qquad \tau = 1, 2, \ldots, \left[\frac{n}{2}\right] \tag{18}$$

and take a nonunitary one-dimensional representation $k \to L_k$ of K of the form

$$L_k = \prod_{p=2}^{n-2\tau} |k_{pp}|^{m_p + i\varrho_p} k_{pp}^{-m_p} \prod_{q=1}^{\tau} |\lambda_q|^{m_q' + i\sigma_q' + \sigma_q''} \lambda^{-m_q'} |\mu_q|^{m_q' + i\sigma_q' - \sigma_q''} \mu_q^{-m_q'}, \tag{19}$$

$$\varrho_p,\ \sigma_q',\ \sigma_q'' \text{ real}.$$

Clearly, if $\sigma_q'' = 0$, for $q = 1, 2, \ldots, \tau$, then L_k becomes a unitary representation. It is also evident that the number τ represents 'the degree of nonunitarity' of the representation $k \to L_k$; for $\tau = 0$ we obtain a unitary representation of P; for $\tau = \left[\frac{n}{2}\right]$ we have a 'maximally' nonunitary one.

The action of the representation $g \to U_q^L$ for supplementary series is given by the standard formula 3(3). It remains only the derivation of the form of the invariant scalar product (2). It is similar to the derivation in the case of SL(2, C), but fairly long (cf. Gel'fand and Naimark 1950, ch. IV). Therefore, we restrict ourselves to a presentation of the final results.

THEOREM 2. *The representation* $g \to U_q^L$ *of the supplementary, nondegenerate series of* SL(n, C) *induced by a representation* $k \to L_k$ *of* P *given by eq.* (19) *is defined by the formula*

$$U_g^L \varphi(z) = \sqrt{\frac{d\mu(\tilde{z})}{d\mu(z)}}\, L_{\tilde{k}} \varphi(\tilde{z}), \tag{20}$$

where $\tilde{k} \in P$ *and* $\tilde{z} \in Z$ *are given by eq.* 3(7) *and* R-N *derivative by eq.* 3(10).

The invariant scalar product for the representation (20) *is given by the formula*

$$(\varphi, \psi) = \int K(\dot{z}) \varphi(z) \overline{\psi(\dot{z}z)}\, d\mu(\dot{z})\, d\mu(z), \tag{21}$$

where $\dot{z}$ *is an element of* Z *of the form*

$$\dot{z} = \left[\begin{array}{c|cccc} \underbrace{\begin{matrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \ldots & \ldots & \\ 0 & \ldots & \ldots & 1 \end{matrix}}_{n-2\tau} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} n-2\tau & & 0 & & \\ \hline & \begin{matrix} 1 & 0 \\ z_1 & 1 \end{matrix} & & 0 & \\ & & \begin{matrix} 1 & 0 \\ z_2 & 1 \end{matrix} & 0 & \\ 0 & & & \ddots & \\ & & & & \begin{matrix} 1 & 0 \\ z_\tau & 1 \end{matrix} \end{array}\right] \tag{22}$$

and

$$K(\dot{z}) = \prod_{j=1}^{\tau} |z_j|^{2(\sigma_j''-1)},$$
$$0 < \sigma_j'' < 1, \quad \mathrm{d}\mu(\dot{z}) = \prod_{p=1}^{\tau} \mathrm{d}x_p \mathrm{d}y_p. \tag{23}$$

The invariant numbers, which define a given representation of the supplementary nondegenerate series are the integers $m_1, m_2, \ldots, m_{n-2\tau}, m_1', m_2', \ldots, m_\tau'$ and the real numbers $\varrho_1, \varrho_2 \ldots, \varrho_{n-2\tau}, \sigma_1', \sigma_2', \ldots, \sigma_\tau', \sigma_1'', \sigma_2'', \ldots, \sigma_\tau''$, $0 < \sigma_p'' < 1$, $p = 1, 2, \ldots, \tau$. One readily verifies that for $n = 2$ the formula (20) coincides with (14) and the invariant scalar product (21), after changing variables coincides with (5).

C. *The Supplementary Degenerate Series of* SL(n, C)

The supplementary degenerate series of representations are constructed in a similar manner. We start with a class of functions on SL(n, C) satisfying the condition

$$\varphi(kg) = L_k \varphi(g), \quad k \in P_{n_1, n_2, \ldots, n_r}, \tag{24}$$

where a representation $k \to L_k$ of $P_{n_1, n_2, \ldots, n_r}$ is again a nonunitary one. We write it in the form (cf. eq. 4(5))

$$L_k = \prod_{p=2}^{r-2\tau} |\Lambda_p|^{m_p + \mathrm{i}\varrho_p} \Lambda_p^{-m_p} \prod_{q=1}^{\tau} |\lambda_q|^{m_q' + \mathrm{i}\sigma_q' + \sigma_q''} \lambda_q^{-m_q'} |\mu_q|^{m_q' + \mathrm{i}\sigma_q' - \sigma_q''} \mu^{-m_q'}, \tag{25}$$

where $p = \det k_{pp}$, $p = 2, 3, \ldots, r-2\tau$, and $\lambda_q, \mu_q, q = 1, 2, \ldots, \tau$, are complex numbers which represent the 2τ last diagonal elements of $k \in P_{n_1, n_2, \ldots, n_r}$ (cf. eq. 3(3)). The number τ represents 'the degree of nonunitarity' of representation L_k, as before.

THEOREM 3. *The representation $g \to U_g^L$ of the supplementary degenerate series of* SL(n, C) *induced by the representation $k \to L_k$ of $K_{n_1, n_2, \ldots, n_r}$ given by eq.* (25) *is defined by the formula*

$$U_g^L \varphi(z) = \sqrt{\frac{\mathrm{d}\tilde{\mu}(\tilde{z})}{\mathrm{d}\tilde{\mu}(z)}} L_{\tilde{k}} \varphi(\tilde{z}), \tag{26}$$

where $\tilde{k} \in P_{n_1, n_2, \ldots, n_r}$ and $\tilde{z} \in Z_{n_1, n_2, \ldots, n_r}$ are factors of the canonical decomposition 4(4) *of the element $\tilde{g} = zg$, i.e., $\tilde{g} = zg = \tilde{k}\tilde{z}$ and* $\mathrm{d}\tilde{\mu}(\tilde{z})/\mathrm{d}\mu(z)$ *is given by eq.* 4(10). ▼

The expressions for the invariant scalar product for the representation (26) and for the variable $\dot{z}$ are the same as for the supplementary nondegenerate series, but the element z should be taken from $Z_{n_1, \ldots, n_r}$.

§ 6. Comments and Supplements

1. The construction of irreducible unitary representations of other semisimple Lie groups can be carried out as in case of SL(n, C) using the general construction presented in sec. 1. In particular, the construction of principal supplementary and degenerate series for SO(n, C) and Sp(n, C) was given by Gel'fand and Naimark 1950. The properties of induced irreducible representations of SO(n, 1) groups were considered by Hirai 1962. The problem of construction of all irreducible representations of SO(p, q), SU(p, q) and Sp(p, q) groups is still not completed so far: partial results were given by Graev 1954, Leznov and Fedoseev 1971 and Leznov and Savelev 1976. The construction of various series of representations of semisimple Lie groups, e.g. principal induced from cuspidal parabolic subgroup is discussed by Lipsman 1974.

2. The problem of a classification of irreducible unitary representations of semisimple Lie groups is in general open. Naimark 1954 published two papers in which he claimed that he would give a complete description of all unitary irreducible representations of complex classical groups: however, the next papers have not appeared so far. In the meantime Stein 1967 has shown that by the method of analytic continuation one can construct new irreducible unitary representations which were not contained in Gel'fand and Naimark classification of (1950) paper.

The full classification of all irreducible unitary representation is known only for several low-dimensional groups like SL(2, R) and SL(2, C) (cf. Gel'fand, Graev and Vilenkin 1966). Dixmier in 1961 published a complete classification in case of De Sitter group SO(4, 1).

3. The existence and properties of irreducible representations of so-called discrete series has been attracting considerable attention in recent years: an irreducible representation $g \to U_g$ of G in H is called *discrete* if there exists a non-zero vector u in H such that the matrix element $(u, U_g u)$ is square integrable on G. The set G_d of all discrete inequivalent irreducible representation of G is called the *discrete series*. Harish-Chandra in two long papers in 1965 and 1966 gave a description of discrete series for semisimple Lie groups (cf. also Warner 1972, I and II). He showed in particular that G has a discrete series if and only if $\operatorname{rank} G = \operatorname{rank} K$; this implies in particular that $U(p, q)$ groups have a discrete series. Special class of discrete series representations for $U(p, q)$ groups was constructed by Graev 1954. The most degenerate discrete series representations for $U(p, q)$ and $SO(p, q)$ were constructed by Rączka and Fischer 1966 and by Rączka, Limić and Niederle 1966 respectively.

4. The properties of irreducibility of representations of semisimple Lie groups induced from parabolic subgroup were discussed in the fundamental paper of Bruhat 1956. The extension of these results were obtained by Wallach in a series of papers 1969, 1971.

5. We now give the important results of Scull 1973, 1976, concerning spectra

of generators of the Lie algebra L of a simple Lie group G. Let $\int \lambda \mathrm{d}E(\lambda)$ be the spectral resolution of $-\mathrm{i}U(X), X \in L$ and let the projection-valued measure $E(\cdot)$ be absolutely continuous with respect to Lebesgue measure $\mu(\cdot)$ on R. Then we say that $-\mathrm{i}U(X)$ has *two-sided spectrum* if $\operatorname{supp}\mu$ is $(-\infty, \infty)$ and *one-sided spectrum* if $\operatorname{supp}\mu$ is either $(-\infty, 0)$ or $(0, \infty)$. It turns out that spectrum of a noncompact generator $-\mathrm{i}U(X)$ depends on G but not on X or U. Indeed we have

THEOREM 1. *Let U be a continuous irreducible unitary representation of a connected simple Lie group G, such that G is not a group of automorphisms of an irreducible hermitian symmetric space. Then if $X \in L$ generates a noncompact one-parameter subgroup* $\exp(tX)$, $-\mathrm{i}U(X)$ *has two-sided spectrum.* ▼

We recall that an irreducible symmetric space X is hermitian if it is of the form $X = K\backslash G$ where G is a noncompact simple Lie group with trivial centre and K is a maximal compact subgroup with nondiscrete nontrivial center. The inspection of Table 4.2.I shows that among the classical Lie groups the following ones are groups of automorphisms of irreducible hermitian symmetric spaces: $\mathrm{SL}(2, R)$, $\mathrm{SU}(p, q)$, $\mathrm{SO}_0(p, 2)$, $\mathrm{Sp}(n, R)$ and $\mathrm{SO}^*(2n)$. Hence the corresponding noncompact generators have two-sided spectrum.

In some, rather exceptional cases, a noncompact generator may have a one-sided spectrum. Indeed we have:

THEOREM 2. *Let G be a connected simple Lie group of automorphisms of an irreducible hermitian symmetric space. There exists an $X \in L$ such that* $-\mathrm{i}U(X)$ *has one-sided spectrum for any representation U of the holomorphic discrete series.* ▼

For various generalizations of these results cf. Scull 1973 and 1976. The detailed analysis of holomorphic discrete series of representations is given in the work of Rossi and Vergne 1973.

§ 7. Exercises

§ 1.1.*** Analyze irreducibility properties of representations $U^{\chi L}$ of $\mathrm{SO}(p, q)$ and $\mathrm{SU}(p, q)$ induced from the minimal parabolic subgroups.

Hint: Use th. 1.4.

§ 1.2.*** Classify irreducible unitary representations of the conformal group $\mathrm{SO}(4, 2)$.

Hint: Use results of sec. 1 and extend technique of Dixmier 1961.

§ 3.1. Show that the Casimir operators of $\mathrm{SL}(2, C)$ in the carrier space of irreducible representation $[m, \varrho]$ have the eigenvalues

$$C_2\psi = -\tfrac{1}{2}(m^2-\varrho^2-4)\psi,$$
$$C_2'\psi = m\varrho\psi,$$

where

$$C_2 = \tfrac{1}{2} M_{\mu\nu} M^{\mu\nu} = J^2 - N^2,$$
$$C_2' = -\tfrac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} M^{\alpha\beta} M^{\gamma\delta} = J \cdot N$$

and

$$J = (M_{32}, M_{13}, M_{21}), \quad N = (M_{01}, M_{02}, M_{03}).$$

§ 3.2. Show that the two irreducible representations $[m, \varrho]$ and $[-m, -\varrho]$ of SL(2, C) are equivalent.

§ 3.3. Show that the representation $U^{*}_{g^{-1}}$ conjugate-contragradient to an irreducible representation $U_g^{[m,\varrho]}$ of SL(2, C) is irreducible and is defined by the parameters $[m, -\varrho]$.

Note: Because the representations $[m, \varrho]$ and $[-m, -\varrho]$ are equivalent, the conjugate-contragradient representation $(U_g^{[m,\varrho]})^*$ is equivalent to $U^{[m,\varrho]}$ if and only if either $m = 0$ or $\varrho = 0$.

§ 3.4.* Let $U^{(j_0, j_1)}$ be an irreducible representation of SL(2, C) and let $|j_0, j_1 : JM\rangle \equiv e_{JM}$ be the canonical basis associated with the set C_2, C_2', J^2 and J_3 of commuting operators of SL(2, C).* Show that the matrix elements of generators $J_\pm = J_1 \pm \mathrm{i} J_2$, J_3, $N_\pm = N_1 \pm \mathrm{i} N_2$ and N_3 have the form

$$\begin{aligned}
J_3 e_{JM} &= M e_{JM},\\
J_- e_{JM} &= \sqrt{[(J+M)(J-M+1)]}\, e_{J,M-1},\\
J_+ e_{JM} &= \sqrt{[(J+M+1)(J-M)]}\, e_{J,M+1},\\
M &= -J, -J+1, \ldots, J-1, J.
\end{aligned}$$

$$\begin{aligned}
N_3 e_{JM} &= C_J \sqrt{(J^2-M^2)}\, e_{J-1,M} - A_J M e_{JM} - C_{J+1} \sqrt{[(J+1)^2 - M^2]}\, e_{J+1,M},\\
N_+ e_{JM} &= C_J \sqrt{[(J-M)(J-M-1)]}\, e_{J-1,M+1} -\\
&\quad - A_J \sqrt{[(J-M)(J+M+1)]}\, e_{J,M+1} +\\
&\quad + C_{J+1} \sqrt{[(J+M+1)(J+M+2)]}\, e_{J+1,M+1},\\
N_- e_{JM} &= -C_J \sqrt{[(J+M)(J+M-1)]}\, e_{J-1,M-1} -\\
&\quad - A_J \sqrt{[(J+M)(J-M+1)]}\, e_{J,M-1} -\\
&\quad - C_{J+1} \sqrt{[(J-M+1)(J-M+2)]}\, e_{J+1,M-1},
\end{aligned}$$

where

$$A_J = \frac{i j_0 j_1}{J(J+1)}, \quad C_J = \frac{i}{J} \sqrt{\left[\frac{(J^2-j_0^2)(J^2-j_1^2)}{4J^2-1}\right]}, \quad J = j_0, j_0+1, \ldots,$$
$$M = -J, -J+1, \ldots, J-1, J, \quad J = j_0, j_0+1, \ldots,$$
$$e_{JM} \in H^{(J)}, \quad H = \bigoplus_{J=j_0}^{\infty} H^{(J)}.$$

Show further that (a) for j_1 pure imaginary, j_0 non-negative half integers we have the principal series, (b) for j_1 real, $0 \leqslant j_1 \leqslant 1$, $j_0 = 0$ supplementary series,

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* The parameters j_0 and j_1 which characterize irreducible unitary representations of SL(2, C) are connected with m, ϱ by the formulas:

$$j_0 = \left|\frac{m}{2}\right|, \quad j_1 = -\mathrm{i}\,(\operatorname{sign} m)\frac{\varrho}{2} \quad \text{for } m \neq 0,$$
$$j_0 = 0, \quad j_1 = \pm \mathrm{i}\frac{\varrho}{2} \quad \text{for } m = 0.$$

(c) for $j_1^2 = (j_0+n)^2$ for some integer n, we have finite-dimensional representations.

§ 5.1.*** Let $U^{(0,j_1)} \otimes U^{(0,j_1')}$ be the tensor product of irreducible representations of the supplementary series of SL(2, C). Find the Clebsch–Gordan coefficients

Hint: Use the technique of generalized projection operators developed in 15.4, cf. also Anderson, Rączka, Rashid and Winternitz 1970 a, b for a solution of a similar problem for representations of the principal series.

Chapter 20

Applications of Induced Representations

We present here two interesting applications of the general theory of induced representations. In sec. 1 we discuss the concept of localizability in relativistic quantum mechanics. We derive also the explicit form of the relativistic position operator. In sec. 2 we discuss the problem of the representations of the Heisenberg canonical commutation relations for finite number of degrees of freedom. We show here the uniqueness of the Schrödinger representation of the canonical commutation relations in the global Weyl form. We discuss also the problem of the equivalence of the Heisenberg and the Schrödinger formulations of quantum mechanics.

§ 1. The Relativistic Position Operator

We shall discuss in this section two basic concepts of relativistic quantum mechanics: the localizability and the position operators of relativistic systems. In subsec. A we introduce, using the concept of the imprimitivity system for the Euclidean group, the notion of localizability. In subsec. B we derive the explicit form of the position operators for a relativistic system, which transforms according to an irreducible representation of the Poincaré group.

A. *Localizable Relativistic Systems*

We begin with a review of the properties of the non-relativistic position operator. In non-relativistic quantum mechanics the position observables are defined by the formula

$$(q_k\psi)(x) = x_k\psi(x), \qquad k = 1, 2, 3, \tag{1}$$

where ψ is the wave function of the particle in the Hilbert space $H = L^2(R^3, \mathrm{d}^3x)$. The Fourier transform of the operator q_k is

$$Fq_kF^{-1} = \mathrm{i}\frac{\partial}{\partial p_k}. \tag{2}$$

This operator is hermitian with respect to the scalar product

$$(\varphi, \psi) = \int \varphi(p)\overline{\psi}(p)\,\mathrm{d}^3p.$$

On the other hand, for a relativistic particle of mass m the invariant scalar product in the Hilbert space $L^2(h^m, \mu)$, $\mathrm{d}\mu^{(p)} = \mathrm{d}^3p/p_0$, h^m = mass hyperboloid, $p^2 = m^2$ is given by

$$(\varphi, \psi) = \int\limits_{h^m} \varphi(p)\overline{\psi}(p)\frac{\mathrm{d}^3 p}{p_0}. \tag{3}$$

Hence q_k of eq. (1) is not hermitian with respect to this scalar product, for

$$\begin{aligned}(q_k\varphi, \psi) &= \mathrm{i}\int \left(\frac{\partial}{\partial p^k}\varphi\right)(p)\overline{\psi}(p)\frac{\mathrm{d}^3p}{p_0}\\ &= \int \varphi(p)\left[\left(-\mathrm{i}\frac{\partial}{\partial p^k} + \mathrm{i}\frac{p_k}{p^2+m^2}\right)\overline{\psi}\right](p)\frac{\mathrm{d}^3p}{p_0}\\ &\neq (\varphi, q_k\psi).\end{aligned}$$

Consequently, the operator $q_k = \mathrm{i}\dfrac{\partial}{\partial p^k}$ cannot represent the position operator for a relativistic particle.

Thus, for wave functions defined on the hyperboloid, the position operator cannot be obtained by a simple Fourier transform characteristic of R^3.

We shall extend now the concept of the position operator to relativistic systems using the Imprimitivity Theorem. In fact: suppose that we have found the set of three commuting, self-adjoint operators Q_1, Q_2, Q_3, which represent the position operators of a relativistic particle of mass m and spin J. Then, by the Spectral Theorem, there exists a common spectral measure $E(S)$, $S \subset R^3$, such that every operator Q_i has the representation

$$Q_i = \int\limits_{R^3} x_i \mathrm{d}E(x). \tag{4}$$

If $S \subset R^3$ and $\psi(x)$ represents the state of the particle in the Hilbert space H, then the expression

$$p(S) = \frac{||E(S)\psi||^2}{||\psi||^2}$$

represents the probability of measuring the position of the particle in the state ψ to be inside the set S.

The spectral measure $E(S)$, which defines the operators Q_i, is strongly limited by Euclidean invariance. In fact, let $\mathscr{E}^3$ denote the Euclidean group in R^3 and let $\mathscr{E}^3 \ni g \to U_g$ be a unitary representation of $\mathscr{E}^3$ in the Hilbert space H. Then, the Euclidean invariance of probability $p(S) \to p(gS)$ implies that

$$U_g E(S) U_g^{-1} = E(gS), \quad g \in \mathscr{E}^3. \tag{5}$$

Because the space R^3 is transitive relative to the group $\mathscr{E}$, eq. (5) means that the spectral function $E(\cdot)$ represents a transitive system of imprimitivity based on the space R^3.

Thus the existence of a position operator Q implies the existence of a transitive system of imprimitivity for the unitary representation of the Euclidean subgroup. This suggests the following definition of the localizability of a relativistic system.

DEFINITION 1. A representation U of the Poincaré group Π defines a *localizable system*, if and only if the restriction $U_{\mathscr{E}}$ of U to the Euclidean group $\mathscr{E}^3$ possesses a transitive system of imprimitivity $E(S)$ based on the space R^3. ▼

Let us note that the condition (5) will be satisfied if the representation $U_{\mathscr{E}}$ of $\mathscr{E}^3$ is induced from SO(3): in this case $X = \mathscr{E}^3/\mathrm{SO}(3) \cong R^3$ and $E(S)$ can be taken as the canonical spectral measure 16.3(1). It remains therefore to verify if the reduction of a representation U of the Poincaré group to $\mathscr{E}^3$ gives the representation $U_{\mathscr{E}^3}$ of the Euclidean group, which is induced from a certain representation L of SO(3). The following proposition shows that this is indeed the case for irreducible representations of Π associated with massive particles of arbitrary spin.

PROPOSITION 1. *Let* $(a, \Lambda) \to U^{m,J}_{(a,\Lambda)}$ *be a unitary irreducible representation of the Poincaré group corresponding to a particle of mass m with arbitrary spin J. Let* χD^J *denote the irreducible representation of* $K = T^4 \circledS \mathrm{SO}(3)$ *used for the induction of* $U^{m,J}$. *The restriction* $U^{m,J}_{\mathscr{E}^3}$ *of* $U^{m,J}$ *to* $\mathscr{E}^3$ *is unitarily equivalent to a representation induced by the representation of* SO(3) *given by the direct integral of irreducible representations* L^J *of* SO(3) *with* $L^J \cong D^J$. ▼

PROOF: In th. 18.1.2 take $G = N \circledS M$ to be the Poincaré group and $W = N_0 \circledS M_0$ to be $\mathscr{E}^3 = T^3 \circledS \mathrm{SO}(3)$.

Then $K = T^4 \circledS \mathrm{SO}(3)$ and by 18.1(15) the restriction $U^{m,J}_{\mathscr{E}^3}$ of $U^{m,J}$ is the direct integral

$$U^{m,J}_{\mathscr{E}^3} \cong \int_{\mathscr{D}} U^J_{\mathscr{E}^3}(D)\,\mathrm{d}\nu(D) \tag{9}$$

where $\mathscr{D}$ is the space of double cosets $K{:}\ \mathscr{E}^3$.

To determine the content of the representation $U^{m,J}_{\mathscr{E}^3}$ we use the prescription given by item 2° of th. 18.1.2. In our case $M_{\hat{n}} = \mathrm{SO}(3)$, $M_{\hat{n}m} = m^{-1}M_{\hat{n}}m \sim \mathrm{SO}(3)$ and $M_0 = \mathrm{SO}(3)$: hence $M_{\hat{n}m} \cap M_0 \sim \mathrm{SO}(3)$. It follows from the definition that the representation $L^{(m)}_r = D^J(m)D^J(r)D^J(m^{-1})$. The character being the restriction of $\hat{n} = (m, 0, 0, 0)$ to N_0 is identically zero. Consequently $\chi L^{(m)}$ is equivalent to D^J representation: therefore the representation $\int L^\lambda d\varrho(\lambda)$ in th. 18.1.2.2° is irreducible and equivalent to D^J; this implies that the representation $U_{\mathscr{E}^3}(D)$ given by eq. 18.1(16) is equivalent to the representation of $\mathscr{E}^3$ induced by the representation D^J of SO(3). By virtue of 18.1(15) the representation $U^{m,J}_{\mathscr{E}^3}$ is the direct integral over the space of double cosets $K{:}\ W$ of above (equivalent) representations of $U_{\mathscr{E}^3}(D)$ of $\mathscr{E}^3$. ▼

Finally, theorem 16.2.1 implies that the representation (9) of $\mathscr{E}^3$ is unitary equivalent to a representation of $\mathscr{E}^3$ induced by a representation of SO(3) given by the direct integral over the set of double cosets K: W of representations equivalent to D^J. ▼

Consequently, massive particles are localizable in the sense of the def. 1.

Theorem 18.1.2 also allows us to see the precise content of the reduction $U^{m,J}_{\mathscr{E}^3}$ of other representations $U^{m,J}$ of the Poincaré group ($m^2 \leqslant 0$, or $m^2 < 0$) when restricted to $\mathscr{E}^3$. In these cases $U^{m,J}_{\mathscr{E}^3}$ is not an induced representation from SO(3), hence they are not localizable in the sense of the definition 1, except for the case $m = 0$, $J = 0$ in which case the representation of SO(3) is trivial.

We can and perhaps we should change the def. 1 of localizability in the cases $m^2 = 0$, $m^2 < 0$. Physically these systems cannot be localized in $\mathscr{E}^3/\mathrm{SO}(3) \sim R^3$ but in $\mathscr{E}^3/\mathscr{E}(2)$ or $\mathscr{E}^3/\mathrm{SO}(2,1)$, hence the corresponding imprimitivity system must be based on these spaces. (See exercise 1.3).

Finally the non-uniqueness of E, is determined by the existence of unitary operators V which commute with $U^{m,J}_{\mathscr{E}^3}$ but not with $E(S)$. Then all other systems of imprimitivity are given by $F(S) = VE(S)V^{-1}$. The position operator is unique only under further assumptions of time-reversal invariance and of regularity of $E(S)$.

It follows immediately from Euclidean covariance, eq. (6), and eq. (4), that the transformation property of the position operators Q_i is given by

$$U_{\{a,R\}} Q_i U^{-1}_{\{a,R\}} = D^1_{ij}(R)[Q_j - a_j]. \tag{7}$$

Infinitesimally, eq. (7) means

$$[Q_i, P_j] = \mathrm{i}\delta_{ij}, \tag{8}$$

and

$$[Q_j, J_k] = \mathrm{i}\varepsilon_{jkl} Q_l, \tag{9}$$

which are physically desired relations. Thus the Heisenberg commutation relations (7) may be viewed as one of the implications of the existence of a transitive imprimitivity system for $\mathscr{E}^3$.

B. *Construction of Relativistic Position Operators*

We now derive the explicit form of position operators Q_i, $i = 1, 2, 3$, for a relativistic particle with mass $m > 0$ and spin J. We shall work in momentum representation. The action of the operator $U^{m,J}_{\{a,\Lambda\}}$, $a \in T^4$, $\Lambda \in \mathrm{SL}(2, C)$, in the carrier space $H^{m,J}$ of a particle $[m, J]$ is given by the formula (cf. eq. 17.2 (39))

$$(U^{m,J}_{\{a,\Lambda\}}\psi)(p) = \exp(\mathrm{i}pa) D^J(\Lambda_p^{-1}\Lambda\Lambda_{L_\Lambda^{-1}p})\psi(L_\Lambda^{-1}p). \tag{10}$$

Here, $\psi(p) = \{\psi_k(p)\}_1^{2J+1}$ is a $(2J+1)$-component vector function on the positive mass hyperboloid h^m, $p^2 = m^2$, square integrable with respect to the invariant measure $d\mu(p) = d^3p/p_0$. The element $\Lambda_p \in SL(2, C)$ is defined by the Mackey decomposition of $SL(2, C)$ (cf. eq. 17.2 (33))

$$\Lambda = \Lambda_p r, \quad \Lambda_p = \begin{bmatrix} \lambda & z \\ 0 & \lambda^{-1} \end{bmatrix}, \quad \lambda \in R^1,\ z \in C^1,\ r \in SU(2). \tag{11}$$

Hence, if $\Lambda \in SU(2)$, then $\Lambda_p = I$. Consequently, the restriction $U^{m,J}_{\mathscr{E}^3}$ of the representation $U^{m,J}_{(a,\Lambda)}$ of the Poincaré group Π to the (covering of the) Euclidean group $\mathscr{E}^3$ gives

$$(U^{m,J}_{(a,r)}\psi)(p) = \exp(ip\cdot a)D^J(r)\psi(R_r^{-1}p), \tag{12}$$

where $R_r \in SO(3)$ is the rotation corresponding to an element $r \in SU(2)$.

We shall now introduce the position coordinates x_k as follows.

Consider the operator V defined by

$$(V\psi)(x) \equiv (2\pi)^{-3/2}\int p_0^{1/2}\exp(ipx)\psi(p)\frac{d^3p}{p_0}. \tag{13}$$

Note that the integral (13) is not the ordinary three-dimensional Fourier transform. The operator V is a unitary operator. We show this explicitly for $J = 0$. The inverse transformation V^{-1} is given by

$$(V^{-1}\varphi)(p) = (2\pi)^{-3/2}\int p_0^{1/2}\exp(-ipx)\varphi(x)d^3x. \tag{14}$$

Let F denote the ordinary three-dimensional Fourier transform. Then

$$\|V\psi\|^2 = \int|V\psi|^2d^3x = \int|F(p_0^{-1/2}\psi)|^2d^3x = \int|p_0^{-1/2}\psi|^2d^3p = \int|\psi|^2\frac{d^3p}{p_0} = \|\psi\|^2.$$

Thus, V is the isometric transformation from $L^2(h^m, d\mu(p))$ into $L^2(R^3, d^3x)$. The set of functions $\{p_0^{1/2}H_l(p_1)H_k(p_2)H_n(p_3)\}$ forms a basis for the space L^2 $(h^m, d\mu)$, where H_i is the ith Hermite function of a single variable. Using eq. (13) one obtains

$$\begin{aligned}(Vp_0^{1/2}H_lH_kH_n)(x) &= (2\pi)^{-3/2}\int\exp(ipx)p_0H_lH_kH_n d^3p/p_0 \\ &= i^{l+k+n}H_l(x_1)H_k(x_2)H_n(x_3).\end{aligned} \tag{15}$$

Hence, V has a dense range in $L^2(R^3, d^3x)$. Consequently, V is a unitary mapping of $L^2(h^m, d\mu)$ onto $L^2(R^3, d^3x)$.

For an arbitrary $J > 0$ the proof runs similarly. Thus, the scalar product for $(V\psi)(x)$ is as in the non-relativistic quantum mechanics.

Set now

$$U^L_{(a,r)} \equiv VU_{(a,r)}V^{-1}. \tag{16}$$

Using eqs. (13), (12) and (14), one obtains

$$(U^L_{(a,r)}\psi)(x) = D^J(r)\psi(R_r^{-1}(x-a)). \tag{17}$$

We now construct explicitly a transitive system of imprimitivity $E^L(S)$ based on R^3. For this purpose let us define the canonical spectral measure $R^3 \supset S \to \to E^L(S)$ by the formula

$$\big(E^L(S)\psi\big)(x) \equiv \chi_S(x)\psi(x), \tag{18}$$

where χ_S is the characteristic function of the set S.

Using eqs. (17) and (18) we obtain

$$U_g^L E^L(S) U_g^{L^{-1}} = E^L(gS), \quad g = (a, r). \tag{19}$$

Indeed:

$$\begin{aligned}\big(U_g^L E^L(S) U_{g^{-1}}^L \psi\big)(x) &= \big(D^J(r) E^L(S) U_{g^{-1}}^L \psi\big)\big(R_r^{-1}(x-a)\big) \\ &= D^J(r)\chi_S\big(R_r^{-1}(x-a)\big) U_{g^{-1}}^L \psi\big(R_r^{-1}(x-a)\big) = E^L(gS)\psi(x).\end{aligned}$$

The position operators Q_l are now defined by eq. (4) with the spectral measure (18), and consequently satisfy

$$(VQ_k\psi)(x) = x_k(V\psi)(x). \tag{20}$$

Hence, in the momentum space,

$$\begin{aligned}(Q_k\psi)(p) = (V^{-1}x_k V\psi)(p) &= (2\pi)^{-3}\int \exp(-\mathrm{i}px)x_k p_0^{1/2}\mathrm{d}^3x \exp(\mathrm{i}xp')\psi(p')\frac{\mathrm{d}^3p'}{p_0'^{1/2}} \\ &= \mathrm{i}\left(\frac{\partial}{\partial p^k} - \frac{p_k}{2p_0^2}\right)\psi(p).\end{aligned} \tag{21}$$

This operator is self-adjoint relative to the scalar product (3).

It follows from eqs. (12) and (21) that the transformation properties of the position operators Q_k relative to Euclidean group $\mathscr{E}^3$ are

$$\begin{aligned}U_{(a,0)}^L Q_k U_{(a,0)}^{L^{-1}} &= Q_k - a_k, \\ U_{(0,r)}^L Q_k U_{(0,r)}^{L^{-1}} &= D_{k'k}^1(R_r)Q_{k'}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}[][Q_l, Q_k] &= 0, \\ [Q_k, P_j] &= \mathrm{i}\delta_{kj}.\end{aligned}$$

The time derivative of the position operators (21) in the Heisenberg representation is defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_k = \mathrm{i}[H, Q_k] = \mathrm{i}[p_0, Q_k] = \frac{p_k}{p_0}, \tag{22}$$

i.e., it represents the operator of the velocity of the particle. We see, therefore, that the operators Q_k given by formula (21) satisfy all the *physical requirements* that could be imposed on position operators.

We now consider in more detail the position operator for a scalar particle. The eigenfunctions of the operators Q_k in the momentum representation localized at a point $x \in R^3$ have in this case the form

$$\psi_x(p) = p_0^{1/2}\exp(-\mathrm{i}px). \tag{23}$$

Indeed,

$$Q_k \psi_x(p) = \mathrm{i}\left(\frac{\partial}{\partial p^k} - \frac{p_k}{2p_0^2}\right)\psi_x(p) = x_k \psi_x(p).$$

Let us also remark that by virtue of eq. (23) the formula (13) can be interpreted as the probability amplitude of finding a scalar particle in a state $\psi(p)$ at the position x at $t = 0$.

Now we perform another 'Fourier' transformation of the eigenfunction (23) of the position operators Q_i. Note that the coordinates obtained by this Fourier transformation are different than the x-coordinate in V-transformation of eq. (13).

$$\begin{aligned}\psi_x(\xi, t = 0) &= (2\pi)^{-3/2} \frac{1}{\sqrt{2}} \int p_0^{1/2} \exp[-\mathrm{i}p(x-\xi)] \frac{\mathrm{d}^3 p}{p_0} \\ &= \mathrm{const}\left(\frac{m}{r}\right)^{5/4} H_{5/4}^{(1)}(\mathrm{i}mr),\end{aligned} \tag{24}$$

where $r = |x-\xi|$ and $H_{5/4}^{(1)}$ denotes the Hankel function of the first kind of order 5/4. The space extension of this function is of the order $1/m$ and for large r it falls off as $\exp(-mr)/r$. Notice, however, that by eq. (13)

$$(V\psi_x)(y) = (2\pi)^{-3/2} \int p_0^{1/2} \exp(-\mathrm{i}p\cdot x) p_0^{1/2} \exp(\mathrm{i}p\cdot y) \frac{\mathrm{d}^3 p}{p_0} = \delta^3(x-y),$$

again as in the non-relativistic case.

The relativistic position operators can also be written in the space determined by this Fourier transform. Indeed, taking the inverse Fourier transform of (21) one obtains

$$(Q_k \psi)(\xi) = \xi_k \psi(\xi) + \frac{1}{8\pi} \int \frac{\exp(-m|\xi-\eta|)}{|\xi-\eta|} \frac{\partial \psi(\eta)}{\partial \eta^k} \mathrm{d}^3 \eta. \tag{25}$$

We see that the operators Q_i in this space are represented by non-local operators.

Let us note that position operators Q_k given by eq. (4) are localized at a moment t in the plane determined by the normal vector $n = (1, 0, 0, 0)$; the stability group of this vector is just $\mathscr{E}^3$. This explains why we took $\mathscr{E}^3$ as a covariance group for position operators. Now in case of massless particles $p_\mu p^\mu = 0$, $p^\mu \neq 0$ and the stability subgroup of any point on the cone is $T^4 \otimes \mathscr{E}^2$: the subgroup $\mathscr{E}^2$ acting on M^4 leaves invariant a null hyperplane $\mathring{H}$ determined by a normal vector $n = (1, 0, 0, 1)$: it is on this null hyperplane alone that we must be able to localize any massless particle (think on a localization of a photon in a photographic plate). Consequently, we must take a subgroup $G_0 = T^3 \otimes \mathscr{E}^2$ instead of $\mathscr{E}^3$ as a covariance group of a position operator of a massless particle. If a representation $U^{0,J}$ of Poincaré group restricted to $T^3 \otimes \mathscr{E}^2$ is a representation induced by a representation L of $\mathscr{E}^2$ then there will exist an imprimitivity system $(E(S), U_{G_0}^{0,J})$, $S \subset T^3 \otimes \mathscr{E}^2/\mathscr{E}^2 \sim R^3$ based on R^3 and by formula (4) we shall have three position operators for photon. Surprisingly enough we have:

PROPOSITION 2. *Let $U^{0,J}$ be an irreducible representation of the Poincaré group, corresponding to a massless particle with spin J. The restriction $U^{0,J}_{T^3 \otimes \mathscr{E}}$ is unitarily equivalent to a representation of $T^3 \otimes \mathscr{E}^2$ induced by a reducible representation of $R^1 \otimes \mathscr{E}^2$.* ▼

The proof can be carried out using Induction-Reduction Theorem 18.2.1 as in proposition 1. The alternative proof was given by Angelopoulos, Bayen and Flato 1975.

Proposition 2 implies that on purely group theoretical basis a massless particle has only two position operator and may be localizable in the plane perpendicular to a direction of motion. This mathematical result coincides with an intuitive idea of a photon hitting a photographic plate and reacting to an ion.

§ 2. The Representations of the Heisenberg Commutation Relations

We consider in this section the problem of representations of Heisenberg (canonical) commutation relation

$$[q_j, p_k] = \mathrm{i}\delta_{jk} I, \quad j, k = 1, 2, \ldots, n. \tag{1}$$

The entity q_j has in non-relativistic quantum mechanics the meaning of the position operator and p_k the meaning of the momentum operator of a particle. Hence, it is of great importance to know the number of non-equivalent, irreducible representations of the algebra (1). The best known representation is the Schrödinger representation

$$\begin{aligned} q_j &: u(x) \to x_j u(x), \\ p_k &: u(x) \to \frac{1}{\mathrm{i}} \frac{\partial u}{\partial x^k} \end{aligned} \tag{2}$$

or, in the global form,

$$\begin{aligned} \exp(\mathrm{i}\beta_j q_j) &: u(x) \to \exp(\mathrm{i}\beta_j x_j) u(x), \\ \exp(\mathrm{i}\alpha_k p_k) &: u(x) \to u(x+\alpha), \end{aligned} \tag{3}$$

which is realized in the Hilbert space $H = L^2(R^n)$. We show that any other representation of the canonical commutation relations (1), integrable to a global representation of the corresponding group is equivalent to the Schrödinger representation. To show this, we first bring the Heisenberg relations (1) into the so-called *Weyl form*.

We perform this using the Baker–Hausdorff formula

$$\exp A \exp B = \exp(A+B+\tfrac{1}{2}[A, B]) = \exp([A, B]) \exp B \exp A, \tag{4}$$

valid for operators A and B whose commutator is a c-number. Because we assume integrability of the representation of (1), this formula holds on the invariant dense set of Nelson–Gårding analytic vectors for the global representation (3) (cf. ch. 11, § 7).

Setting $A = \mathrm{i}\alpha_k p_k$ and $B = \mathrm{i}\beta_k q_k$ where α_k and β_k are real numbers, we obtain

$$\exp(\mathrm{i}\alpha p)\exp(\mathrm{i}\beta q) = \exp(\mathrm{i}\alpha\beta)\exp(\mathrm{i}\beta q)\exp(\mathrm{i}\alpha p), \tag{5}$$

where

$$\alpha\beta = \alpha_k\beta_k, \quad k = 1, 2, \ldots, n.$$

The Baker–Hausdorff formula allows us also to find a composition law for the group associated with the Lie algebra (1). Associating with the generators p_k, q_k and I the group parameters α_k β_k and γ_k, respectively, we obtain

$$\begin{aligned}\exp[\mathrm{i}(\alpha p+\beta q+\gamma I)]&\exp[\mathrm{i}(\alpha' p+\beta' q+\gamma' I)]\\ &= \exp[\mathrm{i}\{(\alpha+\alpha')p+(\beta+\beta')q+\big(\tfrac{1}{2}(\alpha\beta'-\alpha'\beta)+\gamma+\gamma'\big)I\}].\end{aligned} \tag{6}$$

This gives the following composition law for group elements

$$\big(\alpha, \beta, \exp(\mathrm{i}\gamma)\big)\big(\alpha', \beta', \exp(\mathrm{i}\gamma')\big) = \big(\alpha+\alpha', \beta+\beta', \exp[\mathrm{i}(\tfrac{1}{2}(\alpha\beta'-\alpha'\beta)+\gamma+\gamma')]\big). \tag{7}$$

Notice that the Lie algebra L determined by eq. (1) is nilpotent. Indeed, we have

$$L_{(1)} = [L, L] = \{I\}, \quad L_{(2)} = [L_{(1)}, L_{(1)}] = \{0\}.$$

Consequently the global group (7) is also nilpotent. One may easily verify this property on the group level using the definition of nilpotent groups given in 3.5.

Note that the Weyl relations (5) imply the Heisenberg relations (1). In fact, taking derivatives $\partial^2/\partial\alpha_k\partial\beta_j$ on both sides of eq. (5), one recovers eq. (1). The converse statement is, however, not true. The derivation of the Weyl relation (5) was based on the fact that the Schrödinger representation (3) is integrable. If the representation of the Lie algebra (1) is not integrable, then one cannot associate with it a corresponding Weyl formula (5). Consequently, the Heisenberg and the Weyl relations are in fact not equivalent.

Now we show, using the imprimitivity theorem, that every integrable representation of (1) is unitarily equivalent to the Schrödinger representation (3). Set

$$V_\alpha \equiv \exp(\mathrm{i}\alpha p), \quad U_\beta \equiv \exp(\mathrm{i}\beta q), \quad \langle\alpha, \beta\rangle \equiv \exp(\mathrm{i}\alpha\beta). \tag{8}$$

Then, eq. (5) takes the form

$$V_\alpha U_\beta = \langle\alpha, \beta\rangle U_\beta V_\alpha. \tag{9}$$

Here $\langle\alpha, \beta\rangle$ plays the role of the character $\beta(\alpha)$ of an abelian group $G = R^n$. We have, moreover,

$$V_\alpha V_{\alpha'} = V_{\alpha+\alpha'}, \quad U_\beta U_{\beta'} = U_{\beta+\beta'},$$

i.e., the maps $\alpha \to V_\alpha$ and $\beta \to U_\beta$ give representations of the abelian groups isomorphic to R^n. This observation is the starting point of the following theorem.

THEOREM 1. *Let G be a separable, locally compact, abelian group and $\hat{G}$ its dual group of characters. Let $\alpha \to V_\alpha$ and $\beta \to U_\beta$ be unitary representations of G and $\hat{G}$, respectively, in the same Hilbert space H, and satisfy the conditions*

(i) $$V_\alpha U_\beta = \langle\alpha, \beta\rangle U_\beta V_\alpha \quad \textit{for all } \alpha \in G,\ \beta \in \hat{G}. \tag{10}$$

(ii) *The set* $\{V_\alpha, \alpha \in G, U_\beta, \beta \in \hat{G}\}$ *is irreducible.*

Then, there exists a unitary isomorphism $S: H \to L^2(G)$, *such that*

$$SV_\alpha S^{-1}\varphi(x) = \varphi(x+a), \quad SU_\beta S^{-1}\varphi(x) = \langle x, \beta\rangle \varphi(x), \quad x \in G. \tag{11}$$

PROOF: By SNAG's theorem we have

$$(U_\beta \varphi, \psi) = \int_G \langle \alpha, \beta\rangle \mathrm{d}\big(E(\alpha)\varphi, \psi\big). \tag{12}$$

Here the symbol $\mathrm{d}\big(E(\alpha)\varphi, \psi\big)$ means that the character $\langle \alpha, \beta\rangle$ is to be integrated as a function of α with respect to the set function $G \supset A \to \big(E(A)\varphi, \psi\big)$. We replace in eq. (12) U_β by $V_{\alpha'} U_\beta V_{\alpha'}^{-1}$. Then, using the fact that

$$V_{\alpha'} U_\beta V_{\alpha'}^{-1} = \langle \alpha', \beta\rangle U_\beta$$

(by supposition (10)), we conclude that

$$\int_G \langle \alpha, \beta\rangle \langle \alpha', \beta\rangle \mathrm{d}\big(E(\alpha)\varphi, \psi\big) = \int_G \langle \alpha+\alpha', \beta\rangle \mathrm{d}\big(E(\alpha)\varphi, \psi\big)$$

$$= \int_G \langle \alpha'', \beta\rangle \mathrm{d}\big(E(\alpha''-\alpha')\varphi, \psi\big) = \int_G \langle \alpha'', \beta\rangle \mathrm{d}\big(V_{\alpha'} E(\alpha'') V_{\alpha'}^{-1}\varphi, \psi\big). \tag{13}$$

The characters separate points of G. Hence, the measures in eq. (13) are equal. This implies

$$E(A\alpha^{-1}) = V_\alpha E(A) V_\alpha^{-1} \tag{14}$$

for all $A \subset G$ and $\alpha \in G$. Thus, $E(A)$ represents a system of imprimitivity based on G.

The supposition (ii) implies that the pair (V, E) is irreducible. In turn, the Imprimitivity Theorem implies that there exists a unitary map S, such that

$$\begin{aligned} SV_\alpha S^{-1} &= V_\alpha^L \quad \text{for all } \alpha \in G, \\ SE(A)S^{-1} &= E^L(A) \quad \text{for all Borel sets } A \subset G, \end{aligned} \tag{15}$$

where V^L is a representation induced by the stability subgroup K and $E^L(A)$ is the canonical set of projections based on G, i.e.,

$$E^L(A)\varphi(\alpha) = \chi_A(\alpha)\varphi(\alpha). \tag{16}$$

Because $K = \{e\}$ and L are irreducible, the representation V^L is the right regular representation, i.e.,

$$(V_{\alpha_0}^L \varphi)(\alpha) = \varphi(\alpha\alpha_0) = \varphi(\alpha+\alpha_0). \tag{17}$$

Finally, eqs. (12) and (16) imply

$$(SU_\beta S^{-1}\varphi)(\alpha) = \int_G \langle \alpha', \beta\rangle \mathrm{d}\big(SE(\alpha')S^{-1}\varphi\big)(\alpha)$$

$$= \int_G \langle \alpha', \beta\rangle \mathrm{d}\big(E^L(\alpha')\varphi\big)(\alpha) = \langle \alpha, \beta\rangle \varphi(\alpha).$$

This concludes the proof of th. 1. ▼