We write the integral in the form

$$I = \int \sin^2 x \sin x \, dx.$$

Let $u = \sin^2 x$ and $dv = \sin x \, dx$. Then

$$du = 2 \sin x \cos x \, dx \quad \text{and} \quad v = -\cos x.$$

Thus

$$I = -(\sin^2 x)(\cos x) - \int -\cos x(2 \sin x \cos x) \, dx$$

$$= -\sin^2 x \cos x + 2 \int \cos^2 x \sin x \, dx.$$

This last integral could then be determined by substitution, for instance

$$t = \cos x \quad \text{and} \quad dt = -\sin x \, dx.$$

The last integral becomes $-2 \int t^2 \, dt$, and hence

$$I = \int \sin^3 x \, dx = -\sin^2 x \cos x - \tfrac{2}{3} \cos^3 x.$$

To deal with an arbitrary positive integer n, we shall show how to reduce the integral $\int \sin^n x \, dx$ to the integral $\int \sin^{n-2} x \, dx$. Proceeding stepwise downwards will give a method for getting the full answer.

Theorem 3.1. *For any integer $n \geqq 2$, we have*

$$\boxed{\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx.}$$

Proof. We write the integral as

$$I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx.$$

Let $u = \sin^{n-1} x$ and $dv = \sin x \, dx$. Then

$$du = (n-1) \sin^{n-2} x \cos x \, dx \quad \text{and} \quad v = -\cos x.$$

Thus

$$\begin{aligned} I_n &= -\sin^{n-1} x \cos x - \int -(n-1)\cos x \sin^{n-2} x \cos x \, dx \\ &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx. \end{aligned}$$

We replace $\cos^2 x$ by $1 - \sin^2 x$ and get

$$I_n = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx.$$

Therefore

$$I_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2} - (n-1)I_n,$$

whence

$$nI_n = -\sin^{n-1} x \cos x + (n-1)I_{n-2}.$$

Dividing by n gives us our formula.

We leave the proof of the analogous formula for cosine as an exercise.

$$\boxed{\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx.}$$

Integrals involving tangents can be done by a similar technique, because

$$\frac{d \tan x}{dx} = 1 + \tan^2 x.$$

These functions are less used than sine and cosine, and hence we don't write out the formulas, to lighten this printed page which would otherwise become oppressive.

Mixed powers of sine and cosine

One can integrate mixed powers of sine and cosine by replacing $\sin^2 x$ by $1 - \cos^2 x$, for instance.

Example. Find $\int \sin^2 x \cos^2 x \, dx$.

Replacing $\cos^2 x$ by $1 - \sin^2 x$, we see that our integral is equal to

$$\int \sin^2 x \, dx - \int \sin^4 x \, dx,$$

which we know how to integrate. We could also use a special trick for this case, making the substitutions at the beginning of the section. Thus

$$(\sin^2 x)(\cos^2 x) = \frac{1 - \cos^2 2x}{4},$$

which reduces the powers inside the integral. Another application of this same type, namely

$$\cos^2 2x = \frac{1 + \cos 4x}{2}$$

reduces our problem still further. Take your pick and work out the integral completely as an exercise.

Warning. Because of trigonometric identities like

$$\sin^2 x = -\tfrac{1}{2}\cos 2x + \tfrac{1}{2}$$

different forms for the answers are possible. They will differ by a constant of integration.

When we meet an integral involving a square root, we can frequently get rid of the square root by making a trigonometric substitution.

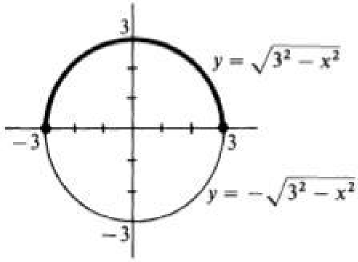
Example. Find the area of a circle of radius 3.

The equation of the circle is

$$x^2 + y^2 = 9,$$

and the portion of the circle in the first quadrant is described by the function

$$y = \sqrt{3^2 - x^2}.$$



One-fourth of the area is therefore given by the integral

$$\int_0^3 \sqrt{3^2 - x^2}\, dx.$$

For such integrals, we want to get rid of the horrible square root sign, so we try to make the expression under the integral into a perfect square. We use the substitution

$$x = 3\sin t \qquad \text{and} \qquad dx = 3\cos t\,dt, \qquad \text{with} \quad 0 \leqq t \leqq \pi/2.$$

When $t = 0$ then $x = 0$ and when $t = \pi/2$, then $x = 3$. Hence our integral becomes

$$\begin{aligned}\int_0^{\pi/2} \sqrt{3^2 - 3^2\sin^2 t}\,3\cos t\,dt &= \int_0^{\pi/2} 9\cos t\cos t\,dt \\ &= 9\int_0^{\pi/2} \cos^2 t\,dt \\ &= \frac{9\pi}{4}.\end{aligned}$$

Note that on the stated interval $0 \leqq t \leqq \pi/2$, the cosine $\cos t$ is positive, and so

$$\sqrt{1 - \sin^2 t} = \sqrt{\cos^2 t} = \cos t.$$

If we picked an interval where the cosine is negative, then when taking the square root, we would need to use an extra minus sign. That is, if $\cos t < 0$, then

$$\sqrt{\cos^2 t} = -\cos t.$$

Since the integral above represented 1/4-th of the area of the circle, it follows that the total area of the circle is 9π.

In general, an integral involving expressions like

$$\sqrt{1 - x^2}$$

can sometimes be evaluated by using the substitution

$$x = \sin\theta, \qquad dx = \cos\theta\,d\theta,$$

because the expression under the square root then becomes a perfect square, namely $1 - \sin^2\theta = \cos^2\theta$.

In making this substitution, we usually let

$$-\pi/2 \leqq \theta \leqq \pi/2 \qquad \text{and} \qquad -1 \leqq x \leqq 1.$$

This is the range where

$$x = \sin\theta \quad \text{has the inverse function } \theta = \arcsin x.$$

Negative powers of sine and cosine

It is usually a pain to integrate negative powers of sine and cosine, although it can be done. You should be aware that the following formula exists.

$$\boxed{\int \frac{1}{\cos\theta}\,d\theta = \int \sec\theta\,d\theta = \log(\sec\theta + \tan\theta).}$$

This is done by substitution. We have

$$\frac{1}{\cos\theta} = \sec\theta = \frac{(\sec\theta)(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}.$$

Let $u = \sec\theta + \tan\theta$. Then the integral is in the form

$$\int \frac{1}{u}\,du.$$

(This is a good opportunity to emphasize that the formula we just obtained is valid on any interval such that $\cos\theta \neq 0$ and

$$\sec\theta + \tan\theta > 0.$$

Otherwise the symbols are meaningless. Determine such an interval as an exercise.) The expression in the above formula is sufficiently complicated that **you should not memorize it**. Plug into it when needed. There is a similar formula for the integral of $1/\sin\theta$, which is obtained by using the prefix co- on the right-hand side. The formula is:

$$\boxed{I = \int \frac{1}{\sin\theta}\,d\theta = \int \csc\theta\,d\theta = -\log(\csc\theta + \cot\theta),}$$

which is similar to the answer given previously for $\int (1/\cos\theta)\,d\theta$. Of course, the answer is over an interval where the expression inside the logarithm is positive. Otherwise, one has to take the absolute value of this expression. The proof is similar, and you can also check it by differentiating the right-hand side to get $1/\sin\theta$.

Warning. On the left-hand side we have sine instead of cosine, so the prefix co- is deleted. On the right-hand side, we have cosecant and cotangent, so the prefix co- is added. *You should remember that there is such a symmetry*, but always check exactly what the correct relation is before using it, because in using this symmetry, *certain minus signs appear* just as a minus sign now appeared on the right-hand side. Do not attempt to memorize when such minus signs occur. Check each time that you need a similar formula, or look it up in integral tables.

Example. Let us evaluate the integral

$$I = \int \frac{1}{x\sqrt{1-x^2}}\,dx.$$

Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$. Then

$$I = \int \frac{1}{\sin\theta\sqrt{\cos^2\theta}}\cos\theta\,d\theta.$$

Over an interval where $\cos\theta$ is positive, we have

$$\sqrt{\cos^2\theta} = \cos\theta,$$

and hence

$$I = \int \frac{1}{\sin\theta}\,d\theta.$$

This integral was evaluated in the above box.

XI, §3. EXERCISES

Find the following integrals.

1. $\int \sin^4 x\,dx$
2. $\int \cos^3 x\,dx$
3. $\int \sin^2 x \cos^3 x\,dx$

Find the area of the region enclosed by the following curves.

4. $x^2 + \dfrac{y^2}{9} = 1$
5. $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$
6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

7. Find the area of a circle of radius $r > 0$.

8. Find the integrals.

(a) $\int \sqrt{1 - \cos\theta}\, d\theta$ (b) $\int \sqrt{1 + \cos\theta}\, d\theta$

[*Hint*: Write $\theta = 2u$. This should help you make the expression under the square root into a perfect square.]

9. For any integers m, n prove the formulas:

$$\sin mx \sin nx = \tfrac{1}{2}[\cos(m - n)x - \cos(m + n)x],$$
$$\sin mx \cos nx = \tfrac{1}{2}[\sin(m + n)x + \sin(m - n)x],$$
$$\cos mx \cos nx = \tfrac{1}{2}[\cos(m + n)x + \cos(m - n)x].$$

[*Hint*: Expand the right-hand side and cancel as much as you can. Use the addition formulas of Chapter IV, §3.]

10. Use the preceding exercise to do this one.
(a) Show that

$$\int_{-\pi}^{\pi} \sin 3x \cos 2x\, dx = 0.$$

(b) Show that

$$\int_{-\pi}^{\pi} \cos 5x \cos 2x\, dx = 0.$$

11. Show in general that for any positive integers m, n we have

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0.$$

12. Show in general that for positive integers m, n,

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n. \end{cases}$$

[*Hint*: If $m \neq n$, use Exercise 9. If $m = n$, use $\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$.]

13. Find $\int \tan x\, dx$.

Find the following integrals.

14. $\int \frac{1}{\sqrt{9 - x^2}}\, dx$

15. $\int \frac{1}{\sqrt{3 - x^2}}\, dx$

16. $\int \frac{1}{\sqrt{2 - 4x^2}}\, dx$

17. $\int \frac{1}{\sqrt{a^2 - b^2x^2}}\, dx$

18. Let f be a continuous function on the interval $[-\pi, \pi]$. We define numbers c_0, a_n, b_n for positive integers n by the formulas

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

These numbers c_0, a_n, b_n are called the **Fourier coefficients** of f.

Example. Let $f(x) = x$. Then the 0-th Fourier coefficient of f is the integral:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\frac{x^2}{2}\bigg|_{-\pi}^{\pi} = 0.$$

The coefficients a_n and b_n are given by the integrals

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx \qquad \text{and} \qquad b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin nx\,dx.$$

You should be able to evaluate these integrals using integration by parts. Compute the Fourier coefficients of the following functions. (If you do 19 first, you might have less work.)

(a) $f(x) = x$ (b) $f(x) = x^2$ (c) $f(x) = |x|$
(d) $f(x) = \cos x$ (e) $f(x) = \sin x$ (f) $f(x) = \sin^2 x$
(g) $f(x) = \cos^2 x$ (h) $f(x) = |\sin x|$ (i) $f(x) = |\cos x|$
(j) $f(x) = 1$

19. (a) Let f be an even function [that is $f(x) = f(-x)$]. Show that its Fourier coefficients b_n are all equal to 0.
(b) Let f be an odd function [that is $f(x) = -f(-x)$]. What can you say about its Fourier coefficients?

XI, §3. SUPPLEMENTARY EXERCISES

1. $\displaystyle\int \frac{\cos^3 x}{\sin x}\,dx$
2. $\displaystyle\int \tan^2 x\,dx$
3. $\displaystyle\int e^x \sin e^x\,dx$
4. $\displaystyle\int \frac{1}{1-\cos x}\,dx$
5. $\displaystyle\int_0^{\pi/2} \cos^2 x\,dx$
6. $\displaystyle\int_0^{\pi/3} \sin^6 x\,dx$
7. $\displaystyle\int_{-\pi}^{\pi} \sin^2 x\cos^2 x\,dx$
8. $\displaystyle\int_0^{2\pi} \sin^3 2x\,dx$

9. $\displaystyle\int_0^{\pi/2} \sin^2 2x \cos^2 2x \, dx$

10. $\displaystyle\int_0^{\pi/4} \cos^4 x \, dx$

11. $\displaystyle\int x^2\sqrt{1 - x^2} \, dx$

12. $\displaystyle\int \frac{1}{(x^2 + 1)^2} \, dx$

13. $\displaystyle\int_0^1 \frac{1}{\sqrt{1 - x^2 \, dx}}$

14. $\displaystyle\int \frac{\sqrt{1 - x^2}}{x^2} \, dx$

15. $\displaystyle\int \frac{x^3}{\sqrt{16 - x^2}} \, dx$

16. $\displaystyle\int \frac{x^3}{\sqrt{1 + x^2}} \, dx$

In the next exercises, we let a be a positive number.

17. $\displaystyle\int \frac{1}{x\sqrt{a^2 - x^2}} \, dx$

18. $\displaystyle\int \frac{x^2}{\sqrt{a^2 - x^2}} \, dx$

19. $\displaystyle\int \frac{1}{x^3\sqrt{a^2 - x^2}} \, dx$

20. $\displaystyle\int \frac{1}{x^2\sqrt{a^2 - x^2}} \, dx$

21. $\displaystyle\int \frac{\sqrt{1 - x^2}}{x} \, dx$

22. $\displaystyle\int \frac{\sqrt{a^2 - x^2}}{x^2} \, dx$

23. $\displaystyle\int \frac{x^2}{(a^2 - x^2)^{3/2}} \, dx$

24. $\displaystyle\int_0^a x^4\sqrt{a^2 - x^2 \, dx}$

XI, §4. PARTIAL FRACTIONS

We want to study the integrals of quotients of polynomials.

Let $f(x)$ and $g(x)$ be two polynomials. We want to investigate the integral

$$\int \frac{f(x)}{g(x)} \, dx.$$

Using long division, one can reduce the problem to the case when the degree of f is less than the degree of g. The following example illustrates this reduction.

Example. Consider the two polynomials $f(x) = x^3 - x + 1$ and

$$g(x) = x^2 + 1.$$

Dividing f by g (you should know how from high school) we obtain a quotient of x with remainder $-2x + 1$. Thus

$$x^3 - x + 1 = (x^2 + 1)x + (-2x + 1).$$

Hence

$$\frac{f(x)}{g(x)} = x + \frac{-2x+1}{x^2+1}.$$

To find the integral of $f(x)/g(x)$ we integrate x, and the quotient on the right, which has the property that the degree of the numerator is less than the degree of the denominator.

From now on, *we assume throughout that when we consider a quotient $f(x)/g(x)$, the degree of f is less than the degree of g.* We assume this because the method we shall describe works only in this case. Factoring out a constant if necessary, we also assume that $g(x)$ can be written

$$g(x) = x^d + \text{lower terms}.$$

We shall begin by discussing special cases, and then describe afterwards how the general case can be reduced to these.

First part. Linear factors in the denominator

Case 1. If a is a number, and n an integer $\geqq 1$, then

$$\boxed{\begin{aligned} \int \frac{1}{(x-a)^n}\,dx &= \frac{1}{-n+1}\frac{1}{(x-a)^{n-1}} \qquad &\text{if } n \neq 1,\\ &= \log(x-a) \qquad &\text{if } n = 1.\end{aligned}}$$

This is an old story. We know how to do it. In fact, we have

$$\int \frac{1}{(x-a)^n}\,dx = \int (x-a)^{-n}\,dx = \int u^{-n}\,du.$$

Suppose $n \neq 1$. Then by substitution $u = x - a$, $du = dx$, we get

$$\int (x-a)^{-n}\,dx = \frac{(x-a)^{-n+1}}{-n+1} = \frac{1}{-n+1}\frac{1}{(x-a)^{n-1}}$$

because

$$u^{-n+1} = u^{-(n-1)} = \frac{1}{u^{n-1}}.$$

Suppose $n = 1$. Then the integral has the form

$$\int \frac{1}{u}\,du = \log u,$$

and hence

$$\int \frac{1}{x-a}\,dx = \log(x-a).$$

Case 2. Next we consider integrals of expressions like

$$\int \frac{1}{(x-2)(x-3)}\,dx \qquad \text{or} \qquad \int \frac{x+1}{(x-1)^2(x-2)}\,dx,$$

where the denominator consists of a product of terms of the form

$$(x-a_1)\cdots(x-a_n)$$

for some numbers, $a_1,\ldots,a_n$ which need not be distinct. The procedure amounts to writing the expression under the integral as a sum of terms, as in Case 1.

Example. We wish to find the integral

$$\int \frac{1}{(x-2)(x-3)}\,dx.$$

To do this, we want to write

$$\frac{1}{(x-2)(x-3)} = \frac{c_1}{x-2} + \frac{c_2}{x-3}$$

with some numbers c_1 and c_2, for which we have to solve. Put the expression on the right over a common denominator. We find

$$\frac{c_1}{x-2} + \frac{c_2}{x-3} = \frac{c_1(x-3) + c_2(x-2)}{(x-2)(x-3)}.$$

Thus $(x-2)(x-3)$ is the common denominator, and

$$\text{numerator} = c_1(x-3) + c_2(x-2) = (c_1 + c_2)x - 3c_1 - 2c_2.$$

We want the fraction to be equal to $1/(x-2)(x-3)$. Thus the numerator must be equal to 1, that is we must have

$$(c_1 + c_2)x - 3c_1 - 2c_2 = 1.$$

Therefore it suffices to solve the simultaneous equations

$$c_1 + c_2 = 0,$$

$$-3c_1 - 2c_2 = 1.$$

Solving for c_1 and c_2 gives $c_2 = 1$ and $c_1 = -1$. Hence

$$\int \frac{1}{(x-2)(x-3)}\,dx = \int \frac{-1}{(x-2)}\,dx + \int \frac{1}{(x-3)}\,dx$$

$$= -\log(x-2) + \log(x-3).$$

Example. Find the integral

$$\int \frac{x+1}{(x-1)^2(x-2)}\,dx.$$

We want to find numbers c_1, c_2, c_3 such that

$$\frac{x+1}{(x-1)^2(x-2)} = \frac{c_1}{x-1} + \frac{c_2}{(x-1)^2} + \frac{c_3}{x-2}.$$

Note that $(x-1)^2$ appears in the denominator of the original quotient. To take this into account, it is necessary to include two terms with $(x-1)$ and $(x-1)^2$ in their denominators, appearing above as

$$\frac{c_1}{x-1} + \frac{c_2}{(x-1)^2}.$$

On the other hand, $(x-2)$ appears only in the first power in the original quotient, so it gives rise only to one term

$$\frac{c_3}{x-2}$$

in the partial fraction decomposition. (The general rule is stated at the end of the section.)

We now describe how to find the constants c_1, c_2, c_3, satisfying the relation

$$\frac{x+1}{(x-1)^2(x-2)} = \frac{c_1}{x-1} + \frac{c_2}{(x-1)^2} + \frac{c_3}{x-2}$$

$$= \frac{c_1(x-1)(x-2) + c_2(x-2) + c_3(x-1)^2}{(x-1)^2(x-2)}.$$

Here we put the fraction on the right over the common denominator

$$(x-1)^2(x-2).$$

We have

$$\begin{aligned} x+1 = \text{numerator} &= c_1(x-1)(x-2) + c_2(x-2) + c_3(x-1)^2 \\ &= (c_1+c_3)x^2 + (-3c_1+c_2-2c_3)x + 2c_1 - 2c_2 + c_3. \end{aligned}$$

Thus to find the constants c_1, c_2, c_3 satisfying the desired relation, we have to solve the simultaneous equations

$$\begin{aligned} c_1 \qquad\quad + \ c_3 &= 0, \\ -3c_1 + \ c_2 - 2c_3 &= 1, \\ 2c_1 - 2c_2 + \ c_3 &= 1. \end{aligned}$$

This is a system of three linear equations in three unknowns, which you can solve to determine c_1, c_2, and c_3. One finds $c_1 = -3$, $c_2 = -2$, $c_3 = 3$. Hence

$$\begin{aligned} \int \frac{x+1}{(x-1)^2(x-2)}\,dx &= \int \frac{-3}{x-1}\,dx + \int \frac{-2}{(x-1)^2}\,dx + \int \frac{3}{(x-2)}\,dx \\ &= -3\log(x-1) + \frac{2}{x-1} + 3\log(x-2). \end{aligned}$$

It is a theorem in algebra that if you follow the above procedure to write a fraction in terms of simpler fractions according to the method illustrated in the examples, you will always be able to solve for the coefficients $c_1, c_2, c_3, \ldots$. The *proof* cannot be given at the level of this course, but in practice, unless you or I make a mistake, we just solve numerically in each case. If we have higher powers of some factor in the denominator, then we have to use higher powers also in the simpler fractions on the right-hand side.

Example. We can decompose

$$\frac{x+1}{(x-1)^3(x-2)} = \frac{c_1}{x-1} + \frac{c_2}{(x-1)^2} + \frac{c_3}{(x-1)^3} + \frac{c_4}{x-2}.$$

Putting the right-hand side over a common denominator, and equating the numerator with $x+1$, we can solve for the coefficients

$$c_1, c_2, c_3, c_4.$$

Second part. Quadratic factors in the denominator

Case 3. We want to find the integral

$$\boxed{I_n = \int \frac{1}{(x^2+1)^n}\,dx}$$

when n is a positive integer. If $n = 1$, there is nothing new, the integral is $\arctan x$. If $n > 1$ we shall use integration by parts. There will be a slight twist on the usual procedure, because if we integrate I_n by parts in the natural way, we find that the exponent n increases by one unit. Let us do the case $n = 1$ as an example, so we start with

$$I_1 = \int \frac{1}{x^2+1}\,dx.$$

Let

$$u = \frac{1}{x^2+1} = (x^2+1)^{-1}, \qquad dv = dx,$$

$$du = \frac{-2x}{(x^2+1)^2}\,dx, \qquad v = x.$$

Then

$$I_1 = \frac{x}{x^2+1} - \int \frac{-2x^2}{(x^2+1)^2}\,dx$$

$$(*) \qquad = \frac{x}{x^2+1} + 2\int \frac{x^2}{(x^2+1)^2}\,dx.$$

In the last integral on the right, write $x^2 = x^2 + 1 - 1$. Then

$$\int \frac{x^2}{(x^2+1)^2}\,dx = \int \frac{x^2+1-1}{(x^2+1)^2}\,dx = \int \frac{1}{x^2+1}\,dx - \int \frac{1}{(x^2+1)^2}\,dx$$

$$= \arctan x - I_2.$$

If we now substitute this in expression (*) we obtain

$$I_1 = \frac{x}{x^2+1} + 2\arctan x - 2I_2.$$

Therefore we can solve for I_2 in terms of I_1, and we find

$$2I_2 = \frac{x}{x^2+1} + 2\arctan x - I_1$$

$$= \frac{x}{x^2+1} + 2\arctan x - \arctan x$$

$$= \frac{x}{x^2+1} + \arctan x.$$

Dividing by 2 yields the value for I_2:

$$\boxed{\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan x.}$$

The same method works in general. We want to reduce I_n to finding I_{n-1}, where

$$I_{n-1} = \int \frac{1}{(x^2+1)^{n-1}}\,dx.$$

Let

$$u = \frac{1}{(x^2+1)^{n-1}} \qquad \text{and} \qquad dv = dx.$$

Then

$$du = -(n-1)\frac{2x}{(x^2+1)^n}\,dx \qquad \text{and} \qquad v = x.$$

Thus

$$I_{n-1} = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{x^2}{(x^2+1)^n}\,dx.$$

We write $x^2 = x^2 + 1 - 1$. We obtain

$$I_{n-1} = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)\int \frac{1}{(x^2+1)^{n-1}}\,dx - 2(n-1)\int \frac{1}{(x^2+1)^n}\,dx$$

or in other words:

$$I_{n-1} = \frac{x}{(x^2+1)^{n-1}} + 2(n-1)I_{n-1} - 2(n-1)I_n.$$

Therefore

$$2(n-1)I_n = \frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1},$$

whence

$$\boxed{\begin{aligned}\int \frac{1}{(x^2+1)^n}\,dx &= \frac{1}{2(n-1)}\frac{x}{(x^2+1)^{n-1}} \\ &\qquad + \frac{(2n-3)}{2(n-1)}\int \frac{1}{(x^2+1)^{n-1}}\,dx,\end{aligned}}$$

or using the abbreviation I_n, we find:

$$\boxed{I_n = \frac{1}{2n-2}\frac{x}{(x^2+1)^{n-1}} + \frac{2n-3}{2n-2}I_{n-1}.}$$

This gives us a recursion formula which lowers the exponent n in the denominator until we reach $n = 1$. In that case, we know that

$$\boxed{\int \frac{1}{x^2+1}\,dx = \arctan x.}$$

If you want to find I_3, use the formula to reduce it to I_2, then use the formula again to reduce it to I_1, which is arctan x. This gives a complete formula for I_3. To get a complete formula for I_n takes n steps. Of course you should not memorize the above formula; you should only remember the method by which it is obtained to apply it to special cases, say to finding I_3, I_4.

Eliminating extra constants by substitution

Sometimes we meet an integral which is a slight variation of the one just considered, with an extra constant. For instance, if b is a number, find

$$\int \frac{1}{(x^2+b^2)^n}\,dx.$$

Using the substitution $x = bz$, $dx = b\,dz$ reduces the integral to

$$\int \frac{1}{(b^2z^2 + b^2)^n}\, b\, dz = \int \frac{b}{b^{2n}(z^2+1)^n}\, dz$$

$$= \frac{1}{b^{2n-1}} \int \frac{1}{(z^2+1)^n}\, dz.$$

We have

$$\int \frac{1}{(z^2+1)^n}\, dz = \int \frac{1}{(x^2+1)^n}\, dx$$

because the two integrals differ only by a change of letters. This shows how to use a substitution to reduce the computation of the integral with b to the integral when $b = 1$ treated above.

Case 4. Find the integral

$$\int \frac{x}{(x^2+b^2)^n}\, dx.$$

This is an old story. We make the substitution

$$u = x^2 + b^2 \qquad \text{and} \qquad du = 2x\, dx.$$

Then

$$\boxed{\int \frac{x}{(x^2+b^2)^n}\, dx = \frac{1}{2}\int \frac{1}{u^n}\, du,}$$

which we know how to evaluate, and thus we find

$$\boxed{\int \frac{x}{(x^2+b^2)^n}\, dx = \begin{cases} \frac{1}{2}\log(x^2+b^2) & \text{if } n = 1, \\ \dfrac{1}{2(-n+1)}\dfrac{1}{(x^2+b^2)^{n-1}} & \text{if } n \neq 1. \end{cases}}$$

Example. Find

$$\int \frac{5x-3}{(x^2+5)^2}\, dx.$$

We write

$$\int \frac{5x-3}{(x^2+5)^2}\, dx = 5\int \frac{x}{(x^2+5)^2}\, dx - 3\int \frac{1}{(x^2+5)^2}\, dx.$$

Then:

$$5\int\frac{x}{(x^2+5)^2}\,dx=\frac{5}{2}\int\frac{2x}{(x^2+5)^2}\,dx=\frac{5}{2}\int\frac{1}{u^2}\,du=\frac{5}{2}\frac{u^{-1}}{-1}=-\frac{5}{2}\frac{1}{x^2+5}.$$

For the second integral on the right, we may put

$$x=\sqrt{5}t \qquad \text{and} \qquad dx=\sqrt{5}\,dt.$$

Then:

$$\int\frac{1}{(x^2+5)^2}\,dx=\int\frac{1}{(5t^2+5)^2}\sqrt{5}\,dt=\frac{\sqrt{5}}{25}\int\frac{1}{(t^2+1)^2}\,dt$$

and we have previously computed

$$\int\frac{1}{(t^2+1)^2}\,dt=\frac{1}{2}\left(\frac{t}{t^2+1}+\arctan t\right).$$

Putting everything together, using $t=x/\sqrt{5}$, we find:

$$\int\frac{5x-3}{(x^2+5)^2}\,dx=-\frac{5}{2}\frac{1}{x^2+5}-3\frac{\sqrt{5}}{25}\frac{1}{2}\left(\frac{x/\sqrt{5}}{(x/\sqrt{5})^2+1}+\arctan\frac{x}{\sqrt{5}}\right).$$

Third part. The general quotient $f(x)/g(x)$

If you are given a polynomial of type x^2+bx+c, then you **factor or complete the square**. The polynomial can thus be written in the form

$$(x-\alpha)(x-\beta) \qquad \text{or} \qquad (x-\alpha)^2+\beta^2$$

with suitable numbers α, β. Two cases arise. For example:

Case 1. $\quad x^2-x-6=(x+2)(x-3).$

Case 2. $\quad x^2-2x+5=(x-1)^2+2^2.$

In Case 1, we have factored the polynomial into two factors, and each factor has degree 1.

In Case 2, we have not factored the polynomial. By a change of variables, we can turn it into an expression t^2+1. Namely, let

$$x-1=2t \qquad \text{so} \qquad x=2t+1.$$

Then

$$(x-1)^2+2^2=2^2t^2+2^2=2^2(t^2+1).$$

We made the change of variables so that 2^2 would come out as a factor. We note that in Case 2, we cannot factor the polynomial any further.

The following general result can be proved, but the proof is long, and cannot be given in this course.

Let $g(x)$ be a polynomial with real numbers as coefficients. Then $g(x)$ can always be written as a product of terms of type

$$(x-\alpha)^n \quad \text{and} \quad [(x-\beta)^2+\gamma^2]^m,$$

n, m being integers $\geqq 0$, and some constant factor.

This can be quite difficult to do explicitly, but in the exercises, the situation is fixed up so that it is easy.

Example. By completing the square, we write

$$x^2+2x+3=(x+1)^2+2=(x+1)^2+(\sqrt{2})^2.$$

We can then evaluate the integral:

$$\int \frac{1}{x^2+2x+3}\,dx$$

Let $x+1=\sqrt{2}t$ and $dx=\sqrt{2}\,dt$. Then

$$\begin{aligned}\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+(\sqrt{2})^2}\,dx &= \frac{1}{2}\int \frac{1}{t^2+1}\sqrt{2}\,dt \\ &= \frac{\sqrt{2}}{2}\arctan t \\ &= \frac{\sqrt{2}}{2}\arctan\frac{x+1}{\sqrt{2}}.\end{aligned}$$

Example. Let us find

$$\int \frac{x}{x^2+2x+3}\,dx.$$

We write

$$\begin{aligned}\int \frac{x}{x^2+2x+3}\,dx &= \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+3}\,dx \\ &= \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx - \int \frac{1}{x^2+2x+3}\,dx.\end{aligned}$$

Then:

$$\frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\log(x^2+2x+3).$$

Putting this together with the previous example, we find:

$$\int \frac{x}{x^2+2x+3}\,dx = \tfrac{1}{2}\log(x^2+2x+3) - \frac{\sqrt{2}}{2}\arctan\left(\frac{x+1}{\sqrt{2}}\right).$$

Example. Find the integral

$$\int \frac{2x+5}{(x^2+1)^2(x-3)}\,dx.$$

We can find numbers $c_1, c_2, \ldots$ such that the quotient is equal to

$$\begin{aligned}\frac{2x+5}{(x^2+1)^2(x-3)} &= \frac{c_1+c_2x}{x^2+1} + \frac{c_3+c_4x}{(x^2+1)^2} + \frac{c_5}{x-3}\\ &= \frac{c_1}{x^2+1} + c_2\frac{x}{x^2+1} + c_3\frac{1}{(x^2+1)^2}\\ &\quad + c_4\frac{x}{(x^2+1)^2} + c_5\frac{1}{x-3}.\end{aligned}$$

It is a theorem of algebra that you can always solve for the constants c_1, c_2, c_3, c_4, c_5 to get such a decomposition of the original fraction into the sum on the right, which is called the **partial fraction decomposition.** Observe that corresponding to the term with x^2+1 you need several terms on the right-hand side, especially those with an x in the numerator. If you do not include these, then you would get an incomplete formula, which would not work out. You could not compute the constants.

We now compute the constants. We put the right-hand side of the decomposition over the common denominator

$$(x^2+1)^2(x-3).$$

The numerator is equal to

$$2x + 5 = c_1(x^2 + 1)(x - 3) + c_2 x(x^2 + 1)(x - 3) + c_3(x - 3)$$
$$+ c_4 x(x - 3) + c_5(x^2 + 1)^2.$$

We equate the coefficients of x^4, x^3, x^2, x and the respective constants, and get a system of five linear equations in five unknowns, which can be solved. It is tedious to do it here and we leave it as an exercise, but we write down the equations:

$$\begin{array}{rl}
c_2 \qquad\qquad + c_5 = 0 & \text{(coefficient of } x^4\text{)},\\
c_1 - 3c_2 \qquad\qquad = 0 & \text{(coefficient of } x^3\text{)},\\
-3c_1 + c_2 \qquad + c_4 + 2c_5 = 0 & \text{(coefficient of } x^2\text{)},\\
c_1 - 3c_2 + c_3 - 3c_4 \qquad = 2 & \text{(coefficient of } x\text{)},\\
-3c_1 \qquad - 3c_3 \qquad + c_5 = 5 & \text{(coefficient of } 1\text{)}.
\end{array}$$

For the integral, we then obtain:

$$\int \frac{2x + 5}{(x^2 + 1)^2(x - 3)}\,dx = c_1 \arctan x + \tfrac{1}{2}c_2 \log(x^2 + 1)$$
$$+ c_3 \int \frac{1}{(x^2 + 1)^2}\,dx - \tfrac{1}{2}c_4 \frac{1}{x^2 + 1} + c_5 \log(x - 3).$$

The integral which we left standing is just that of Case 3, so we have shown how to find the desired integral.

Example. There is a partial fraction decomposition.

$$\frac{x^4 + 2x - 1}{(x^2 + 2)^3(x - 5)^2} = \frac{c_1 + c_2 x}{(x^2 + 2)} + \frac{c_3 + c_4 x}{(x^2 + 2)^2} + \frac{c_5 + c_6 x}{(x^2 + 2)^3}$$
$$+ \frac{c_7}{x - 5} + \frac{c_8}{(x - 5)^2}.$$

It would be tedious to compute the constants, and we don't do it.

The general rule is as follows: Suppose we have a quotient $f(x)/g(x)$ with degree of $f <$ degree of g. We factor g as far as possible into terms like

$$(x-\alpha)^n \quad \text{and} \quad [(x-\beta)^2+\gamma^2]^m,$$

n, m being integers $\geqq 0$. Then

$$\frac{f(x)}{g(x)} = \text{sum of terms of the following type:}$$

$$\frac{c_1}{x-\alpha}+\frac{c_2}{(x-\alpha)^2}+\cdots+\frac{c_n}{(x-\alpha)^n}$$

$$+\frac{d_1+e_1x}{(x-\beta)^2+\gamma^2}+\cdots+\frac{d_m+e_mx}{[(x-\beta)^2+\gamma^2]^m}$$

with suitable constants $c_1, c_2, \ldots, d_1, d_2, \ldots, e_1, e_2, \ldots$.

Once the quotient $f(x)/g(x)$ is written as above, then Cases 1, 2, and 3 allow us to integrate each term. We then find that the integral involves functions of the following type:

A rational function
Log terms
Arctangent terms.

XI, §4. EXERCISES

Find the following integrals.

1. $\displaystyle\int \frac{2x-3}{(x-1)(x+7)}\,dx$
2. $\displaystyle\int \frac{x}{(x^2-3)^2}\,dx$
3. (a) $\displaystyle\int \frac{1}{(x-3)(x+2)}\,dx$ (b) $\displaystyle\int \frac{1}{(x+2)(x+1)}\,dx$ (c) $\displaystyle\int \frac{1}{x^2-1}\,dx$
4. $\displaystyle\int \frac{x}{(x+1)(x+2)(x+3)}\,dx$
5. $\displaystyle\int \frac{x+2}{x^2+x}\,dx$
6. $\displaystyle\int \frac{x}{(x+1)^2}\,dx$
7. $\displaystyle\int \frac{x}{(x+1)(x+2)^2}\,dx$
8. $\displaystyle\int \frac{2x-3}{(x-1)(x-2)}\,dx$
9. Write out in full the integral

$$\int \frac{1}{(x^2+1)^2}\,dx.$$

10. Either by doing the integration by parts repeatedly or by plugging into the general formula in the text, write out in full the following integrals:

(a) $\displaystyle\int \frac{1}{(x^2+1)^3}\,dx$ (b) $\displaystyle\int \frac{1}{(x^2+1)^4}\,dx.$

Find the following integrals.

11. $\displaystyle\int \frac{2x-3}{(x^2+1)^2}\,dx$

12. $\displaystyle\int \frac{x+1}{(x^2+9)^2}\,dx$

13. $\displaystyle\int \frac{4}{(x^2+16)^2}\,dx$

14. $\displaystyle\int \frac{1}{(x+1)(x^2+1)}\,dx$

15. Find the constants in the expression from the example in the text:

$$\frac{2x+5}{(x^2+1)^2(x-3)} = \frac{c_1+c_2x}{x^2+1} + \frac{c_3+c_4x}{(x^2+1)^2} + \frac{c_5}{x-3}.$$

16. Using substitution, prove the two formulas:

(a) $$\boxed{\int \frac{1}{x^2+b^2}\,dx = \frac{1}{b}\arctan\frac{x}{b}.}$$

(b) $$\boxed{\int \frac{1}{(x+a)^2+b^2}\,dx = \frac{1}{b}\arctan\frac{x+a}{b}.}$$

For the next problems, factor x^3-1 and x^4-1 into irreducible factors.

17. (a) $\displaystyle\int \frac{1}{x^4-1}\,dx$ (b) $\displaystyle\int \frac{x}{x^4-1}\,dx$

18. (a) $\displaystyle\int \frac{1}{x^3-1}\,dx$ (b) $\displaystyle\int \frac{1}{x(x^2+x+1)}\,dx$

19. $\displaystyle\int \frac{x^2-2x-2}{x^3-1}\,dx$

XI, §5. EXPONENTIAL SUBSTITUTIONS

This section has several purposes.

First, we expand our techniques of integration, by using the exponential function.

Second, this gives practice in the exponential function and the logarithm in a new context, which will make you learn these functions better for having used them.

Third, we shall introduce two new functions

$$\frac{e^x + e^{-x}}{2} \quad \text{and} \quad \frac{e^x - e^{-x}}{2}.$$

In the next chapter you will see these functions applied to some physical situations, to describe the equation of a hanging cable, or a soap film between two rings. Such functions will also be used to find the integrals which give the length of various curves. Here we just use them systematically to find integrals.

We start by showing how to make a simple substitution.

Example. Let us find

$$I = \int \sqrt{1 - e^x}\, dx.$$

We put $u = e^x$, $du = e^x\, dx$ so that $dx = du/u$. Then

$$I = \int \sqrt{1 - u}\, \frac{1}{u}\, du.$$

Now put $1 - u = v^2$ and $-du = 2v\, dv$ to get rid of the square root sign. Then $u = 1 - v^2$ and we obtain

$$\begin{aligned} I &= \int \frac{v}{1 - v^2}(-2v)\, dv = 2 \int \frac{v^2}{v^2 - 1}\, dv \\ &= 2 \int \frac{v^2 - 1 + 1}{v^2 - 1}\, dv \\ &= 2\left[\int dv + \int \frac{1}{v^2 - 1}\, dv\right] \\ &= 2v + 2 \int \frac{1}{(v + 1)(v - 1)}\, dv. \end{aligned}$$

This last integral can be integrated by partial fractions, to give the final answer.

We have learned how to integrate expressions involving

$$\sqrt{1 - x^2}.$$

We substitute $x = \sin\theta$ to make the expression under the square root sign into a perfect square. But what if we have to deal with an integral like

$$\int \sqrt{1+x^2}\,dx?$$

We need to make a substitution which makes the expression under the square root sign into a perfect square. There are two possible types of functions which we can use. First, let us try to substitute $x = \tan\theta$ to get rid of the square root. We find

$$\sqrt{1+\tan^2\theta} = \sec\theta = \frac{1}{\cos\theta}$$

over an interval where $\cos\theta$ is positive. Already a negative power of cosine is not so nice. Even worse,

$$dx = \sec^2\theta\,d\theta,$$

so

$$\int \sqrt{1+x^2}\,dx = \int \sec^3\theta\,d\theta = \int \frac{1}{\cos^3\theta}\,d\theta,$$

which can be done, but not pleasantly, so we don't do it.

Here we give a better way of getting rid of the horrible square root sign. We need a better pair of functions $f_1(t)$ and $f_2(t)$ such that

$$1 + f_1(t)^2 = f_2(t)^2.$$

Such functions are easily found by using the exponential function e^t. Namely, we let

$$f_1(t) = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad f_2(t) = \frac{e^t + e^{-t}}{2}.$$

If you multiply out, you will find immediately that these functions satisfy the desired relation. These functions have a name: they are called the **hyperbolic sine** and **hyperbolic cosine**, and are denoted by **sinh** and **cosh.** (sinh is pronounced cinch, while cosh is pronounced cosh.) Thus we **define**

$$\boxed{\sinh t = \frac{e^t - e^{-t}}{2} \quad \text{and} \quad \cosh t = \frac{e^t + e^{-t}}{2}.}$$

Carry out the manipulation which shows that

$$\boxed{\cosh^2 t - \sinh^2 t = 1.}$$

Furthermore, the standard rules for differentiation show that

$$\boxed{\frac{d \cosh t}{dt} = \sinh t} \quad \text{and} \quad \boxed{\frac{d \sinh t}{dt} = \cosh t.}$$

These formulas are very similar to those for the ordinary sine and cosine, except for some reversals of sign. They allow us to treat some cases of integrals which could not be done before, and in particular get rid of square root signs as follows.

Example. Find

$$I = \int \sqrt{1 + x^2}\, dx.$$

We make the substitution

$$x = \sinh t \quad \text{and} \quad dx = \cosh t\, dt.$$

Then $1 + \sinh^2 t = \cosh^2 t$, so that $\sqrt{1 + x^2} = \sqrt{\cosh^2 t} = \cosh t$. Hence

$$\begin{aligned} I &= \int \cosh t \cosh t\, dt \\ &= \int \frac{e^t + e^{-t}}{2} \frac{e^t + e^{-t}}{2}\, dt \\ &= \frac{1}{4} \int (e^{2t} + 2 + e^{-2t})\, dt \\ &= \frac{1}{4}\left(\frac{e^{2t}}{2} + 2t - \frac{e^{-2t}}{2}\right). \end{aligned}$$

The answer is, of course, given in terms of t. If we want it in terms of x, then we need to study the *inverse function*, which we may call **arcsinh (hyperbolic arcsine)**, and we may write

$$t = \operatorname{arcsinh} x.$$

At first it seems that we are in a situation similar to that of sine and cosine, when we could not give explicitly a formula for the inverse function. We just called those inverse functions arcsine and arccosine. It is remarkable that here, we can give a formula as follows.

$$\boxed{\textit{If} \quad x = \sinh t \quad \textit{then} \quad t = \log(x + \sqrt{x^2 + 1}).}$$

Proof. We have

$$x = \tfrac{1}{2}(e^t - e^{-t}).$$

Let $u = e^t$. Then

$$x = \frac{1}{2}\left(u - \frac{1}{u}\right).$$

We multiply this equation by $2u$ and get the equation

$$u^2 - 2ux - 1 = 0.$$

We can then solve for u in terms of x by the quadratic formula, and get

$$u = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

so

$$u = x \pm \sqrt{x^2 + 1}.$$

But $u = e^t > 0$ for all t. Since $\sqrt{x^2 + 1} > x$, it follows that we cannot have the minus sign in this relation. Hence finally

$$e^t = u = x + \sqrt{x^2 + 1}.$$

Now we take the log to find

$$t = \log(x + \sqrt{x^2 + 1}).$$

This proves the desired formula.

Thus unlike the case of sine and cosine, we get here an explicit formula for the inverse function of the hyperbolic sine.

If we now substitute $t = \log(x + \sqrt{x^2 + 1})$ in the indefinite integral found above, we get the explicit answer:

$$\int \sqrt{1 + x^2}\, dx = \frac{1}{4}\left[\frac{1}{2}(x + \sqrt{x^2 + 1})^2 + 2\log(x + \sqrt{x^2 + 1}) - \frac{1}{2}(x + \sqrt{x^2 + 1})^{-2}\right].$$

We may also want to find a definite integral.

Example. Let $B > 0$. Find

$$\int_0^B \sqrt{1 + x^2}\, dx.$$

We substitute B in the indefinite integral, we substitute 0, and subtract, to find:

$$\int_0^B \sqrt{1 + x^2}\, dx = \frac{1}{4}\left[\frac{e^{2t}}{2} + 2t - \frac{e^{-2t}}{2}\right]_0^{\log(B + \sqrt{B^2 + 1})}$$

$$= \tfrac{1}{4}[\tfrac{1}{2}(B + \sqrt{B^2 + 1})^2 + 2\log(B + \sqrt{B^2 + 1}) - \tfrac{1}{2}(B + \sqrt{B^2 + 1})^{-2}]$$

because when we substitute 0 for t in the expression in brackets we find 0.

In cases when you have to use an inverse function for cosh, you can rely on the following assertion.

For $t \geqq 0$, the function $x = \cosh t$ has an inverse function, which is given by

$$t = \log(x + \sqrt{x^2 - 1}).$$

This is proved just like the similar statement for sinh. Do Exercises 5 and 6, which are actually worked out in the answer section. But do them before looking up the answer section, you will learn the subject better for doing so.

Remark. Integrals like

$$\int \sqrt{1 + x^3}\, dx \qquad \text{and} \qquad \int \sqrt{1 + x^4}\, dx$$

are much more complicated, and cannot be found by means of the elementary functions of this course.

XI, §5. EXERCISES

Find the integrals.

1. $\displaystyle\int \sqrt{1 + e^x}\, dx$

2. $\displaystyle\int \frac{1}{1 + e^x}\, dx$

3. $\displaystyle\int \frac{1}{e^x + e^{-x}}\, dx$

4. $\displaystyle\int \frac{1}{\sqrt{e^x + 1}}\, dx$

5. Let $f(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x = y$.
   (a) Show that f is strictly increasing for all x.
   (b) Sketch the graph of f.
   Let $x = \operatorname{arcsinh} y$ be the inverse function.
   (c) For which numbers y is $\operatorname{arcsinh} y$ defined?
   (d) Let $g(y) = \operatorname{arcsinh} y$. Show that

$$g'(y) = \frac{1}{\sqrt{1 + y^2}}.$$

It was shown in the text that $x = g(y) = \log(y + \sqrt{y^2 + 1})$.

6. Let $f(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x = y$.
   (a) Show that f is strictly increasing for $x \geqq 0$.
   Then the inverse function exists for this interval. Denote this inverse function by $x = \operatorname{arccosh} y$.
   (b) Sketch the graph of f.
   (c) For which numbers y is $\operatorname{arccosh} y$ defined?
   (d) Let $g(y) = \operatorname{arccosh} y$. Show that

$$g'(y) = \frac{1}{\sqrt{y^2 - 1}}.$$

   (e) Show that $x = g(y) = \log(y + \sqrt{y^2 - 1})$. Thus you can actually give an explicit expression for this inverse function in terms of the logarithm. This is another way in which the hyperbolic functions behave more simply than sine and cosine, because we could not give an explicit formula for the arcsine and arccosine.

Find the following integrals.

7. $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}}\,dx$

8. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

9. $\displaystyle\int \frac{x^2+1}{x-\sqrt{x^2+1}}\,dx$

10. $\displaystyle\int \sqrt{x^2-1}\,dx$

11. Find the area between the x-axis and the hyperbola

$$x^2 - y^2 = 1$$

in the first quadrant between $x = 1$ and $x = B$, with $B > 1$.

For the graph of the hyperbola, see Chapter II, §9.

12. Find the area between the x-axis and the hyperbola

$$y^2 - x^2 = 1$$

in the first quadrant, between $x = 0$ and $x = B$.

13. Let a be a positive number, and let $y = a\cosh(x/a)$. Show that

$$\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

[This is the differential equation of the hanging cable. See the appendix after §3 of the next chapter.]

14. Verify that for any number $a > 0$ we have

$$\boxed{\int \sqrt{a^2+x^2}\,dx = \tfrac{1}{2}\left[x\sqrt{a^2+x^2} + a^2\log\left(x + \sqrt{a^2+x^2}\right)\right].}$$

CHAPTER XII

Applications of Integration

Mathematics consists in discovering and describing certain objects and structures. It is essentially impossible to give an all-encompassing description of these. Hence, instead of such a definition, we simply state that the objects of study of mathematics as we know it are those which you will find described in the mathematical journals of the past two centuries, and leave it at that. There are many reasons for studying these objects, among which are aesthetic reasons (some people like them), and practical reasons (some mathematics can be applied).

Physics, on the other hand, consists in describing the empirical world by means of mathematical structures. The empirical world is the world with which we come into contact through our senses, through experiments, measurements, etc. What makes a good physicist is the ability to choose, among many mathematical structures and objects, the ones which can be used to describe the empirical world. I should of course immediately qualify the above assertion in two ways: First, the description of physical situations by mathematical structures can only be done within the degree of accuracy provided by the experimental apparatus. Second, the description should satisfy certain aesthetic criteria (simplicity, elegance). After all, a complete listing of all results of all experiments performed is a description of the physical world, but is quite a distinct thing from giving at one single stroke a general principle which will account simultaneously for the results of all these experiments.

For psychological reasons, it is impossible (for most people) to learn certain mathematical theories without seeing first a geometric or physical interpretation. Hence in this book, before introducing a mathematical notion, we frequently introduce one of its geometric or physical interpretations. These two, however, should not be confused. Thus we might make two columns, as shown on the following page.

As far as the logical development of our course is concerned, we could omit the second column entirely. The second column is used, however, for many purposes: To motivate the first column (because our brain is made up in such a way that to understand something in the first column, it needs the second). To provide applications for the first column, other than pure aesthetic satisfaction (granting that you like the subject).

Mathematics	Physics and geometry
Numbers	Points on a line
Derivative	Slope of a curve Rate of change
$\dfrac{df}{dx} = Kf(x)$	Exponential decay
Integral	Length Area Volume Work

Nevertheless, it is important to keep in mind that the derivative, as the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and the integral, as a unique number between upper and lower sums, are not to be confused with a slope or an area, respectively. It is simply our mind which interprets the mathematical notion in physical or geometric terms. Besides, we frequently assign several such interpretations to the same mathematical notion (viz. the integral being interpreted as an area, or as the work done by a force).

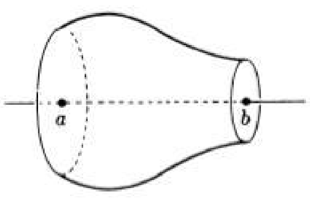
And by the way, the above remarks which are about physics and mathematics belong neither to physics nor to mathematics. They belong to philosophy.

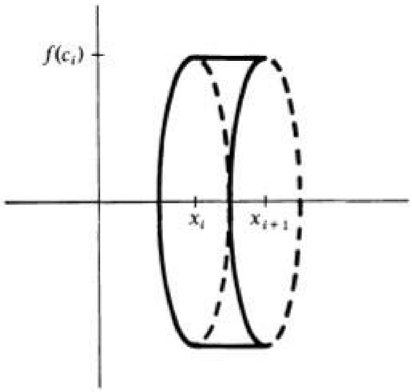
Experience shows that for a course which deals with integration and Taylor's formula in one term, time is lacking to cover *all* the applications of integration given in the book, as well as to cover the computations associated with Taylor's formula and an estimate of its remainder. The basic applications like length of curve, volume of revolution, area in polar coordinates cannot be omitted. One then has to make a choice about the others, which deal with geometric concepts (area of revolution) or physical concepts (work). As stated already in the foreword, my feeling is that except for doing the section on work, if time is lacking, it is best to omit other applications in order to have plenty of time to handle the computations resulting from Taylor's formula.

XII, §1. VOLUMES OF REVOLUTION

We start our applications with volumes of revolutions. The main reason is that the integrals to be evaluated come out easier than in other applications. But ultimately, we derive systematically the lengths, areas, and volumes of all the standard geometric figures.

Let $y = f(x)$ be a continuous function of x on some interval $a \leqq x \leqq b$. Assume that $f(x) \geqq 0$ in this interval. If we revolve the curve $y = f(x)$ around the x-axis, we obtain a solid, whose volume we wish to compute.





Take a partition of $[a, b]$, say

$$a = x_0 \leqq x_1 \leqq \cdots \leqq x_n = b.$$

Let c_i be a minimum of f in the interval $[x_i, x_{i+1}]$ and let d_i be a maximum of f in that interval. Then the solid of revolution in that small interval lies between a small cylinder and a big cylinder. The width of these cylinders is $x_{i+1} - x_i$ and the radius is $f(c_i)$ for the small cylinder and $f(d_i)$ for the big one. Hence the volume of revolution, denoted by V, satisfies the inequalities

$$\sum_{i=0}^{n-1} \pi f(c_i)^2(x_{i+1} - x_i) \leqq V \leqq \sum_{i=0}^{n-1} \pi f(d_i)^2(x_{i+1} - x_i).$$

It is therefore reasonable to define this volume to be

$$\boxed{V = \int_a^b \pi f(x)^2\, dx.}$$

Example. Compute the volume of the sphere of radius 1.

We take the function $y = \sqrt{1 - x^2}$ between 0 and 1. If we rotate this curve around the x-axis, we shall get half the sphere. Its volume is therefore

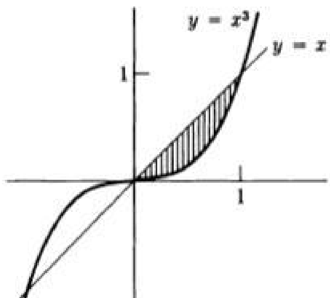
$$\int_0^1 \pi(1 - x^2)\, dx = \tfrac{2}{3}\pi.$$

The volume of the full sphere is therefore $\frac{4}{3}\pi$.

Example. Find the volume obtained by rotating the region between $y = x^3$ and $y = x$ in the first quadrant around the x-axis.

The graph of the region is illustrated on the figure. We take only that part in the first quadrant, so $0 \leqq x \leqq 1$. The required volume V is equal to the difference of the volumes obtained by rotating $y = x$ and $y = x^3$. Let $f(x) = x$ and $g(x) = x^3$. Then

$$\begin{aligned} V &= \pi \int_0^1 f(x)^2\, dx - \pi \int_0^1 g(x)^2\, dx \\ &= \pi \int_0^1 x^2\, dx - \pi \int_0^1 x^6\, dx \\ &= \frac{\pi}{3} - \frac{\pi}{7}. \end{aligned}$$



Example. We can make infinite solid chimneys and see if they have finite volume. Consider the function

$$f(x) = 1/\sqrt{x}.$$

Let

$$0 < a < 1.$$

The volume of revolution of the curve

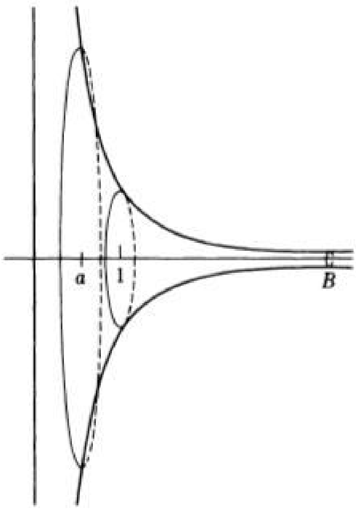
$$y = 1/\sqrt{x}$$

between $x = a$ and $x = 1$ is given by the integral

$$\int_a^1 \pi \frac{1}{x}\, dx = \pi \log x \Big|_a^1$$

$$= -\pi \log a.$$

As a approaches 0, $\log a$ becomes very large negative, so that $-\log a$ becomes very large positive, and the volume becomes arbitrarily large. We illustrate the chimney in the following figure.



However, if you compute the volume of the curve

$$y = \frac{1}{x^{1/4}}$$

between a and 1, you will find that it approaches a limit, as $a \to 0$. Do Exercise 12.

In the above computation, we determined the volume of a chimney near the y-axis. We can also find the volume of the chimney going off to the right, say between 1 and a number $B > 1$. Suppose the chimney is defined by $y = 1/\sqrt{x}$. The volume of revolution between 1 and B is given by the integral

$$\int_1^B \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_1^B \pi \frac{1}{x}\, dx = \pi \log B.$$

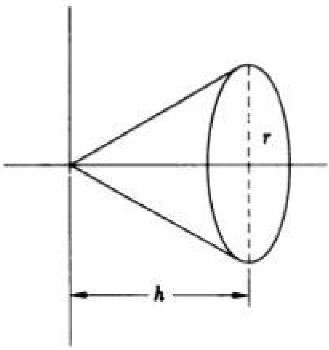
As $B \to \infty$, we see that this volume becomes arbitrarily large. However, using another function, as for instance in Exercise 13, you will find a finite volume for the infinite chimney!

XII, §1. EXERCISES

1. Find the volume of a sphere of radius r.

Find the volumes of revolution of the following:

2. $y = 1/\cos x$ between $x = 0$ and $x = \pi/4$
3. $y = \sin x$ between $x = 0$ and $x = \pi/4$
4. $y = \cos x$ between $x = 0$ and $x = \pi/4$
5. The region between $y = x^2$ and $y = 5x$
6. $y = xe^{x/2}$ between $x = 0$ and $x = 1$
7. $y = x^{1/2}e^{x/2}$ between $x = 1$ and $x = 2$
8. $y = \log x$ between $x = 1$ and $x = 2$
9. $y = \sqrt{1 + x}$ between $x = 1$ and $x = 5$
10. (a) Let B be a number > 1. What is the volume of revolution of the curve $y = e^{-x}$ between 1 and B? Does this volume approach a limit as B becomes large? If so, what limit?
    (b) Same question for the curve $y = e^{-2x}$.
    (c) Same question for the curve $y = \sqrt{x}e^{-x^2}$.
11. Find the volume of a cone whose base has radius r, and of height h, by rotating a straight line passing through the origin around the x-axis. What is the equation of the straight line?



12. Compute the volume of revolution of the curve

$$y = \frac{1}{x^{1/4}}$$

between a and 1. Determine the limit as $a \to 0$.

13. Compute the volume of revolution of the curve

$$y = 1/x^2$$

between $x = 2$ and $x = B$, for any number $B > 2$. Does this volume approach a limit as $B \to \infty$? If yes, what limit?

14. For which numbers $c > 0$ will the volume of revolution of the curve

$$y = 1/x^c$$

between 1 and B approach a limit as $B \to \infty$? Find this limit in terms of c.

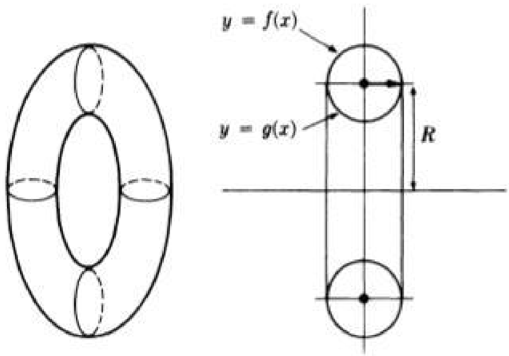
15. For which numbers $c > 0$ will the volume of revolution of the curve

$$y = 1/x^c$$

between a and 1 approach a limit as $a \to 0$? Find this limit in terms of c.

XII, §1. SUPPLEMENTARY EXERCISES

1. Find the volume of a doughnut as shown on the figure. The doughnut is obtained by rotating a circle of radius a about a straight line, say the x-axis.



(a) The doughnut (b) Cross section of the doughnut

Let R be the distance from the line to the center of the disc. We assume $R > a$. You can reduce this problem to the case discussed in the section as follows. Let $y = f(x)$ be the function whose graph is the upper half of the circle, and let $y = g(x)$ be the function whose graph is the lower half of the circle. Write down f and g explicitly. You then have to subtract the volume obtained by rotating the lower semicircle from the volume obtained by rotating the upper semicircle.

Find the volumes of the solid obtained by rotating each region as indicated, around the x-axis.

2. $y = x^2$, between $y = 0$ and $x = 2$

3. $y = \dfrac{4}{x+1}$, $x = -5$, $x = -2$, $y = 0$

4. $y = \sqrt{x}$, the x-axis and $x = 2$

5. $y = 1/x$, $x = 1$, $x = 3$ and the x-axis

6. $y = \sqrt{x}$, $y = x^3$

7. The region bounded by the line $x + y = 1$ and the coordinate axes

8. The ellipse $a^2x^2 + b^2y^2 = a^2b^2$

9. $y = e^{-x}$, between $x = 1$ and $x = 5$

10. $y = \log x$, between $x = 1$ and $x = 2$

11. $y = \tan x$, $x = \pi/3$ and the x-axis

In the next problems, you are asked to find a volume of revolution of a region between certain bounds, and determine whether this volume approaches a limit when the bound B becomes very large. If it does, give this limit.

12. The region bounded by $1/x$, the x-axis, between $x = 1$ and $x = B$ for $B > 1$.

13. The region bounded by $1/x^2$ and the x-axis, between $x = 1$ and $x = B$ for $B > 1$.

14. The region bounded by $y = 1/\sqrt{x}$, the x-axis, between $x = 1$ and $x = B$ for $B > 1$.

In the next problems, find the volume of revolution, determined by bounds involving a number $a > 0$, and find whether this volume approaches a limit as a approaches 0. If it does, state what limit.

15. The region bounded by $y = 1/\sqrt{x}$, the x-axis, between $x = a$ and $x = 1$ for $0 < a < 1$.

16. The region bounded by $y = 1/x$, the x-axis, between $x = a$ and $x = 1$, with $0 < a < 1$.

17. The region bounded by $(\cos x)/\sqrt{\sin x}$, the x-axis, between $x = a$ and $x = \pi/4$, with $0 < a < \pi/4$.

XII, §2. AREA IN POLAR COORDINATES

Suppose we are given a continuous function

$$r = f(\theta)$$

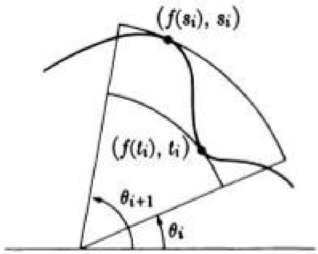
which is defined in some interval $a \leqq \theta \leqq b$. We assume that $f(\theta) \geqq 0$ and $b \leqq a + 2\pi$.

We wish to find an integral expression for the area encompassed by the curve $r = f(\theta)$ between the two bounds a and b.

Let us take a partition of $[a, b]$, say

$$a = \theta_0 \leqq \theta_1 \leqq \cdots \leqq \theta_n = b.$$

The picture between θ_i and θ_{i+1} might look like this:



We let s_i be a number between θ_i and θ_{i+1} such that $f(s_i)$ is a maximum in that interval, and we let t_i be a number such that $f(t_i)$ is a minimum in that interval. In the figure, we have drawn the circles (or rather the sectors) of radius $f(s_i)$ and $f(t_i)$, respectively. Let

A_i = area between $\theta = \theta_i$, $\theta = \theta_{i+1}$, and bounded by the curve

= area of the set of points (r, θ) in polar coordinates such that

$$\theta_i \leqq \theta \leqq \theta_{i+1} \quad \text{and} \quad 0 \leqq r \leqq f(\theta).$$

The area of a sector having angle $\theta_{i+1} - \theta_i$ and radius R is equal to the fraction

$$\frac{\theta_{i+1} - \theta_i}{2\pi}$$

of the total area of the circle of radius R, namely πR^2. Hence we get the inequality

$$\frac{\theta_{i+1} - \theta_i}{2\pi} \pi f(t_i)^2 \leqq A_i \leqq \frac{\theta_{i+1} - \theta_i}{2\pi} \pi f(s_i)^2.$$

Let $G(\theta) = \frac{1}{2}f(\theta)^2$. We see that the sum of the small pieces of area A_i satisfies the inequalities

$$\sum_{i=0}^{n-1} G(t_i)(\theta_{i+1} - \theta_i) \leqq \sum_{i=0}^{n-1} A_i \leqq \sum_{i=0}^{n-1} G(s_i)(\theta_{i+1} - \theta_i).$$

Thus the desired area lies between the upper sum and lower sum associated with the partition. Thus it is reasonable that the area in polar coordinates is given by

$$\boxed{A = \int_a^b \tfrac{1}{2}f(\theta)^2\,d\theta.}$$

Example. Find the area bounded by one loop of the curve

$$r^2 = 2a^2 \cos 2\theta \qquad (a > 0).$$

If $-\dfrac{\pi}{4} \leqq \theta \leqq \dfrac{\pi}{4}$, then $\cos 2\theta \geqq 0$. Thus we can write

$$r = \sqrt{2}a\sqrt{\cos 2\theta}.$$

The area is therefore

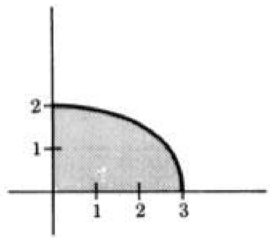
$$\int_{-\pi/4}^{\pi/4} \tfrac{1}{2}2a^2 \cos 2\theta\,d\theta = a^2.$$

Example. Find the area bounded by the curve

$$r = 2 + \cos\theta,$$

in the first quadrant.

First we sketch the area in the first quadrant, i.e. for θ between 0 and $\pi/2$. It looks like this:



The area is given by the integral

$$\tfrac{1}{2}\int_0^{\pi/2} (2 + \cos\theta)^2\,d\theta = \tfrac{1}{2}\int_0^{\pi/2} (4 + 4\cos\theta + \cos^2\theta)\,d\theta.$$

Each term is easily integrated. The final answer is

$$\frac{1}{2}\left(2\pi + 4 + \frac{\pi}{4}\right).$$

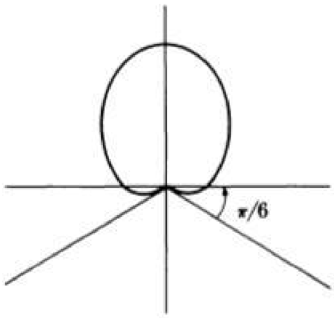
Example. Let us find the area enclosed by the curve given in polar coordinates by

$$r = 1 + 2\sin\theta.$$

Note that for $-\pi/6 \leqq \theta \leqq 7\pi/6$ and only for those θ is

$$1 + 2\sin\theta \geqq 0.$$

The curve looks like the figure.



The area is therefore equal to

$$A = \frac{1}{2}\int_{-\pi/6}^{7\pi/6}(1 + 2\sin\theta)^2\,d\theta$$

$$= 2\cdot\frac{1}{2}\int_{-\pi/6}^{\pi/2}(1 + 4\sin\theta + 4\sin^2\theta)\,d\theta.$$

We use the identity

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}.$$

The integral is then easily evaluated, and we leave this to the reader.

XII, §2. EXERCISES

Find the area enclosed by the following curves:

1. $r = 2(1 + \cos\theta)$
2. $r^2 = a^2 \sin 2\theta \quad (a > 0)$
3. $r = 2a\cos\theta$
4. $r = \cos 3\theta, \quad -\pi/6 \leqq \theta \leqq \pi/6$
5. $r = 1 + \sin\theta$
6. $r = 1 + \sin 2\theta$
7. $r = 2 + \cos\theta$
8. $r = 2\cos 3\theta, \quad -\pi/6 \leqq \theta \leqq \pi/6$

XII, §2. SUPPLEMENTARY EXERCISES

Find the areas of the following regions, bounded by the curve given in polar coordinates.

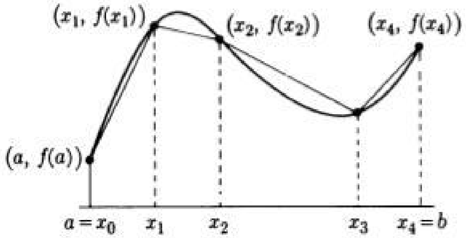
1. $r = 10\cos\theta$
2. $r = 1 - \cos\theta$
3. $r = \sqrt{1 - \cos\theta}$
4. $r = 2 + \sin 2\theta$
5. $r = \sin^2\theta$
6. $r = 1 - \sin\theta$
7. $r = 1 + 2\sin\theta$
8. $r = 1 + \sin 2\theta$
9. $r = \cos 3\theta$
10. $r = 2 + \cos\theta$

Find the area between the following curves, given in rectangular coordinates.

11. $y = 4 - x^2$, $y = 0$, between $x = -2$ and $x = 2$
12. $y = 4 - x^2$, $y = 8 - 2x^2$, between $x = -2$ and $x = 2$
13. $y = x^3 + x^2$, $y = x^3 + 1$, between $x = -1$ and $x = 1$
14. $y = x - x^2$, $y = -x$, between $x = 0$ and $x = 2$
15. $y = x^2$, $y = x + 1$, between the two points where the two curves intersect.
16. $y = x^3$ and $y = x + 6$ between $x = 0$ and the value of $x > 0$ where the two curves intersect.

XII, §3. LENGTH OF CURVES

Let $y = f(x)$ be a differentiable function over some interval $[a, b]$ (with $a < b$) and assume that its derivative f' is continuous. We wish to determine the length of the curve described by the graph. The main idea is to approximate the curve by small line segments and add these up.



Consequently, we consider a partition of our interval:

$$a = x_0 \leqq x_1 \leqq \cdots \leqq x_n = b.$$

For each x_i we have the point $(x_i, f(x_i))$ on the curve $y = f(x)$. We draw the line segments between two successive points. The length of such a segment is the length of the line between

$$(x_i, f(x_i)) \qquad \text{and} \qquad (x_{i+1}, f(x_{i+1})),$$

and is equal to

$$\sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}.$$

By the mean value theorem, we conclude that

$$f(x_{i+1}) - f(x_i) = (x_{i+1} - x_i)f'(c_i)$$

for some number c_i between x_i and x_{i+1}. Using this, we see that the length of our line segment is

$$\sqrt{(x_{i+1} - x_i)^2 + (x_{i+1} - x_i)^2 f'(c_i)^2}.$$

We can factor out $(x_{i+1} - x_i)^2$ and we see that the sum of the length of these line segments is

$$\sum_{i=0}^{n-1} \sqrt{1 + f'(c_i)^2}(x_{i+1} - x_i).$$

Let $G(x) = \sqrt{1 + f'(x)^2}$. Then $G(x)$ is continuous, and we see that the sum we have just written down is

$$\sum_{i=0}^{n-1} G(c_i)(x_{i+1} - x_i).$$

The value $G(c_i)$ satisfies the inequalities

$$\min_{[x_i, x_{i+1}]} G \leqq G(c_i) \leqq \max_{[x_i, x_{i+1}]} G$$

that is $G(c_i)$ lies between the minimum and the maximum of G on the interval $[x_i, x_{i+1}]$. Thus the sum we have written down lies between a lower sum and an upper sum for the function G. We called such sums Riemann sums. This is true for every partition of the interval. We know from the basic theory of integration that there is exactly one number lying between every upper sum and every lower sum, and that number is the definite integral. Therefore it is very reasonable to **define**:

length of our curve between *a* and *b*

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Example. We wish to set up the integral for the length of the curve $y = x^2$ between $x = 0$ and $x = 1$. From the definition above, we see that the integral is

$$\int_0^1 \sqrt{1 + (2x)^2}\, dx = \int_0^1 \sqrt{1 + 4x^2}\, dx.$$

This integral is of the same type as that considered in Chapter XI, §5. First let

$$u = 2x, \qquad du = 2\, dx.$$

When $x = 0$, then $u = 0$, and when $x = 1$, then $u = 2$. Hence

$$\int_0^1 \sqrt{1 + 4x^2}\, dx = \int_0^2 \sqrt{1 + u^2}\, \tfrac{1}{2}\, du = \tfrac{1}{2} \int_0^2 \sqrt{1 + u^2}\, du.$$

The answer then comes from Chapter XI, §5.

Example. We want the length of the curve $y = e^x$ between $x = 1$ and $x = 2$. We have $dy/dx = e^x$ and $(dy/dx)^2 = e^{2x}$ so by the general formula, the length is given by the integral

$$\int_1^2 \sqrt{1 + e^{2x}}\, dx.$$

This can be evaluated more rapidly, and we carry out the computation. Make the substitution

$$1 + e^{2x} = u^2.$$

Then

$$2e^{2x}\,dx = 2u\,du.$$

Since $e^{2x} = u^2 - 1$, we obtain

$$\int \sqrt{1 + e^{2x}}\,dx = \int u \frac{u\,du}{u^2 - 1} = \int \frac{u^2}{u^2 - 1}\,du$$

$$= \int \frac{u^2 - 1 + 1}{u^2 - 1}\,du$$

$$= \int 1\,du + \int \frac{1}{u^2 - 1}\,du.$$

But

$$\frac{1}{u^2 - 1} = \frac{1}{2}\left(\frac{1}{u - 1} - \frac{1}{u + 1}\right).$$

Hence

$$\int \sqrt{1 + e^{2x}}\,dx = u + \tfrac{1}{2}\left[\log \frac{u - 1}{u + 1}\right].$$

When $x = 1$, $u = \sqrt{1 + e^2}$. When $x = 2$, $u = \sqrt{1 + e^4}$. Hence the length of the curve over the given interval is equal to

$$\int_1^2 \sqrt{1 + e^{2x}}\,dx = u + \tfrac{1}{2}\left[\log \frac{u - 1}{u + 1}\right]\Bigg|_{\sqrt{1+e^2}}^{\sqrt{1+e^4}}$$

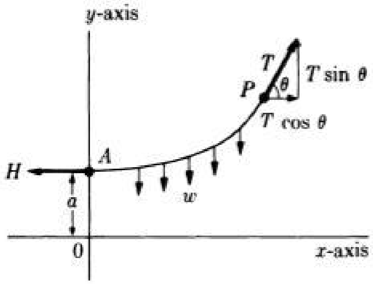
$$= \sqrt{1 + e^4} + \tfrac{1}{2}\log \frac{\sqrt{1 + e^4} - 1}{\sqrt{1 + e^4} + 1}$$

$$- \sqrt{1 + e^2} - \tfrac{1}{2}\log \frac{\sqrt{1 + e^2} - 1}{\sqrt{1 + e^2} + 1}.$$

A little complicated, but it is an explicit answer.

APPENDIX. THE HANGING CABLE

We want to show here how one can determine the equation of a hanging cable as shown on the figure.



We suppose that the cable is fixed on the left to a wall, and is subjected to a tension T in a certain direction at the other end. The cable is fixed to the wall at point A, submitted to a horizontal tension which is constant, and denoted by H. We want to find explicitly the height of the cable

$$y = f(x).$$

The answer is as follows.

> *If a is the height on the left, the units of measurement are such that $a = H/w$, and the cable has horizontal slope where it touches the wall, then*
>
> $$y = a\cosh(x/a).$$

We shall show this by first deriving the differential equation satisfied by the cable. Thus we first show that

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.}$$

Let the weight be w per unit length, and let the length be s. Then the weight W of cable of length s is ws.

The tension at P has to balance the horizontal tension H, and the weight W which pulls down. This tension has a horizontal component and a vertical component, which are given by $T\cos\theta$ and $T\sin\theta$, respectively. Thus we must have

$$T\cos\theta = H, \qquad T\sin\theta = W = ws.$$

Dividing, we get

$$\frac{T\sin\theta}{T\cos\theta} = \tan\theta = \frac{W}{H}.$$

But $\tan\theta$ is the slope of the cable at the point $P = (x, y)$. Therefore

$$\frac{dy}{dx} = \frac{w}{H}s.$$

On the other hand, we know that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Therefore

$$\frac{d^2y}{dx^2} = \frac{w}{H}\frac{ds}{dx} = \frac{w}{H}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

This is the differential equation which we wanted.

You may have done an exercise in Chapter XI, §5 showing that the function

$$y = a\cosh(x/a)$$

satisfies the equation

$$(*) \qquad \frac{d^2y}{dx^2} = \frac{1}{a}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

We now prove the converse.

Theorem 3.1. *If $y = f(x)$ satisfies $(*)$, and also*

$$f(0) = a, \qquad f'(0) = 0,$$

then $y = a\cosh(x/a)$.

The condition $f(0) = a$ means that the cable is hooked on the left at height a above the x-axis. The condition $f'(0) = 0$ means that at this point, the cable is horizontal.

Let

$$\frac{dy}{dx} = u.$$

Our differential equation can then be written

$$\frac{du}{dx} = \frac{1}{a}\sqrt{1 + u^2}.$$

Thus

$$\frac{1}{\sqrt{1 + u^2}}\, du = \frac{1}{a}\, dx$$

and we integrate by a substitution as in Chapter XI, §5. We put

$$u = \sinh t, \qquad du = \cosh t\, dt.$$

We have

$$\sqrt{1 + u^2} = \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t,$$

because $\cosh t > 0$ for all t. Hence

$$\int \frac{1}{\sqrt{1 + u^2}}\, du = \int \frac{\cosh t}{\cosh t}\, dt = t.$$

Therefore

$$t = \frac{1}{a}x + C$$

for some constant C. Hence

$$u = \sinh t = \sinh\left(\frac{x}{a} + C\right).$$

But $u = dy/dx = f'(x)$, and $f'(0) = 0$ by assumption. Hence

$$\sinh C = 0,$$

which means that $C = 0$. Therefore

$$\frac{dy}{dx} = f'(x) = \sinh\left(\frac{x}{a}\right).$$

Integrating once more, we get

$$y = f(x) = a \cosh\left(\frac{x}{a}\right) + K$$

for some constant K. But $f(0) = a$, and $\cosh 0 = 1$, so that

$$a = f(0) = a \cdot 1 + K.$$

Hence $K = 0$, and thus finally

$$y = f(x) = a\cosh(x/a)$$

as desired.

XII, §3. EXERCISES

Find the lengths of the following curves:

1. $y = x^{3/2}$, $0 \leqq x \leqq 4$
2. $y = \log x$, $\frac{1}{2} \leqq x \leqq 2$
3. $y = \log x$, $1 \leqq x \leqq e^2$
4. $y = 4 - x^2$, $-2 \leqq x \leqq 2$
5. $y = e^x$ between $x = 0$ and $x = 1$.
6. $y = x^{3/2}$ between $x = 1$ and $x = 3$.
7. $y = \frac{1}{2}(e^x + e^{-x})$ between $x = -1$ and $x = 1$.
8. $y = \log(1 - x^2)$, $0 \leqq x \leqq \frac{3}{4}$
9. $y = \frac{1}{2}(e^x + e^{-x})$, $-1 \leqq x \leqq 0$
10. $y = \log\cos x$, $0 \leqq x \leqq \pi/3$

XII, §4. PARAMETRIC CURVES

There is one other way in which we can describe a curve. Suppose that we look at a point which moves in the plane. Its coordinates can be given as a function of time t. Thus, when we give two functions of t, say

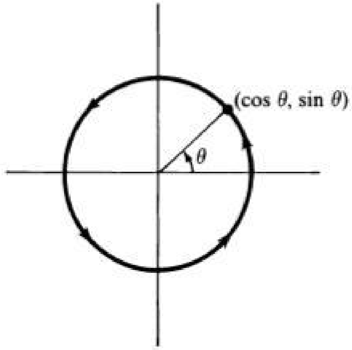
$$x = f(t), \qquad y = g(t),$$

we may view these as describing a point moving along a curve. The functions f and g give the coordinates of the point as functions of t.

Example 1. Let $x = \cos\theta$ and $y = \sin\theta$. Then

$$(x, y) = (\cos\theta, \sin\theta)$$

is a point on the circle.



As θ increases, we view the point as moving along the circle, in counterclockwise direction. The choice of letter θ really does not matter, and we could use t instead. In practice, the angle θ is itself expressed as a function of time. For example, if a bug moves around the circle with uniform (constant) angular speed, then we can write

$$\theta = \omega t,$$

where ω is constant. Then

$$x = \cos(\omega t) \qquad \text{and} \qquad y = \sin(\omega t).$$

This describes the motion of a bug around the circle with angular speed ω.

When (x, y) is described by two functions of t as above, we say that we have a **parametrization of the curve** in terms of the **parameter** t.

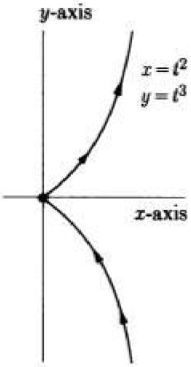
Example 2. Sketch the curve $x = t^2$, $y = t^3$.
We can make a table of values as usual.

t	x	y
0	0	0
1	1	1
2	4	8
-1	1	-1
-2	4	-8

Thus for each number t, we can plot the corresponding point (x, y). We also investigate when x and y are increasing or decreasing functions of t. For instance, taking the derivative, we get

$$\frac{dx}{dt} = 2t, \qquad \text{and} \qquad \frac{dy}{dt} = 3t^2.$$

Thus x increases when $t > 0$ and decreases when $t < 0$. The y-coordinate is increasing since $t^2 > 0$ (unless $t = 0$). Furthermore, the x-coordinate is always positive (unless $t = 0$). Thus the graph looks like this:



The parametric expression for the x- and y-coordinates is often useful to describe a motion of a bug (or a particle), whose coordinates are given as a function of time t. The arrows drawn in the figure suggest such a motion.

We can sometimes transform a curve given parametrically into a curve defined by an equation, possibly with some additional inequalities.

Example 3. The points (t^2, t^3) satisfy an "ordinary" equation

$$y^2 = x^3, \qquad \text{or} \qquad y = x^{3/2}.$$

However, we might also have written down the equation

$$y^4 = x^6,$$

which is satisfied by all points on our curve. In this case, however, there are solutions of this equation which are not given by our parametrization, corresponding to negative values of x, for instance,

$$x = -2, \qquad y = \pm 2\sqrt{2}.$$

Thus if we want to describe the set of all points on the parametrized curve by this latter relation, we must add an inequality $x \geqq 0$. It is then correct to say that the set of all points on the parametrized curve is the set of all solutions of the equation

$$y^4 = x^6$$

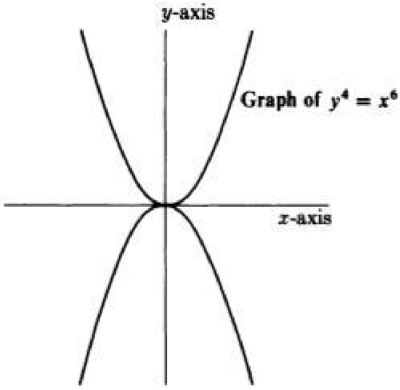
satisfying the inequality $x \geqq 0$.

Similarly, it is also correct to say that the set of all points on the parametrized curve is the set of all solutions of the equation

$$y^8 = x^{12}$$

satisfying the inequality $x \geqq 0$. And so on.

The graph of the equation $y^4 = x^6$ is as shown on the figure.



It is symmetric about both the x-axis and the y-axis. However, in the parametrized curve,

$$x = t^2, \qquad y = t^3,$$

only the right-hand portion of this graph occurs.

Example 4. Let

$$x(t) = \tfrac{1}{2}(e^t + e^{-t}) \qquad \text{and} \qquad y(t) = \tfrac{1}{2}(e^t - e^{-t}).$$

Either you have already verified that

$$x(t)^2 - y(t)^2 = 1,$$

or you should do so now by a straightforward multiplication and subtraction. Then you see that the point

$$(x(t), y(t)) = (\tfrac{1}{2}(e^t + e^{-t}), \tfrac{1}{2}(e^t - e^{-t}))$$

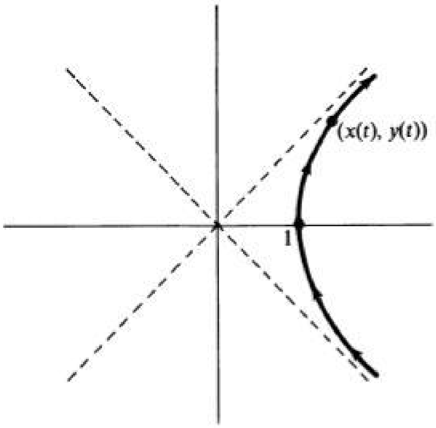
lies on the hyperbola defined by the equation

$$x^2 - y^2 = 1.$$

But note that $x(t) > 0$, in other words the x-coordinate given by the above function of t is always positive. Thus our functions

$$(x(t), y(t)) = \left(\tfrac{1}{2}(e^t + e^{-t}), \tfrac{1}{2}(e^t - e^{-t})\right)$$

describe a point on the right-hand branch of the hyperbola.



When t is large negative, then $x(t)$ is large positive, and $y(t)$ is large negative. When t is large positive, then $x(t)$ is large positive, and $y(t)$ is large positive.

As t increases the y-coordinate $y(t)$ increases from large negative to large positive. Thus a bug moving along the hyperbola according to the above parametrization is moving up, on the right-hand part of the hyperbola.

Length of parametrized curves

We shall now determine the length of a curve given by a parametrization.

Suppose that our curve is given by

$$x = f(t), \qquad y = g(t),$$

with $a \leqq t \leqq b$, and assume that both f, g have continuous derivatives. We consider a partition of the t-interval $[a, b]$:

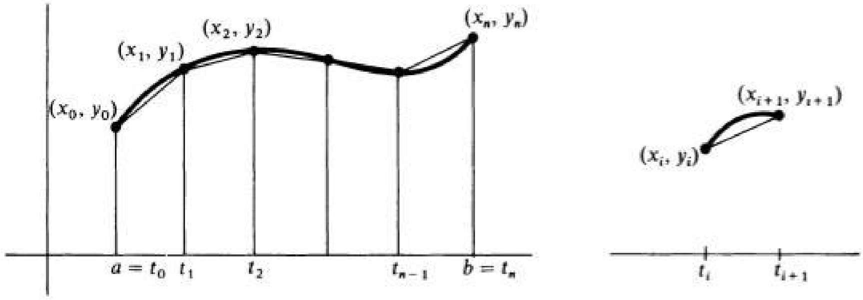
$$a = t_0 \leqq t_1 \leqq \cdots \leqq t_n = b.$$

We then obtain points

$$(x_i, y_i) = (f(t_i), g(t_i))$$

on the curve. The distance between two successive points is

$$\sqrt{(y_{i+1} - y_i)^2 + (x_{i+1} - x_i)^2} = \sqrt{(f(t_{i+1}) - f(t_i))^2 + (g(t_{i+1}) - g(t_i))^2}.$$



The sum of the lengths of the line segments gives an approximation of the length of the curve when the partition is sufficiently fine, that is when the numbers t_i, t_{i+1} are close together. Thus the sum

$$\sum_{i=0}^{n-1} \sqrt{(f(t_{i+1}) - f(t_i))^2 + (g(t_{i+1}) - g(t_i))^2}$$

gives an approximation to the length of the curve. We use the mean value theorem for f and g. There are numbers c_i and d_i between t_i and t_{i+1} such that

$$f(t_{i+1}) - f(t_i) = f'(c_i)(t_{i+1} - t_i),$$

$$g(t_{i+1}) - g(t_i) = g'(d_i)(t_{i+1} - t_i).$$

Substituting these values and factoring out $(t_{i+1} - t_i)$, we see that the sum of the lengths of our line segments is equal to

$$\sum_{i=0}^{n-1} \sqrt{f'(c_i)^2 + g'(d_i)^2}(t_{i+1} - t_i).$$

Let

$$G(t) = \sqrt{f'(t)^2 + g'(t)^2}.$$

Then our sum is almost equal to

$$\sum_{i=0}^{n-1} G(c_i)(t_{i+1} - t_i),$$

which would be a Riemann sum for G. It is not, because it is not necessarily true that $c_i = d_i$. Nevertheless, what we have done makes it very reasonable to **define the length of our curve (in parametric form) to be**

$$\boxed{\ell_a^b = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt.}$$

A complete justification that this integral is a limit, in a suitable sense, of our sums would require some additional theory, which is irrelevant anyway since we just want to make it reasonable that the above integral should represent what we mean physically by length.

Observe that when a curve is given in usual form $y = f(x)$ we can let

$$t = x = g(t) \qquad \text{and} \qquad y = f(t).$$

This shows how to view the usual form as a special case of the parametric form. In that case, $g'(t) = 1$ and the formula for the length in parametric form is seen to be the same as the formula we obtained before for a curve $y = f(x)$.

It is also convenient to put the formula in the other standard notation for the derivative. We have

$$\frac{dx}{dt} = f'(t) \qquad \text{and} \qquad \frac{dy}{dt} = g'(t).$$

Hence the length of the curve can be written in the form

$$\boxed{\ell_a^b = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.}$$

It is customary to let

$$s(t) = \text{length of the curve as function of } t.$$

Thus we may write

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, dt.$$

This yields

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{f'(t)^2 + g'(t)^2}.$$

Sometimes one writes symbolically

$$\boxed{(ds)^2 = (dx)^2 + (dy)^2,}$$

to suggest the Phythagoras theorem.

Example. Find the length of the curve

$$x = \cos t, \qquad y = \sin t$$

between $t = 0$ and $t = \pi$.

The length is the integral

$$\int_0^\pi \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt.$$

In view of the relation $(-\sin t)^2 = (\sin t)^2$ and a basic formula relating sine and cosine, we get

$$\int_0^\pi dt = \pi.$$

If we integrated between 0 and 2π we would get 2π. This is the length of the circle of radius 1.

Example. Find the length of the curve

$$x = \cos^3 \theta, \qquad y = \sin^3 \theta$$

for $0 \leqq \theta \leqq \pi/2$.

We have

$$\frac{dx}{d\theta} = 3\cos^2 \theta(-\sin \theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = 3\sin^2 \theta \cos \theta.$$

Hence

$$\begin{aligned} \ell_0^{\pi/2} &= \int_0^{\pi/2} \sqrt{9\cos^4 \theta \sin^2 \theta + 9 \sin^4 \theta \cos^2 \theta}\, d\theta \\ &= 3\int_0^{\pi/2} \sqrt{\cos^2 \theta \sin^2 \theta}\, d\theta \qquad (\text{because } \cos^2 \theta + \sin^2 \theta = 1) \\ &= 3\int_0^{\pi/2} \sin \theta \cos \theta\, d\theta \qquad (\text{because } \sin \theta \cos \theta \geqq 0 \text{ for } 0 \leqq \theta \leqq \pi/2). \end{aligned}$$

We integrate this by letting $u = \sin \theta$, $du = \cos \theta \, d\theta$, so that the integral is of the form

$$\int u \, du = u^2/2.$$

Then

$$\ell_0^{\pi/2} = 3 \frac{\sin^2 \theta}{2} \Bigg|_0^{\pi/2} = 3/2.$$

Example. Find the length of the same curve as in the preceding example, but for $0 \leqq \theta \leqq 2\pi$.

The same argument as before leads to the length formula

$$\ell_0^{2\pi} = 3 \int_0^{2\pi} \sqrt{\cos^2 \theta \sin^2 \theta} \, d\theta.$$

However, if A is a number, **the formula**

$$\sqrt{A^2} = A$$

is true only if A is positive. If A is negative, then

$$\sqrt{A^2} = -A = |A|.$$

Thus in taking the square root, we must be careful of the intervals where $\cos \theta \sin \theta$ is positive or negative. We have to split the integral into a sum:

$$\ell_0^{2\pi} = 3 \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta - 3 \int_{\pi/2}^{\pi} \cos \theta \sin \theta \, d\theta$$

$$+ 3 \int_{\pi}^{3\pi/2} \cos \theta \sin \theta \, d\theta - 3 \int_{3\pi/2}^{2\pi} \cos \theta \sin \theta \, d\theta.$$

These can now be easily evaluated as before to give the final answer 6. On the other hand, observe that

$$\int_0^{2\pi} \cos \theta \sin \theta \, d\theta = \tfrac{1}{2} \sin^2 \theta \Bigg|_0^{2\pi} = 0.$$

Here you get the value 0 because the function is sometimes positive and sometimes negative over the larger interval $0 \leqq \theta \leqq 2\pi$, and there are cancellations.

Polar coordinates

Let us now find a formula for the length of curves given in polar coordinates. Say the curve is

$$r = f(\theta),$$

with $\theta_1 \leqq \theta \leqq \theta_2$. We know that

$$\boxed{\begin{aligned} x &= r\cos\theta = f(\theta)\cos\theta,\\ y &= r\sin\theta = f(\theta)\sin\theta. \end{aligned}}$$

This puts the curve in parametric form, just as in the preceding considerations. Consequently we can apply the definition as before, and we see that the length is

$$\int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

You can compute $dx/d\theta$ and $dy/d\theta$ using the rule for the derivative of a product. If you do this, you will find that many terms cancel, and you obtain:

The length of a curve expressed in polar coordinates by $r = f(\theta)$ is given by the formula

$$\boxed{\int_{\theta_1}^{\theta_2} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.}$$

You should work it out for yourself, but for the record, we also do it in full here. Try not to look at it before you do it on your own.

We have:

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta,$$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta.$$

Hence

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= f(\theta)^2\sin^2\theta - 2f(\theta)f'(\theta)\sin\theta\cos\theta + f'(\theta)^2\cos^2\theta \\ &\quad + f(\theta)^2\cos^2\theta + 2f(\theta)f'(\theta)\cos\theta\sin\theta + f'(\theta)^2\sin^2\theta \\ &= f(\theta)^2 + f'(\theta)^2 \end{aligned}$$

because $\sin^2\theta + \cos^2\theta = 1$ and the middle terms cancel The formula then follows by plugging in.

Example. Find the length of the curve given in polar coordinates by $r = \sin\theta$, between $\theta = 0$ and $\theta = \pi/2$.

We use the formula just derived, and see that this length is given by the integral

$$\int_0^{\pi/2} \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \int_0^{\pi/2} d\theta = \pi/2.$$

Example. Find the length of the curve given in polar coordinates by $r = 1 - \cos\theta$ for $0 \leqq \theta \leqq \pi/4$.

We let $f(\theta) = 1 - \cos\theta$. The formula gives the length as

$$\begin{aligned}\ell_0^{\pi/4} &= \int_0^{\pi/4} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta \\ &= \int_0^{\pi/4} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta \\ &= \int_0^{\pi/4} \sqrt{2(1-\cos\theta)}\, d\theta.\end{aligned}$$

Put $\theta = 2u$. Recall the formula $1 - \cos 2u = 2\sin^2 u$. Then

$$1 - \cos\theta = 2\sin^2(\theta/2).$$

Hence the integral is

$$\begin{aligned}\ell_0^{\pi/4} &= \int_0^{\pi/4} \sqrt{4\sin^2(\theta/2)}\, d\theta. \\ &= 2\int_0^{\pi/4} \sin\left(\frac{\theta}{2}\right) d\theta \qquad (\text{because} \quad \sin(\theta/2) \geqq 0 \quad \text{if} \quad 0 \leqq \theta \leqq \pi/4) \\ &= -4\cos\left(\frac{\theta}{2}\right)\Big|_0^{\pi/4} = 4\left[1 - \cos\left(\frac{\pi}{8}\right)\right].\end{aligned}$$

XII, §4. EXERCISES

1. Carry out the computation giving the length in polar coordinates.
2. Find the length of a circle of radius r.
3. Find the length of the curve $x = e^t\cos t$, $y = e^t\sin t$ between $t = 1$ and $t = 2$.

4. Find the length of the curve $x = \cos^3 t$, $y = \sin^3 t$ (a) between $t = 0$ and $t = \pi/4$, and (b) between $t = 0$ and $t = \pi$.

Find the length of the following curves in the indicated interval.

5. $x = 2t + 1$, $y = t^2$, $0 \leqq t \leqq 2$

6. $x = 4 + 2t$, $y = \frac{1}{2}t^2 + 3$, $-2 \leqq t \leqq 2$

7. $x = 9t^2$, $y = 9t^3 - 3t$, $0 \leqq t \leqq 1/\sqrt{3}$

8. $x = 3t$, $y = 4t - 1$, $0 \leqq t \leqq 1$

9. $x = 1 - \cos t$, $y = t - \sin t$, $0 \leqq t \leqq 2\pi$

10. $x = a(1 - \cos t)$, $y = a(t - \sin t)$, with $a > 0$, and $0 \leqq t \leqq \pi$.

11. Sketch the curve $r = e^{\theta}$ (in polar coordinates), and also the curve $r = e^{-\theta}$.

12. Find the length of the curve $r = e^{\theta}$ between $\theta = 1$ and $\theta = 2$.

13. In general, give the length of the curve $r = e^{\theta}$ between two values θ_1 and θ_2.

Find the length of the following curves given in polar coordinates.

14. $r = 3\theta^2$ from $\theta = 1$ to $\theta = 2$

15. $r = e^{-4\theta}$, from $\theta = 1$ to $\theta = 2$

16. $r = 3\cos\theta$, from $\theta = 0$ to $\theta = \pi/4$

17. $r = 2/\theta$ from $\theta = \frac{1}{2}$ to $\theta = 4$ [*Hint*: Use $\theta = \sinh t$.]

18. $r = 1 + \cos\theta$ from $\theta = 0$ to $\theta = \pi/4$

19. $r = 1 - \cos\theta$ from $\theta = 0$ to $\theta = \pi$

20. $r = \sin^2 \dfrac{\theta}{2}$ from $\theta = 0$ to $\theta = \pi$

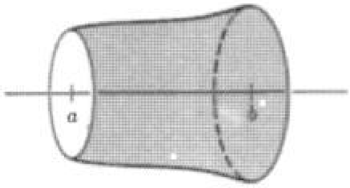
21. Find the length of one loop of the curve $r = 1 + \cos\theta$

22. Same, with $r = \cos\theta$, between $-\pi/2$ and $\pi/2$.

23. Find the length of the curve $r = 2/\cos\theta$ between $\theta = 0$ and $\theta = \pi/3$.

XII, §5. SURFACE OF REVOLUTION

Let $y = f(x)$ be a positive continuously differentiable function on an interval $[a, b]$. We wish to find a formula for the area of the surface of revolution of the graph of f around the x-axis, as illustrated on the figure.

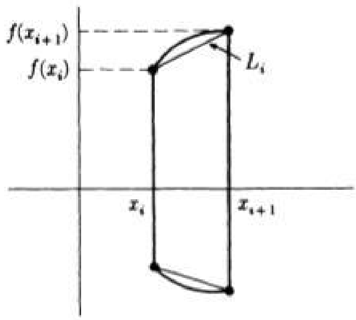


We shall see that the surface area is given by the integral

$$\boxed{S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

The idea again is to approximate the curve by line segments, as illustrated. We use a partition

$$a = x_0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n = b.$$



On the small interval $[x_i, x_{i+1}]$ the curve is approximated by the line segment joining the points $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$. Let L_i be the length of the segment. Then

$$L_i = \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}.$$

The length of a circle of radius y is $2\pi y$. If we rotate the line segment about the x-axis, then the area of the surface of rotation will be between

$$2\pi f(t_i)L_i \qquad \text{and} \qquad 2\pi f(s_i)L_i,$$

where $f(t_i)$ and $f(s_i)$ are the minimum and maximum of f, respectively, on the interval $[x_i, x_{i+1}]$. This is illustrated on Fig. 1.

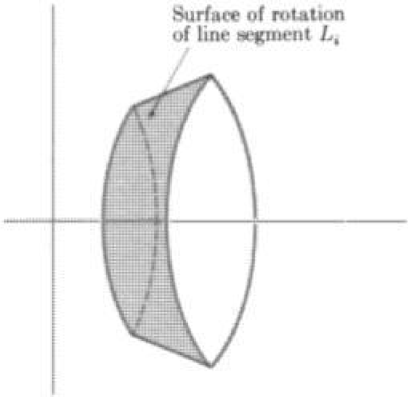


Figure 1

On the other hand, by the mean value theorem, we can write

$$f(x_{i+1}) - f(x_i) = f'(c_i)(x_{i+1} - x_i)$$

for some number c_i between x_i and x_{i+1}. Hence

$$\begin{aligned} L_i &= \sqrt{(x_{i+1} - x_i)^2 + f'(c_i)^2(x_{i+1} - x_i)^2} \\ &= \sqrt{1 + f'(c_i)^2}(x_{i+1} - x_i). \end{aligned}$$

Therefore the expression

$$2\pi f(c_i)\sqrt{1 + f'(c_i)^2}(x_{i+1} - x_i)$$

is an approximation of the surface of revolution of the curve over the small interval $[x_i, x_{i+1}]$. Now take the sum:

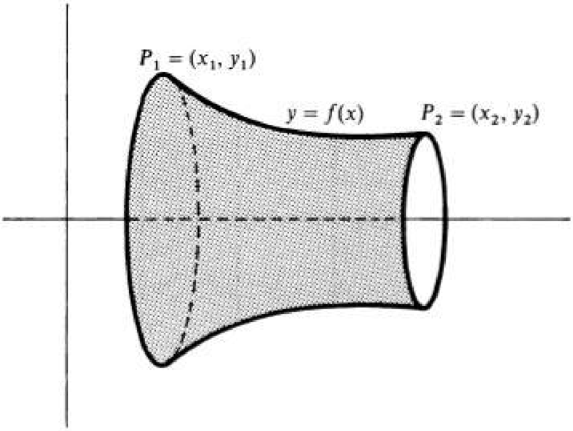
$$\sum_{i=0}^{n-1} 2\pi f(c_i)\sqrt{1 + f'(c_i)^2}(x_{i+1} - x_i).$$

This is a Riemann sum, between the upper and lower sums for the integral

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.}$$

Thus it is reasonable that the surface area should be defined by this integral, as was to be shown.

Physical example. It occurs frequently in practice that one wants to determine a minimal surface of revolution, obtained by rotating a curve between two given points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the plane. This is sometimes called the soap film problem. Indeed, given two rings perpendicular to the x-axis, the problem is to find a soap-film stretching across these two rings.



The soap film will realize the minimal surface of revolution. What is the equation for the curve $y = f(x)$? It turns out to be similar to that of the hanging cable, namely

$$y = b \cosh \frac{x - a}{b},$$

where a, b are constants depending on the two points (x_1, y_1) and (x_2, y_2). Here again we see a use of the cosh function.

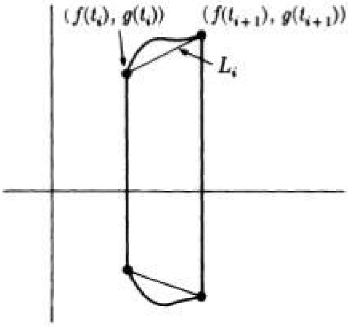
Area of revolution for parametric curves

As with length, we can also deal with curves given in parametric form. Suppose that

$$x = f(t), \qquad y = g(t), \qquad a \leqq t \leqq b.$$

We take a partition

$$a = t_0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n = b.$$



Then the length L_i between $(f(t_i), g(t_i))$ and $(f(t_{i+1}), g(t_{i+1}))$ is given by

$$\begin{aligned} L_i &= \sqrt{(f(t_{i+1}) - f(t_i))^2 + (g(t_{i+1}) - g(t_i))^2} \\ &= \sqrt{f'(c_i)^2 + g'(d_i)^2}(t_{i+1} - t_i), \end{aligned}$$

where c_i, d_i are numbers between t_i and t_{i+1}. Hence

$$2\pi g(c_i)\sqrt{f'(c_i)^2 + g'(d_i)^2}(t_{i+1} - t_i)$$

is an approximation for the surface of revolution of the curve in the small integral $[t_i, t_{i+1}]$. Consequently, it is reasonable that the surface of revolution is given by the integral

$$\boxed{S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.}$$

When $t = x$, this coincides with the formula found previously. It is also useful to write this formula symbolically

$$\boxed{S = \int 2\pi y\, ds,}$$

where symbolically, we had used

$$ds = \sqrt{(dx)^2 + (dy)^2}.$$

When using this symbolic notation, we do not put limits of integration. Only when we use the explicit parameter t over an interval $a \leqq t \leqq b$ do we put the values a, b for t below and above the integral sign. In this case, the surface area is written

$$\boxed{S = \int_a^b 2\pi y \frac{ds}{dt}\, dt.}$$

Example. We wish to find the area of a sphere of radius $a > 0$. It is best to view the sphere as the area of revolution of a circle of radius a, and to express the circle in parametric form,

$$x = a\cos\theta, \qquad y = a\sin\theta, \qquad 0 \leqq \theta \leqq \pi.$$

Then the formula yields:

$$\begin{aligned} S &= \int_0^\pi 2\pi a \sin\theta\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta \\ &= \int_0^\pi 2\pi a^2 \sin\theta\,d\theta \\ &= 2\pi a^2(-\cos\theta)\Big|_0^\pi \\ &= 4\pi a^2. \end{aligned}$$

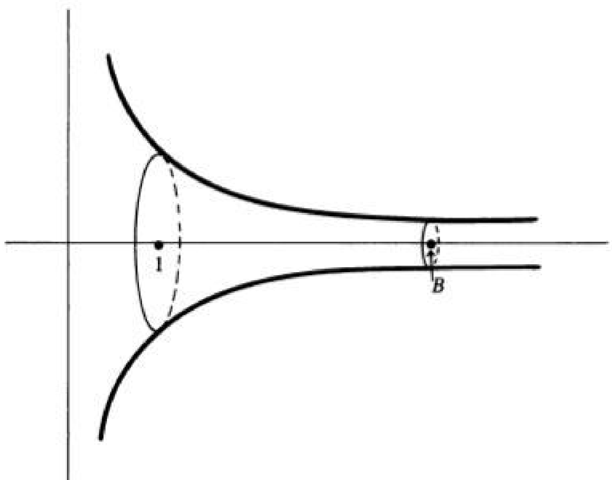
Let us now look at surfaces of revolution in terms of limits. Let $y = f(x)$ be a positive function as above, defined for all positive numbers x. Let:

V_B = volume of revolution of the graph of f between $x = 1$ and $x = B$;

S_B = area of revolution of the graph of f between $x = 1$ and $x = B$.

It is a fact which is usually very surprising that there may be cases when V_B approaches a finite limit when $B \to \infty$ whereas S_B become arbitrarily large when $B \to \infty$!!

Example. Let $f(x) = 1/x$.



Then using the formulas for volumes and surface of revolution, we find:

$$V_B = \int_1^B \pi \frac{1}{x^2}\,dx = \pi\left(1 - \frac{1}{B}\right) \to \pi \qquad \text{as } B \to \infty.$$

$$S_B = \int_1^B 2\pi \frac{1}{x}\sqrt{1 + f'(x)^2}\,dx$$

Now $f'(x)^2$ is a positive number, and so the expression under the square root sign is $\geqq 1$. Therefore

$$S_B \geqq 2\pi \int_1^B \frac{1}{x}\,dx = 2\pi \log B \to \infty \qquad \text{as} \quad B \to \infty.$$

Here we see how the volume approaches the finite limit π, whereas the surface of revolution becomes arbitrarily large.

In terms of a naive interpretation, suppose you have a bucket of paint containing π cubic units of paint. Then you can fill out the funnel inside the surface of revolution with this paint. But seemingly paradoxically, there isn't enough paint to paint the surface of revolution, as $B \to \infty$. This shows how treacherous naive intuition can be.

XII, §5. EXERCISES

1. Find the area of the surface obtained by rotating the curve

$$x = a\cos^3\theta, \qquad y = a\sin^3\theta$$

around the x-axis. [Sketch the curve. There is some symmetry. Determine the appropriate interval of θ.]

2. Find the area of the surface obtained by rotating the curve $y = x^3$ around the x-axis, between $x = 0$ and $x = 1$.

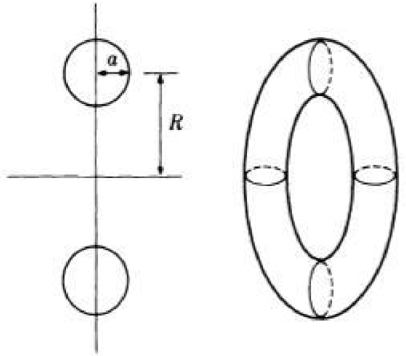
3. Find the area of the surface obtained by rotating the curve

$$x = \tfrac{1}{2}t^2 + t, \qquad y = t + 1$$

around the x-axis, from $t = 0$ to $t = 4$.

4. The circle $x^2 + y^2 = a^2$ is rotated around a line tangent to the circle. Find the area of the surface of rotation. [*Hint*: Set up coordinate axes and a parametrization of the circle in a convenient way. Remember what the curve $r = 2a\sin\theta$ in polar coordinates looks like? What if you rotate this curve around the x-axis?]

5. A circle as shown on the figure is rotated around the x-axis to form a torus (fancy name for a doughnut). What is the area of the torus?



Cross section of torus The torus

6. Find the area of the surface obtained by rotating an arc of the curve $y = x^{1/2}$ between (0, 0) and (4, 2) around the x-axis.

XII, §6. WORK

Suppose a particle moves on a curve, and that the length of the curve is described by a variable u.

Let $f(u)$ be a function. We interpret f as a force acting on the particle, in the direction of the curve. We want to find an integral expression for the work done by the force between two points on the curve.

Whatever our expression will turn out to be, it is reasonable that the work done should satisfy the following properties:

If a, b, c are three numbers, with $a \leqq b \leqq c$, then the work done between a and c is equal to the work done between a and b, plus the work done between b and c. If we denote the work done between a and b by $W_a^b(f)$, then we should have

$$W_a^c(f) = W_a^b(f) + W_b^c(f).$$

Furthermore, if we have a constant force M acting on the particle, it is reasonable to expect that the work done between a and b is

$$M(b - a).$$

Finally, if g is a stronger force than f, say $f(u) \leqq g(u)$, on the interval $[a, b]$, then we shall do more work with g than with f, meaning

$$W_a^b(f) \leqq W_a^b(g).$$

In particular, if there are two constant forces m and M such that

$$m \leqq f(u) \leqq M$$

throughout the interval $[a, b]$, then

$$m(b-a) \leqq W_a^b(f) \leqq M(b-a).$$

We shall see below that the work done by the force f between a distance a and a distance b is given by the integral

$$W_a^b(f) = \int_a^b f(u)\, du.$$

If the particle or object happens to move along a straight line, say along the x-axis, then f is given as a function of x, and our integral is simply

$$\int_a^b f(x)\, dx.$$

Furthermore, if the length of the curve u is given as a function of time t (as it is in practice, cf. §3) we see that the force becomes a function of t by the chain rule, namely $f(u(t))$. Thus between time t_1 and t_2 the work done is equal to

$$\boxed{\int_{t_1}^{t_2} f(u(t)) \frac{du}{dt}\, dt.}$$

This is the most practical expression for the work, since curves and forces are most frequently expressed as functions of time.

Let us now see why the work done is given by the integral. Let P be a partition of the interval $[a, b]$:

$$a = u_0 \leqq u_1 \leqq u_2 \leqq \cdots \leqq u_n = b.$$

Let $f(t_i)$ be a minimum for f on the small interval $[u_i, u_{i+1}]$, and let $f(s_i)$ be a maximum for f on this same small interval. Then the work done by moving the particle from length u_i to u_{i+1} satisfies the inequalities

$$f(t_i)(u_{i+1} - u_i) \leqq W_{u_i}^{u_{i+1}}(f) \leqq f(s_i)(u_{i+1} - u_i).$$

Adding these together we find

$$\sum_{i=0}^{n-1} f(t_i)(u_{i+1} - u_i) \leqq W_a^b(f) \leqq \sum_{i=0}^{n-1} f(s_i)(u_{i+1} - u_i).$$

The expressions on the left and right are lower and upper sums for the integral, respectively. Since the integral is the unique number between the lower sums and upper sums, it follows that

$$\boxed{W_a^b(f) = \int_a^b f(u)\,du.}$$

Example. Find the work done in stretching a spring from its unstretched position to a length of 10 cm longer. You may assume that the force needed to stretch the spring is proportional to the increase in length.

We visualize the spring as being horizontal, on the x-axis. Thus there is a constant K such that the force is given by

$$f(x) = Kx.$$

The work done is therefore

$$\begin{aligned}\int_0^{10} Kx\,dx &= \tfrac{1}{2}K \cdot 100\\ &= 50\,K.\end{aligned}$$

Example. Assume that gravity is a force inversely proportional to the square of the distance from the center of the earth. What work is done lifting a weight of 2 tons from the surface of the earth to a height of 100 mi above the earth? Assume that the radius of the earth is 4000 mi.

By assumption, there is a constant C such that the force of gravity is given by $f(x) = C/(x + 4000)^2$, where x denotes the height above the earth. When $x = 0$, our assumption is that

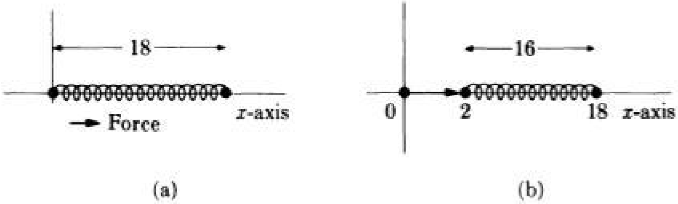
$$f(0) = 2 \text{ tons} = \frac{C}{(4000)^2}.$$

Hence $C = 32 \times 10^6$ tons. The work done is equal to the integral

$$\begin{aligned}\int_0^{100} f(x)\,dx &= 32 \times 10^6\left(-\frac{1}{(x + 4000)}\right)\Bigg|_0^{100}\\ &= 32 \times 10^6\left[\frac{1}{4000} - \frac{1}{4100}\right] \text{ton miles.}\end{aligned}$$

XII, §6. EXERCISES

1. A spring is 18 in. long, and a force of 10 lb is needed to hold the spring to a length of 16 in. If the force is given as $f(x) = kx$, where k is a constant, and x is the decrease in length, what is the constant k? How much work is done in compressing the spring from 16 in. to 12 in.?



2. Assuming that the force is given as $k\sin(\pi x/18)$, in the spring compression problem, answer the two questions of the preceding problem for this force.

3. A particle at the origin attracts another particle with a force inversely proportional to the square of the distance between them. Let C be the proportionality constant. What work is done in moving the second particle along a straight line away from the origin, from a distance r_1 to a distance $r > r_1$ from the origin?

4. In the preceding exercise, determine whether the work approaches a limit as r becomes very large, and find this limit if it exists.

5. Two particles repel each other with a force inversely proportional to the cube of their distance. If one particle is fixed at the origin, what work is done in moving the other along the x-axis from a distance of 10 cm to a distance of 1 cm toward the origin?

6. Assuming that gravity, as usual, is a force inversely proportional to the square of the distance from the center of the earth, what work is done in lifting a weight of 1000 lb from the surface of the earth to a height of 4000 mi above the surface? (Assume the radius of the earth is 4000 mi.)

7. A metal bar has length L and cross section S. If it is stretched x units, then the force $f(x)$ required is given by

$$f(x) = \frac{ES}{L}x$$

where E is a constant. If a bar 12 in. long of uniform cross section 4 in^2 is stretched 1 in. find the work done (in terms of E).

8. A particle of mass M grams at the origin attracts a particle of mass m grams at a point x cm away on the x-axis with a force of CmM/x^2 dynes, where C is a constant. Find the work done by the force
(a) when m moves from $x = 1/100$ to $x = 1/10$;
(b) when m moves from $x = 1$ to $x = 1/10$.

9. A unit positive charge of electricity at 0 repels a positive charge of amount c with a force c/r^2, where r is the distance between the particles. Find the work done by this force when the charge c moves along a straight line through 0 from a distance r_1 to a distance r_2 from 0.

10. Air is confined in a cylindrical chamber fitted with a piston. If the volume of air, at a pressure of 20 pounds/in^2 is 75 in^3, find the work done on the piston when the air expands to twice its original volume. Use the law

$$\text{Pressure} \cdot \text{Volume} = \text{Constant}.$$

XII, §7. MOMENTS AND CENTER OF GRAVITY

Suppose we have masses $m_1, \ldots, m_n$ at points $x_1, \ldots, x_n$ on the x-axis. The total **moment** of these masses is defined to be

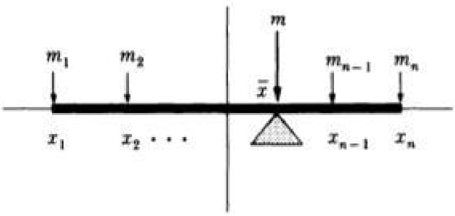
$$m_1 x_1 + \cdots + m_n x_n = \sum_{i=1}^{n} m_i x_i.$$

We may think of these masses as being distributed on some rod of uniform density, as shown on the figure.

The total mass is

$$m = m_1 + \cdots + m_n = \sum_{i=1}^{n} m_i.$$

We wish to find the point of the rod such that if we balance the rod at that point, then the rod will not move either up or down. Call this point $\bar{x}$.



Then $\bar{x}$ is a point such that if the total mass m is placed at $\bar{x}$ it will have the same balancing effect as the other masses m_i at x_i. The equation for this condition is that

$$m\bar{x} = m_1 x_1 + \cdots + m_n x_n = \sum_{i=1}^{n} m_i x_i.$$

Thus we can solve for $\bar{x}$ and get

$$\boxed{\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{1}{m}\sum_{i=1}^{n} m_i x_i.}$$

This point $\bar{x}$ is called the **center of gravity**, or **center of mass** of the masses $m_1, \ldots, m_n$.

Example. Let $m_1 = 4$ be at the point $x_1 = -3$ and let $m_2 = 7$ be at the point $x_2 = 2$. Then the total mass is

$$m = 4 + 7 = 11$$

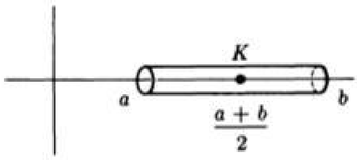
and the moment is

$$4 \cdot (-3) + 7 \cdot 2 = 2.$$

Hence the center of gravity is at the point

$$\bar{x} = 2/11.$$

Now suppose that we have a thin rod, placed along the x-axis on an interval $[a, b]$, as on the figure. Suppose that the rod has constant (uniform) density K.



The length of the rod is $(b - a)$. The total mass of the rod is then the density times the length, namely

$$\text{mass} = K(b - a).$$

It is reasonable to define the moment of the rod to be the same as that of the total mass placed at the center of the rod. This center has coordinates at the midpoint of the interval, namely

$$\frac{a + b}{2}.$$

Hence the **moment** of the rod is

$$M_a^b = K\frac{(a + b)}{2}(b - a).$$

Next suppose the density of the rod is not constant, but varies continuously, and so can be represented by a function $f(x)$. We try to find an approximation of what we mean by the moment of the rod. Thus we take a partition of the interval $[a, b]$,

$$a = x_0 \leqq x_1 \leqq \cdots \leqq x_n = b.$$

On each small interval $[x_i, x_{i+1}]$ the density will not vary much, and an approximation for the moment of the piece of rod along this interval is then given by

$$f(c_i)c_i(x_{i+1} - x_i)$$

where

$$c_i = \frac{x_{i+1} + x_i}{2}$$

is the midpoint of the small interval. Let

$$G(x) = xf(x).$$

Taking the sum of the above approximations yields

$$\sum_{i=0}^{n-1} f(c_i)c_i(x_{i+1} - x_i)$$

which is a Riemann sum for the integral

$$\int_a^b G(x)\,dx = \int_a^b xf(x)\,dx.$$

Consequently it is natural to define the **moment** of the rod with variable density as the integral

$$\boxed{M_a^b(f) = \int_a^b xf(x)\,dx.}$$

Let $\bar{x}$ be the coordinate of the center of gravity of the rod. This means that if the mass of the rod is placed at $\bar{x}$ then it has the same moment as the rod itself, and amounts to the equation

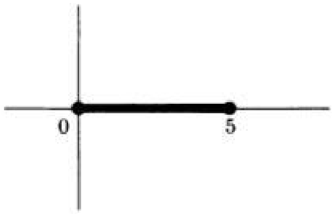
$$\bar{x} \cdot \text{total mass of rod} = \int_a^b xf(x)\,dx.$$

Hence we get an expression for the center of gravity, namely

$$\boxed{\bar{x} = \frac{\int_a^b xf(x)\,dx}{\int_a^b f(x)\,dx}.}$$

Example. Suppose a rod of length 5 cm has density proportional to the distance from one end. Find the center of gravity of the rod.

We suppose the rod laid out so that one end is at the origin, as shown on the figure.



The assumption on the density means that there is a constant C such that the density is given by the function

$$f(x) = Cx.$$

(i) The total mass is

$$\int_0^5 f(x)\,dx = \int_0^5 Cx\,dx = \frac{25C}{2}.$$

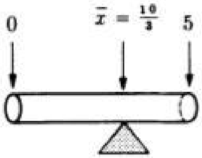
(ii) The moment is

$$\int_0^5 xf(x)\,dx = \int_0^5 Cx^2\,dx = \frac{125C}{3}.$$

Therefore

$$\bar{x} = \frac{125C/3}{25C/2} = \frac{10}{3}.$$

This center of gravity is such that if we balance the rod on a sharp edge at the point $\bar{x}$, then the rod will not lean either way.



A similar discussion can be carried out in higher dimensional space, for plane areas and solid volumes. It is best for this to wait until we discuss double and triple integrals in two or three variables in the next course.

XII, §7. EXERCISES

1. Suppose the density of a rod is proportional to the square of the distance from the origin, and the rod is 10 cm long, lying along the x-axis between 5 and 15 cm from the origin. Find its center of gravity.
2. Same as in Exercise 1, but assume the rod has constant density C.
3. Same as in Exercise 1, but assume the density of the rod is inversely proportional to the distance from the origin.

Part Four

Taylor's Formula and Series

In this part we study the approximation of functions by certain sums, called series. The chapter on Taylor's formula shows how to approximate functions by polynomials, and we estimate the error term to see how good an approximation we can get.

Note that the derivation of Taylor's formula is an application of integration by parts.

CHAPTER XIII

Taylor's Formula

We finally come to the point where we develop a method which allows us to compute the values of the elementary functions like sine, exp, and log. The method is to approximate these functions by polynomials, with an error term which is easily estimated. This error term will be given by an integral, and our first task is to estimate integrals. We then go through the elementary functions systematically, and derive the approximating polynomials.

You should review the estimates of Chapter X, §3, which will be used to estimate our error terms.

XIII, §1. TAYLOR'S FORMULA

Let f be a function which is differentiable on some interval. We can then take its derivative f' on that interval. Suppose that this derivative is also differentiable. We need a notation for its derivative. We shall denote it by $f^{(2)}$. Similarly, if the derivative of the function $f^{(2)}$ exists, we denote it by $f^{(3)}$, and so forth. In this system, the first derivative is denoted by $f^{(1)}$. (Of course, we can also write $f^{(2)} = f''$.)

In the d/dx notation, we also write:

$$f^{(2)}(x) = \frac{d^2 f}{dx^2},$$

$$f^{(3)}(x) = \frac{d^3 f}{dx^3},$$

and so forth.

Taylor's formula gives us a polynomial which approximates the function, in terms of the derivatives of the function. Since these derivatives are usually easy to compute, there is no difficulty in computing these polynomials.

For instance, if $f(x) = \sin x$, then $f^{(1)}(x) = \cos x$, $f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, and $f^{(4)}(x) = \sin x$. From there on, we start all over again.

In the case of e^x, it is even easier, namely $f^{(n)}(x) = e^x$ for all positive integers n.

It is also customary to denote the function f itself by $f^{(0)}$. Thus $f(x) = f^{(0)}(x)$.

We need one more piece of notation before stating Taylor's formula. When we take successive derivatives of functions, the following numbers occur frequently:

$$1, \quad 2\cdot 1, \quad 3\cdot 2\cdot 1, \quad 4\cdot 3\cdot 2\cdot 1, \quad 5\cdot 4\cdot 3\cdot 2\cdot 1, \quad \text{etc.}$$

These numbers are denoted by

$$1! \quad 2! \quad 3! \quad 4! \quad 5! \quad \text{etc.}$$

Thus

$$\begin{aligned} 1! &= 1, & 4! &= 24, \\ 2! &= 2, & 5! &= 120, \\ 3! &= 6, & 6! &= 720. \end{aligned}$$

When n is a positive integer, the symbol $n!$ is read n **factorial**. Thus in general,

$$n! = n(n-1)(n-2)\cdots 2\cdot 1$$

is the product of the first n integers from 1 to n.

It is also convenient to agree that $0! = 1$. This is the convention which makes certain formulas easiest to write.

Let us now look at the case of a polynomial

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

The numbers $c_0, \ldots, c_n$ are called the **coefficients** of the polynomial. We shall now see that these coefficients can be expressed in terms of the derivatives of $P(x)$ at $x = 0$. You should remember what you did in Chapter III, §7 when you computed higher derivatives. Let k be an integer $\geqq 0$. Then the k-th derivative of $P(x)$ is given by

$$P^{(k)}(x) = c_k k! + \text{an expression containing } x \text{ as a factor.}$$

The reason is: if we differentiate k times the terms

$$c_0,\ c_1x, \ldots, c_{k-1}x^{k-1}$$

then we get 0. And if we differentiate k times a power x^j with $j > k$ then some positive power of x will be left. Then if we evaluate the k-th derivative at 0, we get

$$P^{(k)}(0) = c_k k!$$

because when we substitute 0 for x all the other terms give 0. Therefore we find the desired expression of c_k in terms of the k-th derivative:

$$\boxed{c_k = \frac{P^{(k)}(0)}{k!}.}$$

Next, let f be a function which has derivatives up to order n on an interval. We are seeking a polynomial

$$P(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

whose derivatives at 0 (up to order n) are the same as the derivatives of f at 0, in other words

$$P^{(k)}(0) = f^{(k)}(0).$$

What must the coefficients $c_0, c_1, \ldots, c_n$ be to achieve this? The answer is immediate from our computation of the coefficients of a polynomial, namely we must have

$$k!\ c_k = f^{(k)}(0)$$

for each integer $k = 0, 1, \ldots, n$. Hence we have the desired expression for c_k, namely

$$\boxed{c_k = \frac{f^{(k)}(0)}{k!}.}$$

Definition. The **Taylor polynomial** of degree $\leqq n$ for the function f is the polynomial

$$\boxed{P_n(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.}$$

Example. Let $f(x) = \sin x$. It is easy to work out the derivatives (see §3), and you will find that the Taylor polynomials have the form

$$P_{2m+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}.$$

Only odd values of n occur, so we write

$$n = 2m + 1 \qquad \text{with} \quad m \geqq 0.$$

Example. Let $f(x) = e^x$. Then $f^{(k)}(x) = e^x$ for all positive integers k. Hence $f^{(k)}(0) = 1$ for all k, and so the Taylor polynomial has the form

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

We now want to know how good an approximation the polynomial $P_n(x)$ gives to $f(x)$. So we write

$$f(x) = P_n(x) + R_{n+1}(x),$$

where R_{n+1} is called the **remainder**.

We shall have to estimate the remainder term $R_{n+1}(x)$. We shall eventually prove that there is a number c between 0 and x such that

$$\boxed{R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.}$$

Thus the remainder term will look very much like the main terms, except that the coefficient

$$\frac{f^{(n+1)}(c)}{(n+1)!}$$

is taken at some intermediate point c instead of being taken at 0.

Since it is easy to estimate the derivatives of the functions $\sin x$, $\cos x$, e^x we shall be able to see that the Taylor polynomials give good approximations to the function. If you are willing to take for granted the expression

$$\boxed{f(x) = f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!} x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} + R_n(x),}$$

where

$$\boxed{R_n(x) = \frac{f^{(n)}(c)}{n!} x^n}$$

for some number c between 0 and x, then you can read immediately the later sections, §3, and so on, to get immediately into the applications to the elementary functions.

Observe that it does not make any difference whether we write

$$f(x) = P_n(x) + R_{n+1}(x) \qquad \text{or} \qquad f(x) = P_{n-1}(x) + R_n(x).$$

This amounts merely to a change of indices. We use whichever is more convenient.

Of course, the above assertion does not state anything precise about the number c other than c lies between 0 and x. But the point of the formula, and its remainder term, is that we do not need to know anything more precise in order to *estimate* the remainder term. The polynomial preceding the remainder term gives a value at x. We want to know only how close this value is to $f(x)$. For that purpose, we need only give a bound

$$\frac{|f^{(n)}(c)|}{n!} |x|^n \leqq \text{something or other,}$$

and so we need only give a bound for the n-th derivative $|f^{(n)}(c)|$. To give such a bound can be done without knowing an exact value for the n-th derivative at the number c. You can see how this is done in §3 and the following sections, when we deal systematically with all the elementary functions.

We shall now develop the Taylor formula theoretically, and prove that the remainder term has the stated form. It will also be convenient instead of working with the numbers 0 and x to work with arbitrary numbers a and b. We also state Taylor's formula with a somewhat different form for the remainder term, but a form which will arise naturally in the proof. After that, we shall prove that the integral form is equal to the expression stated above.

Theorem 1.1. *Let f be a function defined on a closed interval between two numbers a and b. Assume that the function has n derivatives on this interval, and that all of them are continuous functions. Then*

$$\boxed{\begin{aligned} f(b) = f(a) + \frac{f^{(1)}(a)}{1!}(b-a) + \frac{f^{(2)}(a)}{2!}(b-a)^2 + \cdots \\ + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + R_n, \end{aligned}}$$

where R_n (which is called the remainder term) is the integral

$$\boxed{R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt.}$$

The remainder term looks slightly complicated. In Theorem 2.1 we shall prove that R_n can be expressed in a form very similar to the other terms, namely

$$\boxed{R_n = \frac{f^{(n)}(c)}{n!} (b-a)^n}$$

for some number c between a and b. Taylor's formula with this form of the remainder is then very easy to memorize,

The most important case of Theorem 1.1 occurs when $a = 0$. In that case, the formula reads

$$f(b) = f(0) + \frac{f'(0)}{1!} b + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!} b^{n-1} + R_n.$$

Furthermore, if x is any number between a and b, the same formula remains valid for this number x instead of b, simply by considering the interval between a and x instead of the interval between a and b. Thus if $a = 0$, then the formula reads

$$\boxed{f(x) = f(0) + \frac{f'(0)}{1!} x + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} + R_n(x),}$$

where

$$\boxed{R_n(x) = f^{(n)}(c) \frac{x^n}{n!}}$$

and c is a number between 0 and x. Each derivative $f(0)$, $f'(0), \ldots, f^{(n-1)}(0)$ is a number, and we see that the terms preceding R_n make up a polynomial in x. This is the approximating polynomial.

We shall now prove the theorem. The proof is an application of integration by parts. First to get the idea of the proof, we carry out two special cases.

Special cases. We proceed stepwise. We know that a function is the integral of its derivative. Thus when $n = 1$ we have

$$f(b) - f(a) = \int_a^b f'(t)\,dt.$$

Let $u = f'(t)$ and $dv = dt$. Then $du = f''(t)\,dt$. We are tempted to put $v = t$. This is one case where we choose another indefinite integral, namely $v = -(b - t)$, which differs from t by a constant. We still have $dv = dt$ (the minus signs cancel!). Integrating by parts, we get

$$\begin{aligned}\int_a^b u\,dv &= uv\Big|_a^b - \int_a^b v\,du \\ &= -f'(t)(b-t)\Big|_a^b - \int_a^b -(b-t)f^{(2)}(t)\,dt \\ &= f'(a)(b-a) + \int_a^b (b-t)f^{(2)}(t)\,dt.\end{aligned}$$

This is precisely the Taylor formula when $n = 2$.

We push it one step further, from 2 to 3. We rewrite the integral just obtained as

$$\int_a^b f^{(2)}(t)(b-t)\,dt.$$

Let $u = f^{(2)}(t)$ and $dv = (b - t)\,dt$. Then

$$du = f^{(3)}(t)\,dt \qquad \text{and} \qquad v = \frac{-(b-t)^2}{2} = \int (b-t)\,dt.$$

Thus, integrating by parts, we find that our integral, which is of the form $\int_a^b u\,dv$, is equal to

$$\begin{aligned}uv\Big|_a^b - \int_a^b v\,du &= -f^{(2)}(t)\frac{(b-t)^2}{2}\Big|_a^b - \int_a^b -\frac{(b-t)^2}{2}f^{(3)}(t)\,dt \\ &= f^{(2)}(a)\frac{(b-a)^2}{2} + R_3.\end{aligned}$$

Here, R_3 is the desired remainder, and the term preceding it is just the proper term in the Taylor formula.

If you need it, do the next step yourself, from 3 to 4. We shall now see how the general step goes, from n to $n + 1$.

General case. Suppose that we have already obtained the first $n-1$ terms of the Taylor formula, with a remainder term

$$R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt,$$

which we rewrite

$$R_n = \int_a^b f^{(n)}(t) \frac{(b-t)^{n-1}}{(n-1)!}\, dt.$$

Let

$$u = f^{(n)}(t) \quad \text{and} \quad dv = \frac{(b-t)^{n-1}}{(n-1)!}\, dt.$$

Then

$$du = f^{(n+1)}(t)\, dt \quad \text{and} \quad v = \frac{-(b-t)^n}{n!}.$$

Here we use the fact that b is constant, and

$$\int (b-t)^{n-1}\, dt = -\frac{(b-t)^n}{n}.$$

Note the appearance of a minus sign, from the chain rule. Also we use

$$n(n-1)! = n!$$

to give the stated value of v. Thus the denominator climbs from $(n-1)!$ to $n!$.

Integrating by parts, we find:

$$R_n = uv\Big|_a^b - \int_a^b v\, du = -f^{(n)}(t)\frac{(b-t)^n}{n!}\Big|_a^b - \int_a^b -\frac{(b-t)^n}{n!} f^{(n+1)}(t)\, dt$$

$$= f^{(n)}(a)\frac{(b-a)^n}{n!} + \int_a^b \frac{(b-t)^n}{n!} f^{(n+1)}(t)\, dt.$$

Thus we have split off one more term of the Taylor formula, and the new remainder is the desired R_{n+1}. This concludes the proof.

XIII, §1. EXERCISES

1. Let $f(x) = \log(1+x)$.
   (a) Find a formula for the derivatives of $f(x)$. Start with $f^{(1)}(x) = (x+1)^{-1}$, $f^{(2)}(x) = -(x+1)^{-2}$. Work out $f^{(k)}(x)$ for $k = 3, 4, 5$ and then write down the formula for arbitrary k.

(b) Find $f^{(k)}(0)$ for $k = 1, 2, 3, 4, 5$. Then show in general that

$$f^{(k)}(0) = (-1)^{k+1}(k-1)!.$$

(c) Conclude that the Taylor polynomial $P_n(x)$ for $\log(1+x)$ is

$$P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n}.$$

2. Find the polynomials $P_n(x)$ for the function $f(x) = \cos x$ and the values $n = 1, 2, 3, 4, 5, 6, 7, 8$.

XIII, §2. ESTIMATE FOR THE REMAINDER

Theorem 2.1. *In Taylor's formula of Theorem* 1.1, *there exists a number c between a and b such that the remainder R_n is given by*

$$\boxed{R_n = \frac{f^{(n)}(c)(b-a)^n}{n!}.}$$

If M_n is a number such that $|f^{(n)}(x)| \leqq M_n$ for all x in the interval, i.e. M_n is an upper bound for $|f^{(n)}(x)|$, then

$$|R_n| \leqq \frac{M_n|b-a|^n}{n!}.$$

Proof. The second assertion follows at once from the first, making the estimate

$$|R_n| = \frac{|f^{(n)}(c)|\,|b-a|^n}{n!} \leqq M_n\frac{|b-a|^n}{n!}.$$

Let us prove the first assertion. Since $f^{(n)}$ is continuous on the interval, there exists a point u in the interval such that $f^{(n)}(u)$ is a maximum, and a point v such that $f^{(n)}(v)$ is a minimum for all values of $f^{(n)}$ in our interval.

Let us assume that $a < b$. Then for any t in the interval, $b - t$ is $\geqq 0$, and hence

$$\frac{(b-t)^{n-1}}{(n-1)!}f^{(n)}(v) \leqq \frac{(b-t)^{n-1}}{(n-1)!}f^{(n)}(t) \leqq \frac{(b-t)^{n-1}}{(n-1)!}f^{(n)}(u).$$

Using Theorem 3.1 of Chapter X, §3, we conclude that similar inequalities hold when we take the integral. However, $f^{(n)}(v)$ and $f^{(n)}(u)$ are fixed numbers which can be taken out of the integral sign. Consequently, we obtain

$$f^{(n)}(v) \int_a^b \frac{(b-t)^{n-1}}{(n-1)!}\,dt \leqq R_n \leqq f^{(n)}(u) \int_a^b \frac{(b-t)^{n-1}}{(n-1)!}\,dt.$$

We now perform the integration, which is very easy, and get

$$\int_a^b (b-t)^{n-1}\,dt = -\left.\frac{(b-t)^n}{n}\right|_a^b = \frac{(b-a)^n}{n}.$$

[*Remark*: this is the same integral that already came up in the integration by parts, in the proof of Theorem 1.1.] Therefore

$$f^{(n)}(v)\,\frac{(b-a)^n}{n!} \leqq R_n \leqq f^{(n)}(u)\,\frac{(b-a)^n}{n!}.$$

By the intermediate value theorem, the n-th derivative $f^{(n)}(t)$ takes on all values between its minimum and maximum in the interval. Hence

$$f^{(n)}(t)\,\frac{(b-a)^n}{n!}$$

takes on all values between *its* minimum and maximum in the interval. Hence there is some point c in the interval such that

$$R_n = f^{(n)}(c)\,\frac{(b-a)^n}{n!},$$

which is what we wanted.

The proof in case $b < a$ is similar, except that certain inequalities get reversed. We omit it.

The estimate of the remainder is particularly useful when b is close to a. In that case, let us rewrite Taylor's formula by setting $b - a = h$. We obtain:

Theorem 2.2. *Assumptions being as in Theorem* 1.1, *we have*

$$f(a+h) = f(a) + f'(a)h + \cdots + f^{(n-1)}(a)\,\frac{h^{n-1}}{(n-1)!} + R_n$$

with the estimate

$$|R_n| \leqq M_n\,\frac{|h|^n}{n!},$$

where M_n is a bound for the absolute value of the n-th derivative of f between a and $a + h$.

In the following sections, we give several examples. We shall often take $a = 0$, so that we have

$$\boxed{f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!} x^{n-1} + R_n(x)}$$

with the estimate

$$|R_n(x)| \leqq M_n \frac{|x|^n}{n!}$$

if M_n is a bound for the n-th derivative of f between 0 and x.

This means that we have expressed $f(x)$ in terms of a polynomial, and a remainder term. As we already said, the polynomial

$$P_n(x) = c_0 + c_1 x + \cdots + c_n x^n,$$

where

$$c_k = \frac{f^{(k)}(0)}{k!},$$

is called the **Taylor polynomial of degree** $\leqq n$ of $f(x)$. We call c_k the k-th **Taylor coefficient** of f. These polynomials will be computed explicitly for all the elementary functions in the next sections.

A polynomial is essentially the easiest function to deal with. Thus it is useful if we can prove that the Taylor polynomials give approximations to the given function. For this to be the case, we have to estimate the remainder, and it is true in the case of the elementary functions that the remainder R_n approaches 0 as $n \to \infty$. This means that the Taylor polynomial $P_n(x)$ approaches $f(x)$ as $n \to \infty$, and we therefore get the desired polynomial approximation.

XIII, §3. TRIGONOMETRIC FUNCTIONS

Let $f(x) = \sin x$ and take $a = 0$ in Taylor's formula. We have already mentioned what the derivatives of $\sin x$ and $\cos x$ are. Thus

$$f(0) = 0, \qquad f^{(2)}(0) = 0,$$

$$f'(0) = 1, \qquad f^{(3)}(0) = -1.$$

The Taylor formula for $\sin x$ is therefore as follows:

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{m-1}\frac{x^{2m-1}}{(2m-1)!} + R_{2m+1}(x).}$$

We see that all the even terms are 0 because $\sin 0 = 0$.

We can estimate $\sin x$ and $\cos x$ very simply, because

$$|\sin x| \leqq 1 \quad \text{and} \quad |\cos x| \leqq 1$$

for all x. In Theorem 2.2 we take the bound $M_n = 1$, that is

$$|f^{(n)}(c)| \leqq 1$$

for all n, and

$$\boxed{|R_n(x)| \leqq \frac{|x|^n}{n!}.}$$

Thus if we look at all values of x such that $|x| \leqq 1$, we see that $R_n(x)$ approaches 0 when n becomes very large.

Example 1. Compute $\sin(0.1)$ to 3 decimals.

Here we have $x = 0.1$. We want to find n such that

$$\frac{|x|^n}{n!} \leqq 10^{-3}.$$

By inspection, we see that $n = 3$ will work. Indeed, we have

$$|R_3(0.1)| \leqq \frac{(0.1)^3}{3!} = \frac{10^{-3}}{6}.$$

Such an error term would put us within the required range of accuracy. Hence we can just use Taylor's formula,

$$\sin x = x + R_3(x).$$

We find

$$\sin(0.1) = 0.100 + E,$$

with the error term $E = R_3(0.1)$, such that $|E| \leqq \frac{1}{6}\ 10^{-3}$. We see how efficient this is for computing the sine for small values of x.

Definition. We shall say that some expression has the **value** A **with an accuracy of** 10^{-n}, or to n **decimals** if the expression is equal to $A + E$, with an error term E such that

$$|E| \leqq 10^{-n}.$$

In the preceding example, we may say that $\sin(0.1)$ has the value 0.1 with an accuracy of 10^{-3}, or to 3 decimals.

Warning. Do not write $\sin(0.1) = 0.1$. **This is false.** Always write the error, so write

$$\sin(0.1) = 0.1 + E,$$

and then give an estimate for $|E|$.

Example 2. Let us compute sine of 10° with an accuracy of 10^{-3}. First we must convert degrees to radians, and we have

$$10^\circ = 10\,\frac{\pi}{180} = \frac{\pi}{18} \text{ radians.}$$

Thus we have to compute $\sin(\pi/18)$. We assume that π is approximately $3.14159\ldots$ and in particular, $\pi < 3.2$. This will be shown later. Then

$$\frac{\pi}{18} < \frac{1}{5}.$$

We have to express $\sin(\pi/18)$ with the Taylor polynomial of some degree and a remainder which has to be estimated. This requires trial and error. You should experiment with various possibilities. Here we give right away one that works. We have

$$\sin\left(\frac{\pi}{18}\right) = \frac{\pi}{18} - \frac{1}{6}\left(\frac{\pi}{18}\right)^3 + R_5\left(\frac{\pi}{18}\right).$$

If we know π accurately enough, then we can compute the first two terms to within any desired accuracy, by means of simple arithmetic: addition, subtraction, multiplication and division. Then we have to estimate $R_5(\pi/18)$ to know that it is within the desired accuracy. We have

$$\left|R_5\left(\frac{\pi}{18}\right)\right| \leqq \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 < \frac{1}{120}\left(\frac{1}{5}\right)^5 < \frac{1}{3} \times 10^{-5}$$

by easy arithmetic. Hence the first two terms

$$\frac{\pi}{18} - \frac{1}{6}\left(\frac{\pi}{18}\right)^3$$

give an **approximation of** sine 10° **to an accuracy** of 10^{-5}, which is better than what we wanted originally. Now try to see how good an approximation you would get by using just one term:

$$\sin\left(\frac{\pi}{18}\right) = \frac{\pi}{18} + R_3\left(\frac{\pi}{18}\right).$$

Example 3. Compute $\sin\left(\frac{\pi}{6} + 0.2\right)$ to an accuracy of 10^{-4}.

In this case, we use the Taylor formula for $f(a + h)$. We take

$$a = \frac{\pi}{6} \qquad \text{and} \qquad h = 0.2.$$

By trial and error, and guessing, we try for the remainder R_4. Thus

$$\begin{aligned}\sin(a + h) &= \sin a + \cos(a)\frac{h}{1} - \sin(a)\frac{h^2}{2!} - \cos(a)\frac{h^3}{3!} + R_4\\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}(0.2) - \frac{1}{2}\frac{(0.2)^2}{2} - \frac{\sqrt{3}}{2}\frac{(0.2)^3}{6} + R_4.\end{aligned}$$

For R_4 we have the estimate

$$|R_4| \leqq \frac{(0.2)^4}{4!} = \frac{16 \cdot 10^{-4}}{24} \leqq 10^{-4},$$

which is within the required bounds of accuracy.

Convention. In Examples 2 and 3 we left the answer as a sum of a few terms, plus an error which we estimated. *You do not need to carry out the actual decimal expansion of the first four terms.* If you have a small pocket computer, however, then do it on the computer to see that you can actually get a decimal answer. For such a purpose, the machine becomes better than the brain, but the brain was better to estimate the remainder.

It is still true that the remainder term of the Taylor formula for $\sin x$ approaches 0 when n becomes large, even when x is > 1. For this we need to investigate $x^n/n!$ when x is > 1. The difficulty is that when $x > 1$, then x^n becomes large when $n \to \infty$, and also $n! \to \infty$ when $n \to \infty$. Thus the numerator and denominator fight each other, and we must determine which one wins. Let us deal with an example, just to get a feel for what goes on. Take $x = 2$. What happens to the fraction $2^n/n!$ when $n \to \infty$? Make a table:

n	1	2	3	4	5
$\frac{2^n}{n!}$	2	2	$\frac{8}{6} = \frac{4}{3}$	$\frac{16}{24} = \frac{2}{3}$	$\frac{32}{120} = \frac{4}{15}$

It should now become experimentally clear that

$$\frac{2^n}{n!} \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus we *guess* the answer by experimenting numerically. Next, our task is to *prove* the general result.

Theorem 3.1. *Let c be any number. Then $c^n/n!$ approaches 0 as n becomes very large.*

Proof. We may assume $c > 0$. Let n_0 be an integer such that $n_0 > 2c$. Thus $c < n_0/2$, and $c/n_0 < \frac{1}{2}$. We write

$$\begin{aligned}\frac{c^n}{n!} &= \frac{c \cdot c \cdots c}{1 \cdot 2 \cdots n_0} \frac{c}{(n_0 + 1)} \frac{c}{(n_0 + 2)} \cdots \frac{c}{n} \\ &\leqq \frac{c^{n_0}}{n_0!}\left(\frac{1}{2}\right) \cdots \left(\frac{1}{2}\right) \\ &= \frac{c^{n_0}}{n_0!}\left(\frac{1}{2}\right)^{n - n_0}.\end{aligned}$$

As n becomes large, $(1/2)^{n-n_0}$ becomes small and our fraction approaches 0. Take for instance $c = 10$. We write

$$\frac{10^n}{n!} = \frac{10 \cdots 10}{1 \cdot 2 \cdots 20} \frac{(10) \cdots (10)}{(21) \cdots (n)} < \frac{10^{20}}{20!}\left(\frac{1}{2}\right)^{n-20}$$

and $(1/2)^{n-20}$ approaches 0 as n becomes large.

From the theorem we see that the remainder

$$|R_n(x)| \leqq \frac{|x|^n}{n!}$$

approaches 0 as n becomes large.

Sometimes a definite integral cannot be evaluated from an indefinite one, but we can find simple approximations to it by using Taylor's expansion.

In the next example, and in exercises, we shall now use frequently the estimate for an integral given in Theorems 3.2 and 3.3 of Chapter X, that is: Let $a < b$ and let f be continuous on $[a, b]$. Let M be a number such that $|f(x)| \leqq M$ for all x in the interval. Then

$$\left|\int_a^b f(x)\, dx\right| \leqq \int_a^b |f(x)|\, dx \leqq M(b-a).$$

Ron Infante tells me the numerical computations of integrals like the one in the next example occur frequently in the study of communication networks, in connection with square waves.

Example 4. Compute to two decimals the integral

$$\int_0^1 \frac{\sin x}{x}\, dx.$$

We have

$$\sin x = x - \frac{x^3}{3!} + R_5(x) \qquad \text{and} \qquad |R_5(x)| \leqq \frac{|x|^5}{5!}.$$

Hence

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{R_5(x)}{x} \qquad \text{and} \qquad \left|\frac{R_5(x)}{x}\right| \leqq \frac{|x|^4}{5!}.$$

Hence

$$\int_0^1 \frac{\sin x}{x}\, dx = x - \frac{x^3}{3 \cdot 3!}\bigg|_0^1 + E, \qquad \text{where} \qquad E = \int_0^1 \frac{R_5(x)}{x}\, dx.$$

The error term E satisfies

$$|E| \leqq \int_0^1 \left|\frac{R_5(x)}{x}\right| dx \leqq \int_0^1 \frac{x^4}{5!}\, dx = \frac{x^5}{5 \cdot 5!}\bigg|_0^1 = \frac{1}{600}.$$

Furthermore

$$x - \frac{x^3}{3 \cdot 3!}\bigg|_0^1 = 1 - \frac{1}{18} = \frac{17}{18}.$$

Hence

$$\int_0^1 \frac{\sin x}{x}\,dx = \frac{17}{18} + E \qquad \text{where} \qquad |E| \leqq \frac{1}{600}.$$

Example 5. Let us compute

$$I = \int_0^1 \sin x^2\,dx.$$

We let $u = x^2$. Then

$$\sin u = u - \frac{u^3}{3!} + R_5(u).$$

Hence

$$\begin{aligned} I &= \int_0^1 \left(x^2 - \frac{x^6}{3!}\right) dx + \int_0^1 R_5(x^2)\,dx \\ &= \left[\frac{x^3}{3} - \frac{x^7}{7\cdot 6}\right]_0^1 + E \\ &= \tfrac{1}{3} - \tfrac{1}{42} + E, \end{aligned}$$

where

$$E = \int_0^1 R_5(x^2)\,dx.$$

We know that

$$|R_5(u)| \leqq \frac{|u|^5}{5!}.$$

Since $u = x^2$, we find

$$|E| \leqq \int_0^1 \frac{x^{10}}{5!}\,dx = \frac{1}{11\cdot 120} < 10^{-3}.$$

Hence

$$I = \tfrac{1}{3} - \tfrac{1}{42} + E, \qquad \text{with} \quad |E| < 10^{-3}.$$

Remark. Although the *notation* in the preceding example is *similar* to integration by substitution, it should be emphasized that the procedure followed is *not* what we called previously integration by substitution.

We have discussed the sine. The cosine can be discussed in the same way. We have the Taylor formula

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} + R_{2m+2}(x)}$$

and

$$|R_{2m+2}(x)| \leqq \frac{|x|^{2m+2}}{(2m+2)!}.$$

Observe that only the even terms appear with non-zero coefficient. In the sine formula, only the odd terms appeared. This is because the odd-order derivatives of cosine are equal to 0 at 0, and the even-order derivatives of the sine are equal to 0 at 0.

Example 6. Suppose we want to find the value of

$$\int_0^1 \frac{\cos x - 1}{x}\,dx,$$

to 2 decimals. We write

$$\cos x = 1 - \frac{x^2}{2} + R_4(x).$$

Then

$$\frac{\cos x - 1}{x} = -\frac{x}{2} + \frac{R_4(x)}{x},$$

and for $0 \leqq x \leqq 1$,

$$\left|\frac{R_4(x)}{x}\right| \leqq \frac{x^4}{4!x} = \frac{x^3}{4!}.$$

We obtain

$$\begin{aligned}\int_0^1 \frac{\cos x - 1}{x}\,dx &= -\int_0^1 \frac{x}{2}\,dx + E \qquad \text{where} \qquad E = \int_0^1 \frac{R_4(x)}{x}\,dx.\\ &= -\frac{1}{2}\frac{x^2}{2}\bigg|_0^1 + E\\ &= -\tfrac{1}{4} + E.\end{aligned}$$

We estimate E:

$$|E| \leqq \int_0^1 \frac{x^3}{4!}\,dx = \frac{1}{24}\frac{x^4}{4}\bigg|_0^1 = \frac{1}{96}.$$

We just miss the desired estimate by a few percentage points. This means that to get the desired accuracy, you have to use one more term from the Taylor polynomial of $\cos x$. Thus you write

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + R_6(x)$$

and

$$\frac{\cos x - 1}{x} = -\frac{x}{2} + \frac{x^3}{4!} + \frac{R_6(x)}{x}.$$

Then

$$\int_0^1 \frac{\cos x - 1}{x}\,dx = \int_0^1 \left[-\frac{x}{2} + \frac{x^3}{4!}\right] dx + E,$$

where

$$E = \int_0^1 \frac{R_6(x)}{x}\,dx.$$

Now we use the estimate

$$\left|\frac{R_6(x)}{x}\right| \leqq \frac{x^6}{6!\,x} = \frac{x^5}{6!},$$

and

$$|E| \leqq \int_0^1 \frac{x^5}{6!}\,dx = \frac{1}{720}\,\frac{1}{6}.$$

Thus the error satisfies $|E| < 10^{-3}$, and

$$\int_0^1 \left[-\frac{x}{2} + \frac{x^3}{4!}\right] dx = -\frac{1}{4} + \frac{1}{96}.$$

This gives the desired value, with an accuracy of three decimals.

Warning. The definite integral of the above example **cannot be written as a sum**

$$\int_0^1 \frac{\cos x - 1}{x}\,dx = \int_0^1 \frac{\cos x}{x}\,dx - \int_0^1 \frac{1}{x}\,dx.$$

Although it is true that the integral of a sum is the sum of the integrals, this is true only when the integrals make sense. The integral

$$\int_0^1 \frac{1}{x}\,dx$$

does not make sense. First the function $1/x$ is not continuous on the interval $0 \leqq x \leqq 1$. It blows up when x approaches 0. Even if we try to interpret this as a limiting value,

$$\int_0^1 \frac{1}{x}\,dx = \lim_{h\to 0} \int_h^1 \frac{1}{x}\,dx = \lim_{h\to 0} (\log 1 - \log h),$$

the limit does not exist because $\log h$ becomes large negative when h approaches 0. So we cannot split the desired integral into a sum.

Also, it can be shown that **there is no simple expression giving an indefinite integral**

$$\int \frac{\cos x - 1}{x}\,dx = F(x),$$

with a function $F(x)$ expressible in terms of elementary functions. On the other hand, as we have seen, we can perfectly well evaluate the definite integral to any desired accuracy.

XIII, §3. EXERCISES

Unless otherwise specified, take the Taylor formula with $a = 0$, $b = x$.

In all the computations, *include* an estimate of the remainder (error) term, which shows that your answer is within the required accuracy.

1. Write down the Taylor polynomial of degree 4 for $\cos x$. Prove the Taylor formula stated in the text for $\cos x$.

2. Give the details for the estimate $|R_{2m+2}(x)| \leqq |x|^{2m+2}/(2m+2)!$ for the function $f(x) = \cos x$.

3. Compute $\cos(0.1)$ to 3 decimals.

4. Estimate the remainder R_3 in the Taylor formula for $\cos x$, for the value $x = 0.1$.

5. Estimate the remainder R_4 in the Taylor formula for $\sin x$, for the value $x = 0.2$.

6. Write down the Taylor polynomial of degree 4 for $\tan x$.

7. Estimate the remainder R_5 in the Taylor formula for $\tan x$, for $0 \leqq x \leqq 0.2$.

In Exercises 8, 9, 10, and 11 the Taylor formula is used with $a \neq 0$.

8. Compute the following values to 3 places.
   (a) $\sin 31°$ (b) $\cos 31°$ (c) $\sin 47°$
   (d) $\cos 47°$ (e) $\sin 32°$ (f) $\cos 32°$

9. Compute cosine 31 degrees to 3 places.

10. Compute sine 61 degrees to 3 places.

11. Compute cosine 61 degrees to 3 places.

12. Compute the following integrals to three decimals.

(a) $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ (b) $\displaystyle\int_0^{0.1} \frac{\cos x - 1}{x}\,dx$

(c) $\displaystyle\int_0^1 \sin x^2\,dx$ (d) $\displaystyle\int_0^1 \frac{\sin x^2}{x}\,dx$

(e) $\displaystyle\int_0^1 \cos x^2\,dx$ (f) $\displaystyle\int_0^1 \frac{\sin x^2}{x^2}\,dx$

13. Compute

$$\int_0^{1/2} \frac{\cos x - 1}{x}\,dx$$

to 5 decimals.

XIII, §4. EXPONENTIAL FUNCTION

All derivatives of e^x are equal to e^x and $e^0 = 1$. Hence the Taylor formula for e^x is

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + R_n(x).}$$

The remainder term satisfies

$$\boxed{R_n(x) = e^c\,\frac{x^n}{n!},}$$

where c is a number between 0 and x. Hence

$$\boxed{|R_n(x)| \leqq e^c\,\frac{|x|^n}{n!}.}$$

Note that e^c is always positive, and

$$\text{if } x < 0, \quad \text{then} \quad c < 0 \quad \text{and} \quad 0 < e^c < 1.$$

As with the sine function, Theorem 3.1 shows that the remainder term approaches 0 as n becomes large.

Example. 1. Compute e to 3 decimals.

We have $e = e^1$. From Chapter VIII, §6 we know that $e < 4$. We estimate R_7:

$$|R_7| \leqq e \frac{1}{7!} \leqq 4 \frac{1}{5{,}040} < 10^{-3}.$$

Thus

$$e = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{6!} + R_7$$

$$= 2.718 \ldots.$$

Of course, the smaller x is, the fewer terms of the Taylor series do we need to approximate e^x.

Remark. We had obtained the naive estimate $e < 4$ in Chapter VIII. Now we have the much finer evaluation, which shows in particular that $e < 3$. Thus using a rough estimate and the Taylor formula allows us to get a precise determination of e. Even if we had started with knowing $e < 3$ (which could also be obtained by an estimate similar to that of Chapter VIII), this would not have helped much more to get the more precise value.

Example 2. How many terms of the Taylor formula do you need to compute $e^{1/10}$ to an accuracy of 10^{-3}?

We certainly have $e^{1/10} < 2$. Thus

$$|R_3(1/10)| \leqq 2 \frac{(1/10)^3}{3!} < \frac{1}{2}\, 10^{-3}.$$

Hence we need just 3 terms (including the 0-th term).

XIII, §4. EXERCISES

In the exercises, when asked to compute a quantity with a certain degree of accuracy, always show how you estimate the error term to prove that you get the desired accuracy.

1. Write down the Taylor polynomial of degree 5 for e^{-x}.
2. Estimate the remainder R_3 in the Taylor formula for e^x for $x = 1/2$.
3. Estimate the remainder R_4 for $x = 10^{-2}$.
4. Estimate the remainder R_3 for $x = 10^{-2}$.

5. Compute e to four decimals, then five decimals, then six decimals. In each case write down e as a sum of fractions, plus a remainder term, and estimate the remainder term. [This exercise is meant to give you a practical feeling for the size of the remainder terms, and how many you will need to get a desired accuracy. You can compute the sum of the fractions on a pocket calculator.]

6. Compute $1/e$ to 3 decimals, and show which remainder would give you an accuracy of 10^{-3}.

7. Estimate the remainder R_4 in the Taylor formula for e^x when
   (a) $x = 2$. (b) $x = 3$.

8. Estimate the remainder R_5 in the Taylor formula for e^x when
   (a) $x = 2$. (b) $x = 3$.

9. How many terms of the Taylor formula for e^x would you need to compute e^2 to
   (a) 4 decimals? (b) 6 decimals?

10. Compute e to 10 decimals. First give it as a sum of rational numbers. Then use some calculating machine to get the decimals. Show your estimate of the error term.

11. Compute $1/e^2$ to 4 decimals.

12. Compute the following integrals to 3 or 4 decimals, depending on how much you want to exert yourself.
   (a) $\displaystyle\int_0^1 \frac{e^x - 1}{x}\,dx$ (b) $\displaystyle\int_0^1 e^{-x^2}\,dx$ (c) $\displaystyle\int_0^1 e^{x^2}\,dx$
   (d) $\displaystyle\int_0^{0.1} e^{x^2}\,dx$ (e) $\displaystyle\int_0^{0.1} e^{-x^2}\,dx$

XIII, §5. LOGARITHM

We want to get a Taylor formula for the log. We cannot handle the log just by writing

$$\log x = \log 0 + \log'(0)x + \cdots$$

because $\log 0$ is not defined. Thus for the log, it is better to derive a formula with $a = 1$, so that

$$\log b = \log 1 + \log'(1)(b - 1) + \log''(1)\,\frac{(b - 1)^2}{2!} + \cdots.$$

Experience shows that it is then more convenient to let

$$b = 1 + x$$

so $b - 1 = x$. In that way we get a Taylor formula

$$\log(1 + x) = x - \frac{x^2}{2} + \cdots.$$

We leave it to you (see the exercise in §1) to derive this formula in the usual way by computing the derivatives $f^{(k)}(0)$, where $f(x) = \log(1 + x)$.

Here we shall obtain the result by another method, which will also be applicable in the next section and is in some ways more efficient, and makes the series for the log easy to remember.

You should know from high school the geometric series

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots.$$

We shall derive it again below, but for the moment let us work formally and not worry about what the infinite sum means. Replacing u by $-x$ we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

Integrate the left-hand side, and the right-hand side term by term, again without worrying about what the infinite sum means. Then we get

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

On the right-hand side we see that the signs alternate, and we have just n in the denominator of x^n/n, rather than the $n!$ which occurred for $\sin x$, $\cos x$, and e^x.

Now we must start all over again, to derive the formula with a remainder term which will allow us to estimate values for the log. Also observe that since $\log 0$ is not defined, we shall have to take x in some interval of numbers > -1. It turns out that the Taylor formula will give values for the function only in the interval

$$-1 < x \leqq 1.$$

This contrasts with $\sin x$, $\cos x$, e^x when we obtained values for all x.

Let u be any number $\neq 1$. We wish to justify the series

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots.$$

Don't worry at first what the infinite sum on the right means. Use it formally. If you cross multiply, you obtain

$$\begin{aligned}(1-u)(1+u+u^2+u^3+\cdots) &= 1+u+u^2+u^3+\cdots \\ &\quad\; -u-u^2-u^3-\cdots \\ &= 1.\end{aligned}$$

So we have justified the geometric series formally.

Next let us worry about the infinite sum. We don't know how to add infinitely many numbers together, so we must state some relation analogue to the above, but with a finite number of terms. This is based on the formula

$$\frac{1-u^n}{1-u} = 1+u+\cdots+u^{n-1}$$

for any integer $n > 1$. The proof is again obtained by cross-multiplying:

$$\begin{aligned}(1-u)(1+u+u^2+\cdots+u^{n-1}) &= 1+u+u^2+\cdots+u^{n-1} \\ &\quad\; -u-u^2-\cdots-u^{n-1}-u^n \\ &= 1-u^n.\end{aligned}$$

Since

$$\frac{1-u^n}{1-u} = \frac{1}{1-u} - \frac{u^n}{1-u},$$

we find finally

$$\boxed{\frac{1}{1-u} = 1+u+u^2+\cdots+u^{n-1}+\frac{u^n}{1-u}.}$$

We want to apply this formula to get an expression for $1/(1+t)$, because we want ultimately to get an expression for

$$\log(1+x) = \int_0^x \frac{1}{1+t}\,dt.$$

Thus we substitute $u = -t$. Then we find

$$\boxed{\frac{1}{1+t} = 1-t+t^2-t^3+\cdots+(-1)^{n-1}t^{n-1}+(-1)^n\frac{t^n}{1+t}.}$$

Consider the interval $-1 < x \leqq 1$, and take the integral from 0 to x (in this interval). The integrals of the powers of t are well known to you. The integral

$$\int_0^x \frac{1}{1+t}\,dt = \log(1+x)$$

is computed by the substitution $u = 1 + t$, $du = dt$. Thus we get:

Theorem 5.1. *For* $-1 < x \leqq 1$, *we have*

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + R_{n+1}(x)}$$

where the remainder $R_{n+1}(x)$ *is the integral*

$$R_{n+1}(x) = (-1)^n \int_0^x \frac{t^n}{1+t}\,dt.$$

Observe that it was essential that $x > -1$ because the expression $1/(1+t)$ has no meaning when $t = -1$. The above formula also holds for $x > 1$. However, we shall see that the remainder term approaches 0 only when x lies in the stated interval.

Case 1. $0 < x \leqq 1$.

In that case, $1 + t \geqq 1$. Thus

$$\frac{t^n}{1+t} \leqq t^n,$$

and our integral is bounded by $\int_0^x t^n\,dt$. Thus in that case,

$$\boxed{|R_{n+1}(x)| \leqq \int_0^x t^n\,dt \leqq \frac{x^{n+1}}{n+1}.}$$

In particular, the remainder approaches 0 as n becomes large.

Remark. In the applications we shall use a device which allows us to deal only with Case 1. *Thus you may omit Case 2 if you wish.*

Case 2. $-1 < x < 0$.

In this case, t lies between 0 and x and is negative, but we still have

$$x \leqq t \leqq 0,$$

and $0 < 1 + x < 1 + t$. Hence

$$\left|\frac{t^n}{1+t}\right| = \frac{|t|^n}{1+t} \leqq \frac{(-t)^n}{1+x}.$$

To estimate the absolute value of the integral, we can invert the limits (we do this because $x \leqq 0$) and thus

$$|R_{n+1}(x)| \leqq \int_x^0 \frac{(-t)^n}{1+x}\,dt$$

so

$$|R_{n+1}(x)| \leqq \frac{(-x)^{n+1}}{(n+1)(1+x)} = \frac{|x|^{n+1}}{(n+1)(1+x)}.$$

Therefore the remainder also approaches 0 in that case. However, when x is negative and $-1 < x < 0$ then $1 + x < 1$ and we cannot estimate $1/(1 + x)$ in the same way as in Case 1, because **we do not have** $1/(1 + x) \leqq 1$. Thus Case 2 is disagreeable. That is the reason we avoid it.

Example. Let us compute $\log 2$ to 3 decimals. We know that $\log(1 + x)$ can be computed efficiently only when x is close to 0. But $2 = 1 + 1$. We have to use some auxiliary device, which avoids plugging $x = 1$ in the formula. For this, we write

$$2 = \tfrac{4}{3}\cdot\tfrac{3}{2}.$$

Then

$$\tfrac{4}{3} = 1 + \tfrac{1}{3} \qquad \text{and} \qquad \tfrac{3}{2} = 1 + \tfrac{1}{2}.$$

By using $x = \frac{1}{3}$ and $x = \frac{1}{2}$ we shall achieve what we want. Indeed,

$$\log 2 = \log(\tfrac{4}{3}\cdot\tfrac{3}{2}) = \log(1 + \tfrac{1}{3}) + \log(1 + \tfrac{1}{2}).$$

To find $\log(1 + \frac{1}{3})$ we use $x = \frac{1}{3}$ and

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + R_6(x).$$

By the estimate of Case 1, we get

$$|R_6(\tfrac{1}{3})| \leqq \tfrac{1}{6}\,(\tfrac{1}{3})^6 \leqq \tfrac{1}{4} \times 10^{-3}.$$

Therefore

$$\begin{aligned}\log\frac{4}{3} = \log\left(1+\frac{1}{3}\right) &= \frac{1}{3} - \frac{(1/3)^2}{2} + \frac{(1/3)^3}{3} - \frac{(1/3)^4}{4} + \frac{(1/3)^5}{5} + E_1\\ &= A_1 + E_1\end{aligned}$$

with

$$E_1 = R_6(\tfrac{1}{3}) \qquad \text{and} \qquad |E_1| \leqq \tfrac{1}{4} \times 10^{-3}.$$

Similarly we get $\log\frac{3}{2}$ by

$$\begin{aligned}\log\frac{3}{2} = \log\left(1+\frac{1}{2}\right) &= \frac{1}{2} - \frac{(1/2)^2}{2} + \cdots + \frac{(1/2)^7}{7} + E_2\\ &= A_2 + E_2\end{aligned}$$

with the error term $E_2 = R_8(1/2)$. We have the estimate

$$|E_2| = |R_8(\tfrac{1}{2})| \leqq \tfrac{1}{2} \times 10^{-3}.$$

Therefore

$$\begin{aligned}\log 2 &= \log(1+\tfrac{1}{2}) + \log(1+\tfrac{1}{3})\\ &= A_1 + A_2 + E_1 + E_2,\end{aligned}$$

where A_1, A_2 are simple expressions which can be computed from fractions using addition, multiplication, and subtraction, and the error term is $E = E_1 + E_2$. We can now estimate E, namely

$$|E| \leqq |E_1| + |E_2| \leqq \tfrac{1}{4} \times 10^{-3} + \tfrac{1}{2} \times 10^{-3} < 10^{-3}.$$

This lies within the desired accuracy. If you have a small pocket computer, you can evaluate a numerical decimal for log 2 easily.

In the above computation, we still needed six or eight terms in the polynomial expression approximating the logarithm to get three decimals accuracy. In Exercise 2, following the same general idea, you will see how to get accurate answers using fewer terms.

Example. Suppose we want to compute $\log\frac{3}{4}$ to 3 decimals. Note that $\frac{3}{4} < 1$, and if we tried to evaluate

$$\log\tfrac{3}{4} = \log(1-\tfrac{1}{4}),$$

then we would have to use Case 2. This can be avoided by using the general rule

$$\log \frac{1}{a} = -\log a$$

for any positive number a. In particular,

$$\begin{aligned}\log \tfrac{3}{4} &= -\log \tfrac{4}{3} \\ &= -\frac{1}{3} + \frac{(1/3)^2}{2} - \frac{(1/3)^3}{3} + \frac{(1/3)^4}{4} - \frac{(1/3)^5}{5} + E,\end{aligned}$$

and

$$|E| \leqq \tfrac{1}{4} \times 10^{-3}.$$

Example. Compute log 1.1 to three decimals.

To do this, i.e. compute $\log(1 + 0.1)$, we take $n = 2$, and $x = 0.1$ in Case 1. We find that

$$|R_3(x)| \leqq \tfrac{1}{3} \times 10^{-3}.$$

Hence

$$\log(1.1) = 0.1 - 0.005 + E$$

with an error E such that $|E| \leqq \frac{1}{3} \times 10^{-3}$.

XIII, §5. EXERCISES

1. Compute the following values up to an accuracy of 10^{-3}, estimating the remainder each time.
   (a) $\log 1.2$ (b) $\log 0.9$ (c) $\log 1.05$
   (d) $\log \frac{9}{10}$ (e) $\log \frac{24}{25}$ (f) $\log \frac{26}{25}$

2. (a) Verify the following formulas:

$$\log 2 = 7 \log \tfrac{10}{9} - 2 \log \tfrac{25}{24} + 3 \log \tfrac{81}{80},$$

$$\log 3 = 11 \log \tfrac{10}{9} - 3 \log \tfrac{25}{24} + 5 \log \tfrac{81}{80}.$$

   (b) Compute log 2 and log 3 to five decimals, using these formulas.

You may ask how one finds such formulas. The answer is that someone clever, probably more than 200 years ago, found them by experimenting with numbers, and after that, everybody copies them.

XIII, §6. THE ARCTANGENT

We proceed as with the logarithm, except that we put $u = -t^2$ in the geometric series, and obtain

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^{m-1}t^{2m-2} + (-1)^m \frac{t^{2m}}{1+t^2}.$$

After integration from 0 to any number x, we obtain:

Theorem 6.1. *The arctan has an expansion*

$$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{m-1}\frac{x^{2m-1}}{2m-1} + R_{2m+1}(x),}$$

where

$$R_{2m+1}(x) = (-1)^m \int_0^x \frac{t^{2m}}{1+t^2}\,dt,$$

and

$$\boxed{|R_{2m+1}(x)| \leqq \int_0^{|x|} t^{2m}\,dt \leqq \frac{|x|^{2m+1}}{2m+1}.}$$

When $-1 \leqq x \leqq 1$, *the remainder approaches* 0 *as* n *becomes large.*

Observe how only odd powers of x occur in the Taylor formula for $\arctan x$. This is the reason for writing $2m+1$, or R_{2m+1}. If we put $n = 2m+1$. then we can write the estimate for the remainder in the form

$$\boxed{|R_n(x)| \leqq \frac{|x|^n}{n}.}$$

This estimate is the same as for the log, except that for arctan, we take only odd integers n.

Remark. If x does not lie in the prescribed interval, i.e. if $|x| > 1$, then the remainder term does not tend to 0 as n becomes large. For instance, if $x = 2$, then the remainder term is bounded by

$$\frac{2^{2m+1}}{2m+1}.$$

By computing a few values with $m = 1, 2, 3, \ldots$ you will see that this expression grows large quite fast. You should know this anyhow from your study of the exponential function.

From our theorem, we get a cute expression for $\pi/4$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

from the Taylor formula for arctan 1. However, it takes many terms to get a good approximation to $\pi/4$ by this expression. Roughly, it takes 1000 terms to get accuracy to 10^{-3}, which is very inefficient. By using a more clever approach, however, we can find π much faster. This is done as follows. First we have:

Addition formula for the tangent:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Proof. Using the addition formulas for sine and cosine proved in Chapter 4, Theorem 3.1, we have

$$\tan(x + y) = \frac{\sin(x + y)}{\cos(x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

Now divide numerator and denominator on the right by $\cos x \cos y$. The desired formula drops out.

In the addition formula for the tangent, put $u = \tan x$ and $v = \tan y$, so that $x = \arctan u$ and $y = \arctan v$. Since

$$\arctan(\tan(x + y)) = x + y = \arctan u + \arctan v,$$

we obtain the **addition formula for the arctangent:**

$$\boxed{\arctan u + \arctan v = \arctan \frac{u + v}{1 - uv}.}$$

Example. Consider the special values $u = 1/2$ and $v = 1/3$. Simple arithmetic shows that for these values, we get

$$\frac{u + v}{1 - uv} = \frac{1/2 + 1/3}{1 - 1/6} = 1.$$

Since arctan $1 = \pi/4$, we obtain the formula:

$$\boxed{\frac{\pi}{4} = \arctan 1 = \arctan \frac{1}{2} + \arctan \frac{1}{3}.}$$

Next we use the Taylor formula for arctan,

$$\arctan x = x - \frac{x^3}{3} + R_5(x)$$

with

$$|R_5(x)| \leqq \frac{|x|^5}{5}.$$

Then we obtain

(1) $$\arctan \frac{1}{2} = \frac{1}{2} - \frac{1}{24} + R_5\left(\frac{1}{2}\right),$$

and

$$|R_5(\tfrac{1}{2})| \leqq \tfrac{1}{160}.$$

This is not exceptionally good, but it shows you that with just two terms of the polynomial approximating arctan x, we already get about 2 decimals accuracy. Using a couple of more terms, you should be able to get 4 decimals.

Similarly, we get

(2) $$\arctan \frac{1}{3} = \frac{1}{3} - \frac{(1/3)^3}{3} + R_5\left(\frac{1}{3}\right)$$

and

$$\left|R_5\left(\frac{1}{3}\right)\right| \leqq \frac{(1/3)^5}{5} \leqq \frac{1}{2025} < \frac{1}{2} \times 10^{-3}.$$

Hence

$$\frac{\pi}{4} = \frac{1}{2} - \frac{1}{24} + \frac{1}{3} - \frac{1}{3^4} + E = A + E,$$

where $E = R_5(\tfrac{1}{2}) + R_5(\tfrac{1}{3})$, and

$$|E| \leqq |R_5(\tfrac{1}{2})| + |R_5(\tfrac{1}{3})| < 10^{-2}.$$

The expression A is a sum of fractions which you can easily calculate on a pocket computer. Then

$$\pi = 4(\arctan \tfrac{1}{2} + \arctan \tfrac{1}{3}) = 4A + 4E,$$

where

$$|4E| < 4 \times 10^{-2}.$$

Notice that this final factor of 4 makes the estimate of the error term somewhat worse. Hence in determining which remainders to take, you have to make sure that in the final step, when you multiply with 4, the accuracy lies within the desired bound. Experiment with R_7 and R_9 to acquire a good feeling for the size of these remainders.

XIII, §6. EXERCISES

1. Prove the formulas:

$$2 \arctan u = \arctan \frac{2u}{1-u^2} \qquad \text{and} \qquad 3 \arctan u = \arctan \frac{3u - u^3}{1 - 3u^2}.$$

2. Prove:
   (a) $\arctan \frac{1}{2} = \arctan \frac{1}{3} + \arctan \frac{1}{7}$
   (b) $\arctan \frac{1}{3} = \arctan \frac{1}{5} + \arctan \frac{1}{8}$
   (c) $\pi/4 = 2 \arctan \frac{1}{5} + \arctan \frac{1}{7} + 2 \arctan \frac{1}{8}$

3. Estimating $R_3(x)$, $R_5(x)$, $R_7(x)$, $R_9(x)$ in the Taylor formula for the arctangent, and using the expression

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3},$$

   as well as the expression (c) in Exercise 2, find various decimal approximations for π. Ultimately, do verify that

$$\pi = 3.14159\ldots$$

   to an accuracy of 5 decimals.

4. Prove the formula $\pi/4 = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$. How few terms of the Taylor formula do you now need in order to get the above accuracy for a decimal approximation to π?

XIII, §7. THE BINOMIAL EXPANSION

In high school, you should have learned the expansion of $(a + b)^n$ or $(1 + x)^n$. For instance

$$(1 + x)^2 = 1 + 2x + x^2,$$

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3,$$

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

Just using algebra, one can determine the coefficients for the expansion of $(1+x)^n$ when n is a positive integer. However, here we shall be interested also in powers $(1+x)^s$ when s is not a positive integer. For this we shall use the general method of Taylor's formula, which states:

$$\begin{aligned} f(x) &= f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_{n+1}(x) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + R_{n+1}(x). \end{aligned}$$

We apply this formula to the function

$$f(x) = (1+x)^s.$$

Theorem 7.1. *Let n be a positive integer and $x \neq -1$. Then*

$$\boxed{\begin{aligned} (1+x)^s = 1 + sx + \frac{s(s-1)}{2!}x^2 + \frac{s(s-1)(s-2)}{3!}x^3 + \cdots \\ + \frac{s(s-1)(s-2)\cdots(s-n+1)}{n!}x^n + R_{n+1}(x). \end{aligned}}$$

Proof. Let $f(x) = (1+x)^s$. Then we compute the derivatives:

$$\begin{aligned} f^{(1)}(x) &= s(1+x)^{s-1}, & f^{(1)}(0) &= s, \\ f^{(2)}(x) &= s(s-1)(1+x)^{s-2}, & f^{(2)}(0) &= s(s-1), \\ f^{(3)}(x) &= s(s-1)(s-2)(1+x)^{s-3}, & f^{(3)}(0) &= s(s-1)(s-2), \\ &\vdots & &\vdots \\ f^{(k)}(x) &= s(s-1)\cdots(s-k+1)(1+x)^{s-k}, & f^{(k)}(0) &= s(s-1)\cdots \\ & & &\quad (s-k+1). \end{aligned}$$

Hence

$$\frac{f^{(k)}(0)}{k!} = \frac{s(s-1)\cdots(s-k+1)}{k!}.$$

By the general Taylor formula, this proves the desired expansion for $(1+x)^s$.

The general formula for the remainder term is

$$R_n(x) = \frac{f^{(n)}(c)x^n}{n!}$$

with some number c between 0 and x, and therefore in the present case we find:

$$\boxed{R_n(x) = \frac{s(s-1)\cdots(s-n+1)}{n!}(1+c)^{s-n}x^n.}$$

It can be shown that if $-1 < x < 1$ then $R_n(x) \to 0$ as $n \to \infty$. We shall estimate the remainder when $n = 2$ and $n = 3$. We do not give the proof in general that $R_n(x)$ approaches 0 when $n \to \infty$.

In these estimates, we use repeatedly the fact that

$$|ab| = |a|\,|b|.$$

For example, products like $s(s-1)(s-2)$ occur all the time in these estimates. Then

$$|s(s-1)(s-2)| = |s|\,|s-1|\,|s-2|.$$

If $s = 1/3$, we find

$$\left|\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\right| = \frac{1}{3}\left|\frac{1}{3}-1\right|\left|\frac{1}{3}-2\right| = \frac{1}{3}\,\frac{2}{3}\,\frac{5}{3} = \frac{10}{27}.$$

Examples involving R_2

Now let us look at R_2. Let

$$f(x) = (1+x)^s,$$

where s is not an integer. We have

$$f^{(2)}(x) = s(s-1)(1+x)^{s-2}.$$

The Taylor formula gives

$$\boxed{(1+x)^s = 1 + sx + R_2(x),}$$

where

$$\begin{aligned} R_2(x) &= f^{(2)}(c)\frac{x^2}{2!} \\ &= s(s-1)(1+c)^{s-2}\frac{x^2}{2}, \end{aligned}$$

for some number c between 0 and x.

For small x, this means that $1 + sx$ should be a good approximation to the s power of $1 + x$, if $R_2(x)$ can be proved to be small. This we can do easily. We see that

$$\boxed{|R_2(x)| = \frac{|s(s-1)|}{2}(1+c)^{s-2}|x|^2,}$$

where c is between 0 and x. By an easy estimate one sees, for instance, that $(1 + x)^{1/2}$ is approximately equal to $1 + \frac{1}{2}x$, and $(1 + x)^{1/3}$ is approximately equal to $1 + \frac{1}{3}x$, for small x.

Example 1. Find $\sqrt{1.2}$ to 2 decimals.
We let $x = 0.2 = 2 \times 10^{-1}$ and $s = 1/2$. Then

$$\begin{aligned}\sqrt{1.2} = (1 + 0.2)^{1/2} &= 1 + \tfrac{1}{2}0.2 + R_2(0.2)\\ &= 1 + 0.1 + R_2(\tfrac{1}{5}).\end{aligned}$$

We must estimate $R_2(1/5)$. Since $0 \leqq c \leqq 1/5$ and $s - 2 = -3/2$, we find

$$(1 + c)^{s-2} = \frac{1}{(1 + c)^{3/2}} \leqq 1,$$

because the smaller the denominator, the larger the fraction. The only information we have on c is that $0 \leqq c \leqq 1/5$, and the smallest possible value of the denominator is when $c = 0$. Consequently

$$\begin{aligned}\left|R_2\left(\frac{1}{5}\right)\right| &\leqq \frac{1}{2}\left|\frac{1}{2} - 1\right|\frac{1}{2}(0.2)^2\\ &\leqq \frac{1}{8}4 \times 10^{-2} = \frac{1}{2} \times 10^{-2}.\end{aligned}$$

Therefore the estimate for the error term is within the desired accuracy.

Example 2. Let us compute $\sqrt{0.8}$ to 2 decimals.
We use $s = 1/2$ and $x = -0.2$. Then

$$\sqrt{0.8} = (1 - 0.2)^{1/2} = 1 - \tfrac{1}{2}0.2 + R_2(-0.2) = 0.9 + R_2(-0.2).$$

Here we have $-0.2 \leqq c \leqq 0$. Hence $1/(1 + c)^{3/2} \leqq 1/(0.8)^{3/2}$, and

$$\begin{aligned}|R_2(-0.2)| &\leqq \frac{1}{2}\left|\frac{1}{2} - 1\right|\frac{1}{2!}\frac{1}{(0.8)^{3/2}}(0.2)^2\\ &\leqq \frac{1}{8}\frac{1}{(0.8)^{3/2}}4 \times 10^{-2}.\end{aligned}$$

The presence of the term $(0.8)^{3/2}$, which is < 1, in the denominator makes it slightly more cumbersome to estimate than in the preceding example, but even then it is not so difficult. Without exerting ourselves, to make the estimate simple we replace 3/2 by 2. Then

$$\frac{1}{(0.8)^{3/2}} < \frac{1}{(0.8)^2} = \frac{1}{0.64} < \frac{10}{6} = \frac{5}{3}.$$

Hence

$$|R_2(-0.2)| < \frac{1}{8}\,\frac{5}{3}\,4 \times 10^{-2} < 10^{-2}.$$

Remark. In the two cases of Example 1 and Example 2, we have met the case when $x > 0$ and $x < 0$. In the estimate for R_2, this gives rise to two different cases:

$$\frac{1}{(1+c)^{3/2}} \leqq 1 \qquad \text{whenever} \quad x > 0 \quad \text{and} \quad 0 \leqq c \leqq x;$$

$$\frac{1}{(1+c)^{3/2}} \leqq \frac{1}{(1+x)^{3/2}} \qquad \text{whenever} \quad x < 0 \quad \text{and} \quad x \leqq c \leqq 0.$$

The second case is more annoying to treat.

In the preceding examples, we computed roots of $1 + x$ when x is small. To find roots of an arbitrary number we can often use a trick as in the next example, in order to reduce the problem to a root $(1 + x)^s$ with small x.

Example 3. Find the value $\sqrt{26}$ to two decimals.
For this we write

$$26 = 25 + 1 = 25(1 + \tfrac{1}{25}).$$

Then

$$\sqrt{26} = 5(1 + \tfrac{1}{25})^{1/2},$$

and we can apply the binomial Taylor formula to find

$$\left(1 + \frac{1}{25}\right)^{1/2} = 1 + \frac{1}{50} + R_2(x),$$

with $x = \frac{1}{25}$ and $s = \frac{1}{2}$. We are in the case $c \geqq 0$, so we get

$$\left|R_2\left(\frac{1}{25}\right)\right| \leqq \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\left(\frac{1}{25}\right)^2 \leqq \frac{1}{8}\,\frac{1}{625} = \frac{1}{5000}.$$

Hence

$$\sqrt{26} = 5(1 + \tfrac{1}{50}) + 5R_2(\tfrac{1}{25}) = 5.1 + E$$

where

$$|E| = 5|R_2(\tfrac{1}{25})| \leqq 10^{-3}.$$

Observe the factor 5 which appeared in the last step, and which multiplies the estimate for $R_2(1/25)$. To get the final accuracy up to 10^{-3}, you needed an accuracy of $(1/5) \times 10^{-3}$ for $R_2(1/25)$ because of this factor 5.

An example involving R_3

Example 4. Find the value $\sqrt{26}$ to four decimals.

For this we write $26 = 25(1 + 1/25)$ as before. By the binomial Taylor formula, we find

$$\left(1 + \frac{1}{25}\right)^{1/2} = 1 + \frac{1}{2}\frac{1}{25} + \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{25}\right)^2 + R_3\left(\frac{1}{25}\right).$$

In estimating the remainder, we are in the case with $c \geqq 0$, so

$$\begin{aligned} \left|R_3\left(\frac{1}{25}\right)\right| &\leqq \frac{1}{2}\left|\frac{1}{2} - 1\right|\left|\frac{1}{2} - 2\right|\frac{1}{3!}\left(\frac{1}{25}\right)^3 \\ &\leqq \frac{1}{2}\frac{1}{2}\frac{3}{2}\frac{1}{3!}\left(\frac{1}{25}\right)^3 \\ &< \frac{1}{16}\frac{1}{1.5} \times 10^{-4} \\ &\leqq \frac{1}{24} \times 10^{-4}. \end{aligned}$$

Then

$$\begin{aligned} 26^{1/2} &= 5(1 + 1/25)^{1/2} \\ &= 5\left(1 + \frac{1}{50} - \frac{1}{4}\frac{1}{625}\right) + E, \end{aligned}$$

where $E = 5R_3(1/25)$ and therefore

$$|E| \leqq \frac{5}{24} \times 10^{-4} < 10^{-4}.$$

This is within the desired accuracy. Again note the factor of 5 in the last step.

The method in the above example was to find a perfect square near 26, and then use Taylor's formula for $(1 + x)^{1/2}$ with a small x. In general, one can use a similar method to find the square root of a number. Find a perfect square as close to the number as possible, and then use Taylor's formula. A similar technique works for cube roots or other roots.

The binomial expansion $(1 + x)^n$ and $(a + b)^n$

We conclude this section by showing how the binomial expansion for $(1 + x)^s$ using Taylor's formula becomes simpler when s is a positive integer n. So let n be a positive integer. Let

$$f(x) = (1 + x)^n.$$

We have no difficulty computing the derivatives:

$$\begin{aligned}
f^{(1)}(x) &= n(1 + x)^{n-1}, & f^{(1)}(0) &= n,\\
f^{(2)}(x) &= n(n-1)(1 + x)^{n-2}, & f^{(2)}(0) &= n(n-1),\\
f^{(3)}(x) &= n(n-1)(n-2)x^{n-3}, & f^{(3)}(0) &= n(n-1)(n-2),\\
&\vdots & &\vdots\\
f^{(n)}(x) &= n!, & f^{(n)}(0) &= n!\\
f^{(n+1)}(x) &= 0. & f^{(n+1)}(0) &= 0.
\end{aligned}$$

The new feature here is that $f^{(n+1)}(x) = 0$. Therefore $f^{(k)}(x) = 0$ for all $k \geqq n + 1$, and the remainder after the n-th term is equal to 0. Hence we get the exact expression:

Theorem 7.2. *Let n be a positive integer. For any number x, we have*

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n.$$

The coefficient of x^k on the right-hand side is usually denoted by the symbol

$$\binom{n}{k}$$

and is called a **binomial coefficient**. Thus from the derivatives which we found above, we get

$$\boxed{\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.}$$

The numerator consists of the product of integers in descending order, from n to $n - k + 1$. It differs from $n!$ in that the remaining product from $(n - k)$ down to 1 is missing. In order to have a more symmetric expression for the binomial coefficient, we multiply the numerator and denominator by $(n - k)!$. Observe that

$$n(n-1)(n-2)\cdots(n-k+1)(n-k)! = n!$$

and consequently we can write the binomial coefficient in the form

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

In this formula, we let $0 \leqq k \leqq n$, and **by convention we let**

$$0! = 1.$$

For example:

$$\binom{3}{0} = \frac{3!}{0!\,3!} = 1, \qquad \binom{3}{1} = \frac{3!}{1!\,2!} = 3,$$

$$\binom{3}{2} = \frac{3!}{2!\,1!} = 3, \qquad \binom{3}{3} = \frac{3!}{3!\,0!} = 1.$$

The integers 1, 3, 3, 1 are exactly the coefficients of the expansion for

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3.$$

In exercises, you can work out the coefficients for higher powers.

If we want the expansion of $(a + b)^n$ with arbitrary numbers a and b, and $a \neq 0$, then we let $x = b/a$, and so:

$$(a+b)^n = a^n\left(1 + \frac{b}{a}\right)^n = a^n \sum_{k=0}^{n} \binom{n}{k}\left(\frac{b}{a}\right)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Thus we can write the binomial expansion in the form

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.}$$

We found this expansion in a fancy way, by using Taylor's formula in a case when the remainder is 0. In high school, the expansion should have been derived in a much more elementary way. The point is that we needed this more general technique here in order to compute values $(1 + x)^s$ with a more general exponent s which may not be an integer. For instance, we needed to compute

$$(1 + x)^{1/2} \quad \text{or} \quad (1 + x)^{1/3}.$$

Then we have to use the method of Taylor's formula.

Definition. Let s be any real number. We define the **binomial coefficient**

$$\boxed{\binom{s}{k} = \frac{s(s-1)(s-2)\cdots(s-k+1)}{k!}.}$$

Example. Suppose that $s = 1/3$. Then

$$\binom{1/3}{2} = \frac{1}{3}\left(\frac{1}{3} - 1\right)\frac{1}{2!},$$

$$\binom{1/3}{3} = \frac{1}{3}\left(\frac{1}{3} - 1\right)\left(\frac{1}{3} - 2\right)\frac{1}{3!},$$

$$\binom{1/3}{4} = \frac{1}{3}\left(\frac{1}{3} - 1\right)\left(\frac{1}{3} - 2\right)\left(\frac{1}{3} - 3\right)\frac{1}{4!},$$

and so on.

Observe that when s is not an integer, then we cannot multiply numerator and denominator by $(s - k)!$ which does not make sense. We have to leave the binomial coefficient as in the definition.

Using the summation sign, we can also write

$$\boxed{(1 + x)^s = \sum_{k=0}^{n} \binom{s}{k} x^k + R_{n+1}(x).}$$

XIII, §7. EXERCISES

In each of the following cases, when asked to compute a number, include the estimate for the error term to show that it lies within the desired accuracy

1. Estimate the remainder R_2 in the Taylor series for $(1 + x)^{1/4}$.
   (a) when $x = 0.01$, (b) when $x = 0.2$, (c) when $x = 0.1$.

2. Estimate the remainder R_3 in the Taylor series for $(1 + x)^{1/2}$:
(a) when $x = 0.2$, (b) when $x = -0.2$, (c) when $x = 0.1$.

3. Estimate R_2 in the remainder of $(1 + x)^{1/3}$ for x lying in the interval

$$-0.1 \leqq x \leqq 0.1.$$

4. Estimate the remainder R_2 in the Taylor series of $(1 + x)^{1/2}$
(a) when $x = -0.2$, (b) when $x = 0.1$.

5. Compute the cube roots to 4 decimals:
(a) $\sqrt[3]{126}$, (b) $\sqrt[3]{130}$, (c) $\sqrt[3]{131}$, (d) $\sqrt[3]{220}$.

6. Compute the square roots to 4 decimals:
(a) $\sqrt{97}$, (b) $\sqrt{102}$, (c) $\sqrt{105}$, (d) $\sqrt{28}$.

XIII, §8. SOME LIMITS

Limits of quotients of functions can be reduced to limits of quotients of polynomials by using a few terms from Taylor's expansion.

Let us first look at polynomials.

Example 1. Find the limit

$$\lim_{x \to 0} \frac{3x - 2x^2 + 5x^4}{7x}.$$

We divide the numerator and denominator by the *lowest* power of x occurring in each, so that we find:

$$\begin{aligned} \frac{3x - 2x^2 + 5x^4}{7x} &= \frac{x(3 - 2x + 5x^3)}{x \cdot 7} \\ &= \frac{3 - 2x + 5x^3}{7}. \end{aligned}$$

It is now easy to find the limit as x approaches 0; namely the limit is $\frac{3}{7}$.

Example 2. Find the limit

$$\lim_{x \to 0} \frac{\cos x - 1}{x^2}.$$

We replace $\cos x$ by $1 - x^2/2 + R_4(x)$, so that

$$\frac{\cos x - 1}{x^2} = \frac{-\frac{1}{2}x^2 + R_4(x)}{x^2} = -\frac{1}{2} + \frac{R_4(x)}{x^2}.$$

Since $|R_4(x)| \leqq |x|^4$, it follows that the limit as $x \to 0$ is equal to $-\frac{1}{2}$.

Example 3. Find the limit

$$\lim_{x \to 0} \frac{\sin x - x + x^3/3!}{x^4}.$$

We have

$$\sin x = x - \frac{x^3}{3!} + R_5(x).$$

Hence

$$\sin x - x + \frac{x^3}{3!} = R_5(x) \qquad \text{and} \qquad |R_5(x)| \leqq \frac{|x|^5}{5!}.$$

Hence

$$\left| \frac{\sin x - x + x^3/3!}{x^4} \right| \leqq \frac{|R_5(x)|}{|x^4|} \leqq \frac{|x|}{5!}.$$

The desired limit is therefore equal to 0.

Example 4. Find the limit

$$\lim_{x \to 0} \frac{\sin x^2}{x \tan x}.$$

To do this, we use the fact that

$$\lim_{u \to 0} \frac{\sin u}{u} = 1,$$

and put $u = x^2$. Also

$$\lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \frac{x}{\sin x} \cos x = 1.$$

Hence

$$\lim_{x\to 0} \frac{\sin x^2}{x \tan x} = \lim_{x\to 0} \frac{\sin x^2}{x^2} \frac{x}{\tan x} = 1.$$

Example 5. We want to find the limit

$$\lim_{x\to 0} \frac{\log(1+x)}{\sin x}.$$

By the Taylor formula, we have

$$\log(1+x) = x + R_2(x),$$
$$\sin x = x + S_3(x).$$

(We write S_3 instead of R_3 because it is a different remainder than for the log.) In each case, we have the estimate

$$|R_2(x)| \leqq C|x|^2 \qquad \text{and} \qquad |S_3(x)| \leqq C'|x|^3$$

for some constants C, C', and x sufficiently close to 0. Hence

$$\frac{\log(1+x)}{\sin x} = \frac{x + R_2(x)}{x + S_3(x)}.$$

Dividing numerator and denominator by x shows that this is

$$= \frac{1 + R_2(x)/x}{1 + S_3(x)/x}.$$

As x approaches 0, each quotient $R_2(x)/x$ and $S_3(x)/x$ approach 0. Hence the limit is 1, as we wanted.

Example 6. Find the limit

$$\lim_{x\to 0} \frac{\sin x - e^x + 1}{x}.$$

Again we write the Taylor formula with a few terms:

$$\sin x = x + R_3(x),$$
$$e^x = 1 + x + S_2(x).$$

Then

$$\frac{\sin x - e^x + 1}{x} = \frac{x - 1 - x + 1 + R_3(x) - S_2(x)}{x}$$

$$= \frac{R_3(x) - S_2(x)}{x}.$$

The right-hand side approaches 0, and so the desired limit is 0.

XIII, §8. EXERCISES

Find the following limits as x approaches 0.

1. $\dfrac{\cos x - 1 + x^2/2!}{x^3}$
2. $\dfrac{\cos x - 1 + x^2/2!}{x^4}$
3. $\dfrac{\sin x + e^x - 1}{x}$
4. $\dfrac{\sin x - e^x + 1}{x}$
5. $\dfrac{e^x - 1}{x}$
6. $\dfrac{\sin (x^2)}{(\sin x)^2}$
7. $\dfrac{\tan x}{\sin x}$
8. $\dfrac{\arctan x}{x}$
9. $\dfrac{\log(1 + x)}{x}$
10. $\dfrac{\log(1 + 2x)}{x}$
11. $\dfrac{e^x - (1 + x)}{x^2}$
12. $\dfrac{\sin x - x}{x^2}$
13. $\dfrac{\cos x - 1}{x^2}$
14. $\dfrac{\log(1 + x^2)}{\sin(x^2)}$
15. $\dfrac{\tan(x^2)}{(\sin x)^2}$
16. $\dfrac{\log(1 + x^2)}{(\sin x)^2}$
17. $\dfrac{\sin x - e^x + 1}{x^2}$
18. $\dfrac{\cos x - e^x}{x}$
19. $\dfrac{e^x - e^{-x}}{x}$
20. $\dfrac{\sin x}{e^x - e^{-x}}$
21. $\dfrac{\sin^2 x}{\sin x^2}$
22. $\dfrac{\tan x^2}{\sin^2 x}$
23. $\dfrac{\log(1 - x)}{\sin x}$
24. $\dfrac{e^x + e^{-x} - 2}{x^2}$
25. $\dfrac{e^x + e^{-x} - 2}{x \sin x}$
26. $\dfrac{\sin x - x}{x^2}$
27. $\dfrac{\sin x - x}{x^3}$
28. $\dfrac{e^x - 1 - x}{x}$
29. $\dfrac{e^x - 1 - x}{x^2}$
30. $\dfrac{\log(1 + x^2)}{x^2}$
31. $\dfrac{(1 + x)^{1/2} - 1 - \frac{1}{2}x}{x^2}$
32. $\dfrac{(1 + x)^{1/3} - 1 - \frac{1}{3}x}{x^2}$
33. Let $f(x)$ be a function which has $n + 1$ continuous derivatives in an open interval containing the origin, and assume that the $(n + 1)$-th derivative is

bounded by a constant M on this interval. Let $P_n(x)$ be the Taylor polynomial of degree n for $f(x)$. What are the following limits?

(a) $\displaystyle\lim_{x\to 0} \frac{f(x) - P_n(x)}{x^2}$ (assuming $n \geqq 3$)

(b) $\displaystyle\lim_{x\to 0} \frac{f(x) - P_n(x)}{x^n}$

(c) $\displaystyle\lim_{x\to 0} \frac{f(x) - P_n(x)}{x^{n-1}}$ (assuming $n \geqq 2$)

Determine the following limits as x approaches 0.

34. $\dfrac{\sin x + \cos x - 1}{x}$

35. $\dfrac{\sin x + \cos x - 1 - x}{x^2}$

36. $\dfrac{\sin x - x + x^3/3!}{x^4}$

37. $\dfrac{\sin x - x + x^3/3!}{x^5}$

38. $\dfrac{\cos x - 1 - x^2/2!}{x}$

39. $\dfrac{\cos x - 1 - x^2/2!}{x^2}$

CHAPTER XIV

Series

Series are a natural continuation of our study of functions. In the preceding chapter we found how to approximate our elementary functions by polynomials, with a certain error term. Conversely, one can define arbitrary functions by giving a series for them. We shall see how in the sections below.

In practice, very few tests are used to determine convergence of series. Essentially, the comparison test is the most frequent. Furthermore, the most important series are those which converge absolutely. Thus we shall put greater emphasis on these.

XIV, §1. CONVERGENT SERIES

Suppose that we are given a sequence of numbers

$$a_1, a_2, a_3, \ldots,$$

i.e. we are given a number a_n for each integer $n \geqq 1$. (We picked the starting place to be 1, but we could have picked any integer.) We form the sums

$$s_n = a_1 + a_2 + \cdots + a_n.$$

It appears to be meaningless to form an infinite sum

$$a_1 + a_2 + a_3 + \cdots$$

because we do not know how to add infinitely many numbers. However, if our sums s_n approach a limit, as n becomes large, then we say that the sum of our sequence **converges**, and we now define its **sum** to be that limit.

The symbols

$$\sum_{n=1}^{\infty} a_n$$

will be called a **series**. We shall say that the **series converges** if the sums s_n approach a limit as n becomes large. Otherwise, we say that it does not converge, or **diverges**. If the series converges, we say that the value of the series is

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_1 + \cdots + a_n).$$

The symbols $\lim\limits_{n\to\infty}$ are to be read: "The limit as n becomes large."

Example. Consider the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots,$$

and let us form the sums

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}.$$

You probably know already that these sums approach a limit and that this limit is 2. To prove it, let $r = \frac{1}{2}$. Then

$$(1 + r + r^2 + \cdots + r^n) = \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r} - \frac{r^{n+1}}{1 - r}.$$

As n becomes large, r^{n+1} approaches 0, whence our sums approach

$$\frac{1}{1 - \frac{1}{2}} = 2.$$

Actually, the same argument works if we take for r any number such that

$$-1 < r < 1.$$

In that case, r^{n+1} approaches 0 as n becomes large, and consequently we can write

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Of course, if $|r| > 1$, then the series $\sum r^n$ does not converge. For instance, the partial sums of the series with $r = -3$ are

$$1 - 3 + 3^2 - 3^3 + \cdots + (-1)^n 3^n.$$

Observe that the n-th term $(-1)^n 3^n$ does not even approach 0 as n becomes large.

In view of the fact that the limit of a sum is the sum of the limits, and other standard properties of limits, we get the following theorem.

Theorem 1.1. *Let $\{a_n\}$ and $\{b_n\}$ $(n = 1, 2, \ldots)$ be two sequences and assume that the series*

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

converge. Then $\sum_{n=1}^{\infty} (a_n + b_n)$ also converges, and is equal to the sum of the two series. If c is a number, then

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n.$$

Finally, if

$$s_n = a_1 + \cdots + a_n$$

and

$$t_n = b_1 + \cdots + b_n,$$

then

$$\sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n = \lim_{n \to \infty} s_n t_n.$$

In particular, series can be added term by term. Of course, they cannot be multiplied term by term!

We also observe that a similar theorem holds for the difference of two series.

If a series $\sum a_n$ converges, then the numbers a_n must approach 0 as n becomes large. However, there are examples of sequences $\{a_n\}$ for which the series does not converge, and yet

$$\lim_{n \to \infty} a_n = 0.$$

Consider, for instance,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

We contend that the partial sums s_n become very large when n becomes large. To see this, we look at partial sums as follows:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}} + \cdots.$$

In each bunch of terms as indicated, we replace each term by that farthest to the right. This makes our sums smaller. Thus our expression is

$$\geqq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \cdots + \frac{1}{8}} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}} + \cdots$$

$$\geqq 1 + \frac{1}{2} + \quad \frac{1}{2} \quad + \quad \frac{1}{2} \quad + \quad \frac{1}{2} \quad + \cdots$$

and therefore becomes arbitrarily large when n becomes large.

XIV, §2. SERIES WITH POSITIVE TERMS

Throughout this section, we shall assume that our numbers a_n are $\geqq 0$. Then the partial sums

$$s_n = a_1 + \cdots + a_n$$

are increasing, i.e.

$$s_1 \leqq s_2 \leqq s_3 \leqq \cdots \leqq s_n \leqq s_{n+1} \leqq \cdots.$$

If they are to approach a limit at all, they cannot become arbitrarily large. Thus in that case there is a number B such that

$$s_n \leqq B$$

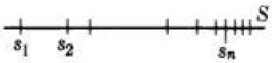
for all n. Such a number B is called an **upper bound**. By a **least upper bound** we mean a number S which is an upper bound, and such that every upper bound B is $\geqq S$. We take for granted that a least upper bound exists. The collection of numbers $\{s_n\}$ has therefore a least upper bound, i.e. there is a smallest number S such that

$$s_n \leqq S$$

for all n. In that case, the partial sums s_n approach S as a limit. In other words, given any positive number $\varepsilon > 0$, we have

$$S - \varepsilon \leqq s_n \leqq S$$

for all n sufficiently large.



This simply expresses the fact that S is the least of all upper bounds for our collection of numbers s_n. We express this as a theorem.

Theorem 2.1. *Let $\{a_n\}$ $(n = 1, 2, \ldots)$ be a sequence of numbers $\geqq 0$ and let*

$$s_n = a_1 + \cdots + a_n.$$

If the sequence of numbers $\{s_n\}$ is bounded, then it approaches a limit S, which is its least upper bound.

Example 1. Prove that the series $\sum_{n=1}^{\infty} 1/n^2$ converges.
Let us look at the series:

$$\frac{1}{1^2} + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}} + \underbrace{\frac{1}{4^2} + \cdots} + \underbrace{\frac{1}{8^2} + \cdots} + \underbrace{\frac{1}{16^2} + \cdots} + \cdots.$$

We look at the groups of terms as indicated. In each group of terms, if we decrease the denominator in each term, then we increase the fraction. We replace 3 by 2, then 4, 5, 6, 7 by 4, then we replace the numbers from 8 to 15 by 8, and so forth. Our partial sums are therefore less than or equal to

$$1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{4^2} + \frac{1}{8^2} + \cdots + \frac{1}{8^2} + \cdots,$$

and we note that 2 occurs twice, 4 occurs four times, 8 occurs eight times, and so forth. Hence the partial sums are less than or equal to

$$1 + \frac{2}{2^2} + \frac{4}{4^2} + \frac{8}{8^2} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

Thus our partial sums are less than or equal to those of the geometric series and are bounded. Hence our series converges.

Theorem 2.1 gives us a very useful criterion to determine when a series with positive terms converges:

Theorem 2.2. *Let*

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n$$

be two series, with $a_n \geqq 0$ for all n and $b_n \geqq 0$ for all n. Assume that there is a number $C > 0$ such that

$$a_n \leqq Cb_n$$

for all n, and that $\sum_{n=1}^{\infty} b_n$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges, and

$$\sum_{n=1}^{\infty} a_n \leqq C \sum_{n=1}^{\infty} b_n.$$

Proof. We have

$$a_1 + \cdots + a_n \leqq Cb_1 + \cdots + Cb_n = C(b_1 + \cdots + b_n) \leqq C \sum_{n=1}^{\infty} b_n.$$

This means that $C \sum_{n=1}^{\infty} b_n$ is a bound for the partial sums

$$a_1 + \cdots + a_n.$$

The least upper bound of these sums is therefore $\leqq C \sum_{n=1}^{\infty} b_n$, thereby proving our theorem.

Theorem 2.2 has an analogue to show that a series does not converge.

Theorem 2.2′. *Let*

$$\sum_{n=1}^{\infty} a_n \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n$$

be two series, with a_n and $b_n \geqq 0$ for all n. Assume that there is a number $C > 0$ such that

$$a_n \geqq Cb_n$$

for all n sufficiently large, and that $\sum_{n=1}^{\infty} b_n$ does not converge. Then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Assume $a_n \geqq Cb_n$ for $n \geqq n_0$. Since $\sum b_n$ diverges, we can make the partial sums

$$\sum_{n=n_0}^{N} b_n = b_{n_0} + \cdots + b_N$$

arbitrarily large as N becomes arbitrarily large. But

$$\sum_{n=n_0}^{N} a_n \geqq \sum_{n=n_0}^{N} Cb_n = C \sum_{n=n_0}^{N} b_n.$$

Hence the partial sums

$$\sum_{n=1}^{N} a_n = a_1 + \cdots + a_N$$

are arbitrarily large as N becomes arbitrarily large, and hence $\sum_{n=1}^{\infty} a_n$ diverges, as was to be shown.

Example 2. Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$$

converges.

We write

$$\frac{n^2}{n^3 + 1} = \frac{1}{n + 1/n^2} = \frac{1}{n}\left(\frac{1}{1 + 1/n^3}\right).$$

Then we see that

$$\frac{n^2}{n^3 + 1} \geqq \frac{1}{2n}.$$

Since $\sum 1/n$ does not converge, it follows that the series of Example 2 does not converge either.

Example 3. The series

$$\sum_{n=1}^{\infty} \frac{n^2+7}{2n^4-n+3}$$

converges. Indeed, we can write

$$\frac{n^2+7}{2n^4-n+3} = \frac{n^2(1+7/n^2)}{n^4(2-(1/n)^3+3/n^4)} = \frac{1}{n^2}\,\frac{1+7/n^2}{2-(1/n)^3+3/n^4}.$$

For n large, the factor

$$\frac{1+7/n^2}{2-(1/n)^3+3/n^4}$$

is certainly bounded, and in fact is near $\frac{1}{2}$. Hence we can compare our series with $1/n^2$ to see that it converges, because $\sum 1/n^2$ converges, and the factor is bounded.

XIV, §2. EXERCISES

1. Show that the series $\sum_{n=1}^{\infty} 1/n^3$ converges.

2. (a) Show that the series $\sum (\log n)/n^3$ converges. [*Hint*: Estimate $(\log n)/n$.]
   (b) Show that the series $\sum (\log n)^2/n^3$ converges.

Test the following series for convergence:

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4+n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+5}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+n+2}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2+1}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2+n}$

XIV, §3. THE RATIO TEST

We continue to consider only series with terms $\geqq 0$. To compare such a series with a geometric series, the simplest test is given by the ratio test.

Ratio test. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with* $a_n > 0$ *for all* n. *Assume that there is a number* c *with* $0 < c < 1$ *such that*

$$\frac{a_{n+1}}{a_n} \leqq c$$

for all n *sufficiently large. Then the series converges.*

Proof. Suppose that there exists some integer N such that

$$\frac{a_{n+1}}{a_n} \leqq c$$

if $n \geqq N$. Then

$$a_{N+1} \leqq ca_N,$$
$$a_{N+2} \leqq ca_{N+1} \leqq c^2 a_N$$

and in general, by induction,

$$a_{N+k} \leqq c^k a_N.$$

Thus

$$\sum_{n=N}^{N+k} a_n \leqq a_N + ca_N + c^2 a_N + \cdots + c^k a_N$$
$$\leqq a_N(1 + c + \cdots + c^k) \leqq a_N \frac{1}{1-c}.$$

Thus in effect, we have compared our series with a geometric series, and we know that the partial sums are bounded. This implies that our series converges.

The ratio test is usually used in the case of a series with positive terms a_n such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c < 1.$$

Example. Show that the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

converges.

We let $a_n = n/3^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+1}}\frac{3^n}{n} = \frac{n+1}{n}\frac{1}{3}.$$

This ratio approaches $\frac{1}{3}$ as $n \to \infty$, and hence the ratio test is applicable: the series converges.

XIV, §3. EXERCISES

Determine whether the following series converge:

1. $\sum n2^{-n}$
2. $\sum n^2 2^{-n}$
3. $\sum \frac{1}{\log n}$
4. $\sum \frac{\log n}{2^n}$
5. $\sum \frac{\log n}{n}$
6. $\sum \frac{n^{10}}{3^n}$
7. $\sum \frac{1}{\sqrt{n(n+1)}}$
8. $\sum \frac{\sqrt{n^3+1}}{e^n}$
9. $\sum \frac{n+1}{\sqrt{n^4+n+1}}$
10. $\sum \frac{n+1}{2^n}$
11. $\sum \frac{n}{(4n-1)(n+15)}$
12. $\sum \frac{1+\cos(\pi n/2)}{e^n}$
13. $\sum \frac{1}{(\log n)^{10}}$
14. $\sum n^2 e^{-n^2}$
15. $\sum n^2 e^{-n^3}$
16. $\sum n^5 e^{-n^2}$
17. $\sum n^4 e^{-n}$
18. $\sum \frac{n^n}{n!3^n}$
19. Let $\{a_n\}$ be a sequence of positive numbers, and assume that

$$\frac{a_{n+1}}{a_n} \geqq 1 - \frac{1}{n}$$

for all n. Show that the series $\sum a_n$ diverges.

20. A ratio test can be applied in the opposite direction to determine when a series diverges. Prove the following statement: Let a_n be a sequence of positive numbers, and let $c \geqq 1$. If $a_{n+1}/a_n \geqq c$ for all n sufficiently large, then the series $\sum a_n$ diverges.

XIV, §4. THE INTEGRAL TEST

You must already have felt that there is an analogy between the convergence of an improper integral and the convergence of a series. We shall now make this precise.

Theorem 4.1. *Let f be a function which is defined and positive for all $x \geqq 1$, and decreasing. The series*

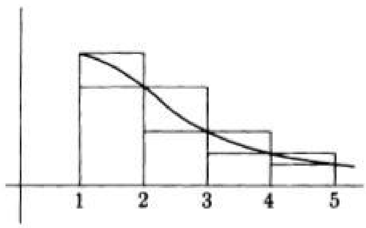
$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the improper integral

$$\int_1^{\infty} f(x)\,dx$$

converges.

We visualize the situation in the following diagram.



Consider the partial sums

$$f(2) + \cdots + f(n)$$

and assume that our improper integral converges. The area under the curve between 1 and 2 is greater than or equal to the area of the rectangle whose height is $f(2)$ and whose base is the interval between 1 and 2. This base has length 1. Thus

$$f(2) \leqq \int_1^2 f(x)\,dx.$$

Again, since the function is decreasing, we have a similar estimate between 2 and 3:

$$f(3) \leqq \int_2^3 f(x)\,dx.$$

We can continue up to n, and get

$$f(2) + f(3) + \cdots + f(n) \leqq \int_1^n f(x)\,dx.$$

As n becomes large, we have assumed that the integral approaches a limit. This means that

$$f(2) + f(3) + \cdots + f(n) \leqq \int_1^\infty f(x)\,dx.$$

Hence the partial sums are bounded, and hence by Theorem 2.1, they approach a limit. Therefore our series converges.

Conversely, assume that the partial sums

$$f(1) + \cdots + f(n)$$

approach a limit as n becomes large.

The area under the graph of f between 1 and n is less than or equal to the sum of the areas of the big rectangles. Thus

$$\int_1^2 f(x)\,dx \leqq f(1)(2-1) = f(1)$$

and

$$\int_2^3 f(x)\,dx \leqq f(2)(3-2) = f(2).$$

Proceeding stepwise, and taking the sum, we see that

$$\int_1^n f(x)\,dx \leqq f(1) + \cdots + f(n-1).$$

The partial sums on the right are less than or equal to their limit. Call this limit L. Then for all positive integers n, we have

$$\int_1^n f(x)\,dx \leqq L.$$

Given any number B, we can find an integer n such that $B \leqq n$. Then

$$\int_1^B f(x)\,dx \leqq \int_1^n f(x)\,dx \leqq L.$$

Hence the integral from 1 to B approaches a limit as B becomes large, and this limit is less than or equal to L. This proves our theorem.

Example. Prove that the series

$$\sum \frac{1}{n^2+1}$$

converges.

Let

$$f(x) = \frac{1}{x^2+1}.$$

Then f is decreasing, and

$$\int_1^B f(x)\,dx = \arctan B - \arctan 1 = \arctan B - \frac{\pi}{4}.$$

As B becomes large, $\arctan B$ approaches $\pi/2$ and therefore has a limit. Hence the integral converges. So does the series, by the theorem.

XIV, §4. EXERCISES

1. Show that the following series diverges: $\sum_{n=2}^{\infty} 1/(n \log n)$.

2. Show that the following series converges: $\sum_{n=1}^{\infty} (n+1)/((n+2)n!)$.

Test for convergence:

3. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^3}$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3+2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n-1}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3-n+5}$

11. Let ε be a number > 0. Show that the series $\sum_{n=1}^{\infty} 1/n^{1+\varepsilon}$ converges.

12. Show that the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^{1+\varepsilon}}$ if $\varepsilon > 0$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(\log n)^2}{n^{3/2}}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(\log n)^3}{n^2}$

13. If $\varepsilon > 0$ show that the series $\sum_{n=2}^{\infty} 1/n(\log n)^{1+\varepsilon}$ converges.

XIV, §5. ABSOLUTE AND ALTERNATING CONVERGENCE

We consider a series $\sum_{n=1}^{\infty} a_n$ in which we do not assume that the terms a_n are $\geqq 0$. We shall say that the series **converges absolutely** if the series

$$\sum_{n=1}^{\infty} |a_n|$$

formed with the absolute values of the terms a_n converges. This is now a series with terms $\geqq 0$, to which we can apply the tests for convergence given in the two preceding sections. This is important, because we have:

Theorem 5.1. *Let $\{a_n\}$ $(n = 1, 2, \ldots)$ be a sequence, and assume that the series*

$$\sum_{n=1}^{\infty} |a_n|$$

converges. Then so does the series $\sum_{n=1}^{\infty} a_n$.

Proof. Let a_n^+ be equal to 0 if $a_n < 0$ and equal to a_n itself if $a_n \geqq 0$. Let a_n^- be equal to 0 if $a_n > 0$ and equal to $-a_n$ if $a_n \leqq 0$. Then both a_n^+ and a_n^- are $\geqq 0$. By assumption and comparison with $\sum |a_n|$, we see that each one of the series

$$\sum_{n=1}^{\infty} a_n^+ \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^-$$

converges. Hence so does their difference

$$\sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-,$$

which is equal to

$$\sum_{n=1}^{\infty} (a_n^+ - a_n^-),$$

which is none other than $\sum_{n=1}^{\infty} a_n$. This proves our theorem.

We shall use one more test for convergence of a series which may have positive and negative terms.

Theorem 5.2. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series such that*

$$\lim_{n\to\infty} a_n = 0,$$

such that the terms a_n *are alternately positive and negative, and such that* $|a_{n+1}| \leqq |a_n|$ *for* $n \geqq 1$. *Then the series is convergent.*

Proof. Let us write the series in the form

$$b_1 - c_1 + b_2 - c_2 + b_3 - c_3 + \cdots,$$

with $b_n, c_n \geqq 0$. Let

$$s_n = b_1 - c_1 + b_2 - c_2 + \cdots + b_n,$$
$$t_n = b_1 - c_1 + b_2 - c_2 + \cdots + b_n - c_n.$$

Since the absolute values of the terms decrease, it follows that

$$s_1 \geqq s_2 \geqq s_3 \geqq \cdots \quad \text{and} \quad t_1 \leqq t_2 \leqq t_3 \leqq \cdots,$$

i.e. that the s_n are decreasing and the t_n are increasing. Indeed,

$$s_{n+1} = s_n - c_n + b_{n+1} \quad \text{and} \quad 0 \leqq b_{n+1} \leqq c_n.$$

Thus we subtract more from s_n by c_n than we add afterwards by b_{n+1}. Hence $s_n \geqq s_{n+1}$. Furthermore, $s_n \geqq t_n$. Hence we may visualize our sequences as follows:

$$s_n \geqq s_{n+1} \geqq \cdots \geqq t_{n+1} \geqq t_n.$$

Note that $s_n - t_n = c_n$, and that c_n approaches 0 as n becomes large. If we let L be the greatest lower bound for the sequence $\{s_n\}$, and M be the least upper bound for the sequence $\{t_n\}$, then

$$s_n \geqq L \geqq M \geqq t_n$$

for all n. Since the difference $s_n - t_n$ becomes arbitrarily small, it follows that $L - M$ is arbitrarily small, and hence equal to 0. Thus $L = M$, and this proves that s_n and t_n approach L as a limit, whence our series $\sum a_n$ converges to L.

Example. The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

is convergent, but not absolutely convergent.

Remark. Not all series which are convergent are either absolutely convergent, or are of the above alternating type. However, these two kinds of series are the ones that arise most frequently in practice, so that we have laid emphasis on them.

XIV, §5. EXERCISES

Determine whether the following series converge absolutely:

1. $\sum \frac{\sin n}{n^3}$
2. $\sum \frac{1 + \cos \pi n}{n!}$
3. $\sum \frac{\sin \pi n + \cos 2\pi n}{n^{3/2}}$
4. $\sum \frac{(-1)^n}{n^2 + 1}$
5. $\sum \frac{(-1)^n + \cos 3n}{n^2 + n}$

Which of the following series converge and which do not?

6. $\sum \frac{(-1)^n}{n}$
7. $\sum \frac{(-1)^n}{n^2}$
8. $\sum (-1)^n \frac{1}{n+1}$
9. $\sum \frac{(-1)^{n+1}}{\log (n+2)}$
10. For each number x, show that the series $\sum (\sin n^2 x)/n^2$ converges absolutely. Let f be the function whose value at x is the above series. Show that f is continuous. Determine whether f is differentiable or not. (Remarkably enough, this was not known for a long time! Cf. J. P. Kahane, Bulletin of the American Mathematical Society, March 1964, p. 199. J. Gerver, a sophomore at Columbia College, showed that the series is differentiable at all points $m\pi/n$, where m, n are odd integers, that the derivative is $-\frac{1}{2}$, and that these are the only points where the function is differentiable. Cf. his articles in *Am. J. Math.*, 1970 and 1971.)

Determine whether the following series converge, and whether they converge absolutely.

11. $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$
12. $\sum \frac{(-1)^{n+2}}{\log n}$
13. $\sum \frac{(-1)^n}{\sqrt[n]{n}}$
14. $\sum (-1)^n \frac{n^2}{n^2 + 1}$

15. $\sum (-1)^n \dfrac{n^2}{n^3+2}$

16. $\sum (-1)^{n+1} \dfrac{n^2+2}{n^3+n-1}$

17. $\sum (-1)^{n+1} \dfrac{\sqrt{n}}{n+2}$

18. $\sum \dfrac{(-1)^n}{n^{5/2}+n}$

19. $\sum (-1)^n \dfrac{n}{n^2+1}$

20. $\sum \dfrac{(-1)^n}{\sqrt{\log n}}$

XIV, §6. POWER SERIES

Perhaps the most important series are power series. Let x be any number and let $\{a_n\}$ $(n = 0, 1, \ldots)$ be a sequence of numbers. Then we can form the series

$$\sum_{n=1}^{\infty} a_n x^n.$$

The partial sums are

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

We have already met such sums when we discussed Taylor's formula.

Example. The power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

converges for all x, absolutely. Indeed, it will suffice to prove that for any number $R > 0$, the above series converges for $0 < x \leqq R$. We use the ratio test. Let

$$b_n = \frac{x^n}{n!}.$$

Then

$$\frac{b_{n+1}}{b_n} = \frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n} = \frac{x}{n+1} \leqq \frac{R}{n+1}.$$

When n is sufficiently large, it follows that $R/(n+1)$ is small, and in particular is $< \frac{1}{2}$, so that we can apply the ratio test to prove our assertion.

Similarly, we could prove that the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

converge absolutely for all x, letting for instance

$$b_n = \frac{x^{2n+1}}{(2n+1)!}$$

for the first one. Then $b_{n+1}/b_n = x^2/[(2n+3)(2n+2)]$, and we can argue as before.

Theorem 6.1. *Assume that there is a number $r \geqq 0$ such that the series*

$$\sum_{n=1}^{\infty} |a_n| r^n$$

converges. Then for all x such that $|x| \leqq r$, the series

$$\sum_{n=1}^{\infty} a_n x^n$$

converges absolutely.

Proof. The absolute value of each term is

$$|a_n||x|^n \leqq |a_n| r^n.$$

Our assertion follows from the comparison Theorem 2.2.

The least upper bound of all numbers r for which we have the convergence stated in the theorem is called the **radius of convergence** of the series. If there is no upper bound for the numbers r such that the power series above converges, then we say that the radius of convergence is **infinity**.

Suppose that there is an upper bound for the numbers r above, and thus let s be the radius of convergence of the series. Then if $x > s$, the series

$$\sum_{n=1}^{\infty} |a_n| x^n$$

does *not* converge. Thus the radius of convergence s is the number such that the series converges absolutely if $0 < x < s$ but does not converge absolutely if $x > s$.

Theorem 6.1 allows us to define a function f; namely, for all numbers x such that $|x| < s$, we define

$$f(x) = \lim_{n \to \infty} (a_0 + a_1 x + \cdots + a_n x^n).$$

Our proofs that the remainder term in Taylor's formula approaches 0 for various functions now allow us to say that these functions are given by their Taylor series. Thus

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

for all x. Furthermore,

$$\log(1 + x) = x - \frac{x^2}{2} + \cdots$$

is valid for $-1 < x < 1$.

(Here we saw that the series converges for $x = 1$, but it does not converge absolutely, cf. §1.)

However, we can now define functions at random by means of a power series, provided we know the power series converges absolutely, for $|x| < r$.

The ratio test usually gives an easy way to determine when a power series converges, or when it diverges.

Example. Prove that the series

$$\sum_{n=2}^{\infty} \frac{\log n}{n^2} x^n$$

converges absolutely for $|x| < 1$, and diverges for $|x| > 1$.

Let $0 < c < 1$ and consider x such that $0 < x \leqq c$. Let

$$b_n = \frac{\log n}{n^2} x^n.$$

Then

$$\frac{b_{n+1}}{b_n} = \frac{\log(n + 1)}{(n + 1)^2} x^{n+1} \frac{n^2}{\log n} \frac{1}{x^n} = \frac{\log(n + 1)}{\log n} \left(\frac{n}{n + 1}\right)^2 x.$$

Since $\log(n+1)/\log n$ and $(n/(n+1))^2$ approach 1 when n becomes very large, it follows that if $c < c_1 < 1$, then for all n sufficiently large

$$\frac{b_{n+1}}{b_n} \leqq c_1$$

and hence our series converges. This is true for every c such that $0 < c < 1$, and hence the series converges absolutely for $|x| < 1$.

Let $c > 1$. If $x \geqq c$ then for all n sufficiently large, it follows that $b_{n+1}/b_n \geqq 1$, whence the series does not converge. This is so for all $c > 1$, and hence the series does not converge if $x > 1$. Hence 1 is the radius of convergence.

If a power series converges absolutely only for $x = 0$, then we agree to say that its radius of convergence is 0. For example, the radius of convergence of the series

$$\sum_{n=1}^{\infty} n!x^n$$

is equal to 0, as one sees by using the ratio test in the divergent case.

Root test. *Let $\sum a_n x^n$ be a power series and assume that*

$$\lim |a_n|^{1/n} = s,$$

where s is a number. If $s \neq 0$ then the radius of convergence of the series is equal to $1/s$. If $s = 0$, then the radius of convergence is infinity. If $|a_n|^{1/n}$ becomes arbitrarily large as n becomes large, then the radius of convergence is 0.

Proof. Without loss of generality, we may assume that $a_n \geqq 0$ for all n. Suppose first that s is a number $\neq 0$, and let $0 \leqq r < 1/s$. Then $sr < 1$. The numbers $a_n^{1/n}r$ approach sr and hence there is some number $\varepsilon > 0$ such that

$$a_n^{1/n}r < 1 - \varepsilon$$

for all n sufficiently large. Hence the series $\sum a_n r^n$ converges by comparison with the geometric series. If on the other hand $r > 1/s$, then $a_n^{1/n}r$ approaches $sr > 1$, and hence we have

$$a_n^{1/n}r \geqq 1 + \varepsilon$$

for all n sufficiently large. Comparison from below shows that the series $\sum a_n r^n$ diverges. We leave the cases $s = 0$ or $s = \infty$ to the reader.

Example. The series $\sum x^n/n^2$ has a radius of convergence equal to 1, because

$$\lim\left(\frac{1}{n^2}\right)^{1/n} = \lim \frac{1}{n^{2/n}} = 1,$$

by Corollary 5.5 of Chapter VIII.

To give other examples, we recall an inequality which you should have worked out in Chapter X, §1, namely

$$\boxed{(n-1)! \leqq n^n e^{-n} e \leqq n!.}$$

From it, we prove:

As n becomes large, the expression

$$\frac{(n!)^{1/n}}{n} = \left[\frac{n!}{n^n}\right]^{1/n}$$

approaches $1/e$.

Proof. Take the n-th root of the inequality $n^n e^{-n} e \leqq n!$. We get

$$ne^{-1}e^{1/n} \leqq (n!)^{1/n}.$$

Dividing by n yields

$$\frac{1}{e}\, e^{1/n} \leqq \frac{(n!)^{1/n}}{n}.$$

On the other hand, multiply both sides of the inequality

$$(n-1)! \leqq n^n e^{-n} e$$

by n. We get $n! \leqq n^n e^{-n} en$. Take an n-th root:

$$(n!)^{1/n} \leqq ne^{-1}e^{1/n}n^{1/n}.$$

Dividing by n yields

$$\frac{(n!)^{1/n}}{n} \leqq \frac{1}{e}\, e^{1/n} n^{1/n}.$$

But we know that both $n^{1/n}$ and $e^{1/n}$ approach 1 as n becomes large. Thus our quotient is squeezed between two numbers approaching $1/e$, and must therefore approach $1/e$.

Example. We have

$$\lim_{n\to\infty}\left[\frac{(3n)!}{n^{3n}}\right]^{1/n}=\frac{27}{e^3}.$$

Proof. We write

$$\frac{(3n)!}{n^{3n}}=\frac{(3n)!}{(3n)^{3n}}\,3^{3n}.$$

The $3n$-th root of this expression is

$$\left[\frac{(3n)!}{(3n)^{3n}}\right]^{1/3n}3.$$

We have seen that

$$\left(\frac{m!}{m^m}\right)^{1/m}\qquad\text{approaches}\qquad\frac{1}{e}$$

as m becomes large. We use $m = 3n$. We conclude that

$$\left(\frac{(3n)!}{n^{3n}}\right)^{1/3n}\qquad\text{approaches}\qquad\frac{3}{e}.$$

Hence

$$\left(\frac{(3n)!}{n^{3n}}\right)^{1/n}\qquad\text{approaches}\qquad\frac{27}{e^3},$$

as desired.

XIV, §6. EXERCISES

1. Use the abbreviation $\lim\limits_{n\to\infty}$ to mean: limit as n becomes very large. Prove that

   (a) $\displaystyle\lim_{n\to\infty}\left[\frac{(3n)!}{n^{3n}}\right]^{1/n}=\frac{27}{e^3}$ (b) $\displaystyle\lim_{n\to\infty}\left[\frac{(3n)!}{n!\,n^{2n}}\right]^{1/n}=\frac{27}{e^2}$

2. Find the limit:

   (a) $\displaystyle\lim_{n\to\infty}\left[\frac{(2n)!}{n^{2n}}\right]^{1/n}$ (b) $\displaystyle\lim_{n\to\infty}\left[\frac{(2n)!(5n)!}{n^{4n}(3n)!}\right]^{1/n}$

Find the radius of convergence of the following series:

3. (a) $\sum n^n x^n$ (b) $\sum \frac{x^n}{n^n}$

4. $\sum \frac{n}{n+5} x^n$
5. $\sum (\log n)x^n$
6. $\sum \frac{1}{\log n} x^n$
7. $\sum (\log n)^2 x^n$
8. $\sum 2^n x^n$
9. $\sum 2^{-n} x^n$
10. $\sum (1+n)^n x^n$
11. $\sum \frac{x^n}{n}$
12. $\sum \frac{x^n}{\sqrt{n}}$
13. $\sum (1+(-2)^n)x^n$
14. $\sum (1+(-1)^n)x^n$
15. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$
16. $\sum_{n=1}^{\infty} \frac{n^n}{n!} x^n$
17. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$
18. $\sum_{n=1}^{\infty} \frac{n^{5n}}{(2n)!n^{3n}} x^n$
19. $\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^2} x^n$
20. $\sum_{n=1}^{\infty} \frac{\sin n\pi/2}{2^n} x^n$
21. $\sum_{n=1}^{\infty} \frac{\log n}{2^n} x^n$
22. $\sum_{n=2}^{\infty} \frac{1+\cos 2\pi n}{3n} x^n$
23. $\sum_{n=2}^{\infty} nx^n$
24. $\sum_{n=1}^{\infty} \frac{\sin 2\pi n}{n!} x^n$
25. $\sum_{n=2}^{\infty} n^2 x^n$
26. $\sum_{n=1}^{\infty} \frac{\cos n^2}{n^n} x^n$
27. $\sum_{n=2}^{\infty} \frac{n}{\log n} x^n$
28. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n!-1} x^n$
29. $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$
30. $\sum_{n=1}^{\infty} \frac{(-1)^n+1}{n!} x^n$

Note: For some of the above radii of convergence, recall that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{1+n}{n}\right)^n = e.$$

XIV, §7. DIFFERENTIATION AND INTEGRATION OF POWER SERIES

If we have a polynomial

$$a_0 + a_1 x + \cdots + a_n x^n$$

with numbers $a_0, a_1, \ldots, a_n$ as coefficients, then we know how to find its derivative. It is $a_1 + 2a_2 x + \cdots + na_n x^{n-1}$. We would like to say that the derivative of a series can be taken in the same way, and that the derivative converges whenever the series does.

Theorem 7.1. *Let r be a number > 0 and let $\sum a_n x^n$ be a series which converges absolutely for $|x| < r$. Then the series $\sum n a_n x^{n-1}$ also converges absolutely for $|x| < r$.*

Proof. Since we are interested in the absolute convergence, we may assume that $a_n \geqq 0$ for all n. Let $0 < x < r$, and let c be a number such that $x < c < r$. Recall that

$$\lim_{n \to \infty} n^{1/n} = 1.$$

We may write

$$n a_n x^n = a_n (n^{1/n} x)^n.$$

Then for all n sufficiently large, we conclude that

$$n^{1/n} x < c$$

because $n^{1/n} x$ comes arbitrarily close to x. Hence for all n sufficiently large, we have

$$n a_n x^n < a_n c^n.$$

We can then compare the series $\sum n a_n x^n$ with $\sum a_n c^n$ to conclude that $\sum n a_n x^n$ converges. Since

$$\sum n a_n x^{n-1} = \frac{1}{x} \sum n a_n x^n$$

we have proved Theorem 7.1.

A similar result holds for integration, but trivially. Indeed, if we have a series $\sum_{n=1}^{\infty} a_n x^n$ which converges absolutely for $|x| < r$, then the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1} = x \sum_{n=1}^{\infty} \frac{a_n}{n+1} x^n$$

has terms whose absolute value is smaller than in the original series.

The preceding results can be expressed by saying that an absolutely convergent power series can be integrated and differentiated term by term and still yield an absolutely convergent power series.

It is natural to expect that if

$$f(x) = \sum_{n=1}^{\infty} a_n x^n,$$

then f is differentiable and its derivative is given by differentiating the series term by term. The next theorem proves this.

Theorem 7.2. *Let*

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

be a power series, which converges absolutely for $|x| < r$. *Then* f *is differentiable for* $|x| < r$, *and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Proof. Let $0 < b < r$. Let $\delta > 0$ be such that $b + \delta < r$. We consider values of x such that $|x| < b$ and values of h such that $|h| < \delta$. We have the numerator of the Newton quotient:

$$f(x+h) - f(x) = \sum_{n=1}^{\infty} a_n(x+h)^n - \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n[(x+h)^n - x^n].$$

By mean value theorem, there exists a number x_n between x and $x + h$ such that

$$(x+h)^n - x^n = n x_n^{n-1} h$$

and consequently

$$f(x+h) - f(x) = \sum_{n=1}^{\infty} n a_n x_n^{n-1} h.$$

Therefore

$$\frac{f(x+h) - f(x)}{h} = \sum_{n=1}^{\infty} n a_n x_n^{n-1}.$$

We have to show that the Newton quotient above approaches the value of the series obtained by taking the derivative term by term. We have

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} - \sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=1}^{\infty} n a_n x_n^{n-1} - \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n [x_n^{n-1} - x^{n-1}].\end{aligned}$$

Using the mean value theorem again, there exists y_n between x_n and x such that the preceding expression is

$$\frac{f(x+h)-f(x)}{h} - \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=2}^{\infty} (n-1)na_n y_n^{n-2}(x_n - x).$$

We have $|y_n| \leqq b + \delta < r$, and $|x_n - x| \leqq |h|$. Consequently,

$$\left|\frac{f(x+h)-f(x)}{h} - \sum_{n=1}^{\infty} na_n x^{n-1}\right| \leqq \sum_{n=2}^{\infty} (n-1)n|a_n||y_n|^{n-2}|h|$$

$$\leqq |h| \sum_{n=2}^{\infty} (n-1)n|a_n|(b+\delta)^{n-2}.$$

By Theorem 7.1 applied twice, we know that the series appearing on the right converges. It is equal to a fixed constant. As h approaches 0, it follows that the expression on the right approaches 0, so the expression on the left also approaches 0. This proves that f is differentiable at x, and that its derivative is equal to $\sum_{n=1}^{\infty} na_n x^{n-1}$, for all x such that $|x| < b$. This is true for all b, $0 < b < r$, and therefore concludes the proof of our theorem.

Theorem 7.3. *Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$ be a power series, which converges absolutely for $|x| < r$. Then the relation*

$$\int f(x)\,dx = \sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is valid in the interval $|x| < r$.

Proof. We know that the series for f integrated term by term converges absolutely in the interval. By the preceding theorem, its derivative term by term is the series for the derivative of the function, thereby proving our assertion.

Example. If we had never heard of the exponential function, we could define a function

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Taking the derivative term by term, we see that

$$f'(x) = f(x).$$

Hence by what we know from Chapter VIII, §1, Exercise 8, we conclude that

$$f(x) = Ke^x$$

for some constant K. Letting $x = 0$ shows that

$$1 = f(0) = K.$$

Thus $K = 1$ and $f(x) = e^x$.

Similarly, if we had never heard of sine and cosine, we could **define** functions

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

Differentiating term by term shows that

$$S'(x) = C(x), \qquad C'(x) = -S(x).$$

Furthermore, $S(0) = 0$ and $C(0) = 1$. It can then be shown easily that any pair of functions $S(x)$ and $C(x)$ satisfying these properties must be the sine and cosine.

XIV, §7. EXERCISES

1. Verify in detail that differentiating term by term the series for the sine and cosine given at the end of the section yields

$$S'(x) = C(x) \quad \text{and} \quad C'(x) = -S(x).$$

2. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Prove that $f''(x) = f(x)$.

3. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(n!)^2}.$$

Prove that

$$x^2 f''(x) + xf'(x) = 4x^2 f(x).$$

4. Let

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

Show that $f'(x) = 1/(1 + x^2)$.

5. Let

$$J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}.$$

Prove that

$$x^2 J''(x) + xJ'(x) + x^2 J(x) = 0.$$

6. For any positive integer k, let

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left(\frac{x}{2}\right)^{2n+k}.$$

Prove that

$$x^2 J_k''(x) + xJ_k'(x) + (x^2 - k^2)J_k(x) = 0.$$

APPENDIX
To the First Four Parts

ε and δ

This appendix is intended to show how the notions of limits and the properties of limits can be explained and proved in terms of the notions and properties of numbers. We therefore assume the latter and carry out the proofs from there.

There remains the problem of showing how the real numbers can be defined in terms of the rational numbers, and the rational numbers in terms of integers. This takes too long to be included in this book.

Aside from the ordinary rules for addition, multiplication, subtraction, division (by non-zero numbers), ordering, positivity, and inequalities, there is one more basic property satisfied by the real numbers. This property is stated in §1. Our proofs then use only these properties.

Warning. The level of abstract understanding and use of language needed to master this appendix is considerably higher than for the rest of the book. We are involved in "proving" properties which are intuitively very clear. Hence you should not take this appendix too seriously unless you are theoretically inclined, or you wish to have an introduction to some essential tools of analysis, i.e. a first acquaintance, which will plant some ideas in your head for future use in higher courses, when the techniques described here become essential because more intricate estimates are needed when dealing with such higher analysis. It is useful to have seen the stuff previously, even though you may not have mastered it the first time. It is part of our psychology that we learn by approximation. Furthermore, knowledge at one level is fully mastered only when you use it at the next level of depth. Hence studying harder things, even though you have a limited understanding of them, makes it possible to understand fully the easier things.

Nevertheless, this appendix should be omitted under ordinary circumstances.

APP., §1. LEAST UPPER BOUND

We meet again the problem of where to jump into the theory. It would be long and tedious to jump in too early. Hence we assume known the contents of Chapter I, §1 and §2. These involve the ordinary operations of addition and multiplication, and the notion of ordering, positivity, negative numbers, and inequalities. Those who are interested in seeing the logical development of these notions are referred to books on analysis.

A collection of numbers will simply be called a **set** of numbers. This is shorter and is the usual terminology. If a set has at least one number in it, we say that it is **non-empty**. A set S' is called a **subset** of S if every element of S' is an element of S. In other words, if S' is part of S.

Let S be a non-empty set of numbers. We shall say that S is **bounded from above** if there exists a number B such that

$$x \leqq B$$

for all x in our set S. We then call B an **upper bound** for S.

A **least upper bound** for S is an upper bound L such that any upper bound B for S satisfies the inequality $B \geqq L$. If M is another least upper bound, then we have $M \geqq L$ and $L \geqq M$, whence $L = M$. Consequently, a least upper bound is unique.

Similarly, we define the notions of bounded from below, and of greatest lower bound. (Do it yourself.)

We shall now give examples, assuming that the reader has an intuitive notion of the real numbers. They make our meaning clearer, but we do not give proofs. Although this is the reverse of the logical order, it is the appropriate psychological order.

Example. The set of positive integers $\{1, 2, 3, \ldots\}$ is not bounded from above. It is bounded from below. The number 1 is a greatest lower bound.

Example. Let S be the set of numbers x such that $0 \leqq x$ and $x^2 < 2$. This set is bounded from below, for instance by 0; it is also bounded from above, and 2 is certainly an upper bound. As a matter of fact, $\sqrt{2}$ is the least upper bound. Note that the least upper bound does not lie in the set S, that is, is not an element of S.

Example. Let T be the set of numbers x such that $0 \leqq x$ and $x^2 \leqq 2$. Again, $\sqrt{2}$ is the least upper bound of T, and is an element of T. It should be intuitively clear that T differs from S only in that it contains the additional element $\sqrt{2}$.

Example. Let U be the set of all numbers $1/n$, where n ranges over the positive integers. Thus U consists of $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Then U is bounded. The number 1 is its least upper bound, and lies in U. The number 0 is the greatest lower bound, and does not lie in U.

The real numbers satisfy a property which is not satisfied by the set of rational numbers, namely:

Fundamental property. *Every non-empty set S of numbers which is bounded from above has a least upper bound. Every non-empty set of numbers S which is bounded from below has a greatest lower bound.*

Proposition 1.1. *Let a be a number such that*

$$0 \leqq a < \frac{1}{n}$$

for every positive integer n. Then $a = 0$. There is no number b such that $b \geqq n$ for every positive integer n.

Proof. Suppose there is a number $a \neq 0$ such that $a < 1/n$ for every positive integer n. Then $n < 1/a$ for every positive integer n. Thus to prove both our assertions, it will suffice to prove the second.

Suppose there is a number b such that $b \geqq n$ for every positive integer n. Let S be the set of positive integers. Then S is bounded, and hence has a least upper bound. Let C be this least upper bound. No number strictly less than C can be an upper bound. Since $0 < 1$, we have $C < C + 1$, whence $C - 1 < C$. Hence there is a positive integer n such that

$$C - 1 < n.$$

This implies that $C < n + 1$ and $n + 1$ is a positive integer. We have contradicted our assumption that C is an upper bound for the set of postive integers, so no such upper bound can exist.

Observe that Proposition 1.1 proves that the set of positive integers is not bounded from above. It is reasonable to ask if this sort of obvious property really needs the least upper bound axiom to be proved, and the answer is *yes*. One can construct systems satisfying all the ordinary rules for addition, multiplication, division (by non-zero elements), inequalities, such that the least upper bound axiom is not satisfied, and such that there exists an element t in the system with the property that $n < t$ for all positive integers n. We don't want to make this appendix too long, and we won't go into the construction of such systems, but it is probably illuminating for you to have it confirmed that the least upper bound property was needed in the proof of Proposition 1.1.

APP., §1. EXERCISES

Determine in each case whether the set is bounded from above, from below, and describe the least upper bound and greatest lower bound if it exists, without giving proofs, just using your intuition of numbers.

1. (a) The set of all positive even integers.
   (b) The set of all positive odd integers.
   (c) The set of all rational numbers.
2. (a) The set of all numbers x such that $0 \leqq x$ and $x^3 < 5$.
   (b) The set of all numbers x such that $0 \leqq x$ and $x^3 \leqq 5$.
   (c) The set of all numbers x such that $x^2 \leqq 4$.
   (d) The set of all numbers x such that $2x - 7 < 4$.
3. Prove that there exists a positive integer N such that if n is an integer $\geqq N$ then $3n > 150$.
4. Let B be a positive number. Prove that there exists a positive integer N such that if n is an integer $\geqq N$ then $5n > B$.
5. Let S be the set of numbers x such that $0 \leqq x$ and $x^2 \leqq 2$. Prove that the least upper bound of S is a number b such that $b^2 = 2$. [*Hint*: Prove that $b^2 > 2$ and $b^2 < 2$ are impossible.]

APP., §2. LIMITS

Let S be a set of numbers and let f be a function defined for all numbers in S. Let x_0 be a number. We shall assume that S is **arbitrarily close to** x_0, i.e. given $\varepsilon > 0$ there exists an element x of S such that $|x - x_0| < \varepsilon$. Let L be a number. We shall say that $f(x)$ **approaches the limit** L **as** x **approaches** x_0 if the following condition is satisfied:

Given a number $\varepsilon > 0$, there exists a number $\delta > 0$ such that for all x in S satisfying

$$|x - x_0| < \delta$$

we have

$$|f(x) - L| < \varepsilon.$$

If that is the case, then we write

$$\lim_{x \to x_0} f(x) = L.$$

We could also rephrase this as follows. We write

$$\lim_{h \to 0} f(x_0 + h) = L$$

and say that **the limit of $f(x_0 + h)$ is L as h approaches** 0 if the following condition is satisfied:

Given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever h is a number with $|h| < \delta$ and $x_0 + h$ in S, then

$$|f(x_0 + h) - L| < \varepsilon.$$

We note that our definition of limit depends on the set S on which f is defined. Thus we should say "limit with respect to S." The next proposition shows that this is really unnecessary.

Proposition 2.1. *Let S be a set of numbers arbitrarily close to x_0 and let S' be a subset of S, also arbitrarily close to x_0. Let f be a function defined on S. If*

$$\lim_{x \to x_0} f(x) = L \qquad \textit{(with respect to } S\textit{)},$$

$$\lim_{x \to x_0} f(x) = M \qquad \textit{(with respect to } S'\textit{)},$$

then $L = M$. In particular, the limit is unique.

Proof. Given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that whenever x is in S' and $|x - x_0| < \delta_1$ we have

$$|f(x) - L| < \frac{\varepsilon}{2},$$

and there exists $\delta_2 > 0$ such that whenever $|x - x_0| < \delta_2$ then

$$|f(x) - M| < \frac{\varepsilon}{2}.$$

Let $\delta = \min(\delta_1, \delta_2)$. If $|x - x_0| < \delta$ then

$$|L - M| \leqq |L - f(x) + f(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $|L - M|$ is less than any $\varepsilon > 0$, and by Proposition 1.1, we must have $|L - M| = 0$, whence

$$L - M = 0$$

and

$$L = M.$$

In practice from now on, we omit to state that x is an element of S. The context makes it clear each time.

Furthermore, in many subsequent proofs, we shall need several simultaneous inequalities to be satisfied, just as in the preceding proof we had inequalities with δ_1 and δ_2. In each case, we use a similar trick, letting δ be the minimum of $\delta_1, \delta_2, \delta_3, \ldots$ needed to make each desired inequality valid. Thus in writing down the proofs, we omit the intermediate $\delta_1, \delta_2, \delta_3 \ldots.$

Remark. Suppose that $\lim_{x \to x_0} f(x) = L$. Then there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ we have

$$|f(x)| < |L| + 1.$$

Indeed, given $1 > 0$ there exists δ such that whenever $|x - x_0| < \delta$ we have

$$|f(x) - L| < 1,$$

so that our assertion follows from standard properties of inequalities.

Also, note that we have trivially

$$\lim_{x \to x_0} C = C$$

for any number C, viewed as a constant function on S. Indeed, given $\varepsilon > 0$,

$$|C - C| < \varepsilon.$$

Remark. We mention a word about limits "**when** x **becomes large**." Let a be a number and f a function defined for all numbers $x \geqq a$. Let L be a number. We shall say that $f(x)$ **approaches** L **as** x **becomes large**, and we write

$$\lim_{x \to \infty} f(x) = L$$

if the following condition is satisfied. Given $\varepsilon > 0$ there exists a number A such that whenever $x > A$ we have

$$|f(x) - L| < \varepsilon.$$

In practice, instead of saying "when x becomes large," we sometimes say "when x approaches ∞." We leave it to you to define the analogous notion "when x becomes large negative," or "x approaches $-\infty$."

In the definition of $\lim\limits_{x \to \infty}$ we took a function f defined for $x \geqq a$. If $a_1 > a$, and we restrict the function to all numbers $\geqq a_1$, then the limit as x becomes very large will be the same.

Let us suppose that $a \geqq 1$. Define a function g for values of x such that

$$0 < x \leqq 1/a$$

by the condition

$$g(x) = f(1/x).$$

Then a second's thought will allow you to prove that

$$\lim_{x \to 0} g(x)$$

exists if and only if

$$\lim_{x \to \infty} f(x)$$

exists, and that they are equal.

Consequently all properties which we prove concerning limits as x approaches 0 (or a number) immediately give rise to similar properties concerning limits as x becomes very large. We leave their formulations to you.

An important case occurs when the function is defined for the positive integers. Then it is called a **sequence**. A sequence of numbers is usually denoted by

$$\{a_1, a_2, \ldots, a_n, \ldots\}$$

or simply $\{a_n\}$.

Example. Let $a_n = f(n) = (-1)^n$. Then

$$a_1 = -1, \qquad a_2 = 1, \qquad a_3 = -1, \qquad a_4 = 1,$$

and so forth. Observe that numbers of the sequence indexed by different integers, for instance a_2 and a_4, may be equal.

Example. Let $a_n = 2n$. This defines the sequence of positive even integers.

Example. Let $a_n = 2n + 1$. This defines the sequence of odd positive integers $\geqq 3$.

Example. We have $\lim_{n \to \infty} 1/nx = 0$ for any number $x \neq 0$. To prove this, say $x > 0$. Given ε, let N be a positive integer such that $1/N < \varepsilon x$. If $n \geqq N$, then

$$\frac{1}{n} \leqq \frac{1}{N} < \varepsilon x,$$

and therefore $1/nx < \varepsilon$. This proves our assertion when $x > 0$. The proof when $x < 0$ is similar. Carry it out completely.

The next theorems, concerning the basic properties of limits, describe limits of sums, products, quotients, inequalities, and composite functions.

Theorem 2.2. *Let S be a set of numbers, and f, g two functions defined for all numbers in S. Let x_0 be a number. If*

$$\lim_{x \to x_0} f(x) = L$$

and

$$\lim_{x \to x_0} g(x) = M,$$

then $\lim_{x \to x_0} (f + g)(x)$ exists and is equal to $L + M$.

Proof. Given $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever $|x - x_0| < \delta$ (and x is in S), we have

$$|f(x) - L| < \frac{\varepsilon}{2},$$

$$|g(x) - M| < \frac{\varepsilon}{2}.$$

We observe that

$$|f(x) + g(x) - L - M| \leqq |f(x) - L| + |g(x) - M| < \varepsilon.$$

This proves that $L + M$ is the limit of $(f + g)(x)$ as x approaches x_0.

Theorem 2.3. *Let S be a set of numbers, and f, g two functions defined for all numbers in S. Let x_0 be a number. If*

$$\lim_{x \to x_0} f(x) = L$$

and

$$\lim_{x \to x_0} g(x) = M,$$

then $\lim_{x \to x_0} f(x)g(x)$ *exists and is equal to* LM.

Proof. Given $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $|x - x_0| < \delta$, we have

$$|f(x) - L| < \frac{1}{2}\frac{\varepsilon}{|M| + 1},$$

$$|g(x) - M| < \frac{1}{2}\frac{\varepsilon}{|L| + 1},$$

$$|f(x)| < |L| + 1.$$

We have

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\
&\leqq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\
&\leqq |f(x)||g(x) - M| + |f(x) - L||M| \\
&< (|L| + 1)\frac{1}{2}\frac{\varepsilon}{|L| + 1} + \frac{1}{2}\frac{\varepsilon}{|M| + 1}|M| \\
&\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\leqq \varepsilon.
\end{aligned}$$

Corollary 2.4. *Let C be a number and let the assumptions be as in the theorem. Then*

$$\lim_{x \to x_0} Cf(x) = CL.$$

Proof. Clear.

Corollary 2.5. *Let the notation be as in Theorem 2.3. Then*

$$\lim_{x \to x_0} [f(x) - g(x)] = L - M.$$

Proof. Clear.

Theorem 2.6. *Let S be a set of numbers, and f a function defined for all numbers in S. Let x_0 be a number. If*

$$\lim_{x \to x_0} f(x) = L$$

and $L \neq 0$, then the limit

$$\lim_{x \to x_0} \frac{1}{f(x)}$$

exists and is equal to $1/L$.

Proof. Given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ we have

$$|f(x) - L| < \frac{|L|}{2}$$

and also

$$|f(x) - L| < \frac{\varepsilon |L|^2}{2}.$$

From the first inequality, we get

$$|f(x)| \geqq |L| - \frac{|L|}{2} = \frac{|L|}{2}.$$

In particular, $f(x) \neq 0$ when $|x - x_0| < \delta$. For such x we get

$$\begin{aligned} \left| \frac{1}{f(x)} - \frac{1}{L} \right| &= \frac{|L - f(x)|}{|f(x)L|} \\ &\leqq \frac{2}{|L|} \frac{|L - f(x)|}{|L|} \\ &< \frac{2}{|L|^2} \frac{\varepsilon |L|^2}{2} = \varepsilon. \end{aligned}$$

Corollary 2.7. *Let the hypotheses be as in Theorem 2.3, and assume that $L \neq 0$. Then*

$$\lim_{x \to x_0} \frac{g(x)}{f(x)}$$

exists and is equal to M/L.

Proof. Use Theorem 2.3 and Theorem 2.6.

Theorem 2.8. *Let S be a set of numbers, and f a function on S. Let x_0 be a number. Let g be a function on S such that $g(x) \leqq f(x)$ for all x in S. Assume that*

$$\lim_{x \to x_0} f(x) = L \qquad \textit{and} \qquad \lim_{x \to x_0} g(x) = M.$$

Then $M \leqq L$.

Proof. Let $\phi(x) = f(x) - g(x)$. Then $\phi(x) \geqq 0$ for all x is S. Also,

$$\lim_{x \to x_0} \phi(x) = L - M$$

by Corollary 2.5. Let K be this limit. We must show $K \geqq 0$. Suppose $K < 0$. Then $-K > 0$ and $|K| = -K$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ we have

$$|\phi(x) - K| < \varepsilon,$$

whence

$$\phi(x) - K < \varepsilon.$$

Since $\phi(x) \geqq 0$, we get $-K < \varepsilon$ for all $\varepsilon > 0$. In particular, for all positive integers n we get $-K < 1/n$. But $-K > 0$. This contradicts Proposition 1.1.

Theorem 2.9. *Let the notation be as in Theorem 2.8 and assume that $M = L$. Let ψ be a function on S such that*

$$g(x) \leqq \psi(x) \leqq f(x)$$

for all x in S. Then

$$\lim_{x \to x_0} \psi(x)$$

exists and is equal to L (or M).

Proof. Given $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ we have

$$|g(x) - L| < \frac{\varepsilon}{4},$$

$$|f(x) - L| < \frac{\varepsilon}{4}.$$

We also have

$$\begin{aligned}|f(x) - \psi(x)| &\leqq |f(x) - g(x)| \\ &\leqq |f(x) - L + L - g(x)| \\ &\leqq |f(x) - L| + |L - g(x)| \\ &< \frac{\varepsilon}{2}.\end{aligned}$$

But

$$\begin{aligned}|L - \psi(x)| &\leqq |L - f(x)| + |f(x) - \psi(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Theorem 2.10. *Let S, T be sets of numbers, and let f, g be functions defined on S and T respectively. Let x_0 be arbitrarily close to S. Assume that for all x in S we have $f(x)$ in T, so that $g \circ f$ is defined. Assume that*

$$\lim_{x \to x_0} f(x)$$

exists and is equal to a number y_0 arbitrarily close to T. Assume that

$$\lim_{y \to y_0} g(y)$$

exists and equals L. Then

$$\lim_{x \to x_0} g(f(x)) = L.$$

Proof. Give ε there exists δ such that whenever y is in T and

$$|y - y_0| < \delta$$

we have

$$|g(y) - L| < \varepsilon.$$

With the above δ being given, there exists δ_1 such that whenever x is in S and $|x - x_0| < \delta_1$ we have $|f(x) - y_0| < \delta$. From this it follows that

$$|g(f(x)) - L| < \varepsilon$$

whence proving our assertion.

[*Note*: Theorem 2.10 justifies the limit procedure used to prove the chain rule.]

This completely proves all the statements about limits we made in Chapter III.

APP., §2. EXERCISES

1. Let g be a bounded function on a set of numbers S. Let f be a function on S such that

$$\lim_{x \to x_0} f(x) = 0.$$

Prove that $\lim_{x \to x_0} f(x)g(x) = 0$.

2. For an arbitrary number x, let

$$f(x) = \lim_{n \to \infty} \frac{1}{1 + nx}.$$

Find the limit explicitly; prove all assertions you make.

3. (a) Let $b > 1$, and write $b = 1 + c$ with $c > 0$. Prove: Given a positive number B, there exists a positive integer N such that if $n \geqq N$, then $b^n > B$.
   (b) Let $0 < x < 1$. Prove that

$$\lim_{n \to \infty} x^n = 0.$$

   (c) If $-1 < x < 0$, is the limit as in (b) still 0? If yes, give a proof. Look at what happens with an example, i.e. write down the values x^n when $x = -1/2$ and $n = 1, 2, 3, 4, 5, 6, 7$.

4. For which numbers x does the following limit exist:

$$f(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n}?$$

Give explicitly the values $f(x)$ for the various x for which the limit exists.

5. For $x \neq -1$, prove that the following limit exists:

$$f(x) = \lim_{n \to \infty} \frac{x^n - 1}{x^n + 1}.$$

   (a) What is $f(1)$, $f(\frac{1}{2})$, $f(2)$?
   (b) What is $\lim_{x \to 1} f(x)$?
   (c) What is $\lim_{x \to -1} f(x)$?

6. Answer the same questions as in Exercise 5 if

$$f(x) = \lim_{n \to \infty} \left(\frac{x^n - 1}{x^n + 1} \right)^2,$$

and $x \neq -1$.

7. Find the following limits, as $n \to \infty$.

(a) $\dfrac{1}{\sqrt{n}}$ (b) $\dfrac{3}{n^2}$ (c) $\dfrac{5}{n^{1/4}}$ (d) $\dfrac{1}{\sqrt{n+1}}$

8. Find the following limits.

(a) $\sqrt{n+1} - \sqrt{n}$ (b) $\sqrt{n+2} - \sqrt{n}$ (c) $\sqrt{n-5} - \sqrt{n}$

[*Hint*: Rationalize the "numerator."]

APP., §3. POINTS OF ACCUMULATION

A **sequence** is a function defined on a set of integers $\geqq 0$. Usually, this set consists of all positive integers. In that case, a sequence amounts to giving numbers

$$a_1, a_2, a_3, \ldots$$

for each positive integer, and we denote the sequence by

$$\{a_n\} \ (n = 1, 2, \ldots).$$

If the set consists of all integers $\geqq 0$, then we denote the sequence by $\{a_n\}$ $(n = 0, 1, 2, \ldots)$.

Let $\{a_n\}$ $(n = 1, 2, \ldots)$ be a sequence. Let C be a number. We say that C is a **point of accumulation** of the sequence if given $\varepsilon > 0$ there exist infinitely many integers n such that

$$|a_n - C| < \varepsilon.$$

Let $\{a_n\}$ $(n = 1, 2, \ldots)$ be a sequence, and L a number. We shall say that L is a **limit of the sequence** if given $\varepsilon > 0$ there exists an integer N such that for all $n > N$ we have

$$|a_n - L| < \varepsilon.$$

The limit is then unique (same type of proof as we had for limits of functions).

We shall say that the sequence $\{a_n\}$ $(n = 1, 2, \ldots)$ is **increasing** if $a_n \leqq a_{n+1}$ for all positive integers n.

Theorem 3.1. *Let* $\{a_n\}$ $(n = 1, 2, \ldots)$ *be an increasing sequence, and assume that it is bounded from above. Then the least upper bound L is a limit of the sequence.*

Proof. Given $\varepsilon > 0$ the number $L - (\varepsilon/2)$ is not an upper bound for the sequence. Hence there exists some number a_N such that

$$L - \frac{\varepsilon}{2} \leqq a_N.$$

This inequality is also satisfied for all $n > N$, since the sequence is increasing. But

$$a_n \leqq L$$

because L is an upper bound. Thus

$$|L - a_n| = L - a_n \leqq \frac{\varepsilon}{2} < \varepsilon$$

for all $n > N$, thereby proving our assertion.

Corollary 3.2. *Let* $\{a_n\}$ $(n = 1, 2, \ldots)$ *be a sequence, and let A, B be two numbers such that* $A \leqq a_n \leqq B$ *for all positive integers n. Then there exists a point of accumulation C of the sequence between A and B.*

Proof. For each integer n we let b_n be the greatest lower bound of the set of numbers $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$. Then $b_n \leqq b_{n+1} \leqq \cdots$, i.e. $\{b_n\}$ $(n = 1, 2, \ldots)$ is an increasing sequence, and B is an upper bound. Let L be its limit, as in Theorem 3.1. We leave it to you as an exercise to prove that this limit is a point of accumulation.

One can reduce the notion of limit of a sequence to that of limits defined previously.

Let S be the set of numbers

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \ldots, \quad \frac{1}{n}, \quad \ldots,$$

i.e. the set of numbers which can be written as $1/n$, where n is a positive integer.

If $\{a_n\}$ $(n = 1, 2, \ldots)$ is a sequence, we let f be a function defined by S by the rule

$$f\left(\frac{1}{n}\right) = a_n.$$

Then you will verify immediately that

$$\lim_{n \to \infty} a_n$$

exists if and only if

$$\lim_{h \to 0} f(h)$$

exists, and in that case the two limits are equal. We say that a sequence $\{a_n\}$ **approaches a number L when n becomes large if**

$$L = \lim_{n \to \infty} a_n.$$

Thus properties concerning limits in the sense of §2 immediately give rise to properties concerning limits of sequences (for instance, limits of sums, products, quotients). We leave their translations to you.

APP., §3. EXERCISES

1. Let $\{I_n\}$ be a sequence of closed intervals, say $I_n = [a_n, b_n]$, where $[a, b]$ means the set of numbers x such that $a \leqq x \leqq b$. Suppose that the left-hand points of this sequence of intervals increase, that is

$$a_1 \leqq \cdots \leqq a_n \leqq a_{n+1} \leqq \cdots$$

and that the right-hand points decrease (that is $b_{n+1} \leqq b_n$ for all positive integers n). Let $L(I_n)$ be the length of the interval I_n, that is

$$L(I_n) = b_n - a_n.$$

If

$$\lim_{n \to \infty} L(I_n) = 0,$$

prove that there exists a point c in each interval I_n such that

$$\begin{aligned} \lim_{n \to \infty} a_n &= \lim_{n \to \infty} b_n \\ &= c. \end{aligned}$$

2. Let c_n be an element of I_n in Exercise 1. Under the same hypothesis as in Exercise 1, prove that

$$\lim_{n \to \infty} c_n = c.$$

APP., §4. CONTINUOUS FUNCTIONS

Let f be a function defined on a set of numbers S. Let x_0 be a number in S. Then S is arbitrarily close to x_0. We say that f is **continuous** at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Note that there may be two numbers a, b with $a < x_0 < b$ such that x_0 is the only point which is in the interval and lies also in S. (In this case, one could say that x_0 is an **isolated** point of S.)

If follows at once from our definition that if $\{a_n\}$ $(n = 1, 2, \ldots)$ is a sequence of numbers in S such that

$$\lim_{n \to \infty} a_n = x_0$$

and f is continuous at x_0, then

$$\lim_{n \to \infty} f(a_n) = f(x_0).$$

It is immediate that the sum, product, quotient of continuous functions are again continuous. (In the quotient, we have to assume that $f(x_0) \neq 0$, of course.) Every constant function is continuous. The function $f(x) = x$ is continuous for all x. This is trivially verified. From the quotient theorem, we see that the function

$$f(x) = \frac{1}{x}$$

(defined for $x \neq 0$) is continuous.

Theorem 4.1. *Let f and g be continuous functions such that the values of f are contained in the domain of definition of g. Then $g \circ f$ is continuous.*

Proof. Let x_0 be a number at which f is defined, and let

$$y_0 = f(x_0).$$

Given $\varepsilon > 0$, since g is continuous at y_0, there exists $\delta_1 > 0$ such that if $|y - y_0| < \delta_1$, then

$$|g(y) - g(y_0)| < \varepsilon.$$

Now with the above δ_1 being given, there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then

$$|f(x) - f(x_0)| < \delta_1.$$

Hence

$$|g(f(x)) - g(f(x_0))| < \varepsilon,$$

thus proving our theorem.

Theorem 4.2. *Let f be a continuous function on a closed interval $a \leqq x \leqq b$. Then there exists a point c in the interval such that $f(c)$ is a maximum, and there exists a point d in the interval such that $f(d)$ is a minimum.*

Proof. We shall first prove that f is bounded, i.e. that there exists a number M such that $|f(x)| \leqq M$ for all x in the interval.

If f is not bounded, then for every positive integer n we can find a number x_n in the interval such that $|f(x_n)| > n$. The sequence of such x_n has a point of accumulation C in the interval. We have

$$\begin{aligned} |f(x_n) - f(C)| &\geqq |f(x_n)| - |f(C)| \\ &\geqq n - f(C). \end{aligned}$$

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever

$$|x_n - C| < \delta$$

we have $|f(x_n) - f(C)| < \varepsilon$. This has to happen for infinitely many n, since C is an accumulation point. Our statements are contradictory, and we therefore conclude that the function is bounded.

Let β be the least upper bound of the set of values $f(x)$ for all x in the interval. Then given a positive integer n, we can find a number z_n in the interval such that

$$|f(z_n) - \beta| < \frac{1}{n}.$$

Let c be a point of accumulation of the sequence of numbers

$$\{z_n\} \qquad (n = 1, 2, \ldots).$$

Then $f(c) \leqq \beta$. We contend that

$$f(c) = \beta$$

(this will prove our theorem).

Given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z_n - c| < \delta$ we have

$$|f(z_n) - f(c)| < \varepsilon.$$

This happens for infinitely many n, since c is a point of accumulation of the sequence $\{z_n\}$. But

$$\begin{aligned}|f(c) - \beta| &\leqq |f(c) - f(z_n)| + |f(z_n) - \beta| \\ &< \varepsilon + \frac{1}{n}.\end{aligned}$$

This is true for every ε and infinitely many positive integers n. Hence $|f(c) - \beta| = 0$ and $f(c) = \beta$.

The proof for the minimum is similar and will be left as an exercise.

Theorem 4.3. *Let f be a continuous function on a closed interval $a \leqq x \leqq b$. Let $\alpha = f(a)$ and $\beta = f(b)$. Let γ be a number such that $\alpha < \gamma < \beta$. Then there exists a number c between a and b such that $f(c) = \gamma$.*

Proof. Let S be the set of numbers x in the interval such that $f(x) \leqq \gamma$. Then S is not empty because a is in it, and b is an upper bound for S. Let c be its least upper bound. Then c is in our interval. We contend that $f(c) = \gamma$. If $f(c) < \gamma$, then $c \neq b$, and $f(x) < \gamma$ for all $x > c$ sufficiently close to c, because f is continuous at c. This contradicts the fact that c is an upper bound for S. If $f(c) > \gamma$, then $c \neq a$, and $f(x) > \gamma$ for all $x < c$ sufficiently close to c, again because f is continuous at c. This contradicts the fact that c is a least upper bound for S. We conclude that $f(c) = \gamma$, as was to be shown.

Part Five

Functions of Several Variables

In the first chapter of this part, we consider vectors, which form the basic algebraic tool in investigating functions of several variables. The differentiation aspects of them which we take up are those which can be handled up to a point by "one variable" methods. The reason for this is that in higher dimensional space, we can join two points by a curve, and study a function by looking at its values only on this curve. This reduces many higher dimensional problems to problems of a one-dimensional situation.

CHAPTER XV

Vectors

The concept of a vector is basic for the study of functions of several variables. It provides geometric motivation for everything that follows. Hence the properties of vectors, both algebraic and geometric, will be discussed in full.

One significant feature of all the statements and proofs of this part is that they are neither easier nor harder to prove in 3-space than they are in 2-space.

XV, §1. DEFINITION OF POINTS IN SPACE

We know that a number can be used to represent a point on a line, once a unit length is selected.

A pair of numbers (i.e. a couple of numbers) (x, y) can be used to represent a point in the plane.

These can be pictured as follows:

(a) Point on a line (b) Point in a plane

Figure 1

We now observe that a triple of numbers (x, y, z) can be used to represent a point in space, that is 3-dimensional space, or 3-space. We simply introduce one more axis. Figure 2 illustrates this.

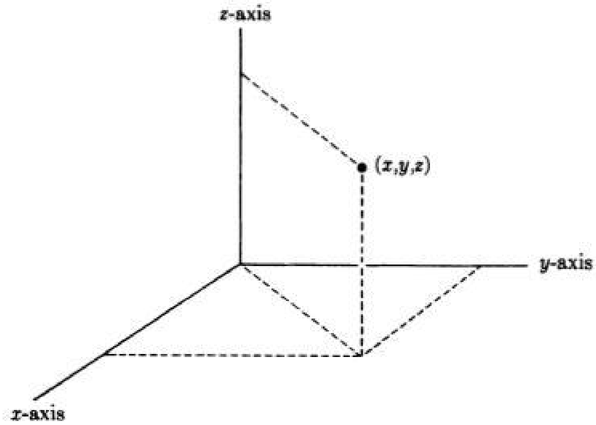


Figure 2

Instead of using x, y, z we could also use (x_1, x_2, x_3). The line could be called 1-space, and the plane could be called 2-space.

Thus we can say that a single number represents a point in 1-space. A couple represents a point in 2-space. A triple represents a point in 3-space.

Although we cannot draw a picture to go further, there is nothing to prevent us from considering a quadruple of numbers.

$$(x_1, x_2, x_3, x_4)$$

and decreeing that this is a point in 4-space. A quintuple would be a point in 5-space, then would come a sextuple, septuple, octuple, ...

We let ourselves be carried away and **define a point in n-space** to be an n-tuple of numbers

$$(x_1, x_2, \ldots, x_n),$$

if n is a positive integer. We shall denote such an n-tuple by a capital letter X, and try to keep small letters for numbers and capital letters for points. We call the numbers $x_1, \ldots, x_n$ the **coordinates** of the point X. For example, in 3-space, 2 is the first coordinate of the point $(2, 3, -4)$, and -4 is its third coordinate. We denote n-space by $\mathbf{R}^n$.

Most of our examples will take place when $n = 2$ or $n = 3$. Thus the reader may visualize either of these two cases throughout the book. However, three comments must be made.

First, we have to handle $n = 2$ and $n = 3$, so that in order to avoid a lot of repetitions, it is useful to have a notation which covers both these cases simultaneously, even if we often repeat the formulation of certain results separately for both cases.

Second, no theorem or formula is simpler by making the assumption that $n = 2$ or 3.

Third, the case $n = 4$ does occur in physics.

Example 1. One classical example of 3-space is of course the space we live in. After we have selected an origin and a coordinate system, we can describe the position of a point (body, particle, etc.) by 3 coordinates. Furthermore, as was known long ago, it is convenient to extend this space to a 4-dimensional space, with the fourth coordinate as time, the time origin being selected, say, as the birth of Christ—although this is purely arbitrary (it might be more convenient to select the birth of the solar system, or the birth of the earth as the origin, if we could determine these accurately). Then a point with negative time coordinate is a BC point, and a point with positive time coordinate is an AD point.

Don't get the idea that "time is *the* fourth dimension", however. The above 4-dimensional space is only one possible example. In economics, for instance, one uses a very different space, taking for coordinates, say, the number of dollars expended in an industry. For instance, we could deal with a 7-dimensional space with coordinates corresponding to the following industries:

1. Steel	2. Auto	3. Farm products	4. Fish
5. Chemicals	6. Clothing	7. Transportation.	

We agree that a megabuck per year is the unit of measurement. Then a point

$$(1{,}000,\ 800,\ 550,\ 300,\ 700,\ 200,\ 900)$$

in this 7-space would mean that the steel industry spent one billion dollars in the given year, and that the chemical industry spent 700 million dollars in that year.

The idea of regarding time as a fourth dimension is an old one. Already in the *Encyclopédie* of Diderot, dating back to the eighteenth century, d'Alembert writes in his article on "dimension":

> Cette manière de considérer les quantités de plus de trois dimensions est aussi exacte que l'autre, car les lettres peuvent toujours être regardées comme représentant des nombres rationnels ou non. J'ai dit plus haut qu'il n'était pas possible de concevoir plus de trois dimensions. Un homme d'esprit de ma connaissance croit qu'on pourrait cependant regarder la durée comme une quatrième dimension, et que le produit temps par la solidité serait en quelque manière un produit de quatre dimensions; cette idée peut être contestée, mais elle a, ce me semble, quelque mérite, quand ce ne serait que celui de la nouveauté.
>
> *Encyclopédie*, Vol. 4 (1754), p. 1010

Translated, this means:

> This way of considering quantities having more than three dimensions is just as right as the other, because algebraic letters can always be viewed as representing numbers, whether rational or not. I said above that it was not possible to conceive more than three dimensions. A clever gentleman with whom I am acquainted believes that nevertheless, one could view duration as a fourth dimension, and that the product time by solidity would be somehow a product of four dimensions. This idea may be challenged, but it has, it seems to me, some merit, were it only that of being new.

Observe how d'Alembert refers to a "clever gentleman" when he apparently means himeself. He is being rather careful in proposing what must have been at the time a far out idea, which became more prevalent in the twentieth century.

D'Alembert also visualized clearly higher dimensional spaces as "products" of lower dimensional spaces. For instance, we can view 3-space as putting side by side the first two coordinates (x_1, x_2) and then the third x_3. Thus we write

$$\mathbf{R}^3 = \mathbf{R}^2 \times \mathbf{R}^1.$$

We use the product sign, which should not be confused with other "products", like the product of numbers. The word "product" is used in two contexts. Similarly, we can write

$$\mathbf{R}^4 = \mathbf{R}^3 \times \mathbf{R}^1.$$

There are other ways of expressing $\mathbf{R}^4$ as a product, namely

$$\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2.$$

This means that we view separately the first two coordinates (x_1, x_2) and the last two coordinates (x_3, x_4). We shall come back to such products later.

We shall now define how to add points. If A, B are two points, say in 3-space,

$$A = (a_1, a_2, a_3) \quad \text{and} \quad B = (b_1, b_2, b_3)$$

then we **define** $A + B$ to be the point whose coordinates are

$$A + B = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3).$$

Example 2. In the plane, if $A = (1, 2)$ and $B = (-3, 5)$, then

$$A + B = (-2, 7).$$

In 3-space, if $A = (-1, \pi, 3)$ and $B = (\sqrt{2}, 7, -2)$, then

$$A + B = (\sqrt{2} - 1,\ \pi + 7,\ 1).$$

Using a neutral n to cover both the cases of 2-space and 3-space, the points would be written

$$A = (a_1, \dots, a_n), \qquad B = (b_1, \dots, b_n),$$

and we **define** $A + B$ to be the point whose coordinates are

$$(a_1 + b_1, \dots, a_n + b_n).$$

We observe that the following rules are satisfied:

1. $(A + B) + C = A + (B + C)$.
2. $A + B = B + A$.
3. If we let

$$O = (0, 0, \dots, 0)$$

be the point all of whose coordinates are 0, then

$$O + A = A + O = A$$

for all A.
4. Let $A = (a_1, \dots, a_n)$ and let $-A = (-a_1, \dots, -a_n)$. Then

$$A + (-A) = O.$$

All these properties are very simple, and are true because they are true for numbers, and addition of n-tuples is defined in terms of addition of their components, which are numbers.

Note. Do not confuse the number 0 and the n-tuple $(0, \dots, 0)$. We usually denote this n-tuple by O, and also call it zero, because no difficulty can occur in practice.

We shall now interpret addition and multiplication by numbers geometrically in the plane (you can visualize simultaneously what happens in 3-space).

Example 3. Let $A = (2, 3)$ and $B = (-1, 1)$. Then

$$A + B = (1, 4).$$

The figure looks like a **parallelogram** (Fig. 3).

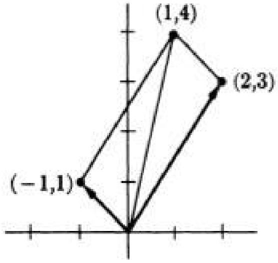


Figure 3

Example 4. Let $A = (3, 1)$ and $B = (1, 2)$. Then

$$A + B = (4, 3).$$

We see again that the geometric representation of our addition looks like a **parallelogram** (Fig. 4).

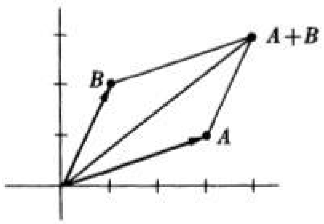


Figure 4

The reason why the figure looks like a **parallelogram** can be given in terms of plane geometry as follows. We obtain $B = (1, 2)$ by starting from the origin $O = (0, 0)$, and moving 1 unit to the right and 2 up. To get $A + B$, we start from A, and again move 1 unit to the right and 2 up. Thus the line segments between O and B, and between A and $A + B$ are the hypotenuses of right triangles whose corresponding legs are of the same length, and parallel. The above segments are therefore parallel and of the same length, as illustrated in Fig. 5.

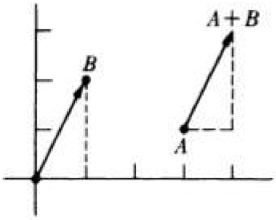


Figure 5

Example 5. If $A = (3, 1)$ again, then $-A = (-3, -1)$. If we plot this point, we see that $-A$ has opposite direction to A. We may view $-A$ as the reflection of A through the origin.

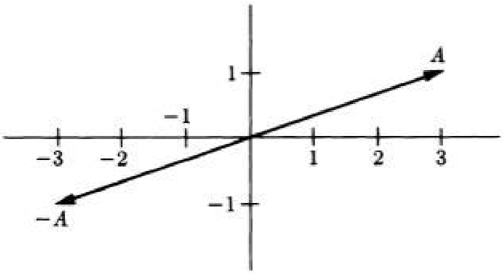


Figure 6

We shall now consider multiplication of A by a number. If c is any number, we **define** cA to be the point whose coordinates are

$$(ca_1, \ldots, ca_n).$$

Example 6. If $A = (2, -1, 5)$ and $c = 7$, then $cA = (14, -7, 35)$.

It is easy to verify the rules:

5. $c(A + B) = cA + cB$.
6. If c_1, c_2 are numbers, then

$$(c_1 + c_2)A = c_1A + c_2A \qquad \text{and} \qquad (c_1c_2)A = c_1(c_2A).$$

Also note that

$$(-1)A = -A.$$

What is the geometric representation of multiplication by a number?

Example 7. Let $A = (1, 2)$ and $c = 3$. Then

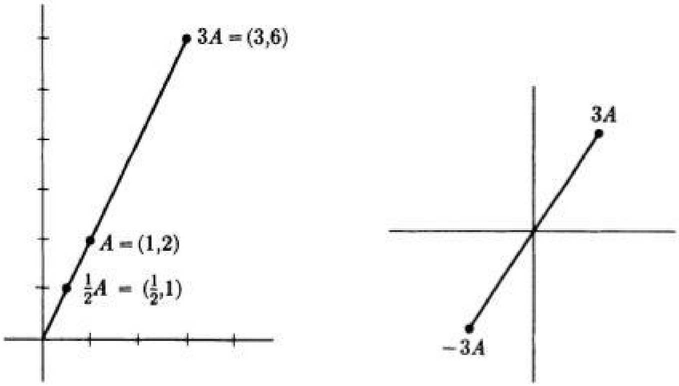
$$cA = (3, 6)$$

as in Fig. 7(a).

Multiplication by 3 amounts to stretching A by 3. Similarly, $\frac{1}{2}A$ amounts to stretching A by $\frac{1}{2}$, i.e. shrinking A to half its size. In general, if t is a number, $t > 0$, we interpret tA as a point in the same direction as A from the origin, but t times the distance. In fact, we define A and

B to have the **same direction** if there exists a number $c > 0$ such that $A = cB$. We emphasize that this means A and B have the same direction **with respect to the origin**. For simplicity of language, we omit the words "with respect to the origin".

Mulitiplication by a negative number reverses the direction. Thus $-3A$ would be represented as in Fig. 7(b).



(a) (b)

Figure 7

We define two vectors A, B (neither of which is zero) to have **opposite directions** if there is a number $c < 0$ such that $cA = B$. Thus when $B = -A$, then A, B have opposite direction.

XV, §1. EXERCISES

Find $A + B$, $A - B$, $3A$, $-2B$ in each of the following cases. Draw the points of Exercises 1 and 2 on a sheet of graph paper.

1. $A = (2, -1)$, $B = (-1, 1)$
2. $A = (-1, 3)$, $B = (0, 4)$
3. $A = (2, -1, 5)$, $B = (-1, 1, 1)$
4. $A = (-1, -2, 3)$, $B = (-1, 3, -4)$
5. $A = (\pi, 3, -1)$, $B = (2\pi, -3, 7)$
6. $A = (15, -2, 4)$, $B = (\pi, 3, -1)$
7. Let $A = (1, 2)$ and $B = (3, 1)$. Draw $A + B$, $A + 2B$, $A + 3B$, $A - B$, $A - 2B$, $A - 3B$ on a sheet of graph paper.
8. Let A, B be as in Exercise 1. Draw the points $A + 2B$, $A + 3B$, $A - 2B$, $A - 3B$, $A + \frac{1}{2}B$ on a sheet of graph paper.
9. Let A and B be as drawn in Fig. 8. Draw the point $A - B$.

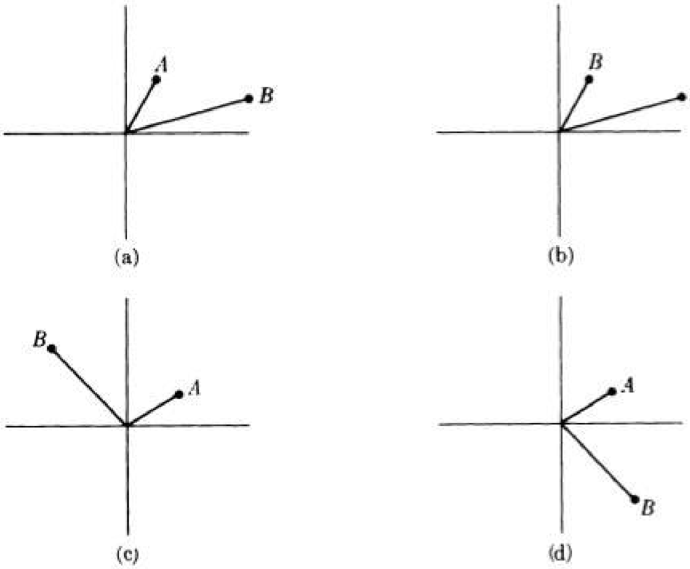


Figure 8

XV, §2. LOCATED VECTORS

We define a **located vector** to be an ordered pair of points which we write $\overrightarrow{AB}$. (This is *not* a product.) We visualize this as an arrow between A and B. We call A the **beginning point** and B the **end point** of the located vector (Fig. 9).

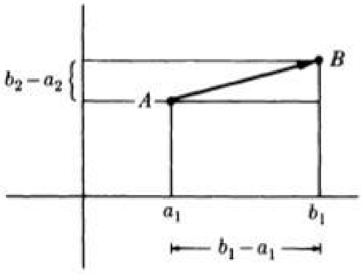


Figure 9

We observe that in the plane,

$$b_1 = a_1 + (b_1 - a_1).$$

Similarly,

$$b_2 = a_2 + (b_2 - a_2).$$

This means that

$$B = A + (B - A)$$

Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two located vectors. We shall say that they are **equivalent** if $B - A = D - C$. Every located vector $\overrightarrow{AB}$ is equivalent to one whose beginning point is the origin, because $\overrightarrow{AB}$ is equivalent to $\overrightarrow{O(B - A)}$. Clearly this is the only located vector whose beginning point is the origin and which is equivalent to $\overrightarrow{AB}$. If you visualize the parellogram law in the plane, then it is clear that equivalence of two located vectors can be interpreted geometrically by saying that the lengths of the line segments determined by the pair of points are equal, and that the "directions" in which they point are the same.

In the next figures, we have drawn the located vectors $\overrightarrow{O(B - A)}$, $\overrightarrow{AB}$, and $\overrightarrow{O(A - B)}$, $\overrightarrow{BA}$.

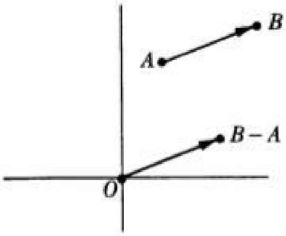


Figure 10

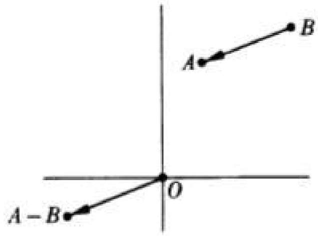


Figure 11

Example 1. Let $P = (1, -1, 3)$ and $Q = (2, 4, 1)$. Then $\overrightarrow{PQ}$ is equivalent to $\overrightarrow{OC}$, where $C = Q - P = (1, 5, -2)$. If

$$A = (4, -2, 5) \qquad \text{and} \qquad B = (5, 3, 3),$$

then $\overrightarrow{PQ}$ is equivalent to $\overrightarrow{AB}$ because

$$Q - P = B - A = (1, 5, -2).$$

Given a located vector $\overrightarrow{OC}$ whose beginning point is the origin, we shall say that it is **located at the origin.** Given any located vector $\overrightarrow{AB}$, we shall say that it is **located at** A.

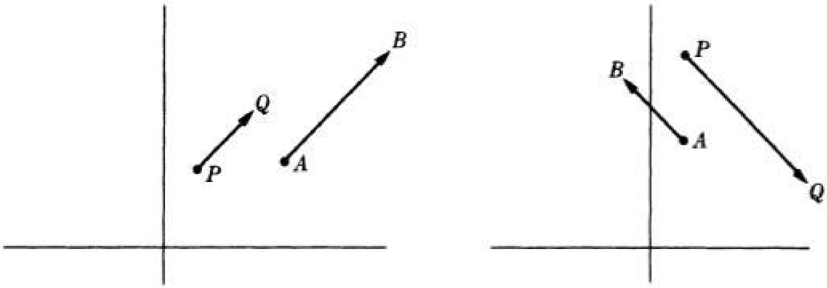
A located vector at the origin is entirely determined by its end point. In view of this, we shall call an n-tuple either a point or a **vector**, depending on the interpretation which we have in mind.

Two located vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are said to be **parallel** if there is a number $c \neq 0$ such that $B - A = c(Q - P)$. They are said to have the

same direction if there is a number $c > 0$ such that $B - A = c(Q - P)$, and have **opposite direction** if there is a number $c < 0$ such that

$$B - A = c(Q - P).$$

In the next pictures, we illustrate parallel located vectors.



(a) Same direction (b) Opposite direction

Figure 12

Example 2. Let

$$P = (3, 7) \quad \text{and} \quad Q = (-4, 2).$$

Let

$$A = (5, 1) \quad \text{and} \quad B = (-16, -14).$$

Then

$$Q - P = (-7, -5) \quad \text{and} \quad B - A = (-21, -15).$$

Hence $\overrightarrow{PQ}$ is parallel to $\overrightarrow{AB}$, because $B - A = 3(Q - P)$. Since $3 > 0$, we even see that $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ have the same direction.

In a similar manner, any definition made concerning n-tuples can be carried over to located vectors. For instance, in the next section, we shall define what it means for n-tuples to be perpendicular.

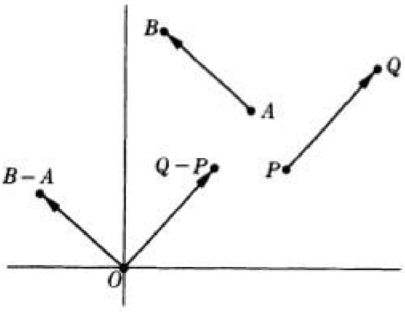


Figure 13

Then we can say that two located vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are **perpendicular** if $B - A$ is perpendicular to $Q - P$. In Fig. 13, we have drawn a picture of such vectors in the plane.

XV, §2. EXERCISES

In each case, determine which located vectors $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are equivalent.

1. $P = (1, -1)$, $Q = (4, 3)$, $A = (-1, 5)$, $B = (5, 2)$.
2. $P = (1, 4)$, $Q = (-3, 5)$, $A = (5, 7)$, $B = (1, 8)$.
3. $P = (1, -1, 5)$, $Q = (-2, 3, -4)$, $A = (3, 1, 1)$, $B = (0, 5, 10)$.
4. $P = (2, 3, -4)$, $Q = (-1, 3, 5)$, $A = (-2, 3, -1)$, $B = (-5, 3, 8)$.

In each case, determine which located vectors $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are parallel.

5. $P = (1, -1)$, $Q = (4, 3)$, $A = (-1, 5)$, $B = (7, 1)$.
6. $P = (1, 4)$, $Q = (-3, 5)$, $A = (5, 7)$, $B = (9, 6)$.
7. $P = (1, -1, 5)$, $Q = (-2, 3, -4)$, $A = (3, 1, 1)$, $B = (-3, 9, -17)$.
8. $P = (2, 3, -4)$, $Q = (-1, 3, 5)$, $A = (-2, 3, -1)$, $B = (-11, 3, -28)$.
9. Draw the located vectors of Exercises 1, 2, 5, and 6 on a sheet of paper to illustrate these exercises. Also draw the located vectors $\overrightarrow{QP}$ and $\overrightarrow{BA}$. Draw the points $Q - P$, $B - A$, $P - Q$, and $A - B$.

XV, §3. SCALAR PRODUCT

It is understood that throughout a discussion we select vectors always in the same n-dimensional space. You may think of the cases $n = 2$ and $n = 3$ only.

In 2-space, let $A = (a_1, a_2)$ and $B = (b_1, b_2)$. We define their **scalar product** to be

$$A \cdot B = a_1 b_1 + a_2 b_2.$$

In 3-space, let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. We define their **scalar product** to be

$$A \cdot B = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

In n-space, covering both cases with one notation, let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be two vectors. We define their **scalar** or **dot product** $A \cdot B$ to be

$$a_1 b_1 + \cdots + a_n b_n.$$

This product is a **number**. For instance, if

$$A = (1, 3, -2) \qquad \text{and} \qquad B = (-1, 4, -3),$$

then

$$A \cdot B = -1 + 12 + 6 = 17.$$

For the moment, we do not give a geometric interpretation to this scalar product. We shall do this later. We derive first some important properties. The basic ones are:

SP 1. *We have* $A \cdot B = B \cdot A$.

SP 2. *If* A, B, C *are three vectors, then*

$$A \cdot (B + C) = A \cdot B + A \cdot C = (B + C) \cdot A.$$

SP 3. *If* x *is a number, then*

$$(xA) \cdot B = x(A \cdot B) \qquad \textit{and} \qquad A \cdot (xB) = x(A \cdot B).$$

SP 4. *If* $A = O$ *is the zero vector, then* $A \cdot A = 0$, *and otherwise*

$$A \cdot A > 0.$$

We shall now prove these properties.

Concerning the first, we have

$$a_1 b_1 + \cdots + a_n b_n = b_1 a_1 + \cdots + b_n a_n,$$

because for any two numbers a, b, we have $ab = ba$. This proves the first property.

For **SP 2**, let $C = (c_1, \ldots, c_n)$. Then

$$B + C = (b_1 + c_1, \ldots, b_n + c_n)$$

and

$$\begin{aligned} A \cdot (B + C) &= a_1(b_1 + c_1) + \cdots + a_n(b_n + c_n) \\ &= a_1 b_1 + a_1 c_1 + \cdots + a_n b_n + a_n c_n. \end{aligned}$$

Reordering the terms yields

$$a_1 b_1 + \cdots + a_n b_n + a_1 c_1 + \cdots + a_n c_n.$$

which is none other than $A \cdot B + A \cdot C$. This proves what we wanted. We leave property **SP 3** as an exercise.

Finally, for **SP 4**, we observe that if one coordinate a_i of A is not equal to 0, then there is a term $a_i^2 \neq 0$ and $a_i^2 > 0$ in the scalar product

$$A \cdot A = a_1^2 + \cdots + a_n^2.$$

Since every term is $\geqq 0$, it follows that the sum is > 0, as was to be shown.

In much of the work which we shall do concerning vectors, we shall use only the ordinary properties of addition, multiplication by numbers, and the four properties of the scalar product. We shall give a formal discussion of these later. For the moment, observe that there are other objects with which you are familiar and which can be added, subtracted, and multiplied by numbers, for instance the continuous functions on an interval $[a, b]$ (cf. Exercise 6).

Instead of writing $A \cdot A$ for the scalar product of a vector with itself, it will be convenient to write also A^2. (This is the only instance when we allow ourselves such a notation. Thus A^3 has no meaning.) As an exercise, verify the following identities:

$$(A + B)^2 = A^2 + 2A \cdot B + B^2,$$

$$(A - B)^2 = A^2 - 2A \cdot B + B^2.$$

A dot product $A \cdot B$ may very well be equal to 0 without either A or B being the zero vector. For instance, let

$$A = (1, 2, 3) \qquad \text{and} \qquad B = (2, 1, -\tfrac{4}{3}).$$

Then

$$A \cdot B = 0$$

We define two vectors A, B to be **perpendicular** (or as we shall also say, **orthogonal**), if $A \cdot B = 0$. For the moment, it is not clear that in the plane, this definition coincides with our intuitive geometric notion of perpendicularity. We shall convince you that it does in the next section. Here we merely note an example. Say in $\mathbf{R}^3$, let

$$E_1 = (1, 0, 0), \qquad E_2 = (0, 1, 0), \qquad E_3 = (0, 0, 1)$$

be the three unit vectors, as shown on the diagram (Fig. 14).

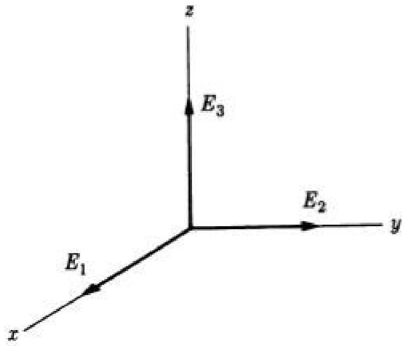


Figure 14

Then we see that $E_1 \cdot E_2 = 0$, and similarly $E_i \cdot E_j = 0$ if $i \neq j$. And these vectors look perpendicular. If $A = (a_1, a_2, a_3)$, then we observe that the i-th component of A, namely

$$a_i = A \cdot E_i$$

is the dot product of A with the i-th unit vector. We see that A is perpendicular to E_i (according to our definition of perpendicularity with the dot product) if and only if its i-th component is equal to 0.

XV, §3. EXERCISES

1. Find $A \cdot A$ for each of the following n-tuples.
   (a) $A = (2, -1)$, $B = (-1, 1)$ (b) $A = (-1, 3)$, $B = (0, 4)$
   (c) $A = (2, -1, 5)$, $B = (-1, 1, 1)$ (d) $A = (-1, -2, 3)$, $B = (-1, 3, -4)$
   (e) $A = (\pi, 3, -1)$, $B = (2\pi, -3, 7)$ (f) $A = (15, -2, 4)$, $B = (\pi, 3, -1)$
2. Find $A \cdot B$ for each of the above n-tuples.
3. Using only the four properties of the scalar product, verify in detail the identities given in the text for $(A + B)^2$ and $(A - B)^2$.
4. Which of the following pairs of vectors are perpendicular?
   (a) $(1, -1, 1)$ and $(2, 1, 5)$ (b) $(1, -1, 1)$ and $(2, 3, 1)$
   (c) $(-5, 2, 7)$ and $(3, -1, 2)$ (d) $(\pi, 2, 1)$ and $(2, -\pi, 0)$
5. Let A be a vector perpendicular to every vector X. Show that $A = O$.

XV, §4. THE NORM OF A VECTOR

We define the **norm** of a vector A, and denote by $\|A\|$, the number

$$\|A\| = \sqrt{A \cdot A}.$$

Since $A \cdot A \geqq 0$, we can take the square root. The norm is also sometimes called the **magnitude** of A.

When $n = 2$ and $A = (a, b)$, then

$$\|A\| = \sqrt{a^2 + b^2},$$

as in the following picture (Fig. 15).

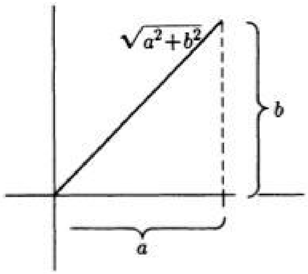


Figure 15

Example 1. If $A = (1, 2)$, then

$$\|A\| = \sqrt{1 + 4} = \sqrt{5}.$$

When $n = 3$ and $A = (a_1, a_2, a_3)$, then

$$\|A\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Example 2. If $A = (-1, 2, 3)$, then

$$\|A\| = \sqrt{1 + 4 + 9} = \sqrt{14}.$$

If $n = 3$, then the picture looks like Fig. 16, with $A = (x, y, z)$.

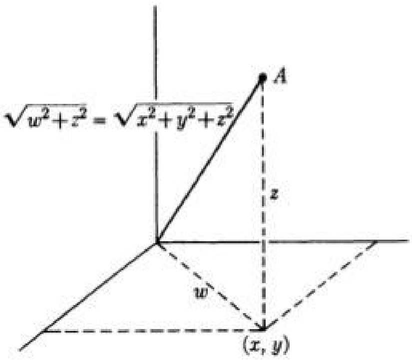


Figure 16

If we first look at the two components (x, y), then the length of the segment between $(0, 0)$ and (x, y) is equal to $w = \sqrt{x^2 + y^2}$, as indicated.
Then again the norm of A by the Pythagoras theorem would be

$$\sqrt{w^2 + z^2} = \sqrt{x^2 + y^2 + z^2}.$$

Thus when $n = 3$, our definition of norm is compatible with the geometry of the Pythagoras theorem.

In terms of coordinates, $A = (a_1, \dots, a_n)$ we see that

$$\|A\| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

If $A \neq O$, then $\|A\| \neq 0$ because some coordinate $a_i \neq 0$, so that $a_i^2 > 0$, and hence $a_1^2 + \cdots + a_n^2 > 0$, so $\|A\| \neq 0$.

Observe that for any vector A we have

$$\boxed{\|A\| = \|-A\|.}$$

This is due to the fact that

$$(-a_1)^2 + \cdots + (-a_n)^2 = a_1^2 + \cdots + a_n^2,$$

because $(-1)^2 = 1$. Of course, this is as it should be from the picture:

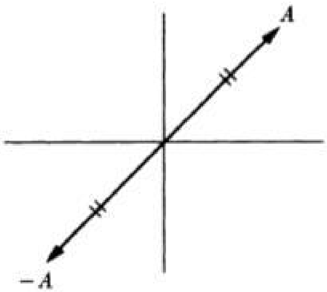


Figure 17

Recall that A and $-A$ are said to have **opposite direction**. However, they have the same norm (magnitude, as is sometimes said when speaking of vectors).

Let A, B be two points. We define the **distance** between A and B to be

$$\|A - B\| = \sqrt{(A - B)\cdot(A - B)}.$$

This definition coincides with our geometric intuition when A, B are points in the plane (Fig. 18). It is the same thing as the length of the located vector $\overrightarrow{AB}$ or the located vector $\overrightarrow{BA}$.

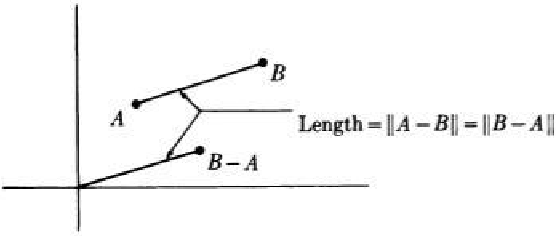


Figure 18

Example 3. Let $A = (-1, 2)$ and $B = (3, 4)$. Then the length of the located vector $\overrightarrow{AB}$ is $\|B - A\|$. But $B - A = (4, 2)$. Thus

$$\|B - A\| = \sqrt{16 + 4} = \sqrt{20}.$$

In the picture, we see that the horizontal side has length 4 and the vertical side has length 2. Thus our definitions reflect our geometric intuition derived from Pythagoras.

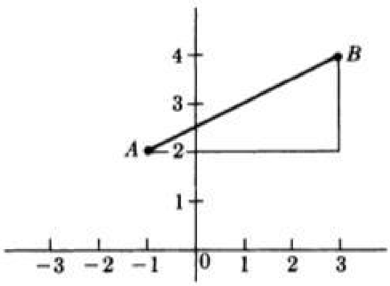


Figure 19

Let P be a point in the plane, and let a be a number > 0. The set of points X such that

$$\|X - P\| < a$$

will be called the **open disc** of radius a centered at P. The set of points X such that

$$\|X - P\| \leqq a$$

will be called the **closed disc** of radius a and center P. The set of points X such that

$$\|X - P\| = a$$

is called the circle of radius a and center P. These are illustrated in Fig. 20.

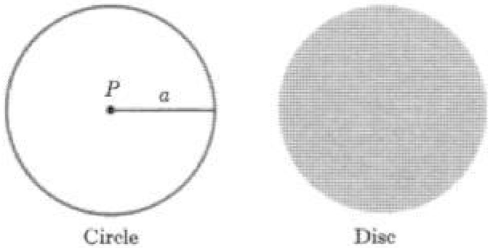


Figure 20

In 3-dimensional space, the set of points X such that

$$\|X - P\| < a$$

will be called the **open ball** of radius a and center P. The set of points X such that

$$\|X - P\| \leqq a$$

will be called the **closed ball** of radius a and center P. The set of points X such that

$$\|X - P\| = a$$

will be called the **sphere** of radius a and center P. In higher dimensional space, one uses this same terminology of ball and sphere.

Figure 21 illustrates a sphere and a ball in 3-space.

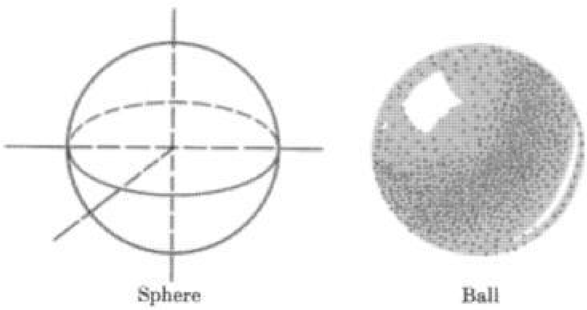


Figure 21

The sphere is the outer shell, and the ball consists of the region inside the shell. The open ball consists of the region inside the shell excluding the shell itself. The closed ball consists of the region inside the shell *and* the shell itself.

From the geometry of the situation, it is also reasonable to expect that if $c > 0$, then $\|cA\| = c\|A\|$, i.e. if we stretch a vector A by multiplying by a positive number c, then the length stretches also by that amount. We verify this formally using our definition of the length.

Theorem 4.1 *Let x be a number. Then*

$$\|xA\| = |x|\ \|A\|$$

(absolute value of x times the norm of A).

Proof. By definition, we have

$$\|xA\|^2 = (xA)\cdot(xA),$$

which is equal to

$$x^2(A\cdot A)$$

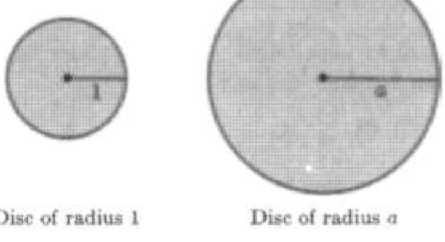
by the properties of the scalar product. Taking the square root now yields what we want.

Let S_1 be the sphere of radius 1, centered at the origin. Let a be a number > 0. If X is a point of the sphere S_1, then aX is a point of the sphere of radius a, because

$$\|aX\| = a\|X\| = a.$$

In this manner, we get all points of the sphere of radius a. (Proof?) Thus the sphere of radius a is obtained by stretching the sphere of radius 1, through multiplication by a.

A similar remark applies to the open and closed balls of radius a, they being obtained from the open and closed balls of radius 1 through multiplication by a.



Disc of radius 1 Disc of radius a

Figure 22

We shall say that a vector E is a **unit** vector if $\|E\| = 1$. Given any vector A, let $a = \|A\|$. If $a \neq 0$, then

$$\frac{1}{a} A$$

is a unit vector, because

$$\left\| \frac{1}{a} A \right\| = \frac{1}{a} a = 1.$$

We say that two vectors A, B (neither of which is O) have the **same direction** if there is a number $c > 0$ such that $cA = B$. In view of this definition, we see that the vector

$$\frac{1}{\|A\|} A$$

is a unit vector in the direction of A (provided $A \neq O$).

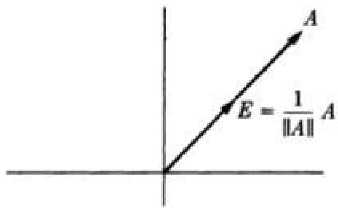


Figure 23

If E is the unit vector in the direction of A, and $\|A\| = a$, then

$$A = aE.$$

Example 4. Let $A = (1, 2, -3)$. Then $\|A\| = \sqrt{14}$. Hence the unit vector in the direction of A is the vector

$$E = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right).$$

Warning. There are as many unit vectors as there are directions. The three **standard unit vectors** in 3-space, namely

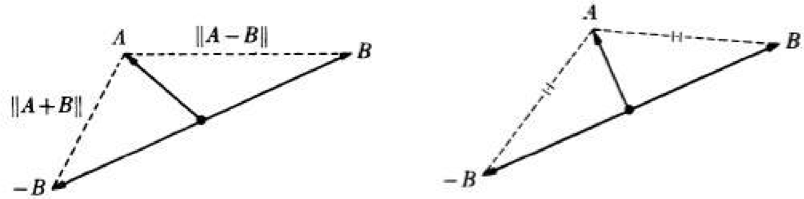
$$E_1 = (1, 0, 0), \qquad E_2 = (0, 1, 0), \qquad E_3 = (0, 0, 1)$$

are merely the three unit vectors in the directions of the coordinate axes.

We are also in the position to justify our definition of perpendicularity. Given A, B in the plane, the condition that

$$\|A + B\| = \|A - B\|$$

(illustrated in Fig. 24(b)) coincides with the geometric property that A should be perpendicular to B.



(a) (b)

Figure 24

We shall prove:

$$\boxed{\|A + B\| = \|A - B\| \text{ if and only if } A \cdot B = 0.}$$

Let $\Leftrightarrow$ denote "if and only if". Then

$$\begin{aligned}
\|A + B\| = \|A - B\| &\Leftrightarrow \|A + B\|^2 = \|A - B\|^2 \\
&\Leftrightarrow A^2 + 2A \cdot B + B^2 = A^2 - 2A \cdot B + B^2 \\
&\Leftrightarrow 4A \cdot B = 0 \\
&\Leftrightarrow A \cdot B = 0.
\end{aligned}$$

This proves what we wanted.

General Pythagoras theorem. *If A and B are perpendicular, then*

$$\|A + B\|^2 = \|A\|^2 + \|B\|^2.$$

The theorem is illustrated on Fig. 25.

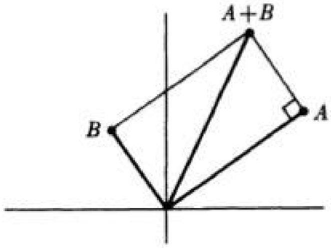


Figure 25

To prove this, we use the definitions, namely

$$\begin{aligned}\|A + B\|^2 &= (A + B)\cdot(A + B) = A^2 + 2A\cdot B + B^2\\ &= \|A\|^2 + \|B\|^2,\end{aligned}$$

because $A \cdot B = 0$, and $A\cdot A = \|A\|^2$, $B \cdot B = \|B\|^2$ by definition.

Remark. If A is perpendicular to B, and x is any number, then A is also perpendicular to xB because

$$A\cdot xB = xA\cdot B = 0.$$

We shall now use the notion of perpendicularity to derive the notion of **projection**. Let A, B be two vectors and $B \neq O$. Let P be the point on the line through $\overrightarrow{OB}$ such that $\overrightarrow{PA}$ is perpendicular to $\overrightarrow{OB}$, as shown on Fig. 26(a).

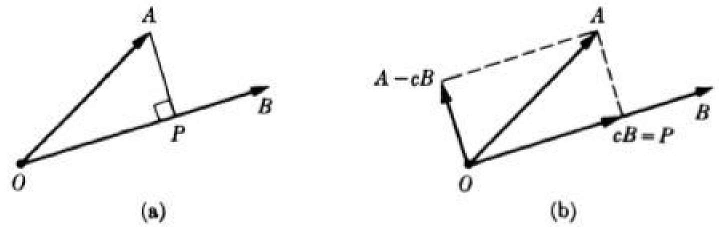


Figure 26

We can write

$$P = cB$$

for some number c. We want to find this number c explicitly in terms of A and B. The condition $\overrightarrow{PA} \perp \overrightarrow{OB}$ means that

$$A - P \text{ is perpendicular to } B,$$

and since $P = cB$ this means that

$$(A - cB)\cdot B = 0,$$

in other words,

$$A\cdot B - cB\cdot B = 0.$$

We can solve for c, and we find $A\cdot B = cB\cdot B$, so that

$$\boxed{c = \frac{A\cdot B}{B\cdot B}.}$$

Conversely, if we take this value for c, and then use distributivity, dotting $A - cB$ with B yields 0, so that $A - cB$ is perpendicular to B. Hence we have seen that there is a unique number c such that $A - cB$ is perpendicular to B, and c is given by the above formula.

Definition. The **component** of A along B is the number $c = \dfrac{A \cdot B}{B \cdot B}$.

The **projection** of A along B is the vector $cB = \dfrac{A \cdot B}{B \cdot B} B$.

Example 5. Suppose

$$B = E_i = (0, \dots, 0, 1, 0, \dots, 0)$$

is the i-th unit vector, with 1 in the i-th component and 0 in all other components.

If $A = (a_1, \dots, a_n)$, *then* $A \cdot E_i = a_i$.

Thus $A \cdot E_i$ is the ordinary i-th component of A.

More generally, if B is a unit vector, not necessarily one of the E_i, then we have simply

$$c = A \cdot B$$

because $B \cdot B = 1$ by definition of a unit vector.

Example 6. Let $A = (1, 2, -3)$ and $B = (1, 1, 2)$. Then the component of A along B is the **number**

$$c = \frac{A \cdot B}{B \cdot B} = \frac{-3}{6} = -\frac{1}{2}.$$

Hence the projection of A along B is the **vector**

$$cB = (-\tfrac{1}{2}, -\tfrac{1}{2}, -1).$$

Our construction gives an immediate geometric interpretation for the scalar product. Namely, assume $A \neq O$ and look at the angle θ between A and B (Fig. 27). Then from plane geometry we see that

$$\cos \theta = \frac{c\|B\|}{\|A\|},$$

or substituting the value for c obtained above.

$$\boxed{A \cdot B = \|A\| \ \|B\| \cos \theta} \quad \text{and} \quad \boxed{\cos \theta = \frac{A \cdot B}{\|A\| \ \|B\|}}.$$

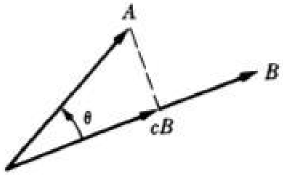


Figure 27

In some treatments of vectors, one takes the relation

$$A \cdot B = \|A\| \ \|B\| \cos \theta$$

as definition of the scalar product. This is subject to the following disadvantages, not to say objections:

(a) The four properties of the scalar product **SP 1** through **SP 4** are then by no means obvious.
(b) Even in 3-space, one has to rely on geometric intuition to obtain the cosine of the angle between A and B, and this intuition is less clear than in the plane. In higher dimensional space, it fails even more.
(c) It is extremely hard to work with such a definition to obtain further properties of the scalar product.

Thus we prefer to lay obvious algebraic foundations, and then recover very simply all the properties. We used plane geometry to see the expression

$$A \cdot B = \|A\| \ \|B\| \cos \theta.$$

After working out some examples, we shall prove the inequality which allows us to justify this in n-space.

Example 7. Let $A = (1, 2, -3)$ and $B = (2, 1, 5)$. Find the cosine of the angle θ between A and B.

By definition,

$$\cos \theta = \frac{A \cdot B}{\|A\| \ \|B\|} = \frac{2 + 2 - 15}{\sqrt{14}\sqrt{30}} = \frac{-11}{\sqrt{420}}.$$

Example 8. Find the cosine of the angle between the two located vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ where

$$P = (1, 2, -3), \qquad Q = (-2, 1, 5), \qquad R = (1, 1, -4).$$

The picture looks like this:

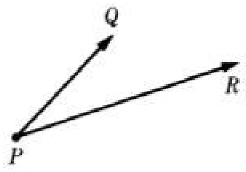


Figure 28

We let

$$A = Q - P = (-3, -1, 8) \qquad \text{and} \qquad B = R - P = (0, -1, -1).$$

Then the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is the same as that between A and B. Hence its cosine is equal to

$$\cos \theta = \frac{A \cdot B}{\|A\| \; \|B\|} = \frac{0 + 1 - 8}{\sqrt{74}\sqrt{2}} = \frac{-7}{\sqrt{74}\sqrt{2}}.$$

We shall prove further properties of the norm and scalar product using our results on perpendicularity. First note a special case. If

$$E_i = (0, \ldots, 0, 1, 0, \ldots, 0)$$

is the i-th unit vector of $\mathbf{R}^n$, and

$$A = (a_1, \ldots, a_n),$$

then

$$A \cdot E_i = a_i$$

is the i-th component of A, i.e. the component of A along E_i. We have

$$|a_i| = \sqrt{a_i^2} \leqq \sqrt{a_1^2 + \cdots + a_n^2} = \|A\|,$$

so that the absolute value of each component of A is at most equal to the length of A.

We don't have to deal only with the special unit vector as above. Let E be any unit vector, that is a vector of norm 1. Let c be the component of A along E. We saw that

$$c = A \cdot E.$$

Then $A - cE$ is perpendicular to E, and

$$A = A - cE + cE.$$

Then $A - cE$ is also perpendicular to cE, and by the Pythagoras theorem, we find

$$\|A\|^2 = \|A - cE\|^2 + \|cE\|^2 = \|A - cE\|^2 + c^2.$$

Thus we have the inequality $c^2 \leqq \|A\|^2$, and $|c| \leqq \|A\|$.

In the next theorem, we generalize this inequality to a dot product $A \cdot B$ when B is not necessarily a unit vector.

Theorem 4.2. *Let A, B be two vectors in $\mathbf{R}^n$. Then*

$$|A \cdot B| \leqq \|A\| \; \|B\|.$$

Proof. If $B = O$, then both sides of the inequality are equal to 0, and so our assertion is obvious. Suppose that $B \neq O$. Let c be the component of A along B, so $c = (A \cdot B)/(B \cdot B)$. We write

$$A = A - cB + cB.$$

By Pythagoras,

$$\|A\|^2 = \|A - cB\|^2 + \|cB\|^2 = \|A - cB\|^2 + c^2\|B\|^2.$$

Hence $c^2\|B\|^2 \leqq \|A\|^2$. But

$$c^2\|B\|^2 = \frac{(A \cdot B)^2}{(B \cdot B)^2}\|B\|^2 = \frac{|A \cdot B|^2}{\|B\|^4}\|B\|^2 = \frac{|A \cdot B|^2}{\|B\|^2}.$$

Therefore

$$\frac{|A \cdot B|^2}{\|B\|^2} \leqq \|A\|^2.$$

Multiply by $\|B\|^2$ and take the square root to conclude the proof.

In view of Theorem 4.2, we see that for vectors A, B in n-space, the number

$$\frac{A \cdot B}{\|A\| \; \|B\|}$$

has absolute value $\leqq 1$. Consequently,

$$-1 \leqq \frac{A \cdot B}{\|A\| \; \|B\|} \leqq 1,$$

and there exists a unique angle θ such that $0 \leqq \theta \leqq \pi$, and such that

$$\cos \theta = \frac{A \cdot B}{\|A\| \, \|B\|}.$$

We define this angle to be the **angle between** A **and** B.

The inequality of Theorem 4.2 is known as the **Schwarz inequality**.

Theorem 4.3. *Let A, B be vectors. Then*

$$\|A + B\| \leqq \|A\| + \|B\|.$$

Proof. Both sides of this inequality are positive or 0. Hence it will suffice to prove that their squares satisfy the desired inequality, in other words,

$$(A + B) \cdot (A + B) \leqq (\|A\| + \|B\|)^2.$$

To do this, we consider

$$(A + B) \cdot (A + B) = A \cdot A + 2A \cdot B + B \cdot B.$$

In view of our previous result, this satisfies the inequality

$$\leqq \|A\|^2 + 2\|A\| \, \|B\| + \|B\|^2,$$

and the right-hand side is none other than

$$(\|A\| + \|B\|)^2.$$

Our theorem is proved.

Theorem 4.3 is known as the **triangle inequality**. The reason for this is that if we draw a triangle as in Fig. 29, then Theorem 4.3 expresses the fact that the length of one side is $\leqq$ the sum of the lengths of the other two sides.

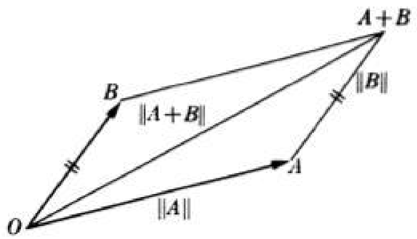


Figure 29

Remark. All the proofs do not use coordinates, only properties **SP 1** through **SP 4** of the dot product. In n-space, they give us inequalities which are by no means obvious when expressed in terms of coordinates. For instance, the Schwarz inequality reads, in terms of coordinates:

$$|a_1b_1 + \cdots + a_nb_n| \leqq (a_1^2 + \cdots + a_n^2)^{1/2}(b_1^2 + \cdots + b_n^2)^{1/2}.$$

Just try to prove this directly, without the "geometric" intuition of Pythagoras, and see how far you get.

XV, §4. EXERCISES

1. Find the norm of the vector A in the following cases.
   (a) $A = (2, -1)$, $B = (-1, 1)$
   (b) $A = (-1, 3)$, $B = (0, 4)$
   (c) $A = (2, -1, 5)$, $B = (-1, 1, 1)$
   (d) $A = (-1, -2, 3)$, $B = (-1, 3, -4)$
   (e) $A = (\pi, 3, -1)$, $B = (2\pi, -3, 7)$
   (f) $A = (15, -2, 4)$, $B = (\pi, 3, -1)$

2. Find the norm of vector B in the above cases.

3. Find the projection of A along B in the above cases.

4. Find the projection of B along A in the above cases.

5. Find the cosine between the following vectors A and B.
   (a) $A = (1, -2)$ and $B = (5, 3)$
   (b) $A = (-3, 4)$ and $B = (2, -1)$
   (c) $A = (1, -2, 3)$ and $B = (-3, 1, 5)$
   (d) $A = (-2, 1, 4)$ and $B = (-1, -1, 3)$
   (e) $A = (-1, 1, 0)$ and $B = (2, 1, -1)$

6. Determine the cosine of the angles of the triangle whose vertices are
   (a) $(2, -1, 1)$, $(1, -3, -5)$, $(3, -4, -4)$.
   (b) $(3, 1, 1)$, $(-1, 2, 1)$, $(2, -2, 5)$.

7. Let $A_1, \ldots, A_r$ be non-zero vectors which are mutually perpendicular, in other words $A_i \cdot A_j = 0$ if $i \neq j$. Let $c_1, \ldots, c_r$ be numbers such that

   $$c_1A_1 + \cdots + c_rA_r = O.$$

   Show that all $c_i = 0$.

8. For any vectors A, B, prove the following relations:
   (a) $\|A + B\|^2 + \|A - B\|^2 = 2\|A\|^2 + 2\|B\|^2$.
   (b) $\|A + B\|^2 = \|A\|^2 + \|B\|^2 + 2A \cdot B$.
   (c) $\|A + B\|^2 - \|A - B\|^2 = 4A \cdot B$.
   Interpret (a) as a "parallelogram law".

9. Show that if θ is the angle between A and B, then

$$\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\| \, \|B\| \cos \theta.$$

10. Let A, B, C be three non-zero vectors. If $A \cdot B = A \cdot C$, show by an example that we do not necessarily have $B = C$.

XV, §5. PARAMETRIC LINES

We define the **parametric equation** or **parametric representation** of a straight line passing through a point P in the direction of a vector $A \neq O$ to be

$$X = P + tA,$$

where t runs through all numbers (Fig. 30).

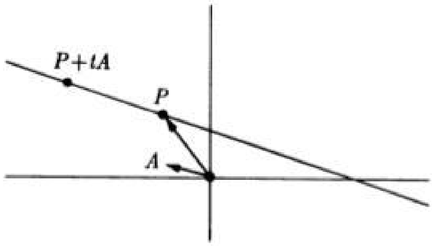


Figure 30

When we give such a parametric representation, we may think of a bug starting from a point P at time $t = 0$, and moving in the direction of A. At time t, the bug is at the position $P + tA$. Thus we may interpret physically the parametric representation as a description of motion, in which A is interpreted as the velocity of the bug. At a given time t, the bug is at the point.

$$X(t) = P + tA,$$

which is called the **position** of the bug at time t.

This parametric representation is also useful to describe the set of points lying on the line segment between two given points. Let P, Q be two points. Then the **segment** between P and Q consists of all the points

$$S(t) = P + t(Q - P) \qquad \text{with} \qquad 0 \leqq t \leqq 1.$$

Indeed, $\overrightarrow{O(Q - P)}$ is a vector having the same direction as $\overrightarrow{PQ}$, as shown on Fig. 31.

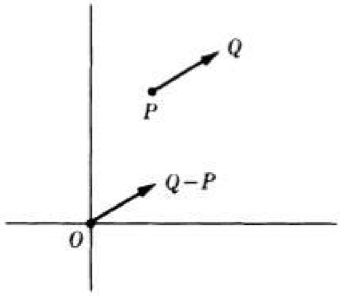


Figure 31

When $t = 0$, we have $S(0) = P$, so at time $t = 0$ the bug is at P. When $t = 1$, we have

$$S(1) = P + (Q - P) = Q,$$

so when $t = 1$ the bug is at Q. As t goes from 0 to 1, the bug goes from P to Q.

Example 1. Let $P = (1, -3, 4)$ and $Q = (5, 1, -2)$. Find the coordinates of the point which lies one third of the distance from P to Q.

Let $S(t)$ as above be the parametric representation of the segment from P to Q. The desired point is $S(1/3)$, that is:

$$\begin{aligned} S\left(\frac{1}{3}\right) &= P + \frac{1}{3}(Q - P) = (1, -3, 4) + \frac{1}{3}(4, 4, -6) \\ &= \left(\frac{7}{3}, \frac{-5}{3}, 2\right). \end{aligned}$$

Warning. The desired point in the above example is *not* given by

$$\frac{P + Q}{3}.$$

Example 2. Find a parametric representation for the line passing through the two points $P = (1, -3, 1)$ and $Q = (-2, 4, 5)$.

We first have to find a vector in the direction of the line. We let

$$A = P - Q,$$

so

$$A = (3, -7, -4).$$

The parametric representation of the line is therefore

$$X(t) = P + tA = (1, -3, 1) + t(3, -7, -4).$$

Remark. It would be equally correct to give a parametric representation of the line as

$$Y(t) = P + tB \qquad \text{where} \qquad B = Q - P.$$

Interpreted in terms of the moving bug, however, one parametrization gives the position of a bug moving in one direction along the line, starting from P at time $t = 0$, while the other parametrization gives the position of another bug moving in the **opposite** direction along the line, also starting from P at time $t = 0$.

We shall now discuss the relation between a parametric representation and the ordinary equation of a line in the plane.

Suppose that we work in the plane, and write the coordinates of a point X as (x, y). Let $P = (p, q)$ and $A = (a, b)$. Then in terms of the coordinates, we can write

$$x = p + ta, \qquad y = q + tb.$$

We can then eliminate t and obtain the usual equation relating x and y.

Example 3. Let $P = (2, 1)$ and $A = (-1, 5)$. Then the parametric representation of the line through P in the direction of A gives us

$$(*) \qquad x = 2 - t, \qquad y = 1 + 5t.$$

Multiplying the first equation by 5 and adding yields

$$(**) \qquad 5x + y = 11,$$

which is the familiar equation of a line.

This elimination of t shows that every pair (x, y) which satisfies the parametric representation $(*)$ for some value of t also satisfies equation $(**)$. Conversely, suppose we have a pair of numbers (x, y) satisfying $(**)$. Let $t = 2 - x$. Then

$$y = 11 - 5x = 11 - 5(2 - t) = 1 + 5t.$$

Hence there exists some value of t which satisfies equation (*). Thus we have proved that the pairs (x, y) which are solutions of (**) are exactly the same pairs of numbers as those obtained by giving arbitrary values for t in (*). Thus the straight line can be described parametrically as in (*) or in terms of its usual equation (**). Starting with the ordinary equation

$$5x + y = 11,$$

we let $t = 2 - x$ in order to recover the specific parametrization of (*).

When we parametrize a straight line in the form

$$X = P + tA,$$

we have of course infinitely many choices for P on the line, and also infinitely many choices for A, differing by a scalar multiple. We can always select at least one. Namely, given an equation

$$ax + by = c$$

with numbers a, b, c, suppose that $a \neq 0$. We use y as parameter, and let

$$y = t.$$

Then we can solve for x, namely

$$x = \frac{c}{a} - \frac{b}{a}t.$$

Let $P = (c/a, 0)$ and $A = (-b/a, 1)$. We see that an arbitrary point (x, y) satisfying the equation

$$ax + by = c$$

can be expressed parametrically, namely

$$(x, y) = P + tA.$$

In higher dimensions, starting with a parametric representation

$$X = P + tA,$$

we cannot eliminate t, and thus the parametric representation is the only one available to describe a straight line.

XV, §5. EXERCISES

1. Find a parametric representation for the line passing through the following pairs of points.
   (a) $P_1 = (1, 3, -1)$ and $P_2 = (-4, 1, 2)$
   (b) $P_1 = (-1, 5, 3)$ and $P_2 = (-2, 4, 7)$

Find a parametric representation for the line passing through the following points.

2. $(1, 1, -1)$ and $(-2, 1, 3)$
3. $(-1, 5, 2)$ and $(3, -4, 1)$
4. Let $P = (1, 3, -1)$ and $Q = (-4, 5, 2)$. Determine the coordinates of the following points:
   (a) The midpoint of the line segment between P and Q.
   (b) The two points on this line segment lying one-third and two-thirds of the way from P to Q.
   (c) The point lying one-fifth of the way from P to Q.
   (d) The point lying two-fifths of the way from P to Q.
5. If P, Q are two arbitrary points in n-space, give the general formula for the midpoint of the line segment between P and Q.

XV, §6. PLANES

We can describe planes in 3-space by an equation analogous to the single equation of the line. We proceed as follows.

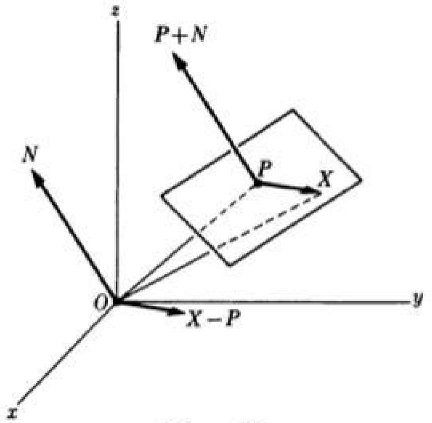


Figure 32

Let P be a point in 3-space and consider a located vector $\overrightarrow{ON}$. We define the **plane passing through P perpendicular to $\overrightarrow{ON}$** to be the collection of all points X such that the located vector $\overrightarrow{PX}$ is perpendicular to $\overrightarrow{ON}$. According to our definitions, this amounts to the condition

$$(X - P) \cdot N = 0,$$

which can also be written as

$$\boxed{X \cdot N = P \cdot N.}$$

We shall also say that this plane is the one perpendicular to N, and consists of all vectors X such that $X - P$ is perpendicular to N. We have drawn a typical situation in 3-spaces in Fig. 32.

Instead of saying that N is **perpendicular** to the plane, one also says that N is **normal** to the plane.

Let t be a number $\neq 0$. Then the set points X such that

$$(X - P) \cdot N = 0$$

coincides with the set of points X such that

$$(X - P) \cdot tN = 0.$$

Thus we may say that our plane is the plane passing through P and perpendicular to the **line** in the direction of N. To find the equation of the plane, we could use any vector tN (with $t \neq 0$) instead of N.

Example 1. Let

$$P = (2, 1, -1) \quad \text{and} \quad N = (-1, 1, 3).$$

Let $X = (x, y, z)$. Then

$$X \cdot N = (-1)x + y + 3z.$$

Therefore the equation of the plane passing through P and perpendicular to N is

$$-x + y + 3z = -2 + 1 - 3$$

or

$$-x + y + 3z = -4.$$

Observe that in 2-space, with $X = (x, y)$, the formulas lead to the equation of the line in the ordinary sense.

Example 2. The equation of the line in the (x, y)-plane, passing through $(4, -3)$ and perpendicular to $(-5, 2)$ is

$$-5x + 2y = -20 - 6 = -26.$$

We are now in position to interpret the coefficients $(-5, 2)$ of x and y in this equation. They give rise to a vector perpendicular to the line. **In any equation**

$$ax + by = c$$

the vector (a, b) **is perpendicular to the line determined by the equation.** Similarly, in 3-space, the vector (a, b, c) is perpendicular to the plane determined by the equation

$$ax + by + cz = d.$$

Example 3. The plane determined by the equation

$$2x - y + 3z = 5$$

is perpendicular to the vector $(2, -1, 3)$. If we want to find a point in that plane, we of course have many choices. We can give arbitrary values to x and y, and then solve for z. To get a concrete point, let $x = 1$, $y = 1$. Then we solve for z, namely

$$3z = 5 - 2 + 1 = 4,$$

so that $z = \frac{4}{3}$. Thus

$$(1, 1, \tfrac{4}{3})$$

is a point in the plane.

In n-space, the equation $X \cdot N = P \cdot N$ is said to be the equation of a **hyperplane**. For example,

$$3x - y + z + 2w = 5$$

is the equation of a hyperplane in 4-space, perpendicular to $(3, -1, 1, 2)$.

Two vectors A, B are said to be parallel if there exists a number $c \neq 0$ such that $cA = B$. Two lines are said to be **parallel** if, given two distinct points P_1, Q_1 on the first line and P_2, Q_2 on the second, the vectors

$$P_1 - Q_1$$

and

$$P_2 - Q_2$$

are parallel.

Two planes are said to be **parallel** (in 3-space) if their normal vectors are parallel. They are said to be **perpendicular** if their normal vectors are perpendicular. The **angle** between two planes is defined to be the angle between their normal vectors.

Example 4. Find the cosine of the angle θ between the planes.

$$2x - y + z = 0,$$
$$x + 2y - z = 1.$$

This cosine is the cosine of the angle between the vectors.

$$A = (2, -1, 1) \qquad \text{and} \qquad B = (1, 2, -1).$$

Therefore

$$\cos\theta = \frac{A \cdot B}{\|A\|\,\|B\|} = -\frac{1}{6}.$$

Example 5. Let

$$Q = (1, 1, 1) \qquad \text{and} \qquad P = (1, -1, 2).$$

Let

$$N = (1, 2, 3)$$

Find the point of intersection of the line through P in the direction of N, and the plane through Q perpendicular to N.

The parametric representation of the line through P in the direction of N is

(1) $$X = P + tN.$$

The equation of the plane through Q perpendicular to N is

(2) $$(X - Q) \cdot N = 0.$$

We visualize the line and plane as follows:

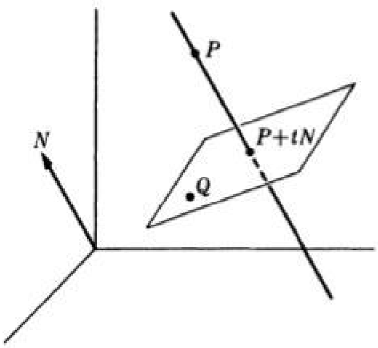


Figure 33

We must find the value of t such that the vector X in (1) also satisfies (2), that is

$$(P + tN - Q)\cdot N = 0,$$

or after using the rules of the dot product,

$$(P - Q)\cdot N + tN\cdot N = 0.$$

Solving for t yields

$$t = \frac{(Q - P)\cdot N}{N\cdot N} = \frac{1}{14}.$$

Thus the desired point of intersection is

$$P + tN = (1, -1, 2) + \tfrac{1}{14}(1, 2, 3) = (\tfrac{15}{14}, -\tfrac{12}{14}, \tfrac{31}{14}).$$

Example 6. Find the equation of the plane passing through the three points

$$P_1 = (1, 2, -1), \qquad P_2 = (-1, 1, 4), \qquad P_3 = (1, 3, -2).$$

We visualize schematically the three points as follows:

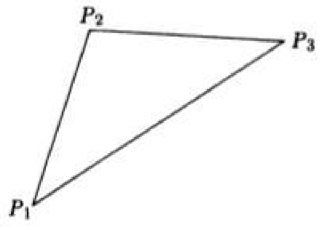


Figure 34

Then we find a vector N perpendicular to $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, or in other words, perpendicular to $P_2 - P_1$ and $P_3 - P_1$. We have

$$P_2 - P_1 = (-2, -1, +5),$$
$$P_3 - P_1 = (0, 1, -1).$$

Let $N = (a, b, c)$. We must solve

$$N\cdot(P_2 - P_1) = 0 \qquad \text{and} \qquad N\cdot(P_3 - P_1) = 0,$$

in other words,

$$\begin{aligned} -2a - b + 5c &= 0, \\ b - c &= 0. \end{aligned}$$

We take $b = c = 1$ and solve for $a = 2$. Then

$$N = (2, 1, 1)$$

satisfies our requirements. The plane perpendicular to N, passing through P_1 is the desired plane. Its equation is therefore $X \cdot N = P_1 \cdot N$, that is

$$2x + y + z = 2 + 2 - 1 = 3.$$

Distance between a point and a plane. Consider a plane defined by the equation

$$(X - P) \cdot N = 0,$$

and let Q be an arbitrary point. We wish to find a formula for the distance between Q and the plane. By this we mean the length of the segment from Q to the point of intersection of the perpendicular line to the plane through Q, as on the figure. We let Q' be this point of intersection.

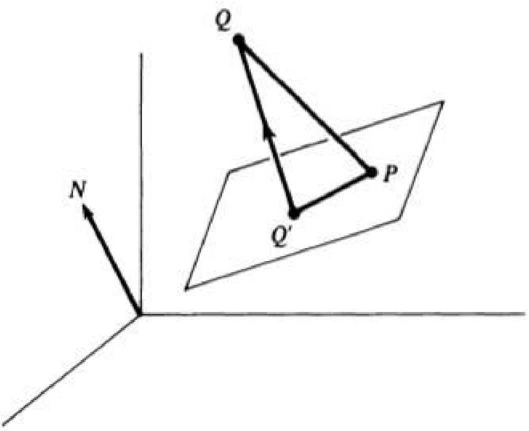


Figure 35

From the geometry, we have:

length of the segment $\overline{QQ'}$ = length of the projection of $\overline{QP}$ on $\overline{QQ'}$.

We can express the length of this projection in terms of the dot product as follows. A unit vector in the direction of N, which is perpendicular to the plane, is given by $N/\|N\|$. Then

$$\begin{aligned}\text{length of the projection of } \overline{QP} \text{ on } \overline{QQ'} \\ &= \text{norm of the projection of } Q - P \text{ on } N/\|N\| \\ &= \left|(Q-P)\cdot\frac{N}{\|N\|}\right|.\end{aligned}$$

This can also be written in the form:

$$\boxed{\text{distance between } Q \text{ and the plane} = \frac{|(Q-P)\cdot N|}{\|N\|}.}$$

Example 7. Let

$$Q = (1, 3, 5), \qquad P = (-1, 1, 7) \qquad \text{and} \qquad N = (-1, 1, -1).$$

The equation of the plane is

$$-x + y - z = -5.$$

We find $\|N\| = \sqrt{3}$,

$$Q - P = (2, 2, -2) \qquad \text{and} \qquad (Q-P)\cdot N = -2 + 2 + 2 = 2.$$

Hence the distance between Q and the plane is $2/\sqrt{3}$.

XV, §6. EXERCISES

1. Show that the lines $2x + 3y = 1$ and $5x - 5y = 7$ are not perpendicular.
2. Let $y = mx + b$ and $y = m'x + c$ be the equations of two lines in the plane. Write down vectors perpendicular to these lines. Show that these vectors are perpendicular to each other if and only if $mm' = -1$.

Find the equation of the line in 2-space, perpendicular to N and passing through P, for the following values of N and P.

3. $N = (1, -1)$, $P = (-5, 3)$
4. $N = (-5, 4)$, $P = (3, 2)$
5. Show that the lines

$$3x - 5y = 1, \qquad 2x + 3y = 5$$

are not perpendicular.

6. Which of the following pairs of lines are perpendicular?
   (a) $3x - 5y = 1$ and $2x + y = 2$
   (b) $2x + 7y = 1$ and $x - y = 5$
   (c) $3x - 5y = 1$ and $5x + 3y = 7$
   (d) $-x + y = 2$ and $x + y = 9$

7. Find the equation of the plane perpendicular to the given vector N and passing through the given point P.
   (a) $N = (1, -1, 3)$, $P = (4, 2, -1)$
   (b) $N = (-3, -2, 4)$, $P = (2, \pi, -5)$
   (c) $N = (-1, 0, 5)$, $P = (2, 3, 7)$

8. Find the equation of the plane passing through the following three points.
   (a) $(2, 1, 1)$, $(3, -1, 1)$, $(4, 1, -1)$
   (b) $(-2, 3, -1)$, $(2, 2, 3)$, $(-4, -1, 1)$
   (c) $(-5, -1, 2)$, $(1, 2, -1)$, $(3, -1, 2)$

9. Find a vector perpendicular to $(1, 2, -3)$ and $(2, -1, 3)$, and another vector perpendicular to $(-1, 3, 2)$ and $(2, 1, 1)$.

10. Find a vector parallel to the line of intersection of the two planes

$$2x - y + z = 1, \qquad 3x + y + z = 2.$$

11. Same question for the planes,

$$2x + y + 5z = 2, \qquad 3x - 2y + z = 3.$$

12. Find a parametric representation for the line of intersection of the planes of Exercises 10 and 11.

13. Find the cosine of the angle between the following planes:
   (a) $x + y + z = 1$, $x - y - z = 5$
   (b) $2x + 3y - z = 2$, $x - y + z = 1$
   (c) $x + 2y - z = 1$, $-x + 3y + z = 2$
   (d) $2x + y + z = 3$, $-x - y + z = \pi$

14. (a) Let $P = (1, 3, 5)$ and $A = (-2, 1, 1)$. Find the intersection of the line through P in the direction of A, and the plane $2x + 3y - z = 1$.
   (b) Let $P = (1, 2, -1)$. Find the point of intersection of the plane

$$3x - 4y + z = 2,$$

with the line through P, perpendicular to that plane.

15. Let $Q = (1, -1, 2)$, $P = (1, 3, -2)$, and $N = (1, 2, 2)$. Find the point of the intersection of the line through P in the direction of N, and the plane through Q perpendicular to N.

16. Find the distance between the indicated point and plane.
   (a) $(1, 1, 2)$ and $3x + y - 5z = 2$
   (b) $(-1, 3, 2)$ and $2x - 4y + z = 1$
   (c) $(3, -2, 1)$ and the yz-plane
   (d) $(-3, -2, 1)$ and the yz-plane

17. Draw the triangle with vertices $A = (1, 1)$, $B = (2, 3)$, and $C = (3, -1)$. Draw the point P such that $\overrightarrow{AP} \perp \overrightarrow{BC}$ and P belongs to the line passing through the points B and C.
    (a) Find the cosine of the angle of the triangle whose vertex is at A.
    (b) What are the coordinates of P?

18. (a) Find the equation of the plane M passing through the point $P = (1, 1, 1)$ and perpendicular to the vector $\overrightarrow{ON}$, where $N = (1, 2, 0)$.
    (b) Find a parametric representation of the line L passing through

$$Q = (1, 4, 0)$$

    and perpendicular to the plane M.
    (c) What is the distance from Q to the plane M?

19. Find the cosine of the angle between the planes

$$2x + 4y - z = 5 \qquad \text{and} \qquad x - 3y + 2z = 0.$$

CHAPTER XVI

Differentiation of Vectors

XVI, §1. DERIVATIVE

Consider a bug moving along some curve in 3-dimensional space. The position of the bug at time t is given by the three coordinates

$$(x(t),\ y(t),\ z(t)),$$

which depend on t. We abbreviate these by $X(t)$. For instance, the position of a bug moving along a straight line was seen in the preceding chapter to be given by

$$X(t) = P + tA,$$

where P is the starting point, and A gives the direction of the bug. However, we can give examples when the bug does not move on a straight line. First we look at an example in the plane.

Example 1. Let $X(\theta) = (\cos\theta,\ \sin\theta)$. Then the bug moves around a circle of radius 1 in counterclockwise direction.

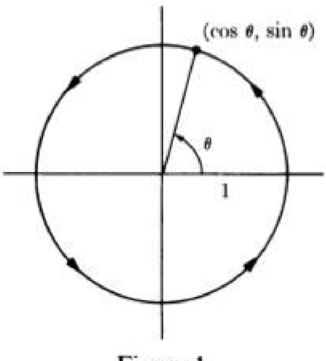


Figure 1

Here we used θ as the variable, corresponding to the angle as shown on the figure. Let ω be the angular speed of the bug, and assume ω constant. Thus $d\theta/dt = \omega$ and

$$\theta = \omega t + \text{a constant}.$$

For simplicity, assume that the constant is 0. Then we can write the position of the bug as

$$X(\theta) = X(\omega t) = (\cos \omega t, \sin \omega t).$$

If the angular speed is 1, then we have simply the representation

$$X(t) = (\cos t,\ \sin t).$$

Example 2. If the bug moves around a circle of radius 2 with angular speed equal to 1, then its position at time t is given by

$$X(t) = (2 \cos t,\ 2 \sin t).$$

More generally, if the bug moves around a circle of radius r, then the position is given by

$$X(t) = (r \cos t,\ r \sin t).$$

In these examples, we assume of course that at time $t = 0$ the bug starts at the point $(r, 0)$, that is

$$X(0) = (r, 0),$$

where r is the radius of the circle.

Example 3. Suppose the position of the bug is given in 3-space by

$$X(t) = (\cos t,\ \sin t,\ t).$$

Then the bug moves along a spiral. Its coordinates are given as functions of t by

$$x(t) = \cos t,$$

$$y(t) = \sin t,$$

$$z(t) = t.$$

The position at time t is obtained by plugging in the special value of t. Thus:

$$X(\pi) = (\cos \pi, \sin \pi, \pi) = (-1, 0, \pi)$$

$$X(1) = (\cos 1, \sin 1, 1).$$

We may now give the definition of a curve in general.

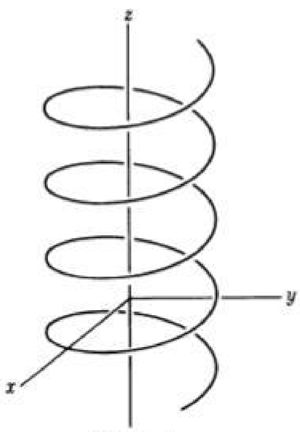


Figure 2

Definition. Let I be an interval. A **parametrized curve** (defined on this interval) is an association which to each point of I associates a vector. If X denotes a curve defined on I, and t is a point of I, then $X(t)$ denotes the vector associated to t by X. We often write the association $t \mapsto X(t)$ as an arrow

$$X : I \to \mathbf{R}^n.$$

We also call this association the **parametrization** of a curve. We call $X(t)$ the **position vector** at time t. It can be written in terms of coordinates,

$$X(t) = (x_1(t), \ldots, x_n(t)),$$

each $x_i(t)$ being a function of t. We say that this curve is **differentiable** if each function $x_i(t)$ is a differentiable function of t.

Remark. We take the intervals of definition for our curves to be open, closed, or also half-open or half-closed. When we define the derivative of a curve, it is understood that the interval of definition contains more than one point. In that case, at an end point the usual limit of

$$\frac{f(a+h) - f(a)}{h}$$

is taken for those h such that the quotient makes sense, i.e. $a + h$ lies in the interval. If a is a left end point, the quotient is considered only for $h > 0$. If a is a right end point the quotient is considered only for $h < 0$. Then the usual rules for differentiation of functions are true in this greater generality, and thus Rules 1 through 4 below, and the chain rule of §2 remain true also. [An example of a statement which is not always true for curves defined over closed intervals is given in Exercise 11(b).]

Let us try to differentiate curves. We consider the Newton quotient

$$\frac{X(t+h) - X(t)}{h}.$$

Its numerator is illustrated in Fig. 3.

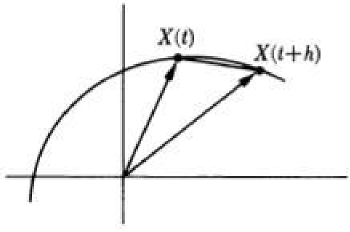


Figure 3

As h approaches 0, we see geometrically that

$$\frac{X(t+h)-X(t)}{h}$$

should approach a vector pointing in the direction of the curve. We can write the Newton quotient in terms of coordinates,

$$\frac{X(t+h)-X(t)}{h}=\left(\frac{x_1(t+h)-x_1(t)}{h},\dots,\frac{x_n(t+h)-x_n(t)}{h}\right)$$

and see that each component is a Newton quotient for the corresponding coordinate. We assume that each $x_i(t)$ is differentiable. Then each quotient

$$\frac{x_i(t+h)-x_i(t)}{h}$$

approaches the derivatives dx_i/dt. For this reason, we define the **derivative** dX/dt to be

$$\frac{dX}{dt}=\left(\frac{dx_1}{dt},\dots,\frac{dx_n}{dt}\right).$$

In fact, we could also say that the vector

$$\left(\frac{dx_1}{dt},\dots,\frac{dx_n}{dt}\right)$$

is the limit of the Newton quotient

$$\frac{X(t+h)-X(t)}{h}$$

as h approaches 0. Indeed, as h approaches 0, each component

$$\frac{x_i(t+h)-x_i(t)}{h}$$

approaches dx_i/dt. Hence the Newton quotient approaches the vector

$$\left(\frac{dx_1}{dt},\dots,\frac{dx_n}{dt}\right).$$

Example 4. If $X(t) = (\cos t, \sin t, t)$ then

$$\frac{dX}{dt} = (-\sin t, \cos t, 1).$$

Physicists often denote dX/dt by $\dot{X}$; thus in the previous example, we could also write

$$\dot{X}(t) = (-\sin t, \cos t, 1) = X'(t).$$

We define the **velocity vector** of the curve at time t to be the vector $X'(t)$.

Example 5. When $X(t) = (\cos t, \sin t, t)$, then

$$X'(t) = (-\sin t, \cos t, 1);$$

the velocity vector at $t = \pi$ is

$$X'(\pi) = (0, -1, 1),$$

and for $t = \pi/4$ we get

$$X'(\pi/4) = (-1/\sqrt{2}, 1/\sqrt{2}, 1).$$

The velocity vector is located at the origin, but when we translate it to the point $X(t)$, then we visualize it as tangent to the curve, as in the next figure.

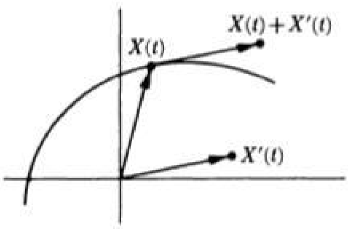


Figure 4

We define the **tangent line** to a curve X at time t to be the line passing through $X(t)$ in the direction of $X'(t)$, provided that $X'(t) \neq O$. Otherwise, we don't define a tangent line. We have therefore given two interpretations for $X'(t)$:

$$\boxed{\begin{array}{l} X'(t) \textit{ is the velocity at time } t; \\ X'(t) \textit{ is parallel to a tangent vector at time } t. \end{array}}$$

By abuse of language, we sometimes call $X'(t)$ a tangent vector, although strictly speaking, we should refer to the located vector $\overrightarrow{X(t)(X(t) + X'(t))}$ as the tangent vector. However, to write down this located vector each time is cumbersome.

Example 6. Find a parametric equation of the tangent line to the curve $X(t) = (\sin t, \cos t)$ at $t = \pi/3$.

We have $X'(t) = (\cos t, -\sin t)$, so that at $t = \dfrac{\pi}{3}$ we get

$$X'\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \quad \text{and} \quad X\left(\frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right).$$

Let $P = X(\pi/3)$ and $A = X'(\pi/3)$. Then a parametric equation of the tangent line at the required point is

$$L(t) = P + tA = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)t.$$

(We use another letter L because X is already occupied.) In terms of the coordinates $L(t) = (x(t), y(t))$, we can write the tangent line as

$$x(t) = \frac{\sqrt{3}}{2} + \frac{1}{2}t,$$

$$y(t) = \frac{1}{2} - \frac{\sqrt{3}}{2}t.$$

Example 7. Find the equation of the plane perpendicular to the spiral

$$X(t) = (\cos t, \sin t, t)$$

when $t = \pi/3$.

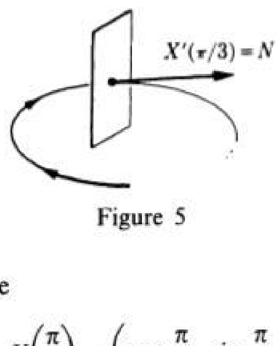


Figure 5

Let the given point be

$$P = X\left(\frac{\pi}{3}\right) = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}, \frac{\pi}{3}\right),$$

so that more simply,

$$P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right).$$

We must then find a vector N perpendicular to the plane at the given point P.

We have $X'(t) = (-\sin t, \cos t, 1)$, so

$$X'\left(\frac{\pi}{3}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\right) = N.$$

The equation of the plane through P perpendicular to N is

$$X \cdot N = P \cdot N,$$

so the equation of the desired plane is

$$\begin{aligned} -\frac{\sqrt{3}}{2}x + \frac{1}{2}y + z &= -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + \frac{\pi}{3} \\ &= \frac{\pi}{3}. \end{aligned}$$

We define the **speed** of the curve $X(t)$ to be the norm of the velocity vector. If we denote the speed by $v(t)$, then by definition we have

$$\boxed{v(t) = \|X'(t)\|,}$$

and thus

$$\boxed{v(t)^2 = X'(t)^2 = X'(t) \cdot X'(t).}$$

We can also omit the t from the notation, and write

$$v^2 = X' \cdot X' = X'^2.$$

Example 8. The speed of the bug moving on the circle

$$X(t) = (\cos t, \sin t)$$

is the norm of the velocity $X'(t) = (-\sin t, \cos t)$, and so is

$$v(t) = \sqrt{(-\sin t)^2 + (\cos^2 t)} = 1.$$

Example 9. The speed of the bug moving on the spiral

$$X(t) = (\cos t, \sin t, t)$$

is the norm of the velocity $X'(t) = (-\sin t, \cos t, 1)$, and so is

$$\begin{aligned} v(t) &= \sqrt{(-\sin t)^2 + (\cos^2 t) + 1} \\ &= \sqrt{2}. \end{aligned}$$

We define the **acceleration vector** to be the derivative

$$\frac{dX'(t)}{dt} = X''(t),$$

provided of course that X' is differentiable. We shall also denote the acceleration vector by $X''(t)$ as above.

We shall now discuss acceleration. There are two possible definitions for a **scalar acceleration**:

First there is the *rate of change of the speed*, that is

$$\frac{dv}{dt} = v'(t).$$

Second, there is the *norm of the acceleration vector*, that is

$$\|X''(t)\|.$$

Warning. These two are usually not equal. Almost any example will show this.

Example 10. Let

$$X(t) = (\cos t, \sin t).$$

Then:

$$v(t) = \|X'(t)\| = 1 \qquad \text{so} \qquad dv/dt = 0.$$

$$X''(t) = (-\cos t, -\sin t) \qquad \text{so} \qquad \|X''(t)\| = 1.$$

Thus if and when we need to refer to scalar acceleration, we must always say which one we mean. One could use the notation $a(t)$ for scalar acceleration, but one must specify which of the two possibilities $a(t)$ denotes.

The fact that the above two quantities are not equal reflects the physical interpretation. A bug moving around a circle at uniform speed has

$dv/dt = 0$. However, the acceleration vector is not O, because the velocity vector is constantly changing. Hence the norm of the acceleration vector is not equal to 0.

We shall list the rules for differentiation. These will concern sums, products, and the chain rule which is postponed to the next section.

The derivative of a curve is defined componentwise. Thus the rules for the derivative will be very similar to the rules for differentiating functions.

Rule 1. *Let $X(t)$ and $Y(t)$ be two differentiable curves (defined for the same values of t). Then the sum $X(t) + Y(t)$ is differentiable, and*

$$\frac{d(X(t) + Y(t))}{dt} = \frac{dX}{dt} + \frac{dY}{dt}.$$

Rule 2. *Let c be a number, and let $X(t)$ be differentiable. Then $cX(t)$ is differentiable, and*

$$\frac{d(cX(t))}{dt} = c\,\frac{dX}{dt}.$$

Rule 3. *Let $X(t)$ and $Y(t)$ be two differentiable curves (defined for the same values of t). Then $X(t)\cdot Y(t)$ is a differentiable function whose derivative is*

$$\frac{d}{dt}\,[X(t)\cdot Y(t)] = X(t)\cdot Y'(t) + X'(t)\cdot Y(t).$$

(This is formally analogous to the derivative of a product of functions, namely **the first times the derivative of the second plus the second times the derivative of the first,** except that the product is now a scalar product.)

As an example of the proofs we shall give the third one in detail, and leave the others to you as exercises.

Let for simplicity

$$X(t) = (x_1(t), x_2(t)) \quad \text{and} \quad Y(t) = (y_1(t), y_2(t)).$$

Then

$$\begin{aligned}\frac{d}{dt}\,X(t)\cdot Y(t) &= \frac{d}{dt}\,[x_1(t)y_1(t) + x_2(t)y_2(t)]\\ &= x_1(t)\frac{dy_1(t)}{dt} + \frac{dx_1}{dt}\,y_1(t) + x_2(t)\frac{dy_2}{dt} + \frac{dx_2}{dt}\,y_2(t)\\ &= X(t)\cdot Y'(t) + X'(t)\cdot Y(t),\end{aligned}$$

by combining the appropriate terms.

The proof for 3-space or n-space is obtained by replacing 2 by 3 or n, and inserting... in the middle to take into account the other coordinates.

Example 11. The square $X(t)^2 = X(t) \cdot X(t)$ comes up frequently in applications, for instance because it can be interpreted as the square of the distance of $X(t)$ from the origin. Using the rule for the derivative of a product, we find the formula

$$\boxed{\frac{d}{dt} X(t)^2 = 2X(t) \cdot X'(t).}$$

You should memorize this formula by repeating it out loud.

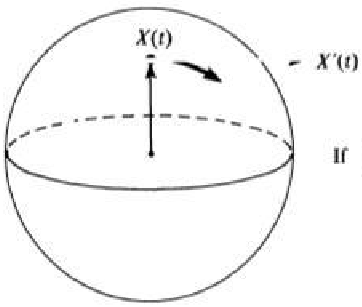
Suppose that $\|X(t)\|$ is constant. This means that $X(t)$ lies on a sphere of constant radius k. Taking the square yields

$$X(t)^2 = k^2$$

that is, $X(t)^2$ is also constant. Differentiate both sides with respect to t. Then we obtain

$$2X(t) \cdot X'(t) = 0 \qquad \text{and therefore} \qquad X(t) \cdot X'(t) = 0$$

Interpretation. *Suppose a bug moves along a curve $X(t)$ which remains at constant distance from the origin, i.e. $\|X(t)\| = k$ is constant. Then the position vector $X(t)$ is perpendicular to the velocity $X'(t)$.*



Curve on a sphere

If $X(t)$ is a curve and $f(t)$ is a function, defined for the same values of t, then we may also form the product $f(t)X(t)$ of the number $f(t)$ by the vector $X(t)$.

Example 12. Let $X(t) = (\cos t, \sin t, t)$ and $f(t) = e^t$, then

$$f(t)X(t) = (e^t \cos t, e^t \sin t, e^t t),$$

and

$$f(\pi)X(\pi) = (e^\pi(-1), e^\pi(0), e^\pi \pi) = (-e^\pi, 0, e^\pi \pi).$$

If $X(t) = (x(t), y(t), z(t))$, then

$$f(t)X(t) = (f(t)x(t), f(t)y(t), f(t)z(t)).$$

We have a rule for such differentiation analogous to Rule 3.

Rule 4. *If both $f(t)$ and $X(t)$ are defined over the same interval, and are differentiable, then so is $f(t)X(t)$, and*

$$\frac{d}{dt} f(t)X(t) = f(t)X'(t) + f'(t)X(t).$$

The proof is just the same as for Rule 3.

Example 13. Let A be a fixed vector, and let f be an ordinary differentiable function of one variable. Let $F(t) = f(t)A$. Then $F'(t) = f'(t)A$. For instance, if $F(t) = (\cos t)A$ and $A = (a, b)$ where a, b are fixed numbers, then

$$F(t) = (a \cos t, b \cos t)$$

and thus

$$F'(t) = (-a \sin t, -b \sin t) = (-\sin t)A.$$

Similarly, if A, B are fixed vectors, and

$$G(t) = (\cos t)A + (\sin t)B,$$

then

$$G'(t) = (-\sin t)A + (\cos t)B.$$

XVI, §1. EXERCISES

Find the velocity of the following curves.

1. $(e^t, \cos t, \sin t)$
2. $(\sin 2t, \log(1 + t), t)$
3. $(\cos t, \sin t)$
4. $(\cos 3t, \sin 3t)$
5. (a) In Exercises 3 and 4, show that the velocity vector is perpendicular to the position vector. Is this also the case in Exercises 1 and 2?

   (b) In Exercises 3 and 4, show that the acceleration vector is in the opposite direction from the position vector.
6. Let A, B be two constant vectors. What is the velocity vector of the curve

$$X = A + tB?$$

7. Let $X(t)$ be a differentiable curve. A plane or line which is perpendicular to the velocity vector $X'(t)$ at the point $X(t)$ is said to be **normal** to the curve at the point t or also at the point $X(t)$. Find the equation of a line normal to the curves of Exercises 3 and 4 at the point $\pi/3$.

8. (a) Find the equation of a plane normal to the curve

$$(e^t, t, t^2)$$

at the point $t = 1$.
(b) Same question at the point $t = 0$.

9. Let P be the point $(1, 2, 3, 4)$ and Q the point $(4, 3, 2, 1)$. Let A be the vector $(1, 1, 1, 1)$. Let L be the line passing through P and parallel to A.
(a) Given a point X on the line L, compute the distance between Q and X (as a function of the parameter t).
(b) Show that there is precisely one point X_0 on the line such that this distance achieves a minimum, and that this minimum is $2\sqrt{5}$.
(c) Show that $X_0 - Q$ is perpendicular to the line.

10. Let P be the point $(1, -1, 3, 1)$ and Q the point $(1, 1, -1, 2)$. Let A be the vector $(1, -3, 2, 1)$. Solve the same questions as in the preceding problem, except that in this case the minimum distance is $\sqrt{146/15}$.

11. Let $X(t)$ be a differentiable curve defined on an open interval. Let Q be a point which is not on the curve.
(a) Write down the formula for the distance between Q and an arbitrary point on the curve.
(b) If t_0 is a value of t such that the distance between Q and $X(t_0)$ is at a minimum, show that the vector $Q - X(t_0)$ is normal to the curve, at the point $X(t_0)$. [*Hint*: Investigate the minimum of the square of the distance.]
(c) If $X(t)$ is the parametric representation of a straight line, show that there exists a unique value t_0 such that the distance between Q and $X(t_0)$ is a minimum.

12. Let N be a non-zero vector, c a number, and Q a point. Let P_0 be the point of intersection of the line passing through Q, in the direction of N, and the plane $X \cdot N = c$. Show that for all points P of the plane, we have

$$\|Q - P_0\| \leqq \|Q - P\|.$$

13. Prove that if the speed is constant, then the acceleration is perpendicular to the velocity.

14. Prove that if the acceleration of a curve is always perpendicular to its velocity, then its speed is constant.

15. Let B be a non-zero vector, and let $X(t)$ be such that $X(t) \cdot B = t$ for all t. Assume also that the angle between $X'(t)$ and B is constant. Show that $X''(t)$ is perpendicular to $X'(t)$.

16. Write a parametric representation for the tangent line to the given curve at the given point in each of the following cases.
    (a) $(\cos 4t, \sin 4t, t)$ at the point $t = \pi/8$
    (b) $(t, 2t, t^2)$ at the point $(1, 2, 1)$
    (c) $(e^{3t}, e^{-3t}, 3\sqrt{2}\, t)$ at $t = 1$
    (d) (t, t^3, t^4) at the point $(1, 1, 1)$

17. Let A, B be fixed non-zero vectors. Let

$$X(t) = e^{2t}A + e^{-2t}B.$$

Show that $X''(t)$ has the same direction as $X(t)$.

18. Show that the two curves $(e^t, e^{2t}, 1 - e^{-t})$ and $(1 - \theta, \cos\theta, \sin\theta)$ intersect at the point $(1, 1, 0)$. What is the angle between their tangents at that point?

19. At what points does the curve $(2t^2, 1 - t, 3 + t^2)$ intersect the plane

$$3x - 14y + z - 10 = 0?$$

20. Let $X(t)$ be a differentiable curve.
    (a) Suppose that $X'(t) = O$ for all t throughout its interval of definition I. What can you say about the curve?
    (b) Suppose $X'(t) \neq O$ but $X''(t) = O$ for all t in the interval. What can you say about the curve?

21. Let $X(t) = (a\cos t, a\sin t, bt)$, where a, b are constant. Let $\theta(t)$ be the angle which the tangent line at a given point of the curve makes with the z-axis. Show that $\cos\theta(t)$ is the constant $b/\sqrt{a^2 + b^2}$.

22. Show that the velocity and acceleration vectors of the curve in Exercise 21 have constant norms (magnitudes).

23. Let B be a fixed unit vector, and let $X(t)$ be a curve such that $X(t)\cdot B = e^{2t}$ for all t. Assume also that the velocity vector of the curve has a constant angle θ with the vector B, with $0 < \theta < \pi/2$.
    (a) Show that the speed is $2e^{2t}/\cos\theta$.
    (b) Determine the dot product $X'(t)\cdot X''(t)$ in terms of t and θ.

24. Let

$$X(t) = \left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2}, 1\right).$$

Show that the cosine of the angle between $X(t)$ and $X'(t)$ is constant.

25. Suppose that a bug moves along a differentiable curve $B(t) = (x(t), y(t), z(t))$, lying in the surface $z^2 = 1 + x^2 + y^2$. (This means that the coordinates (x, y, z) of the curve satisfy this equation.)

(a) Show that

$$2x(t)x'(t) = B(t)\cdot B'(t).$$

(b) Assume that the cosine of the angle between the vector $B(t)$ and the velocity vector $B'(t)$ is always positive. Show that the distance of the bug to the yz-plane increases whenever its x-coordinate is positive.

26. A bug is moving in space on a curve given by

$$X(t) = (t,\ t^2,\ \tfrac{2}{3}t^3),$$

(a) Find a parametric representation of the tangent line at $t = 1$.
(b) Write the equation of the normal plane to the curve at $t = 1$.

27. Let a particle move in the plane so that its position at time t is

$$C(t) = (e^t \cos t,\ e^t \sin t).$$

Show that the tangent vector to the curve makes a constant angle of $\pi/4$ with the position vector.

XVI, §2. LENGTH OF CURVES

Suppose a bug travels along a curve $X(t)$. The rate of change of the distance traveled is equal to the speed, so we may write the equation

$$\frac{ds(t)}{dt} = v(t).$$

Consequently it is reasonable to make the following definition.

We define the **length** of a curve X between two values a, b of t ($a \leqq b$) in the interval of definition of the curve to be the integral of the speed:

$$\int_b^a v(t)\,dt = \int_a^b \|X'(t)\|\,dt.$$

By definition, we can rewrite this integral in the form

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad \text{when} \qquad X(t) = (x(t), y(t)),$$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \qquad \text{when} \qquad X(t) = (x(t), y(t), z(t)),$$

$$\int_a^b \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}\, dt \qquad \text{when} \qquad X(t) = (x_1(t), \ldots x_n(t)).$$

Example 1. Let the curve be defined by

$$X(t) = (\sin t,\ \cos t).$$

Then $X'(t) = (\cos t, -\sin t)$ and $v(t) = \sqrt{\cos^2 t + \sin^2 t} = 1$. Hence the length of the curve between $t = 0$ and $t = 1$ is

$$\int_0^1 v(t)\, dt = t \Big|_0^1 = 1.$$

In this case, of course, the integral is easy to evaluate. There is no reason why this should always be the case.

Example 2. Set up the integral for the length of the curve

$$X(t) = (e^t,\ \sin t,\ t)$$

between $t = 1$ and $t = \pi$.

We have $X'(t) = (e^t, \cos t, 1)$. Hence the desired integral is

$$\int_1^\pi \sqrt{e^{2t} + \cos^2 t + 1}\, dt.$$

In this case, there is no easy formula for the integral. In the exercises, however, the functions are adjusted in such a way that the integral can be evaluated by elementary techniques of integration. Don't expect this to be the case in real life, though. The presence of the square root sign usually makes it impossible to evaluate the length integral by elementary functions.

XVI, §2. EXERCISES

1. Find the length of the spiral $(\cos t, \sin t, t)$ between $t = 0$ and $t = 1$.

2. Find the length of the spirals.
   (a) $(\cos 2t, \sin 2t, 3t)$ between $t = 1$ and $t = 3$.
   (b) $(\cos 4t, \sin 4t, t)$ between $t = 0$ and $t = \pi/8$.

3. Find the length of the indicated curve for the given interval:
   (a) $(t, 2t, t^2)$ between $t = 1$ and $t = 3$. [*Hint*: You will get at some point the integral $\int \sqrt{1 + u^2}\, du$. The easiest way of handling that is to let

$$u = \frac{e^t - e^{-t}}{2} = \sinh t, \qquad \text{so} \qquad 1 + \sinh^2 t = \cosh^2 t,$$

where

$$\cosh t = \frac{e^t + e^{-t}}{2}.$$

This makes the expression under the square root sign into a perfect square. This method will in fact prove the general formula

$$\boxed{\int \sqrt{a^2 + x^2}\, dx = \frac{1}{2}\left[x\sqrt{a^2 + x^2} + a^2 \log(x + \sqrt{a^2 + x^2})\right].}$$

Of course, you can check the formula by differentiating the right-hand side, and just use it for the exercise.
   (b) $(e^{3t}, e^{-3t}, 3\sqrt{2}\, t)$ between $t = 0$ and $t = \frac{1}{3}$.

   [*Hint*: At some point you will meet a square root.

$$\sqrt{e^{6t} + e^{-6t} + 2}.$$

   The expression under the square root is a perfect square. Try squaring $(e^{3t} + e^{-3t})$. What do you get?]
4. Find the length of the curve defined by

$$X(t) = (t - \sin t,\ 1 - \cos t)$$

between (a) $t = 0$ and $t = 2\pi$, (b) $t = 0$ and $t = \pi/2$.

[*Hint*: Remember the identity

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}.$$

Therefore letting $t = 2\theta$ gives

$$1 - \cos t = 2 \sin^2(t/2).$$

The expression under the integral sign will then be a perfect square.]

5. Find the length of the curve $X(t) = (t, \log t)$ between:
   (a) $t = 1$ and $t = 2$, (b) $t = 3$ and $t = 5$. [*Hint*: Substitute $u^2 = 1 + t^2$ to evaluate the integral. Use partial fractions.]

6. Find the length of the curve defined by $X(t) = (t, \log \cos t)$ between $t = 0$ and $t = \pi/4$.

7. Let $X(t) = (t, t^2, \frac{2}{3}t^3)$.
   (a) Find the speed of this curve.
   (b) Find the length of the curve between $t = 0$ and $t = 1$.

8. Let $X(t) = (6t, 2t^3, 3\sqrt{2}\, t^2)$. Find the length of the curve between $t = 0$ and $t = 1$.

CHAPTER XVII

Functions of Several Variables

We view functions of several variables as functions of points in space. This appeals to our geometric intuition, and also relates such functions more easily with the theory of vectors. The gradient will appear as a natural generalization of the derivative. In this chapter we are mainly concerned with basic definitions and notions. We postpone the important theorems to the next chapter.

XVII, §1. GRAPHS AND LEVEL CURVES

In order to conform with usual terminology, and for the sake of brevity, a collection of objects will simply be called a **set**. In this chapter, we are mostly concerned with sets of points in space.

Let S be a set of points in n-space. A **function** (defined on S) is an association which to each element of S associates a **number**. For instance, if to each point we associate the numerical value of the temperature at that point, we have the temperature function.

Remark. In the previous chapter, we considered parametrized curves, associating a vector to a point. We do **not** call these functions. Only when the values of the association are **numbers** do we use the word **function**. We find this to be the most useful convention for this course.

In practice, we sometimes omit mentioning explicitly the set S, since the context usually makes it clear for which points the function is defined.

Example 1. In 2-space (the plane) we can define a function f by the rule

$$f(x, y) = x^2 + y^2.$$

It is defined for all points (x, y) and can be interpreted geometrically as the square of the distance between the origin and the point.

Example 2. Again in 2-space, we can define a function f by the formula

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \qquad \text{for all} \quad (x, y) \neq (0, 0).$$

We do not define f at $(0, 0)$ (also written O).

Example 3. In 3-space, we can define a function f by the rule

$$f(x, y, z) = x^2 - \sin(xyz) + yz^3.$$

Since a point and a vector are represented by the same thing (namely an n-tuple), we can think of a function such as the above also as a function of vectors. When we do not want to write the coordinates, we write $f(X)$ instead of $f(x_1, \ldots, x_n)$. As with numbers, we call $f(X)$ the **value** of f at the point (or vector) X.

Just as with functions of one variable, we define the **graph** of a function f of n variables $x_1, \ldots, x_n$ to be the set of points in $(n + 1)$-space of the form

$$(x_1, \ldots, x_n, f(x_1, \ldots, x_n)),$$

the $(x_1, \ldots, x_n)$ being in the domain of definition of f.

When $n = 1$, the graph of a function f is a set of points $(x, f(x))$. Thus the graph itself is in 2-space.

When $n = 2$, the graph of a function f is the set of points

$$(x, y, f(x, y)).$$

When $n = 2$, it is already difficult to draw the graph since it involves a figure in 3-space. The graph of a function of two variables may look like this:

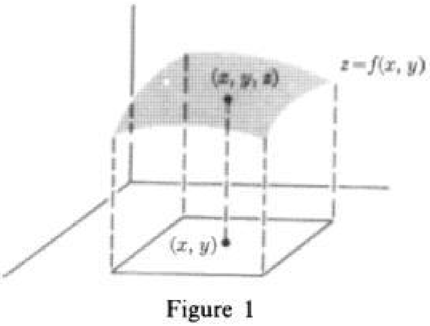


Figure 1

For each number c, the equation $f(x, y) = c$ is the equation of a curve in the plane. We have considerable experience in drawing the graphs of such curves, and we may therefore assume that we know how to draw this graph in principle. This curve is called the **level curve** of f at c. It gives us the set of points (x, y) where f takes on the value c. By drawing a number of such level curves, we can get a good description of the function.

Example 4. Let $f(x, y) = x^2 + y^2$. The level curves are described by equations

$$x^2 + y^2 = c.$$

These have a solution only when $c \geqq 0$. In that case, they are circles (unless $c = 0$ in which case the circle of radius 0 is simply the origin). In Fig. 2, we have drawn the level curves for $c = 1$ and 4.

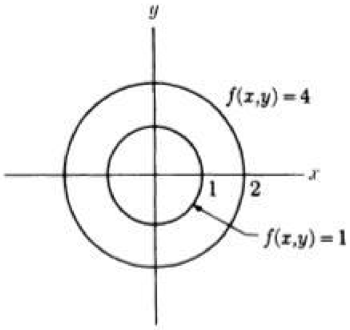


Figure 2

The graph of the function $z = f(x, y) = x^2 + y^2$ is then a figure in 3-space, which we may represent as follows.

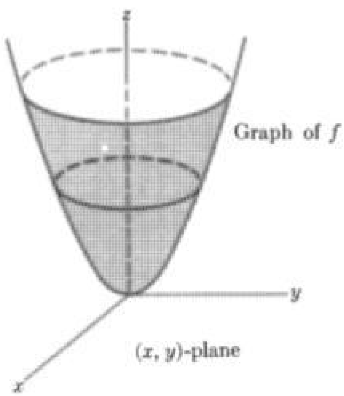


Figure 3

Example 5. Let the elevation of a mountain in meters be given by the formula

$$f(x, y) = 4{,}000 - 2x^2 - 3y^4.$$

We see that $f(0, 0) = 4{,}000$ is the highest point of the mountain. As x, y increase, the altitude decreases. The mountain and its level curves might look like this.

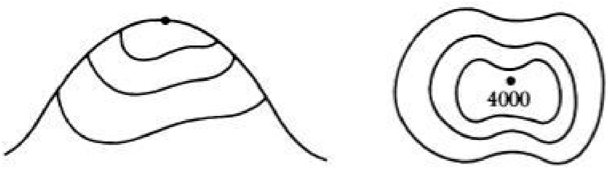


Figure 4

In this case, the highest point is at the origin, and the level curves indicate decreasing altitude as they move away from the origin.

If we deal with a function of three variables, say $f(x, y, z)$, then $(x, y, z) = X$ is a point in 3-space. In that case, the set of points satisfying the equation

$$f(x, y, z) = c$$

for some constant c is a surface. The notion analogous to that of level curve is that of **level surface**.

Example 6. Let $f(x, y, z) = x^2 + y^2 + z^2$. Then f is the square of the distance from the origin. The equation

$$x^2 + y^2 + z^2 = c$$

is the equation of a sphere for $c > 0$, and the radius is of course $\sqrt{c}$. If $c = 0$ this is the equation of a point, namely the origin itself. If $c < 0$ there is no solution. Thus the level surfaces for the function f are spheres.

Example 7. Let $f(x, y, z) = 3x^2 + 2y^2 + z^2$. Then the level surfaces for f are defined by the equations

$$3x^2 + 2y^2 + z^2 = c.$$

They have the same shape as ellipses, and called ellipsoids, for $c > 0$.

It is harder to draw figures in 3 dimensions than in 2 dimensions, so we restrict ourselves to drawing level curves.

The graph of a function of three variables is the set of points

$$(x, y, z, f(x, y, z))$$

in 4-dimensional space. Not only is this graph hard to draw, it is impossible to draw. It is, however, possible to define it as we have done by writing down coordinates of points.

In physics, a function f might be a potential function, giving the value of the potential energy at each point of space. The level surfaces are then sometimes called surfaces of **equipotential**. The function f might also give a temperature distribution (i.e. its value at a point X is the temperature at X). In that case, the level surfaces are called **isothermal** surfaces.

XVII, §1. EXERCISES

Sketch the level curves for the functions $z = f(x, y)$, where $f(x, y)$ is given by the following expressions.

1. $x^2 + 2y^2$
2. $y - x^2$
3. $y - 3x^2$
4. $x - y^2$
5. $3x^2 + 3y^2$
6. xy
7. $(x - 1)(y - 2)$
8. $(x + 1)(y + 3)$
9. $\dfrac{x^2}{4} + \dfrac{y^2}{16}$
10. $2x - 3y$
11. $\sqrt{x^2 + y^2}$
12. $x^2 - y^2$
13. $y^2 - x^2$
14. $(x - 1)^2 + (y + 3)^2$
15. $(x + 1)^2 + y^2$

XVII, §2. PARTIAL DERIVATIVES

In this section and the next, we discuss the notion of differentiability for functions of several variables. When we discussed the derivative of functions of one variable, we assumed that such a function was defined on an interval. We shall have to make a similar assumption in the case of several variables, and for this we need to introduce a new notion.

Let U be a set in the plane. We shall say that U is an **open set** if the following condition is satisfied. Given a point P in U, there exists an open disc D of radius $a > 0$ which is centered at P and such that D is contained in U.

Let U be a set in space. We shall say that U is an **open set** in space if given a point P in U, there exists an open ball B of radius $a > 0$ which is centered at P and such that B is contained in U.

A similar definition is given of an open set in n-space.

Given a point P in an open set, we can go in all directions from P by a small distance and still stay within the open set.

Example 1. In the plane, the set consisting of the first quadrant, excluding the x- and y-axes, is an open set.

The x-axis is not open in the plane (i.e. in 2-space). Given a point on the x-axis, we cannot find an open disc centered at the point and contained in the x-axis.

Example 2. Let U be the open ball of radius $a > 0$ centered at the origin. Then U is an open set. This is illustrated on Fig. 5.

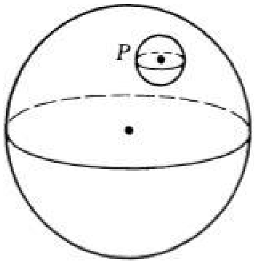


Figure 5

In the next picture we have drawn an open set in the plane, consisting of the region inside the curve, but not containing any point of the boundary. We have also drawn a point P in U, and a ball (disc) around P contained in U.

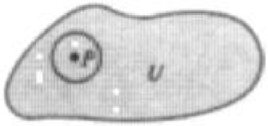


Figure 6

When we defined the derivative as a limit of

$$\frac{f(x+h) - f(x)}{h},$$

we needed the function f to be defined in some open interval around the point x.

Now let f be a function of n variables, defined on an open set U. Then for any point X in U, the function f is also defined at all points which are close to X, namely all points which are contained in an open ball centered at X and contained in U. We shall obtain the partial derivative of f by keeping all but one variable fixed, and taking the ordinary derivative with respect to the one variable.

Let us start with two variables. Given a function $f(x, y)$ of two variables x, y, let us keep y constant and differentiate with respect to x. We are then led to consider the limit as h approaches 0 of

$$\frac{f(x + h, y) - f(x, y)}{h}$$

Definition. If this limit exists, we call it the **derivative** of f **with respect to the first variable**, or also the **first partial derivative** of f, and denote it by

$$(D_1 f)(x, y).$$

This notation allows us to use any letters to denote the variables. For instance,

$$\lim_{h \to 0} \frac{f(u + h, v) - f(u, v)}{h} = D_1 f(u, v).$$

Note that $D_1 f$ is a single function. We often omit the parentheses, writing

$$D_1 f(u, v) = (D_1 f)(u, v)$$

for simplicity.

Also, if the variables x, y are agreed upon, then we write

$$\boxed{D_1 f(x, y) = \frac{\partial f}{\partial x}.}$$

Similarly, we define

$$\boxed{D_2 f(x, y) = \lim_{h \to 0} \frac{f(x, y + k) - f(x, y)}{h}}$$

and also write

$$\boxed{D_2 f(x, y) = \frac{\partial f}{\partial y}.}$$

Example 3. Let $f(x, y) = x^2 y^3$. Then

$$\frac{\partial f}{\partial x} = 2xy^3 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 3x^2 y^2.$$

We observe that the partial derivatives are themselves functions. This is the reason why the notation $D_i f$ is sometimes more useful than the notation $\partial f/\partial x_i$. It allows us to write $D_i f(P)$ for any point P in the set where the partial is defined. There cannot be any ambiguity or confusion with a (meaningless) symbol $D_i(f(P))$, since $f(P)$ is a number. Thus $D_i f(P)$ means $(D_i f)(P)$. It is the value of the function $D_i f$ at P.

Example 4. Let $f(x, y) = \sin xy$. To find $D_2 f(1, \pi)$, we first find $\partial f/\partial y$, or $D_2 f(x, y)$, which is simply

$$D_2 f(x, y) = (\cos xy)x.$$

Hence

$$D_2 f(1, \pi) = (\cos \pi) \cdot 1 = -1.$$

Also,

$$D_2 f\left(3, \frac{\pi}{4}\right) = \left(\cos \frac{3\pi}{4}\right) \cdot 3 = -\frac{1}{\sqrt{2}} \cdot 3 = -\frac{3}{\sqrt{2}}.$$

A similar definition of the partial derivatives is given in 3-space. Let f be a function of three variables (x, y, z), defined on an open set U in 3-space. We define, for instance,

$$(D_3 f)(x, y, z) = \frac{\partial f}{\partial z} = \lim_{h \to 0} \frac{f(x, y, z + h) - f(x, y, z)}{h},$$

and similarly for the other variables.

Example 5. Let $f(x, y, z) = x^2 y \sin(yz)$. Then

$$D_3 f(x, y, z) = \frac{\partial f}{\partial z} = x^2 y \cos(yz) y = x^2 y^2 \cos(yz).$$

Let $X = (x, y, z)$ for abbreviation. Let

$$E_1 = (1, 0, 0), \qquad E_2 = (0, 1, 0), \qquad E_3 = (0, 0, 1)$$

be the three standard unit vectors in the directions of the coordinate axes. Then we can abbreviate the Newton quotient for the partial derivatives by writing

$$D_i f(X) = \frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(X + hE_i) - f(X)}{h}.$$

Indeed, observe that

$$hE_1 = (h, 0, 0) \qquad \text{so} \qquad f(X + hE_1) = f(x + h, y, z),$$

and similarly for the other two variables.

In a similar fashion we can define the partial derivatives in n-space, by a definition which applies simultaneously to 2-space and 3-space. Let f be a function defined on an open set U in n-space. Let the variables be $(x_1, \dots, x_n)$.

For small values of h, the point

$$(x_1 + h, x_2, \dots, x_n)$$

is contained in U. Hence the function is defined at that point, and we may form the quotient

$$\frac{f(x_1 + h, x_2, \dots, x_n) - f(x_1, \dots, x_n)}{h}.$$

If the limit exists as h tends to 0, then we call it the **first partial derivative** of f and denote it by

$$\boxed{D_1 f(x_1, \dots, x_n), \quad \text{or} \quad D_1 f(X), \quad \text{or also by} \quad \frac{\partial f}{\partial x_1}.}$$

Similarly, we let

$$\begin{aligned} D_i f(X) &= \frac{\partial f}{\partial x_i} \\ &= \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h} \end{aligned}$$

if it exists, and call it the i-th **partial derivative**.

Let

$$E_i = (0, \dots, 0, 1, 0, \dots, 0)$$

be the i-th vector in the direction of the i-th coordinate axis, having components equal to 0 except for the i-th component which is 1. Then we have

$$(D_i f)(X) = \lim_{h \to 0} \frac{f(X + hE_i) - f(X)}{h}.$$

This is a very useful brief notation which applies simultaneously to 2-space, 3-space, or n-space.

Definition. Let f be a function of two variables (x, y). We define the **gradient** of f, written **grad f**, to be the vector

$$\boxed{\operatorname{grad} f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).}$$

Example 6. Let $f(x, y) = x^2y^3$. Then

$$\operatorname{grad} f(x, y) = (2xy^3, 3x^2y^2),$$

so that in this case,

$$\operatorname{grad} f(1, 2) = (16, 12).$$

Thus the gradient of a function f associates a **vector** to a point X.

If f is a function of three variables (x, y, z), then we define the gradient to be

$$\boxed{\operatorname{grad} f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).}$$

Example 7. Let $f(x, y, z) = x^2y \sin(yz)$. Find grad $f(1, 1, \pi)$. First we find the three partial derivatives, which are:

$$\frac{\partial f}{\partial x} = 2xy \sin(yz),$$

$$\frac{\partial f}{\partial y} = x^2[y \cos(yz)z + \sin(yz)],$$

$$\frac{\partial f}{\partial z} = x^2y \cos(yz)y = x^2y^2 \cos(yz).$$

We then substitute $(1, 1, \pi)$ for (x, y, z) in these partials, and get

$$\operatorname{grad} f(1, 1, \pi) = (0, -\pi, -1).$$

Let f be defined in an open set U in n-space, and assume that the partial derivatives of f exist at each point X of U. We define the **gradient of f at X** to be the vector

$$\operatorname{grad} f(X) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = (D_1 f(X), \ldots, D_n f(X)),$$

whose components are the partial derivatives. One must read this

$$(\operatorname{grad} f)(X),$$

but we shall usually omit the parentheses around grad f. Sometimes one also writes ∇f instead of grad f. Thus in 2-space we also write

$$\nabla f(x, y) = (\nabla f)(x, y) = (D_1 f(x, y), D_2 f(x, y)),$$

and similarly in 3-space,

$$\nabla f(x, y, z) = (\nabla f)(x, y, z) = (D_1 f(x, y, z), D_2 f(x, y, z), D_3 f(x, y, z)).$$

So far, we defined the gradient only by a formula with partial derivatives. We shall give a geometric interpretation for the gradient in Chapter XVIII, §3. There we shall see that it gives the direction of maximal increase of the function, and that its magnitude is the rate of increase in that direction.

Using the formula for the derivative of a sum of two functions, and the derivative of a constant times a function, we conclude at once that the gradient satisfies the following properties:

Theorem 2.1. *Let f, g be two functions defined on an open set U, and assume that their partial derivatives exist at every point of U. Let c be a number. Then*

$$\operatorname{grad}(f + g) = \operatorname{grad} f + \operatorname{grad} g,$$

$$\operatorname{grad}(cf) = c \operatorname{grad} f.$$

We shall give later several geometric and physical interpretations for the gradient.

XVII, §2. EXERCISES

Find the partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \text{and} \quad \frac{\partial f}{\partial z},$$

for the following functions $f(x, y)$ or $f(x, y, z)$.

1. $xy + z$ 2. $x^2y^5 + 1$ 3. $\sin(xy) + \cos z$

4. $\cos(xy)$ 5. $\sin(xyz)$ 6. e^{xyz}

7. $x^2 \sin(yz)$ 8. xyz 9. $xz + yz + xy$

10. $x \cos(y - 3z) + \arcsin(xy)$

11. Find grad $f(P)$ if P is the point $(1, 2, 3)$ in Exercises 1, 2, 6, 8, and 9.

12. Find grad $f(P)$ if P is the point $(1, \pi, \pi)$ in Exercises 4, 5, 7.

13. Find grad $f(P)$ if

$$f(x, y, z) = \log(z + \sin(y^2 - x))$$

and

$$P = (1, -1, 1).$$

14. Find the partial derivatives of x^y. [*Hint*: $x^y = e^{y \log x}$.]

Find the gradient of the following functions at the given point.

15. $f(x, y, z) = e^{-2x} \cos(yz)$ at $(1, \pi, \pi)$

16. $f(x, y, z) = e^{3x+y} \sin(5z)$ at $(0, 0, \pi/6)$

XVII, §3. DIFFERENTIABILITY AND GRADIENT

Let f be a function defined on an open set U. Let X be a point of U. For all vectors H such that $\|H\|$ is small (and $H \neq O$), the point $X + H$ also lies in the open set. However, we **cannot** form a quotient

$$\frac{f(X + H) - f(x)}{H}$$

because it is **meaningless to divide by a vector**. In order to define what we mean for a function f to be differentiable, we must therefore find a way which does not involve dividing by H.

We reconsider the case of functions of one variable. Let us fix a number x. We had defined the derivative to be

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Let

$$\varphi(h) = \frac{f(x + h) - f(x)}{h} - f'(x).$$

Then $\varphi(h)$ is not defined when $h = 0$, but

$$\lim_{h \to 0} \varphi(h) = 0.$$

We can write

$$f(x + h) - f(x) = f'(x)h + h\varphi(h).$$

This relation has meaning so far only when $h \neq 0$. However, we observe that if we define $\varphi(0)$ to be 0, then the preceding relation is obviously true when $h = 0$ (because we just get $0 = 0$).

Let

$$g(h) = \varphi(h) \quad \text{if} \quad h > 0,$$

$$g(h) = -\varphi(h) \quad \text{if} \quad h < 0.$$

Then we have shown that if f is differentiable, there exists a function g such that

$$f(x + h) - f(x) = f'(x)h + |h|g(h), \tag{1}$$

and

$$\lim_{h \to 0} g(h) = 0.$$

Conversely, suppose that there exists a number a and a function $g(h)$ such that

$$f(x + h) - f(x) = ah + |h|g(h). \tag{1a}$$

and

$$\lim_{h \to 0} g(h) = 0.$$

We find for $h \neq 0$,

$$\frac{f(x + h) - f(x)}{h} = a + \frac{|h|}{h} g(h).$$

Taking the limit as h approaches 0, we observe that

$$\lim_{h \to 0} \frac{|h|}{h} g(h) = 0.$$

Hence the limit of the Newton quotient exists and is equal to a. Hence f is differentiable, and its derivative $f'(x)$ is equal to a.

Therefore, the existence of a number a and a function g satisfying (1a) above could have been used as the definition of differentiability in the case of functions of one variable. The great advantage of (1) is that no h appears in the denominator. It is this relation which will suggest to us how to define differentiability for functions of several variables, and how to prove the chain rule for them.

Let us begin with two variables. We let

$$X = (x, y) \qquad \text{and} \qquad H = (h, k).$$

Then the notion corresponding to $x + h$ in one variable is here

$$X + H = (x + h, y + k).$$

We wish to compare the values of a function f at X and $X + H$, i.e. we wish to investigate the difference

$$f(X + H) - f(X) = f(x + h, y + k) - f(x, y).$$

Definition. We say that f is **differentiable** at X if the partial derivatives

$$\frac{\partial f}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial y}$$

exist, and if there exists a function g (defined for small H) such that

$$\lim_{H \to O} g(H) = 0$$

and

$$\boxed{f(x + h, y + k) - f(x, y) = \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k + \|H\| g(H).} \tag{2}$$

We view the term

$$\frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k$$

as an approximation to $f(X + H) - f(X)$, depending in a particularly simple way on h and k.

If we use the abbreviation

$$\operatorname{grad} f = \nabla f,$$

then formula (2) can be written

$$\boxed{f(X + H) - f(X) = \nabla f(x) \cdot H + \|H\| g(H).}$$

As with grad f, one must read $(\nabla f)(X)$ and not the meaningless $\nabla(f(X))$ since $f(X)$ is a number for each value of X, and thus it makes no sense to apply ∇ to a number. The symbol ∇ is applied to the function f, and $(\nabla f)(X)$ is the value of ∇f at X.

We now consider a function of n variables.

Let f be a function defined on an open set U. Let X be a point of U. If $H = (h_1, \ldots, h_n)$ is a vector such that $\|H\|$ is small enough, then $X + H$ will also be a point of U and so $f(X + H)$ is defined. Note that

$$X + H = (x_1 + h_1, \ldots, x_n + h_n).$$

This is the generalization of the $x + h$ with which we dealt previously in one variable, or the $(x + h,\ y + k)$ in two variables. For three variables, we already run out of convenient letters, so we may as well write n instead of 3.

Definition. We say that f is **differentiable** at X if the partial derivatives $D_1 f(X), \ldots, D_n f(X)$ exist, and if there exists a function g (defined for small H) such that

$$\lim_{H \to O} g(H) = 0 \qquad \left(\text{also written} \quad \lim_{\|H\| \to 0} g(H) = 0\right)$$

and

$$f(X + H) - f(X) = D_1 f(X) h_1 + \cdots + D_n f(x) h_n + \|H\| g(H).$$

With the other notation for partial derivatives, this last relation reads:

$$f(X + H) - f(X) = \frac{\partial f}{\partial x_1} h_1 + \cdots + \frac{\partial f}{\partial x_n} h_n + \|H\| g(H).$$

We say that f is **differentiable** in the open set U if it is differentiable at every point of U, so that the above relation holds for every point X in U.

In view of the definition of the gradient in §2, we can rewrite our fundamental relation in the form

$$(3) \qquad \boxed{f(X + H) - f(X) = (\operatorname{grad} f(X)) \cdot H + \|H\| g(H).}$$

The term $\|H\| g(H)$ has an order of magnitude smaller than the previous term involving the dot product. This is one advantage of the present notation. We know how to handle the formalism of dot products and are accustomed to it, and its geometric interpretation. This will help us later in interpreting the gradient geometrically.

Example 1. Suppose that we consider values for H pointing only in the direction of the standard unit vectors. In the case of two variables, consider for instance $H = (h, 0)$. Then for such H, the condition for differentiability reads:

$$f(X + H) = f(x + h, y) = f(x, y) + \frac{\partial f}{\partial x} h + |h| g(H).$$

In higher dimensional space, let $E_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the i-th unit vector. Let $H = hE_i$ for some number h, so that

$$H = (0, \ldots, 0, h, 0, \ldots, 0).$$

Then for such H,

$$f(X + H) = f(X + hE_i) = f(X) + \frac{\partial f}{\partial x_i} h + |h| g(H),$$

and therefore if $h \neq 0$, we obtain

$$\frac{f(X + H) - f(x)}{h} = D_i f(X) + \frac{|h|}{h} g(H).$$

Because of the special choice of H, we can divide by the *number* h, but we are *not* dividing by the vector H.

The functions which we meet in practice are differentiable. The next theorem gives a criterion which shows that this is true. A function $\varphi(X)$ is said to be **continuous** if

$$\lim_{H \to O} \varphi(X + H) = \varphi(X),$$

for all X in the domain of definition of the function.

Theorem 3.1. *Let f be a function defined on some open set U. Assume that its partial derivatives exist for every point in this open set, and that they are continuous. Then f is differentiable.*

We shall omit the proof. Observe that in practice, the partial derivatives of a function are given by formulas from which it is clear that they are continuous.

XVII, §3. EXERCISES

1. Let $f(x, y) = 2x - 3y$. What is $\partial f/\partial x$ and $\partial f/\partial y$?

2. Let $A = (a, b)$ and let f be the function on $\mathbf{R}^2$ such that $f(X) = A \cdot X$. Let $X = (x, y)$. In terms of the coordinates of A, determine $\partial f/\partial x$ and $\partial f/\partial y$.

3. Let $A = (a, b, c)$ and let f be the function on $\mathbf{R}^3$ such that $f(X) = A \cdot X$. Let $X = (x, y, z)$. In terms of the coordinates of A, determine $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

4. Generalize the above two exercises to n-space.

5. Let f be defined on an open set U. Let X be a point of U. Let A be a vector, and let g be a function defined for small H, such that

$$\lim_{H \to O} g(H) = 0.$$

Assume that

$$f(X + H) - f(X) = A \cdot H + \|H\| g(H).$$

Prove that $A = \operatorname{grad} f(X)$. You may do this exercise in 2 variables first and then in 3 variables, and let it go at that. Use coordinates, e.g. let $A = (a, b)$ and $X = (x, y)$. Use special values of H, as in Example 1.

CHAPTER XVIII

The Chain Rule and the Gradient

In this chapter, we prove the chain rule for functions of several variables and give a number of applications. Among them will be several interpretations for the gradient. These form one of the central points of our theory. They show how powerful the tools we have accumulated turn out to be.

XVIII, §1. THE CHAIN RULE

Let f be a function defined on some open set U. Let $C(t)$ be a curve such that the values $C(t)$ are contained in U. Then we can form the composite function $f \circ C$, which is a function of t, given by

$$(f \circ C)(t) = f(C(t)).$$

Example 1. Take $f(x, y) = e^x \sin(xy)$. Let $C(t) = (t^2, t^3)$. Then

$$f(C(t)) = e^{t^2} \sin(t^5).$$

The expression on the right is obtained by substituting t^2 for x and t^3 for y in $f(x, y)$. This is a function of t in the old sense of functions of one variable. If we interpret f as the temperature, then $f(C(t))$ is the temperature of a bug traveling along the curve $C(t)$ at time t.

The chain rule tells us how to find the derivative of this function, provided we know the gradient of f and the derivative C'. Its statement is as follows.

Chain rule. *Let f be a function which is defined and differentiable on an open set U. Let C be a differentiable curve (defined for some interval of numbers t) such that the values $C(t)$ lie in the open set U. Then the function*

$$f(C(t))$$

is differentiable (as a function of t), and

$$\boxed{\frac{df(C(t))}{dt} = (\operatorname{grad} f(C(t))) \cdot C'(t).}$$

Memorize this formula by repeating it out loud.

In the notation dC/dt, this also reads

$$\frac{df(C(t))}{dt} = (\operatorname{grad} f)(C(t)) \cdot \frac{dC}{dt}.$$

Proof of the Chain Rule. By definition, we must investigate the quotient

$$\frac{f(C(t+h)) - f(C(t))}{h}.$$

Let

$$K = K(t, h) = C(t+h) - C(t).$$

Then our quotient can be rewritten in the form

$$\frac{f(C(t) + K) - f(C(t))}{h}.$$

Using the definition of differentiability for f, we have

$$f(X + K) - f(X) = (\operatorname{grad} f)(X) \cdot K + \|K\| g(K)$$

and

$$\lim_{\|K\| \to 0} g(K) = 0.$$

Replacing K by what it stands for, namely $C(t+h) - C(t)$, and dividing by h, we obtain:

$$\frac{f(C(t+h)) - f(C(t))}{h} = (\operatorname{grad} f)(C(t)) \cdot \frac{C(t+h) - C(t)}{h}$$

$$\pm \left\| \frac{C(t+h) - C(t)}{h} \right\| g(K).$$

As h approaches 0, the first term of the sum approaches what we want, namely

$$(\operatorname{grad} f)(C(t)) \cdot C'(t).$$

The second term approaches

$$\pm \|C'(t)\| \lim_{h \to 0} g(K),$$

and when h approaches 0, so does $K = C(t+h) - C(t)$. Hence the second term of the sum approaches 0. This proves our chain rule.

To use the chain rule for certain computations, it is convenient to reformulate it in terms of components, and in terms of the two notations we have used for partial derivatives

$$\frac{\partial f}{\partial x} = D_1 f(x, y), \qquad \frac{\partial f}{\partial y} = D_2 f(x, y)$$

when the variables are x, y.

Suppose $C(t)$ is given in terms of coordinates by

$$C(t) = (x_1(t), \ldots, x_n(t)),$$

then

$$\boxed{\frac{d(f(C(t)))}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}.}$$

If f is a function of two variables (x, y) then

$$\boxed{\frac{df(C(t))}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.}$$

In the D_1, D_2 notation, we can write this formula in the form

$$\boxed{\frac{d}{dt}(f(x(t), y(t))) = (D_1 f)(x, y)\frac{dx}{dt} + (D_2 f)(x, y)\frac{dy}{dt},}$$

and similarly for several variables. For simplicity we usually omit the parentheses around $D_1 f$ and $D_2 f$. Also on the right-hand side we have abbreviated $x(t)$, $y(t)$ to x, y, respectively. Without any abbreviation, the formula reads:

$$\frac{d}{dt}(f(x(t), y(t))) = D_1 f(x(t), y(t))\frac{dx}{dt} + D_2 f(x(t), y(t))\frac{dy}{dt}.$$

Example 2. Let $C(t) = (e^t, t, t^2)$ and let $f(x, y, z) = x^2yz$. Then putting

$$x = e^t, \qquad y = t, \qquad z = t^2$$

we get:

$$\begin{aligned}\frac{d}{dt}f(C(t)) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\\ &= 2xyze^t + x^2z + x^2y2t.\end{aligned}$$

If we want this function entirely in terms of t, we substitute back the values for x, y, z in terms of t, and get

$$\begin{aligned}\frac{d}{dt}f(C(t)) &= 2e^t tt^2e^t + e^{2t}t^2 + e^{2t}t2t\\ &= 2t^3e^{2t} + t^2e^{2t} + 2t^2e^{2t}.\end{aligned}$$

In some cases, as in the next example, one does not use the chain rule in several variables, just the old one from one-variable calculus.

Example 3. Let

$$f(x, y, z) = \sin(x^2 - 3zy + xz).$$

Then keeping y and z constant, and differentiating with respect to x, we find

$$\frac{\partial f}{\partial x} = \cos(x^2 - 3zy + xz)\cdot(2x + z).$$

More generally, let

$$f(x, y, z) = g(x^2 - 3zy + xz),$$

where g is a differentiable function of one variable. [In the special case above, we have $g(u) = \sin u$.] Then the chain rule gives

$$\frac{\partial f}{\partial x} = g'(x^2 - 3zy + xz)(2x + z).$$

We denote the derivative of g by g' as usual. We do *not* write it as dg/dx, because x is a letter which is already occupied for other purposes. We could let

$$u = x^2 - 3zy + xz,$$

in which case it would be all right to write

$$\boxed{\frac{\partial f}{\partial x} = \frac{dg}{du}\frac{\partial u}{\partial x},}$$

and we would get the same answer as above.

XVIII, §1. EXERCISES

1. Let P, A be constant vectors. If $g(t) = f(P + tA)$, show that

$$g'(t) = (\operatorname{grad} f)(P + tA) \cdot A.$$

2. Suppose that f is a function such that

$$\operatorname{grad} f(1, 1, 1) = (5, 2, 1).$$

Let $C(t) = (t^2, t^{-3}, t)$. Find

$$\frac{d}{dt}(f(C(t))) \quad \text{at} \quad t = 1.$$

3. Let $f(x, y) = e^{9x+2y}$ and $g(x, y) = \sin(4x + y)$. Let C be a curve such that $C(0) = (0, 0)$. Given:

$$\frac{d}{dt}f(C(t))\bigg|_{t=0} = 2 \qquad \text{and} \qquad \frac{d}{dt}g(C(t))\bigg|_{t=0} = 1,$$

Find $C'(0)$.

4. (a) Let P be a constant vector. Let $g(t) = f(tP)$, where f is some differentiable function. What is $g'(t)$?
   (b) Let f be a differentiable function defined on all of space. Assume that $f(tP) = tf(P)$ for all numbers t and all points P. Show that for all P we have

$$f(P) = \operatorname{grad} f(O) \cdot P.$$

5. Let f be a differentiable function of two variables and assume that there is an integer $m \geqq 1$ such that

$$f(tx, ty) = t^m f(x, y)$$

for all numbers t and all x, y. Prove **Euler's relation**

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = mf(x, y).$$

[*Hint*: Let $C(t) = (tx, ty)$. Differentiate both sides of the given equation with respect to t, keeping x and y **constant**. Then put $t = 1$.]

6. Generalize Exercise 5 to n variables, namely let f be a differentiable function of n variables and assume that there exists an integer $m \geqq 1$ such that $f(tX) = t^m f(X)$ for all numbers t and all points X in $\mathbf{R}^n$. Show that

$$x_1\frac{\partial f}{\partial x_1} + \cdots + x_n\frac{\partial f}{\partial x_n} = mf(X),$$

which can also be written $X \cdot \operatorname{grad} f(X) = mf(X)$.

7. (a) Let $f(x, y) = (x^2 + y^2)^{1/2}$. Find $\partial f/\partial x$ and $\partial f/\partial y$.
   (b) Let $f(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$. Find $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

8. Let $r = (x_1^2 + \cdots + x_n^2)^{1/2}$. What is $\partial r/\partial x_i$?

9. Find the derivatives with respect to x and y of the following functions.
   (a) $\sin(x^3y + 2x^2)$ (b) $\cos(3x^2y - 4x)$
   (c) $\log(x^2y + 5y)$ (d) $(x^2y + 4x)^{1/2}$

XVIII, §2. TANGENT PLANE

We begin by an example analyzing a function along a curve where the values of the function are constant. This gives rise to a very important principle of perpendicularity.

Example 1. Let f be a function on $\mathbf{R}^3$. Let us interpret f as giving the temperature, so that at any point X in $\mathbf{R}^3$, the value of the function $f(X)$ is the temperature at X. Suppose that a bug moves in space along a differentiable curve, which we may denote in parametric form by

$$B(t).$$

Thus $B(t) = (x(t), y(t), z(t))$ is the position of the bug at time t. Let us assume that the bug starts from a point where it feels that the temperature is comfortable, and therefore that the temperature is constant along the path on which it moves. In other words, f is constant along the curve $B(t)$. This means that for all values of t, we have

$$f(B(t)) = k,$$

where k is constant. Differentiating with respect to t, and using the chain rule, we find that

$$\operatorname{grad} f(B(t)) \cdot B'(t) = 0.$$

This means that the gradient of f is perpendicular to the velocity vector at every point of the curve.

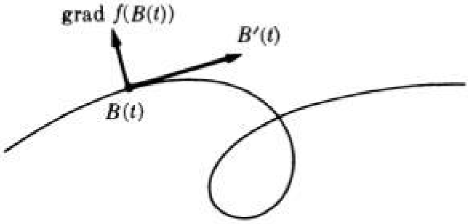


Figure 1

Let f be a differentiable function defined on an open set U in 3-space, and let k be a number. The set of points X such that

$$f(X) = k \quad \text{and} \quad \operatorname{grad} f(X) \neq O$$

is called a **surface**. It is the level surface of level k, for the function f. For the applications we have in mind, we impose the additional condition that grad $f(X) \neq O$. It can be shown that this eliminates the points where the surface is not smooth.

Let $C(t)$ be a differentiable curve. We shall say that the curve **lies on the surface** if, for all t, we have

$$f(C(t)) = k.$$

This simply means that all the points of the curve satisfy the equation of the surface. For instance, let the surface be defined by the equation

$$x^2 + y^2 + z^2 = 1.$$

The surface is the sphere of radius 1, centered at the origin, and here we have $f(x, y, z) = x^2 + y^2 + z^2$. Let

$$C(t) = (x(t), y(t), z(t))$$

be a curve, defined for t in some interval. Then $C(t)$ lies on the surface means that

$$x(t)^2 + y(t)^2 + z(t)^2 = 1 \quad \text{for all } t \text{ in the interval.}$$

In other words,

$$f(C(t)) = 1, \quad \text{or also} \quad C(t)^2 = 1.$$

For theoretical purposes, it is neater to write $f(C(t)) = 1$. For computational purposes, we have to go back to coordinates if we want specific numerical values in a given problem.

Now suppose that a curve $C(t)$ lies on a surface $f(X) = k$. Thus we have

$$f(C(t)) = k \quad \text{for all } t.$$

If we differentiate this relation, we get from the chain rule:

$$\operatorname{grad} f(C(t)) \cdot C'(t) = 0.$$

Let P be a point of the surface, and let $C(t)$ be a curve on the surface passing through P. This means that there is a number t_0 such that $C(t_0) = P$. For this value t_0, we obtain

$$\operatorname{grad} f(P) \cdot C'(t_0) = 0.$$

Thus the gradient of f at P is perpendicular to the tangent vector of the curve at P. [We assume that $C'(t_0) \neq O$.] This is true for **every** differentiable curve on the surface passing through P. It is therefore very reasonable to make the following

Definition. The **tangent plane** to the surface $f(X) = k$ at the point P is the plane through P, perpendicular to grad $f(P)$.

We know from Chapter XV how to find such a plane. The definition applies only when grad $f(P) \neq O$. If

$$\operatorname{grad} f(P) = O,$$

then we do not define the notion of tangent plane.

The fact that grad $f(P)$ is perpendicular to every curve passing through P on the surface also gives us an interpretation of the gradient as being perpendicular to the surface

$$f(X) = k.$$

which is one of the level surfaces for the function f (Fig. 2).

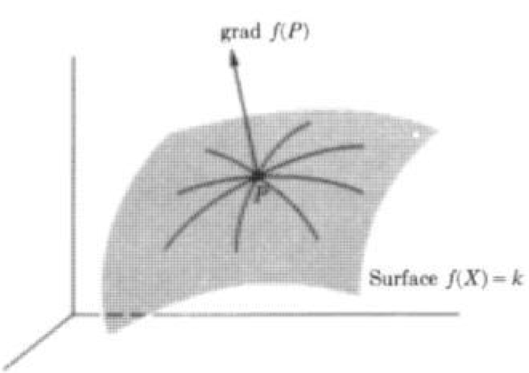


Figure 2

Example 2. Find the tangent plane to the surface

$$x^2 + y^2 + z^2 = 3$$

at the point (1, 1, 1).

Let $f(X) = x^2 + y^2 + z^2$. Then at the point $P = (1, 1, 1)$,

$$\operatorname{grad} f(P) = (2, 2, 2).$$

The equation of a plane passing through P and perpendicular to a vector N is

$$X \cdot N = P \cdot N.$$

In the present case, this yields

$$2x + 2y + 2z = 2 + 2 + 2 = 6.$$

Observe that our arguments also give us a means of finding a vector perpendicular to a curve in 2-space at a given point, simply by applying the preceding discussion to the plane instead of 3-space. A curve is defined by an equation $f(x, y) = k$, and in this case, grad $f(x_0, y_0)$ is perpendicular to the curve at the point (x_0, y_0) on the curve.

Example 3. Find the tangent line to the curve

$$x^2y + y^3 = 10$$

at the point $P = (1, 2)$, and find a vector perpendicular to the curve at that point.

Let $f(x, y) = x^2y + y^3$. Then

$$\operatorname{grad} f(x, y) = (2xy, x^2 + 3y^2),$$

and so

$$\operatorname{grad} f(P) = \operatorname{grad} f(1, 2) = (4, 13).$$

Let $N = (4, 13)$. Then N is perpendicular to the curve at the given point. The tangent line is given by $X \cdot N = P \cdot N$, and thus its equation is

$$4x + 13y = 4 + 26 = 30.$$

Example 4. A surface may also be given in the form $z = g(x, y)$ where g is some function of two variables. In this case, the tangent plane is determined by viewing the surface as expressed by the equation

$$g(x, y) - z = 0.$$

For instance, suppose the surface is given by $z = x^2 + y^2$. We wish to determine the tangent plane at (1, 2, 5). Let $f(x, y, z) = x^2 + y^2 - z$. Then

$$\operatorname{grad} f(x, y, z) = (2x, 2y, -1) \quad \text{and} \quad \operatorname{grad} f(1, 2, 5) = (2, 4, -1).$$

The equation of the tangent plane at $P = (1, 2, 5)$ perpendicular to

$$N = (2, 4, -1)$$

is

$$2x + 4y - z = P \cdot N = 5.$$

This is the desired equation.

Example 5. Find a parametric equation for the tangent line to the curve of intersection of the two surfaces

$$x^2 + y^2 + z^2 = 6 \quad \text{and} \quad x^3 - y^2 + z = 2,$$

at the point $P = (1, 1, 2)$.

The tangent line to the curve is the line in common with the tangent planes of the two surfaces at the point P. We know how to find these tangent planes, and in Chapter XV, we learned how to find the parametric representation of the line common to two planes, so we know how to do this problem. We carry out the numerical computation in full.

The first surface is defined by the equation $f(x, y, z) = 6$. A vector N_1 perpendicular to this first surface at P is given by

$$N_1 = \operatorname{grad} f(P), \quad \text{where} \quad \operatorname{grad} f(x, y, z) = (2x, 2y, 2z).$$

Thus for $P = (1, 1, 2)$ we find

$$N_1 = (2, 2, 4).$$

The second surface is given by the equation $g(x, y, z) = 2$, and

$$\operatorname{grad} g(x, y, z) = (3x^2, -2y, 1).$$

Thus a vector N_2 perpendicular to the second surface at P is

$$N_2 = \operatorname{grad} g(1, 1, 2) = (3, -2, 1).$$

A vector $A = (a, b, c)$ in the direction of the line of intersection is perpendicular to both N_1 and N_2. To find A, we therefore have to solve the equations

$$A \cdot N_1 = 0 \quad \text{and} \quad A \cdot N_2 = 0.$$

This amounts to solving

$$2a + 2b + 4c = 0,$$

$$3a - 2b + c = 0.$$

Let, for instance, $a = 1$. Solving for b and c yields

$$a = 1, \qquad b = 1, \qquad c = -1.$$

Thus $A = (1, 1, -1)$. Finally, the parametric representation of the desired line is

$$P + tA = (1, 1, 2) + t(1, 1, -1).$$

XVIII, §2. EXERCISES

1. Find the equation of the tangent plane and normal line to each of the following surfaces at the specific point.
   (a) $x^2 + y^2 + z^2 = 49$ at $(6, 2, 3)$
   (b) $xy + yz + zx - 1 = 0$ at $(1, 1, 0)$
   (c) $x^2 + xy^2 + y^3 + z + 1 = 0$ at $(2, -3, 4)$
   (d) $2y - z^3 - 3xz = 0$ at $(1, 7, 2)$
   (e) $x^2y^2 + xz - 2y^3 = 10$ at $(2, 1, 4)$
   (f) $\sin xy + \sin yz + \sin xz = 1$ at $(1, \pi/2, 0)$

2. Let $f(x, y, z) = z - e^x \sin y$, and $P = (\log 3, 3\pi/2, -3)$. Find:
   (a) grad $f(P)$,
   (b) the normal line at P to the level surface for f which passes through P,
   (c) the tangent plane to this surface at P.

3. Find a parametric representation of the tangent line to the curve of intersection of the following surfaces at the indicated point.
   (a) $x^2 + y^2 + z^2 = 49$ and $x^2 + y^2 = 13$ at $(3, 2, -6)$
   (b) $xy + z = 0$ and $x^2 + y^2 + z^2 = 9$ at $(2, 1, -2)$
   (c) $x^2 - y^2 - z^2 = 1$ and $x^2 - y^2 + z^2 = 9$ at $(3, 2, 2)$
   [*Note*: The tangent line above may be defined to be the line of intersection of the tangent planes of the given point.]

4. Let $f(X) = 0$ be a differentiable surface. Let Q be a point which does not lie on the surface. Given a differentiable curve $C(t)$ on the surface, defined on an open interval, give the formula for the distance between Q and a point $C(t)$. Assume that this distance reaches a minimum for $t = t_0$. Let $P = C(t_0)$. Show that the line joining Q to P is perpendicular to the curve at P.

5. Find the equation of the tangent plane to the surface $z = f(x, y)$ at the given point P when f is the following function:
   (a) $f(x, y) = x^2 + y^2$, $P = (3, 4, 25)$
   (b) $f(x, y) = x/(x^2 + y^2)^{1/2}$, $P = (3, -4, \frac{3}{5})$
   (c) $f(x, y) = \sin(xy)$ at $P = (1, \pi, 0)$

6. Find the equation of the tangent plane to the surface $x = e^{2y-z}$ at $(1, 1, 2)$.

7. Let $f(x, y, z) = xy + yz + zx$. (a) Write down the equation of the level surface for f through the point $P = (1, 1, 0)$. (b) Find the equation of the tangent plane to this surface at P.

8. Find the equation of the tangent plane to the surface

$$3x^2 - 2y + z^3 = 9$$

at the point $(1, 1, 2)$

9. Find the equation of the tangent plane to the surface

$$z = \sin(x + y)$$

at the point where $x = 1$ and $y = 2$.

10. Find the tangent plane to the surface $x^2 + y^2 - z^2 = 18$ at the point $(3, 5, -4)$.

11. (a) Find a unit vector perpendicular to the surface

$$x^3 + xz = 1$$

at the point $(1, 2, -1)$.
(b) Find the equation of the tangent plane at that point.

12. Find the cosine of the angle between the surfaces

$$x^2 + y^2 + z^2 = 3 \quad \text{and} \quad x - z^2 - y^2 = -3$$

at the point $(-1, 1, -1)$. (This angle is the angle between the normal vectors at the point.)

13. (a) A differentiable curve $C(t)$ lies on the surface

$$x^2 + 4y^2 + 9z^2 = 14,$$

and is so parametrized that $C(0) = (1, 1, 1)$. Let

$$f(x, y, z) = x^2 + 4y^2 + 9z^2$$

and let $h(t) = f(C(t))$. Find $h'(0)$.
(b) Let $g(x, y, z) = x^2 + y^2 + z^2$ and let $k(t) = g(C(t))$. Suppose in addition that $C'(0) = (4, -1, 0)$, find $k'(0)$.

14. Find the equation of the tangent plane to the level surface

$$(x + y + z)e^{xyz} = 3e$$

at the point $(1, 1, 1)$.

XVIII, §3. DIRECTIONAL DERIVATIVE

Let f be defined on an open set and assume that f is differentiable. Let P be a point of the open set, and let A be a **unit vector** (i.e. $\|A\| = 1$). Then $P + tA$ is the parametric representation of a straight line in the direction of A and passing through P. We observe that

$$\frac{d(P + tA)}{dt} = A.$$

For instance, if $n = 2$ and $P = (p, q)$, $A = (a, b)$, then

$$P + tA = (p + ta, q + tb),$$

or in terms of coordinates,

$$x = p + ta, \qquad y = q + tb.$$

Hence

$$\frac{dx}{dt} = a \qquad \text{and} \qquad \frac{dy}{dt} = b$$

so that

$$\frac{d(P + tA)}{dt} = (a, b) = A.$$

The same argument works in higher dimensions.

We wish to consider the rate of change of f in the direction of A. It is natural to consider the values of f on the line $P + tA$, that is to consider the values

$$f(P + tA).$$

The rate of change of f along this line will then be given by taking the derivative of this expression, which we know how to do. We illustrate the line $P + tA$ in the figure.

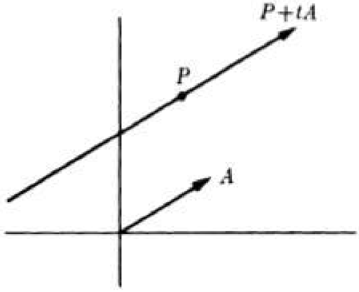


Figure 3

If f represents a temperature at the point P, we look at the variation of temperature in the direction of A, starting from the point P. The value $f(P + tA)$ gives the temperature at the point $P + tA$. This is a function of t, say

$$g(t) = f(P + tA).$$

The rate of change of this temperature function is $g'(t)$, the derivative with respect to t, and $g'(0)$ is the rate of change at time $t = 0$, i.e. the rate of change of f at the point P, in the direction of A.

By the chain rule, if we take the derivative of the function

$$g(t) = f(P + tA),$$

which is defined for small values of t, we obtain

$$\frac{df(P + tA)}{dt} = \operatorname{grad} f(P + tA) \cdot A.$$

When t is equal to 0, this derivative is equal to

$$\operatorname{grad} f(P) \cdot A.$$

For obvious reasons, we now make the

Definition. Let A be a unit vector. The **directional derivative of f in the direction of A at P** is the number

$$D_A f(P) = \operatorname{grad} f(P) \cdot A.$$

We interpret this directional derivative as the rate of change of f along the straight line in the direction of A, at the point P. Thus if we agree on the notation $D_A f(P)$ for the directional derivative of f at P in the direction of the unit vector A, then we have

$$\boxed{D_A f(P) = \left.\frac{df(P + tA)}{dt}\right|_{t=0} = \operatorname{grad} f(P) \cdot A.}$$

In using this formula, the reader should remember that A is taken to be a **unit vector.** When a direction is given in terms of a vector whose norm is not 1, then one must first divide this vector by its norm before applying the formula.

Example 1. Let $f(x, y) = x^2 + y^3$ and let $B = (1, 2)$. Find the directional derivative of f in the direction of B, at the point $(-1, 3)$.

We note that B is not a unit vector. Its norm is $\sqrt{5}$. Let

$$A = \frac{1}{\sqrt{5}} B.$$

Then A is a unit vector having the same direction as B. Let

$$P = (-1, 3).$$

Then grad $f(P) = (-2, 27)$. Hence by our formula, the directional derivative is equal to:

$$\operatorname{grad} f(P) \cdot A = \frac{1}{\sqrt{5}}(-2 + 54) = \frac{52}{\sqrt{5}}.$$

Consider again a differentiable function f on an open set U.

Let P be a point of U. *Let us assume that* grad $f(P) \neq O$, and let A be a unit vector. We know that

$$D_A f(P) = \operatorname{grad} f(P) \cdot A = \|\operatorname{grad} f(P)\| \|A\| \cos \theta,$$

where θ is the angle between grad $f(P)$ and A. Since $\|A\| = 1$, we see that the directional derivative is equal to

$$\boxed{D_A f(P) = \|\operatorname{grad} f(P)\| \cos \theta.}$$

We remind the reader that this formula holds only when A is a unit vector.

The value of $\cos \theta$ varies between -1 and $+1$ when we select all possible unit vectors A.

The maximal value of $\cos \theta$ is obtained when we select A such that $\theta = 0$, i.e. when we select A to have the same direction as grad $f(P)$. In that case, the directional derivative is equal to the norm of the gradient.

Thus we have obtained another interpretation for the gradient:

The direction of the gradient is that of maximal increase of the function.

The norm of the gradient is the rate of increase of the function in that direction (i.e. in the direction of maximal increase).

The directional derivative in the direction of A is at a minimum when $\cos\theta = -1$. This is the case when we select A to have opposite direction to grad $f(P)$. That direction is therefore the direction of maximal decrease of the function.

For example, f might represent a temperature distribution in space. At any point P, a particle which feels cold and wants to become warmer fastest should move in the direction of grad $f(P)$. Another particle which is warm and wants to cool down fastest should move in the direction of $-\text{grad } f(P)$.

Example 2. Let $f(x, y) = x^2 + y^3$ again, and let $P = (-1, 3)$. Find the directional derivative of f at P, in the direction of maximal increase of f.

We have found previously that grad $f(P) = (-2, 27)$. The directional derivative of f in the direction of maximal increase is precisely the norm of the gradient, and so is equal to

$$\|\text{grad } f(P)\| = \|(-2, 27)\| = \sqrt{4 + 27^2} = \sqrt{733}.$$

XVIII, §3. EXERCISES

1. Let $f(x, y, z) = z - e^x \sin y$, and $P = (\log 3, 3\pi/2, -3)$. Find:
   (a) the directional derivative of f at P in the direction of $(1, 2, 2)$,
   (b) the maximum and minimum values for the directional derivative of f at P.

2. Find the directional derivatives of the following functions at the specified points in the specified directions.
   (a) $\log(x^2 + y^2)^{1/2}$ at $(1, 1)$, direction $(2, 1)$
   (b) $xy + yz + zx$ at $(-1, 1, 7)$, direction $(3, 4, -12)$
   (c) $4x^2 + 9y^2$ at $(2, 1)$ in the direction of maximum directional derivative

3. A temperature distribution in space is given by the function

   $$f(x, y) = 10 + 6\cos x \cos y + 3\cos 2x + 4\cos 3y.$$

   At the point $(\pi/3, \pi/3)$, find the direction of greatest increase of temperature, and the direction of greatest decrease of temperature.

4. In what direction are the following functions of X increasing most rapidly at the given point?
   (a) $x/\|X\|^{3/2}$ at $(1, -1, 2)$ $\quad (X = (x, y, z))$
   (b) $\|X\|^5$ at $(1, 2, -1, 1)$ $\quad (X = (x, y, z, w))$

5. (a) Find the directional derivative of the function

   $$f(x, y) = 4xy + 3y^2$$

   in the direction of $(2, -1)$, at the point $(1, 1)$.

(b) Find the directional derivative in the direction of maximal increase of the function.

6. Let $f(x, y, z) = (x + y)^2 + (y + z)^2 + (z + x)^2$. What is the direction of greatest increase of the function at the point $(2, -1, 2)$? What is the directional derivative of f in this direction at that point?

7. Let $f(x, y) = x^2 + xy + y^2$. What is the direction in which f is increasing most rapidly at the point $(-1, 1)$? Find the directional derivative of f in this direction.

8. Suppose the temperature in (x, y, z)-space is given by

$$f(x, y, z) = x^2y + yz - e^{xy}.$$

Compute the rate of change of temperature at the point $P = (1, 1, 1)$ in the direction of $\overrightarrow{PO}$.

9. (a) Find the directional derivative of the function

$$f(x, y, z) = \sin(xyz)$$

at the point $P = (\pi, 1, 1)$ in the direction of $\overrightarrow{OA}$ where A is the unit vector $(0, 1/\sqrt{2}, -1/\sqrt{2})$.

(b) Let U be a unit vector whose direction is *opposite* to that of

$$(\text{grad } f)\,(P).$$

What is the value of the directional derivative of f at P in the direction of U?

10. Let f be a differentiable function defined on an open set U. Suppose that P is a point of U such that $f(P)$ is a maximum, i.e. suppose we have

$$f(P) \geqq f(X) \qquad \text{for all} \quad X \text{ in } U.$$

Show that grad $f(P) = O$.

XVIII, §4. FUNCTIONS DEPENDING ONLY ON THE DISTANCE FROM THE ORIGIN

The first such function which comes to mind is the distance function. In 2-space, it is given by

$$r = \sqrt{x^2 + y^2}.$$

In 3-space, it is given by

$$r = \sqrt{x^2 + y^2 + z^2}.$$

In n-space, it is given by

$$r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Let us find its gradient. For instance, in 2-space,

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{1}{2}(x^2 + y^2)^{-1/2}2x \\ &= \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}.\end{aligned}$$

Differentiating with respect to y instead of x you will find

$$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

Hence

$$\boxed{\operatorname{grad} r = \left(\frac{x}{r}, \frac{y}{r}\right).}$$

This can also be written

$$\boxed{\operatorname{grad} r = \frac{X}{r}.}$$

Thus the gradient of r is the unit vector in the direction of the position vector. It points outward from the origin.

If we are dealing with functions on 3-space, so

$$r = \sqrt{x^2 + y^2 + z^2}$$

then the chain rule again gives

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \text{and} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}}$$

so again

$$\boxed{\operatorname{grad} r = \frac{X}{r}.}$$

Warning: Do **not** write $\partial r/\partial X$. This suggests dividing by a vector X and is therefore bad notation. The notation $\partial r/\partial x$ was correct and good notation since we differentiate only with respect to the single variable x. Information coming from differentiating with respect to all the variables is correctly expressed by the formula grad $r = X/r$ in the box.

In n-space, let

$$r = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Then

$$\frac{\partial r}{\partial x_i} = \tfrac{1}{2}(x_1^2 + \cdots + x_n^2)^{-1/2}2x_i$$

so

$$\boxed{\frac{\partial r}{\partial x_i} = \frac{x_i}{r}.}$$

By definition of the gradient, it follows that

$$\boxed{\operatorname{grad} r = \frac{X}{r}.}$$

We now come to other functions depending on the distance. Such functions arise frequently. For instance, a temperature function may be inversely proportional to the distance from the source of heat. A potential function may be inversely proportional to the square of the distance from a certain point. The gradient of such functions has special properties which we discuss further.

Example 1. Let

$$f(x, y) = \sin r = \sin \sqrt{x^2 + y^2}.$$

Then $f(x, y)$ depends only on the distance r of (x, y) from the origin. By the chain rule,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{d \sin r}{dr} \cdot \frac{\partial r}{\partial x} \\ &= (\cos r)\tfrac{1}{2}(x^2 + y^2)^{-1/2}2x \\ &= (\cos r)\frac{x}{r}.\end{aligned}$$

Similarly, $\partial f/\partial y = (\cos r)y/r$. Consequently

$$\begin{aligned}\operatorname{grad} f(x, y) &= \left((\cos r)\frac{x}{r}, (\cos r)\frac{y}{r}\right)\\ &= \frac{\cos r}{r}(x, y)\\ &= \frac{\cos r}{r}X.\end{aligned}$$

The same use of the chain rule as in the special case

$$f(x, y) = \sin r$$

which we worked out in Example 1 shows:

Let g be a differentiable function of one variable, and let $f(X) = g(r)$. Then

$$\boxed{\operatorname{grad} f(X) = \frac{g'(r)}{r}X.}$$

Work out all the examples given in Exercise 2. You should memorize and keep in mind this simple expression for the gradient of a function which depends only on the distance. Such dependence is expressed by the function g.

Exercises 9 and 10 give important information concerning functions which depend only on the distance from the origin, and should be seen as essential complements of this section. They will prove the following result.

A differentiable function $f(X)$ depends only on the distance of X from the origin if and only if grad $f(X)$ *is parallel to X, or O.*

In this situation, the gradient grad $f(X)$ may point towards the origin, or away from the origin, depending on whether the function is decreasing or increasing as the point moves away from the origin.

Example 2. Suppose a heater is located at the origin, and the temperature at a point decreases as a function of the distance from the origin, say is inversely proportional to the square of the distance from the origin. Then temperature is given as

$$h(X) = g(r) = k/r^2$$

for some constant $k > 0$. Then the gradient of temperature is

$$\operatorname{grad} h(X) = -2k \frac{1}{r^3} \frac{X}{r} = -\frac{2k}{r^4} X.$$

The factor $2k/r^4$ is positive, and we see that grad $h(X)$ points in the direction of $-X$. Each circle centered at the origin is a level curve for temperature. Thus the gradient may be drawn as on the following figure. The gradient is parallel to X but in opposite direction. A bug traveling along the circle will stay at constant temperature. If it wants to get warmer fastest, it must move toward the origin.

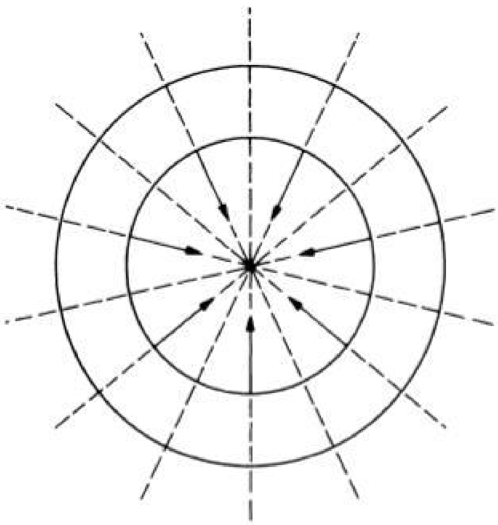


Figure 4

The dotted lines indicate the path of the bug when moving in the direction of maximal increase of the function. These lines are perpendicular to the circles of constant temperature.

XVIII, §4. EXERCISES

1. Let g be a function of r, let $r = \|X\|$, and $X = (x, y, z)$. Let $f(X) = g(r)$. Show that
$$\left(\frac{dg}{dr}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2.$$

2. Let g be a function of r, and $r = \|X\|$. Let $f(X) = g(r)$. Find grad $f(X)$ for the following functions.

(a) $g(r) = 1/r$ (b) $g(r) = r^2$ (c) $g(r) = 1/r^3$
(d) $g(r) = e^{-r^2}$ (e) $g(r) = \log 1/r$ (f) $g(r) = 4/r^m$
(g) $g(r) = \cos r$

You may either work out each exercise separately, writing

$$r = \sqrt{x_1^2 + \cdots + x_n^2},$$

and use the chain rule, finding $\partial f/\partial x_i$ in each case, or you may apply the general formula obtained in Example 1, that if $f(X) = g(r)$, we have

$$\operatorname{grad} f(X) = \frac{g'(r)}{r} X.$$

Probably you should do both for a while to get used to the various notations and situations which may rise.

The next five exercises concern certain parametrizations, and some of the results from them will be used in Exercise 9.

3. Let A. B be two unit vectors such that $A \cdot B = 0$. Let

$$F(t) = (\cos t)A + (\sin t)B.$$

Show that $F(t)$ lies on the sphere of radius 1 centered at the origin, for each value of t. [*Hint*: What is $F(t) \cdot F(t)$?]

4. Let P, Q be two points on the sphere of radius 1, centered at the origin. Let $L(t) = P + t(Q - P)$, with $0 \leqq t \leqq 1$. If there exists a value of t in $[0, 1]$ such that $L(t) = O$, show that $t = \frac{1}{2}$, and that $P = -Q$.

5. Let P, Q be two points on the sphere of radius 1. Assume that $P \neq -Q$. Show that there exists a curve joining P and Q on the sphere of radius 1, centered at the origin. By this we mean there exists a curve $C(t)$ such that $C(t)^2 = 1$, or if you wish $\|C(t)\| = 1$ for all t, and there are two numbers t_1 and t_2 such that $C(t_1) = P$ and $C(t_2) = Q$. [*Hint*: Divide $L(t)$ in Exercise 4 by its norm.]

6. If P, Q are two unit vectors such that $P = -Q$, show that there exists a differentiable curve joining P and Q on the sphere of radius 1, centered at the origin. You may assume that there exists a unit vector A which is perpendicular to P. Then use Exercise 3.

7. Parametrize the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ by a differentiable curve.

8. Let f be a differentiable function (in two variables) such that $\operatorname{grad} f(X) = cX$ for some constant c and all X in 2-space. Show that f is constant on any circle of radius $a > 0$, centered at the origin. [*Hint*: Put $x = a\cos t$ and $y = a \sin t$ and find df/dt.]

Exercise 8 is a special case of a general phenomenon, stated in Exercise 9.

9. Let f be a differentiable function in n variables, and assume that there exists a function h such that $\operatorname{grad} f(X) = h(X)X$. Show that f is constant on the sphere of radius $a > 0$ centered at the origin.

[That f is constant on the sphere of radius a means that given any two points P, Q on this sphere, we must have $f(P) = f(Q)$. To prove this, use the fact proved in Exercises 5 and 6 that given two such points, there exists a curve $C(t)$ joining the two points, i.e. $C(t_1) = P$, $C(t_2) = Q$, and $C(t)$ lies on the sphere for all t in the interval of definition, so

$$C(t) \cdot C(t) = a^2.$$

The hypothesis that grad $f(X)$ can be written in the form $h(X)X$ for some function h means that grad $f(X)$ is *parallel* to X (or O). Indeed, we know that grad $f(X)$ parallel to X means that grad $f(X)$ is equal to a scalar multiple of X, and this scalar may depend on X, so we have to write it as a function $h(X)$.]

10. Let $r = \|X\|$. Let g be a differentiable function of one variable whose derivative is never equal to 0. Let $f(X) = g(r)$. Show that grad $f(X)$ is parallel to X for $X \neq O$.

[This statement is the converse of Exercise 9. The proof is quite easy, cf. Example 1. The function $h(X)$ of Exercise 9 is then seen to be equal to $g'(r)/r$.]

XVIII, §5. CONSERVATION LAW

Definition. Let U be an open set. By a **vector field** on U we mean an association which to every point of U associates a vector of the same dimension.

If F is a vector field on U, and X a point of U, then we denote by $F(X)$ the vector associated to X by F and call it the **value of F at X**, as usual.

Example 1. Let $F(x, y) = (x^2y, \sin xy)$. Then F is a vector field which to the point (x, y) associates $(x^2y, \sin xy)$, having the same number of coordinates, namely two of them in this case.

A vector field in physics is often interpreted as a field of forces. A vector field may be visualized as a field of arrows, which to each point associates an arrow as shown on the figure.

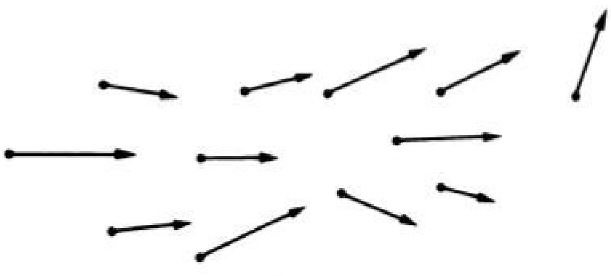


Figure 5

Each arrow points in the direction of the force, and the length of the arrow represents the magnitude of the force.

If f is a differentiable function on U, then we observe that grad f is a vector field, which associates the vector grad $f(P)$ to the point P of U.

If F is a vector field, and if there exists a differentiable function f such that $F = \operatorname{grad} f$, then the vector field is called **conservative**. Since

$$-\operatorname{grad} f = \operatorname{grad}(-f)$$

it does not matter whether we use f or $-f$ in the definition of conservative.

Let us assume that F is a conservative field on U, and let ψ be a differentiable function such that for all points X in U we have

$$F(X) = -\operatorname{grad} \psi.$$

In physics, one interprets ψ as the **potential energy**. Suppose that a particle of mass m moves on a differentiable curve $C(t)$ in U. **Newton's law** states that

$$\boxed{F(C(t)) = mC''(t)}$$

for all t where $C(t)$ is defined. Newton's law says that force equals mass times acceleration.

Physicists define the **kinetic energy** to be

$$\tfrac{1}{2}mC'(t)^2 = \tfrac{1}{2}mv(t)^2.$$

Conservation Law. *Assume the vector field F is conservative, that is $F = -\operatorname{grad} \psi$, where ψ is the potential energy. Assume that a particle moves on a curve satisfying Newton's law. Then the sum of the potential energy and kinetic energy is constant.*

Proof. We have to prove that

$$\psi(C(t)) + \tfrac{1}{2}mC'(t)^2$$

is constant. To see this, we differentiate the sum. By the chain rule, we see that the derivative is equal to

$$\operatorname{grad} \psi(C(t)) \cdot C'(t) + mC'(t) \cdot C''(t).$$

By Newton's law, $mC''(t) = F(C(t)) = -\text{grad } \psi(C(t))$. Hence this derivative is equal to

$$\text{grad } \psi(C(t)) \cdot C'(t) - \text{grad } \psi(C(t)) \cdot C'(t) = 0.$$

This proves what we wanted.

It is not true that all vector fields are conservative. We shall discuss the problem of determining which ones are conservative in the next book.

The fields of classical physics are for the most part conservative.

Example 2. Consider a force $F(X)$ which is inversely proportional to the square of the distance from the point X to the origin, and in the direction of X. Then there is a constant k such that for $X \neq O$ we have

$$F(X) = k \frac{1}{\|X\|^2} \frac{X}{\|X\|},$$

because $X/\|X\|$ is the unit vector in the direction of X. Thus

$$F(X) = k \frac{1}{r^3} X,$$

where $r = \|X\|$. A potential energy for F is given by

$$\psi(X) = \frac{k}{r}.$$

This is immediately verified by taking the partial derivatives of this function.

If there exists a function $\varphi(X)$ such that

$$F(X) = (\text{grad } \varphi)(X), \qquad \text{that is} \quad F = \text{grad } \varphi,$$

then we shall call such a function φ a **potential function** for F. Our conventions are such that a potential function is equal to *minus* the potential energy.

XVIII, §5. EXERCISES

1. Find a potential function for a force field $F(X)$ that is inversely proportional to the distance from the point X to the origin and is in the direction of X.
2. Same question, replacing "distance" with "cube of the distance."

3. Let k be an integer $\geqq 1$. Find a potential function for the vector field F given by

$$F(X) = \frac{1}{r^k} X, \qquad \text{where} \quad r = \|X\|.$$

[*Hint*: Recall the formula that if $\varphi(X) = g(r)$, then

$$\operatorname{grad} \varphi(X) = \frac{g'(r)}{r} X.$$

Set $F(X)$ equal to the right-hand side and solve for g.]

Answers to Exercises

I am much indebted to Anthony Petrello for some of the answers to the exercises.

I, §2, p. 13

1. $-3 < x < 3$ **2.** $-1 \leqq x \leqq 0$ **3.** $-\sqrt{3} \leqq x \leqq -1$ or $1 \leqq x \leqq \sqrt{3}$
4. $x < 3$ or $x > 7$ **5.** $-1 < x < 2$ **6.** $x < -1$ or $x > 1$ **7.** $-5 < x < 5$
8. $-1 \leqq x \leqq 0$ **9.** $x \geqq 1$ or $x = 0$ **10.** $x \leqq -10$ or $x = 5$
11. $x \leqq -10$ or $x = 5$ **12.** $x \geqq 1$ or $x = -\frac{1}{2}$ **13.** $x < -4$
14. $-5 < x < -3$ **15.** $-3 < x < -2$ and $-2 < x < -1$ **16.** $-2 < x < 2$
17. $-2 < x < 8$ **18.** $2 < x < 4$ **19.** $-4 < x < 10$ **20.** $x < -4$ and $x > 10$
21. $x < -10$ and $x > 4$

I, §3, p. 17

1. $-\frac{3}{2}$ **2.** $\dfrac{1}{(2x+1)}$ **3.** 0, 2, 108 **4.** $2z - z^2$, $2w - w^2$
5. $x \neq \sqrt{2}$ or $-\sqrt{2}$. $f(5) = \frac{1}{23}$ **6.** All x. $f(27) = 3$
7. (a) 1 (b) 1 (c) -1 (d) -1 **8.** (a) 1 (b) 4 (c) 0 (d) 0
9. (a) -2 (b) -6 (c) $x^2 + 4x - 2$ **10.** $x \geqq 0, 2$
11. (a) odd (b) even (c) odd (d) odd

I, §4, p. 20

1. 8 and 9 **2.** $\frac{1}{5}$ and -1 **3.** $\frac{1}{16}$ and 2 **4.** $\frac{1}{9}$ and $2^{1/3}$ **5.** $\frac{1}{16}$ and $\frac{1}{2}$ **6.** 9 and 8
7. $-\frac{1}{3}$ and -1 **8.** $\frac{1}{4}$ and $\frac{1}{4}$ **9.** 1 and $-\frac{1}{4}$ **10.** $-\frac{1}{512}$ and $\frac{1}{3}$

11. Yes. Suppose a is negative, so write $a = -b$ where b is positive. Let c be a positive number such that $c^n = b$. Then $(-c)^n = a$ because $(-1)^n = -1$ since n is odd.

II, §1, p. 24

3. x negative, y positive **4.** x negative, y negative

II, §3, p. 33

5. $y = -\frac{8}{3}x + \frac{5}{3}$ **6.** $y = -\frac{3}{2}x + 5$ **7.** $x = \sqrt{2}$

8. $y = \dfrac{9}{\sqrt{3}+3}\,x + 4 - \dfrac{9\sqrt{3}}{\sqrt{3}+3}$ **9.** $y = 4x - 3$ **10.** $y = -2x + 2$

11. $y = -\frac{1}{2}x + 3 + \dfrac{\sqrt{2}}{2}$ **12.** $y = \sqrt{3}x + 5 + \sqrt{3}$ **19.** $-\frac{1}{4}$ **20.** -8

21. $2 + \sqrt{2}$ **22.** $\frac{1}{6}(3 + \sqrt{3})$ **23.** $y = (x - \pi)\left(\dfrac{2}{\sqrt{2} - \pi}\right) + 1$

24. $y = (x - \sqrt{2})\left(\dfrac{\pi - 2}{1 - \sqrt{2}}\right) + 2$ **25.** $y = -(x + 1)\left(\dfrac{3}{\sqrt{2} + 1}\right) + 2$

26. $y = (x + 1)(3 + \sqrt{2}) + \sqrt{2}$ **29.** (a) $x = -4$, $y = -7$ (b) $x = \frac{3}{2}$, $y = \frac{5}{2}$
(c) $x = -\frac{1}{3}$, $y = \frac{7}{3}$ (d) $x = -6$, $y = -5$

II, §4, p. 35

1. $\sqrt{97}$ **2.** $\sqrt{2}$ **3.** $\sqrt{52}$ **4.** $\sqrt{13}$ **5.** $\frac{1}{2}\sqrt{5}$ **6.** $(4, -3)$ **7.** 5 and 5 **8.** $(-2, 5)$
9. 5 and 7

II, §8, p. 51

5. $(x - 2)^2 + (y + 1)^2 = 25$ **6.** $x^2 + (y - 1)^2 = 9$ **7.** $(x + 1)^2 + y^2 = 3$
8. $y + \frac{25}{8} = 2(x + \frac{1}{4})^2$ **9.** $y - 1 = (x + 2)^2$ **10.** $y + 4 = (x - 1)^2$
11. $(x + 1)^2 + (y - 2)^2 = 2$ **12.** $(x - 2)^2 + (y - 1)^2 = 2$
13. $x + \frac{25}{8} = 2(y + \frac{1}{4})^2$ **14.** $x - 1 = (y + 2)^2$

III, §1, p. 61

1. 4 **2.** -2 **3.** 2 **4.** $\frac{3}{4}$ **5.** $-\frac{1}{4}$ **6.** 0 **7.** 4 **8.** 6 **9.** 3 **10.** 12 **11.** 2
12. 3 **13.** a

III, §2, p. 70

	Tangent line at $x = 2$	Slope at $x = 2$
1. $2x$	$y = 4x - 3$	4
2. $3x^2$	$y = 12x - 16$	12
3. $6x^2$	$y = 24x - 32$	24
4. $6x$	$y = 12x - 12$	12
5. $2x$	$y = 4x - 9$	4
6. $4x + 1$	$y = 9x - 8$	9
7. $4x - 3$	$y = 5x - 8$	5
8. $\dfrac{3x^2}{2} + 2$	$y = 8x - 8$	8
9. $-\dfrac{1}{(x+1)^2}$	$y = -\frac{1}{9}x + \frac{5}{9}$	$-\frac{1}{9}$
10. $-\dfrac{2}{(x+1)^2}$	$y = -\frac{2}{9}x + \frac{10}{9}$	$-\frac{2}{9}$

III, §3, p. 75

1. $4x + 3$ **2.** $-\dfrac{2}{(2x+1)^2}$ **3.** $\dfrac{1}{(x+1)^2}$ **4.** $2x + 1$ **5.** $-\dfrac{1}{(2x-1)^2}$ **6.** $9x^2$

7. $4x^3$ **8.** $5x^4$ **9.** $6x^2$ **10.** $\dfrac{3x^2}{2} + 1$ **11.** $-2/x^2$ **12.** $-3/x^2$

13. $-2/(2x-3)^2$ **14.** $-3/(3x+1)^2$ **15.** $-1/(x+5)^2$ **16.** $-1/(x-2)^2$
17. $-2x^{-3}$ **18.** $-2(x+1)^{-3}$

III, §4, p. 78

1. $x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ **2.** $4x^3$
3. (a) $\frac{2}{3}x^{-1/3}$ (b) $-\frac{3}{2}x^{-5/2}$ (c) $\frac{7}{6}x^{1/6}$ **4.** $y = 9x - 8$ **5.** $y = \frac{1}{3}x + \frac{4}{3}$, slope $\frac{1}{3}$

6. $y = \dfrac{-3}{2^9}x + \dfrac{7}{32}$, slope $\dfrac{-3}{2^9}$ **7.** $y = \dfrac{1}{2\sqrt{3}}x + \dfrac{\sqrt{3}}{2}$, slope $\dfrac{1}{2\sqrt{3}}$

8. (a) $\frac{1}{4}5^{-3/4}$ (b) $-\frac{1}{4}7^{-5/4}$ (c) $\sqrt{2}(10^{\sqrt{2}-1})$ (d) $\pi 7^{\pi-1}$

III, §5, p. 89

1. (a) $\frac{2}{3}x^{-2/3}$ (b) $\frac{9}{4}x^{-1/4}$ (c) x (d) $\frac{9}{4}x^2$
2. (a) $55x^{10}$ (b) $-8x^{-3}$ (c) $\frac{4}{3}x^3 - 15x^2 + 2x$
3. (a) $-\frac{3}{8}x^{-7/4}$ (b) $3 - 6x^2$ (c) $20x^4 - 21x^2 + 2$
4. (a) $21x^2 + 8x$ (b) $\frac{8}{3}x^{-1/3} + 20x^3 - 3x^2 + 3$
5. (a) $-25x^{-2} + 6x^{-1/2}$ (b) $6x^2 + 35x^6$ (c) $16x^3 - 21x^2 + 1$
6. (a) $\frac{6}{5}x - 16x^7$ (b) $12x^3 - 4x + 1$ (c) $7\pi x^6 - 40x^4 + 1$
7. $(x^3 + x) + (3x^2 + 1)(x - 1)$ **8.** $(2x^2 - 1)4x^3 + 4x(x^4 + 1)$
9. $(x + 1)(2x + \frac{15}{2}x^{1/2}) + (x^2 + 5x^{3/2})$
10. $(2x - 5)(12x^3 + 5) + 2(3x^4 + 5x + 2)$

11. $(x^{-2/3}+x^2)\left(3x^2-\dfrac{1}{x^2}\right)+(-\frac{2}{3}x^{-5/3}+2x)\left(x^3+\dfrac{1}{x}\right)$

12. $(2x+3)(-2x^{-3}-x^{-2})+2(x^{-2}+x^{-1})$ **13.** $\dfrac{9}{(x+5)^2}$ **14.** $\dfrac{(-2x^2+2)}{(x^2+3x+1)^2}$

15. $\dfrac{(t+1)(t-1)(2t+2)-(t^2+2t-1)2t}{(t^2-1)^2}$

16. $\dfrac{(t^2+t-1)(-5/4)t^{-9/4}-t^{-5/4}(2t+1)}{(t^2+t-1)^2}$

17. $\frac{5}{49}$, $y=\frac{5}{49}t+\frac{4}{49}$ **18.** $\frac{1}{2}$, $y=\frac{1}{2}t$

III, §5, Supplementary Exercises, p. 89

1. $9x^2-4$ **3.** $2x+1$ **5.** $\frac{5}{2}x^{3/2}-\frac{5}{2}x^{-7/2}$ **7.** $x^2-1+(x+5)(2x)$
9. $(\frac{3}{2}x^{1/2}+2x)(x^4-99)+(x^{3/2}+x^2)(4x^3)$

11. $(4x)\left(\dfrac{1}{x^2}+4x+8\right)+(2x^2+1)\left(\dfrac{-2}{x^3}+4\right)$

13. $(x+2)(x+3)+(x+1)(x+3)+(x+1)(x+2)$
15. $3x^2(x^2+1)(x+1)+x^3(2x)(x+1)+(x^3)(x^2+1)$

17. $\dfrac{-2}{(2x+3)^2}$ **19.** $\dfrac{5(3x^2+4x)}{(x^3+2x^2)^2}$ **21.** $\dfrac{-2(x+1)+2x}{(x+1)^2}$

23. $\dfrac{(x+1)(x-1)3(\frac{1}{2}x^{-1/2})-3x^{1/2}[(x-1)+(x+1)]}{(x+1)^2(x-1)^2}$

25. $\dfrac{(x^2+1)(x+7)(5x^4)-(x^5+1)((x^2+1)+(2x)(x+7))}{(x^2+1)^2(x+7)^2}$

27. $\dfrac{(1-x^2)(3x^2)-x^3(-2x)}{(1-x^2)^2}$ **29.** $\dfrac{(x^2+1)(2x-1)-(x^2-x)(2x)}{(x^2+1)^2}$

31. $\dfrac{(x^2+x-4)(2)-(2x+1)(2x+1)}{(x^2+x-4)^2}$

33. $\dfrac{(x^2+2)(4-3x^2)-(4x-x^3)(2x)}{(x^2+2)^2}$ **35.** $\dfrac{-5x-(1-5x)}{x^2}$

37. $\dfrac{(x+1)(x-2)(2x)-x^2((x-2)+(x+1))}{(x+1)^2(x-2)^2}$

39. $\dfrac{(4x^3-x^5+1)(12x^3+\frac{5}{4}x^{1/4})-(3x^4+x^{5/4})(12x^2-5x^4)}{(4x^3-x^5+1)^2}$

41. $(y-18)=\frac{25}{32}(x-16)$ **43.** $(y+12)=19x$ **45.** $(y-10)=14(x-1)$

47. $y-\dfrac{4}{9}=\dfrac{-12}{81}(x-2)$ **49.** $y-\dfrac{4}{3}=\dfrac{-4}{9}(x-2)$

51. Point of tangency: $(3,-3)$. Both curves intersect here and have slope -1.
53. Both curves have the point $(1, 3)$ in common and have slope 6 at this point.
55. Tangent line $(y-7)=16x$ at $(0,7)$; tangent line $(y-19)=16(x-1)$ at $(1,19)$; tangent line $(y+13)=16(x+1)$ at $(-1,-13)$.

III, §6, p. 99

1. $8(x+1)^7$ **2.** $\frac{1}{2}(2x-5)^{-1/2}\cdot 2$ **3.** $3(\sin x)^2\cos x$ **4.** $5(\log x)^4\left(\frac{1}{x}\right)$

5. $(\cos 2x)2$ **6.** $\frac{1}{x^2+1}(2x)$ **7.** $e^{\cos x}(-\sin x)$ **8.** $\frac{1}{e^x+\sin x}(e^x+\cos x)$

9. $\cos\left[\log x+\frac{1}{x}\right]\left(\frac{1}{x}-\frac{1}{x^2}\right)$ **10.** $\frac{\sin 2x-(x+1)(\cos 2x)2}{(\sin 2x)^2}$

11. $3(2x^2+3)^2(4x)$ **12.** $-[\sin(\sin 5x)](\cos 5x)5$ **13.** $\frac{1}{\cos 2x}(-\sin 2x)2$

14. $[\cos(2x+5)^2](2(2x+5))(2)$. **15.** $[\cos(\cos(x+1))](-\sin(x+1))$

16. $(\cos e^x)e^x$ **17.** $-\frac{1}{(3x-1)^8}[4(3x-1)^3]\cdot 3$ **18.** $-\frac{1}{(4x)^6}\cdot 3(4x)^2\cdot 4$

19. $-\frac{1}{(\sin 2x)^4}2(\sin 2x)(\cos 2x)\cdot 2$ **20.** $-\frac{1}{(\cos 2x)^4}2(\cos 2x)(-\sin 2x)2$

21. $-\frac{1}{(\sin 3x)^2}(\cos 3x)\cdot 3$ **22.** $-\sin^2 x+\cos^2 x$ **23.** $(x^2+1)e^x+2xe^x$

24. $(x^3+2x)(\cos 3x)\cdot 3+(3x^2+2)\sin 3x$

25. $-\frac{1}{(\sin x+\cos x)^2}(\cos x-\sin x)$ **26.** $\frac{2e^x\cos 2x-(\sin 2x)e^x}{e^{2x}}$

27. $\frac{(x^2+3)/x-(\log x)(2x)}{(x^2+3)^2}$ **28.** $\frac{\cos 2x-(x+1)(-\sin 2x)\cdot 2}{\cos^2 2x}$

29. $(2x-3)(e^x+1)+2(e^x+x)$ **30.** $(x^3-1)(e^{3x}\cdot 3+5)+3x^2(e^{3x}+5x)$

31. $\frac{(x-1)3x^2-(x^3+1)}{(x-1)^2}$ **32.** $\frac{(2x+3)2x-(x^2-1)2}{(2x+3)^2}$

33. $2(x^{4/3}-e^x)+(\frac{4}{3}x^{1/3}-e^x)(2x+1)$

34. $(\sin 3x)\frac{1}{4}x^{-3/4}+3(\cos 3x)(x^{1/4}-1)$ **35.** $[\cos(x^2+5x)](2x+5)$

36. $e^{3x^2+8}(6x)$ **37.** $\frac{-1}{[\log(x^4+1)]^2}\cdot\frac{1}{x^4+1}\cdot 4x^3$

38. $\frac{-1}{[\log(x^{1/2}+2x)]^2}\frac{1}{(x^{1/2}+2x)}(\frac{1}{2}x^{-1/2}+2)$ **39.** $\frac{2e^x-2xe^x}{e^{2x}}$

40. $\frac{2x}{1+x^6}$; $\frac{4}{65}$

III, §6, Supplementary Exercises, p. 100

1. $2(2x+1)2$ **3.** $7(5x+3)^6 5$ **5.** $3(2x^2+x-5)^2(4x+1)$
7. $\frac{1}{2}(3x+1)^{-1/2}(3)$ **9.** $-2(x^2+x-1)^{-3}(2x+1)$ **11.** $-\frac{5}{3}(x+5)^{-8/3}$
13. $(x-1)3(x-5)^2+(x-5)^3$ **15.** $4(x^3+x^2-2x-1)^3(3x^2+2x-2)$

17. $\frac{(x-1)^{1/2}(\frac{3}{4})(x+1)^{-1/4}-(x+1)^{3/4}(\frac{1}{2})(x-1)^{-1/2}}{x-1}$

19. $\dfrac{(3x+2)^9(\frac{5}{2})(2x^2+x-1)^{3/2}(4x+1)-(2x^2+x-1)^{5/2}(9)(3x+2)^8(3)}{(3x+2)^{18}}$

21. $\frac{1}{2}(2x+1)^{-1/2}(2)$ **23.** $\frac{1}{2}(x^2+x+5)^{-1/2}(2x+1)$
25. $3x^2\cos(x^3+1)$ **27.** $(e^{x^3+1})(3x^2)$ **29.** $(\cos(\cos x))(-\sin x)$
31. $(e^{\sin(x^3+1)})(3x^2\cos(x^3+1))$
33. $[\cos((x+1)(x^2+2))][(x+1)(2x)+(x^2+2)]$
35. $(e^{(x+1)(x-3)})((x+1)+(x-3))$ **37.** $2\cos(2x+5)$

39. $\dfrac{2}{2x+1}$ **41.** $\left(\cos\dfrac{x-5}{2x+4}\right)\left(\dfrac{(2x+4)-(x-5)2}{(2x+4)^2}\right)$

43. $(e^{2x^2+3x+1})(4x+3)$ **45.** $\dfrac{1}{2x+1}[\cos(\log 2x+1)]2$

47. $-(6x-2)\sin(3x^2-2x+1)$ **49.** $80(2x+1)^{79}(2)$
51. $49(\log x)^{48}(x^{-1})$ **53.** $5(e^{2x+1}-x)^4(2e^{2x+1}-1)$

55. $\frac{1}{2}(3\log(x^2+1)-x^3)^{-1/2}\left(\dfrac{3}{x^2+1}(2x)-3x^2\right)$

57. $\dfrac{2(\cos 3x)(\cos 2x)-3(\sin 2x)(-\sin 3x)}{(\cos 3x)^2}$

59. $\dfrac{(\sin x^3)(1/2x^2)4x-(\log 2x^2)(\cos x^3)3x^2}{(\sin x^3)^2}$

61. $\dfrac{(\cos 2x)(4x^3)-2(x^4+4)(-\sin 2x)}{(\cos 2x)^2}$

63. $\dfrac{(\cos x^3)(4)(2x^2+1)^3(4x)+(2x^2+1)^4(\sin x^3)(3x^2)}{\cos^2 x^3}$

65. $-3e^{-3x}$ **67.** $(e^{-4x^2+x})(-8x+1)$

69. $\dfrac{e^{-x}[2x/(x^2+2)]-[\log(x^2+2)](e^{-x})(-1)}{e^{-2x}}$

III, §7, p. 103

1. $18x$ **2.** $5(x^2+1)^4\cdot 2+20(x^2+1)^3 4x^2$ **3.** 0 **4.** 7! **5.** 0 **6.** 6
7. $-\cos x$ **8.** $\cos x$ **9.** $-\sin x$ **10.** $-\cos x$ **11.** $\sin x$ **12.** $\cos x$

In Problems 7 through 12, there is a pattern. Note that the derivatives of $\sin x$ are:

$$\begin{aligned} f(x) &= \sin x; \\ f^{(1)}(x) &= \cos x; \\ f^{(2)}(x) &= -\sin x; \\ f^{(3)}(x) &= -\cos x; \\ f^{(4)}(x) &= \sin x. \end{aligned}$$

Then the derivatives repeat. Thus every fourth derivative is the same. Hence to find the n-th derivative, we just divide n by 4, and if r is the remainder, so

$n = 4q + r$, then

$$f^{(n)}(x) = f^{(r)}(x).$$

13. (a) 5! (b) 7! (c) 13! **14.** (a) $k!$ (b) $k!$ (c) 0 (d) 0

III, §8, p. 106

1. $-(2x + y)/x$ **2.** $\dfrac{3-x}{y+1}$ **3.** $\dfrac{y-3x^2}{3y^2-x}$ **4.** $\dfrac{6x^2}{3y^2+1}$ **5.** $\dfrac{1-2y}{2x+2y-1}$

6. $-y^2/x^2$ **7.** $-\dfrac{1+4xy}{2(y+x^2)}$ **8.** $\dfrac{x(y^2-1)}{y(1-x^2)}$ **9.** $(y-3) = 3(x+1)$

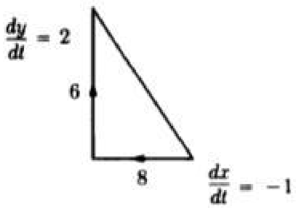
10. $y + 1 = 4(x - 3)$ **11.** $(y - 2) = \frac{3}{4}(x - 6)$ **12.** $y + 2 = -\frac{3}{4}(x - 1)$
13. $(y + 4) = \frac{3}{4}(x - 3)$ **14.** $y - 2 = -\frac{1}{8}(x + 4)$ **15.** $y - 3 = \frac{7}{4}(x - 2)$

III, §9, p. 114

1. (a) 1/6 (b) 0 (c) Impossible **2.** 0 **3.** 320 ft/sec^2 **4.** 0 **5.** 240 m^3/sec

6. 36π cm^3/sec **7.** $2\pi r, \dfrac{\pi d}{2}, \dfrac{c}{2\pi}$ **8.** $\frac{3}{16}$ units/sec

9. (a) $\frac{4}{3}$ ft/sec (b) 1 ft/sec **10.** (a) $\frac{5}{3}$ ft/sec (b) 2 ft/sec
11. $(\frac{1}{2}, \frac{1}{4})$
12. The picture is as follows.



We are given $dx/dt = -1$ and $dy/dt = 2$. The area is $A = \frac{1}{2}xy$, so

$$\frac{dA}{dt} = \frac{1}{2}\left[x\frac{dy}{dt} + y\frac{dx}{dt}\right] = \frac{1}{2}[2x - y].$$

We are given that at some time, $x = 8$. Since the speed is uniform toward the origin, after 2 min we find $x = 8 - 2 = 6$. Also after 2 min we find $y = 6 + 4 = 10$. Hence after 2 min we get

$$\frac{dA}{dt} = \frac{1}{2}[12 - 10] = 1 \text{ cm}^2/\text{min}.$$

13. 90 cm^2/sec **14.** $-8/5$ ft/sec **15.** 3/200 ft/min **16.** $5/4\pi$ ft/min
17. $t = \frac{1}{4}$, acc $= 4$

18. Both x and y are functions of time t. Differentiating each side of

$$y = x^2 - 6x$$

with respect to t, we find

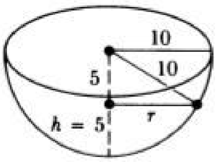
$$\frac{dy}{dt} = 2x\frac{dx}{dt} - 6\frac{dx}{dt}.$$

When $dy/dt = 4\,dx/dt$ this yields

$$4\frac{dx}{dt} = 2x\frac{dx}{dt} - 6\frac{dx}{dt}.$$

Canceling dx/dt yields $4 = 2x - 6$, so $x = 5$, and $y = -5$.

19. $4/75\pi$ ft/min. Draw the picture



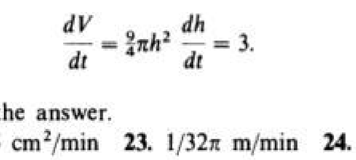
When $h = 5$, then the distance from the top of the water to the top of the hemisphere is also 5, so by Pythagoras,

$$5^2 + r^2 = 10^2.$$

You can then solve for r. Use that $dV/dt = 4$ to find dh/dt.

20. $\frac{1}{2}(50^2 \cdot 7 + 60^2 \cdot 3)(50^2 \cdot 3.5^2 + 60^2 \cdot 1.5^2)^{-1/2}$

21. $1/12\pi$ ft/min



The assumption about the diameter implies that $r = 3h/2$ so $V = \frac{3}{4}\pi h^3$. Then

$$\frac{dV}{dt} = \frac{9}{4}\pi h^2\frac{dh}{dt} = 3.$$

When $h = 4$ this gives the answer.

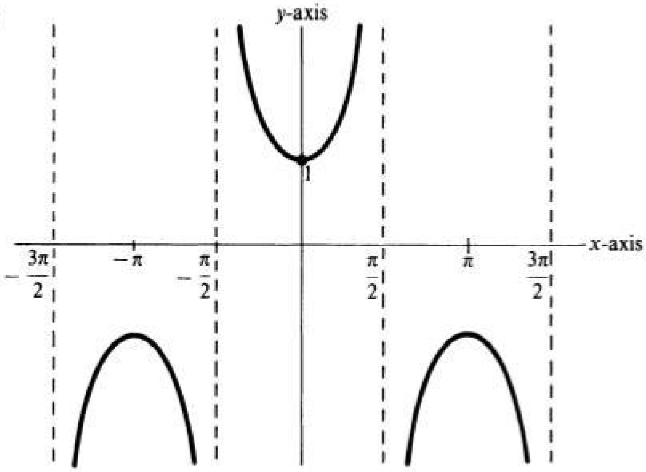
22. $-3/400$ cm/min, $-6\pi/5$ cm^2/min **23.** $1/32\pi$ m/min **24.** 100π ft^3/sec

IV, §1, p. 131

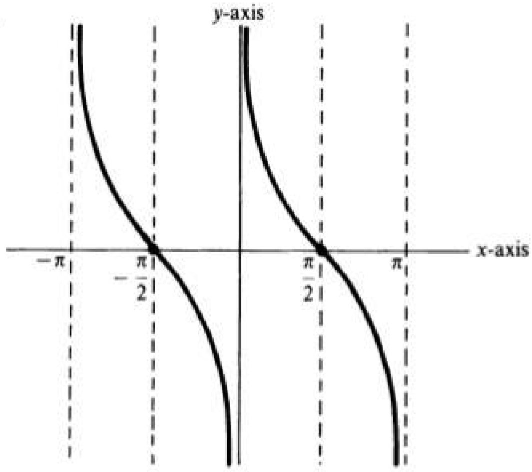
1. $\sqrt{2}/2$ **2.** $\sqrt{3}/2$ **3.** $\sqrt{3}/2$ **4.** $\frac{1}{2}$ **5.** $-\sqrt{3}/2$ **6.** $-\frac{1}{2}$ **7.** $\sqrt{3}/2$ **8.** $-\sqrt{2}/2$
9. 1 **10.** $\sqrt{3}$ **11.** 1 **12.** -1 **13.** $-\frac{1}{2}$ **14.** $-\sqrt{3}/2$ **15.** $-1/2$ **16.** $\sqrt{3}/2$
17. $-\sqrt{3}/2$ **18.** $\frac{1}{2}$

IV, §2, p. 135

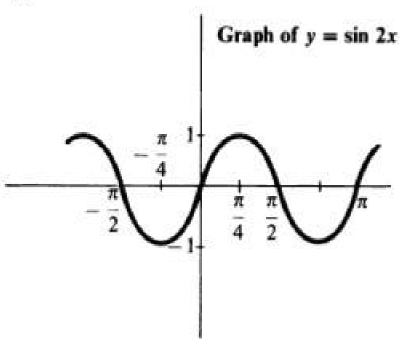
2.



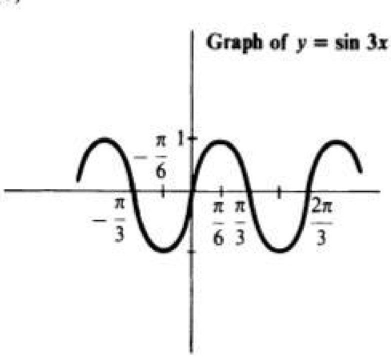
3.



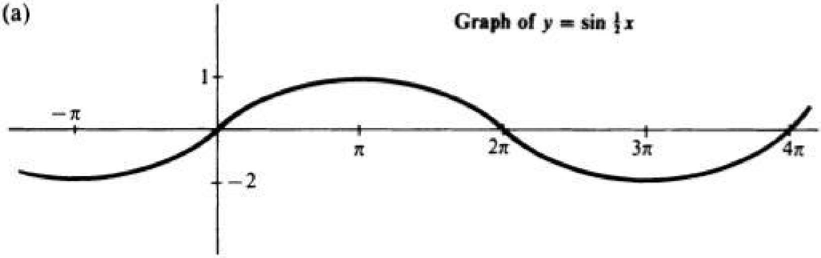
4. (a)



(b)

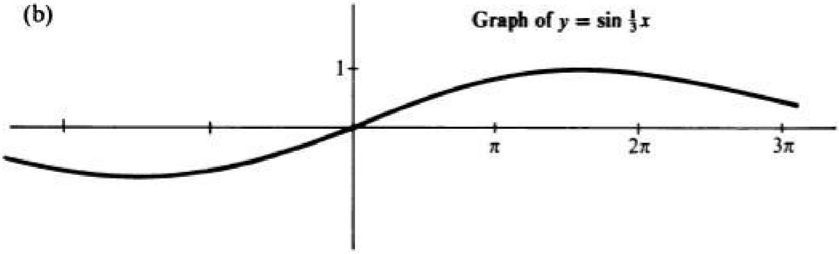


5. (a)



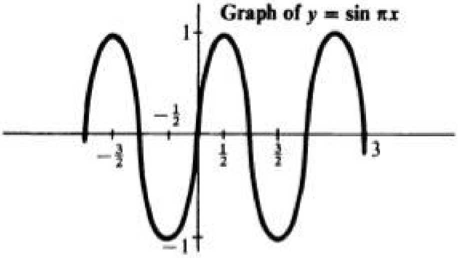
Graph of $y = \sin \frac{1}{2}x$

(b)



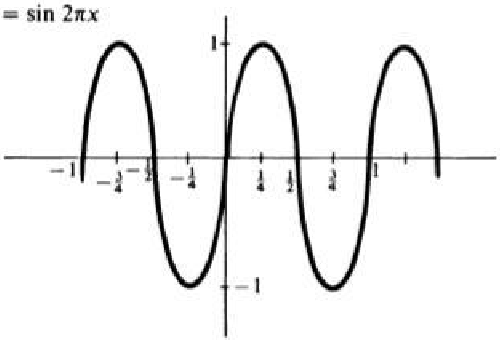
Graph of $y = \sin \frac{1}{3}x$

6. (a)

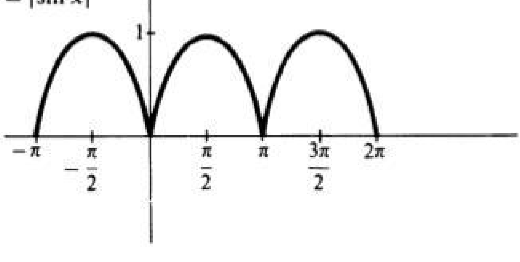


Graph of $y = \sin \pi x$

(c) $y = \sin 2\pi x$



7. (a) $y = |\sin x|$



IV, §3, p. 140

1. $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$ **2.** $\frac{-\sqrt{2}}{4}(\sqrt{3}-1)$

3. (a) $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$ (b) $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$ (c) $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$ (d) $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$

(e) $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$ (f) $\frac{-\sqrt{2}}{4}(\sqrt{3}+1)$ (g) $\frac{1}{2}$ (h) $-\sqrt{3}/2$

5. $\sin 3x = 3\sin x - 4\sin^3 x$, $\cos 3x = 4\cos^3 x - 3\cos x$

IV, §4, p. 145

1. $-\csc^2 x$ **2.** $3\cos 3x$ **3.** $-5\sin 5x$
4. $(8x+1)\cos(4x^2+x)$ **5.** $(3x^2)\sec^2(x^3-5)$ **6.** $(4x^3-3x^2)\sec^2(x^4-x^3)$
7. $\cos x \sec^2(\sin x)$ **8.** $\sec^2 x \cos(\tan x)$ **9.** $-\sec^2 x \sin(\tan x)$ **10.** -1
11. 0 **12.** $\sqrt{3}/2$ **13.** $-\sqrt{2}$ **14.** 2 **15.** $-2\sqrt{3}$

16. (a) $y = 1$ (b) $\left(y-\frac{\sqrt{3}}{2}\right)=\frac{-1}{2}\left(x-\frac{\pi}{6}\right)$

(c) $y = 1$ (d) $(y+1)=6\left(x-\frac{\pi}{4}\right)$

(e) $y = 1$ (f) $(y-\sqrt{2})=-\sqrt{2}\left(x-\frac{\pi}{4}\right)$

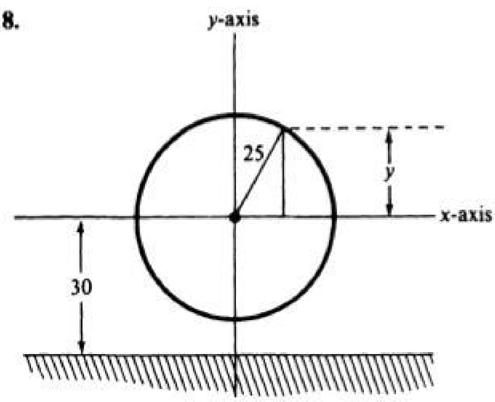
(g) $(y-1)=-2\left(x-\frac{\pi}{4}\right)$ (h) $y-1=x-\frac{\pi}{2}$

(i) $(y-\frac{1}{2})=\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{3}\right)$ (j) $(y-\frac{1}{2})=\frac{-\pi\sqrt{3}}{6}(x-1)$

(k) $y = 1$ (l) $\left(y-\frac{1}{\sqrt{3}}\right)=\frac{4\pi}{3}\left(x-\frac{1}{6}\right)$

17. (a) -1.6 (b) $\frac{2}{3}$ (c) $\sqrt{3}/30$

18.



One revolution every two minutes is half a revolution per minute. Hence $d\theta/dt = \pi$ (in radians per minute). But

$$y = 25 \sin \theta$$

so

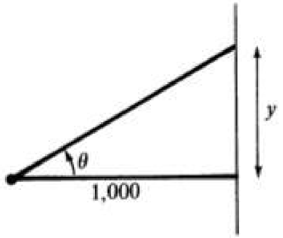
$$\frac{dy}{dt} = 25 \frac{d \sin \theta}{d\theta} \frac{d\theta}{dt} = 25(\cos \theta)\pi.$$

When the height of a point on the wheel is 42.5 then $y = 42.5 - 30 = 12.5$. Therefore $\sin \theta = 12.5/25 = \frac{1}{2}$ and $\theta = \pi/6$. Hence

$$\left.\frac{dy}{dt}\right|_{\theta=\pi/6} = 25\left(\cos \frac{\pi}{6}\right)\pi = \tfrac{25}{2} \sqrt{3}\pi \text{ ft/min.}$$

19. (a) 180 ft/sec (b) 360 ft/sec (c) 2250 ft/sec (d) 9000/91 ft/sec (e) 1530 ft/sec **20.** 25 rad/hr

21.



We are given $d\theta/dt = 4\pi$ (two revolutions $= 4\pi$ radians). Then

$$\tan \theta = \frac{y}{1000} \qquad \text{so} \quad y = 1000 \tan \theta.$$

Using the chain rule,

$$\frac{dy}{dt} = 1000(1 + \tan^2 \theta) \frac{d\theta}{dt}.$$

(a) The point on the wall nearest to the light is when $\theta = 0$. Then $\tan 0 = 0$, so

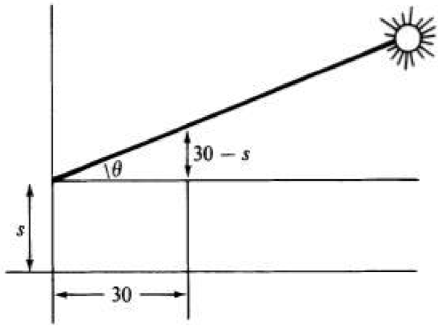
$$\left.\frac{dy}{dt}\right|_{\theta=0} = 1000 \cdot 4\pi = 4{,}000\pi \text{ ft/min.}$$

(b) When $y = 500$ then $\tan \theta = \frac{1}{2}$, so we substitute $\frac{1}{2}$ for $\tan \theta$ and get

$$\left.\frac{dy}{dt}\right|_{y=500} = 1000(\tfrac{5}{4})4\pi = 5000\pi \text{ ft/min.}$$

22. $8000\pi/27$ ft/sec

23. Let s be the length of the shadow.



We are given $d\theta/dt = \pi/10$. We also have

$$\tan\theta = \frac{30 - s}{30} = 1 - \frac{s}{30}.$$

Differentiating with respect to t and using the chain rule gives:

$$(1 + \tan^2\theta)\frac{d\theta}{dt} = -\frac{1}{30}\frac{ds}{dt}.$$

If $\theta = \pi/6$ then $\tan(\pi/6) = 1/\sqrt{3}$. Substituting yields:

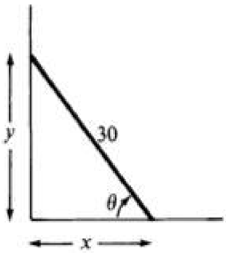
$$\left(1 + \frac{1}{3}\right)\frac{\pi}{10} = -\frac{1}{30}\left.\frac{ds}{dt}\right|_{\theta=\pi/6}.$$

We can solve for ds/dt, namely

$$\left.\frac{ds}{dt}\right|_{\theta=\pi/6} = -30\left(\frac{4}{3}\right)\frac{\pi}{10} = -4\pi \text{ ft/hr}.$$

24. $54/5\pi$ deg/min

25. We are given $dx/dt = 3$. Find $d\theta/dt$ when $x = 15$.



We have

$$\cos\theta = \frac{x}{30}.$$

Hence by the chain rule,

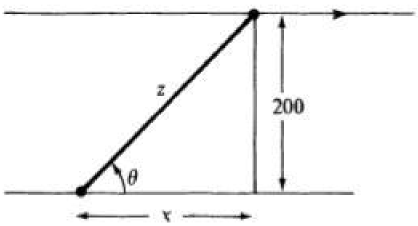
$$-\sin\theta\,\frac{d\theta}{dt}=\frac{1}{30}\,\frac{dx}{dt}.$$

When $x=15$ we have $\cos\theta=1/2$ and so $\theta=\pi/3$. Hence $\sin\theta=\sqrt{3}/2$. Then

$$\left.\frac{d\theta}{dt}\right|_{x=15}=-\frac{1}{\sqrt{3}/2}\,\frac{1}{30}\,3$$
$$=\frac{-1}{5\sqrt{3}}\text{ rad/sec.}$$

26. $9/2\pi$ deg/sec

27.



We are given $dx/dt=20$. We want to find $d\theta/dt$ when $z=400$. We have

$$\tan\theta=\frac{200}{x},$$

so differentiating with respect to t and using the chain rule,

$$(1+\tan^2\theta)\,\frac{d\theta}{dt}=200\left(\frac{-1}{x^2}\right)\frac{dx}{dt}.$$

When $z=400$, then $\sin\theta=200/400=1/2$. Hence $\theta=\pi/6$ and therefore $\tan\theta=1/\sqrt{3}$. Also $x=200\sqrt{3}$. This gives

$$\left(1+\frac{1}{3}\right)\left.\frac{d\theta}{dt}\right|_{z=400}=200\cdot\frac{-1}{200^2\cdot 3}\,20$$
$$=\frac{-1}{30}.$$

Hence finally

$$\left.\frac{d\theta}{dt}\right|_{z=400}=-\frac{1}{30}\,\frac{3}{4}=-\frac{1}{40}\text{ ft/sec.}$$

28. $-1/25$ rad/sec

IV, §6, p. 157

3. (a) $(\sqrt{2}, \pi/4)$ (b) $(\sqrt{2}, 5\pi/4)$ (c) $(6, \pi/3)$ (d) $(1, \pi)$

4. (a) $(y-1)^2 + x^2 = 1$ (b) $(x - \frac{3}{2})^2 + y^2 = \frac{9}{4}$

5. (a) $\left(y - \dfrac{a}{2}\right)^2 + x^2 = \left(\dfrac{a}{2}\right)^2$ (b) $\left(x - \dfrac{a}{2}\right)^2 + y^2 = \left(\dfrac{a}{2}\right)^2$

(c) $x^2 + (y-a)^2 = a^2$ (d) $(x-a)^2 + y^2 = a^2$

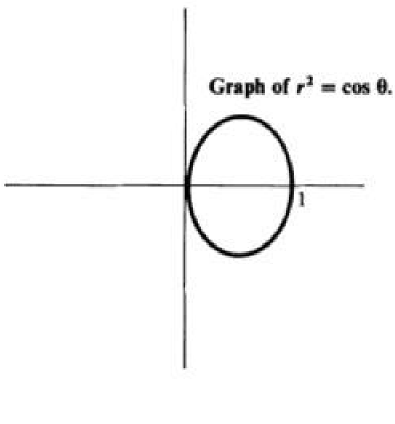
6. $r^2 = \cos\theta$. This is equivalent with $r = \sqrt{\cos\theta}$. Only values of θ such that $\cos\theta \geqq 0$ will give a contribution to r. Also, since $\cos(-\theta) = \cos\theta$ the curve is symmetric with respect to the x-axis. We make a table.

θ	r
0 to $\pi/2$	dec. 1 to 0
$-\pi/$ to 0	inc. 0 to 1

In these intervals, we have $0 \leqq \cos\theta \leqq 1$, and hence

$$\sqrt{\cos\theta} \geqq \cos\theta, \qquad \text{(watch out!)}$$

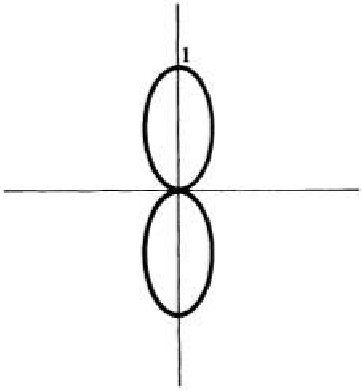
with equality only at the end points. Since $r = \cos\theta$ is the equation of a circle (similarly to $r = \sin\theta$, see Problem 5), the graph looks like this.



8. (a) $r = \sin^2\theta$. The right-hand side is always $\geqq 0$ so there is a value of r for each value of θ. Since

$$-1 \leqq \sin\theta \leqq 1,$$

it follows that $\sin^2\theta \leqq |\sin\theta|$. Also the regions of increase and decrease are over intervals of length $\pi/2$. You should make a table of these, and then see that the graph looks like this.



The ovals are thinner than circles, contrary to Problem 6, where they were more expanded than circles.

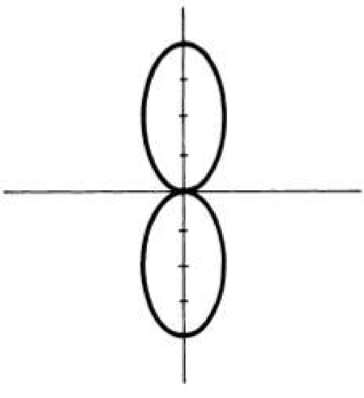
9. $r = 4\sin^2\theta$. Note that the right-hand side is always $\geqq 0$, and so there is a value of r for each value of θ. Also

$$\sin(-\theta) = -\sin\theta \qquad \text{and} \qquad \sin^2(-\theta) = \sin^2\theta$$

so the graph is symmetric with respect to the x-axis. We make a table.

θ	r
0 to $\pi/2$	inc. 0 to 4
$\pi/2$ to π	dec. 4 to 0.

Also observe that $\sin^2\theta \leqq |\sin\theta|$ because $|\sin\theta| \leqq 1$. Hence the graph is something like this:



10. $x^2 + y^2 = 25$ (circle) **11.** $x^2 + y^2 = 16$ (circle)

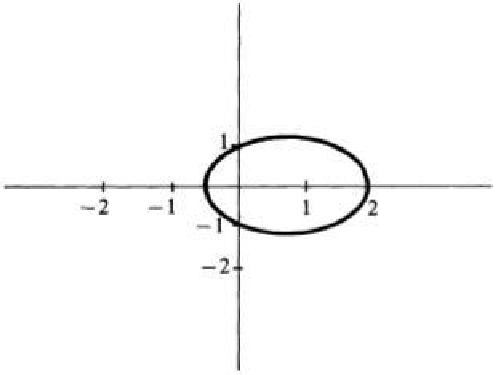
12. (a) $r\cos\theta = 1$ is equivalent with $x = 1$, which is a vertical line!

13. $r = 3/\cos\theta$. This is defined for $\cos\theta \neq 0$. In this case, the equation is equivalent with $r\cos\theta = 3$, and $x = r\cos\theta$, so the equation in rectangular coordinates is $x = 3$, which is a vertical line.

16. $(x^2 + x + y^2)^2 = x^2 + y^2$ **17.** $(x^2 + y^2 + 2y)^2 = x^2 + y^2$ **27.** $y^2 = 2x + 1$

28. $r = \dfrac{2}{2 - \cos\theta}$. We make a table:

θ	$\cos\theta$	$2 - \cos\theta$	r
inc. 0 to $\pi/2$	dec. 1 to 0	inc. 1 to 2	dec. 2 to 1
inc. $\pi/2$ to π	dec. 0 to -1	inc. 2 to 3	dec. 1 to 2/3
inc. π to $3\pi/2$	inc. -1 to 0	dec. 3 to 2	inc. 2/3 to 1
inc. $5\pi/2$ to 2π	inc. 0 to 1	dec. 2 to 1	inc. 1 to 2



One can see that this equation is an ellipse by converting to (x, y)-coordinates. The equation is equivalent with

$$r(2 - \cos\theta) = 2, \qquad \text{that is} \qquad 2\sqrt{x^2 + y^2} - x = 2.$$

By algebra, this is equivalent to $4(x^2 + y^2) = (x + 2)^2$, that is

$$3x^2 - 4x + 4y^2 = 4.$$

By completing the square this is the equation of an ellipse

$$3\left(x - \frac{2}{3}\right)^2 + 4y^2 = \frac{16}{3}.$$

29. $\sqrt{x^2 + y^2} = 4 - 2x$. Since $r \geqq 0$ by assumption, we must have $4 - 2x \geqq 0$, or equivalently $x \leqq 2$. Conversely, for $x \leqq 2$ the relation is equivalent with what we obtain when we square both sides, and the equation becomes

$$x^2 + y^2 = 16 - 16x + 4x^2,$$

or equivalently

$$3x^2 - 16x - y^2 = -16.$$

This is the equation of a hyperbola. Thus the equation in polar coordinates is equivalent with the equation of a hyperbola, together with the additional condition $x \leqq 2$.

30. $r = \tan\theta = \sin\theta/\cos\theta$. Multiply both sides by $\cos\theta$ to see that this equation is equivalent to $r\cos\theta = \sin\theta$, that is

$$x = \sin\theta.$$

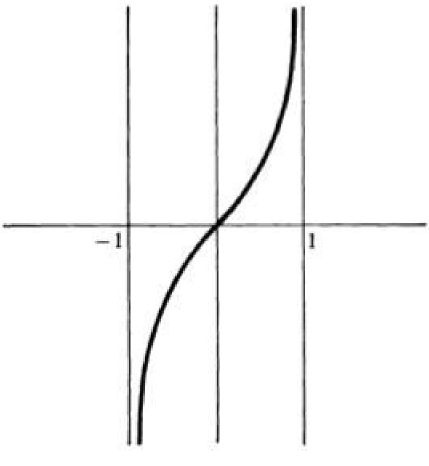
Of course, the function is not defined when $\cos\theta = 0$. Since

$$-1 < \sin\theta < 1,$$

it follows that $-1 < x < 1$. We make a small table:

θ	x
0	0
$\pi/4$	$1/\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$
inc. 0 to $\pi/2$	inc. 0 to 1

As θ approaches $\pi/2$, x approaches 1. But $\cos\theta$ approaches 0 and so $r = \tan\theta$ becomes very large positive. Hence the graph looks as in the following figure for $0 \leqq x < 1$.



The graph also has a symmetry. Since $r = y/x$ and $r \geqq 0$, both y and x must have the same sign, that is both $x, y > 0$ or both $x, y < 0$, unless $x = 0$.

The next interval of θ for which $\tan\theta$ is positive is then

$$\pi \leqq \theta < \frac{3\pi}{2}.$$

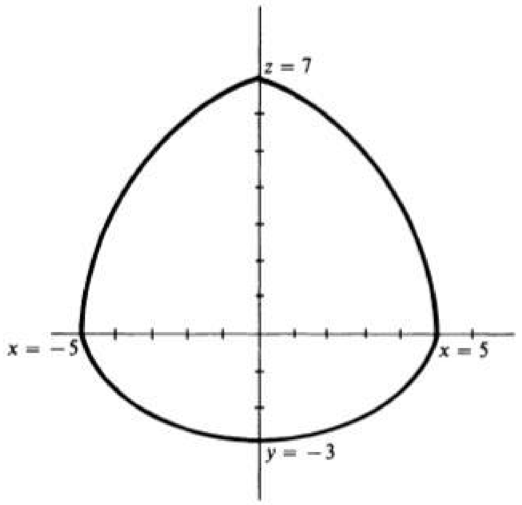
Either by making a table again, or by symmetry, using

$$\tan(\theta + \pi) = \tan\theta$$

you see that the graph is as shown for $-1 < x \leqq 0$.

31. $r = 5 + 2\sin\theta$. We make a table:

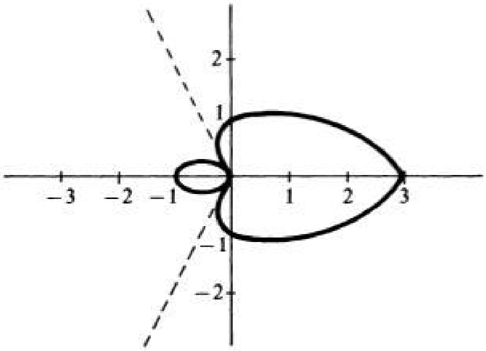
θ	$\sin\theta$	r
0 to $\pi/2$	inc. 0 to 1	inc. 5 to 7
$\pi/2$ to π	dec. 1 to 0	dec. 7 to 5
π to $3\pi/2$	dec. 0 to -1	dec. 5 to 3
$3\pi/2$ to 2π	inc. -1 to 0	inc. 3 to 5



32. $r = |1 + 2\cos\theta|$. Again we make a table. The absolute value sign makes the right-hand side positive, so we get a value of r for every value of θ. However, we want to choose intervals which take into account changes of sign of $1 + 2\cos\theta$, this is when $\cos\theta = -1/2$. We make the table accordingly, when $\theta = 2\pi/3$ or $\theta = 4\pi/3$.

θ	$\cos\theta$	r
0 to $\pi/2$	dec. 1 to 0	dec. 3 to 1
$\pi/2$ to $2\pi/3$	dec. 0 to $-1/2$	dec. 1 to 0
$2\pi/3$ to π	dec. $-1/2$ to -1	inc. 0 to 1

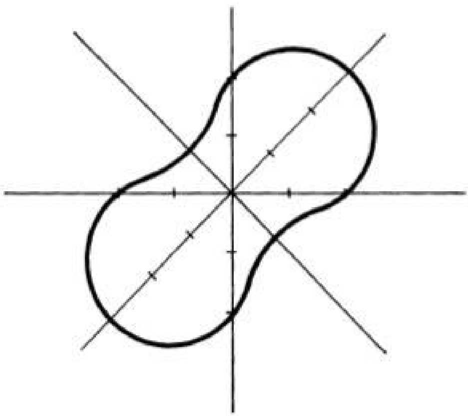
Since $\cos(-\theta) = \cos\theta$, the graph is symmetric with respect to the x-axis, and looks like this:



33. (a) $r = 2 + \sin 2\theta$. Since $\sin 2\theta$ lies between -1 and $+1$, it follows that the right-hand side is positive for all θ and so there is an r corresponding to every value of θ. We make a table, choosing the intervals to reflect the regions of increase of $\sin 2\theta$, so by intervals of length $\pi/4$.

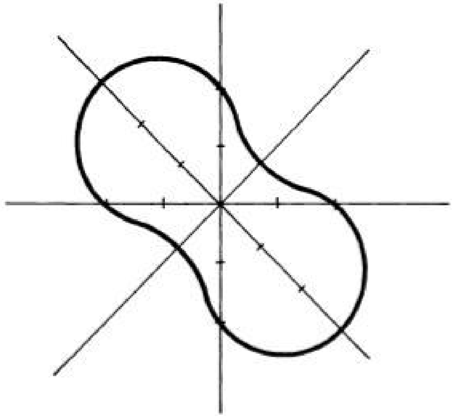
θ	2θ	$\sin 2\theta$	r
0 to $\pi/4$	0 to $\pi/2$	inc. 0 to 1	inc. 2 to 3
$\pi/4$ to $\pi/2$	$\pi/2$ to π	dec. 1 to 0	dec. 3 to 2
$\pi/2$ to $3\pi/4$	π to $3\pi/2$	dec. 0 to -1	dec. 2 to 1
$3\pi/4$ to π	$3\pi/2$ to 2π	inc. -1 to 0	inc. 1 to 2

The graph looks like this.



33. (b) Make a table. We use intervals of length $\pi/4$ because $\sin\theta$ changes its behavior on intervals of length $\pi/2$, so $\sin 2\theta$ changes its behavior on intervals of length $\pi/4$.

θ	2θ	$\sin 2\theta$	$r = 2 - \sin 2\theta$
inc. 0 to $\pi/4$	inc. 0 to $\pi/2$	inc. 0 to 1	dec. 2 to 1
inc. $\pi/4$ to $\pi/2$	inc. $\pi/2$ to π	dec. 1 to 0	inc. 1 to 2
inc. $\pi/2$ to $3\pi/4$	inc. π to $3\pi/2$	dec. 0 to -1	inc. 2 to 3
inc. $3\pi/4$ to π	inc. $3\pi/2$ to 2π	inc. -1 to 0	dec. 3 to 2



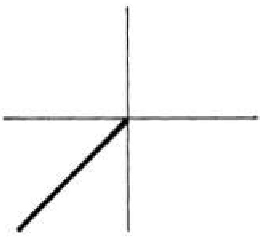
The part for $\pi \leqq \theta \leqq 2\pi$ is obtained similarly, or by symmetry.

34. $x \leqq 0$, $y = 0$ (negative x-axis)

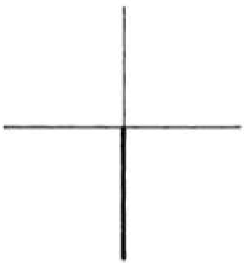
35. $x = 0$, $y \geqq 0$ (positive y-axis)

36. $x = 0$, $y \leqq 0$ (negative y-axis)

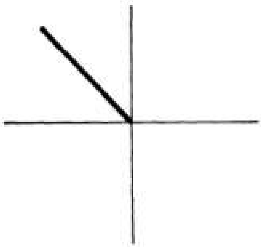
37.



38.



39.



V, §1, p. 165

1. 1 **2.** $\frac{3}{4}$ **3.** $\frac{1}{6}$ **4.** 1 **5.** $\frac{3}{4}$ **6.** 0 **7.** ± 1

8. $\dfrac{\pi}{4} + 2n\pi$ and $\dfrac{5\pi}{4} + 2n\pi$, $n =$ integer. **9.** $n\pi$, $n =$ interger

10. $\dfrac{\pi}{2} + n\pi$, $n =$ integer

V, §2, p. 175

1. Increasing for all x.
2. Decreasing for $x \leqq \frac{1}{2}$, increasing for $x \geqq \frac{1}{2}$.
3. Increasing all x.
4. Decreasing $x \leqq -\sqrt{2/3}$ and $x \geqq \sqrt{2/3}$. Increasing $-\sqrt{2/3} \leqq x \leqq \sqrt{2/3}$.
5. Increasing all x.
6. Decreasing for $x \leqq 0$. Increasing for $x \geqq 0$.
17. Min: 1; max: 4
19. Max: -1; min: 4
21. Min: -1; max: -2, 1
24. Let $f(x) = \tan x - x$. Then $f'(x) = 1 + \tan^2 x - 1 = \tan^2 x$. But $\tan^2 x$ is a square, and so is > 0 for $0 < x < \pi/2$. Hence f is strictly increasing. Since $f(0) = 0$ it follows that $f(x) > 0$ for all x with $0 < x < \pi/2$.
26. Base $= \sqrt{C/3}$, height $= \sqrt{C/12}$ **27.** Radius $= \sqrt{C/3\pi}$, height $= \sqrt{C/3\pi}$
28. Base $= \sqrt{C/6}$, height $= \sqrt{C/6}$; Radius $= \sqrt{C/6\pi}$, height $= 2\sqrt{C/6\pi}$
30. (a) $f(t) = -3t + C$ (b) $f(t) = 2t + C$ **31.** $f(t) = -3t + 1$
32. $f(t) = 2t - 5$ **33.** $x(t) = 7t - 61$ **34.** $H(t) = -2t + 30$

VI, §1, p. 187

1. 0, 0 **2.** 0, 0 **3.** 0, 0 **4.** $\dfrac{1}{\pi}, \dfrac{1}{\pi}$ **5.** 0, 0 **6.** $\infty, -\infty$ **7.** $-\infty, \infty$
8. $-\frac{1}{2}, -\frac{1}{2}$ **9.** $-\infty, +\infty$ **10.** 0, 0 **11.** $\infty, -\infty$ **12.** $-\infty, \infty$
13. ∞, ∞ **14.** $-\infty, -\infty$ **15.** $\infty, -\infty$ **16.** $-\infty, \infty$
17. ∞, ∞ **18.** $-\infty, -\infty$

19.

n	a_n	$x \to \infty$	$x \to -\infty$
Odd	>0	$f(x) \to \infty$	$f(x) \to -\infty$
Odd	<0	$f(x) \to -\infty$	$f(x) \to \infty$
Even	>0	$f(x) \to \infty$	$f(x) \to \infty$
Even	<0	$f(x) \to -\infty$	$f(x) \to -\infty$

20. Suppose a polynomial has odd degree, say

$$f(x) = ax^n + \text{lower terms},$$

and $a \neq 0$. Suppose first $a > 0$. If $x \to \infty$ then $f(x) \to \infty$ and in particular, $f(x) > 0$ for some x. If $x \to -\infty$ then $f(x) \to -\infty$, and in particular, $f(x) < 0$ for some x. By the intermediate value theorem, there is some number c such that $f(c) = 0$. The same argument works if $a < 0$.

VI, §2, p. 191

1. For $\sin x$: $0, \pi$, and add $n\pi$ with any integer n.

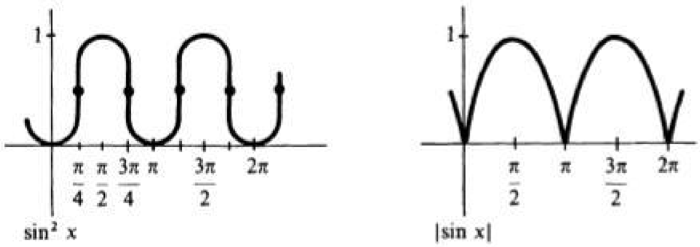
2. For $\cos x$: $\frac{\pi}{2}, \frac{3\pi}{2}$, and add $n\pi$.

3. Let $f(x) = \tan x$. Then $f'(x) = 1 + \tan^2 x$, and

$$f''(x) = \frac{d}{dx}(1 + \tan^2 x) = 2(\tan x)(1 + \tan^2 x).$$

The expression $1 + \tan^2 x$ is always > 0, and $f''(x) = 0$ if and only if $x = 0$ (in the given interval). If $x > 0$ then $\tan x > 0$ and if $x < 0$ then $\tan x < 0$. Hence $x = 0$ is the inflection point.

4. Sketch of graphs of $\sin^2 x$ and $|\sin x|$.



Observe that the function $\sin^2 x$ is differentiable, and its derivative is 0 at $x = 0, \pi/2, \pi$, etc., so the curve is flat at these points. On the other hand, $|\sin x|$ is not differentiable at $0, \pi, 2\pi$, etc., where it is "pointed." For instance, the graph of $|\sin x|$ for $\pi \leqq x \leqq 2\pi$ is obtained by reflecting the graph of $\sin x$ across the x-axis.

Let $f(x) = \sin^2 x$. Then $f'(x) = 2 \sin x \cos x = \sin 2x$. Also $f''(x) = 2 \cos 2x$. These allow you to find easily the regions of increase, decrease, and the inflection points, when $f''(x) = 0$. The inflection points are when $\cos 2x = 0$, that is $2x = \pi/2 + n\pi$ with an integer n, so $x = \pi/4 + n\pi/2$ with an integer n.

6. Bending up for $x > 0$; down for $x < 0$.

7. Bending up for $x > \sqrt{3}$, $-\sqrt{3} < x \leqq 0$. Down for $x < -\sqrt{3}$, $0 \leqq x < \sqrt{3}$.

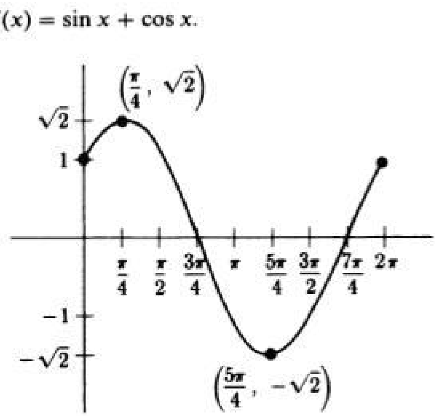
8. Bending up for $x > 1$, $-1 < x \leqq 0$. Down for $x < -1$ and $0 \leqq x < 1$.

9. Max at $x = \pi/4$. Min at $x = 5\pi/4$.
Strictly increasing for $0 \leqq x \leqq \pi/4$, $5\pi/4 \leqq x \leqq 8\pi/4$. Decreasing for

$$\pi/4 \leqq x \leqq 5\pi/4.$$

Inflection points: $3\pi/4$ and $7\pi/4$.

Sketch of curve $f(x) = \sin x + \cos x$.



VI, §3, p. 196

1. Let $f(x) = ax^3 + bx^2 + cx + d$. Then

$$f''(x) = 6ax + 2b.$$

There is a unique solution $x = -b/3a$. Furthermore, if $a > 0$:

$$f''(x) = 6ax + 2b > 0 \quad \Leftrightarrow \quad x > -b/3a,$$
$$f''(x) = 6ax + 2b < 0 \quad \Leftrightarrow \quad x < -b/3a.$$

Hence $x = -b/3a$ is an inflection point, and is the only one. If $a < 0$, dividing an inequality by a changes the direction of the inequality, but the argument is the same.

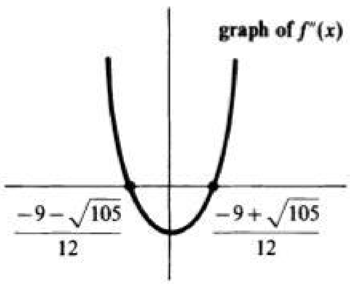
17. (a), (e) **18.** (c)

19. $f''(x) = 12x^2 + 18x - 2$. So $\frac{1}{2}f''(x) = 6x^2 + 9x - 1$, and $f''(x) = 0$ if and only if

$$x = \frac{-9 - \sqrt{105}}{12} \quad \text{or} \quad x = \frac{-9 + \sqrt{105}}{12}$$

Furthermore the graph of f'' is a parabola bending up (because the coefficient 12 of x^2 is positive) and so

$$f''(x) < 0 \quad \text{if and only if} \quad \frac{-9 - \sqrt{105}}{12} < x < x = \frac{-9 + \sqrt{105}}{12}.$$



Therefore $f''(x)$ changes sign at the two roots of $f''(x)$, whence these two roots are inflection points of f.

We have $f'(x) = 4x^3 + 9x^2 - 2x = x(4x^2 + 9x - 2)$. The critical points of f are the roots of f', that is

$$x_1 = 0, \qquad x_2 = \frac{-9 - \sqrt{113}}{8}, \qquad x_3 = \frac{-9 + \sqrt{113}}{8}.$$

Note that $f(-2) < 0$ (by direct calculation, so $f(x)$ is negative at some $x < 0$. Also

$$\text{if } x \to -\infty \text{ then } f(x) \to \infty,$$

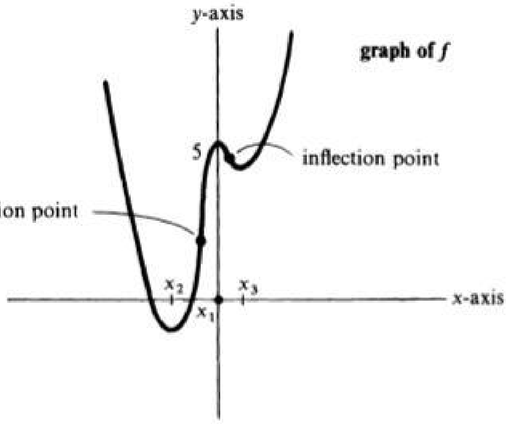
$$\text{if } x \to \infty \text{ then } f(x) \to \infty.$$

If $x \geqq 0$ then $f(x) > 0$. Indeed, if $0 < x \leqq 1$ then $-x^2 + 5 > 0$ and the other two terms $x^4 + 3x^3$ are both positive, so $f(x) > 0$. If $x \geqq 1$, then

$$x^4 - x^2 > 0,$$

and $3x^3 + 5 > 0$ so again $f(x) > 0$.

We can now sketch the graph of f.



21. We have $f'(x) = 6x^5 - 6x^3 + \frac{9}{8}x = 3x(2x^4 - 2x^2 + \frac{3}{8})$. Let $u = x^2$. Then the roots of $f'(x)$ (which are the critical points of f) are $x = 0$, and those values coming from the quadratic formula applied to u, namely

$$2u^2 - 2u + \tfrac{3}{8} = 0.$$

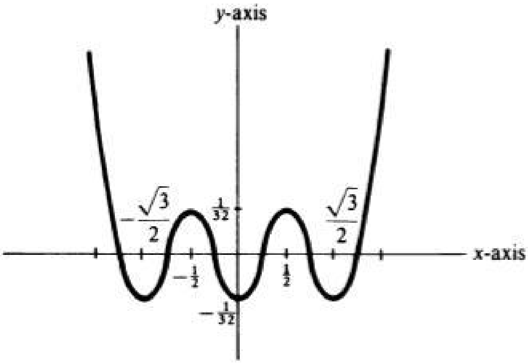
This yields $u = 1/4$ or $u = 3/4$, or in terms of x,

$$x = \pm 1/2 \qquad \text{and} \qquad x = \pm\sqrt{3}/2.$$

Observe that $f(x)$ can also be written in terms of $u = x^2$, namely

$$f(x) = u^3 - \tfrac{3}{2}u^2 + \tfrac{9}{16}u - \tfrac{1}{32}.$$

Then you will find that the values of f at the critical points are all equal to $1/32$ or $-1/32$. (Neat?) The graph looks like this.



The inflection points can also be found easily. We have

$$f''(x) = 30x^4 - 18x^2 + \tfrac{9}{8} = 30u^2 - 18u + \tfrac{9}{8}.$$

Hence $f''(x) = 0$ if and only if

$$u = \frac{-18 \pm \sqrt{189}}{60}.$$

This gives two values for u, and then $x = \pm\sqrt{u}$ are the inflection points.

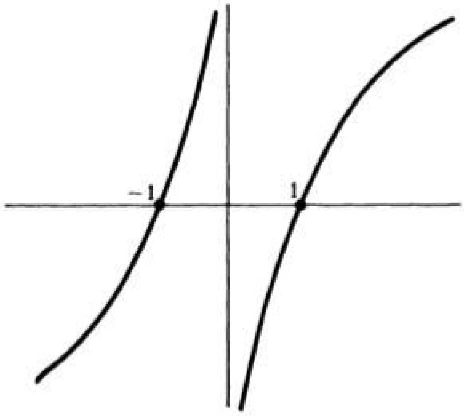
VI, §4, p. 201

Also see graphs below.

	c.p.	Increasing	Decreasing
1.	$3 \pm \sqrt{11}$	$x \leqq 3 - \sqrt{11}$ and $x \geqq 3 + \sqrt{11}$	$3 - \sqrt{11} \leqq x < 3$ and $3 < x \leqq 3 + \sqrt{11}$
2.	$3 \pm \sqrt{10}$	$3 - \sqrt{10} \leqq x \leqq 3 + \sqrt{10}$	$3 + \sqrt{10} \leqq x$ and $x \leqq 3 - \sqrt{10}$
3.	$-1 \pm \sqrt{2}$	$-1 - \sqrt{2} \leqq x \leqq -1 + \sqrt{2}$	$x \leqq -1 - \sqrt{2}$ and $x \geqq -1 + \sqrt{2}$
For **4** and **5**, see graphs below.			
6.	0	$x \leqq 0$	$\sqrt{2} < x$ and $0 \leqq x < \sqrt{2}$
7.	None	$x < -\frac{1}{3}$, $x > -\frac{1}{3}$	

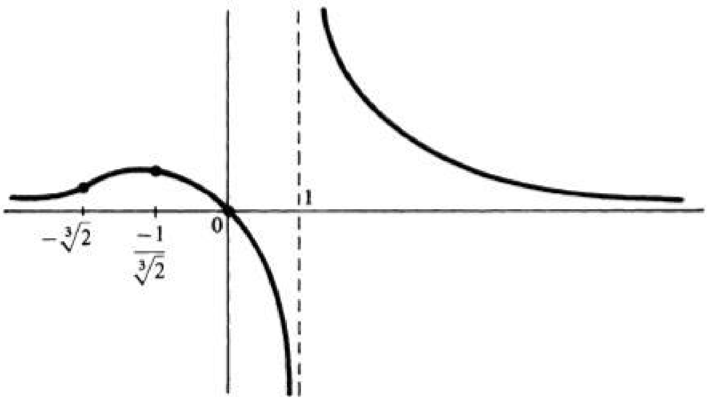
	Rel. Max.	Rel. Min.	Increasing	Decreasing
9.	$-\sqrt{3}$	$\sqrt{3}$	$x < -\sqrt{3}$, $x > \sqrt{3}$	$-\sqrt{3} < x < 0$, $0 < x < \sqrt{3}$
12.	None	None	Nowhere	$x < 5/3$, $x > 5/3$
14.	None	0	$x > 0$	$-1 < x < 0$
16.	None	None	Nowhere	$x < -\sqrt{5}$, $-\sqrt{5} < x < \sqrt{5}$, $x > \sqrt{5}$
18.	0	None	$x < -2$, $-2 < x < 0$	$0 < x < 2$, $x > 2$

4. $y = f(x) = x - 1/x$; no critical point. Function strictly increasing on every interval where defined.



5. $y = f(x) = x/(x^3 - 1)$; $f'(x) = -(2x^3 + 1)/(x^3 - 1)^2$

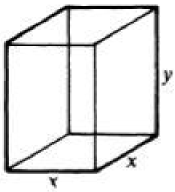
$$f''(x) = 6x^2(x^3 - 1)(x^3 + 2)/(x^3 - 1)^4$$
$$= ((x^3 + 2)/(x^3 - 1)) \cdot \text{square}.$$



Critical point at $x = -1/\sqrt[3]{2}$. Inflection point at $x = -\sqrt[3]{2}$.
Local max at critical point.

VI, §5, p. 211

1. $r\sqrt{2}$ by $\frac{1}{2}r\sqrt{2}$ where r is the radius of the semicircle.
2. Let x be the side of the base and y the other side, as shown on the figure.



Let $A(x)$ be the combined area. Then

$$A(x) = x^2 + 4xy = 48, \qquad \text{whence} \qquad y = \frac{48 - x^2}{4x}.$$

If the volume is V, then

$$V = x^2y = 12x - \frac{x^3}{4}.$$

Then $V'(x) = 12 - 3x^2/4$ and $V'(x) = 0$ if and only if $x = 4$, and $y = 2$ by substituting back in the expression for y in terms of x. We have for $x > 0$:

$$V'(x) > 0 \iff 12 - 3x^2/4 > 0 \iff x < 4,$$
$$V'(x) < 0 \iff 12 - 3x^2/4 < 0 \iff x > 4.$$

Hence $x = 4$ is a maximum point.
4. The total cost $f(x)$ is given by

$$f(x) = \frac{300}{x}\left(2 + \frac{x^2}{600}\right)\frac{30}{100} + D\,\frac{300}{x} = \frac{300D + 180}{x} + \frac{3x}{20}.$$

Then $f'(x) = 0$ if and only if $x^2 = 20(300D + 180)/3$. Answers: (a) $x = 20\sqrt{3}$ (b) $x = 40\sqrt{2}$ (c), (d), (e) $x = 60$ because the critical points are larger than 60, namely $20\sqrt{13}$, $60\sqrt{2}$ and $20\sqrt{23}$ in the respective cases.
5. $4\sqrt{2}$ by $8\sqrt{2}$
6. Answer: $x = 8/3$. For $x, y \geqq 0$ we require $x + y = 4$ and we want to minimize $x^2 + y^3$. Since $y = 4 - x$ we must minimize

$$f(x) = x^2 + (4 - x)^3.$$

Then $f'(x) = 2x - 3(4 - x)^2$, and $f'(x) = 0$ if and only if $x = (26 \pm 10)/6$. The solution 36/6 is beyond the range $0 \leqq x \leqq 4$. The critical point in this range is therefore $x = 16/6 = 8/3$. By a direction computation you can see that $f(8/3)$ is smaller than $f(0)$ or $f(4)$. Since there is only one critical point in the given interval, it must be the required minimum. You could also graph $f'(x)$ (parabola) to see that $f'(x) < 0$ if $0 < x < 8/3$ and $f'(x) > 0$ if $8/3 < x < 4$, so $f(x)$ is decreasing to the left of the critical point, and increasing to the right of the critical point, whence the critical point is a minimum in the given interval.

7. Minimum: use $24\pi/(4 + \pi)$ cm for circle; $96/(4 + \pi)$ cm for square.
Maximum: use whole wire for circle.
We show how to set up Exercise 7. Let x be the side of the square. Then $4x$ is the perimeter of the square, and $0 \leqq 4x \leqq 24$, so $0 \leqq x \leqq 6$. Also, $24 - 4x$ is the length of the circle, i.e. the circumference. But

$$24 - 4\pi = 2\pi r, \qquad \text{where } r \text{ is the radius,}$$

and πr^2 is the area of the circle. The sum of the areas is

$$\pi r^2 + x^2.$$

This is expressed in terms of two variables, x and r, but we have the relation

$$r = \frac{24 - 4x}{2\pi},$$

so that the sum of the areas can be expressed in terms of one variable,

$$f(x) = \pi\left(\frac{24 - 4x}{2\pi}\right)^2 + x^2.$$

You can now minimize or maximize by finding first the critical points, and then investigate if they are maxima, minima, or whether such extrema occur when $4x = 24$ or $4x = 0$. Note that the graph of f is a parabola, bending up. What is $f(0)$? What is $f(6)$? The possible values for x are in the interval $0 \leqq x \leqq 6$. The maximum or minimum of f in this interval is either $f(0)$, $f(6)$, or at the critical point x such that $f'(x) = 0$. Draw the graph of f to get the idea of what happens.

8. Answer: $(1, 2)$. The square of the distance of (x, y) to $(2, 1)$ is

$$(x - 2)^2 + (y - 1)^2.$$

Since $y^2 = 4x$, we get $x = y^2/4$ and so we have to minimize

$$f(y) = \left(\frac{y^2}{4} - 2\right)^2 + (y - 1)^2.$$

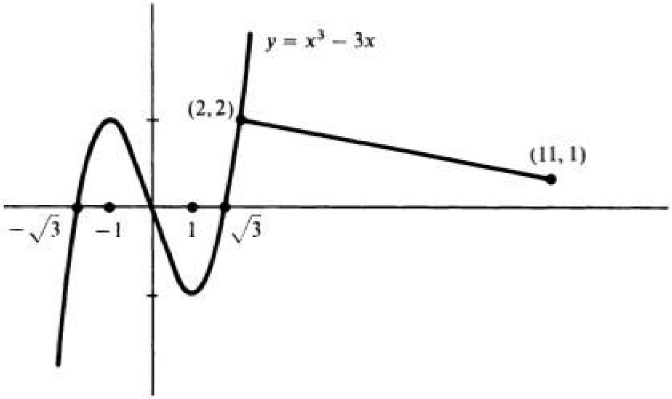
But $f'(y) = (y^3/4) - 2$ and $f'(y) = 0$ if and only if $y = 2$, so $x = 1$. You can verify for yourself that $f'(y) > 0$ if $y > 2$ and $f'(y) < 0$ if $y < 2$, so $y = 2$ is a minimum for $f(y)$.

9. $(\sqrt{5}/2, \frac{1}{2})$, $(-\sqrt{5}/2, \frac{1}{2})$

10. The square of the distance between $(11, 1)$ and (x, y) is

$$(x - 11)^2 + (y - 1)^2 = (x - 11)^2 + (x^3 - 3x - 1)^2 = f(x),$$

which is a function of x alone. The picture is as shown roughly.



We have to minimize $f(x)$. As x becomes large positive or negative, $f(x)$ becomes large positive because

$$f(x) = x^6 + \text{lower degree terms.}$$

Hence a minimum occurs at a critical point. We have

$$f'(x) = 2(x - 11) + 2(x^3 - 3x - 1)(3x^2 - 3)$$

so

$$\tfrac{1}{2} f'(x) = 3x^5 - 12x^3 - 3x^2 + 10x - 8.$$

Plugging 2 directly shows that $f'(2) = 0$, and therefore 2 is a critical point. If we could show that 2 is the only critical point of f, then we would be done. Let us get more information on $f'(x)$. By long division, we obtain a factoring

$$\tfrac{1}{2} f'(x) = (x - 2)g(x) \qquad \text{where} \qquad g(x) = 3x^4 + 6x^3 - 3x + 4.$$

The function $g(x)$ is still complicated. If $g(x) \neq 0$ for all x we would be done, but it looks hard, even if true. Let us avoid technical complications and let us simplify our original problem by inspection. The picture suggests that all points (x, y) on the curve $y = x^3 - 3x$ such that $x \leqq 1$ will be at further distance from $(11, 1)$ than $(2, 2)$. This is actually easily proved, for $f(2) = 82$, and if $x \leqq 1$ then

$$f(x) \geqq (x - 11)^2 \geqq 100 > 82.$$

Therefore it suffices to prove that $f(2)$ is a minimum for $f(x)$ when $x > 1$, and it suffices to prove that 2 is the only critical point of $f(x)$ for $x > 1$. Thus it suffices to prove that $g(x) \neq 0$ for $x > 1$. This is easy because $3x^4 + 4$ is positive, and

$$6x^3 - 3x = 3(2x^3 - 1) > 0 \qquad \text{for} \quad x > 1.$$

This proves that $g(x) > 0$ for $x > 1$. Therefore $x = 2$ is the only critical point of f for $x > 1$, and finally we have proved that the minimum of $f(x)$ is at $x = 2$.

11. $(5, 3), (-5, 3)$ **12.** $(-1/2, 1\sqrt{2})$ **13.** $(-1, 0)$

14. Answer $(1, 2)$. The square of the distance between (x, y) and $(9, 0)$ is $(x - 9)^2 + y^2$. Since $y = 2x^2$, we have to minimize

$$f(x) = (x - 9)^2 + (2x^2)^2 = (x - 9)^2 + 4x^4.$$

Then $f'(x) = 16x^3 + 2x - 18$, and $f'(1) = 0$. You can graph $f'(x)$ as usual. Since $f''(x) = 48x^2 + 2 > 0$ for all x, we see that $f'(x)$ is strictly increasing, so $f'(x) = 0$ only for $x = 1$, which is the only critical point of f. But $f(x)$ becomes large when x becomes large positive or negative, and so $f(x)$ has a minimum. Since there is only one critical point for f, the minimum is equal to the critical point, thus giving the answer.

15. $F = 2\sqrt{3}Q/9b^2$

16. Answer: $y = 2h/3$. We have $F'(y) = y\frac{1}{2}(h - y)^{-1/2}(-1) + (h - y)^{1/2}$, so

$$F'(y) = \frac{2h - 3y}{2(h - y)^{1/2}}.$$

Thus $F'(y) = 0$ if and only if $y = 2h/3$. But $F(0) = F(h) = 0$ and $F(y) > 0$ for all y in the interval $0 < y < h$. Hence the critical point must be the maximum.

17. $(2, 0)$

18. $\max x = 0$ (no triangle); $\min x = \dfrac{L}{2\pi\left(\dfrac{1}{6\sqrt{3}} + \dfrac{1}{2\pi}\right)}$. Cut the wire with x devoted to the triangle and $L - x$ to the circle. Let s be the side of the triangle so $3s = x$; let h be the height of the triangle, and r the radius of the circle. Then

$$L = x + 2\pi r \qquad \text{and} \qquad A = \tfrac{1}{2}sh + \pi r^2.$$

We need other relations to make A a function of one variable x, namely

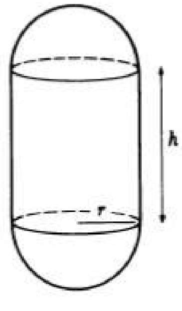
$$h = s\sqrt{3}/2 = x\sqrt{3}/6 \qquad \text{and} \qquad r = (L - x)/2\pi.$$

Then $A(x) = x^2\sqrt{3}/36 + ((L - x)/2\pi)^2$ and $A'(x) = x\sqrt{3}/18 - (L - x)/2\pi$. Thus the critical point is as given in the answer. Since the graph of $A(x)$ is a parabola bending up, the critical point is a minimum. The maximum occurs at the end point of the interval $0 \leqq x \leqq L$. To find out which end point, evaluate $A(0)$ and $A(L)$ and compare the two values to see that $A(0)$ is bigger.

19. $4\left[1 + \left(\dfrac{13.5}{4}\right)^{2/3}\right]^{3/2} = \dfrac{13\sqrt{13}}{2}$

20. We just set it up. Let r be the radius of the base of the cylinder, and h the height. Then the total volume, which is constant, is

$$V = \pi r^2 h + \tfrac{4}{3}\pi r^3.$$



This allows to solve for h, namely

$$h = \frac{V - 4\pi r^3/3}{\pi r^2}.$$

The cost of material is a constant times:

(Area of cylinder) + (twice area of sphere).

We can express this cost as a function of r and h, namely

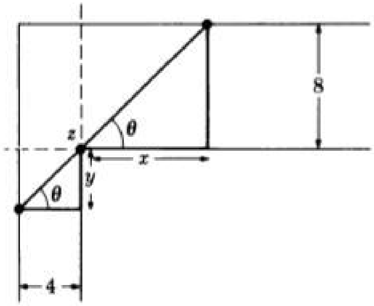
$$2\pi rh + 2 \cdot 4\pi r^2.$$

Since h is expressed as a function of r above, we get the total cost expressed as a function of r only, namely

$$f(r) = C\left[2\pi r\left(\frac{V - 4\pi r^3/3}{\pi r^2}\right) + 8\pi r^2\right] = C\left[\frac{2V}{r} + \frac{16}{3}\pi r^2\right].$$

From here on, you can find $f'(r)$ and proceed in the usual way to find when $f'(r) = 0$. There is only one critical point, and $f(r)$ becomes large when r approaches 0 or r becomes large. Hence the critical point is a minimum.

21. The picture is as follows:



The longest rod is that which will fit the minimal distance labeled z in the figure. We let x, y be as shown in the figure. We have

$$z = \sqrt{4^2 + y^2} + \sqrt{x^2 + 8^2}.$$

This depends on the two variables x, y. But we can find a relation between them by using similar triangles, namely

$$\frac{8}{x} = \frac{y}{4},$$

so that

$$y = \frac{32}{x}.$$

Hence

$$z = f(x) = \sqrt{16 + \left(\frac{32}{x}\right)^2} + \sqrt{x^2 + 64}.$$

You then have to minimize $f(x)$. The answer comes out

$$x = 4\sqrt[3]{4}, \qquad z = \sqrt{4^2 + \left(\frac{32}{4\sqrt[3]{4}}\right)^2} + \sqrt{(4\sqrt[3]{4})^2 + 8^2}$$

22. From the figure in the text, we have $0 \leqq x \leqq a$, and

$$\operatorname{dist}(P, R) = \sqrt{x^2 + y_1^2},$$
$$\operatorname{dist}(R, Q) = \sqrt{(a - x)^2 + y_2^2}.$$

Hence the sum of the distances is

$$f(x) = \sqrt{x^2 + y_1^2} + \sqrt{(a - x)^2 + y_2^2}.$$

Then

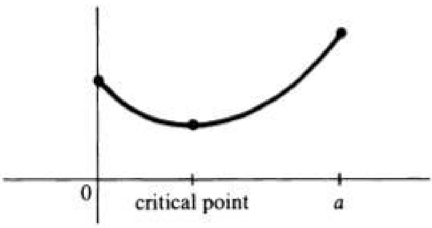
$$f'(x) = \frac{x}{\sqrt{x^2 + y_1^2}} - \frac{(a - x)}{\sqrt{(a - x)^2 + y_2^2}}.$$

We have $f'(x) = 0$ if and only if

$$\frac{x}{\sqrt{x^2 + y_1^2}} = \frac{a - x}{\sqrt{(a - x)^2 + y_2^2}},$$

which is the desired cosine relation.

In particular, we see that there is only one critical point. Hence the minimum of f is either at the end points $x = 0$ or $x = a$, or at the critical point. We shall now prove that the critical point is the minimum. We are trying to prove that the graph of f looks something like this.



When x is near 0 and $x > 0$ then the first term in $f'(x)$ is small and the second term is near

$$-\frac{a}{\sqrt{a^2 + y_2^2}}$$

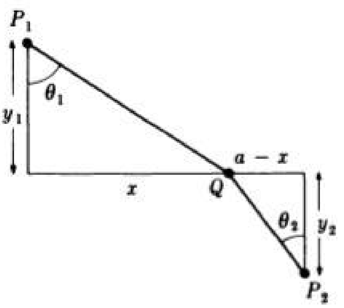
which is negative. Hence $f'(x) < 0$ when x is near 0, and the function is decreasing when x is near 0.

On the other hand, suppose x is near a. Then the second term in $f'(x)$ is near 0 and the first term is near

$$\frac{a}{\sqrt{a^2 + y_1^2}} > 0.$$

Consequently $f'(x) > 0$ when x is near a. Therefore f is increasing when x is near a. Since $f(x)$ is decreasing for x near 0 and increasing for x near a it follows that the minimum cannot be at the end points, and hence is somewhere in the middle. Hence the minimum is a critical point, and we have seen that there is only one critical point. Hence the minimum is at the critical point. This proves what we wanted.

23. Let x and $a - x$ be as on the figure. Then $0 \leqq x \leqq a$.



Let t_1 be the time needed to travel from P_1 to Q, and t_2 the time needed to travel from Q to P_2. Then

$$t_1 = \frac{\text{dist}(P_1, Q)}{v_1} \quad \text{and} \quad t_2 = \frac{\text{dist}(Q, P_2)}{v_2}.$$

Then

$$t_1 + t_2 = f(x) = \frac{1}{v_1}\sqrt{x^2 + y_1^2} + \frac{1}{v_2}\sqrt{(a - x)^2 + y_2^2}.$$

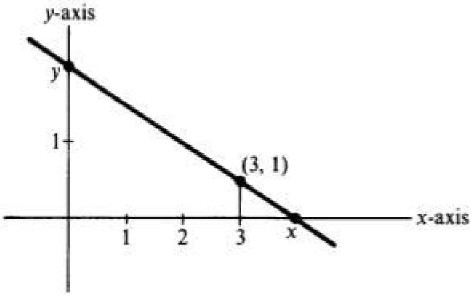
Both v_1, v_2 are given as constant. Hence again we have to take $f'(x)$ and set it equal to 0. This is similar to Exercise 22, and we find exactly the relation that is to be proved.

24. $p = s/n$. Since $L(0) = L(1) = 0$ and $L(p) > 0$ for $0 < p < 1$, it follows that the maximum is at a critical point. But

$$\begin{aligned} L'(p) &= p^s(n - s)(1 - p)^{n-s-1}(-1) + sp^{s-1}(1 - p)^{n-s} \\ &= p^{s-1}(1 - p)^{n-s-1}[p(s - n) + s(1 - p)] \\ &= p^{s-1}(1 - p)^{n-s-1}(-np + s). \end{aligned}$$

For $0 < p < 1$, the factor $p^{s-1}(1 - p)^{n-s-1}$ is not 0, so $L'(p) = 0$ if and only if $-np + s = 0$, that is $p = s/n$. Hence there is only one critical point, so the maximum is at the critical point.

25. (a)



Let x, y be the intercepts of the line with the axes. Then the area of the triangle is equal to

$$A = \tfrac{1}{2}xy.$$

We want to minimize the area. By similar triangles, we know that

$$\frac{y}{x} = \frac{1}{x-3} \qquad \text{so} \qquad y = \frac{x}{x-3}.$$

Then the area is given by

$$A(x) = \tfrac{1}{2}x\,\frac{x}{x-3} = \frac{1}{2}\,\frac{x^2}{x-3}.$$

From the physical considerations, we are limited to the interval $x > 3$. As x approaches 3 and $x > 3$ the denominator approaches 0 and is positive. Since x^2 approaches 9, it follows that $A(x)$ becomes large positive. Also as $x \to \infty$, $A(x) \to \infty$. Hence the minimum of A will occur at a critical point.

We find the critical points. We have

$$2A'(x) = \frac{(x-3)2x - x^2}{(x-3)^2} = \frac{x^2 - 6x}{(x-3)^2}.$$

Then $A'(x) = 0$ if and only if $x = 6$ ($x = 0$ is excluded because $x > 3$). Hence the desired line passes through the point $(6, 0)$. The equation is then

$$y - 0 = \frac{1-0}{3-6}(x-6) = -\tfrac{1}{3}(x-6).$$

or also

$$y - 2 = -\tfrac{1}{3}x.$$

25. (b) $3y = -2x + 12$

26. $x = (a_1 + \cdots + a_n)/n$. We are given

$$f(x) = (x - a_1)^2 + \cdots + (x - a_n)^2.$$

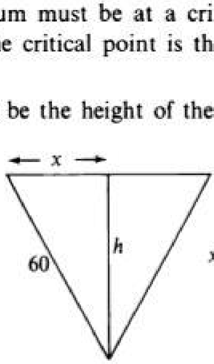
Then

$$\begin{aligned} f'(x) &= 2(x - a_1) + \cdots + 2(x - a_n) \\ &= 2x - 2a_1 + 2x - 2a_2 + \cdots + 2x - 2a_n \\ &= n2x - 2(a_1 + \cdots + a_n). \end{aligned}$$

So $f'(x) = 0$ if and only if $nx = a_1 + \cdots + a_n$. Divide by n to get the critical point of f. Since $f(x) \to \infty$ as $x \to \pm\infty$ because for instance just one square term $(x - a_1)^2$ becomes large when x becomes large positive or negative, it follows that the minimum must be at a critical point, and there is only one critical point. Hence the critical point is the minimum.

27. $25/\sqrt{2}$

28. Answer: $\theta = \pi/2$. Let h be the height of the triangle as shown on the picture.



It suffices to maximize the area given by

$$A(x) = \tfrac{1}{2}xh = \tfrac{1}{2}x\sqrt{60^2 - x^2} \qquad \text{for} \quad 0 < x < 60.$$

Then $A'(x) = (-2x^2 + 60^2)/2(60^2 - x^2)^{1/2}$ and $A'(x) = 0$ if and only if $x = 60/\sqrt{2}$. But $A(x) \geqq 0$ and $A(0) = A(60) = 0$. Hence the critical point is the maximum. The above value for x implies that $\theta = \pi/2$, because the triangle is similar to the triangle with sides 1, 1, $\sqrt{2}$.

29. $\pi/3$. We work it out. Let the depth be y. Then $\sin\theta = y/100$. Maximum capacity occurs when the area of the cross section is maximum. This area is equal to

$$A = 100y + 2(\tfrac{1}{2}y\ 100\cos\theta).$$

You have a choice whether to express A entirely in terms of y or entirely in terms of θ. Suppose we do it in terms of θ. Then

$$A(\theta) = 10^4 \sin\theta + 10^4 \sin\theta\cos\theta = 10^4[\sin\theta + \tfrac{1}{2}\sin 2\theta].$$

So $A'(\theta) = 10^4[\cos\theta + \cos 2\theta]$, and $0 \leqq \theta \leqq \pi/2$. But in this interval, $\cos\theta$ and $\cos 2\theta$ are strictly decreasing so $A'(\theta)$ is strictly decreasing. We have $A'(\theta) = 0$ precisely when $\cos\theta + \cos 2\theta = 0$, which occurs when $\theta = \pi/3$. Thus $A'(\theta) > 0$ if $0 < \theta < \pi/3$ and $A'(\theta) < 0$ if $\pi/3 < \theta < \pi/2$. Hence $A(\theta)$ is increasing for $0 \leqq \theta \leqq \pi/3$ and decreasing for $\pi/3 \leqq \theta \leqq \pi/2$. Hence the maximum occurs when $\theta = \pi/3$.

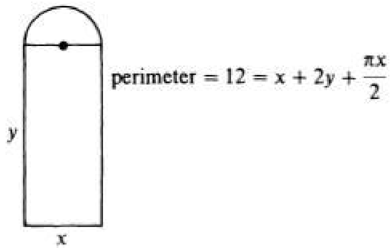
30. (a) $a = 16$ (b) $a = -54$. To see this, note that

$$f'(x) = 2x - a/x^2,$$

and $f'(x) = 0$ if and only if $a = 2x^3$. For $x = 2$ and $x = -3$, this gives the desired value for a. You can check that this is a minimum directly by determining when $f'(x) > 0$ or $f'(x) < 0$. As for part (c), this is one of the rare cases when taking the second derivative is useful. The second derivative is $f''(x) = 2 + 2a/x^3$, and the critical point has been determined to be when $a = 2x^3$; so if x is the critical point we get $f''(x) = 6 > 0$. Hence the critical point must be a local minimum.

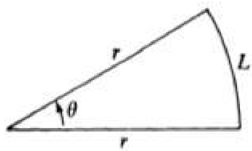
31. $\dfrac{c}{1 + \sqrt[3]{a/b}}$ away from b

32. base $= 24/(4 + \pi)$ and height $= 12/(4 + \pi y)$. The picture is as follows.



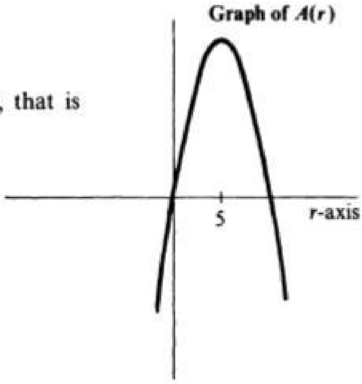
Area $= xy + \frac{1}{2}\pi r^2 = xy + \frac{1}{2}\pi(x/2)^2$. You can solve for y in terms of x by using the perimeter equation, so the area is given as a function of x, $A(x)$. Finding $A'(x) = 0$ yields the desired value for x. It is a critical point, the only critical point, and $A(x)$ is a parabola which bends down, so the critical point is a maximum.

33. (a) Find the radius and angle of a circular sector of maximum area if the perimeter is 20 cm.



Let r be the radius of the sector, and L the length of the circular arc. Then $L = (\theta/2\pi)2\pi r = \theta r$. The perimeter is

$$P = 2r + L = 2r + \theta r = 20.$$



Hence we can solve for θ in terms of r, that is

$$\theta = \frac{20}{r} - 2.$$

The area of the sector is

$$A = \frac{\theta}{2\pi}\,\pi r^2 = \frac{\theta r^2}{2}$$

so in terms of r alone:

$$A(r) = \left(\frac{20}{r} - 2\right)\frac{r^2}{2} = 10r - r^2.$$

The graph of $A(r)$ is a parabola bending down, so the maximum is at the critical point. But

$$A'(r) = 10 - 2r$$

so the maximum is when $2r = 10$ so $r = 5$. Then

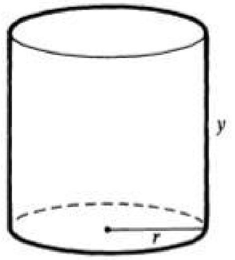
$$\theta = \frac{20}{5} - 2 = 2.$$

33. (b) radius = 4 cm, angle = 2 radians

34. Don't make things more complicated than they need to be. Observe that $2\sin\theta\cos\theta = \sin 2\theta$. This is a maximum when $\theta = \pi/4$.

35. $2(1 + \sqrt[3]{36})^{3/2}$

36. $r = (V/\pi)^{1/3} = y$. Note that $V = \pi r^2 y$ so $y = V/\pi r^2$.



Let S = surface area so $S = \pi r^2 + 2\pi r y$. Then

$$S(r) = \pi r^2 + 2V/r.$$

We have $S'(r) = 2\pi r - 2V/r^2 = 0$ if and only if $r = (V/\pi)^{1/3}$. There is only one critical point. But $S(r)$ becomes large when r approaches 0 or also when r becomes large. There is a minimum since $S(r) > 0$ for $r > 0$, and so the minimum is equal to the single critical point.

37.

$$P = 2r + L = 2r + r\theta.$$

From $A = \theta r^2/2$ we get $\theta = 2A/r^2$ so P can be expressed in terms of r only by

$$P(r) = 2r + 2A/r.$$

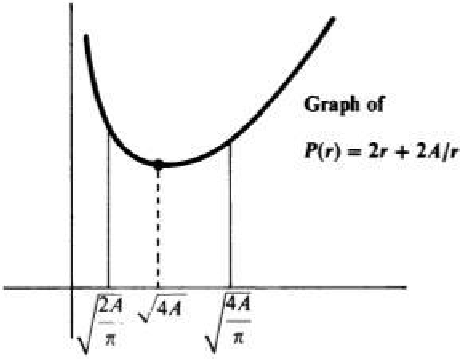
We have $P'(r) = 2 - 2A/r^2$, and $P'(r) = 0$ if and only if $r = A^{1/2}$. So P has only one critical point, and $P(r) \to \infty$ as $r \to \infty$ and as $r \to 0$. Hence P has a minimum and that minimum, is at the critical point. This is a minimum for all values of $r > 0$. In part (a), we have $\theta \leqq \pi$ so $r^2 \geqq 2A/\pi$, and the data limits us to the interval

$$\sqrt{2A/\pi} \leqq r.$$

Hence in part (a) the minimum is at the critical point. In part (b), we have $\theta \leqq \pi/2$ so $r^2 \geqq 4A/\pi$, which limits us to the interval

$$\sqrt{4A/\pi} \leqq r.$$

Since $\sqrt{4A/\pi} > \sqrt{A}$, the minimum is at the end point $r = \sqrt{4A/\pi}$.



38. $x = 20$. The profits are given by

$$P(x) = 50x - f(x).$$

Then $P'(x) = -3x^2 + 90x - 600$. By the quadratic formula, $P'(x) = 0$ if and only if $x = 20$ or $x = 10$. But the graph of $P(x)$ is that of a cubic, which you should know how to do. Also $P(10)$ is negative, and $P(0)$ is also negative. So the maximum is at $x = 20$.

39. 18 units, daily profit \$266

40. $(50 + 5\sqrt{94})/3$. Same method as Problems 38 and 39.

41. 30 units; \$8900

42. 20. Let $g(x)$ be the profits. Since $p = 1000 - 10x$, we get

$$g(x) = x(1000 - 10x) - f(x) = -20x^2 + 800x - 6000.$$

The graph of $g(x)$ is a parabola bending down, and $g'(x) = 0$ if and only if $x = 20$, which gives the maximum for $g(x)$.

VII, §1, p. 221

1. Yes; all real numbers **3.** Yes; all real numbers **5.** Yes; for $y < 1$
7. Yes; for $y \geqq 1$ **9.** Yes; for $y \leqq -1$ **11.** Yes; for $y \geqq 2$
13. Yes; for $-1 \leqq y \leqq 1$

VII, §2, p. 224

0. Let $f(x) = -x^3 + 2x + 1$. Then

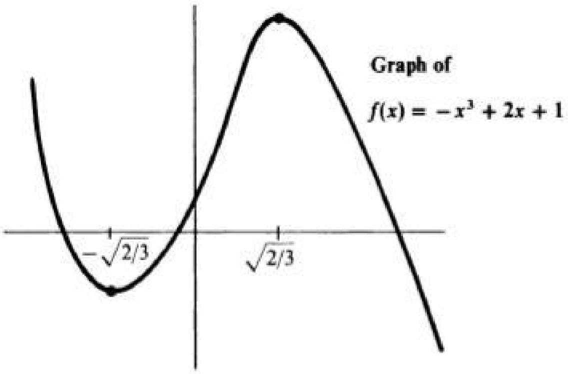
$$\begin{aligned} f'(x) &= -3x^2 + 2 \\ &= 0 \quad \text{if and only if} \quad 3x^2 = 2 \\ &\quad\quad \text{if and only if} \quad x = \sqrt{2/3} \quad \text{and} \quad x = -\sqrt{2/3}. \end{aligned}$$

These are the critical points of f. Also

$$\begin{aligned} f'(x) > 0 \quad &\Leftrightarrow \quad -3x^2 + 2 > 0 \\ &\Leftrightarrow \quad 3x^2 < 2 \\ &\Leftrightarrow \quad |x| < \sqrt{2/3}, \\ f'(x) < 0 \quad &\Leftrightarrow \quad x^2 > 2/3, \\ &\Leftrightarrow \quad x > \sqrt{2/3} \quad \text{and} \quad x < -\sqrt{2/3}. \end{aligned}$$

There are three maximal intervals where an inverse function of f could be defined (excluding the end points):

$$x < -\sqrt{2/3}, \qquad -\sqrt{2/3} < x < \sqrt{2/3}, \qquad x > \sqrt{2/3}.$$



Over each such interval, there is an inverse function of f whose value at 2 is some point in the interval. We work out two out of the three cases, but you only had to pick one of them.

Case 1. Observe that $f(1) = 2$, and $1 > \sqrt{2/3}$. Therefore in this case, if g is the inverse function, then

$$g'(2) = \frac{1}{f'(1)} = \frac{1}{-3+2} = -1.$$

Case 2. Take the interval $-\sqrt{2/3} < x < \sqrt{2/3}$. We want to solve $f(x) = 2$, that is

$$-x^3 + 2x + 1 = 2, \qquad \text{or} \qquad x^3 - 2x + 1 = 0.$$

Factoring, this is the same as

$$(x-1)(x^2 + x - 1) = 0.$$

In the given interval, $x = 1$ is not a solution. There are two other possible solutions:

$$x = \frac{-1+\sqrt{5}}{2} \qquad \text{and} \qquad x = \frac{-1-\sqrt{5}}{2}.$$

But $(-1-\sqrt{5})/2$ is not in the present interval of definition. Hence there is only one possible x in this case, namely $x_1 = (-1+\sqrt{5})/2$. If g is the inverse function for the given interval, then

$$g'(2) = \frac{1}{f'(x_1)} = \frac{1}{-3x_1^2 + 2}.$$

(Alternative answers depend on the choice of intervals.)

1. $\frac{1}{3}$ **2.** $\frac{1}{11}$ **3.** $\frac{1}{3}$ or $-\frac{1}{3}$ **4.** ± 1 **5.** ± 1 **6.** $\pm\frac{1}{2}$ or $\pm\dfrac{1}{2\sqrt{2}}$

7. $\frac{1}{4}$ **8.** -1 or $\frac{1}{2} \pm \frac{3}{10}\sqrt{5}$ **9.** $\frac{1}{24}$ **10.** $\dfrac{1}{10\sqrt{2}}$ or $\dfrac{-1}{10\sqrt{2}}$

11. $$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))},$$

$$g''(y) = \frac{-1}{f'(g(y))^2} f''(g(y))g'(y) = \frac{-1}{f'(x)^2} f''(x)g'(y).$$

If $f'(x) > 0$ then $g'(y) > 0$ by the first formula, and $g''(y) < 0$ by the second.

VII, §3, p. 229

1 and 2. View the cosine as defined on the interval

$$0 \leqq x \leqq \pi.$$

On this interval, the cosine is strictly decreasing, and for $0 < x < \pi$ we have

$$\frac{d \cos x}{dx} = -\sin x < 0.$$

Hence the inverse function $x = g(y)$ exists, and is called the **arccosine**. Since $y = \cos x$ decreases from 1 to -1, the arccosine is defined on the interval $[-1, 1]$. Its derivative is given by $g'(y) = 1/f'(x)$, so that

$$\frac{d \arccos y}{dy} = g'(y) = \frac{1}{-\sin x}.$$

But we have the relationship

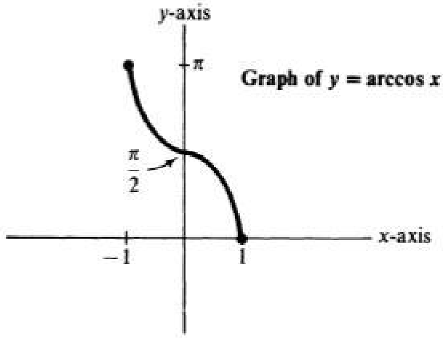
$$\sin^2 x = 1 - \cos^2 x,$$

and for $0 < x < \pi$ we have $\sin x > 0$ so that

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - y^2}.$$

Consequently $g'(y) = \dfrac{-1}{\sqrt{1-y^2}}$.

The graph of arccos looks like this.



3. (a) $2/\sqrt{3}$ (b) $\sqrt{2}$ (c) $\pi/6$ (d) $\pi/4$ (e) 2 (f) $\pi/3$

4. $-2/\sqrt{3}$, $-\sqrt{2}$, $\pi/3$, $\pi/4$

5. Let $y = \sec x$ on interval $0 < x < \pi/2$. Then $x = \operatorname{arcsec} y$ is defined on $1 < y$, and $dx/dy = \dfrac{1}{y\sqrt{y^2-1}}$.

6. $-\pi/2$ **7.** 0 **8.** $\pi/2$ **9.** $\pi/2$ **10.** $-\pi/4$ **11.** $\dfrac{1}{\sqrt{1-(x^2-1)^2}}\,2x$

12. $\dfrac{-1}{\sqrt{-(x^2+5x+6)}}$ **13.** $\dfrac{-1}{(\arcsin x)^2\sqrt{1-x^2}}$ **14.** $\dfrac{4}{\sqrt{1-4x^2(\arccos 2x)^2}}$

VII, §4, p. 233

1. $\pi/4$, $\pi/6$, $-\pi/4$, $\pi/3$ **2.** $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ **3.** $1/(1+y^2)$

4. (a) $-\pi/4$ (b) 0 (c) $-\pi/6$ (d) $\pi/6$ **5.** $\dfrac{3}{1+9x^2}$ **7.** 0 **9.** $\dfrac{2\cos 2x}{1+\sin^2 2x}$

11. $\dfrac{(\cos x)(\arcsin x) - (\sin x)/\sqrt{1-x^2}}{(\arcsin x)^2}$ **13.** $\dfrac{-1}{1+x^2}$

15. $\dfrac{9}{\sqrt{1-9x^2}}(1+\arcsin 3x)^2$ **17.** $\left(y - \dfrac{\pi}{4}\right) = \sqrt{2}\left(x - \dfrac{1}{\sqrt{2}}\right)$

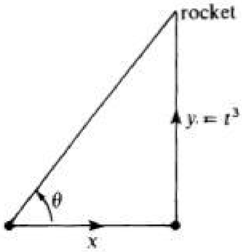
19. $\left(y - \dfrac{\pi}{3}\right) = \dfrac{1}{2}\left(x - \dfrac{\sqrt{3}}{2}\right)$ **21.** $\left(y + \dfrac{\pi}{6}\right) = \dfrac{2}{\sqrt{5}}\left(x + \dfrac{1}{2}\right)$ **22.** 2/25 rad/sec

23. 440 ft/sec **24.** 3/26 rad/sec **25.** 0.02 rad/sec **27.** $\dfrac{d\theta}{dt} = \frac{1}{6}\sin\theta\tan\theta$

28. $\dfrac{2}{25\sqrt{21}}$ rad/sec **29.** $\frac{1}{82}$ rad/sec

31. (a) $\dfrac{1500}{(400)^2 + (\frac{225}{4})^2}$ rad/sec (b) $\dfrac{1500}{(600)^2 + (\frac{375}{4})^2}$ rad/sec

33. Picture:



We are given $dx/dt = -50$. When $t = 0$ we have $x(0) = 300$, so in general

$$x(t) = 300 - 50t.$$

Furthermore, $dy/dt = 3t^2$. We want to find $d\theta/dt$. We have

$$\tan\theta = y/x \qquad \text{so} \qquad \theta = \arctan y/x.$$

Then

$$\frac{d\theta}{dt} = \frac{1}{1 + (y/x)^2}\,\frac{x\,dy/dt - y\,dx/dt}{x^2}.$$

But $y(5) = 125$ and $x(5) = 300 - 250 = 50$. Hence

$$\left.\frac{d\theta}{dt}\right|_{t=5} = \frac{1}{1 + \left(\dfrac{125}{50}\right)^2}\,\frac{50\cdot 3\cdot 25 - 125\cdot(-50)}{50^2}$$

$$= 16/29 \text{ rad/sec}.$$

VIII, §1, p. 244

1. (a) $y = 2e^2x - e^2$ (b) $y = 2e^{-4}x + 5e^{-4}$ (c) $y = 2x + 1$
2. (a) $y = \frac{1}{2}e^{-2}x + 3e^{-2}$ (b) $y = \frac{1}{2}e^{1/2}x + \frac{1}{2}e^{1/2}$ (c) $y = \frac{1}{2}x + 1$
3. $y = 3e^2x - 4e^2$ **4.** (a) $e^{\sin 3x}(\cos 3x)3$ (b) $\cos(e^x + \sin x)(e^x + \cos x)$
(c) $\cos(e^{x+2})e^{x+2}$ (d) $4\cos(e^{4x-5})e^{4x-5}$

5. (a) $\dfrac{1}{1 + e^{2x}}\,e^x$ (b) $e^x(-\sin(3x + 5))3 + e^x\cos(3x + 5)$

(c) $2(\cos 2x)e^{\sin 2x}$ (d) $-\dfrac{1}{\sqrt{1 - x^2}}\,e^{\arccos x}$ (e) $-e^{-x}$

(h) $e^{-\arcsin x}\left(\dfrac{-1}{\sqrt{1 - x^2}}\right)$ (i) $e^x\sec^2 e^x$ (j) $\dfrac{1}{1 + e^{4x}}\,2e^{2x}$ (k) $\dfrac{-1}{\sin^2 e^x}(\cos e^x)e^x$

(l) $\dfrac{1}{\sqrt{1 - (e^x + x)^2}}(e^x + 1)$ (m) $\sec^2(xe^{\tan x})$ (n) $(1 + \tan^2 e^x)e^x$

6. (c) Let $f(x) = xe^x$. Suppose you have already proved that

$$f^{(n)}(x) = (x + n)e^x.$$

Then

$$\begin{aligned} f^{(n+1)}(x) &= \frac{d}{dx}[(x+n)e^x] \\ &= (x+n)e^x + e^x \qquad \text{(derivative of a product)} \\ &= (x+n+1)e^x. \end{aligned}$$

This proves the formula for the $(n + 1)$-th derivative.

8. (c) Differentiate $f(x)/e^{h(x)}$ by the rule for quotients and the chain rule. We get:

$$\begin{aligned} \frac{d}{dx}\left(\frac{f(x)}{e^{h(x)}}\right) &= \frac{e^{h(x)}f'(x) - f(x)e^{h(x)}h'(x)}{e^{2h(x)}} \\ &= \frac{e^{h(x)}h'(x)f(x) - f(x)e^{h(x)}h'(x)}{e^{2h(x)}} \\ &= 0. \end{aligned}$$

Hence $f(x)/e^{h(x)}$ is constant, so there is a constant C such that

$$\frac{f(x)}{e^{h(x)}} = C.$$

Now cross multiply to get $f(x) = Ce^{h(x)}$.

9. $(y - e^2) = 2e^2(x - 1)$ **10.** $y - 2e^2 = 3e^2(x - 2)$ **11.** $y - 5e^5 = 6e^5(x - 5)$

12. $y = x$ **13.** $(y - 1) = -x$ **14.** $y - e^{-1} = e^{-1}(x - 1)$

15. Let $f(x) = e^x + x$. Then $f'(x) = e^x + 1 > 0$ for all x. Hence f is strictly increasing. We have $f(0) = 1$, and $f(-1) = 1/e - 1 < 0$ because $1/e < 1$. By the intermediate value theorem, there exists some x such that $f(x) = 0$, and this value of x is unique because f is strictly increasing.

16. See the proof of Theorem 5.1.

17. If $x = 1$ in Exercise 16(b) we get $2 < e$. If $x = 1$ in Exercise 16(c) we get $2.5 < e$.

18. See Theorem 5.2.

19. (a) Let $f_1(x) = e^{-x} - (1 - x)$. Then $f_1'(x) = -e^{-x} + 1$ and since $e^x > 1$ for $x > 0$ we get

$$f_1'(x) = -\frac{1}{e^x} + 1 > 0 \qquad \text{for} \quad x > 0.$$

Hence f is strictly increasing for $x \geqq 0$. Since $f_1(0) = 0$ we conclude $f_1(x) > 0$ for $x > 0$, in other words

$$e^{-x} - (1 - x) > 0 \qquad \text{for} \quad x > 0,$$

and therefore $e^{-x} > 1 - x$ for $x > 0$, as desired.

(b) Let $f_2(x) = 1 - x + x^2/2 - e^{-x}$. Then

$$f_2'(x) = -1 + x + e^{-x} = f_1(x).$$

By part (a), we know that $f_1(x) > 0$ for $x > 0$. Hence f_2 is strictly increasing for $x \geqq 0$. Since $f_2(0) = 0$ we conclude that $f_2(x) > 0$ for $x > 0$, whence (b) follows at once.

(c) Let $f_3(x) = e^{-x} - \left(1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{3 \cdot 2}\right)$. Then $f_3'(x) = f_2(x)$. Use part (b) and similar arguments as before to conclude $f_3(x) > 0$ for $x > 0$, whence (c) follows.

(d) Left to you.

20. If we put $x = 1/2$ in Exercise 19(a), then we find $\frac{1}{2} < e^{-1/2}$, or in other words, $\frac{1}{2} < 1/e^{1/2}$. Hence $e^{1/2} < 2$ and $e < 4$. If we put $x = 1$ in Exercise 19(c), then we find

$$\frac{1}{2} - \frac{1}{6} < e^{-1} = \frac{1}{e}, \qquad \text{that is} \qquad \frac{1}{3} < \frac{1}{e},$$

whence $e < 3$.

21. $\cosh^2(t) = \frac{1}{4}(e^t + e^{-t})^2 = \frac{1}{4}(e^{2t} + 2e^te^{-t} + e^{-2t})$

$$= \tfrac{1}{4}(e^{2t} + 2 + e^{-2t}).$$

Similarly,

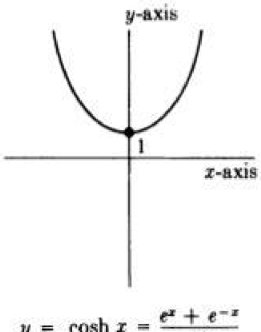
$$\sinh^2(t) = \tfrac{1}{4}(e^{2t} - 2 + e^{-2t}).$$

Subtracting yields $\cosh^2 - \sinh^2 = 1$.

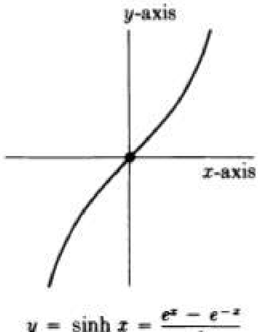
As for the derivative, $\cosh'(t) = \frac{1}{2}(e^t - e^{-t})$; don't forget how to use the chain rule: let $u = -t$. Then

$$\frac{de^{-t}}{dt} = \frac{de^u}{du}\frac{du}{dt} = -e^{-t}.$$

22.



$y = \cosh x = \dfrac{e^x + e^{-x}}{2}$



$y = \sinh x = \dfrac{e^x - e^{-x}}{2}$

VIII, §2, p. 255

1. (a) $(y - \log 2) = \frac{1}{2}(x - 2)$
(b) $(y - \log 5) = \frac{1}{5}(x - 5)$
(c) $(y - \log \frac{1}{2}) = 2(x - \frac{1}{2})$

2. (a) $(y - \log 2) = -(x + 1)$
(b) $(y - \log 5) = \frac{4}{5}(x - 2)$
(c) $(y - \log 10) = \dfrac{-3}{5}(x + 3)$

3. (a) $\dfrac{\cos x}{\sin x}$ (b) $\cos(\log(2x + 3))\dfrac{1}{2x + 3} \cdot 2$ (c) $\dfrac{1}{x^2 + 5} \cdot 2x$
(d) $\dfrac{(\sin x)/x - (\log 2x)\cos x}{\sin^2 x}$

4. $(y - \log 4) = \frac{1}{4}(x - 3)$ **5.** $(y - \log 3) = \frac{2}{3}(x - 4)$ **7.** $(y - 1) = \dfrac{1}{e}(x - e)$

8. $(y - e) = 2(x - e)$ **9.** $(y - 2\log 2) = (1 + \log 2)(x - 2)$

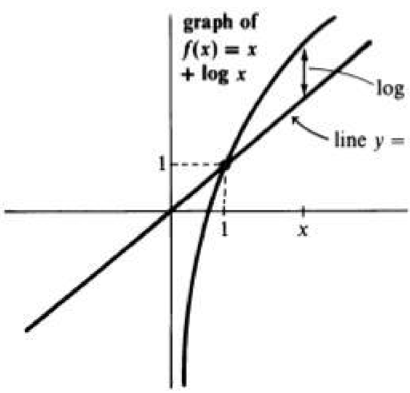
10. $(y - 3) = \dfrac{3}{e}(x - e)$ **11.** $(y - 1) = \dfrac{-1}{e}(x - e)$

12. $\left(y - \dfrac{1}{\log 2}\right) = \dfrac{-1}{2(\log 2)^2}(x - 2)$

14. $\dfrac{2x}{x^2 + 3}$ **15.** $\dfrac{-1}{x(\log x)^2}$ **16.** $\dfrac{\log x - 1}{(\log x)^2}$ **17.** $\frac{1}{3}(\log x)^{-2/3} + (\log x)^{1/3}$

18. $\dfrac{-x}{1 - x^2}$

19. Let $f(x) = x + \log x$. Then $f'(x) = 1 + 1/x > 0$ for $x > 0$. Hence f is strictly increasing. Also $f''(x) = -1/x^2$, so f is bending down. Note that $f(1) = 1$. If $x \to \infty$ then $f(x) \to \infty$ because both x and $\log x$ become large. In fact $f(x)$ lies at a distance $\log x$ above the line $y = x$. As $x \to 0$, $\log x \to -\infty$ (think of $x = 1/e^z = e^{-z}$ where $z \to \infty$). Hence the graph looks like this:



VIII, §3, p. 261

1. $10^x \log 10$, $7^x \log 7$ **2.** $3^x \log 3$, $\pi^x \log \pi$
5. $(y - 1) = (\log 10)x$ **6.** $y - \pi^2 = \pi^2 \log \pi(x - 2)$
7. (a) $e^{x \log x}[\log x + 1]$ (b) $x^{(x^x)}[x^{x-1} + (\log x)x^x(1 + \log x)]$ In (b), we write

$$x^{(x^x)} = e^{(x^x)\log x}$$

and use the chain rule. The derivative of $(x^x)\log x$ is found by the rule for differentiating a product, and we have

$$\frac{d}{dx}\, e^{(x^x)\log x} = e^{(x^x)\log x}\left[x^x \cdot \frac{1}{x} + \frac{d(x^x)}{dx} \log x\right].$$

The derivative of x^x was found in (a), and the answer drops out.
8. (a) $y - 1 = x - 1$ (b) $y - 4 = 2(1 + \log 2)(x - 2)$
(c) $y - 27 = 27(1 + \log 3)(x - 3)$
9. (a) Let $f(x) = x^{x^{1/2}} = e^{x^{1/2}\log x}$. Then

$$f'(x) = x^{\sqrt{x}}\left[\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}} \log x\right]$$

so

$$f'(2) = 2^{\sqrt{2}}\left[\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log 2\right].$$

The tangent line at $x = 2$ is

$$y - 2^{\sqrt{2}} = 2^{\sqrt{2}}\left[\frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \log 2\right](x - 2).$$

(b) $y - 5^{\sqrt{5}} = 5^{\sqrt{5}}\left[\dfrac{1}{\sqrt{5}} + \dfrac{1}{2\sqrt{5}} \log 5\right](x - 5)$

10. Let $f(x) = x^{\sqrt[3]{x}} = e^{x^{1/3}\log x}$. Then

$$f'(x) = x^{\sqrt[3]{x}}\left[x^{1/3}\frac{1}{x} + \tfrac{1}{3}x^{-2/3}\log x\right].$$

(a) Tangent line at $x = 2$ is

$$y - 2^{2^{1/3}} = 2^{2^{1/3}}[2^{-2/3} + \tfrac{1}{3}2^{-2/3} \log 2](x - 2).$$

(b) Tangent line at $x = 5$ is

$$y - 5^{5^{1/3}} = 5^{5^{1/3}}[5^{-2/3} + \tfrac{1}{3}5^{-2/3} \log 5](x - 5).$$

11. Let $f(x) = x^a - 1 - a(x - 1)$. Then $f(1) = 0$, and

$$f'(x) = ax^{a-1} - a.$$

If $x > 1$, then $f'(x) > 0$ so $f(x)$ is strictly increasing. If $x < 1$ then $f'(x) < 0$ so $f(x)$ is strictly decreasing. Hence $f(1)$ is as minimum value, so that for all $x > 0$ and $x \neq 1$ we get $f(x) > 0$.

12. $x = 0$ and $x = -2/\log a$

13. We have $\left(1 + \dfrac{r}{x}\right)^x = \left(1 + \dfrac{1}{y}\right)^{yr}$. If $y \to \infty$, then $\left(1 + \dfrac{1}{y}\right)^y$ approaches e, by Limit 3, so its r-th power approaches e^r. This uses the fact that the r-th power function is continuous. If z approaches z_0, then z^r approaches z_0^r. Here

$$z = \left(1 + \frac{1}{y}\right)^y$$

and z approaches e by Limit 3.

14. (a) We can write $\dfrac{a^h - 1}{h} = \dfrac{f(h) - f(0)}{h}$ where $f(x) = a^x$. Hence the limit is equal to $f'(0)$. Since $f'(x) = a^x \log a$ we find $f'(0) = \log a$.

(b) Putting $h = 1/n$ we have $n(a^{1/n} - 1) = \dfrac{a^h - 1}{h}$ so the limit comes from part (a).

VIII, §4, p. 266

1. $-\log 25$ **2.** $5e^{-4}$ **3.** $e^{-(\log 10)10^{-6}t}$ **4.** $20/e$ **5.** $-(\log 2)/K$ **6.** $(\log 3)/4$

7. $12 \log 10/\log 2$ **8.** $\dfrac{-3 \log 2}{\log 9 - \log 10}$ **10.** 1984: $(50{,}000)2^{84/50}$; 2000: $2 = 10^5$

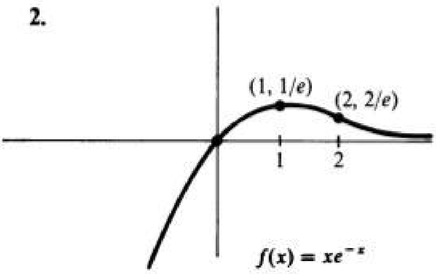
11. $30[\frac{4}{5}]^{5/3}$ **12.** $4\left[\dfrac{\log \frac{1}{20}}{\log \frac{1}{3}}\right]$ **13.** $2\left[\dfrac{\log \frac{5}{8}}{\log \frac{1}{2}}\right]$

14. (a) $40\left[\dfrac{\log \frac{7}{10}}{\log \frac{2}{5}}\right]$ (b) $40\left[\dfrac{\log \frac{4}{25}}{\log \frac{2}{5}}\right]$ (c) $100[\frac{2}{5}]^{1/2}$ **15.** $\log 2$

16. (a) $\frac{1}{5568} \log \frac{1}{2}$ (b) $5568(\log 4/5)/(\log 1/2)$

VIII, §5, p. 274

2.



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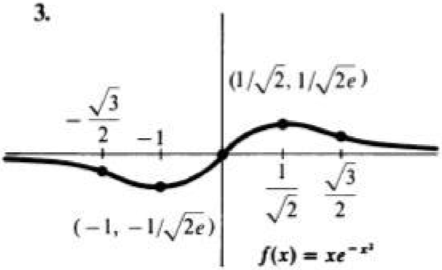

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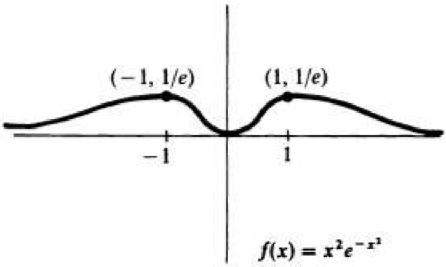
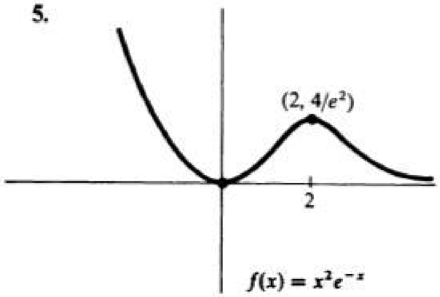
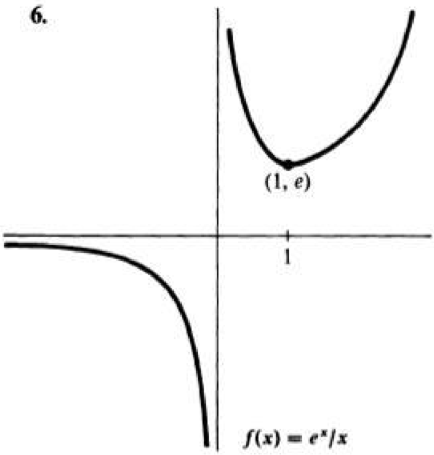

$(-1, 1/e)$
$(1, 1/e)$
-1
1
$f(x) = x^2e^{-x^2}$

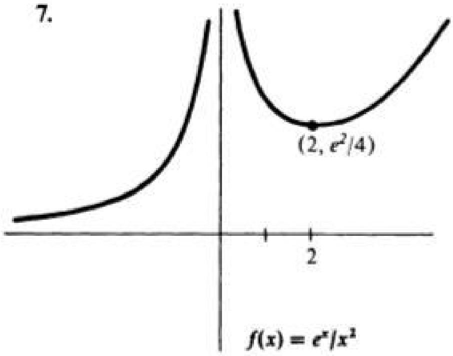
5.


$(2, 4/e^2)$
2
$f(x) = x^2e^{-x}$

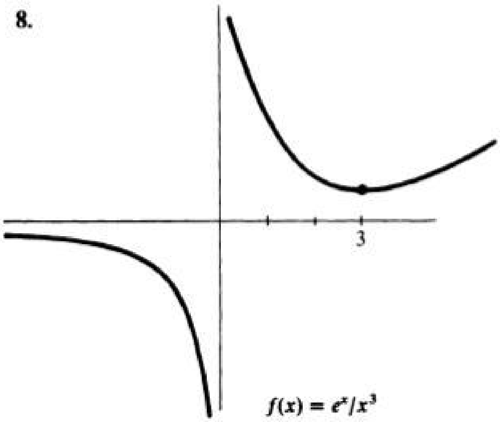
6.


$(1, e)$
1
$f(x) = e^x/x$

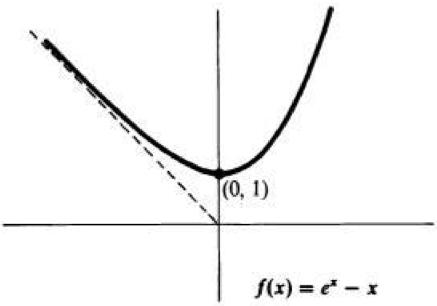
7.


$(2, e^2/4)$
2
$f(x) = e^x/x^2$

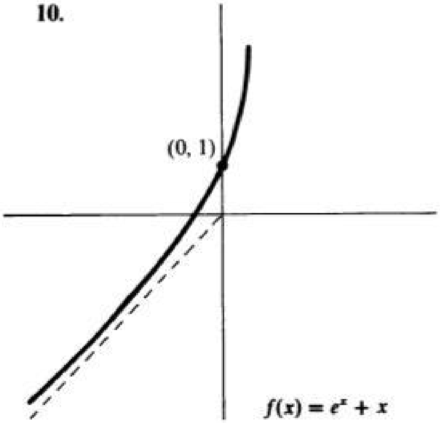
8.


3
$f(x) = e^x/x^3$

9.



10.



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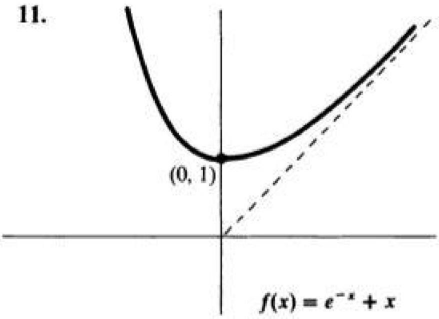
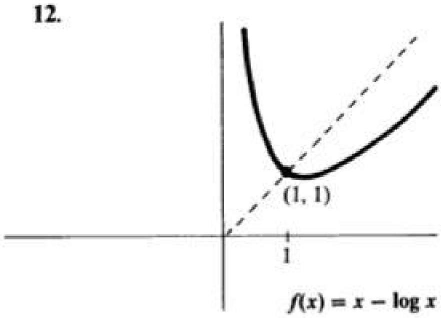


12.



13. Suppose first $a < 0$. Let $f(x) = e^x - ax$. When x is large positive, then

$$f(x) = e^x\left(1 - \frac{ax}{e^x}\right)$$

is large positive. When x is large negative, then e^x is close to 0 and $-ax$ is large negative, so $f(x)$ is negative. By the intermediate value theorem, the equation $f(x) = 0$ has a solution.

Suppose next that $a \geqq e$. Then $f(1) = e - a \leqq 0$, and again $f(x)$ is large when x is large positive. The intermediate value theorem again provides a solution.

14. (a) $-\dfrac{n}{2^n}\log 2$

(b) Limit is 0 in both cases. For instance, let $x = e^{-y}$. As x approaches 0, then y becomes large, and

$$x \log x = -ye^{-y} = -\frac{y}{e^y},$$

which approaches 0 by Theorem 5.1. Also $x^2 \log x = x(x \log x)$, and the product of the limits is the limit of the product of x and $x \log x$, so is equal to 0.

15. Let $x = e^{-y}$. Then $\log x = -y$ and

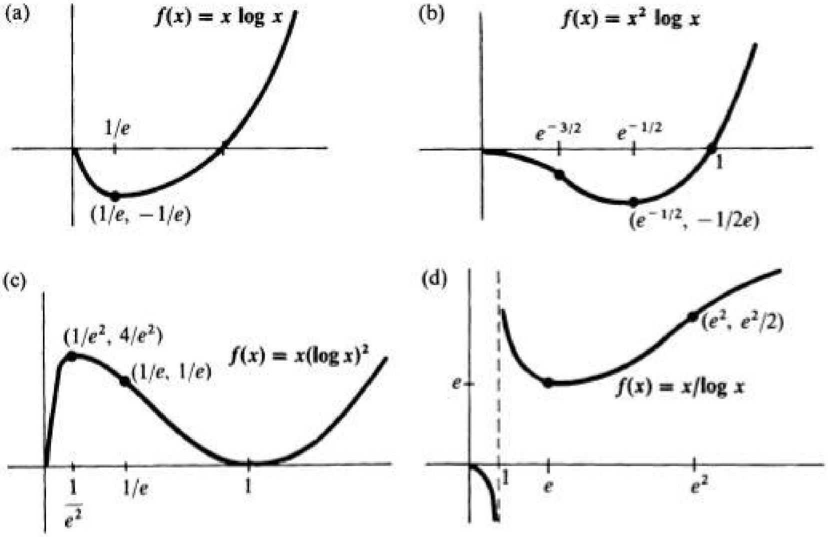
$$x(\log x)^n = e^{-y}y^n = \frac{y^n}{e^y}.$$

As $x \to 0$, $y \to \infty$ so $x(\log x)^n \to 0$ by Theorem 5.1.

16. Let $x = e^y$. As $x \to \infty$, $y \to \infty$ so

$$\frac{(\log x)^n}{x} = \frac{y^n}{e^y} \to 0 \qquad \text{by Theorem 5.1.}$$

17.



17. (a) $f(x) = x \log x$, defined for $x > 0$. Then:

$$f'(x) = x \cdot \frac{1}{x} + \log x = 1 + \log x.$$

We have:

$$f'(x) = 0 \iff \log x = -1 \iff x = e^{-1},$$

$$f'(x) > 0 \iff \log x > -1 \iff x > e^{-1},$$

$$f'(x) < 0 \iff \log x < -1 \iff x < e^{-1}.$$

So there is only one critical point, and the regions of increase and decrease are given by the regions where $f' > 0$ and $f' < 0$.

We also get $f''(x) = 1/x > 0$ for all $x > 0$, so f is bending up.

$$\text{If} \quad x \to \infty \quad \text{then} \quad \log x \to \infty \quad \text{also, so} \quad f(x) \to \infty.$$

$$\text{If} \quad x \to 0 \quad \text{then} \quad f(x) \to 0 \quad \text{by Exercise 14(b).}$$

This justifies all the items in the graph.

17. (b) We carry out the details of the graph for $f(x) = x^2 \log x$. We have

$$f'(x) = x + 2x \log x = x(1 + 2 \log x).$$

Since $x > 0$, it follows that $f'(x) > 0$ if and only if $1 + 2\log x > 0$, and

$$\begin{aligned} 1 + 2\log x > 0 \quad &\text{if and only if} \quad \log x > -1/2 \\ &\text{if and only if} \quad x > e^{-1/2}. \end{aligned}$$

Thus f is strictly increasing for $x \geqq e^{-1/2}$ and is strictly decreasing for

$$0 < x \leqq e^{-1/2}.$$

From Exercise 14 we know that $x \log x$ approaches 0 as x approaches 0. Hence $f'(x)$ approaches 0 as x approaches 0, which means the curve looks flat near 0. We have

$$f''(x) = (1 + 2 \log x) + 2 = 3 + 2 \log x.$$

Then $f''(x) = 0$ if and only if $3 + 2\log x = 0$, or, in other words, $\log x = -3/2$ and $x = e^{-3/2}$. Thus the inflection point occurs for $x = e^{-3/2}$. This explains all the indicated features of the graph.

17. (c) $f(x) = x(\log x)^2$ for $x > 0$. Then

$$\begin{aligned} f'(x) &= x \cdot 2(\log x)\frac{1}{x} + (\log x)^2 \\ &= (\log x)(2 + \log x). \end{aligned}$$

The signs of the two factors $\log x$ and $2 + \log x$ will vary according to the intervals when either factor is 0. We have

$$\begin{aligned} f'(x) = 0 \quad &\Leftrightarrow \quad \log x = 0 \quad \text{or} \quad 2 + \log x = 0 \\ &\Leftrightarrow \quad x = 1 \quad \text{or} \quad \log x = -2 \\ &\Leftrightarrow \quad x = 1 \quad \text{or} \quad x = e^{-2}. \end{aligned}$$

So there are two critical points at $x = 1$ and $x = e^{-2}$. We make a table of regions of increase and decrease corresponding to the intervals between critical points.

interval	$\log x$	$2 + \log x$	$f'(x)$	f
$0 < x < e^{-2}$	neg.	neg.	pos.	s.i.
$e^{-2} < x < 1$	neg.	pos.	neg.	s.d.
$1 < x$	pos.	pos.	pos.	s.i.

The second derivative is not too bad:

$$f''(x) = (\log x)\frac{1}{x} + \frac{1}{x}(2 + \log x) = \frac{2}{x}(1 + \log x).$$

For $x > 0$ we have $2/x > 0$ so $f''(x) = 0$ if and only if $x = e^{-1}$. Since

$$f''(x) > 0 \quad \Leftrightarrow \quad \log x > -1 \quad \Leftrightarrow \quad x > e^{-1},$$
$$f''(x) < 0 \quad \Leftrightarrow \quad \log x < -1 \quad \Leftrightarrow \quad x < e^{-1},$$

it follows that $x = e^{-1}$ is an inflection point. The graph bends up if $x > e^{-1}$ and bends down if $x < e^{-1}$.

If $x \to \infty$ then $\log x \to \infty$ so $f(x) \to \infty$.

If $x \to 0$ then $f(x) \to 0$ by Exercise 15.

This justifies all features of the graph as drawn.

17. (d) We carry out the details. Let $f(x) = x/\log x$ for $x > 0$, and $x \neq 1$. Then

$$f'(x) = \left(\log x - x \cdot \frac{1}{x}\right)/(\log x)^2 = \frac{\log x - 1}{(\log x)^2}.$$

For $x \neq 1$ the denominator is positive (being a square). Hence

$$f'(x) = 0 \quad \Leftrightarrow \quad \log x = 1 \quad \Leftrightarrow \quad x = e,$$
$$f'(x) > 0 \quad \Leftrightarrow \quad \log x > 1 \quad \Leftrightarrow \quad x > e,$$
$$f'(x) < 0 \quad \Leftrightarrow \quad \log x < 1 \quad \Leftrightarrow \quad x < e.$$

Next we list the behavior as x becomes large positive, x approaches 0, and x approaches 1 (since the denominator is not defined at $x = 1$).

If $x \to \infty$ then $x/\log x \to \infty$ by Theorem 4.3.

If $x \to 0$, then $x/\log x \to 0$.

This is because $\log x$ becomes large negative, but is in the denominator, so dividing by a large negative number contributes to the fraction approaching 0.

If $x \to 1$ and $x > 1$ then $x/\log x \to \infty$. *Proof*: The numerator x approaches 1. The denominator $\log x$ approaches 0, and is positive for $x > 1$. So $x/\log x \to \infty$.

If $x \to 1$ and $x < 1$ then $x/\log x \to -\infty$. *Proof*: Again the numerator x approaches 1, and the denominator $\log x$ approaches 0 but is negative for $x < 1$, so $x/\log x \to -\infty$.

This already justifies the graph as drawn in so far as regions of increase and decrease are concerned, and for the critical point (there is only one critical point). Let us now look at the regions of bending up and down. We write the first derivative in the form

$$f'(x) = \frac{1}{\log x} - \frac{1}{(\log x)^2}.$$

Then

$$f''(x) = \frac{-1}{(\log x)^2}\frac{1}{x} - (-2)(\log x)^{-3}\frac{1}{x}$$

$$= \frac{-1}{(\log x)^3}\frac{1}{x}(\log x - 2).$$

Therefore:

$$f''(x) = 0 \quad \Leftrightarrow \quad \log x = 2 \quad \Leftrightarrow \quad x = e^2.$$

We shall now analyze the sign of $f''(x)$ in various intervals, taken between the points 0, 1, and e^2 which are the points where the factors of $f''(x)$ change sign. Note that the sign of $f''(x)$ (plus or minus) will be determined by the signs of $\log x$, x, and $\log x - 2$, together with the minus sign in front.

If $x > e^2$ then $f''(x) < 0$ and the graph bends down, because $\log x - 2 > 0$, both $\log x$ and x are positive, and the minus sign in front makes $f''(x)$ negative.

If $1 < x < e^2$ then $f''(x) > 0$ because $\log x - 2 < 0$, both $\log x$ and x are positive, and the minus sign in front together with the fact that $\log x - 2$ is negative make $f''(x)$ positive. Hence the graph bends up for $1 < x < e^2$.

If $0 < x < 1$ then $f''(x) < 0$ because x is positive, $\log x$ is negative, $\log x - 2$ is negative, and the minus sign in front combines with the other signs to make $f''(x)$ negative. Hence the graph bends down for $0 < x < 1$.

18. Let $f(x) = x^x = e^{x \log x}$. Then

$$f'(x) = e^{x \log x}\left(x \cdot \frac{1}{x} + \log x\right) = e^{x \log x}(1 + \log x).$$

Since $e^u > 0$ for all numbers u, we have $e^{x \log x} > 0$ for all $x > 0$, so

$$f'(x) = 0 \quad \Leftrightarrow \quad \log x = -1 \quad \Leftrightarrow \quad x = e^{-1},$$

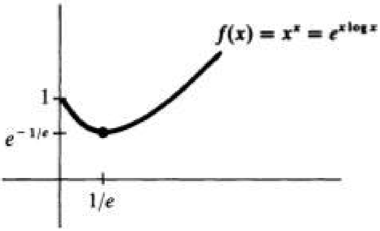
$$f'(x) > 0 \quad \Leftrightarrow \quad \log x > -1 \quad \Leftrightarrow \quad x > e^{-1}.$$

This already takes care of Exercise 18.

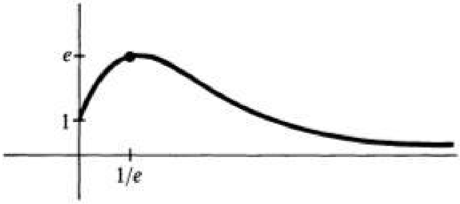
19. Note that $f(e^{-1}) = (e^{-1})^{1/e} = e^{-1/e} = 1/e^{1/e}$. Also

$$f'(x) < 0 \quad \Leftrightarrow \quad \log x < -1 \quad \Leftrightarrow \quad x < e^{-1}.$$

Finally $f''(x) = e^{x \log x}[1/x + (1 + \log x)^2] > 0$ for all $x > 0$ so the graph bends up.



20. Note that $x^{-x} = 1/x^x$. But you should go through the rigamarole about taking the derivative etc. to see the graph as follows.



21. Let $f(x) = 2^x x^x = e^{x \log 2} e^{x \log x} = e^{x(\log 2 + \log x)}$. Then

$$f'(x) = e^{x(\log 2 + \log x)}\left[x \cdot \frac{1}{x} + \log 2 + \log x\right]$$
$$= 2^x x^x (1 + \log 2 + \log x).$$

Since $2^x x^x > 0$ for all $x > 0$, we get

$$f'(x) > 0 \iff 1 + \log 2 + \log x > 0$$
$$\iff \log x > -1 - \log 2$$
$$\iff x > e^{-1-\log 2}$$

and $e^{-1-\log 2} = e^{-1}e^{-\log 2} = 1/2e$, as was to be shown.

22. (a) 1 (b) 1 (c) $1/e$ (d) 1

IX, §1, p. 291

1. $-(\cos 2x)/2$ **2.** $\dfrac{\sin 3x}{3}$ **3.** $\log(x + 1)$, $x > -1$ **4.** $\log(x + 2)$, $x > -2$

IX, §3, p. 296

1. 156 **2.** 2 **3.** 2 **4.** $\log 2$ **5.** $\log 3$ **6.** $\frac{2}{5}$ **7.** $e - 1$

IX, §4, p. 307

1. (a) $U_1^2 = \frac{9}{4}(\frac{3}{2} - 1) + 4(2 - \frac{3}{2})$
$L_1^2 = 1(\frac{3}{2} - 1) + \frac{9}{4}(2 - \frac{3}{2})$

(b) $U_1^2 = \frac{16}{9}(\frac{4}{3} - 1) + \frac{25}{9}(\frac{5}{3} - \frac{4}{3}) + 4(2 - \frac{5}{3})$
$L_1^2 = 1(\frac{4}{3} - 1) + \frac{16}{9}(\frac{5}{3} - \frac{4}{3}) + \frac{25}{9}(2 - \frac{5}{3})$

(c) $U_1^2 = \frac{25}{16}(\frac{5}{4} - 1) + \frac{9}{4}(\frac{3}{2} - \frac{5}{4}) + \frac{49}{16}(\frac{7}{4} - \frac{3}{2}) + 4(2 - \frac{7}{4})$
$L_1^2 = 1(\frac{5}{4} - 1) + \frac{25}{16}(\frac{3}{2} - \frac{5}{4}) + \frac{9}{4}(\frac{7}{4} - \frac{3}{2}) + \frac{49}{16}(2 - \frac{7}{4})$

(d) $$U_1^2 = \frac{1}{n}\cdot\left[\left(1 + \frac{1}{n}\right)^2 + \left(1 + \frac{2}{n}\right)^2 + \cdots + \left(1 + \frac{n}{n}\right)^2\right]$$
$$L_1^2 = \frac{1}{n}\left[1 + \left(1 + \frac{1}{n}\right)^2 + \cdots + \left(1 + \frac{n-1}{n}\right)^2\right]$$

2. (a) $U_1^3 = 1(\frac{3}{2} - 1) + \frac{2}{3}(2 - \frac{3}{2}) + \frac{1}{2}(\frac{5}{2} - 2) + \frac{2}{5}(3 - \frac{5}{2})$
$L_1^3 = \frac{2}{3}(\frac{3}{2} - 1) + \frac{1}{2}(2 - \frac{3}{2}) + \frac{2}{5}(\frac{5}{2} - 2) + \frac{1}{3}(3 - \frac{5}{2})$

(b) $U_1^3 = 1(\frac{4}{3} - 1) + \frac{3}{4}(\frac{5}{3} - \frac{4}{3}) + \frac{3}{5}(\frac{6}{3} - \frac{5}{3}) + \frac{3}{6}(\frac{7}{3} - \frac{6}{3}) + \frac{3}{7}(\frac{8}{3} - \frac{7}{3})$
$+ \frac{3}{8}(\frac{9}{3} - \frac{8}{3})$
$L_1^3 = \frac{3}{4}(\frac{4}{3} - 1) + \frac{3}{5}(\frac{5}{3} - \frac{4}{3}) + \frac{3}{6}(\frac{6}{3} - \frac{5}{3}) + \frac{3}{7}(\frac{7}{3} - \frac{6}{3}) + \frac{3}{8}(\frac{8}{3} - \frac{7}{3}) + \frac{3}{9}(\frac{9}{3} - \frac{8}{3})$

(c) $U_1^3 = 1(\frac{5}{4} - \frac{4}{4}) + \frac{4}{5}(\frac{6}{4} - \frac{5}{4}) + \frac{4}{6}(\frac{7}{4} - \frac{6}{4}) + \frac{4}{7}(\frac{8}{4} - \frac{7}{4})$
$+ \frac{4}{8}(\frac{9}{4} - \frac{8}{4}) + \frac{4}{9}(\frac{10}{4} - \frac{9}{4}) + \frac{4}{10}(\frac{11}{4} - \frac{10}{4}) + \frac{4}{11}(\frac{12}{4} - \frac{11}{4})$
$L_1^3 = \frac{4}{5}(\frac{5}{4} - \frac{4}{4}) + \frac{4}{6}(\frac{6}{4} - \frac{5}{4}) + \frac{4}{7}(\frac{7}{4} - \frac{6}{4}) + \frac{4}{8}(\frac{8}{4} - \frac{7}{4})$
$+ \frac{4}{9}(\frac{9}{4} - \frac{8}{4}) + \frac{4}{10}(\frac{10}{4} - \frac{9}{4}) + \frac{4}{11}(\frac{11}{4} - \frac{10}{4}) + \frac{4}{12}(\frac{12}{4} - \frac{11}{4})$

(d) $$U_1^3 = \frac{1}{n}\left[1 + \frac{1}{\left(1 + \frac{1}{n}\right)} + \cdots + \frac{1}{\left(1 + \frac{2n-1}{n}\right)}\right]$$
$$= \left[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n-1}\right]$$
$$L_1^3 = \frac{1}{n}\left[\frac{1}{\left(1 + \frac{1}{n}\right)} + \cdots + \frac{1}{\left(1 + \frac{2n}{n}\right)}\right] = \left[\frac{1}{n+1} + \cdots + \frac{1}{3n}\right]$$

3. (a) $U_0^2 = \frac{1}{2}(\frac{1}{2} - 0) + 1(1 - \frac{1}{2}) + \frac{3}{2}(\frac{3}{2} - 1) + 2(2 - \frac{3}{2})$
$L_0^2 = 0(\frac{1}{2} - 0) + \frac{1}{2}(1 - \frac{1}{2}) + 1(\frac{3}{2} - 1) + \frac{3}{2}(2 - \frac{3}{2})$

(b) $U_0^2 = \frac{1}{3}(\frac{1}{3} - 0) + \frac{2}{3}(\frac{2}{3} - \frac{1}{3}) + 1(\frac{3}{3} - \frac{2}{3}) + \frac{4}{3}(\frac{4}{3} - 1)$
$+ \frac{5}{3}(\frac{5}{3} - \frac{4}{3}) + 2(\frac{6}{3} - \frac{5}{3})$

$L_0^2 = 0(\frac{1}{3} - 0) + \frac{1}{2}(\frac{2}{3} - \frac{1}{3}) + \frac{2}{3}(\frac{3}{3} - \frac{2}{3}) + 1(\frac{4}{3} - 1)$
$+ \frac{4}{3}(\frac{5}{3} - \frac{4}{3}) + \frac{5}{3}(\frac{6}{3} - \frac{5}{3})$

(c) $U_0^2 = \frac{1}{4}(\frac{1}{4} - 0) + \frac{1}{2}(\frac{1}{2} - \frac{1}{4}) + \frac{3}{4}(\frac{3}{4} - \frac{1}{2}) + 1(1 - \frac{3}{4})$
$+ \frac{5}{4}(\frac{5}{4} - 1) + \frac{3}{2}(\frac{3}{2} - \frac{5}{4}) + \frac{7}{4}(\frac{7}{4} - \frac{3}{2}) + 2(2 - \frac{7}{4})$

$L_0^2 = 0(\frac{1}{4} - 0) + \frac{1}{4}(\frac{1}{2} - \frac{1}{4}) + \frac{1}{2}(\frac{3}{4} - \frac{1}{2}) + \frac{3}{4}(1 - \frac{3}{4})$
$+ 1(\frac{5}{4} - 1) + \frac{5}{4}(\frac{3}{2} - \frac{5}{4}) + \frac{3}{2}(\frac{7}{4} - \frac{3}{2}) + \frac{7}{4}(2 - \frac{7}{4})$

(d) $$U_0^2 = \frac{1}{n}\left[\frac{1}{n} + \frac{2}{n} + \cdots + \frac{2n}{n}\right] \quad L_0^2 = \frac{1}{n}\left[0 + \frac{1}{n} + \cdots + \frac{2n-1}{n}\right]$$

4. (a) $U_0^2 = \frac{1}{4}(\frac{1}{2} - 0) + 1(1 - \frac{1}{2}) + \frac{9}{4}(\frac{3}{2} - 1) + 4(2 - \frac{3}{2})$

$L_0^2 = 0(\frac{1}{2} - 0) + \frac{1}{4}(1 - \frac{1}{2}) + 1(\frac{3}{2} - 1) + \frac{9}{4}(2 - \frac{3}{2})$

(b) $U_0^2 = \frac{1}{9}(\frac{1}{3} - 0) + \frac{4}{9}(\frac{2}{3} - \frac{1}{3}) + 1(1 - \frac{2}{3}) + \frac{16}{9}(\frac{4}{3} - 1)$
$+ \frac{25}{9}(\frac{5}{3} - \frac{4}{3}) + 4(2 - \frac{5}{3})$

$L_0^2 = 0(\frac{1}{3} - 0) + \frac{1}{9}(\frac{2}{3} - \frac{1}{3}) + \frac{4}{9}(1 - \frac{2}{3}) + 1(\frac{4}{3} - 1)$
$+ \frac{16}{9}(\frac{5}{3} - \frac{4}{3}) + \frac{25}{9}(2 - \frac{5}{3})$

(c) $U_0^2 = \frac{1}{16}(\frac{1}{4} - 0) + \frac{1}{4}(\frac{1}{2} - \frac{1}{4}) + \frac{9}{16}(\frac{3}{4} - \frac{1}{2}) + 1(1 - \frac{3}{4})$
$+ \frac{25}{16}(\frac{5}{4} - 1) + \frac{9}{4}(\frac{3}{2} - \frac{5}{4}) + \frac{49}{16}(\frac{7}{4} - \frac{3}{2}) + 4(2 - \frac{7}{4})$

$L_0^2 = 0(\frac{1}{4} - 0) + \frac{1}{16}(\frac{1}{2} - \frac{1}{4}) + \frac{1}{4}(\frac{3}{4} - \frac{1}{2}) + \frac{9}{16}(1 - \frac{3}{4})$
$+ 1(\frac{5}{4} - 1) + \frac{25}{16}(\frac{3}{2} - \frac{5}{4}) + \frac{9}{4}(\frac{7}{4} - \frac{3}{2}) + \frac{49}{16}(2 - \frac{7}{4})$

(d) $$U_0^2 = \frac{1}{n}\left[\frac{1}{n^2} + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{2n}{n}\right)^2\right]$$

$$L_0^2 = \frac{1}{n}\left[0 + \frac{1}{n^2} + \cdots + \left(\frac{2n-1}{n}\right)^2\right]$$

5. $$U_1^2 = \frac{1}{n}\left[1 + \frac{1}{\left(1 + \frac{1}{n}\right)} + \cdots + \frac{1}{\left(1 + \frac{n-1}{n}\right)}\right]$$
$$= \left[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}\right]$$
$$L_1^2 = \frac{1}{n}\left[\frac{1}{\left(1 + \frac{1}{n}\right)} + \frac{1}{\left(1 + \frac{2}{n}\right)} + \cdots + \frac{1}{\left(1 + \frac{n}{n}\right)}\right]$$
$$= \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right]$$

6. The area under the curve $y = 1/x$ between $x = 1$ and $x = 2$ is $\log 2 - \log 1 = \log 2$. Write down that this area is less than an upper sum and greater than a lower sum, and use Exercise 5 to get the desired inequalities.

7. $U_1^n = [\log 2 + \log 3 + \cdots + \log n] = \log n!$
$L_1^n = [\log 1 + \log 2 + \cdots + \log(n-1)] = \log(n-1)!$

X, §1, p. 317

1. $\frac{63}{6}$ **2.** 0 **3.** 0 **4.** 0

5. (b) and (c) Let $f(x) = x^{-1/2}$, and let the partition be $P = \{1, 2, 3, \ldots, n\}$ for the interval $[1, n]$. Then compare the integral

$$\int_1^n f(x)\,dx = \left.\frac{x^{1/2}}{1/2}\right|_1^n = 2(\sqrt{n} - 1)$$

with the upper and lower sums.

6. (a) Use $f(x) = x^2$ and $P = \{0, 2, \ldots, n\}$, with interval $[0, n]$. The lower sum could start with 0, but this 0 may be omitted since $0 + A = A$ for all numbers A.
(b) Let $f(x) = x^3$, interval $[0, n]$, partition $P = \{0, \ldots, n\}$.
(c) Let $f(x) = x^{1/4}$, interval $[0, n]$, partition $P = \{0, \ldots, n\}$.

7. (d) Use $f(x) = 1/x^4$ over the interval $[1, n]$ with the partition $\{1, \ldots, n\}$ consisting of the positive integers from 1 to n. Then

$$\int_1^n f(x)\,dx = \frac{x^{-3}}{-3}\Bigg|_1^n = -\frac{1}{3}\left(\frac{1}{n^3} - 1\right) = \frac{1}{3} - \frac{1}{3n^3}.$$

Comparing with the lower and upper sum yields

$$\frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{n^4} \leqq \frac{1}{3} - \frac{1}{3n^3} \leqq 1 + \frac{1}{2^4} + \cdots + \frac{1}{(n-1)^4}.$$

8. Let $f(x) = \dfrac{1}{1 + x^2}$. Then $\displaystyle\int_0^1 f(x)\ dx = \arctan x\Big|_0^1 = \pi/4$. On the interval $[0, 1]$ use the partition $P = \left\{0, \dfrac{1}{n}, \dfrac{2}{n}, \ldots, \dfrac{n}{n}\right\}$. Draw the picture.

9. Let $f(x) = x^2$. Then $\displaystyle\int_0^1 f(x)\,dx = \frac{x^3}{3}\Big|_0^1 = 1/3$. On the interval $[0, 1]$ use the partition $P = \{0, 1/n, \ldots, n/n\}$. Draw the picture.

10. $\displaystyle\int_1^n \log x\,dx = (x \log x - x)\Big|_1^n = n \log n - n + 1.$

The lower sum is $\log 1 + \log 2 + \cdots + \log(n-1) = \log(n-1)!$ and

$$e^{\text{lower sum}} = (n-1)!, \qquad \text{because} \qquad e^{\log u} = u.$$

On the other hand,

$$e^{n \log n - n + 1} = e^{n \log n} e^{-n} e^{1} = n^n e^{-n} e.$$

Since lower sum $\leqq$ integral, we get $e^{\text{lower sum}} \leqq e^{\text{integral}}$, so

$$(n-1)! \leqq n^n e^{-n} e.$$

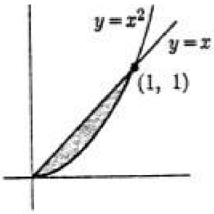
The upper sum is $\log 2 + \log 3 + \cdots + \log n = \log n!$ and so $e^{\text{upper sum}} = n!$ Since integral $\leqq$ upper sum, we have $e^{\text{integral}} \leqq e^{\text{upper sum}}$. Since the integral is $n \log n - n + 1$, we get

$$n^n e^{-n} e \leqq n!.$$

X, §2, p. 325

1. x^4 **2.** $3x^5/5 - x^6/6$ **3.** $-2\cos x + 3\sin x$ **4.** $\frac{9}{5}x^{5/3} + 5\sin x$
5. $5e^x + \log x$ **6.** 0 **7.** 0 **8.** $e^2 - e^{-1}$ **9.** $4 \cdot 28/3$

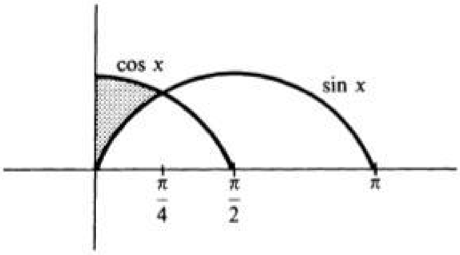
10. In this problem, the curves intersect at $x = 0$ and $x = 1$.]



Hence the area is

$$\int_0^1 (x - x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 = \frac{1}{2} - \frac{1}{3}.$$

11. $\frac{1}{2}$ from -1 to 1 **12.** $\frac{1}{12}$
13. $\frac{1}{2}$ **14.** $\frac{8}{3} + \frac{5}{12}$.
15. $\sqrt{2} - 1$. In this problem the graph is as follows.



The first point of intersection is when $x = \pi/4$. Hence the area between the curves is

$$\int_0^{\pi/4} (\cos x - \sin x)\,dx.$$

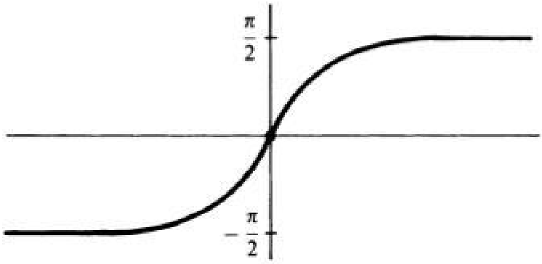
16. $9/2$ **17.** $\frac{1}{3} - \frac{1}{2}$ **18.** 0 **19.** $\pi^2/2 - 2$ **20.** 4 **21.** 4 **22.** 1 **23.** 4 **24.** $-\pi^2$
25. 4 **26.** (a) 14 (b) 14 (c) $2n$

X, §4, p. 334

1. Yes, $\sqrt{2}$ **2.** No
3. Yes, $\pi/2$. We have

$$\int_0^B \frac{1}{1 + x^2}\,dx = \arctan x\bigg|_0^B = \arctan B - \arctan 0$$
$$= \arctan B.$$

As $B \to \infty$, $\arctan B \to \pi/2$. Remember the graph of $\arctan x$ which is as follows.



4. No. Let $0 < b < 5$. Then

$$\int_0^b \frac{1}{5-x}\,dx = -\log(5-x)\Big|_0^b = -[\log(5-b) - \log 5]$$
$$= \log 5 - \log(5-b).$$

As b approaches 5, $5-b$ approaches 0, and $\log(5-b)$ becomes large negative. Hence the integral from 0 to 5 does not exist.

5. Let $2 < a < 3$. Then

$$\int_a^3 \frac{1}{x-2}\,dx = \log(x-2)\Big|_a^3 = \log 1 - \log(a-2)$$
$$= -\log(a-2).$$

As $a \to 2$, $\log(a-2) \to -\infty$, so $-\log(a-2) \to \infty$ and the integral from 2 to 3 does not exist.

6. No.

7. Here we have $0 \leqq x < 2$. Remember that we have the indefinite integral.

$$\int \frac{1}{x}\,dx = \log(-x) \qquad \text{if} \quad x < 0.$$

Therefore, if $x < 2$ then

$$\int \frac{1}{x-2}\,dx = \log(2-x).$$

You can verify this by the chain rule, differentiating the right-hand side. Don't forget the -1. Note that when $x < 2$ then $2 - x > 0$. The log is not defined at negative numbers. Now you can find the definite integral for $0 < a < 2$. We have

$$\int_0^a \frac{1}{x-2}\,dx = \log(2-x)\Big|_0^a = \log(2-a) - \log 2.$$

The right-hand side approaches $-\infty$ as $a \to 2$, so the integral

$$\int_0^2 \frac{1}{x-2}\,dx = \lim_{a\to 2}\int_0^a \frac{1}{x-2}\,dx$$

does not exist.

8. Improper integral does not exist. Let $2 < c < 3$. Then

$$\int_c^3 \frac{1}{(x-2)^2}\,dx = \int_c^3 (x-2)^{-2}\,dx = -(x-2)^{-1}\Big|_c^3$$
$$= -\left[1 - \frac{1}{c-2}\right].$$

As $c \to 2$, the quotient $1/(c-2)$ becomes arbitrarily large.

9. Does not exist.

10. Exists. Let $1 < c < 4$. Then

$$\int_c^4 (x-1)^{-2/3}\,dx = 3(x-1)^{1/3}\Big|_c^4 = 3[3^{1/3} - (c-1)^{1/3}].$$

As $c \to 1$, $(c-1)^{1/3} \to 0$, and so

$$\lim_{c\to 1}\int_c^4 (x-1)^{-2/3}\,dx = 3\cdot 3^{1/3}.$$

11. Integral exists for $s < 1$. Let $0 < a < 1$. Then for $s \neq 1$, we get

$$\int_a^1 x^{-s}\,dx = \frac{x^{-s+1}}{-s+1}\Big|_a^1 = \frac{1}{1-s} - \frac{a^{1-s}}{1-s}.$$

If $s < 1$, then $1 - s > 0$ and a^{1-s} approaches 0 as a approaches 0. Hence the limit of the right-hand side as $a \to 0$ exists and is equal to $1/(1-s)$. On the other hand, if $s > 1$, then

$$a^{1-s} = \frac{1}{a^{s-1}},$$

and $s - 1 > 0$, so $a^{s-1} \to 0$ as $a \to 0$, and $1/a^{s-1} \to \infty$, so the integral from 0 to 1 does not exist. When $s = 1$,

$$\int_a^1 \frac{1}{x}\,dx = \log x\Big|_a^1 = \log 1 - \log a = -\log a.$$

When $a \to 0$, $\log a \to -\infty$ so the integral does not exist.

12. The integral $\int_1^\infty x^{-s}\,dx$ exists for $s > 1$ and does not exist for $s < 1$. Evaluate

$$\int_1^B x^{-s}\,dx = \frac{B^{-s+1}}{1-s} - \frac{1}{1-s}.$$

If $s > 1$ then we write $B^{-s+1} = 1/B^{s-1}$ and $s - 1 > 0$, so $1/B^{s-1} \to 0$ as $B \to \infty$. The limit of the integral exists and is equal to $1/(s-1)$. If $s < 1$, then $B^{-s+1} = B^{1-s}$ becomes arbitrarily large when B becomes large.

13. Yes. Let $B > 1$. Then

$$\int_1^B e^{-x}\,dx = -e^{-x}\Big|_1^B = -[e^{-B} - e^{-1}] = \frac{1}{e} - \frac{1}{e^B}.$$

As B becomes large, $1/e^B$ approaches 0, and the integral from 1 to B approaches $1/e$. It exists.

14. Does not exist.

15. $-\frac{1}{2}e^{-2B} + \frac{1}{2}e^{-4}$. Yes, $\frac{1}{2}e^{-4}$.

XI, §1, p. 339

1. $e^{x^2}/2$ **2.** $-\frac{1}{4}e^{-x^4}$ **3.** $\frac{1}{6}(1 + x^3)^2$ **4.** $(\log x)^2/2$

5. $\dfrac{(\log x)^{-n+1}}{1-n}$ if $n \neq 1$, and $\log(\log x)$ if $n = 1$. **6.** $\log(x^2 + x + 1)$

7. $x - \log(x + 1)$ **8.** $\dfrac{\sin^2 x}{2}$ **9.** $\dfrac{\sin^3 x}{3}$ **10.** 0 **11.** $\frac{2}{5}$ **12.** $-\arctan(\cos x)$

13. $\frac{1}{2}(\arctan x)^2$ **14.** 2/15. Let $u = 1 - x^2$. **15.** $-\frac{1}{4}\cos(\pi^2/2) + \frac{1}{4}$

16. (a) $-(\cos 2x)/2$ (b) $(\sin 2x)/2$ (c) $-(\cos 3x)/3$ (d) $(\sin 3x)/3$ (e) $e^{4x}/4$ (f) $e^{5x}/5$ (g) $-e^{-5x}/5$ **17.** $-\frac{1}{2}e^{-B^2} + \frac{1}{2}$. Yes, $\frac{1}{2}$ **18.** $-\frac{1}{3}e^{-B^3} + \frac{1}{3}$. Yes, $\frac{1}{3}$

XI, §1, Supplementary Exercises, p. 340

1. $\frac{1}{4}\log(x^4 + 2)$ **3.** $\dfrac{\sin^5 x}{5}$ **5.** $\sqrt{x^2 - 1}$ **7.** $\dfrac{-1}{6(3x^2 + 5)}$ **9.** $\dfrac{-1}{2\sin^2 x}$

11. $\dfrac{1}{3}\left[\dfrac{u^{17/5}}{17/5} - \dfrac{u^{12/5}}{12/5}\right]$ where $u = x^3 + 1$. **13.** $\dfrac{-\cos 3x}{3}$ **15.** $-\cos e^x$

17. $\log(\log x)$ **19.** $\log(e^x + 1)$ **21.** $\frac{1}{4}$ **23.** $\dfrac{4\sqrt{2}}{3} - \dfrac{2}{3}$ **25.** $\dfrac{\pi}{4}$ **27.** $\dfrac{\pi^2}{72}$ **29.** $e - \dfrac{1}{e}$

XI, §2, p. 344

1. $x \arcsin x + \sqrt{1 - x^2}$. Let

$$u = \arcsin x, \qquad dv = dx,$$

$$du = \frac{1}{\sqrt{1-x^2}}\,dx, \qquad v = x.$$

Then

$$\int \arcsin x \, dx = \int u \, dv = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx.$$

Now the integral $\int \frac{x}{\sqrt{1-x^2}} \, dx$ can be done by substitution, letting $u = 1 - x^2$ and $du = -2x \, dx$, so

$$\int \frac{x}{\sqrt{1-x^2}} \, dx = \frac{-1}{2} \int u^{-1/2} \, du.$$

2. $x \arctan x - \frac{1}{2}\log(x^2 + 1)$. Let $u = \arctan x$.

3. $\frac{e^{2x}}{13}(2 \sin 3x - 3 \cos 3x)$. Let $I = \int e^{2x} \sin 3x \, dx$. Let

$$u = e^{2x}, \qquad dv = \sin 3x \, dx,$$
$$du = 2e^{2x} \, dx, \qquad v = -\tfrac{1}{3} \cos 3x.$$

Then

$$I = -\tfrac{1}{3}e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x \, dx.$$

Apply the same procedure to this second integral, with

$$u = e^{2x}, \qquad dv = \cos 3x \, dx,$$
$$du = 2e^{2x} \, dx, \qquad v = \tfrac{1}{3} \sin 3x.$$

Then

$$\begin{aligned} I &= -\tfrac{1}{3}e^{2x} \cos 3x + \tfrac{2}{3}[\tfrac{1}{3}e^{2x} \sin 3x - \tfrac{2}{3}I] \\ &= -\tfrac{1}{3}e^{2x} \cos 3x + \tfrac{2}{9}e^{2x} \sin 3x - \tfrac{4}{9}I. \end{aligned}$$

Thus we see I appearing on the right-hand side with some constant factor. We can solve for I to get

$$\tfrac{13}{9}I = \tfrac{2}{9}e^{2x} \sin 3x - \tfrac{1}{3}e^{2x} \cos 3x.$$

Multiply by 9 and divide by 13 to get the answer.

4. $\frac{1}{10}e^{-4x} \sin 2x - \frac{1}{5}e^{-4x} \cos 2x$

5. $x(\log x)^2 - 2x\log x + 2x$. Let $u = (\log x)^2$, so $du = 2(\log x)\dfrac{1}{x}\,dx$. Let $dv = dx$, $v = x$. Then

$$\int (\log x)^2\,dx = x(\log x)^2 - \int \frac{x}{x}\log x\,dx.$$

Then $x/x = 1$, and you are reduced to $\int \log x\,dx = x\log x - x$.

6. $(\log x)^3x - 3\int(\log x)^2\,dx$ **7.** $x^2e^x - 2xe^x + 2e^x$

8. $-x^2e^{-x} - 2xe^{-x} - 2e^{-x}$ **9.** $-x\cos x + \sin x$ **10.** $x\sin x + \cos x$

11. $-x^2\cos x + 2\int x\cos x\,dx$ **12.** $x^2\sin x - 2\int x\sin x\,dx$

13. $\frac{1}{2}[x^2\sin x^2 + \cos x^2]$. Write

$$I = \int x^3\cos x^2\,dx = \int x^2(\cos x^2)\,xdx = \frac{1}{2}\int x^2(\cos x^2)\,2xdx.$$

First let $u = x^2$, $du = 2xdx$ and use substitution to get

$$I = \frac{1}{2}\int u\cos u\,du.$$

Then use integration by parts.

14. $-\frac{1}{3}(1-x^2)^{3/2} + \frac{2}{5}(1-x^2)^{5/2} - \frac{1}{7}(1-x^2)^{7/2}$. Let $u = 1 - x^2$:

$$\begin{aligned}\int x^5\sqrt{1-x^2}\,dx &= \int x^4\sqrt{1-x^2}\,x\,dx\\ &= -\frac{1}{2}\int(1-u)^2u^{1/2}\,du\\ &= -\frac{1}{2}\int(u^{1/2} - 2u^{3/2} + u^{5/2})\,du\end{aligned}$$

15. $\frac{1}{3}x^3\log x - \frac{1}{9}x^3$. Let $u = \log x$ and $dv = x^2\,dx$. Then $du = (1/x)\,dx$ and $v = x^3/3$ so

$$\int x^2\log x\,dx = \tfrac{1}{3}x^3\log x - \frac{1}{3}\int x^2\,dx.$$

16. $(\log x)\dfrac{x^4}{4} - \dfrac{x^4}{16}$

17. $(\log x)^2\dfrac{x^3}{3} - \dfrac{2}{3}\displaystyle\int x^2\log x\,dx$. Repeat the procedure to get the complete answer.

18. $-\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2}$. Let $u = x^2$ first.

19. $\dfrac{1}{4}\left(\dfrac{1}{1-x^4}\right) + \frac{1}{4}\log(1-x^4)$ **20.** -4π

21. $-Be^{-B} - e^{-B} + 1$. Yes, 1. First evaluate the indefinite integral $\int xe^{-x}\,dx$ by parts. Let

$$u = x, \qquad dv = e^{-x}\,dx,$$
$$du = dx, \qquad v = -e^{-x}.$$

Then the integral is equal to

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}.$$

Now put in the limits of integration:

$$\int_0^B xe^{-x}\,dx = -xe^{-x}\Big|_0^B - e^{-x}\Big|_0^B = -Be^{-B} - [e^{-B} - 1].$$

As $B \to \infty$ the two terms with B approach 0, so you get the answer.

22. Yes, $5/e$ **23.** Yes, $16/e$

24. $-\dfrac{1}{\log B} + \dfrac{1}{\log 2}$. Yes, $1/\log 2$. Let $u = \log x$. When $x = 2$, $u = \log 2$. When $x = B$, $u = \log B$. Then

$$\int_2^B \frac{1}{x(\log x)^2}\,dx = \int_{\log 2}^{\log B} u^{-2}\,du.$$

25. Yes, $1/3(\log 3)^3$

26. $2 \log 2 - 2$. The indefinite integral is $x \log x - x$. Let $0 < a < 2$. Then

$$\int_a^2 \log x\,dx = x \log x - x\Big|_a^2 = 2 \log 2 - 2 - (a \log a - a).$$

As $a \to 0$ we know from Chapter VIII, §5, Exercise 14 that $a \log a$ approaches 0. Hence $a \log a - a$ approaches 0 as a approaches 0, which gives the answer.

XI, §2, Supplementary Exercises, p. 346

1. $\frac{1}{2}(x^2 \arctan x + \arctan x - x)$. Use $u = \arctan x$ and $dv = x\,dx$. Also use the trick

$$\int \frac{x^2}{x^2+1}\,dx = \int \frac{x^2+1-1}{x^2+1}\,dx = \int 1\,dx - \int \frac{1}{x^2+1}\,dx.$$

2. (a) If I is the integral, then

$$I = x\sqrt{1-x^2} + \arcsin x - I$$

so $2I = x\sqrt{1-x^2} + \arcsin x$, and dividing by 2 yields the answer.

(b) Integrating by parts reduces the integral to (a): $u = \arcsin x$, $dv = x\,dx$. Use a trick as in Exercise 1.

3. $\frac{1}{4}(2x^2 \arccos x - \arccos x - x\sqrt{1-x^2})$ **5.** $1 - \frac{\pi}{2}$ **7.** $\frac{\pi}{32}$ **9.** $\frac{2}{e}$

10.

$$\int_0^1 x^3\sqrt{1-x^2}\,dx = \int_0^1 x^2(1-x^2)^{1/2}x\,dx.$$

Let $u = 1 - x^2$, $du = -2x\,dx$. When $x = 0$, $u = 1$ and when $x = 1$, $u = 0$. Then

$$\begin{aligned}\int_0^1 x^2(1-x^2)^{1/2}\,dx &= \frac{-1}{2}\int_1^0 (1-u)u^{1/2}\,du = \frac{-1}{2}\int_1^0 [u^{1/2} - u^{3/2}]\,du \\ &= \frac{-1}{2}\left[\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]\Bigg|_1^0 \\ &= \frac{-1}{2}\left[0 - \left(\frac{2}{3} - \frac{2}{5}\right)\right] = \frac{2}{15}.\end{aligned}$$

11. $-2e^{-\sqrt{x}}(x^{3/2} + 3x + 6x^{1/2} + 6)$. Let $x = u^2$ first. Then

$$\int xe^{-\sqrt{x}}\,dx = \int u^2e^{-u}2u\,du = 2\int u^3e^{-u}\,du.$$

13. Let $u = (\log x)^n$ and $dv = dx$.
14. Let $u = x^n$ and $dv = e^x\,dx$.
15. Let $u = (\log x)^n$ and $dv = x^m\,dx$. Then

$$du = n(\log x)^{n-1}\frac{1}{x}\,dx \qquad \text{and} \qquad v = x^{m+1}/(m+1).$$

16. First we find the indefinite integral by parts with $u = x^n$, $dv = e^{-x}\,dx$, so

$$\int x^ne^{-x}\,dx = -x^ne^{-x} + n\int x^{n-1}e^{-x}\,dx.$$

Then we have the definite integral

$$\int_0^B x^ne^{-x}\,dx = -B^ne^{-B} + n\int_0^B x^{n-1}e^{-x}\,dx.$$

Taking the limit as $B \to \infty$ and using that $B^ne^{-B} \to 0$, we find:

$$\int_0^\infty x^ne^{-x}\,dx = n\int_0^\infty x^{n-1}e^{-x}\,dx.$$

Let $I_n = \int_0^\infty x^n e^{-x}\,dx$. This last equality can be rewritten in the form

$$I_n = nI_{n-1}.$$

Thus we have reduced the evaluation of the integral to the next step. For instance, $I_{10} = 10I_9$; $I_9 = 9I_8$; $I_8 = 8I_7$; and so on. Continuing in this way, it takes n steps to get

$$I_n = n!I_0 = n!\int_0^\infty e^{-x}\,dx,$$

where $n! = n(n-1)(n-2)\cdots 3.2.1$ is the product of the first n integers. This final integral is easily evaluated, namely

$$\int_0^\infty e^{-x}\,dx = \lim_{B\to\infty}\int_0^B e^{-x}\,dx = \lim_{B\to\infty} -e^{-x}\Big|_0^B$$
$$= \lim_{B\to\infty} -[e^{-B}-1] = 1.$$

XI, §3, p. 354

1. $-\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x$ **2.** $\frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x$

3. $\dfrac{\sin^3 x}{3} - \dfrac{\sin^5 x}{5}$ **4.** 3π **5.** 8π **6.** πab (if $a, b > 0$) **7.** πr^2

8. (a) $-2\sqrt{2}\cos\theta/2$ (b) $2\sqrt{2}\sin\theta/2$ **13.** $-\log\cos x$ **14.** $\arcsin\dfrac{x}{3}$

15. $\arcsin\dfrac{x}{\sqrt{3}}$ **16.** $\frac{1}{2}\arcsin(\sqrt{2}x)$ **17.** $\dfrac{1}{b}\arcsin\dfrac{bx}{a}$. Let $x = au/b$, $dx = (a/b)\,du$.

18. (a) $c_0 = a_n = 0$ all n, $b_n = -(2/n)\cos n\pi$.
(b) $c_0 = \pi^2/3$, $a_n = -(4/n^2)\cos n\pi$, $b_n = 0$ all n.
(c) $c_0 = \pi/2$, $a_n = 2(\cos n\pi - 1)/\pi n^2$, $b_n = 0$ all n.

19. (b) all a_n and $c_0 = 0$

XI, §3, Supplementary Exercises, p. 356

1. $\log\sin x - \dfrac{\sin^2 x}{2}$

2. Write $\tan^2 x = \tan^2 x + 1 - 1$ and note that $d\tan x/dx = \tan^2 x + 1$.

3. $-\cos e^x$ **4.** Let $x = 2u$, $dx = 2du$ **5.** $\dfrac{\pi}{4}$ **7.** $\dfrac{\pi}{4}$ **9.** $\dfrac{\pi}{16}$

11. $\frac{1}{8}\arcsin x - \frac{1}{8}x(1-2x^2)\sqrt{1-x^2}$ **13.** $\dfrac{\pi}{2}$ **14.** $-\arcsin x - \dfrac{1}{x}\sqrt{1-x^2}$

15. $-16u^{1/2} + \frac{1}{3}u^{3/2}$ where $u = 16 - x^2$

16. Let $u + 1 + x^2$. Then

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int \frac{x^2\,2x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int \frac{u-1}{u^{1/2}}\,du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right].$$

The rest of the exercises are done by letting $x = \sin\theta$ or $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$. We give the answers, but work out Exercise 19 in full.

17. $\dfrac{-1}{a}\log\left[\dfrac{a+\sqrt{a^2-x^2}}{x}\right]$ **18.** $\dfrac{a^2}{2}\arcsin(x/a) - \frac{1}{2}x\sqrt{a^2-x^2}$

19. $\dfrac{-\sqrt{a^2-x^2}}{2a^2x^2} - \dfrac{1}{2a^3}\log\left[\dfrac{a+\sqrt{a^2-x^2}}{x}\right]$. We have a choice of whether to let $x = a\cos\theta$ or $x = a\sin\theta$. The principle is the same. Let us do as usual,

$$x = a\sin\theta, \qquad dx = a\cos\theta\,d\theta.$$

Then

$$\int \frac{1}{x^3\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a^3\sin^3\theta(a\cos\theta)}\,a\cos\theta\,d\theta = \frac{1}{a^3}\int \frac{1}{\sin^3\theta}\,d\theta.$$

It's a pain, but we show how to do it. Recall that to integrate positive powers of sine, we used integration by parts. We try a similar method here. Thus let

$$I = \int \frac{1}{\sin^3\theta}\,d\theta = \int \frac{1}{\sin\theta}\,\frac{1}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin\theta}\csc^2\theta\,d\theta.$$

In analogy with the tangent, we have

$$\frac{d\cot\theta}{d\theta} = -\csc^2\theta,$$

so we let

$$u = \frac{1}{\sin\theta}, \qquad dv = \csc^2\theta\,d\theta,$$

$$du = -\frac{1}{\sin^2\theta}\cos\theta\,d\theta, \qquad v = -\cot\theta.$$

Then

$$I = -\frac{\cot\theta}{\sin\theta} - \int \frac{\cos^2\theta}{\sin^3\theta}\,d\theta = -\frac{\cos\theta}{\sin^2\theta} - \int \frac{1-\sin^2\theta}{\sin^3\theta}\,d\theta$$

and so

$$I = -\frac{\cos\theta}{\sin^2\theta} - I + \int \frac{1}{\sin\theta}\,d\theta,$$

whence

$$I = \frac{1}{2}\left[-\frac{\cos\theta}{\sin^2\theta} - \log(\csc\theta + \cot\theta)\right].$$

You may leave the answers in terms of θ, this is usually done. But if you want the answer in terms of x, then use:

$$\sin\theta = \frac{x}{a}, \qquad \cos\theta = \sqrt{1-\sin^2\theta} = \frac{1}{a}\sqrt{a^2 - x^2},$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{a}{x}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{a^2-x^2}}{x}.$$

20. $-\frac{1}{a^2}\cot\theta$ where $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$.

21. $\sqrt{1-x^2} - \log\left(\frac{1+\sqrt{1+x^2}}{x}\right)$. The method is the same as Exercise 19.

22. Let $x = at$, $dx = a\,dt$, and reduce to Exercise 14.

23. $\frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a}$

XI, §4, p. 370

1. $-\frac{1}{8}\log(x-1) + \frac{17}{8}\log(x+7)$

2. $\frac{-1}{2(x^2-3)}$. Don't use partial fractions here, use the substitution $u = x^2 - 3$ and $du = 2x\,dx$.

3. (a) $\frac{1}{5}[\log(x-3) - \log(x+2)]$ (b) $\log(x+1) - \log(x+2)$

4. $-\frac{1}{2}\log(x+1) + 2\log(x+2) - \frac{3}{2}\log(x+3)$

5. $2\log x - \log(x+1)$ **6.** $\log(x+1) + \frac{1}{x+1}$

7. $-\log(x+1) + \log(x+2) - \frac{2}{x+2}$

8. $\log(x-1) + \log(x-2)$ **9.** $\frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x$

10. (a) $\frac{1}{4}\frac{x}{(x^2+1)^2} + \frac{3}{8}\frac{x}{(x^2+1)} + \frac{3}{8}\arctan x$

11. $\frac{-1}{x^2+1} - 3\left[\frac{x}{2(x^2+1)} + \frac{1}{2}\arctan x\right]$

12. $\frac{1}{2}\frac{-1}{x^2+9} + \frac{1}{18}\frac{x}{x^2+9} + \frac{1}{54}\arctan\frac{x}{3}$ **13.** $\frac{1}{8}\frac{x}{x^2+16} + \frac{1}{32}\arctan\frac{x}{4}$

14. $\frac{1}{4}\log\frac{(x+1)^2}{x^2+1} + \frac{1}{2}\arctan x$. Factorization:

$$x^3 - 1 = (x-1)(x^2+x+1) \quad\text{and}\quad x^4 - 1 = (x+1)(x-1)(x^2+1).$$

15. $C_1 = -\frac{33}{100}$, $C_2 = -\frac{11}{100}$, $C_3 = -\frac{130}{100}$, $C_4 = -\frac{110}{100}$, $C_5 = \frac{11}{100}$

16. (a) Let $x = bt$, $dx = b\,dt$ (b) Let $x + a = bt$, $dx = b\,dt$.

17. (a) $-\frac{1}{2}\arctan x + \frac{1}{4}\log\left(\frac{x-1}{x+1}\right)$ (b) $\frac{1}{4}[\log(x^2-1) - \log(x^2+1)]$

18. (a) $\frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$

(b) $\frac{1}{2}\log\frac{x^2}{x^2+x+1} - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$

19. $-\log(x-1) + \log(x^2+x+1)$

XI, §5, p. 377

1. $2\sqrt{1+e^x} - \log(\sqrt{1+e^x}+1) + \log(\sqrt{1+e^x}-1)$ **2.** $x - \log(1+e^x)$

3. $\arctan(e^x)$ **4.** $-\log(\sqrt{1+e^x}+1) + \log(\sqrt{1+e^x}-1)$

5. Let $y = f(x) = \frac{1}{2}(e^x - e^x) = \sinh x$. Then

$$f'(x) = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$$

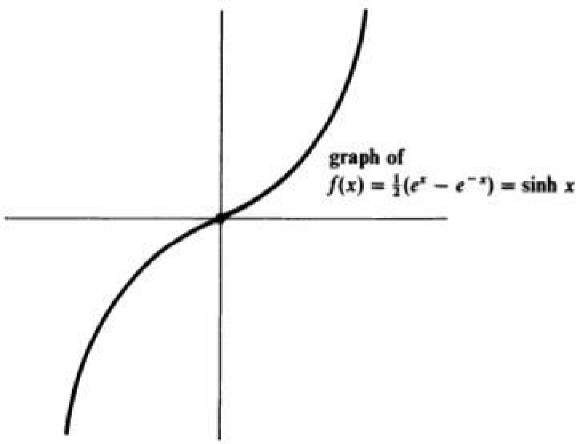
But $\cosh x > 0$ for all x, so f is strictly increasing for all x. If x is large negative, then e^x is small, and e^{-x} is large positive, so $f(x)$ is large positive. If x is large positive, then e^x is large positive, and e^{-x} is small. Hence

$$f(x) \to \infty \quad \text{as} \quad x \to \infty,$$

$$f(x) \to -\infty \quad \text{as} \quad x \to -\infty.$$

By the intermediate value theorem, the values of $f(x)$ consist of all numbers. Hence the inverse function $x = g(y)$ is defined for all numbers y. We have

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\cosh x} = \frac{1}{\sqrt{\sinh^2 x + 1}} = \frac{1}{\sqrt{y^2+1}}.$$



6. Let $y = f(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$. Then

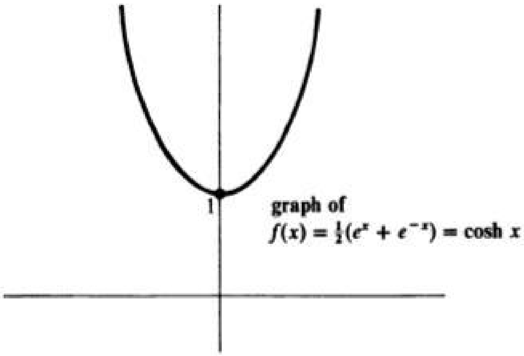
$$f'(x) = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x.$$

If $x > 0$ then $e^x > 1$ and $0 < e^{-x} < 1$ so $f'(x) > 0$ for all $x > 0$. Hence f is strictly increasing, and the inverse function

$$x = g(y) = \operatorname{arccosh} y$$

exists. We have $f(0) = 1$. As $x \to \infty$, $e^x \to \infty$ and $e^{-x} \to 0$, so $f(x) \to \infty$. Hence the values of $f(x)$ consist of all numbers $\geqq 1$ when $x \geqq 0$. Hence the inverse function g is defined for all numbers $\geqq 1$. We have

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{\sinh x} = \frac{1}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\sqrt{y^2 - 1}}.$$



Finally, let $u = e^x$. Then $y = \frac{1}{2}(u + 1/u)$. Multiply by $2u$ and solve the quadratic equation to get

$$e^x = u = y + \sqrt{y^2 - 1} \qquad \text{or} \qquad e^x = u = y - \sqrt{y^2 - 1}.$$

The graph of $f(x) = \cosh x$ for all numbers x bends as shown on the figure, and there are two possible inverse functions depending on whether we look at the interval

$$x \leqq 0 \qquad \text{or} \qquad x \geqq 0.$$

Taking

$$x = \log(y + \sqrt{y^2 - 1})$$

is the inverse function for $x \geqq 0$ and taking

$$x = \log(y - \sqrt{y^2 - 1})$$

is the inverse functions for $x \leqq 0$. Indeed suppose we take the solution with the minus sign. Then by simple algebra, you can see that

$$y - \sqrt{y^2 - 1} \leqq 1.$$

[*Proof*: you have to check that $y - 1 \leqq \sqrt{y^2 - 1}$. Since $y \geqq 1$, it suffices to check that $(y-1)^2 \leqq y^2 - 1$, which amounts to

$$y^2 - 2y + 1 \leqq y^2 - 1,$$

or $y \geqq 1$, which checks.]

Thus $y - \sqrt{y^2 - 1} \leqq 1$, whence

$$\log(y - \sqrt{y^2 - 1}) \leqq 0.$$

For $x \geqq 0$ it follows that we have to use the solution of the quadratic equation for u in terms of y with the plus sign, that is

$$e^x = u = y + \sqrt{y^2 - 1} \qquad \text{and} \qquad x = \log(y + \sqrt{y^2 - 1}).$$

7. Let

$$I = \int \frac{x^2}{\sqrt{x^2 + 4}}\,dx.$$

Let $x = 2\sinh t$, $dx = 2\cosh t\,dt$. Then $x^2 + 4 = 4\cosh^2 t$. Hence

$$\begin{aligned} I &= \int \frac{4\sinh^2 t}{2\cosh t}\,2\cosh t\,dt \\ &= 4\int \sinh^2 t\,dt = \frac{4}{4}\int (e^{2t} - 2 + e^{-2t})\,dt \\ &= \tfrac{1}{2}e^{2t} - 2t - \tfrac{1}{2}e^{-2t}. \end{aligned}$$

8. $\log(x + \sqrt{x^2 + 1})$

9. Let

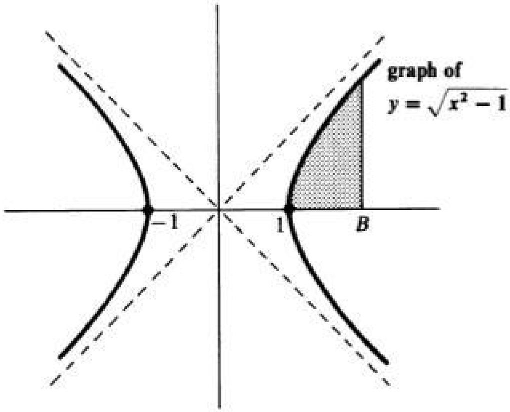
$$I = \int \frac{x^2 + 1}{x - \sqrt{x^2 + 1}}\,dx.$$

Let $x = \sinh t$, $dx = \cosh t\,dt$. Then

$$\begin{aligned} I &= \int \frac{\cosh^2 t}{\sinh t - \cosh t}\cosh t\,dt \\ &= \int \frac{\cosh^3 t}{\dfrac{e^t - e^{-t}}{2} - \dfrac{e^t + e^{-t}}{2}}\,dt \\ &= \int \frac{\left(\dfrac{e^t + e^{-t}}{2}\right)^3}{-e^{-t}}\,dt \\ &= -\frac{1}{8}\int e^t(e^{3t} + 3e^t + 3e^{-t} + e^{-3t})\,dt \\ &= -\frac{1}{8}\left(\frac{e^{4t}}{4} + \frac{3e^{2t}}{2} + 3t + \frac{e^{-2t}}{-2}\right). \end{aligned}$$

10. $-\frac{1}{2}\log(x + \sqrt{x^2 - 1}) + \frac{1}{2}x\sqrt{x^2 - 1}$ (see Exercise 11)

11. $-\frac{1}{2}\log(B + \sqrt{B^2 - 1}) + \frac{1}{2}B\sqrt{B^2 - 1}$. The graph of the equation $x^2 - y^2 = 1$ is a hyperbola as drawn.



The top part of the hyperbola in the first quadrant is the graph of the function

$$y = \sqrt{x^2 - 1},$$

with the positive square root, and $x \geqq 1$. Hence the area under the graph between 1 and B is

$$\text{Area} = \int_1^B \sqrt{x^2 - 1}\, dx.$$

We want to make the expression under the square into a perfect square. We use the substitution

$$x = \cosh t \qquad \text{and} \qquad dx = \sinh t\, dt.$$

Then you get into an integral consisting of powers of e^t and e^{-t} which is easy to evaluate. Change the limits of integration in the way explained in the last example of the section. Let $u = e^t$ and solve a quadratic equation for u. You will find the given answer.

12. $\log(B + \sqrt{B^2 + 1}) + B\sqrt{B^2 + 1}$

13. Let $y = a \cosh(x/a)$. Then

$$\frac{dy}{dx} = a \sinh(x/a) \cdot \frac{1}{a} = \sinh(x/a),$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \cosh(x/a) \cdot \frac{1}{a} = \frac{1}{a}\cosh(x/a) \\ &= \frac{1}{a}\sqrt{1 + \sinh^2(x/a)} \\ &= \frac{1}{a}\sqrt{1 + (dy/dx)^2}.\end{aligned}$$

14. Let $x = az$, $dx = adz$, and reduce to the worked-out case.

XII, §1 p. 384

1. $\frac{4}{3}\pi r^3$ **2.** π **3.** $\frac{\pi^2}{8} - \frac{\pi}{4}$ **4.** $\frac{\pi^2}{8} + \frac{\pi}{4}$ **5.** $\frac{2 \cdot 5^4 \pi}{3}$ **6.** $\pi(e-2)$ **7.** πe^2

8. $\pi[2(\log 2)^2 - 4 \log 2 + 2]$. Integrate $\int (\log x)^2\,dx$ by parts, $u = (\log x)^2$, $du = \frac{2 \log x}{x}\,dx$, $dv = dx$

9. 16π **10.** (a) $\frac{\pi}{2}\left(\frac{1}{e^2} - \frac{1}{e^{2B}}\right)$; yes $\frac{\pi}{2e^2}$

(b) $\frac{\pi}{4}\left(\frac{1}{e^4} - \frac{1}{e^{4B}}\right)$; yes $\frac{\pi}{4e^4}$ (c) $\frac{\pi}{4}\left(\frac{1}{e^2} - \frac{1}{e^{2B^2}}\right)$; yes $\frac{\pi}{4e^2}$

11. Equation of the line is $y = \frac{r}{h}x$. Volume is $\frac{\pi r^2 h}{3}$ **12.** $2\pi(1 - \sqrt{a})$, 2π

13. $\frac{\pi}{24} - \frac{\pi}{3B^3}$; yes $\frac{\pi}{24}$ **14.** For all $c > 1/2$, $\pi/(2c-1)$

15. For all $c < 1/2$, $\pi/(1-2c)$

XII, §1, Supplementary Exercises, p. 385

1. $f(x) = R + \sqrt{a^2 - x^2}$ and $g(x) = R - \sqrt{a^2 - x^2}$. The volume is

$$V = \pi \int_{-a}^{a} f(x)^2\,dx - \pi \int_{-a}^{a} g(x)^2\,dx$$

which after easy algebra comes out equal to

$$4\pi R \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = 2\pi^2 R a^2.$$

2. $\frac{32\pi}{5}$ **3.** 12π **4.** 2π **5.** $\frac{2\pi}{3}$ **6.** $\frac{5\pi}{14}$ **7.** $\frac{\pi}{3}$ **8.** $\frac{4}{3}\pi a^2 b$ **9.** $\frac{\pi}{2}(e^{-2} - e^{-10})$

10. $\pi[2(\log 2)^2 - 4 \log 2 + 2]$ **11.** $\pi\left[\sqrt{3} - \frac{\pi}{3}\right]$ **12.** $\pi\left(1 - \frac{1}{B}\right)$, π as $B \to \infty$

13. $\frac{\pi}{3}\left(1 - \frac{1}{B^3}\right)$, $\frac{\pi}{3}$ as $B \to \infty$ **14.** $\pi \log B$

15. $\pi \log \frac{1}{a}$. No limit as $a \to 0$. The volume increases without bound.

16. $\pi\left(\frac{1}{a} - 1\right)$. No limit as $a \to 0$. The volume increases without bound.

17. $\pi\left[\frac{\sqrt{2}}{2} - \cos a + \log(\sqrt{2} - 1) - \log(\csc a - \cot a)\right]$

No limit as $a \to 0$. The volume increases without bound.

XII, §2, p. 390

1. 6π **2.** a^2 (using symmetry and values of θ such that $\sin 2\theta \geqq 0$, the problem reduces to $\int_0^{\pi/2} a^2 \sin 2\theta\,d\theta$)

3. πa^2 **4.** $\dfrac{\pi}{12}$ **5.** $3\pi/2$ **6.** $3\pi/2$ **7.** $9\pi/2$ **8.** $\pi/3$

XII, §2, Supplementary Exercises, p. 390

1. 25π **2.** $\dfrac{3\pi}{2}$ **3.** π **4.** $\dfrac{9\pi}{2}$ **5.** $\dfrac{3\pi}{8}$ **6.** $\dfrac{3\pi}{2}$ **7.** $2\pi + \dfrac{3\sqrt{3}}{2}$ **8.** $\dfrac{3\pi}{2}$

9. $\dfrac{\pi}{4}$ **10.** $\dfrac{9\pi}{2}$ **11.** $10\frac{2}{3}$ **12.** $10\frac{2}{3}$ **13.** $\frac{4}{3}$ **14.** $\frac{4}{3}$ **15.** $\dfrac{5\sqrt{5}}{6}$ **16.** 10

XII, §3, p. 397

1. $\frac{8}{27}(10^{3/2} - 1)$ **2.** $\dfrac{\sqrt{5}}{2} + \log\left(\dfrac{4 + 2\sqrt{5}}{1 + \sqrt{5}}\right)$

3. $\sqrt{e^4 + 1} + 2 - \sqrt{2} + \log\left(\dfrac{1 + \sqrt{2}}{1 + \sqrt{e^4 + 1}}\right)$ **4.** $2\sqrt{17} + \log\left(\dfrac{\sqrt{17} + 4}{\sqrt{17} - 4}\right)^{1/4}$

5. $\sqrt{1 + e^2} + \frac{1}{2} \log \dfrac{\sqrt{1 + e^2} - 1}{\sqrt{1 + e^2} + 1} - \left(\sqrt{2} + \frac{1}{2} \log \dfrac{\sqrt{2} - 1}{\sqrt{2} + 1}\right)$

6. $\frac{1}{27}(31^{3/2} - 13^{3/2})$ **7.** $e - \dfrac{1}{e}$

8. We work out Exercise 8 in full.

$$\begin{aligned}
\text{length} &= \int_0^{3/4} \sqrt{1 + \left(\frac{-2x}{1 - x^2}\right)^2}\, dx \\
&= \int_0^{3/4} \sqrt{1 + \frac{4x^2}{(1 - x^2)^2}}\, dx \\
&= \int_0^{3/4} \frac{\sqrt{1 - 2x^2 + x^4 + 4x^2}}{1 - x^2}\, dx \\
&= \int_0^{3/4} \frac{\sqrt{(1 + x^2)^2}}{1 - x^2}\, dx \\
&= \int_0^{3/4} \frac{1 + x^2}{1 - x^2}\, dx \\
&= \int_0^{3/4} \frac{2}{1 - x^2}\, dx + \int_0^{3/4} \frac{x^2 - 1}{1 - x^2}\, dx \\
&= 2\int_0^{3/4} \frac{1}{2}\left(\frac{1}{1 - x} + \frac{1}{1 + x}\right) dx - \int_0^{3/4} dx \\
&= -\log(1 - x) + \log(1 + x)\Big|_0^{3/4} - \frac{3}{4} \\
&= \log\left(\frac{1 + 3/4}{1 - 3/4}\right) - \tfrac{3}{4} = \log 7 - 3/4
\end{aligned}$$

9. $\frac{1}{2}\left(e - \frac{1}{e}\right)$ **10.** $\log(2 + \sqrt{3})$

XII, §4, p. 407

2. $2\pi r$ **3.** $\sqrt{2}(e^2 - e)$ **4.** (a) $\frac{3}{4}$ (b) 3 **5.** $2\sqrt{5} + \log(\sqrt{5} + 2)$

6. $4\sqrt{2} + 2\log\frac{\sqrt{2}+1}{\sqrt{2}-1}$ **7.** $2\sqrt{3}$ **8.** 5 **9.** 8 **10.** $4a$ **12.** $\sqrt{2}(e^2 - e)$

13. $\sqrt{2}(e^{\theta_2} - e^{\theta_1})$ **14.** $(8^{3/2} - 5^{3/2})$ **15.** $\frac{\sqrt{17}}{4}(e^{-4} - e^{-8})$ **16.** $\frac{3\pi}{4}$

17. $\sqrt{5} - \frac{\sqrt{17}}{4} + \log\left[\frac{8 + 2\sqrt{17}}{1 + \sqrt{5}}\right]$ **18.** $4\sin\frac{\pi}{8} = 2\sqrt{2 - \sqrt{2}}$ **19.** 4 **20.** 2

21. 8 **22.** π **23.** $2\sqrt{3}$

XII, §5, p. 415

1. $12\pi a^2/5$ **2.** $\frac{\pi}{27}(10\sqrt{10} - 1)$ **3.** $\frac{2\pi}{3}(26\sqrt{26} - 2\sqrt{2})$ **4.** $4\pi^2 a^2$ **5.** $4\pi^2 aR$

6. $\frac{\pi}{6}(17\sqrt{17} - 1)$

XII, §6, p. 418

1. 5 lb/in.; 80 in.-lb **2.** $\frac{10}{\sin\frac{\pi}{9}}$ lb/in.; $\frac{180}{\pi}\left[\cot\frac{\pi}{9} - \frac{1}{2}\csc\frac{\pi}{9}\right]$ in.-lb

3. $c\left[\frac{1}{r_1} - \frac{1}{r}\right]$ **4.** yes; $\frac{c}{r_1}$ **5.** $\frac{99c}{200}$ where c is the constant of proportionality

6. 2×10^6 pound-miles **7.** $\frac{E}{6}$ in.-lb

8. (a) -90 CmM dyne-cm (b) 9 CmM dyne-cm **9.** $c\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

10. 1500 log 2 in.-pounds. If A is the area of the cross section of the cylinder, and $P(x)$ is the pressure, then

$$P(x) \cdot Ax = C = \text{constant}.$$

If a is the length of the cylinder when the volume is $75\,\text{in}^3$, then

$$\text{Work} = \int_a^{2a} \text{Force}\,dx = \int_a^{2a} P(x)A\,dx = \int_a^{2a} \frac{C}{x}\,dx = C\log 2.$$

But $C = 20 \cdot 75 = 1{,}500$ from initial data. This gives the answer.

XII, §7, p. 423

1. $\frac{3}{4}\left(\frac{15^4 - 5^4}{15^3 - 5^3}\right)$ **2.** 10 **3.** 10/log 3

XIII, §1, p. 434

1. (a) $f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{(1+x)^k}$

(b) $f^{(k)}(0) = (-1)^{k+1}(k-1)!$

(c) Since $(k-1)!/k! = 1/k$, we get from (b)

$$\frac{f^{(k)}(0)}{k!} = \frac{(-1)^{k+1}(k-1)!}{k!} = \frac{(-1)^{k+1}}{k}$$

and

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.$$

This proves that when $f(x) = \log(1 + x)$ then the n-th Taylor polynomial is given by

$$P_n(x) = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k} x^k.$$

2. For $f(x) = \cos x$, $f^{(n)}(x) = f^{(n+4)}(x)$. Use this and the formula for $P_n(x)$ to derive $P_n(x)$ for the function $f(x) = \cos x$.

XIII, §3, p. 446

1. $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

2. $|f^{(n)}(c)| \leqq 1$ for all n and all numbers c so the estimate follows from Theorem 2.1.

3. $1 - \frac{0.01}{2} + R_4(0.1) = 0.995 + R_4(0.1)$ **4.** $|R_3| \leqq \frac{(0.1)^3}{3!} = \frac{1}{6} \times 10^{-3}$.

5. $|R_4| \leqq \frac{2}{3}10^{-4}$

6. $P_4(x) = x + \frac{x^3}{3}$. Let $f(x) = \tan x$. You have to find first all derivatives $f^{(1)}(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$, $f^{(4)}(x)$, and then $f^{(1)}(0), \ldots, f^{(4)}(0)$. Then use the general formula for the Taylor polynomial

$$P_4(x) = f(0) + f^{(1)}(0)x + \cdots + f^{(4)}(0)\frac{x^4}{4!}.$$

7. $|R_5| \leqq 10^{-4}$ by crude estimates

8. (a) $\sin\left(\frac{\pi}{6}+\frac{\pi}{180}\right)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right)+E$ and $|E|<10^{-3}$

(b) $\cos\left(\frac{\pi}{6}+\frac{\pi}{180}\right)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(\frac{\pi}{180}\right)+E$

(c) $\sin\left(\frac{\pi}{4}+\frac{2\pi}{180}\right)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\left(\frac{2\pi}{180}\right)+E$

(d) $\cos\left(\frac{\pi}{4}+\frac{2\pi}{180}\right)=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\left(\frac{2\pi}{180}\right)+E$

(e) $\sin\left(\frac{\pi}{6}+\frac{2\pi}{180}\right)=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{2\pi}{180}\right)+E$

(f) $\cos\left(\frac{\pi}{6}+\frac{2\pi}{180}\right)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(\frac{2\pi}{180}\right)+E$

We carry out the steps for part (e) in full.

$$\begin{aligned}\sin 32^\circ &= \sin\left(\frac{\pi}{6}+\frac{2\pi}{180}\right)=\sin\left(\frac{\pi}{6}+\frac{\pi}{90}\right)=\sin(a+h)\\ &= \sin\frac{\pi}{6}+\left(\cos\frac{\pi}{6}\right)\frac{\pi}{90}+R_2(h)\\ &= \frac{1}{2}+\frac{\sqrt{3}}{2}\frac{\pi}{90}+R_2\left(\frac{\pi}{90}\right),\end{aligned}$$

and

$$\left|R_2\left(\frac{\pi}{90}\right)\right|\leqq\left(\frac{\pi}{90}\right)^2\frac{1}{2}\leqq\left(\frac{3.15}{90}\right)^2\frac{1}{2}\leqq(3.5\times10^{-2})^2\frac{1}{2}\leqq10^{-3}.$$

9. $\cos\left(\frac{\pi}{6}+\frac{\pi}{180}\right)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(\frac{\pi}{180}\right)+E$

10. $\sin\left(\frac{\pi}{3}+\frac{\pi}{180}\right)=\frac{\sqrt{3}}{2}+\frac{1}{2}\left(\frac{\pi}{180}\right)+E$

11. $\cos\left(\frac{\pi}{3}+\frac{\pi}{180}\right)=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right)+E$

12. (a) $\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{R_5(x)}{x}$ so

$$\int_0^1\frac{\sin x}{x}\,dx=1-\frac{1}{3\cdot3!}+E\qquad\text{where}\qquad|E|\leqq\frac{1}{5\cdot5!}.$$

(b) $-\frac{(0.1)^2}{4}+E$ where $|E|\leqq\frac{10^{-4}}{4\cdot4!}$

(c) Write $\sin u = u - \frac{u^3}{3!} + R_5(u)$, so that for $u = x^2$ we get

$$\sin x^2 = x^2 - \frac{x^6}{3!} + R_5(x^2).$$

We have, for $u \geqq 0$,

$$|R_5(u)| \leqq \frac{u^5}{5!} = \frac{x^{10}}{5!}.$$

Hence

$$\int_0^1 \sin x^2\,dx = \frac{1}{3} - \frac{1}{7\cdot 3!} + E,$$

where

$$|E| \leqq \int_0^1 \frac{x^{10}}{5!}\,dx = \frac{1}{11\cdot 5!} \leqq 10^{-3}.$$

(d) $\frac{1}{2} - \frac{1}{6\cdot 3!} + E$, and $|E| \leqq \frac{1}{10\cdot 5!}$

(e) $1 - \frac{1}{5\cdot 2!} + \frac{1}{9\cdot 4!} + E$, and $|E| \leqq \frac{1}{13\cdot 6!}$

(f) $1 - \frac{1}{5\cdot 3!} + E$, and $|E| \leqq \frac{1}{9\cdot 5!}$

13.

$$\begin{aligned}\int_0^{1/2} \frac{\cos x - 1}{x}\,dx &= \int_0^{1/2} \frac{1 - \frac{x^2}{2} + \frac{x^4}{4!} + R_6(x) - 1}{x}\,dx \\ &= \int_0^{1/2}\left(-\frac{x}{2} + \frac{x^3}{4!} + \frac{R_6(x)}{x}\right)dx \\ &= -\frac{x^2}{4} + \frac{x^4}{96}\Big|_0^{1/2} + E \\ &= -\frac{1}{16} + \frac{1}{16\cdot 96} + E,\end{aligned}$$

and

$$|E| \leqq \frac{1}{6!}\int_0^{1/2} x^5\,dx = \frac{1}{6!}\frac{x^6}{6}\Big|_0^{1/2} = \frac{1}{6!\cdot 6\cdot 2^6} < 10^{-5}.$$

XIII, §4, p. 448

1. $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$ **2.** $|R_3| \leqq e^{1/2}\frac{(\frac{1}{2})^3}{3!} \leqq \frac{2(\frac{1}{8})}{6} = \frac{1}{24}$

3. $|R_4| \leqq e^{10^{-2}}\frac{(10^{-2})^4}{4!} \leqq \frac{2(10^{-8})}{4!} = \frac{10^{-8}}{12}$

4. $|R_3| \leqq e^{10^{-2}} \dfrac{(10^{-2})^3}{3!} \leqq \dfrac{2(10^{-6})}{6} = \dfrac{10^{-6}}{3}$

6. $|R_7| \leqq e^{-1} \dfrac{|-1|^7}{7!} \leqq \dfrac{1}{7!} = \dfrac{1}{5040}$

7. (a) $|R_4| \leqq e^2 \dfrac{|2|^4}{4!} \leqq 6$ (b) $|R_4| \leqq e^3 \dfrac{|3|^4}{4!} \leqq 214$

8. (a) $|R_5| \leqq e^2 \dfrac{|2|^5}{5!} \leqq \dfrac{64}{15}$ (b) $|R_5| \leqq e^3 \dfrac{|3|^5}{5!} \leqq \dfrac{648}{5}$

9. (a) $|R_{13}| \leqq \dfrac{16 \cdot 2^{12}}{12!}$ using $e < 4$ (b) $|R_{16}| \leqq \dfrac{16 \cdot 2^{16}}{16!}$ using $e < 4$

10. $e = 1 + \displaystyle\sum_{n=1}^{13} \frac{1}{n!} + E$ **11.** $e^{-2} = \displaystyle\sum_{k=0}^{12} \frac{(-2)^k}{k!} + E$ and $|E| \leqq \dfrac{2^{13}}{13!}$

12. (a) $1 + \dfrac{1}{2 \cdot 2!} + \dfrac{1}{3 \cdot 3!} + \cdots + \dfrac{1}{7 \cdot 7!} + E$, where $|E| \leqq \dfrac{e}{8 \cdot 8!}$

(b) Write $e^u = 1 + u + \cdots + \dfrac{u^4}{4!} + R_5(u)$. For $u \leqq 0$ we have $e^u \leqq 1$, and hence $|R_5(u)| \leqq \dfrac{|u|^5}{5!}$. Now put $u = -x^2$. Then

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + R_5(-x^2).$$

Integrate the first part consisting of powers of x term by term. We get

$$\int_0^1 e^{-x^2}\, dx = 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} + E,$$

and

$$|E| \leqq \int_0^1 |R_5(-x^2)|\, dx \leqq \int_0^1 \frac{x^{10}}{5!}\, dx = \frac{1}{11 \cdot 5!} \leqq 10^{-3}.$$

(c) $1 + \dfrac{1}{3} + \dfrac{1}{5 \cdot 2!} + \dfrac{1}{7 \cdot 3!} + \dfrac{1}{9 \cdot 4!} + \dfrac{1}{11 \cdot 5!} + E$,

where $|E| \leqq \dfrac{e}{13 \cdot 6!} < \dfrac{1}{3} \times 10^{-3}$.

(d) $e^u = 1 + u + R_2(u)$, and we put $u = x^2$. For $0 \leqq x \leqq 0.1$ we find $e^u \leqq 2$ (generous estimate). Hence

$$|R_2(u)| \leqq \frac{2u^2}{2!} = u^2.$$

We have $e^{x^2} = 1 + x^2 + R_2(x^2)$, and $|R_2(x^2)| \leqq x^4$. Hence

$$\int_0^{0.1} e^{x^2}\, dx = 0.1 + \frac{(0.1)^3}{3} + E, \qquad \text{where} \qquad |E| \leqq \int_0^{0.1} x^4\, dx \leqq \frac{10^{-5}}{5}.$$

(e) $0.1 - (\frac{1}{3})10^{-3} + E$, where $|E| \leqq 10^{-6}$.

XIII, §5, p. 455

1. (a) $\log 1.2 = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + R_4$, $|R_4| \leqq 4 \cdot 10^{-4}$

(b) $\log 0.9 = -\log 10/9 = -\log\left(1 + \frac{1}{9}\right)$

$$= -\left[\frac{1}{9} - \frac{(1/9)^2}{2}\right] - R_3(1/9) = -\frac{1}{9} + \frac{1}{2\cdot 81} - R_3(1/9)$$

and $|-R_3(1/9)| < \frac{1}{2} \times 10^{-3}$

(c) $\log 1.05 = 0.05 - \frac{(0.05)^2}{2} + R_3$ and $|R_3| \leqq \frac{5^3}{3} \cdot 10^{-6}$

(d) $\log 9/10 = -\log 10/9$ as in (b).

(e) $\log 24/25 = -\log 25/24 = -\log\left(1 + \frac{1}{24}\right)$

$$= \frac{-1}{24} + \frac{1}{2(24)^2} - R_3(1/24),$$

and $|R_3(1/24)| < 10^{-3}$.

(f) $\log 26/25 = 0.04 + R_2(1/25)$, $|R_2| \leqq 8 \times 10^{-4}$

2. (a) We transform the right-hand side until it is equal to the left-hand side. We have:

$$\begin{aligned} -7\log\frac{9}{10} + 2\log\frac{24}{25} + 3\log\frac{81}{80} &= \log\left(\frac{10}{9}\right)^7\left(\frac{24}{25}\right)^2\left(\frac{81}{80}\right)^3 \\ &= \log\frac{2^7 5^7 (2^3\cdot 3)^2 (3^4)^3}{(3^2)^7 5^4 (2^4)^3 5^3} \\ &= \log\frac{2^7 2^6 3^2 3^{12} 5^7}{2^{12} 3^{14} 5^7} \\ &= \log 2. \end{aligned}$$

The case of log 3 is done in the same way. Note that each fraction 9/10, 24/25, 81/80 is close to 1, and hence the value of the log is approximated very well by just a few terms from the Taylor formula. For instance,

$$\log\frac{9}{10} = -\log\frac{10}{9} = -\log\left(1 + \frac{1}{9}\right).$$

Then

$$\log\left(1 + \frac{1}{9}\right) = \frac{1}{9} - \frac{(1/9)^2}{2} + \frac{(1/9)^3}{3} - \frac{(1/9)^4}{4} + \frac{(1/9)^5}{5} + R_6(1/9),$$

and

$$|R_6(1/9)| \leqq \frac{(1/9)^6}{9} < \frac{1}{2} \times 10^{-6}.$$

Hence

$$7 \log \frac{10}{9} = 7A_1 + E_1, \tag{1}$$

where $E_1 = 7R_6(1/9)$ and

$$|E_1| = 7|R_6(1/9)| < 3.5 \times 10^{-6}.$$

Observe the factor of 7 which comes in throughout at this point.

Next, we have

$$\begin{aligned} \log \frac{25}{24} = \log\left(1 + \frac{1}{24}\right) &= \frac{1}{24} - \frac{1}{24^2} + \frac{1}{24^3} + R_4(1/24) \\ &= A_2 + R_4(1/24), \end{aligned}$$

and

$$|R_4(1/24)| \leqq \frac{(1/24)^4}{4} < 10^{-6}.$$

Hence

$$-2 \log \frac{25}{24} = -2A_2 - 2R_4(1/24) = -2A_2 + E_2, \tag{2}$$

where

$$|E_2| = |2R_4(1/24)| < 2 \times 10^{-6}.$$

Thirdly, we have

$$\begin{aligned} \log \frac{81}{80} = \log\left(1 + \frac{1}{80}\right) &= \frac{1}{80} - \frac{1}{80^2} + R_3(1/80) \\ &= A_3 + R_3(1/80), \end{aligned}$$

and

$$|R_3(1/80)| \leqq \frac{(1/80)^3}{3} < \frac{1}{1.5} \times 10^{-6}.$$

Hence

$$3 \log \frac{81}{80} = 3A_3 + E_3, \tag{3}$$

where

$$|E_3| = 3|R_3(1/80)| < \frac{3}{1.5} \times 10^{-6} \leqq 2 \times 10^{-6}.$$

We may now put together the computations of the three terms, and we find:

$$\log 2 = 7A_1 - 2A_2 + 3A_3 + E, \qquad \text{where} \quad E = E_1 + E_2 + E_3$$

and

$$|E| \leqq |E_1| + |E_2| + |E_3| < 3.5 \times 10^{-6} + 2 \times 10^{-6} + 2 \times 10^{-6}$$
$$< 10^{-5}.$$

This concludes the computation of log 2.

The computation for log 3 is similar. In each case, note that the factors 7, -2, 3 for log 2 and 11, -3, 5 for log 3 have to be taken into account, and contribute to the error term.

XIII, §6, p. 459

1. $$2 \arctan u = \arctan u + \arctan u = \arctan \frac{u + u}{1 - u^2} = \arctan \frac{2u}{1 - u^2},$$

$$3 \arctan u = \arctan u + 2 \arctan u = \arctan u + \arctan \frac{2u}{1 - u^2}.$$

Now let $v = 2u/(1 - u^2)$ and use the formula for $\arctan u + \arctan v$.

2. (a) Let $u = 1/2$ and $v = 1/3$ in the formula for $\arctan u + \arctan v$.
(b) Let $u = 1/5$ and $v = 1/8$ in this same formula.
(c) We have to apply the addition formula repeatedly. We start with

$$2 \arctan \frac{1}{5} = \arctan \frac{1}{5} + \arctan \frac{1}{5} = \arctan \frac{1/5 + 1/5}{1 - 1/25}$$
$$= \arctan \frac{5}{12}.$$

Next,

$$\arctan \frac{5}{12} + \arctan \frac{1}{7} = \arctan \frac{5/12 + 1/7}{1 - 5/84} = \arctan \frac{47}{79}.$$

Next,

$$2 \arctan \frac{1}{8} = \arctan \frac{1}{8} + \arctan \frac{1}{8} = \arctan \frac{1/8 + 1/8}{1 - 1/64}$$
$$= \arctan \frac{16}{63}.$$

Then finally

$$2 \arctan \frac{1}{5} + \arctan \frac{1}{7} + 2 \arctan \frac{1}{8} = \arctan \frac{47}{79} + \arctan \frac{16}{63}$$
$$= \arctan \frac{47/79 + 16/63}{1 - 47 \cdot 16/79 \cdot 63}$$
$$= \arctan 1 = \pi/4.$$

3. By Taylor's formula, we have

$$\arctan\frac{1}{5}=\frac{1}{5}-\frac{1}{3}\left(\frac{1}{5}\right)^3+R_5\left(\frac{1}{5}\right)$$

so

$$8\cdot\arctan\frac{1}{5}=\frac{8}{5}-\frac{8}{3}\left(\frac{1}{5}\right)^3+E_1, \tag{1}$$

where

$$E_1=8R_5\left(\frac{1}{5}\right)\qquad\text{and}\qquad |E_1|\leqq 8\,\frac{1}{5}\left(\frac{1}{5}\right)^5=\frac{16}{3}\times 10^{-4}.$$

Second, we have

$$\arctan\frac{1}{7}=\frac{1}{7}-\frac{1}{3}\left(\frac{1}{7}\right)^3+R_5\left(\frac{1}{7}\right)$$

so

$$4\cdot\arctan\frac{1}{7}=\frac{4}{7}-\frac{4}{3}\left(\frac{1}{7}\right)^3+E_2, \tag{2}$$

where

$$E_2=4R_5\left(\frac{1}{7}\right)\qquad\text{and}\qquad |E_2|\leqq 4\,\frac{1}{5}\left(\frac{1}{7}\right)^5<\frac{2}{5}\times 10^{-4}.$$

Third, we have

$$\arctan\frac{1}{8}=\frac{1}{8}-\frac{1}{3}\left(\frac{1}{8}\right)^3+R_5\left(\frac{1}{8}\right)$$

so

$$8\cdot\arctan\frac{1}{8}=1-\frac{8}{3}\left(\frac{1}{8}\right)^3+E_3 \tag{3}$$

where

$$E_3=8R_5\left(\frac{1}{8}\right)\qquad\text{and}\qquad |E_3|\leqq 8\,\frac{1}{5}\left(\frac{1}{8}\right)^5<\frac{8}{15}\times 10^{-4}.$$

Adding the three expressions, we find by Exercise 2(c):

$$\begin{aligned}\pi&=8\arctan\frac{1}{5}+4\arctan\frac{1}{7}+8\arctan\frac{1}{8}\\&=\frac{8}{5}-\frac{8}{3}\left(\frac{1}{5}\right)^3+\frac{4}{7}-\frac{4}{3}\left(\frac{1}{7}\right)^3+1-\frac{8}{3}\left(\frac{1}{8}\right)^3+E,\end{aligned}$$

where

$$E=E_1+E_2+E_3\qquad\text{and}\qquad |E|\leqq|E_1|+|E_2|+|E_3|<10^{-3}.$$

4. $\arctan(1/5) + \arctan(1/5) = \arctan 5/12$ using $u = v = 1/5$,
$\arctan(1/5) + \arctan(5/12) = \arctan(37/55)$ using $u = 1/5$ and $v = 5/12$,
$\arctan(1/5) + \arctan(37/55) = \arctan(120/119)$.

This last value is equal to $4\arctan(1/5)$. Then

$$4\arctan(1/5) - \arctan(1/239) = \arctan(120/119) + \arctan(-1/239).$$

Let $u = 120/119$ and $v = -1/239$ and use arithmetic to get $\arctan 1$.

XIII, §7, p. 467

In the answers, we give only the approximating value, except in a couple of cases to illustrate an estimate for the error term. But you should include the estimate in your work.

1. (a) $|R_2| \leqq \frac{3}{32}\cdot 10^{-4} < 10^{-5}$. We use $s = 1/4$ and

$$R_2(x) = \frac{1}{4}\left|-\frac{3}{4}\right|\frac{1}{2}(1+c)^{-7/4}|x|^2.$$

Since $(1+c)^{-7/4} \leqq 1$, we get

$$|R_2(0.1)| \leqq \frac{3}{32}(0.1)^2 \leqq \frac{3}{32}10^{-4} < 10^{-5}.$$

(b) $|R_2| \leqq \frac{3}{8}\cdot 10^{-2}$ (c) $|R_2| \leqq \frac{3}{32}\cdot 10^{-2}$

2. (a) $|R_3| \leqq 5\cdot 10^{-4}$ (b) $|R_3| \leqq \frac{1}{16}(0.8)^{-5/2}(0.2)^3 \leqq \frac{1}{2\cdot 8^3} \leqq 10^{-3}$

(c) $|R_3| \leqq \frac{1}{16}\cdot 10^{-4}$

3. Estimate $R_2(x)$ for $(1+x)^{1/3}$ and $-0.1 \leqq x \leqq 0.1$. The general expression for R_2 with $s = 1/3$ is

$$|R_2(x)| = \left|\frac{(1/3)(1/3-1)}{2}\right|(1+c)^{1/3-2}|x|^2$$

so the term $(1+c)^{-5/3} = 1/(1+c)^{5/3}$ will be biggest when $x = -0.1$. Also $|x|^2$ is biggest when $x = 0.1$. Hence

$$\begin{aligned}|R_2(x)| &\leqq \frac{1}{3}\frac{2}{3}\frac{1}{2}(0.9)^{-5/3}(0.1)^2\\ &\leqq \frac{1}{9}\left(\frac{10}{9}\right)^{5/3}10^{-2}\\ &\leqq \frac{1}{9}\left(\frac{10}{9}\right)^{2}10^{-2} = \frac{1}{729} < \frac{1}{7}\times 10^{-2}.\end{aligned}$$

4. (a) $|R_2| \leqq \frac{1}{2}\cdot(0.8)^{-3/2}\cdot 10^{-2} \leqq \frac{1}{2}\cdot\frac{1}{(0.8)^2}\cdot 10^{-2} \leqq 10^{-2}$ (b) $|R_2| \leqq \frac{1}{8}\cdot 10^{-2}$

5. (a) $5\left(1+\frac{1}{3}\cdot\frac{1}{125}\right)+E$ (b) $5\left(1+\frac{1}{3}\cdot\frac{1}{25}-\frac{1}{9}\cdot\frac{1}{625}\right)+E$

(c) $5\left(1+\frac{1}{3}\cdot\frac{6}{125}-\frac{1}{9}\cdot\frac{6^2}{125^2}\right)+E.$

In this part we include the estimate for the error. We write

$$131 = 125 + 6 = 125\left(1+\frac{6}{125}\right)$$

so

$$(131)^{1/3} = 5\left(1+\frac{6}{125}\right)^{1/3}.$$

Then

$$\left|R_3\left(\frac{6}{125}\right)\right| \leqq \frac{1}{3}\,\frac{2}{3}\,\frac{5}{3}\,\frac{1}{3!}\left(\frac{6}{125}\right)^3 \leqq \frac{1}{9}\times 10^{-4}.$$

Hence

$$(131)^{1/3} = 5\left(1+\frac{1}{3}\,\frac{6}{125}-\frac{1}{9}\,\frac{6^2}{(125)^2}\right)+E,$$

where

$$|E| = \left|5R_3\left(\frac{6}{125}\right)\right| \leqq \frac{5}{9}\times 10^{-4} < 10^{-4}.$$

(d) $6\left(1+\frac{1}{3}\cdot\frac{4}{6^3}\right)+E$

6. (a) $10\left(1-\frac{1}{2}\cdot\frac{3}{100}-\frac{1}{8}\cdot\frac{3^2}{100^2}\right)+E$ (b) $10\left(1+\frac{1}{2}\cdot\frac{2}{100}\right)+E$

(c) $10\left(1+\frac{1}{2}\cdot\frac{5}{100}-\frac{1}{8}\cdot\frac{5^2}{100^2}\right)+E$

(d) $5\left(1+\frac{1}{2}\cdot\frac{3}{25}-\frac{1}{8}\cdot\frac{3^2}{25^2}+\frac{1}{3!}\cdot\frac{3}{8}\cdot\frac{3^3}{25^3}\right)+E$

By writing $28 = 25 + 3 = 25(1 + 3/25)$ you can apply the same method as in the examples, and $E = 5R_4(3/25)$, so we have to estimate $R_4(3/25)$. We have:

$$\left|R_4\left(\frac{3}{25}\right)\right| \leqq \frac{1}{2}\,\frac{1}{2}\,\frac{3}{2}\,\frac{5}{2}\,\frac{1}{4!}\left(\frac{3}{25}\right)^4 \leqq \frac{1}{8}\times 10^{-4}$$

so $|E| \leqq (5/8)\times 10^{-4}$ which is within the desired accuracy.

XIII, §8, p. 471

1. 0 **2.** 1/4! **3.** 2 **4.** 0 **5.** 1 **6.** 1 **7.** 1 **8.** 1 **9.** 1 **10.** 2 **11.** $\frac{1}{2}$ **12.** 0 **13.** $-\frac{1}{2}$ **14.** 1 **15.** 1 **16.** 1 **17.** $-\frac{1}{2}$ **18.** -1 **19.** 2 **20.** $\frac{1}{2}$ **21.** 1 **22.** 1

23. -1 **24.** 1 **25.** 1 **26.** 0 **27.** $-\frac{1}{6}$ **28.** 0 **29.** $\frac{1}{2}$ **30.** 1 **31.** $-\frac{1}{8}$ **32.** $-\frac{1}{9}$
33. (a) 0 (b) 0 (c) 0 **34.** 1 **35.** $-\frac{1}{2}$ **36.** 0 **37.** $\frac{1}{5!}$ **38.** 0 **39.** -1

XIV, §2, p. 480

3. No **4.** Yes **5.** No **6.** No **7.** No **8.** Yes **9.** Yes

XIV, §3, p. 482

1. Yes **2.** Yes **3.** No **4.** Yes **5.** No **6.** Yes **7.** No **8.** Yes **9.** No
10. Yes **11.** No **12.** Yes **13.** No **14.** Yes **15.** Yes **16.** Yes **17.** Yes
18. Yes

XIV, §4, p. 485

3. Yes **4.** Yes **5.** Yes **6.** Yes **7.** Yes **8.** Yes **9.** Yes **10.** Yes

XIV, §5, p. 488

1. Yes **2.** Yes **3.** Yes **4.** Yes **5.** Yes
6. Converges, but not absolutely **7.** Yes **8.** Converges, but not absolutely
9. Converges, but not absolutely **11.** Converges, but not absolutely
12. Converges, but not absolutely
13. Does not converge; does not converge absolutely
14. Does not converge **15.** Converges, but not absolutely
17. Converges, but not absolutely **18.** Yes
19. Converges, but not absolutely **20.** Converges, but not absolutely

XIV, §6, p. 494

2. (a) $4/e^2$ (b) $2^2 5^5 e^{-4}/3^3$ **3.** (a) 0 (b) ∞ **4.** 1 **5.** 1 **6.** 1 **7.** 1 **8.** $\frac{1}{2}$
9. 2 **10.** 0 **11.** 1 **12.** 1 **13.** $\frac{1}{2}$ **14.** 1 **15.** $\frac{1}{4}$ **16.** $\frac{1}{e}$ **17.** 27 **18.** $\frac{4}{e^2}$ **19.** 0
20. 2 **21.** 2 **22.** 3 **23.** 1 **24.** ∞ **25.** 1 **26.** ∞ **27.** 1 **28.** ∞ **29.** e
30. ∞

App., §1, p. 504

1. (a) glb is 2; lub does not exist. (b) glb is 1; lub does not exist.
(c) glb does not exist; lub does not exist.
2. (a) glb is 0; lub is $\sqrt[3]{5}$. (b) glb is 0; lub is $\sqrt[3]{5}$. (c) glb is -2; lub is 2.
(d) glb does not exist; lub is $\frac{11}{2}$.

App., §2, p. 513

4. $f(x)$ exist for all x; $f(x) = 1$, $|x| \geqq 1$; $f(x) = 0$, $|x| < 1$

5. (a) $f(1) = 0$, $f(\frac{1}{2}) = -1$, $f(2) = 1$

(b) $\lim_{x \to 1} f(x)$ does no exist (c) $\lim_{x \to -1} f(x)$ does not exist

6. (a) $f(1) = 0$, $f(\frac{1}{2}) = 1$, $f(2) = 1$

(b) $\lim_{x \to 1} f(x) = 1$ (c) $\lim_{x \to -1} f(x) = 1$

7. (a) 0 (b) 0 (c) 0 (d) 0 **8.** (a) 0 (b) 0 (c) 0

XV, §1, p. 530

	$A + B$	$A - B$	$3A$	$-2B$
1.	$(1, 0)$	$(3, -2)$	$(6, -3)$	$(2, -2)$
2.	$(-1, 7)$	$(-1, -1)$	$(-3, 9)$	$(0, -8)$
3.	$(1, 0, 6)$	$(3, -2, 4)$	$(6, -3, 15)$	$(2, -2, -2)$
4.	$(-2, 1, -1)$	$(0, -5, 7)$	$(-3, -6, 9)$	$(2, -6, 8)$
5.	$(3\pi, 0, 6)$	$(-\pi, 6, -8)$	$(3\pi, 9, -3)$	$(-4\pi, 6, -14)$
6.	$(15 + \pi, 1, 3)$	$(15 - \pi, -5, 5)$	$(45, -6, 12)$	$(-2\pi, -6, 2)$

XV, §2, p. 534

1. No **2.** Yes **3.** No **4.** Yes **5.** No **6.** Yes **7.** Yes **8.** No

XV, §3, p. 537

1. (a) 5 (b) 10 (c) 30 (d) 14 (e) $\pi^2 + 10$ (f) 245

2. (a) -3 (b) 12 (c) 2 (d) -17 (e) $2\pi^2 - 16$ (f) $15\pi - 10$

4. (b) and (d)

XV, §4, p. 551

1. (a) $\sqrt{5}$ (b) $\sqrt{10}$ (c) $\sqrt{30}$ (d) $\sqrt{14}$ (e) $\sqrt{10 + \pi^2}$ (f) $\sqrt{245}$

2. (a) $\sqrt{2}$ (b) 4 (c) $\sqrt{3}$ (d) $\sqrt{26}$ (e) $\sqrt{58 + 4\pi^2}$ (f) $\sqrt{10 + \pi^2}$

3. (a) $(\frac{3}{2}, -\frac{3}{2})$ (b) $(0, 3)$ (c) $(-\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ (d) $(\frac{17}{26}, -\frac{51}{26}, \frac{34}{13})$

(e) $\dfrac{\pi^2 - 8}{2\pi^2 + 29}(2\pi, -3, 7)$ (f) $\dfrac{15\pi - 10}{10 + \pi^2}(\pi, 3, -1)$

4. (a) $(-\frac{6}{5}, \frac{3}{5})$ (b) $(-\frac{6}{5}, \frac{18}{5})$ (c) $(\frac{2}{15}, -\frac{1}{15}, \frac{1}{3})$ (d) $-\frac{17}{14}(-1, -2, 3)$

(e) $\dfrac{2\pi^2 - 16}{\pi^2 + 10}(\pi, 3, -1)$ (f) $\dfrac{3\pi - 2}{49}(15, -2, 4)$

5. (a) $\dfrac{-1}{\sqrt{5}\sqrt{34}}$ (b) $\dfrac{-2}{\sqrt{5}}$ (c) $\dfrac{10}{\sqrt{14}\sqrt{35}}$ (d) $\dfrac{13}{\sqrt{21}\sqrt{11}}$ (e) $\dfrac{-1}{\sqrt{12}}$

6. (a) $\dfrac{35}{\sqrt{41\cdot 35}}, \dfrac{6}{\sqrt{41\cdot 6}}, 0$ (b) $\dfrac{1}{\sqrt{17\cdot 26}}, \dfrac{16}{\sqrt{41\cdot 17}}, \dfrac{25}{\sqrt{26\cdot 41}}$

7. Let us dot the sum

$$c_1A_1 + \cdots + c_rA_r = O$$

with A_i. We find

$$c_1A_1\cdot A_i + \cdots + c_iA_i\cdot A_i + \cdots + c_rA_r\cdot A_i = O\cdot A_i = 0.$$

Since $A_j\cdot A_i = 0$ if $j \neq i$ we find

$$c_iA_i\cdot A_i = 0.$$

But $A_i\cdot A_i \neq 0$ by assumption. Hence $c_i = 0$, as was to be shown.

8. (a) $\|A + B\|^2 + \|A - B\|^2 = (A + B)\cdot(A + B) + (A - B)\cdot(A - B)$

$$= A^2 + 2A\cdot B + B^2 + A^2 - 2A\cdot B + B^2$$

$$= 2A^2 + 2B^2 = 2\|A\|^2 + 2\|B\|^2$$

9. $\|A - B\|^2 = A^2 - 2A\cdot B + B^2 = \|A\|^2 - 2\|A\|\ \|B\|\cos\theta + \|B\|^2$

XV, §5, p. 556

1. (a) Let $A = P_2 - P_1 = (-5, -2, 3)$. Parametric representation of the line is $X(t) = P_1 + tA = (1, 3, -1) + t(-5, -2, 3)$.
(b) $(-1, 5, 3) + t(-1, -1, 4)$

2. $X = (1, 1, -1) + t(3, 0, -4)$ **3.** $X = (-1, 5, 2) + t(-4, 9, 1)$

4. (a) $(-\frac{3}{2}, 4, \frac{1}{2})$ (b) $(-\frac{2}{3}, \frac{11}{3}, 0), (-\frac{7}{3}, \frac{13}{3}, 1)$ (c) $(0, \frac{17}{5}, -\frac{2}{5})$ (d) $(-1, \frac{19}{5}, \frac{1}{5})$

5. $P + \frac{1}{2}(Q - P) = \dfrac{P + Q}{2}$

XV, §6, p. 562

1. The normal vectors $(2, 3)$ and $(5, -5)$ are not perpendicular because their dot product $10 - 15 = -5$ is not 0.

2. The normal vectors are $(-m, 1)$ and $(-m', 1)$, and their dot product is $mm' + 1$. The vectors are perpendicular if and only if this dot product is 0, which is equivalent with $mm' = -1$.

3. $y = x + 8$ **4.** $4y = 5x - 7$ **6.** (c) and (d)

7. (a) $x - y + 3z = -1$ (b) $3x + 2y - 4z = 2\pi + 26$ (c) $x - 5z = -33$

8. (a) $2x + y + 2z = 7$ (b) $7x - 8y - 9z = -29$ (c) $y + z = 1$

9. $(3, -9, -5)$, $(1, 5, -7)$ (Others would be constant multiples of these.)

10. $(-2, 1, 5)$ **11.** $(11, 13, -7)$

12. (a) $X = (1, 0, -1) + t(-2, 1, 5)$
(b) $X = (-10, -13, 7) + t(11, 13, -7)$ or also $(1, 0, 0) + t(11, 13, -7)$

13. (a) $-\frac{1}{3}$ (b) $-\dfrac{2}{\sqrt{42}}$ (c) $\dfrac{4}{\sqrt{66}}$ (d) $-\dfrac{2}{\sqrt{18}}$

14. (a) $(-4, \frac{11}{2}, \frac{15}{2})$ (b) $(\frac{25}{13}, \frac{10}{13}, -\frac{9}{13})$ **15.** $(1, 3, -2)$

16. (a) $\dfrac{8}{\sqrt{35}}$ (b) $\dfrac{13}{\sqrt{21}}$

17. (a) $-2/\sqrt{40}$ (b) $(41/17, 23/17)$ **18.** (a) $x + 2y = 3$ (c) $6/\sqrt{5}$

19. $-12/7\sqrt{6}$

XVI, §1, p. 575

1. $(e^t, -\sin t, \cos t)$ **2.** $\left(2\cos 2t, \dfrac{1}{1+t}, 1\right)$ **3.** $(-\sin t, \cos t)$

4. $(-3\sin 3t, 3\cos 3t)$ **6.** B

7. $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right) + t\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$. $(-1, 0) + t(-1, 0)$, or $y = \sqrt{3}x$, $y = 0$

8. (a) $ex + y + 2z = e^2 + 3$ (b) $x + y = 1$

11. $\sqrt{(X(t) - Q)\cdot(X(t) - Q)}$

If t_0 is a value of t which minimizes the distance, then it also minimizes the square of the distance, which is easier to work with because it does not involve the square root sign. Let $f(t)$ be the square of the distance, so

$$f(t) = (X(t) - Q)^2 = (X(t) - Q)\cdot(X(t) - Q).$$

At a minimum, the derivative must be 0, and the derivative is

$$f'(t) = 2(X(t) - Q)\cdot X'(t).$$

Hence at a minimum, we have $(X(t_0) - Q)\cdot X'(t_0) = 0$, and hence $X(t_0) - Q$ is perpendicular to $X'(t_0)$, i.e. is perpendicular to the curve. If $X(t) = P + tA$ is the parametric representation of a line, then $X'(t) = A$, so we find

$$(P + t_0A - Q)\cdot A = 0.$$

Solving for t_0 yields $(P - Q)\cdot A + t_0 A\cdot A = 0$, whence

$$t_0 = \frac{(Q - P)\cdot A}{A\cdot A}.$$

13. Differentiate $X'(t)^2 = \text{constant}$ to get

$$2X'(t)\cdot X''(t) = 0.$$

14. Let $v(t) = \|X'(t)\|$. To show $v(t)$ is constant, it suffices to prove that $v(t)^2$ is constant, and $v(t)^2 = X'(t) \cdot X'(t)$. To show that a function is constant it suffices to prove that its derivative is 0, and we have

$$\frac{d}{dt} v(t)^2 = 2X'(t) \cdot X''(t).$$

By assumption, $X'(t)$ is perpendicular to $X''(t)$, so the right-hand side is 0, as desired.

15. Differentiate the relation $X(t) \cdot B = t$, you get

$$X'(t) \cdot B = 1,$$

so $\|X'(t)\| \, \|B\| \cos\theta = 1$. Hence $\|X'(t)\| = 1/\|B\| \cos\theta$ is constant. Hence the square $X'(t)^2$ is constant. Differentiate, you get

$$2X'(t) \cdot X''(t) = 0,$$

so $X'(t) \cdot X''(t) = 0$, and $X'(t)$ is perpendicular to $X''(t)$, as desired.

16. (a) $(0, 1, \pi/8) + t(-4, 0, 1)$ (b) $(1, 2, 1) + t(1, 2, 2)$
(c) $(e^3, e^{-3}, 3\sqrt{2}) + t(3e^{-3}, -3e^{-3}, 3\sqrt{2})$ (d) $(1, 1, 1) + t(1, 3, 4)$

18. Let $X(t) = (e^t, e^{2t}, 1 - e^{-t})$ and $Y(\theta) = (1 - \theta, \cos\theta, \sin\theta)$. Then the two curves intersect when $t = 0$ and $\theta = 0$. Also

$$X'(t) = (e^t, 2e^{2t}, e^{-t}) \quad \text{and} \quad Y'(\theta) = (-1, -\sin\theta, \cos\theta)$$

so

$$X'(0) = (1, 2, 1) \quad \text{and} \quad Y'(0) = (-1, 0, 1).$$

The angle between their tangents at the point of intersection is the angle between $X'(0)$ and $Y'(0)$, which is $\pi/2$, because

$$\text{cosine of the angle} = \frac{X'(0) \cdot Y'(0)}{\|X'(0)\| \, \|Y'(0)\|} = 0.$$

19. $(18, 4, 12)$ when $t = -3$ and $(2, 0, 4)$ when $t = 1$.
By definition, a point $X(t) = (x(t), y(t), z(t))$ lies on the plane if and only if

$$3x(t) - 14y(t) + z(t) - 10 = 0.$$

In the present case, this means that

$$3(2t^2) - 14(1 - t) + (3 + t^2) - 10 = 0.$$

This is a quadratic equation for t, which you solve by the quadratic formula. You will get the two values $t = -3$ or $t = 1$, which you substitute back in the parametric curve $(2t^2, 1 - t, 3 + t^2)$ to get the two points.

20. (a) Each coordinate of $X(t)$ has derivative equal to 0, so each coordinate is constant, so $X(t) = A$ for some constant A.
(b) $X(t) = tA + B$ for constant vectors $A \neq O$ and B.

21. Let $E = (0, 0, 1)$ be the unit vector in the direction of the z-axis. Then $X'(t) = (-a \sin t, a \cos t, b)$ and

$$\cos \theta(t) = \frac{X'(t) \cdot E}{\|X'(t)\|} = \frac{b}{\sqrt{a^2 + b^2}}.$$

23. Differentiate the relation $X(t) \cdot B = e^{2t}$, you get

$$X'(t) \cdot B = 2e^{2t} = \|X'(t)\| \, \|B\| \cos \theta.$$

But $\|B\| = 1$ by assumption, so the speed is $v(t) = \|X'(t)\| = 2e^{2t}/\cos \theta$. Square this and differentiate. You find

$$X'(t) \cdot X''(t) = \frac{8e^{4t}}{\cos^2 \theta}.$$

25. (a) To say that $B(t)$ lies on the surface means that the coordinates of $B(t)$ satisfy the equation of the surface, that is

$$z(t)^2 = 1 + x(t)^2 - y(t)^2.$$

Differentiate. You get

$$2z(t)z'(t) = 2x(t)x'(t) - 2y(t)y'(t),$$

which after dividing by 2 yields

$$(*) \qquad z(t)z'(t) = x(t)x'(t) - y(t)y'(t).$$

Now

$$B(t) \cdot B'(t) = x(t)x'(t) + y(t)y'(t) + z(t)z'(t).$$
$$= 2x(t)x'(t) \quad \text{by } (*).$$

(b) Given any point (x, y, z) the distance of this point to the yz-plane is just $|x|$. So if x is positive, the distance is x itself. We use the derivative test: if $x'(t) \geqq 0$ for all t then x is increasing. We have:

$$2x(t)x'(t) = B(t) \cdot B'(t) \quad \text{by (a)}$$
$$= \|B(t)\| \, \|B'(t)\| \cos \theta(t).$$

By assumption, $\cos \theta(t)$ is positive, and the norms $\|B(t)\|$, $\|B'(t)\|$ are $\geqq 0$, so if $x(t) > 0$, dividing by $2x(t)$ shows that $x'(t) \geqq 0$, whence $x(t)$ is increasing, as was to be shown.

26. (a) $(1, 1, \frac{2}{3}) + t(1, 2, 2)$ (b) $x + 2y + 2z = 1$

27. We have $C'(t) = (-e^t \sin t + e^t \cos t, e^t \cos t + e^t \sin t)$. Let θ be the angle between $C(t)$ and $C'(t)$ (the position vector). Then

$$\cos \theta = \frac{C(t) \cdot C'(t)}{\|C(t)\| \, \|C'(t)\|}$$

and a little algebra will show you it is independent of t.

XVI, §2, p. 579

1. $\sqrt{2}$ **2.** (a) $2\sqrt{13}$ (b) $\frac{\pi}{8}\sqrt{17}$

3. (a) $\frac{3}{2}(\sqrt{41}-1)+\frac{5}{4}\left(\log\frac{6+\sqrt{41}}{5}\right)$ (b) $e-\frac{1}{e}$

4. (a) 8 (b) $4-2\sqrt{2}$

The integral for the length is $L(t)=\int_a^b\sqrt{2-2\cos t}\,dt$. Use the formula

$$\sin^2 u=\frac{1-\cos 2u}{2},$$

with $t=2u$.

5. (a) $\sqrt{5}-\sqrt{2}+\log\frac{2+2\sqrt{2}}{1+\sqrt{5}}=\sqrt{5}-\sqrt{2}+\frac{1}{2}\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\,\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

The speed is $\|X'(t)\|=\sqrt{1+(1/t)^2}$ so the length is

$$L=\int_1^2\frac{1}{t}\sqrt{1+t^2}\,dt=\int_{\sqrt{2}}^{\sqrt{5}}\frac{u^2}{u^2-1}\,du$$
$$=\int_{\sqrt{2}}^{\sqrt{5}}\frac{u^2-1+1}{u^2-1}\,du=\int_{\sqrt{2}}^{\sqrt{5}}du+\int_{\sqrt{2}}^{\sqrt{5}}\frac{1}{u^2-1}\,du.$$

But

$$\frac{1}{u^2-1}=\frac{1}{2}\left(\frac{1}{u-1}-\frac{1}{u+1}\right).$$

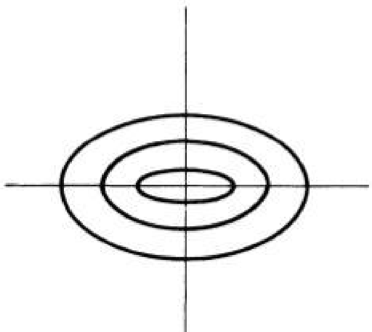
These last integrals give you logs, with appropriate numbers in front.

(b) $\sqrt{26}-\sqrt{10}+\frac{1}{2}\log\left(\frac{\sqrt{26}-1}{\sqrt{26}+1}\cdot\frac{\sqrt{10}+1}{\sqrt{10}-1}\right)=\sqrt{26}-\sqrt{10}+\log\frac{5}{3}\left(\frac{1+\sqrt{10}}{1+\sqrt{26}}\right)$

6. $\log(\sqrt{2}+1)$ **7.** 5/3 **8.** 8

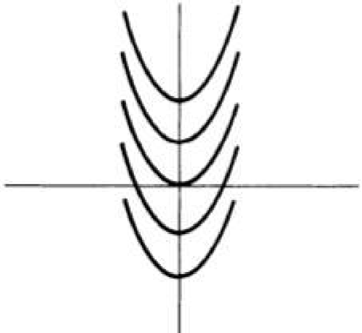
XVII, §1, p. 586

1.



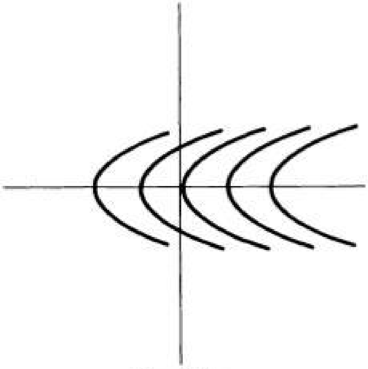
Ellipses

2.



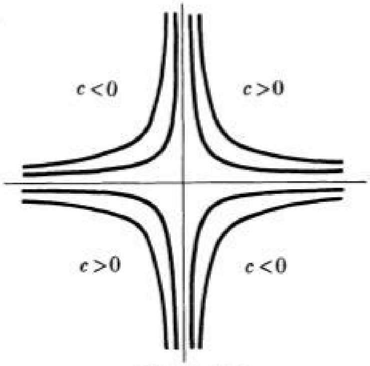
Parabolas

4.



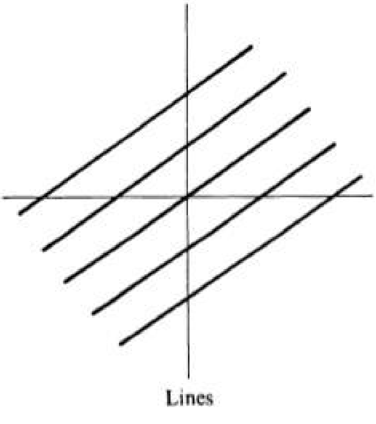
Parabolas

6.



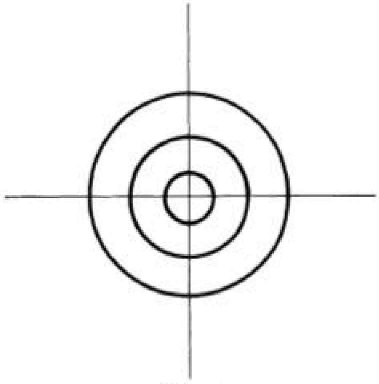
Hyperbolas

10.



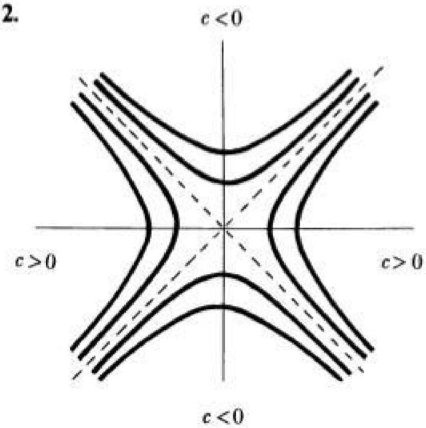
Lines

11.



Circles

12.



Hyperbolas

XVII, §2, p. 592

	$\partial f/\partial x$	$\partial f/\partial y$	$\partial f/\partial z$
1.	y	x	1
2.	$2xy^5$	$5x^2y^4$	0
3.	$y\cos(xy)$	$x\cos(xy)$	$-\sin(z)$
4.	$-y\sin(xy)$	$-x\sin(xy)$	0
5.	$yz\cos(xyz)$	$xz\cos(xyz)$	$xy\cos(xyz)$
6.	yze^{xyz}	xze^{xyz}	xye^{xyz}
7.	$2x\sin(yz)$	$x^2z\cos(yz)$	$x^2y\cos(yz)$
8.	yz	xz	xy
9.	$z+y$	$z+x$	$x+y$
10.	$\cos(y-3z)+\dfrac{y}{\sqrt{1-x^2y^2}}$	$-x\sin(y-3z)+\dfrac{x}{\sqrt{1-x^2y^2}}$	$3x\sin(y-3z)$

11. (1) $(2, 1, 1)$ (2) $(64, 80, 0)$ (6) $(6e^6, 3e^6, 2e^6)$ (8) $(6, 3, 2)$ (9) $(5, 4, 3)$
12. (4) $(0, 0, 0)$ (5) $(\pi^2\cos\pi^2, \pi\cos\pi^2, \pi\cos\pi^2)$
(7) $(2\sin\pi^2, \pi\cos\pi^2, \pi\cos\pi^2)$

13. $(-1, -2, 1)$ **14.** $\dfrac{\partial x^y}{\partial x} = yx^{y-1}$ $\dfrac{\partial x^y}{\partial y} = x^y\log x$

15. $(-2e^{-2}\cos\pi^2, -\pi e^{-2}\sin\pi^2, -\pi e^{-2}\sin\pi^2)$ **16.** $\left(\frac{3}{2}, \frac{1}{2}, -\dfrac{5\sqrt{3}}{2}\right)$

XVII, §3, p. 598

1. $2, -3$ **2.** a, b **3.** a, b, c

5. Select first $H = (h, 0) = hE_1$. Then $A \cdot H = ha_1$ if $A = (a_1, a_2)$. Divide both sides of the relation

$$f(X+H) - f(X) = a_1h + |h|g(H)$$

by $h \neq 0$ and take the limit to see that $\alpha_1 = D_1f(X)$. Similarly use $H = (0, h) = hE_2$ to see that $\alpha_2 = D_2f(x, y)$. Similar argument for three variables.

XVIII, §1, p. 603

1. $\dfrac{d}{dt}(P + tA) = A$, so this follows directly from the chain rule.

2. 5. Indeed, $C'(t) = (2t, -3t^{-4}, 1)$ and $C'(1) = (2, -3, 1)$. Dot this with given grad $f(1, 1, 1)$ to find 5.

3. $C'(0) = (0, 1)$

Let $C'(0) = (a, b)$. Now $\operatorname{grad} f(C(0)) = (9, 2)$ and $\operatorname{grad} g(C(0)) = (4, 1)$, so using the chain rule on the functions f and g, respectively, we obtain

$$2 = \frac{d}{dt} f(C(t))\bigg|_{t=0} = (9, 2)\cdot(a, b) = 9a + 2b,$$

$$1 = \frac{d}{dt} g(C(t))\bigg|_{t=0} = (4, 1)\cdot(a, b) = 4a + b.$$

Solving for the above simultaneous equations yields $C'(0) = (0, 1)$.

4. (a) $\operatorname{grad} f(tP)\cdot P$.

(b) Use 4(a) and let $t = 0$.

5. Viewing x, y as constant, put $P = (x, y)$ and use Exercise 4(a). Then put $t = 1$. If you expand out, you will find the stated answer.

7. (a) $\partial f/\partial x = x/r$ and $\partial f/\partial y = y/r$ if $r = \sqrt{x^2 + y^2}$.

(b) $\dfrac{\partial f}{\partial x} = \dfrac{x}{(x^2 + y^2 + z^2)^{1/2}}$, $\dfrac{\partial f}{\partial y} = \dfrac{y}{(x^2 + y^2 + z^2)^{1/2}}$, $\dfrac{\partial f}{\partial z} =$ guess what?

8. $\dfrac{\partial r}{\partial x_i} = \dfrac{x_i}{r}$

9. (a) $\partial f/\partial x = (3x^2y + 4x)\cos(x^3y + 2x^2)$

$\partial f/\partial y = x^3 \cos(x^3y + 2x^2)$

(b) $\partial f/\partial x = -(6xy - 4)\sin(3x^2y - 4x)$

$\partial f/\partial y = -3x^2 \sin(3x^2y - 4x)$

(c) $\partial f/\partial x = \dfrac{2xy}{(x^2y + 5y)}$, $\dfrac{\partial f}{\partial y} = \dfrac{x^2 + 5}{x^2y + 5y} = \dfrac{1}{y}$

(d) $\partial f/\partial x = \frac{1}{2}(2xy + 4)(x^2y + 4x)^{-1/2}$

$\partial f/\partial y = \frac{1}{2}x^2(x^2y + 4x)^{-1/2}$

XVIII, §2, p. 609

1.	Plane	Line
(a)	$6x + 2y + 3z = 49$	$X = (6, 2, 3) + t(12, 4, 6)$
(b)	$x + y + 2z = 2$	$X = (1, 1, 0) + t(1, 1, 2)$
(c)	$13x + 15y + z = -15$	$X = (2, -3, 4) + t(13, 15, 1)$
(d)	$6x - 2y + 15z = 22$	$X = (1, 7, 2) + t(-6, 2, -15)$
(e)	$4x + y + z = 13$	$X = (2, 1, 4) + t(8, 2, 2)$
(f)	$z = 0$	$X = (1, \pi/2, 0) + t(0, 0, \pi/2 + 1)$

2. (a) $(3, 0, 1)$ (b) $X = \left(\log 3, \dfrac{3\pi}{2}, -3\right) + t(3, 0, 1)$

(c) $3x + z = 3\log 3 - 3$

3. (a) $X = (3, 2, -6) + t(2, -3, 0)$ (b) $X = (2, 1, -2) + t(-5, 4, -3)$

(c) $X = (3, 2, 2) + t(2, 3, 0)$

4. $\|C(t) - Q\|$ and see Exercise 11 of Chapter II, §1.
5. (a) $6x + 8y - z = 25$ (b) $16x + 12y - 125z = -75$
(c) $\pi x + y + z = 2\pi$ **6.** $x - 2y + z = 1$
7. (b) $x + y + 2z = 2$ **8.** $3x - y + 6z = 14$
9. $(\cos 3)x + (\cos 3)y - z = 3\cos 3 - \sin 3.$

10. $3x + 5y + 4z = 18$ **11.** (a) $\dfrac{1}{\sqrt{27}}(5, 1, 1)$ (b) $5x + y + z - 6 = 0$

12. $\dfrac{-10}{3\sqrt{12}}$ **13.** (a) 0 (b) 6 **14.** $4ex + 4ey + 4ez = 12e$

XVIII, §3, p. 614

1. (a) $\frac{5}{3}$ (b) $\max = \sqrt{10}$, $\min = -\sqrt{10}$

2. (a) $\dfrac{3}{2\sqrt{5}}$ (b) $\frac{48}{13}$ (c) $2\sqrt{145}$

3. Increasing $\left(-\dfrac{9\sqrt{3}}{2}, -\dfrac{3\sqrt{3}}{2}\right)$, decreasing $\left(\dfrac{9\sqrt{32}}{2}, \dfrac{3\sqrt{3}}{2}\right)$

4. (a) $\left(\dfrac{9}{2\cdot 6^{7/4}}, \dfrac{3}{2\cdot 6^{7/4}}, -\dfrac{6}{2\cdot 6^{7/4}}\right)$ (b) $(1, 2, -1, 1)$

5. (a) $-2/\sqrt{5}$ (b) $\sqrt{116}$ **6.** $\dfrac{1}{\sqrt{54}}(5, 2, 5)$, $6\sqrt{6}$ **7.** $\left(\dfrac{-1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$, $\sqrt{2}$

8. $\dfrac{1}{\sqrt{3}}(2e - 5)$ **9.** (a) 0 (b) $-\sqrt{1 + 2\pi^2}$

10. For any unit vector A, the function of t given by $f(P + tA)$ has a maximum at $t = 0$ (for small values of t), and hence its derivative is 0 at $t = 0$. But its derivative is $\operatorname{grad} f(P + tA)\cdot A$, which at $t = 0$ is $\operatorname{grad} f(P)\cdot A$. This is true for all A, whence $\operatorname{grad} f(P) = O$. (For instance, let A be any one of the standard unit vectors in the directions of the coordinate axes.)

Although the above argument is the one which will work in Problem 11, there is a basically easier way to see the assertion. Fix all but one variable, and say x_1 is the variable. Let

$$g(x) = f(x, a_2, \ldots, a_n), \quad \text{where} \quad P = (a_1, \ldots, a_n).$$

Then g is a function of one variable, which has a maximum at $x = a_1$. Hence $g'(a_1) = 0$ by last year's calculus. But

$$g'(a_1) = D_1 f(a_1, \ldots, a_n).$$

Similarly $D_i f(P) = 0$ for all i, as asserted.

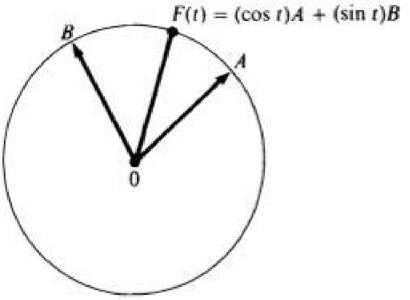
XVIII, §4, p. 619

1. $\dfrac{\partial f}{\partial x} = \dfrac{dg}{dr}\dfrac{\partial r}{\partial x} = \dfrac{dg}{dr}\dfrac{x}{r}$. Replace x by y and z. Square each term and add. You can factor

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1.$$

2. (a) $-X/r^3$ (b) $2X$ (c) $-3X/r^5$ (d) $-2e^{-r^2}X$ (e) $-X/r^2$
(f) $-4mX/r^{m+2}$ (g) $-(\sin r)X/r$

3. $F(t)^2 = (\cos t)^2 A^2 + 2(\cos t)(\sin t)\, A \cdot B + (\sin t)^2 B^2 = 1$,
because $A^2 = B^2 = 1$ since A, B are unit vectors and $A \cdot B = 0$ by assumption. Hence $\|F(t)\| = 1$, so $F(t)$ lies on the sphere of radius 1.



4. Note that $L(t) = (1 - t)P + tQ$. If $L(t) = 0$ for some value of t, then

$$(1 - t)P = -tQ$$

Square both sides, use $P^2 = Q^2 = 1$ to get $(1 - t)^2 = t^2$. It follows that $t = 1/2$, so $\frac{1}{2}P = \frac{-1}{2}Q$, whence $P = -Q$.

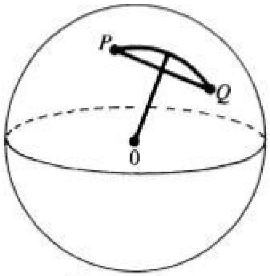
5. By Exercise 4, $L(t) \neq O$ if $0 \leqq t \leqq 1$. Then $L(t)/\|L(t)\|$ is a unit vector, and this expression is composed of differentiable expressions so is differentiable. Furthermore, we have

$$L(0) = P \quad \text{and} \quad L(1) = Q.$$

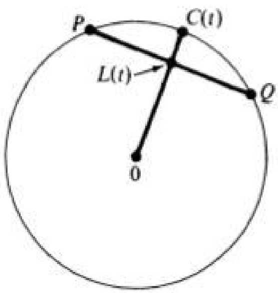
Thus if we put $C(t) = L(t)/\|L(t)\|$, then $\|C(t)\| = 1$ for all t, and the curve $C(t)$ lies on the sphere. Also

$$C(0) = P \quad \text{and} \quad C(1) = Q.$$

Hence $C(t)$ is a curve on the sphere which joins P and Q.
The picture looks as follows.



On the sphere



cross section

Note that $C(t)$ is the unit vector in the direction of $L(t)$.

6. Suppose P, Q are two points on the sphere, but $P = -Q$. In this case we cannot apply Exercise 5, but we can apply Exercise 3. We let

$$C(t) = (\cos t)P + (\sin t)A,$$

where A is a unit vector perpendicular to P. Then $C(t)^2 = 1$, so $C(t)$ lies on the sphere, and we have

$$C(0) = P, \qquad C(\pi) = -P.$$

Thus $C(t)$ is a curve on the sphere joining P and $-P$.

7. Let $x = a \cos t$ and $y = b \sin t$.

9. Let P, Q be two points on the sphere of radius a. It suffices to prove that $f(P) = f(Q)$. By Exercises 5 and 6, there exists a curve $C(t)$ on the sphere which joins P and Q, that is $C(t)$ is defined on an interval, and there are two numbers t_1 and t_2 such that $C(t_1) = P$ and $C(t_2) = Q$. In those exercises, we did it only for the sphere or radius 1, but you can do it for a sphere of arbitrary radius a by considering $aC(t)$ instead of the $C(t)$ in Exercises 5 or 6. Now, it suffices to prove that the function $f(C(t))$ is constant (as function of t). Take its derivative, get by the chain rule

$$\frac{d}{dt} f(C(t)) = \operatorname{grad} f(C(t)) \cdot C'(t) = h(C(t))C(t) \cdot C'(t).$$

But $C(t)^2 = a^2$ because $C(t)$ is on the sphere of radius a. Differentiating this with respect to t yields $2C(t) \cdot C'(t) = 0$, so $C(t) \cdot C'(t) = 0$, which you plug in above to see that the derivative of $f(C(t)) = 0$. Hence $f(C(t_1)) = f(C(t_2))$ so $f(P) = f(Q)$.

10. $\operatorname{grad} f(X) = \left(g'(r)\dfrac{x}{r}, g'(r)\dfrac{y}{r}, g'(r)\dfrac{z}{r}\right) = \dfrac{g'(r)}{r} X$ (say in three variables), and $g'(r)/r$ is a scalar factor of X, so $\operatorname{grad} f(X)$ and X are parallel.

XVIII, §5, p. 623

1. $k \log \|X\|$ **2.** $-\dfrac{k}{2r^2}$ **3.** $\begin{cases} \log r, & k = 2 \\ \dfrac{1}{(2-k)r^{k-2}}, & k \neq 2 \end{cases}$

Exercises 1 and 2 are special cases of 3. Let.

$$F(X) = \frac{1}{r^k} X.$$

We have to find a function $g(r)$ such that if we put $f(X) = g(r)$ then $F(X) = \operatorname{grad} f(X)$. This means we must solve the equation

$$\frac{1}{r^k} X = \frac{g'(r)}{r} X,$$

or in other words

$$g'(r) = r^{1-k}.$$

Then

$$g(r) = \int r^{1-k}\, dr,$$

which is an integral in one variable. You should know how to find it.

Index

A Brief Table of Integrals

1. $\int u\,dv = uv - \int v\,du$

2. $\int a^u\,du = \frac{a^u}{\ln a} + C, \quad a \neq 1, \quad a > 0$

3. $\int \cos u\,du = \sin u + C$

4. $\int \sin u\,du = -\cos u + C$

5. $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, \quad n \neq -1$

6. $\int (ax+b)^{-1}\,dx = \frac{1}{a}\ln|ax+b| + C$

7. $\int x(ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a^2}\left[\frac{ax+b}{n+2} - \frac{b}{n+1}\right] + C, \quad n \neq -1, -2$

8. $\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$

9. $\int x(ax+b)^{-2}\,dx = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

10. $\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$

11. $\int (\sqrt{ax+b})^n\,dx = \frac{2}{a}\frac{(\sqrt{ax+b})^{n+2}}{n+2} + C, \quad n \neq -2$

12. $\int \frac{\sqrt{ax+b}}{x}\,dx = 2\sqrt{ax+b} + b\int \frac{dx}{x\sqrt{ax+b}}$

13. (a) $\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\tan^{-1}\sqrt{\frac{ax+b}{-b}} + C, \quad \text{if } b < 0$

    (b) $\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + C, \quad \text{if } b > 0$

14. $\int \frac{\sqrt{ax+b}}{x^2}\,dx = -\frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}} + C$

15. $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}} + C$

16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

17. $\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

18. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

19. $\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{2a^2}\int \frac{dx}{a^2-x^2}$

20. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C = \ln|x+\sqrt{a^2+x^2}| + C$

21. $\displaystyle\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + C$

22. $\displaystyle\int x^2\sqrt{a^2+x^2}\,dx = \frac{x(a^2+2x^2)\sqrt{a^2+x^2}}{8} - \frac{a^4}{8}\sinh^{-1}\frac{x}{a} + C$

23. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a\sinh^{-1}\left|\frac{a}{x}\right| + C$

24. $\displaystyle\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx = \sinh^{-1}\frac{x}{a} - \frac{\sqrt{a^2+x^2}}{x} + C$

25. $\displaystyle\int \frac{x^2}{\sqrt{a^2+x^2}}\,dx = -\frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x\sqrt{a^2+x^2}}{2} + C$

26. $\displaystyle\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

27. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x} + C$

28. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

29. $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

30. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx = \frac{a^4}{8}\sin^{-1}\frac{x}{a} - \frac{1}{8}x\sqrt{a^2-x^2}\,(a^2-2x^2) + C$

31. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

32. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\sin^{-1}\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{x} + C$

33. $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \frac{1}{2}x\sqrt{a^2-x^2} + C$

34. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

35. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

36. $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

37. $\displaystyle\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + C$

38. $\displaystyle\int \left(\sqrt{x^2-a^2}\right)^n dx = \frac{x\left(\sqrt{x^2-a^2}\right)^n}{n+1} - \frac{na^2}{n+1}\int\left(\sqrt{x^2-a^2}\right)^{n-2}dx, \qquad n \neq -1$

39. $\displaystyle\int \frac{dx}{\left(\sqrt{x^2-a^2}\right)^n} = \frac{x\left(\sqrt{x^2-a^2}\right)^{2-n}}{(2-n)a^2} - \frac{n-3}{(n-2)a^2}\int\frac{dx}{\left(\sqrt{x^2-a^2}\right)^{n-2}}, \qquad n \neq 2$

40. $\displaystyle\int x\left(\sqrt{x^2-a^2}\right)^n dx = \frac{\left(\sqrt{x^2-a^2}\right)^{n+2}}{n+2} + C, \qquad n \neq -2$

41. $\displaystyle\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2} - \frac{a^4}{8}\cosh^{-1}\frac{x}{a} + C$

42. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}\,dx = \sqrt{x^2-a^2} - a\sec^{-1}\left|\frac{x}{a}\right| + C$

43. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^2}\,dx = \cosh^{-1}\frac{x}{a} - \frac{\sqrt{x^2-a^2}}{x} + C$

Continued overleaf.

44. $\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{x^2-a^2} + C$

45. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C = \frac{1}{a}\cos^{-1}\left|\frac{a}{x}\right| + C$

46. $\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$

47. $\int \frac{dx}{\sqrt{2ax-x^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$

48. $\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) + C$

49. $\int (\sqrt{2ax-x^2})^n\,dx = \frac{(x-a)(\sqrt{2ax-x^2})^n}{n+1} + \frac{na^2}{n+1}\int (\sqrt{2ax-x^2})^{n-2}\,dx,$

50. $\int \frac{dx}{(\sqrt{2ax-x^2})^n} = \frac{(x-a)(\sqrt{2ax-x^2})^{2-n}}{(n-2)a^2} + \frac{(n-3)}{(n-2)a^2}\int \frac{dx}{(\sqrt{2ax-x^2})^{n-2}}$

51. $\int x\sqrt{2ax-x^2}\,dx = \frac{(x+a)(2x-3a)\sqrt{2ax-x^2}}{6} + \frac{a^3}{2}\sin^{-1}\frac{x-a}{a} + C$

52. $\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\sin^{-1}\frac{x-a}{a} + C$

53. $\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -2\sqrt{\frac{2a-x}{x}} - \sin^{-1}\left(\frac{x-a}{a}\right) + C$

54. $\int \frac{x\,dx}{\sqrt{2ax-x^2}} = a\sin^{-1}\frac{x-a}{a} - \sqrt{2ax-x^2} + C$

55. $\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{1}{a}\sqrt{\frac{2a-x}{x}} + C$

56. $\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$

57. $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

58. $\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C$

59. $\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} + C$

60. $\int \sin^n ax\,dx = \frac{-\sin^{n-1} ax\cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax\,dx$

61. $\int \cos^n ax\,dx = \frac{\cos^{n-1} ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx$

62. (a) $\int \sin ax\cos bx\,dx = -\frac{\cos(a+b)x}{2(a+b)} - \frac{\cos(a-b)x}{2(a-b)} + C, \quad a^2 \neq b^2$

(b) $\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)}, \quad a^2 \neq b^2$

(c) $\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)}, \quad a^2 \neq b^2$

63. $\int \sin ax\cos ax\,dx = -\frac{\cos 2ax}{4a} + C$

64. $\int \sin^n ax\cos ax\,dx = \frac{\sin^{n+1} ax}{(n+1)a} + C, \quad n \neq -1$

This table is continued on the endpapers at the back.

65. $\displaystyle\int \frac{\cos ax}{\sin ax}\,dx = \frac{1}{a}\ln|\sin ax| + C$

66. $\displaystyle\int \cos^n ax \sin ax\,dx = -\frac{\cos^{n+1} ax}{(n+1)a} + C, \qquad n \neq -1$

67. $\displaystyle\int \frac{\sin ax}{\cos ax}\,dx = -\frac{1}{a}\ln|\cos ax| + C$

68. $\displaystyle\int \sin^n ax \cos^m ax\,dx = -\frac{\sin^{n-1} ax \cos^{m+1} ax}{a(m+n)} + \frac{n-1}{m+n}\int \sin^{n-2} ax \cos^m ax\,dx,$

$n \neq -m \qquad$ (If $n = -m$, use No. 86.)

69. $\displaystyle\int \sin^n ax \cos^m ax\,dx = \frac{\sin^{n+1} ax \cos^{m-1} ax}{a(m+n)} + \frac{m-1}{m+n}\int \sin^n ax \cos^{m-2} ax\,dx,$

$m \neq -n \qquad$ (If $m = -n$, use No. 87.)

70. $\displaystyle\int \frac{dx}{b + c\sin ax} = \frac{-2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)\right] + C, \qquad b^2 > c^2$

71. $\displaystyle\int \frac{dx}{b + c\sin ax} = \frac{-1}{a\sqrt{c^2-b^2}}\ln\left|\frac{c + b\sin ax + \sqrt{c^2-b^2}\cos ax}{b + c\sin ax}\right| + C, \qquad b^2 < c^2$

72. $\displaystyle\int \frac{dx}{1 + \sin ax} = -\frac{1}{a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + C$

73. $\displaystyle\int \frac{dx}{1 - \sin ax} = \frac{1}{a}\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) + C$

74. $\displaystyle\int \frac{dx}{b + c\cos ax} = \frac{2}{a\sqrt{b^2-c^2}}\tan^{-1}\left[\sqrt{\frac{b-c}{b+c}}\tan\frac{ax}{2}\right] + C, \qquad b^2 > c^2$

75. $\displaystyle\int \frac{dx}{b + c\cos ax} = \frac{1}{a\sqrt{c^2-b^2}}\ln\left|\frac{c + b\cos ax + \sqrt{c^2-b^2}\sin ax}{b + c\cos ax}\right| + C, \qquad b^2 < c^2$

76. $\displaystyle\int \frac{dx}{1 + \cos ax} = \frac{1}{a}\tan\frac{ax}{2} + C$

77. $\displaystyle\int \frac{dx}{1 - \cos ax} = -\frac{1}{a}\cot\frac{ax}{2} + C$

78. $\displaystyle\int x\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax + C$

79. $\displaystyle\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax + C$

80. $\displaystyle\int x^n \sin ax\,dx = -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$

81. $\displaystyle\int x^n \cos ax\,dx = \frac{x^n}{a}\sin ax - \frac{n}{a}\int x^{n-1}\sin ax\,dx$

82. $\displaystyle\int \tan ax\,dx = -\frac{1}{a}\ln|\cos ax| + C$

83. $\displaystyle\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$

84. $\displaystyle\int \tan^2 ax\,dx = \frac{1}{a}\tan ax - x + C$

85. $\displaystyle\int \cot^2 ax\,dx = -\frac{1}{a}\cot ax - x + C$

86. $\displaystyle\int \tan^n ax\,dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int \tan^{n-2} ax\,dx, \qquad n \neq 1$

87. $\displaystyle\int \cot^n ax\,dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int \cot^{n-2} ax\,dx, \qquad n \neq 1$

88. $\displaystyle\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$

89. $\displaystyle\int \csc ax\,dx = -\frac{1}{a}\ln|\csc ax + \cot ax| + C$

Continued overleaf.

90. $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$

91. $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$

92. $\int \sec^n ax\,dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\,dx, \qquad n \neq 1$

93. $\int \csc^n ax\,dx = -\frac{\csc^{n-2} ax \cot ax}{a(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2} ax\,dx, \qquad n \neq 1$

94. $\int \sec^n ax \tan ax\,dx = \frac{\sec^n ax}{na} + C, \qquad n \neq 0$

95. $\int \csc^n ax \cot ax\,dx = -\frac{\csc^n ax}{na} + C, \qquad n \neq 0$

96. $\int \sin^{-1} ax\,dx = x\sin^{-1} ax + \frac{1}{a}\sqrt{1-a^2x^2} + C$

97. $\int \cos^{-1} ax\,dx = x\cos^{-1} ax - \frac{1}{a}\sqrt{1-a^2x^2} + C$

98. $\int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\ln(1+a^2x^2) + C$

99. $\int x^n \sin^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1$

100. $\int x^n \cos^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\cos^{-1} ax + \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1$

101. $\int x^n \tan^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\tan^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{1+a^2x^2}, \qquad n \neq -1$

102. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$

103. $\int b^{ax}\,dx = \frac{1}{a}\frac{b^{ax}}{\ln b} + C, \qquad b > 0, \quad b \neq 1$

104. $\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$

105. $\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

106. $\int x^n b^{ax}\,dx = \frac{x^n b^{ax}}{a\ln b} - \frac{n}{a\ln b}\int x^{n-1}b^{ax}\,dx, \qquad b > 0, \quad b \neq 1$

107. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

108. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

109. $\int \ln ax\,dx = x\ln ax - x + C$

110. $\int x^n \ln ax\,dx = \frac{x^{n+1}}{n+1}\ln ax - \frac{x^{n+1}}{(n+1)^2} + C, \qquad n \neq -1$

111. $\int x^{-1}\ln ax\,dx = \frac{1}{2}(\ln ax)^2 + C$

112. $\int \frac{dx}{x\ln ax} = \ln|\ln ax| + C$

113. $\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$

114. $\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$

115. $\int \sinh^2 ax\,dx = \frac{\sinh 2ax}{4a} - \frac{x}{2} + C$

116. $\int \cosh^2 ax\,dx = \frac{\sinh 2ax}{4a} + \frac{x}{2} + C$

117. $\int \sinh^n ax\,dx = \frac{\sinh^{n-1} ax \cosh ax}{na} - \frac{n-1}{n}\int \sinh^{n-2} ax\,dx, \qquad n \neq 0$

118. $\int \cosh^n ax\,dx = \dfrac{\cosh^{n-1} ax \sinh ax}{na} + \dfrac{n-1}{n}\int \cosh^{n-2} ax\,dx, \qquad n \neq 0$

119. $\int x \sinh ax\,dx = \dfrac{x}{a}\cosh ax - \dfrac{1}{a^2}\sinh ax + C$

120. $\int x \cosh ax\,dx = \dfrac{x}{a}\sinh ax - \dfrac{1}{a^2}\cosh ax + C$

121. $\int x^n \sinh ax\,dx = \dfrac{x^n}{a}\cosh ax - \dfrac{n}{a}\int x^{n-1}\cosh ax\,dx$

122. $\int x^n \cosh ax\,dx = \dfrac{x^n}{a}\sinh ax - \dfrac{n}{a}\int x^{n-1}\sinh ax\,dx$

123. $\int \tanh ax\,dx = \dfrac{1}{a}\ln(\cosh ax) + C$

124. $\int \coth ax\,dx = \dfrac{1}{a}\ln|\sinh ax| + C$

125. $\int \tanh^2 ax\,dx = x - \dfrac{1}{a}\tanh ax + C$

126. $\int \coth^2 ax\,dx = x - \dfrac{1}{a}\coth ax + C$

127. $\int \tanh^n ax\,dx = -\dfrac{\tanh^{n-1} ax}{(n-1)a} + \int \tanh^{n-2} ax\,dx, \qquad n \neq 1$

128. $\int \coth^n ax\,dx = -\dfrac{\coth^{n-1} ax}{(n-1)a} + \int \coth^{n-2} ax\,dx, \qquad n \neq 1$

129. $\int \operatorname{sech} ax\,dx = \dfrac{1}{a}\sin^{-1}(\tanh ax) + C$

130. $\int \operatorname{csch} ax\,dx = \dfrac{1}{a}\ln\left|\tanh\dfrac{ax}{2}\right| + C$

131. $\int \operatorname{sech}^2 ax\,dx = \dfrac{1}{a}\tanh ax + C$

132. $\int \operatorname{csch}^2 ax\,dx = -\dfrac{1}{a}\coth ax + C$

133. $\int \operatorname{sech}^n ax\,dx = \dfrac{\operatorname{sech}^{n-2} ax \tanh ax}{(n-1)a} + \dfrac{n-2}{n-1}\int \operatorname{sech}^{n-2} ax\,dx, \qquad n \neq 1$

134. $\int \operatorname{csch}^n ax\,dx = -\dfrac{\operatorname{csch}^{n-2} ax \coth ax}{(n-1)a} - \dfrac{n-2}{n-1}\int \operatorname{csch}^{n-2} ax\,dx, \qquad n \neq 1$

135. $\int \operatorname{sech}^n ax \tanh ax\,dx = -\dfrac{\operatorname{sech}^n ax}{na} + C, \qquad n \neq 0$

136. $\int \operatorname{csch}^n ax \coth ax\,dx = -\dfrac{\operatorname{csch}^n ax}{na} + C, \qquad n \neq 0$

137. $\int e^{ax}\sinh bx\,dx = \dfrac{e^{ax}}{2}\left[\dfrac{e^{bx}}{a+b} - \dfrac{e^{-bx}}{a-b}\right] + C, \qquad a^2 \neq b^2$

138. $\int e^{ax}\cosh bx\,dx = \dfrac{e^{ax}}{2}\left[\dfrac{e^{bx}}{a+b} + \dfrac{e^{-bx}}{a-b}\right] + C, \qquad a^2 \neq b^2$

139. $\int_0^\infty x^{n-1}e^{-x}\,dx = \Gamma(n) = (n-1)!, \qquad n > 0.$

140. $\int_0^\infty e^{-ax^2}\,dx = \dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}, \qquad a > 0$

141. $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\cdot\dfrac{\pi}{2}, & \text{if } n \text{ is an even integer} \geq 2, \\ \dfrac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n}, & \text{if } n \text{ is an odd integer} \geq 3 \end{cases}$