# THEORY OF GROUP REPRESENTATIONS AND APPLICATIONS

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*Dedicated to*

*OLA and PIERRETTE*

# Preface

This book is written primarily for physicists, but also for other scientists and mathematicians to acquaint them with the more modern and powerful methods and results of the theory of topological groups and of group representations and to show the remarkable wide scope of applications. In this respect it is markedly different than, and goes much beyond, the standard books on group theory in quantum mechanics. Although we aimed at a mathematically rigorous level, we tried to make the exposition very explicit, the language less abstract, and have illustrated the results by many examples and applications.

During the past two decades many investigations by physicists and mathematicians have brought a certain degree of maturity and completeness to the theory of group representations. We have in mind the new results in the development of the general theory, as well as many explicit constructions of representations of specific groups. At the same time new applications of, in particular, noncompact groups revealed interesting structures in the symmetry, as well as in the dynamics, of quantum theory. The mathematical sophistication and knowledge of the physicists have also markedly increased. For all these reasons it is timely to collect the new results and to present a book on a much higher level than before, in order to facilitate further developments and applications of group representations.

There is no other comparable book on group representations, neither in mathematical nor in physical literature, and we hope that it will prove to be useful in many areas of research.

Many of the results appear, to our knowledge, for the first time in book form. These include, in particular, a systematic exposition of the theory and applications of induced representations, the classification of all finite-dimensional irreducible representations of arbitrary Lie groups, the representation theory of Lie and enveloping algebras by means of unbounded operators, new integrability conditions for representations of Lie algebras and harmonic analysis on homogeneous spaces.

In the domain of applications, we have discussed the general problem of symmetries in quantum theory, in particular, relativistic invariance, group theoretical derivation of relativistic wave equations, as well as various applications of group representations to dynamical problems in quantum theory.

We have tried to achieve a certain amount of completeness so that the book can be used as a textbook for an advanced course in mathematical physics on Lie algebras, Lie groups and their representations. Some of the standard topics can be found scattered in various texts but, so far, not all under a single cover.

A book in the border area of theoretical physics and pure mathematics is always problematic. And so this book may seem to be too difficult, detailed and abstract to some physicists, and not detailed and complete enough for some mathematicians, as we have deliberately omitted a number of proofs. Fortunately the demand for knowledge of modern mathematics among physicists is on the rise. And to give the proofs of all theorems in such a wide area of mathematics is impossible even in a large volume as this one. Where too long technical details would cloud the clarity and when the steps of the proof did not seem to be essential for further development of the subject we have omitted the proofs.

The material collected in this book originated from lectures given by the authors over many years in Warsaw, Trieste, Schladming, Istanbul, Göteborg and Boulder. It has passed several rewritings. We are especially grateful to many friends and colleagues who read, corrected and commented on parts of the manuscript. We would like to thank Dr. S. Woronowicz for his careful and patient reading of the entire manuscript and pointing out numerous improvements and corrections. We have discussed parts of the manuscript with many of our friends and colleagues who made constructive criticism, in particular S. Dymus, M. Flato, B. Kostant, G. Mackey, K. Maurin, L. Michel, I. Segal, D. Sternheimer, S. Ström, A. Sym, I. Szczyrba and A. Wawrzyńczyk.

A considerable part of this book contains the results of the research carried out under collaboration between Colorado University in Boulder and Institute for Nuclear Research in Warsaw. This collaboration was partially supported by National Science Foundation under the contract No. GF-41958. The authors are particularly grateful to Dr. C. Zalar, Program Manager for Europe and North America for his kind and effective support for American-Polish scientific collaboration.

Finally, we would like to express our gratitude to Mr J. Panz, editor in the Polish Scientific Publishers, for his great help in preparing this manuscript for printing. We are also obliged to Mrs Z. Osek for her kind help in all phases of preparing the manuscript for publication.

*A. O. Barut and R. Rączka*

Boulder and Warsaw, August 1976

# Contents

# Outline of the Book

The book begins with a long chapter on Lie algebras. This is a self-contained detailed exposition of the theory and applications of Lie algebras. The theory of Lie algebras is an independent discipline in its own right and the chapter can be read independently of others. We give, after basic concepts, the structure and theory of arbitrary Lie algebras, a description of nilpotent and solvable algebras and a complete classification of both complex and real simple Lie algebras. Another feature is the detailed discussion of decomposition theorems of Lie algebras, i.e., Gauss, Cartan and Iwasawa decompositions.

Ch. 2 begins with a review of the properties of topological spaces, in order to introduce the concepts of topological groups. The general properties of topological groups such as compactness, connectedness and metric properties are treated. We discuss further integration over the group manifold, i.e., the invariant measure (Haar measure) on the group. The fundamental Mackey decomposition theorem of topological groups is also given.

Ch. 3 begins with a review of differentiable manifolds, their analytic structures and tangent spaces. With these preparations on topological groups and differentiable manifolds we introduce Lie groups as topological groups with an analytic structure and derive the basic relations between Lie groups and Lie algebras. The remaining sections of ch. 3 are devoted to the composition and decomposition properties of groups (i.e., Levi–Malcev, Gauss, Cartan, Iwasawa decompositions), to the classification of Lie groups and to some results on the structure of Lie groups and to the construction of invariant measure and of invariant metric.

In the next chapter, 4, we introduce the concepts of homogeneous and symmetric spaces on which groups act. These concepts play an important role in the modern theory of group representations and in physical applications. We further give a classification of globally symmetric Riemannian spaces associated with the classical simple Lie groups. Also discussed in this chapter is the concept of quasi-invariant measure, because invariant measures do not exist in general on homogeneous spaces.

The theory of group representations, the main theme of the book, begins in ch. 5 where we first give the definitions, the general properties of representations, irreducibility, equivalence, tensor and direct product of representations. We further treat the Mautner and the Gel'fand–Raikov theorems on the decomposition and completeness of group representations.

The detailed group representation theory is then developed in successive steps beginning with the simplest case of commutative groups, in ch. 6, followed by the representations of compact groups, in ch. 7. For completeness we also review here, as a special case, the representations of finite groups. The representation theory of compact groups is complete and we give the general theorems (the Peter–Weyl and Weyl approximation theorems) of this theory. With a view towards applications, we discuss also the projection operators, decomposition of the representations and of tensor products.

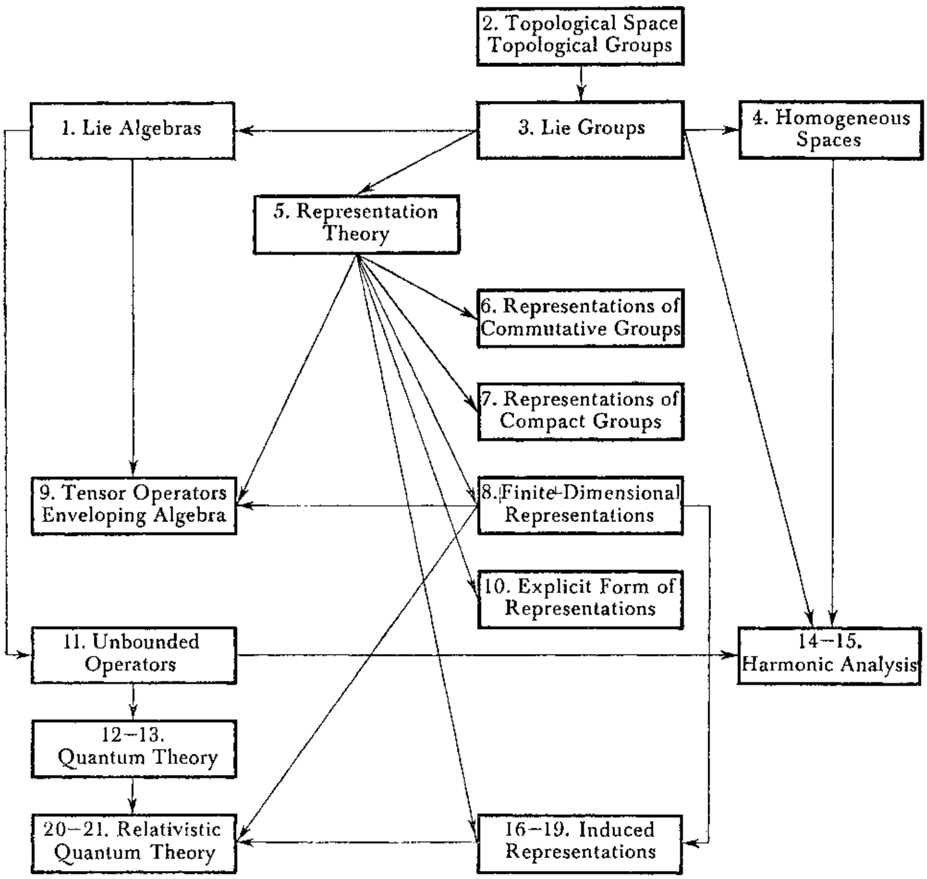
Next comes the description of all finite-dimensional irreducible representations of arbitrary Lie groups (compact or non-compact) (ch. 8). Here we give a more complete treatment of the properties of representations of semisimple groups than is available, to our knowledge, in any other book. The methods for the explicit construction of the finite-dimensional representations are treated in ch. 10, after a necessary discussion of tensor operators, enveloping algebras and invariant or Casimir operators and their spectra in ch. 9. (These concepts are used to specify and label the representations.) Among the methods we give the Gel'fand–Zetlin method, the tensor method, the method of harmonic functions and the method of creation and annihilation operators.

Ch. 11 deals with the representation theory of Lie and enveloping algebras by unbounded operators and the related questions of integrability of Lie algebra representations to the representations of the corresponding Lie groups. This is one of the most important chapters of the book. The theory of unbounded operators is also important for applications because most of the observables in quantum theory are represented by unbounded operators. More specifically, the theory of analytic vectors for Lie groups and Lie algebras is presented.

In chs. 12 and 13 we give a treatment of the role that the theory of group representation plays in all areas of quantum theory and specific applications. The mathematical structure of group representations in the Hilbert space is particularly adapted to quantum theory. In fact, we can base the framework of quantum theory solely on the concept of group representations. Historically, also, the concepts of Hilbert space and representation of groups in the Hilbert space had their origin in quantum theory. We also discuss the concepts of kinematical and dynamical symmetries, a classification of basic symmetries of physics and the use of group representations in solving dynamical problems in quantum mechanics.

The next two chapters (14 and 15) are devoted to harmonic analysis on Lie groups and on homogeneous and symmetric spaces. Here the theory encompasses a generalization of the Fourier expansion for non-commutative groups, the corresponding spectral synthesis and Plancherel formulas. We discuss the general theory as well as specific applications to some simple and semi-direct product groups.

The following four chapters, 16–19, are devoted to the theory of induced representations, one of the most important themes of the book. Already in ch. 8 we have used induced representations to obtain a classification as well as the explicit form of all irreducible finite-dimensional representations of Lie groups. Here the general theory is presented.




Ch. 16 deals with the basic properties of induced representations and the fundamental imprimitivity theorem. In the next chapter, 17, the induced representations of semi-direct product of groups is given, with a derivation of the complete classification of all representations of the Poincaré group. The further properties of induced representations (the induction-reduction theorem, the tensor product theorem and the Frobenius reciprocity theorem) are discussed in ch. 18. In ch. 19 the theory is applied to derive explicitly the induced irreducible unitary, hence infinite-dimensional, representations of principal and supplementary series of complex classical Lie groups.

Finally, in chs. 20–21, we take up applications of the imprimitivity theorem and induced representations of the Poincaré group in quantum physics: first to the concept of relativistic position operator and to the proof of equivalence of Heisenberg and Schrödinger descriptions in non-relativistic quantum mechanics (in ch. 20), next, in ch. 21, to the classification of all finite-dimensional relativistic wave equations, to applications of imaginary mass representations, to Gel'fand–Yaglom type and infinite component relativistic wave equations, and to the problem of group extension of the representations of the relativity group by discrete operations and by other symmetry groups.

A number of mathematical concepts which are not so familiar to physicists and which are essential for the book have been collected in the appendices on functional analysis, and on other results from algebra, topology, integration theory, etc.

Each chapter contains at the end notes on further developments of the subject as well as exercises.

# Notations

Our space-time metric $g_{\mu\nu}$ is such that $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. The symbol $a^*$ denotes the hermitian conjugation of a matrix or an operator $a$. The symbols $\bar{a}$ and $a^T$ denote the complex conjugation and the transposition of a matrix $a$. The symbol $\blacktriangledown$ denotes the end of the proof of a theorem or of an example. The direct sum of vector spaces $V_i$ is written as $V_1 \dot{+} V_2 \dot{+} \ldots$ and the direct sum of Lie algebras $L_i$ as $L_1 \oplus L_2 \oplus \ldots$ The semidirect sum of two Lie algebras is denoted by $L_1 \niplus L_2$ and the semidirect product of two groups as $G_1 \rtimes G_2$, while the direct product of two groups is written as $G_1 \times G_2$. The expression 'th. 8.6.3' means theorem 3 of chapter 8 in section 6. The expression 'exercise 9.7.3.1' means exercise § 3.1 in chapter 9, section 7. The quotation, say 'Lunn 1969', denotes the reference to the paper of the author Lunn from the year 1969: if there are several papers of the same author in a given year we have additional index a, b, ..., etc.

The symbol

$$\left[\frac{1}{2}n\right] = \begin{cases} \frac{1}{2}n & \text{if } n = 2r, \\ \frac{1}{2}(n-1) & \text{if } n = 2r+1; \end{cases}$$

the symbol

$$\left\{\frac{1}{2}n\right\} = \begin{cases} \frac{1}{2}n & \text{if } n = 2r, \\ \frac{1}{2}(n+1) & \text{if } n = 2r+1. \end{cases}$$

We use throughout Einstein summation convention unless stated otherwise.

For the sake of simplicity we use the symbol $\surd(\ldots)$ instead of $\sqrt{(\ldots)}$ for roots.

# Chapter 1

# Lie Algebras

For didactic reasons we have found it advantageous to begin with the discussion of Lie algebras first then go over to the topological concepts and to Lie groups. The theory of Lie algebras has become a discipline in its own right.

## § 1. Basic Concepts and General Properties

### A. *Lie Algebras*

Let $L$ be a finite-dimensional vector space over the field $K$ of real or complex numbers. The vector space $L$ is called a *Lie algebra over K* if there is a rule of composition $(X, Y) \to [X, Y]$ in $L$ which satisfies the following axioms:

$$[\alpha X+\beta Y, Z] = \alpha[X, Z]+\beta[Y, Z] \quad \text{for } \alpha, \beta \in K \quad \text{(linearity)}, \tag{1}$$

$$[X, Y] = -[Y, X] \quad \text{for all } X, Y \in L \quad \text{(antisymmetry)}, \tag{2}$$

$$[X, [Y, Z]]+[Y, [Z, X]]+[Z, [X, Y]] = 0 \quad \text{for all } X, Y, Z \in L. \tag{3}$$

The third axiom is the Jacobi identity (or Jacobi associativity). The operation [ , ] is called *Lie multiplication*. From axiom (3) it follows that this Lie multiplication is, in general, non-associative. If $K$ is the field of real (complex) numbers, then $L$ is called a *real* (*complex*) *Lie algebra*. A Lie algebra is said to be *abelian* or *commutative* if for any $X, Y \in L$ we have $[X, Y] = 0$.

Consider two subsets $M$ and $N$ of vectors of the Lie algebra $L$ and denote by $[M, N]$ the linear hull of all vectors of the form $[X, Y]$, $X \in M$, $Y \in N$. If $M$ and $N$ are linear subspaces of an algebra $L$, then the following relations hold:

$$[M_1+M_2, N] \subset [M_1, N]+[M_2, N], \tag{4a}$$

$$[M, N] = [N, M], \tag{4b}$$

$$[L, [M, N]] \subset [M, [N, L]]+[N, [L, M]]. \tag{4c}$$

These relations can be readily verified using the axioms (1)–(3). A subspace $N$ of the algebra $L$ is a *subalgebra*, if $[N, N] \subset N$, and an *ideal*, if $[L, N] \subset N$. Clearly, an ideal is automatically a subalgebra. A *maximal ideal* $N$, which satisfies the condition $[L, N] = 0$ is called the *center* of $L$, and because $[N, N] = 0$, the center is always commutative.

Let $e_1, \ldots, e_n$ be a basis in our vector space $L$. Then, because of linearity, the commutator $Z = [X, Y]$, when expressed in terms of coordinates, (i.e. $X = x^i e_i$, etc.) takes the form*

$$z^i = [X, Y]^i = c_{jk}{}^i x^j y^k, \quad i, j, k = 1, 2, \ldots, n, \tag{5}$$

with $[e_j, e_k] = c_{jk}{}^i e_i$. The numbers $c_{jk}{}^i$ are called the *structure constants*, and $n$ the *dimension* of the Lie algebra $L$. It follows from axioms (2) and (3) that the structure constants $c_{jk}{}^i$ satisfy the conditions:

$$c_{jk}{}^i = -c_{kj}{}^i, \tag{6}$$

$$c_{is}{}^p c_{jk}{}^s + c_{js}{}^p c_{ki}{}^s + c_{ks}{}^p c_{ij}{}^s = 0. \tag{7}$$

The existence of subalgebras or ideals of a Lie algebra $L$ is reflected in certain definite restrictions on the structure constants. If $e_1, e_2, \ldots, e_k$ are the basis elements of a subalgebra, then the structure constants must satisfy the relations

$$c_{ij}{}^s = 0 \quad \text{for } i, j \leqslant k,\ s > k, \tag{8}$$

and, if they are the basis elements of an ideal, then

$$c_{ij}{}^s = 0 \quad \text{for } i \leqslant k,\ s > k \text{ and an arbitrary } j. \tag{9}$$

The structure constants are not constants as their name might imply. In fact, it follows from definition (5) that under a change of basis in the algebra $L$, the $c_{ij}{}^k$ transform as a third rank tensor with one contravariant and two covariant indices.

EXAMPLE 1. Let $L$ be the set of all skew-hermitian traceless $2\times 2$ matrices. Clearly, $L$ is of (real) dimension three. Let us choose in $L$ the basis

$$e_1 = \frac{1}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \quad e_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad e_3 = \frac{1}{2}\begin{bmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix},$$

and define the commutator $[X, Y]$ in $L$ as follows:

$$[X, Y] = XY - YX, \quad X, Y \in L. \tag{10}$$

It is readily verified that this commutator satisfies the axioms (1)–(3) for the Lie multiplication. Using (10) we find that $e_i$ satisfy the following commutation relations

$$[e_i, e_k] = \varepsilon_{ikl} e_l, \quad i, k, l = 1, 2, 3,$$

where $\varepsilon_{ikl}$ is the totally antisymmetric tensor in $R^3$. The elements of $L$ are linear combinations of $e_i$ with real coefficients. The matrices $\sigma_k = 2\mathrm{i}\, e_k$ are called Pauli matrices and satisfy $[\sigma_i, \sigma_k] = 2\mathrm{i}\,\varepsilon_{ikl}\sigma_l$.

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* We use the summation convention over repeated indices throughout the book.

Hence $L$ is the three-dimensional, real Lie algebra with the structure constants $c_{ik}{}^l$ given by

$$c_{ikl} = \varepsilon_{ikl}.$$

The algebra $L$ so defined is denoted by the symbol su(2) (or o(3)), in anticipation of the classification of Lie algebras discussed in §§ 4 and 5.

*Remark:* If $A$ is any finite-dimensional associative algebra with the multiplication law $(X, Y) \to X \cdot Y$ one can obtain a Lie algebra by interpreting the Lie composition rule $[X, Y]$ as$(X \cdot Y - Y \cdot X)$. (See also Ado's theorem in § 2.)

EXAMPLE 2. Let $L$ be the vector space of all $n \times n$ real matrices $\{x_{ik}\}$, $i, k = 1, 2, \ldots, n$, over the field $R$ of real numbers. This vector space $L$ with the Lie multiplication (10) is again a real Lie algebra. It is the full real linear Lie algebra and is denoted by the symbol $\mathrm{gl}(n, R)$.

The subset $M$ consisting of all skew-symmetric matrices $X$ satisfying $X^T = -X$ is also closed under the Lie multiplication (10). Therefore $M$ is a subalgebra which is denoted by the symbol $\mathrm{o}(n)$. The subset $N$ of matrices of the form $\lambda I$, multiples of identity, obeys

$$[\mathrm{gl}(n, R), N] = 0.$$

Hence, $N$ is a one-dimensional subalgebra contained in the center of $\mathrm{gl}(n, R)$.

We can introduce the so-called *Weyl basis* in $\mathrm{gl}(n, R)$ by taking as the basis elements $e_{ij}$, $i, j = 1, 2, \ldots, n$, the $n \times n$ matrices of the form

$$(e_{ij})_{lk} = \delta_{il}\,\delta_{jk}, \tag{11}$$

which satisfy the following commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}, \quad i, j, k, l = 1, 2, \ldots, n. \tag{12}$$

From (11) and (12) one can immediately read off the structure constants:

$$c_{sm,\,kr}{}^{ij} = \delta_s{}^i \delta_{mk} \delta_r{}^j - \delta_k{}^i \delta_{rs} \delta_m{}^j. \tag{13}$$

The basis vectors $\tilde{e}_{ik}$ for the $\mathrm{o}(n)$ subalgebra can be taken to be of the form

$$\tilde{e}_{ik} = e_{ik} - e_{ki}, \quad i, k, = 1, 2, \ldots, n. \;\blacktriangledown \tag{14}$$

The complex extension $V_c$ of a real vector space $V$ is the complex vector space consisting of all elements $z$ of the form $z = x + \mathrm{i}y$, $x, y \in V$. The multiplication of an element $z \in V_c$ by a complex number $\gamma = \alpha + \mathrm{i}\beta \in C$ is defined by

$$\gamma z = \alpha x - \beta y + \mathrm{i}(\alpha y + \beta x).$$

The *complex extension* $L^c$ of a real Lie algebra $L$ is a complex Lie algebra which satisfies the following conditions:

(i) $L^c$ is the complex extension of the real vector space $L$.

(ii) The Lie multiplication in $L^c$ is

$$\begin{aligned} Z = [Z_1, Z_2] &= [X_1 + \mathrm{i}Y_1, X_2 + \mathrm{i}Y_2] \\ &= [X_1, X_2] - [Y_1, Y_2] + \mathrm{i}[X_1, Y_2] + \mathrm{i}[Y_1, X_2] \\ &\equiv X + \mathrm{i}Y. \end{aligned} \tag{15}$$

A complex Lie algebra $L$ of dimension $n$ with a basis $\{e_i\}_1^n$ can also be considered to be a real Lie algebra of dimension $2n$ with the basis vectors $e_1, ie_1, \dots, e_n, ie_n$. The real Lie algebra so defined will be denoted by the symbol $L^R$. Conversely, a real form $L^r$ of a complex Lie algebra $L^c$ is a real Lie algebra whose complex extension is $L^c$.

### The Algebras $A_n$, $B_n$, $C_n$ and $D_n$

The complex extension of $\mathrm{gl}(n, R)$ is the set of all complex $n\times n$ matrices with the Lie multiplication (10). It is called the *full complex linear Lie algebra* and is denoted by $\mathrm{gl}(n, C)$. The subset of all $n\times n$ complex matrices with trace zero is a subalgebra of $\mathrm{gl}(n, c)$ and is denoted by $\mathrm{sl}(n, c)$ or $A_{n-1}$.

Other sequences of complex algebras are associated with various bilinear forms. Let $\Phi(\xi, \eta)$ be a bilinear form defined in an $m$-dimensional complex vector space $V^m$. The linear transformations $X$ which act on $V^m$ and satisfy the condition

$$\Phi(X\xi, \eta)+\Phi(\xi, X\eta) = 0, \qquad \xi, \eta \in V^m,$$

generate a linear Lie algebra $L$. Indeed, if the above relation is true for $X$ and $Y$, then

$$\begin{aligned}\Phi\big([X, Y]\xi, \eta\big) &= \Phi(XY\xi, \eta)-\Phi(YX\xi, \eta)\\ &= -\Phi\big(\xi, [X, Y]\eta\big),\end{aligned}$$

where the Lie multiplication of $X$ and $Y$ is defined as $[X, Y] = XY-YX$.

If the bilinear form $\Phi(\xi, \eta)$ is non-singular* and symmetric (e.g. $\Phi \equiv \xi_i\eta_i$), then $L$ is called an *orthogonal Lie algebra.* For $m = 2n+1$, $n = 1, 2, \dots$ the sequence of corresponding algebras is denoted by $\mathrm{o}(2n+1, C)$ or $B_n$, and for $m = 2n$, by $\mathrm{o}(2n, C)$ or $D_n$.

The algebras associated with non-singular skew-symmetric bilinear forms are called *symplectic Lie algebras.* It is known from elementary algebra, that skew-symmetric forms in odd-dimensional spaces are always singular (det $= 0$). Therefore the symplectic algebras can only be realized in even-dimensional complex spaces $V^{2n}$ and are denoted by $\mathrm{sp}(n, C)$ or $C_n$.

The algebras $A_n$, $B_n$, $C_n$ and $D_n$, $n = 1, 2, \dots,$ form the set of the classical complex Lie algebras.

### Direct Sums and Quotient Algebras

Let $V_i$, $i = 1, 2, \dots, k$, be subspaces of a vector space $V$ and let

$$D = \sum_{i=1}^{k} V_i \tag{16}$$

---

* A bilinear form $\Phi(\xi, \eta)$ is non-singular, if for every $\xi_0 \in V^m$, the linear form $\Phi(\xi_0, \eta)$ is not identically 0 in $\eta$. In coordinate form, $\Phi(\xi, \eta) = \xi^i a_{ij}\eta^j$ is non-singular if and only if its matrix $a_{ij}]$ is non-singular: $\det[a_{ij}] \neq 0$.

be the collection of all vectors of the form

$$d = \sum_{i=1}^{k} v_i, \quad v_i \in V_i, \ i = 1, 2, \ldots, k. \tag{17}$$

If each vector $d \in D$ has a unique representation in the form (17), then we say that $D$ is the *direct sum* of subspaces $V_i$, $i = 1, 2, \ldots, k$, and we write

$$D = V_1 \dot{+} V_2 \dot{+} \ldots \dot{+} V_k = \sum_{i=1}^{k} \dot{+} V_i. \tag{18}$$

If a Lie algebra $L$, as a vector space, can be written as a direct sum in the form (18), i.e. $L = L_1 \dot{+} L_2 \dot{+} \ldots \dot{+} L_k$, and, if in addition

$$[L_i, L_i] \subset L_i, \quad [L_i, L_j] = 0, \quad i, j = 1, 2, \ldots, k, \tag{19}$$

then $L$ is said to be *decomposed* into a direct sum of Lie algebras $L_1, L_2, \ldots, L_k$ and is denoted by $L = L_1 \oplus L_2 \oplus \ldots \oplus L_k$.

Clearly the subalgebras $L_i$, $i = 1, 2, \ldots, k$, are ideals of $L$, because

$$[L, L_i] = [L_i, L_i] \subset L_i. \tag{20}$$

Moreover if $N$ is an ideal of a subalgebra $L_i$, then $N$ is also an ideal of the algebra $L$.

Let $N$ be a subalgebra of some Lie algebra $L$. We introduce in the space $L$ the relation

$$X \simeq Y \pmod N, \tag{21}$$

if $X - Y \in N$, that is a vector in $X$ is a sum of a vector in $Y$ *and* a vector $n$ in $N$. This relation satisfies

1° $X \simeq X$,

2° if $X \simeq Y$, then $Y \simeq X$,

3° if $X \simeq Y$ and $Y \simeq Z$, then $X \simeq Z$,

and, therefore, is an equivalence relation. The whole algebra $L$ decomposes into the disjoint classes $K_x = X + N$ of equivalent elements. The set $\{K_x\}$ of all classes does not form in general a Lie algebra: in fact, if

$$X_1 \simeq Y_1 \pmod N, \quad \text{i.e.,} \quad X_1 = Y_1 + n_1,$$
$$X_2 \simeq Y_2 \pmod N, \quad \text{i.e.,} \quad X_2 = Y_2 + n_2$$

then

$$[X_1, X_2] = [Y_1, Y_2] + [Y_1, n_2] + [n_1, Y_2] + [n_1, n_2]. \tag{22}$$

Therefore, in general, the relation

$$[X_1, X_2] \simeq [Y_1, Y_2] \pmod N \tag{22a}$$

does not hold. However, if the subalgebra $N$ is in addition an ideal, then the last three terms in eq. (22) are contained in $N$ and the condition (22a) is satisfied.

The resulting Lie algebra is called the *quotient Lie algebra* of $L$ with respect to $N$ and is denoted by $L/N$.

EXAMPLE 3. Let $P$ be the Poincaré algebra with the commutation relations

$$\left.\begin{aligned}[M_{\mu\nu}, M_{\varrho\sigma}] &= g_{\mu\varrho}M_{\nu\sigma}+g_{\nu\sigma}M_{\mu\varrho}-g_{\nu\varrho}M_{\mu\sigma}-g_{\mu\sigma}M_{\nu\varrho},\\ M_{\mu\nu} &= -M_{\nu\mu},\end{aligned}\right\} \tag{23a}$$

$$[M_{\mu\nu}, P_\sigma] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu, \tag{23b}$$

$$[P_\mu, P_\nu] = 0, \tag{23c}$$

where $\mu, \nu, \ldots = 0, 1, 2, 3$, $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ and $g_{\mu\nu} = 0$ for $\mu \neq \nu$.

The set $t^4$ of linear combinations of $P_\nu$, $\nu = 0, 1, 2, 3$ (the generators of translation), is an ideal of $P$. If we introduce an equivalence relation

$$X \simeq Y\,(\text{mod}\, t^4), \qquad X, Y \in P,$$

then the set of classes $K_x = X+t^4$, $X \in P$, of equivalent elements forms a six-dimensional (quotient) Lie algebra which is isomorphic to the Lorentz algebra so(3, 1) generated by $M_{\mu\nu}$ of eq. (23a). On the other hand the equivalence relation

$$X \simeq Y\,\big((\text{mod}\,\text{so}(3,1)\big) \qquad X, Y \in P,$$

defines a four-dimensional quotient vector space (but not a quotient Lie algebra). ▼

## B. *Operations over Lie Algebras*

We shall now discuss the properties of various operations defined over Lie algebras. Let $L$ and $L'$ be two arbitrary Lie algebras over the set of real or complex numbers and let $\varphi$ be a map of $L$ into $L'$. A map $\varphi$ is called a *homomorphism* if

$$\varphi(\alpha X+\beta Y) = \alpha\varphi(X)+\beta\varphi(Y), \qquad X, Y \in L,\ \alpha, \beta \in K, \tag{24a}$$

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)], \qquad X, Y \in L. \tag{24b}$$

The set $N$

$$N = \{X \in L\colon \varphi(X) = 0\}$$

is called the *kernel* of the homomorphism $\varphi$. It is an ideal of $L$. In fact, if $X \in L$ and $Y \in N$ then,

$$\varphi([X, Y]) = [\varphi(X), 0] = 0,$$

i.e., $[X, Y] \in N$. One can readily verify that $L/N$ is isomorphic with $\varphi(L)$.

Let $N$ be an ideal of a Lie algebra $L$. The map

$$\varphi\colon X \to X+N$$

is called the *natural homomorphism of L onto L/N*. A one-to-one homomorphism of one algebra onto another is called an *isomorphism*, and the corresponding algebras $L$, and $L'$ are said to be *isomorphic*: in this case we shall write $L \sim L'$

An isomorphic map of $L$ onto itself is called an *automorphism*. An automorphism $\varphi$ of a Lie algebra $L$ is called *involutive* if $\varphi^2 = I$.

A map $\sigma$ of a complex Lie algebra into itself which satisfies the conditions

$$\sigma(\lambda X+\mu Y) = \bar{\lambda}\sigma(X)+\bar{\mu}\sigma(Y), \qquad \sigma[X, Y] = [\sigma(X), \sigma(Y)], \qquad \sigma^2 = I, \quad (25)$$

is called a *conjugation*. For instance if $L^c$ is the complex extension of a real Lie algebra $L$, then the map

$$\sigma: X+\mathrm{i}Y \to X-\mathrm{i}Y, \qquad X, Y \in L,$$

defines a conjugation in $L^c$. Note that a conjugation $\sigma$ is not an automorphism of $L$ because it is antilinear.

A *derivation** $D$ of a Lie algebra $L$ is a linear mapping of $L$ into itself satisfying

$$D([X, Y]) = [D(X), Y]+[X, D(Y)], \qquad X, Y \in L. \quad (26)$$

It is evident that if $D_1$ and $D_2$ are two derivations of $L$, then $\alpha D_1+\beta D_2$ is also a derivation. Moreover, if $D_1$ and $D_2$ are derivations, then

$$\begin{aligned} D_1 D_2([X, Y]) &= D_1\{[D_2 X, Y]+[X, D_2 Y]\} \\ &= [D_1 D_2 X, Y]+[D_2 X, D_1 Y]+[D_1 X, D_2 Y]+[X, D_1 D_2 Y]. \end{aligned}$$

Interchanging indices 1 and 2 and subtracting, we get

$$[D_1, D_2]([X, Y]) = [[D_1, D_2]X, Y]+[X, [D_1, D_2]Y], \quad (27)$$

i.e., the commutator of two derivations is again a derivation. Therefore, the set $L_A$ of all derivations forms a Lie algebra itself, the *derivation algebra* $L_A$. It is interesting to note that the algebra $L_A$ is the Lie algebra of the group of all automorphisms $G_A$ of the original algebra $L$. In fact, if $\varphi_t = \exp(\mathrm{i}At)$ is a one-parameter group of automorphism of $L$, that is,

$$\varphi_t([X, Y]) = [\varphi_t(X), \varphi_t(Y)] \in L, \qquad X, Y \in L, \quad (28)$$

then differentiation with respect to $t$ gives for $t = 0$

$$A([X, Y]) = [AX, Y]+[X, AY],$$

i.e., the generator $A$ of the one-parameter subgroup $\varphi(t)$ of automorphisms is a derivation. Conversely, one can show also that if $A$ satisfies eq. (26), then the corresponding one-parameter subgroup satisfies eq. (28) (cf. exercise 1.8).

Let $L$ be a Lie algebra over the real numbers $R$ or the complex numbers $C$. Consider the linear map $\mathrm{ad}X$ of $L$ into itself defined by

$$\mathrm{ad}X(Y) \equiv [X, Y], \qquad X, Y \in L. \quad (29)$$

Using the Jacobi identity (3), we get

$$\mathrm{ad}X([Y, Z]) = [\mathrm{ad}X(Y), Z]+[Y, \mathrm{ad}X(Z)], \quad (30)$$

* Also called an *infinitesimal automorphism* of $L$ in older publications.

i.e., the map $\operatorname{ad}X$ represents a derivation of $L$. Furthermore, using (29) and the Jacobi identity we obtain

$$\operatorname{ad}[X, Y](Z) = [\operatorname{ad}X, \operatorname{ad}Y](Z). \tag{31}$$

Hence the set $L_a = \{\operatorname{ad}X, X \in L\}$ is a linear Lie algebra, a subalgebra of the Lie algebra $L_A$ of all derivations and is called the *adjoint algebra.* The map $\psi$: $X \to \operatorname{ad}X$ is the homomorphism of $L$ onto $L_a$. Clearly the kernel of the homomorphism $\psi$ is the center of $L$.

The Lie algebra $L_a$ is moreover an ideal of the Lie algebra $L_A$ of all derivations. In fact, if $D \in L_A$ and $Y \in L$, we have

$$[D, \operatorname{ad}X](Y) = D[X, Y]-[X, DY] = [DX, Y] = \operatorname{ad}DX(Y), \tag{32}$$

i.e.,

$$[D, \operatorname{ad}X] \in L_a.$$

Note finally that if $\varphi$ is any automorphism of $L$, then, by (29) and (24b), we have:

$$\operatorname{ad}\varphi(X)(Y) = [\varphi(X), Y] = \varphi([X, \varphi^{-1}Y]) = \varphi\{\operatorname{ad}X[\varphi^{-1}(Y)]\},$$

i.e.,

$$\operatorname{ad}\varphi(X) = \varphi \operatorname{ad}X\varphi^{-1}. \tag{33}$$

We refer further to ch. 3.3 for the groups $G_A\,(G_a)$ of all (inner) automorphisms of $L$ and their Lie algebras.

EXAMPLE 4. Referring to the three-dimensional Lie algebra of example 1, we see that $\operatorname{ad}e_1(e_1) = 0$, $\operatorname{ad}e_1(e_2) = e_3$, $\operatorname{ad}e_1(e_3) = -e_2$, etc. Hence $L_a$ is again a three-dimensional Lie algebra, and can be represented in the basis $\{e_i\}$ by matrices

$$\operatorname{ad}e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \operatorname{ad}e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \operatorname{ad}e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e., $L_a$ is the set of three-dimensional skew-symmetric matrices, i.e., $o(3)$.

Similarly, the adjoint algebra for $gl(n, R)$, example 2, is given by the relations $\operatorname{ad}e_{ij}(e_{kl}) = \delta_{jk}e_{il}-\delta_{il}e_{kj}$ (dimension $n^2$), and that of $o(n)$ is given by the set of $n(n-1)/2$-dimensional skew-symmetric matrices. ▼

We introduce now the important concept of a *semidirect sum* of two Lie algebras. Let $T$ and $M$ be two Lie algebras and let $D$ be the homomorphism of $M$ into the set of linear operators in the vector space $T$ such that every operator $D(X)$, $X \in M$, is a derivation of $T$. We endow the direct sum of vector spaces $T \dotplus M$ with a Lie algebra structure by using the given Lie brackets of $T$ and $M$ in each subspace, and for the Lie brackets between the two subspaces, we set

$$[X, Y] = \big(D(X)\big)(Y) \quad \text{for } X \in M,\ Y \in T. \tag{34}$$

It is evident that all the axioms (1)–(3) for a Lie algebra are satisfied: in particular, by virtue of eq. (34), we have for $X \in M$, $Y_i \in T$

$$[X, [Y_1, Y_2]]+[Y_2, [X, Y_1]]+[Y_1, [Y_2, X]] \\ = D(X)([Y_1, Y_2]) - [D(X)Y_1, Y_2] - [Y_1, D(X)Y_2],$$

which is zero since $D(X)$ is a derivation. The Lie algebra so obtained is called the *semidirect sum* of $T$ and $M$. The subalgebra $T$, by virtue of eq. (34), is an ideal of the semidirect sum. In other words, a Lie algebra $L$ is a semidirect sum of subalgebras $T$ and $M$, if $L = T \dotplus M$, and,

$$[T, T] \subset T, \quad [M, M] \subset M, \quad [M, T] \subset T. \tag{35}$$

We shall use the symbol $T \oplus M$ for a semidirect sum, writing the ideal $T$ first, and the subalgebra $M$ second.

Example 5. The Poincaré algebra of example 3 is the semidirect sum of the ideal $t^4$ and the Lorentz algebra so(3, 1) with

$$D(M_{\mu\nu})(P_\sigma) = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu,$$

i.e.

$$P = t^4 \oplus \mathrm{so}(3, 1). \tag{36}$$

## C. *Representations of Lie Algebras*

Let $L$ be a Lie algebra over the field $K$ and let $H$ be a linear space. A *representation* of $L$ in $H$ is a homomorphism $X \to T(X)$ of $L$ into the set of linear operators in $H$, i.e., for $X$, $Y$ in $L$ and $\alpha$, $\beta$ in $K$, we have

$$\alpha X+\beta Y \to \alpha T(X)+\beta T(Y), \tag{37}$$

$$[X, Y] \to [T(X), T(Y)] \equiv T(X)T(Y)-T(Y)T(X). \tag{38}$$

Notice that by virtue of eq. (38) the Jacobi identity (3) is automatically satisfied.

If the carrier space $H$ is infinite-dimensional, then we assume in addition that the operators $T(X)$ for all $X \in L$ have a common linear invariant domain $D$ which is dense in $H$. (Cf. ch. 11.1 for an explicit construction of the domain $D$.)

Example 6. Let $L$ be an arbitrary Lie algebra with the commutation relations

$$[X_l, X_k] = c_{lk}{}^j X_j, \quad l, k = 1, 2, \dots, n,$$

where the structure constants $c_{lk}{}^j$ are taken to be real.

Then the map

$$X_l \to T(X_l) = C_l \equiv \{-c_{lk}{}^j\} \tag{39}$$

of $L$ into $n \times n$ matrices in $R^n$ provides a finite-dimensional representation of $L$. Indeed, by virtue of eqs. (6) and (7) we have

$$[T(X_i), T(X_j)] = C_iC_j - C_jC_i = c_{ij}{}^s T(X_s). \tag{40}$$

The representation $X \to T(X)$ of the Lie algebra $L$ given by eq. (39) is called the *adjoint representation* of $L$.

EXAMPLE 7. Let $L$ be a Lie algebra defined by the following set of commutation relations

$$[P_l, Q_k] = \frac{1}{i}\delta_{lk} Z, \quad [P_l, Z] = 0 = [Q_k, Z], \quad l, k = 1, 2, \ldots, n. \tag{41}$$

Let $H = L^2(R^n)$ and $D = C_0^\infty(R^n)$. Then the map

$$\begin{gathered} Q_l \to x_l, \quad l = 1, 2, \ldots, n, \\ P_k \to \frac{1}{i}\frac{\partial}{\partial x_k}, \quad k = 1, 2, \ldots, n, \\ Z \to I \end{gathered} \tag{42}$$

defines a representation of $L$ in the carrier space $H$ with $D$ as a common linear invariant domain dense in $H$. ▼

## § 2. Solvable, Nilpotent, Semisimple and Simple Lie Algebras

### A. *Theorem of Ado*

One of the central problems in the theory of Lie algebras is the determination and classification of all non-isomorphic Lie algebras. We saw in § 1.A that the matrix algebras $A_n$, $B_n$, $C_n$ and $D_n$ provide large classes of Lie algebras. One could therefore conjecture that perhaps the matrix algebras exhaust all the possible Lie algebras. This is indeed true due to the following fundamental result:

THEOREM 1. *Every Lie algebra over the field of complex numbers $C$ is isomorphic to some matrix algebra** (for the proof see Ado 1947).

The theorem is also true for real Lie algebras. Indeed if $L$ is a real Lie algebra then its complex extension $L^c$ is a matrix Lie algebra by th. 1; consequently, the real contraction of $L^c$ to $L$ is also a matrix Lie algebra.

Ado's theorem says in fact that every abstract Lie algebra may be considered to be a subalgebra of the full linear Lie algebra gl$(n, C)$, $n = 1, 2, \ldots$ Therefore the problem of classification of all non-isomorphic abstract Lie algebras may be reduced to the more tractable problem of the enumeration of all non-isomorphic linear Lie subalgebras of gl$(n, C)$.

We now review the most important classes of Lie algebras:

### B. *Solvable and Nilpotent Algebras*

If $N$ is an ideal of an algebra $L$, then $[N, N]$ is also an ideal of $L$. In fact, by the formula 1(4c), we have

$$[L, [N, N]] \subset [N, [N, L]] + [N, [L, N]] \subset [N, N]. \tag{1}$$

* The corresponding theorem for Lie groups is not true globally, but locally (cf. Birkhoff 1936).

In particular $L$ is an ideal of $L$, and therefore $[L, L]$ is again an ideal which may be smaller than $L$, and it may happen that the sequence of ideals

$$L^{(0)} = L, \quad L^{(1)} = [L^0, L^0], \quad \dots, \quad L^{(n+1)} = [L^{(n)}, L^{(n)}], \quad n = 0, 1, 2, \dots \tag{2}$$

terminates, i.e. $L^{(n)} = 0$, for some $n$.

DEFINITION 1. A Lie algebra $L$ is called *solvable* if, for some positive integer $n$, $L^{(n)} = 0$.

EXAMPLE 1. Consider the Lie algebra $e(2)$ of the group of motions of the two-dimensional real plane consisting of the two-dimensional translations and rotations around an axis perpendicular to the plane. The generators of this group satisfy the following commutation relations

$$[X_1, X_2] = 0, \quad [X_1, X_3] = X_2, \quad [X_3, X_2] = X_1.$$

We see that

$$L^{(1)} = t^2 \text{ (the Lie algebra with basis elements } X_1 \text{ and } X_2),$$
$$L^{(2)} = 0.$$

Hence $e(2)$ is solvable. ▼

The property of solvability of a Lie algebra is hereditary, i.e. every subalgebra $L_s$ is also solvable. In fact,

$$L_s^{(1)} = [L_s^{(0)}, L_s^{(0)}]$$

is an ideal of $L_s$ which satisfies $L_s^{(1)} \subset L^{(1)}$. Hence $L_s^{(n)} \subset L^{(n)} = 0$, i.e. $L_s$ is a solvable algebra. It is evident also that every homomorphic image of a solvable algebra is solvable. Moreover, if a Lie algebra $L$ contains a solvable ideal $N$ such that the quotient algebra $L/N$ is solvable, then $L$ is also solvable.

Every solvable algebra contains a commutative ideal. In fact, if $L^{(n)} = 0$ and $L^{(n-1)} \neq 0$, then $L^{(n-1)}$ is an ideal of $L$, and $[L^{(n-1)}, L^{(n-1)}] = 0$.

Next we introduce the following sequence of ideals:

$$L_{(0)} = L, \quad L_{(1)} = [L_{(0)}, L], \quad \dots, \quad L_{(n+1)} = [L_{(n)}, L]. \tag{3}$$

DEFINITION 2. A Lie algebra is called *nilpotent* if for some positive integer $n$, $L_{(n)} = 0$.

It is easily verified by induction that $L^{(n)} \subset L_{(n)}$. In fact $L^{(0)} = L_0$ and if $L^{(n)} \subset L_{(n)}$ then

$$L^{(n+1)} = [L^{(n)}, L^{(n)}] \subset [L_{(n)}, L] \subset L_{(n+1)}.$$

Therefore a nilpotent algebra is solvable. The converse is not true: for instance, the two-dimensional non-commutative Lie algebra defined by the commutation relation

$$[X, Y] = X$$

is solvable, but not nilpotent. Similarly, the Lie algebra $e(2)$ of example 1 above is solvable, but not nilpotent.

It is evident from the definition that every subalgebra and every homomorphic image of nilpotent algebra is nilpotent.

Every nilpotent algebra has a nontrivial center. Indeed, if $L_{(n)} = 0$ and $L_{(n-1)} \neq 0$, then by eq. (3), $[L_{(n-1)}, L] = 0$, i.e., $L_{(n-1)}$ is the center of $L$.

According to th. 1, every Lie algebra is isomorphic to some linear subalgebra of the full linear algebra gl$(n, C)$. It is instructive to see the form of these linear matrix algebras corresponding to solvable and nilpotent algebras.

Let $T^{(m)}$ denote the vector space of all $m \times m$ upper triangular matrices, and $S^{(m)}$ the vector space of all $m \times m$ upper triangular matrices with equal diagonal elements. Let $S^{(m_1, m_2, \ldots, m_k)}$ denote the set of all linear transformations $A$ acting in the space

$$V = V_1 \dot{+} V_2 \dot{+} \ldots \dot{+} V_k$$

in such a way that

(i) $A \in S^{(m_1, m_2, \ldots, m_k)}$ leaves the subspaces $V_i$, $i = 1, 2, \ldots, k$, invariant,

(ii) in each subspace $V_i$ with the basis $\xi_1^{(i)}, \xi_2^{(i)}, \ldots, \xi_{m_i}^{(i)}$, $A \in S^{(m_i)}$ has the form

$$\begin{bmatrix} \lambda_i & & a_{jk}^{(i)} & \\ & \lambda_i & & \\ & & \ddots & \\ 0 & & & \lambda_i \end{bmatrix}.$$

The commutators of triangular matrices are again triangular matrices. Therefore he vector spaces $T^{(m)}$ and $S^{(m_1, m_2, \ldots, m_k)}$ represent Lie algebras. Moreover, the following theorem holds:

THEOREM 2. *An arbitrary, solvable Lie algebra of linear transformations is isomorphic to a subalgebra of some Lie algebra $T^{(m)}$. An arbitrary nilpotent linear Lie algebra is isomorphic to a subalgebra of some Lie algebra $S^{(m_1, m_2, \ldots, m_k)}$.*

(For the proof see Dynkin 1947, § 2).

Previous statements about nilpotent and solvable algebras can now easily be verified in terms of triangular matrices.

### C. *The Killing Form*

We introduced in 1(29) the homomorphism $X \to \mathrm{ad}X$ by the relation

$$\mathrm{ad}X(Y) = [X, Y].$$

In terms of coordinates we have

$$(\mathrm{ad}X(Y))^i = [X, Y]^i = c_{lk}{}^i x^l y^k,$$

i.e.

$$(\mathrm{ad}X)_k{}^i = c_{lk}{}^i x^l. \tag{4}$$

We define now a '*scalar product*' in a Lie algebra by setting

$$(X, Y) = \mathrm{Tr}(\mathrm{ad}X \mathrm{ad}Y). \tag{5}$$

The scalar product (5) has the following properties:

(i) symmetry: $(X, Y) = (Y, X)$, (6a)

(ii) bilinearity: $(\alpha X+\beta Y, Z) = \alpha(X, Z)+\beta(Y, Z)$ for all $X, Y, Z \in L$ and $\alpha$, $\beta$ real or complex numbers, (6b)

(iii) $\big(\operatorname{ad}X(Y), Z\big)+\big(Y, \operatorname{ad}X(Z)\big) = 0$, or $([X, Y], Z)+(Y, [X, Z]) = 0$. (6c)

These properties follow directly from the properties of the trace. For example, let

$$\begin{aligned} a = \big(\operatorname{ad}X(Y), Z\big) &= \operatorname{Tr}\{\operatorname{ad}([X, Y])\operatorname{ad}Z\} \\ &= \operatorname{Tr}(\operatorname{ad}X\operatorname{ad}Y\operatorname{ad}Z)-\operatorname{Tr}(\operatorname{ad}Y\operatorname{ad}X\operatorname{ad}Z), \\ b = \big(Y, \operatorname{ad}X(Z)\big) &= \operatorname{Tr}\{\operatorname{ad}Y(\operatorname{ad}[X, Z])\} \\ &= \operatorname{Tr}(\operatorname{ad}Y\operatorname{ad}X\operatorname{ad}Z)-\operatorname{Tr}(\operatorname{ad}Y\operatorname{ad}Z\operatorname{ad}X), \end{aligned}$$

then by equality $\operatorname{Tr}(ABC) = \operatorname{Tr}(CAB)$, we have $a+b = 0$, i.e. (6c).

The symmetric bilinear form (5) on $L\times L$ is called the *Killing form*. In terms of the coordinates in some basis, from (4), we have

$$(X, Y) = \operatorname{Tr}\big((\operatorname{ad}X)_k{}^i \operatorname{ad}(Y)_i{}^s\big) = c_{lk}{}^i x^l c_{si}{}^k y^s = g_{ls}x^l y^s, \tag{7}$$

where the symmetric second rank tensor

$$g_{ls} = c_{lk}{}^i c_{si}{}^k \tag{8}$$

is called the *Cartan metric tensor* of the Lie algebra $L$. Note that for some algebras (e.g. commutative) the Killing form (5) and consequently the metric tensor (8) can be degenerate, i.e., $\det[g_{kl}] = 0$.

For an arbitrary automorphism $\psi$ of a given Lie algebra $L$ we have by eq. 1(33)

$$\operatorname{ad}\psi(X) = \psi \operatorname{ad}X\psi^{-1}.$$

Therefore,

$$\big(\psi(X), \psi(Y)\big) = (X, Y), \tag{9}$$

i.e. the Killing form is invariant under the action of the group $G_A$ of all automorphisms of the algebra $L$.

The Killing form (5) and the associated Cartan tensor (8) play a fundamental role in the theory of Lie algebras and their representations.

For example, a simple criterion for the solvability of Lie algebras in terms of the Killing form is given in

THEOREM 3. If $(X, X) = 0$ *for each* $X \in L$ *then* $L$ *is a solvable Lie algebra**
*If an algebra* $L$ *is nilpotent, then* $(X, X) = 0$ *for all* $X \in L$.

(For the proof see Dynkin 1947, th. V.) ▼

We prove now three useful lemmas.

* Note that the converse is not true.

LEMMA 4. *Let $(\ ,\ )$, $(\ ,\ )^c$ and $(\ ,\ )^R$ denote the Killing forms of the real algebra $L$, its complex extension $L^c$ and the real form $(L^c)^R$ of the complex algebra $L^c$. Then,*

$$(X, Y) = (X, Y)^c \quad \text{for } X, Y \in L, \tag{10}$$

$$(X, Y)^R = 2\,\mathrm{Re}\big((X, Y)^c\big) \quad \text{for } X, Y \in (L^c)^R. \tag{11}$$

PROOF: We can choose the same basis (i.e. the same set of structure constants) for $L$ and $L^c$. Then the Cartan metric tensors in $L$ and $L^c$ coincide. This proves eq. (10). In order to prove eq. (11), we consider a linear transformation $A$ and a basis $e_1, \ldots, e_n$ in $L^c$. Let $A = B + \mathrm{i}C$ be the decomposition of the transformation $A$ on the real and the imaginary parts. Then, in the basis $e_1, \ldots, e_n, \mathrm{i}e_1, \ldots, \mathrm{i}e_n$ of $L^R$ we have

$$A(e_k) = Be_k + C(\mathrm{i}e_k), \qquad k = 1, 2, \ldots, n,$$
$$A(\mathrm{i}e_k) = -Ce_k + B(\mathrm{i}e_k), \quad k = 1, 2, \ldots, n.$$

Hence, the transformation $\tilde{A}$ in $L^R$ induced by the transformation $A$ in $L^c$ has the form $\tilde{A} = \begin{bmatrix} B & C \\ -C & B \end{bmatrix}$. Putting $A = \mathrm{ad}X\mathrm{ad}Y$ and using the definition (5) we obtain eq. (11). ▼

LEMMA 5. *Let $N$ be an ideal of a Lie algebra $L$. If $X, Y \in N$ then*

$$(X, Y)_N = (X, Y)_L, \tag{12}$$

*i.e. the value of the Killing form on $N$ taken with respect to $N$ is the same as with respect to $L$.*

PROOF: If $e_{\bar{1}}, e_{\bar{2}}, \ldots, e_{\bar{r}}, e_{r+1}, \ldots, e_n$ is a basis of $L$ such that $e_{\bar{1}}, e_{\bar{2}}, \ldots, e_{\bar{r}}$, $r \leqslant n$, is a basis of $N$ (i.e. the barred indices refer to the ideal $N$), then for $X, Y \in N$ we have, by 1(9),

$$(X, Y)_L = \mathrm{Tr}_L(\mathrm{ad}X\mathrm{ad}Y) = c_{\bar{l}k}{}^{s} x^{\bar{l}} c_{\bar{t}s}{}^{k} y^{\bar{t}}$$
$$= c_{\bar{l}\bar{k}}{}^{\bar{s}} x^{\bar{l}} c_{\bar{t}\bar{s}}{}^{\bar{k}} y^{\bar{t}} = \mathrm{Tr}_N(\mathrm{ad}X\mathrm{ad}Y) = (X, Y)_N. \ ▼$$

LEMMA 6. *The orthogonal complement (with respect to the Killing form) of an ideal $N \subset L$ is also an ideal.*

PROOF: Let $X \in N^\perp \equiv \{X \in L\colon (X, N) = 0\}$. Then, for every $Y \in N$ and $Z \in L$, we have from (6c)

$$\big(\mathrm{ad}Z(X), Y\big) = -\big(X, \mathrm{ad}Z(Y)\big) = 0.$$

Therefore for an arbitrary $Z$, $\mathrm{ad}Z(X) \in N^\perp$, i.e. $N^\perp$ is an ideal of $L$. ▼

EXAMPLE 1. Let us calculate explicitly the Killing form of the Lie algebra $\mathrm{sl}(n, C)$. Using formula 1(13) for the structure constants of $\mathrm{gl}(n, C)$, we first evaluate the Cartan metric tensor (8)

$$g_{sm,s'm'} = c_{sm,kr}{}^{ij} c_{s'm',ij}{}^{kr} = 2n\delta_{sm'}\delta_{ms'} - 2\delta_{sm}\delta_{s'm'}.$$

Therefore the Killing from for gl($n$, $C$) is

$$(X, Y) = g_{sm,s'm'}x_{sm}y_{s'm'} = 2n\mathrm{Tr}(X\cdot Y) - 2\mathrm{Tr}X\mathrm{Tr}Y. \tag{13}$$

The set sl($n$, $C$), by definition, consists of elements of gl($n$, $C$) satisfying the condition $\mathrm{Tr}X = 0$; it is an ideal of gl($n$, $C$). Therefore by lemma 5 and formula (13) we have

$$(X, Y)_{\mathrm{sl}(n,C)} = 2n\mathrm{Tr}(X\cdot Y), \qquad X, Y \in \mathrm{sl}(n, C). \tag{14}$$

The set $N = \{\lambda I\}$, $\lambda \in C$, is also an ideal of gl($n$, $C$). The Killing form (13) is zero when $X$ or $Y \in N$. Hence the scalar product (13) for gl($n$, $C$) is degenerate. We will see in subsection D that the Killing form is always degenerate for a Lie algebra which contains a non-zero commutative ideal.

The subset of sl($n$, $C$) consisting of all real matrices generates the real subalgebra sl($n$, $R$) whose complex extension is the algebra sl($n$, $C$). Therefore, by lemma 4 and eq. (14), the Killing form for sl($n$, $R$) is

$$(X, Y)_{\mathrm{sl}(n,R)} = 2n\mathrm{Tr}(X\cdot Y), \qquad X, Y \in \mathrm{sl}(n, R). \tag{15}$$

In particular, for $n = 2$ with the basis given in examples 1.1 and 1.4, we find $(e_i, e_j) = -2\delta_{ij}$, $i, j = 1, 2, 3$. ▼

## D. *Simple and Semisimple Lie Algebras*

We have separated the class of solvable and nilpotent algebras from the set of all Lie algebras. In this section we define the class of simple and semisimple Lie algebras, which play a fundamental role in the study of the structure and classification of Lie algebras.

DEFINITION 3. A Lie algebra $L$ is *semisimple* if it has no non-zero commutative ideal.

The criterion for semisimplicity is given by the following theorem:

THEOREM 7 (Cartan). *A Lie algebra L is semisimple if and only if its Killing form is non-degenerate.*

PROOF: If the algebra $L$ is not semisimple, then it has a commutative ideal $N$. If $X_{\bar{1}}, X_{\bar{2}}, \ldots, X_{\bar{r}}$ are basis elements of the ideal $N$, then the structure constants satisfy the condition 1(9) (the barred indices refer to the ideal $N$), i.e.

$$c_{i\bar{l}}{}^{s} = 0 \quad \text{for } \bar{l} \leqslant r,\ s > r \text{ and } i = 1, 2, \ldots, n,$$

$$c_{\bar{m}\bar{l}}{}^{\bar{t}} = 0 \quad \text{for } \bar{m}, \bar{l}, \bar{t} \leqslant r.$$

Therefore we obtain

$$g_{i\bar{m}} = c_{is}{}^{t}c_{\bar{m}t}{}^{s} = c_{i\bar{s}}{}^{t}c_{\bar{m}t}{}^{\bar{s}} = c_{i\bar{s}}{}^{\bar{t}}c_{\bar{m}\bar{t}}{}^{\bar{s}} = 0.$$

Due to these vanishing components in the metric tensor, $\det[g_{il}]$ vanishes, i.e., the Killing form (5) is degenerate. In order to prove the second half of the theorem we suppose that the orthogonal complement $L^\perp$ of the algebra $L$ is nontrivial. Because $L$ is an ideal of $L$, the orthogonal complement $L^\perp$ is also an ideal of $L$ by lemma 6. If $X \in L^\perp$, then $(X, X) = 0$. Hence, by th. 3, $L^\perp$ is a solvable ideal. Consequently, $L^\perp$ contains a nontrivial, commutative ideal, which is at the same time an ideal of $L$. Hence we obtain a contradiction because $L$ is semisimple, and it has no commutative ideal. Therefore, $L^\perp = 0$, and consequently $(X, Y)$ is non-degenerate. ▼

DEFINITION 4. A Lie algebra $L$ is *simple* if it has no ideals other than $\{0\}$ and $L$, and if $L^{(1)} = [L, L] \neq 0$. ▼

We shall see in § 4 that the classes of algebras $A_n$, $B_n$, $C_n$ and $D_n$ are simple algebras. There are only five more simple algebras.

The condition $L^{(1)} \neq 0$ eliminates the Lie algebras of dimension one, which would be simple but not semisimple. For instance, the algebra of example 1.1 is simple. A solvable algebra $L$ cannot contain a simple subalgebra; this follows from the fact that if $L'$ is any simple subalgebra, then the ideal $[L', L']$ equals $L'$, by def. 4; hence, the sequence of ideals $L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$, $L^{(0)} = L$ of algebra $L$ would always contain $L'$ and therefore would never terminate. Thus, if $L$ contains a simple subalgebra, it cannot be solvable.

A Lie algebra $L$ is said to be *compact* if there exists in $L$ a positive definite quadratic form $(\cdot, \ \cdot)$ satisfying the condition*

$$\big([X, Y], Z\big) + \big(Y, [X, Z]\big) = 0. \tag{16}$$

All remaining Lie algebras are called *noncompact*. The Killing form (5) satisfies the condition (16). Hence, if a Cartan metric tensor of a semisimple Lie algebra $L$ is positive (or negative) definite, then $L$ is compact.

In a complex Lie algebra any invariant quadratic form is indefinite. Hence, every complex Lie algebra is noncompact; a compact Lie algebra is a certain real form $L^r$ of the complex Lie algebra $L$ (cf. § 5). We now show that for a compact semisimple Lie algebra $L$ the structure constants $c_{rs}{}^t$ may be represented by a third-order totally antisymmetric covariant tensor; indeed, if we use the Cartan metric tensor $g_{tl}$ in $L$ for lowering the indices of contravariant tensors, then the tensor

$$c_{rsl} \equiv c_{rs}{}^t g_{tl}, \tag{17}$$

by virtue of eq. (8), may be written in the form

$$\begin{aligned} c_{rsl} &= c_{rs}{}^t c_{tm}{}^n c_{ln}{}^m = -c_{sm}{}^t c_{tr}{}^n c_{ln}{}^m - c_{mr}{}^t c_{ts}{}^n c_{ln}{}^m, \qquad \text{by eq. 1(7)},\\ &= c_{sm}{}^t c_{rt}{}^n c_{ln}{}^m + c_{mr}{}^t c_{ts}{}^n c_{nl}{}^m, \qquad \text{by eq. 1(6)}. \end{aligned}$$

The last expression is invariant under cyclic permutations of the indices and is skew in $r$ and $s$, by eq. 1(6); hence, the tensor (17) is totally antisymmetric.

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* We show in ch. 3.8 that a Lie algebra of a compact Lie group is compact. This justifies the extension of the notion of compactness from groups to algebras.

On the other hand in a compact Lie algebra $L$ the Cartan metric tensor may be taken to be in the form $g_{tt} = \delta_{tt}$; hence, by eq. (17),

$$c_{rst} = c_{rs}{}^{t}, \tag{18}$$

i.e. the structure constant $c_{rs}{}^{l}$ and the components $c_{rst}$ of tensor (17) coincide.

## § 3. The Structure of Lie Algebras

The class of solvable Lie algebras complements in some sense the class of semisimple ones: indeed, every solvable Lie algebra contains a commutative ideal, while, on the other hand, a semisimple Lie algebra has no commutative ideal. The following theorems show that in a certain sense the classification of all Lie algebras is reduced to a classification of solvable and semisimple Lie algebras.

We start with the analysis of the structure of compact Lie algebras. We show that an arbitrary compact Lie algebra is a direct sum $N \oplus S$ of two ideals, where $N$ is the center of $L$ and $S$ is semisimple. This fundamental result is obtained in two steps:

PROPOSITION 1. *Let $L$ be a compact Lie algebra. Every ideal $N$ of $L$ is a simple summand, i.e., there exists another ideal $S$ in $L$ such that*

$$N \cap S = 0, \quad N \oplus S = L \quad (\textit{direct sum of ideals}). \tag{1}$$

PROOF: Let $(\cdot, \cdot)$ be a positive definite, quadratic form in $L$ satisfying condition 2(6c). Denote by $S$ the orthogonal complement of the space $N$ in $L$ in the sense of the metric induced by the form $(\cdot, \cdot)$. Clearly, $N \cap S = 0$ and $N \oplus S = L$. It remains to show that $S$ is an ideal of $L$. Indeed, for an arbitrary $l \in L$, $n \in N$ and $s \in S$, according to eq. 2(6c), we have

$$(n, [l, s]) = -([l, n], s) = 0,$$

because $[l, n] \in N$. Hence, $[l, s] \in S$, i.e. $S$ is an ideal of $L$. ▼

The main structure theorem for compact Lie algebras is embodied in

THEOREM 2. *A compact Lie algebra $L$ is a direct sum*

$$L = N \oplus S = N \oplus S_1 \oplus S_2 \oplus \ldots \oplus S_n \tag{2}$$

*of ideals, where $N$ is the center of $L$, $S$ is semisimple and $S_i$ are simple algebras.*

PROOF: The center $N$ of $L$ is an ideal of $L$, hence, by proposition 1, $L$ decomposes into a direct sum of its center $N$ and an ideal $S$ without a center. If $S$ is not simple, then, again by proposition 1, $S$ decomposes into a direct sum $S' \oplus S''$ of ideals; every summand must be noncommutative because $S$ has no center. Repeating successively this procedure, we obtain a decomposition of $L$ into a direct sum of its center $N$ and noncommutative simple ideals $S_i$. ▼

The direct sum of simple ideals is semisimple.

Clearly, the center $N$ of $L$ is commutative. Hence, the problem of classification of all compact Lie algebras is reduced in fact to the problem of classification of all real compact simple Lie algebras. We give a solution of this latter problem in § 5.

We turn now to the structure theorems for arbitrary Lie algebras and prove first the following important property for an arbitrary Lie algebra.

PROPOSITION 3. *Let $L$ be a Lie algebra over $R$ or $C$. There exists in $L$ a maximal solvable ideal $N$ such that any other solvable ideal of $L$ is contained in $N$.*

PROOF: Let $N$ be a solvable ideal of $L$ which is not contained in any other solvable ideal, and let $M$ be an arbitrary solvable ideal of $L$. Let $\varphi$ be the natural homomorphism of $N+M$ onto $(N+M)/M$. Then, $\varphi(N) = (N+M)/M$ and the kernel of the homomorphism $\varphi$ restricted to $N$ is $N \cap M$; consequently $(N+M)/M$ and $N/(N \cap M)$ are isomorphic.

Now because $N \cap M$ is solvable, the quotient Lie algebra $N/(N \cap M)$ is also solvable; therefore the isomorphic algebra $(N+M)/M$ is also solvable. Because $(N+M)/M$ and $M$ are solvable, $N+M$ is a solvable ideal of $L$. Hence, $M \subset N$. ▼

The maximal solvable ideal $N$, which contains any other solvable ideal of a Lie algebra $L$ is called the *radical*.

For a semisimple Lie algebra $L$ the radical $N$ must be zero; indeed, if $N \neq 0$, then $N^{(k)} = [N^{(k-1)}, N^{(k-1)}]$, $N^{(0)} = N$, are also ideals of $L$ by virtue of eq. 2(2); therefore, if $N^{(n-1)} \neq 0$ and $N^{(n)} = 0$, then, $N^{(n-1)}$ would be a non-zero commutative ideal of $L$. Consequently, if $L$ is semisimple $N$ must be zero.

Thus, the solvable and semisimple algebras form two disjoint classes of Lie algebras.

It can be guessed at this point that if we separate the radical $N$ from a given Lie algebra $L$, the resulting Lie algebra is semisimple. Indeed, we have

PROPOSITION 4. *Let $L$ be a Lie algebra over $R$ or $C$. If $N$ is the radical of $L$, then the quotient algebra $L/N$ is semisimple.*

PROOF: Let $\varphi$ be the natural homomorphism of $L$ onto $L/N$. Suppose that $S$ is a solvable non-zero ideal of $L/N$ and let $\tilde{S} = \varphi^{-1}(S)$. Clearly, since $\varphi(N) = 0$, the ideal $\tilde{S}$ is larger than $N$ and contains $N$. The algebras $\tilde{S}/N$ and $N$ are solvable. Therefore, $\tilde{S}$ is also solvable and contains $N$. This, however, contradicts the maximality of the radical $N$. Thus, $S = \{0\}$. Consequently, $L/N$ does not contain a commutative ideal and therefore is semisimple.

EXAMPLE 1. Let $L$ be the Poincaré Lie algebra $P$. It follows from the commutation relations 1(23) that the set $\{t^4\}$ of translation generators $P_\mu$, $\mu = 0, 1, 2, 3$, represents the maximal solvable ideal of $P$. The quotient algebra

$$M = P/t^4$$

is the Lorentz algebra, which is semisimple. ▼

Proposition 4 states in fact that an arbitrary Lie algebra $L$ consists of two pieces: a radical $N$ and a semisimple algebra $L/N$. The following fundamental theorem gives a fuller description of this decomposition:

THEOREM 5 (Levi–Malcev theorem). *Let L be an arbitrary Lie algebra over R or C with the radical N. Then, there exists a semisimple subalgebra S of L such that*

$$L = N \niplus S. \tag{3}$$

*Any two decompositions of L of the form* (3) *are related by an automorphism of the algebra L.* ▼

(For the proof cf. Chevalley 1955, vol. III, ch. V, § 4, th. 4.)

The formula (3) is called the *Levi decomposition* of $L$, and the subalgebra $S$ is called the *Levi factor*.

Theorem 5 implies that

$$[N, N] \subset N, \qquad [S, S] \subset S, \qquad [N, S] \subset N, \tag{4}$$

i.e. any Lie algebra $L$ is a semidirect sum $N \niplus S$ of the maximal solvable ideal $N$ and a semisimple subalgebra $S$. For instance, the Poincaré algebra $P$, given by 1(23) has the following Levi decomposition

$$P = t^4 \niplus M \qquad M = \mathrm{so}(3, 1) \tag{5}$$

with

$$[t^4, t^4] = 0, \qquad [M, M] \subset M, \qquad [t^4, M] \subset t^4. \tag{6}$$

The Levi–Malcev theorem allows us to reduce the problem of classification of all Lie algebras to the following ones:

(i) Classification of all solvable Lie algebras.

(ii) Classification of all semisimple Lie algebras.

(iii) Classification of all derivations 1(26) of solvable Lie algebras implied by the classification of semisimple Lie algebras.

At the present time a complete solution exists only for the problem (ii), and this is one of the most remarkable and important results in the theory of Lie algebras. For problems (i) and (ii) there are only partial solutions.

In the second step, the problem of classification of all semisimple Lie algebras is reduced to the problem of classification of simple Lie algebras. Indeed, we have

THEOREM 6 (Cartan). *A semisimple complex or real Lie algebra can be decomposed into a direct sum of pairwise orthogonal simple subalgebras. This decomposition is unique.*

PROOF: Suppose $N$ is a non-zero ideal of $L$. By lemma 2.6 we know that the orthogonal complement $N^\perp$ is again an ideal of $L$. It is evident that $N \cap N^\perp$ is also an ideal of $L$; hence, if $X \in N \cap N^\perp$, then $(X, X) = 0$. Consequently, the ideal $N \cap N^\perp$ is solvable by virtue of lemma 2.5 and th. 2.3. The assumed semisimplicity of $L$ implies $N \cap N^\perp = 0$. Therefore, the algebra $L$ has the decomposition

$$L = N \oplus N^\perp, \quad \text{where } [N, N^\perp] = 0 \text{ and } (N, N^\perp) = 0.$$

If $N$ or $N^{\perp}$ is still semisimple, we repeat the procedure until the semisimple algebra $L$ is decomposed onto a direct sum of simple, pairwise orthogonal non-commutative subalgebras:

$$L = N_1 \oplus N_2 \oplus \dots \oplus N_k, \quad [N_i, N_j] = 0, \tag{7}$$

$$(N_i, N_j) = 0, \quad i, j = 1, \dots, k, \quad i \neq j.$$

Let now $M_1 \oplus M_2 \oplus \dots \oplus M_s$ be another decomposition of $L$ onto simple ideals. Let $M_k$ be a simple ideal which does not occur among the ideals $N_i$. Then, since $M_k$ and $N_i$ are different simple ideals of $L$ we have

$$[M_k, N_i] \subset M_k \cap N_i = \{0\}.$$

Hence $M_k$ belongs to the center of $L$ which is zero, because $L$ is semisimple. Consequently the decomposition (7) is unique (up to permutation). ▼

The classification of all simple complex and real Lie algebras is treated in the next two sections.

## § 4. Classification of Simple, Complex Lie Algebras

In this section we introduce the important concept of a root system associated with a semisimple complex Lie algebra. Next, we give Dynkin's concept of simple roots, which provide a basis for the classification of all simple complex Lie algebras. Finally, we enumerate the classical and exceptional simple complex Lie algebras.

### A. *Root System*

It is well known that the Lie algebra commutation relations of so(3) can be written as $[J_3, J_{\pm}] = \pm J_{\pm}$, $[J_+, J_-] = J_0$ which is often used in physics. In this section we shall give a generalization of this procedure to arbitrary semisimple Lie algebras, which is also of great theoretical importance.

Let $V$ be a vector space. A subspace $W \subset V$ is called *invariant* under a set $T$ of linear transformations of the vector space $V$ if for each $\tau \in T$ we have $\tau W \subset W$. A set $T$ of linear transformations is called *semisimple* if the complement of every invariant subspace of $V$ with respect to $T$ is also an invariant subspace.

DEFINITION 1. A subalgebra $H$ of a semisimple algebra $L$ is called a *Cartan subalgebra* if

1° $H$ is a maximal abelian subalgebra in $L$.

2° For an arbitrary $X \in H$ the transformation ad$X$ of the space $L$ is semisimple. ▼

Let $\alpha$ be a linear function on a complex vector space $H \subset L$ where $H$ is a Cartan subalgebra of $L$. Denote by $L^{\alpha}$ the linear subspace of $L$ defined by the condition

$$L^{\alpha} \equiv \{Y \in L: [X, Y] = \alpha(X) Y \text{ for all } X \in H\}. \tag{1}$$

If $L^\alpha \neq \{0\}$ then $\alpha$ is called a *root** and $L^\alpha$ the root subspace, actually a *root vector*, as we shall see presently. It follows from the Jacobi identity, that

$$[L^\alpha, L^\beta] \subset L^{\alpha+\beta} \tag{2}$$

for arbitrary complex linear functions $\alpha, \beta$ on $H$. The properties of roots and root subspaces are described by the following theorem.

THEOREM 1. *Let $L$ be a semisimple complex Lie algebra and let $\Delta$ denote the set of non-zero roots. Then*

1° $L = H \dot{+} \sum_{\alpha\in\Delta} \dot{+} L^\alpha$.

2° *For every $\alpha \in \Delta$,* $\dim L^\alpha = 1$ (*i.e. roots are non-degenerate, except* $\alpha = 0$).

3° *If roots $\alpha, \beta$ satisfy $\alpha+\beta \neq 0$, then $(L^\alpha, L^\beta) = 0$.*

4° *The restriction of the Killing form on the Cartan subalgebra, i.e. on $H\times H$, is non-degenerate. For every root $\alpha \in \Delta$ there exists a unique vector $H_\alpha \in H$, such that*

$$(X, H_\alpha) = \alpha(X) \quad \textit{for all } X \in H. \tag{3}$$

5° *If $\alpha \in \Delta$ then $-\alpha \in \Delta$, and if $X_\alpha \in L^\alpha$, $X_{-\alpha} \in L^{-\alpha}$, then*

$$[X_\alpha, X_{-\alpha}] = (X_\alpha, X_{-\alpha})H_\alpha, \qquad \alpha(H_\alpha) \neq 0.$$

6° *If $\alpha, \beta \in \Delta$ and $\alpha+\beta \neq 0$, then $[L^\alpha, L^\beta] = L^{\alpha+\beta}$.* ▼

(For the proof cf. e.g. Helgason 1962, ch. III, § 4.)

Thus $H$ and the root vectors $L^\alpha$ provide a suitable basis for $L$. By item 4° of th. 1 we have a one-to-one correspondence between roots $\alpha$ and elements $H_\alpha$ of the Cartan subalgebra $H$. Clearly a vector $H_\alpha \in H$ corresponds to a root if and only if there exists in $L$ a root vector $E_\alpha$ satisfying the relation

$$[X, E_\alpha] = (X, H_\alpha)E_\alpha \qquad \text{for each } X \in H.$$

In what follows we shall for brevity denote the scalar product $(H_\alpha, H_\beta)$ by $(\alpha, \beta)$.

We illustrate th. 1 for the $A_n$-Lie algebra.

EXAMPLE 1. Let $L = \mathrm{sl}(nC)$. This algebra is spanned by the basis vectors $e_{ik}$, 1(11), satisfying the commutation relations 1(12). Let $\lambda_i$, $i = 1, 2, \ldots, n$, be complex numbers such that $\sum_{i=1}^{n} \lambda_i = 0$. Then by virtue of 1(12) the elements

$$A_{\lambda_1, \lambda_2, \ldots, \lambda_n} = \sum_{i=1}^{n} \lambda_i e_{ii} \tag{4}$$

span a maximal commutative subalgebra $H$ of $\mathrm{sl}(n, C)$. Using 1(12) we obtain

$$[A_{\lambda_1, \lambda_2, \ldots, \lambda_n}, e_{ik}] = (\lambda_i - \lambda_k)e_{ik}. \tag{5}$$

---

* The name 'root' is due to the fact that $[X, Y] = \alpha Y$ is an eigenvalue equation and $\alpha$'s can be obtained in a coordinate system by the solution of the secular equation $\det[X^i C_{ij}^k - \alpha\delta_j^k] = 0$.

Hence every one-dimensional subspace $E_{ik}$ of $\mathrm{sl}(n, C)$ spanned by the vector $e_{ik}$ is invariant under the operation $\mathrm{ad}X$, for $X \in H$. Therefore the complement of $E_{ik}$ in $\mathrm{sl}(n, C)$ for a given $i$ and $k$ is also invariant under $\mathrm{ad}X$, $X \in H$. Consequently the transformations $\mathrm{ad}X$, $X \in H$, are semisimple; thus, $H$ is the Cartan subalgebra of $\mathrm{sl}(n, C)$, by virtue of def. 1.

Furthermore, by eqs. (1), and (5), the complex linear forms

$$\alpha_{ik}(A_{\lambda_1, \lambda_2, \dots, \lambda_n}) \equiv \lambda_i - \lambda_k, \quad i, k = 1, 2, \dots, n, \ i \neq k$$

are non-zero roots of $\mathrm{sl}(n, C)$. The rays $E_{ik} = ((e_{ik}))$ are one-dimensional root subspaces $L^{\alpha_{ik}}$. The decomposition 1° of th. 1 takes, in the present case, the form

$$\mathrm{sl}(n, C) = H \dot{+} \sum_{\substack{i,k=1\\ i \neq k}}^{n} \dot{+} L^{\alpha_{ik}}, \quad L^{\alpha_{ik}} = ((e_{ik})).$$

The basis of the subalgebra $H$ can be chosen as

$$H_i = e_{ii} - e_{i+1, i+1}, \quad i = 1, 2, \dots, n-1. \tag{6}$$

Then $\mathrm{Tr}(H_i) = 0$. Next we determine the explicit form of the root system $\Delta$. By eq. (3) we have

$$\alpha_{ik}(X) = (X, H_{\alpha_{ik}})$$

for an arbitrary $X = A_{\lambda_1, \lambda_2, \dots, \lambda_n} \in H$. In order to determine the unknown vector $H_{\alpha_{ik}} \in H$ we represent it in the form (4), i.e.

$$H_{\alpha_{ik}} = \sum_{s=1}^{n} \mu_s e_{ss}, \quad \sum_{s=1}^{n} \mu_s = 0; \tag{7}$$

we get

$$\alpha_{ik}(A_{\lambda_1, \lambda_2, \dots, \lambda_n}) = (A_{\lambda_1, \lambda_2, \dots, \lambda_n}, A_{\mu_1, \mu_2, \dots, \mu_n}) = \lambda_i - \lambda_k.$$

Using eq. 2(14) and (4), we obtain on the other hand

$$(A_{\lambda_1, \lambda_2, \dots, \lambda_n}, A_{\mu_1, \mu_2, \dots, \mu_n}) = 2n \mathrm{Tr}(A_{\lambda_1, \lambda_2, \dots, \lambda_n} \cdot A_{\mu_1, \mu_2, \dots, \mu_n}) = 2n \sum_{s=1}^{n} \lambda_s \mu_s.$$

Thus the equation

$$2n \sum_{s=1}^{n} \lambda_s \mu_s = \lambda_i - \lambda_k \tag{8}$$

has to be satisfied for arbitrary $\lambda_s$, $s = 1, 2, \dots, n$, provided that $\sum_{s=1}^{n} \lambda_s = 0$. It is readily verified that the equation (8) holds if and only if

$$\mu_s = \begin{cases} \dfrac{1}{2n}, & s = i, \\[2ex] -\dfrac{1}{2n}, & s = k, \\[2ex] 0, & s \neq i, k. \end{cases} \tag{9}$$

From (7) and (9) the final form of vectors $H_{\alpha_{ik}} \in H$ corresponding to roots $\alpha_{ik}$ is given by

$$H_{\alpha_{ik}} = \frac{1}{2n}(e_{ii}-e_{kk}), \quad i, k = 1, 2, \dots, n, \ i \neq k. \tag{10}$$

If we set

$$\tilde{H}_i = \frac{1}{2n} e_{ii}, \tag{11}$$

then the $\Delta$-system of $\mathrm{sl}(n, C)$ is finally

$$\Delta\big(\mathrm{sl}(n, C)\big) = \{\tilde{H}_i - \tilde{H}_k, \ i, k = 1, 2, \dots, n, \ i \neq k\}. \tag{12}$$

The reader can easily verify the statements 5° and 6° of the theorem. ▼

The following theorem describes the basic properties of the root system for semisimple complex Lie algebras.

THEOREM 2. 1° *If* $\alpha \in \Delta$ *then* $-\alpha \in \Delta$, *but for* $k \neq \pm 1$, $k\alpha \notin \Delta$.

2° *Suppose* $\alpha, \beta \in \Delta$, $\alpha \neq \pm\beta$. *If* $\beta_k = \beta + k\alpha$ *and* $\beta_k \in \Delta$ *for integers* $k$, $p \leqslant k \leqslant q$, *but* $\beta_{p-1} \notin \Delta$, $\beta_{q+1} \notin \Delta$, *then*

$$\frac{2(\beta, \alpha)}{(\alpha, \alpha)} = -(p+q).$$

3° $(\beta, \alpha) = \dfrac{-2(p_{\beta,\alpha}+q_{\beta,\alpha})}{\sum\limits_{\varphi\in\Delta}(p_{\varphi,\alpha}+q_{\varphi,\alpha})}$, *where* $p_{\varrho\sigma}(q_{\varrho\sigma})$ *is the smallest (largest) number in the series* $\varrho_k = \varrho + k\sigma$, $\varrho, \sigma, \varrho_k \in \Delta$ *defined in* 2°.

4° *The Killing form defines on the linear space*

$$H^* = \sum_{\alpha\in\Delta} r_\alpha H_\alpha, \quad r_\alpha \in R,$$

*a real positive definite metric. Moreover* $H = H^* \dot{+} \mathrm{i}H^*$. ▼

(For the proof cf. e.g. Helgason 1962, ch. III, § 4.)

If we choose as the basis of the Lie algebra the Cartan subalgebra and the root vectors, we obtain the so-called *Cartan–Weyl set of commutation relations* for a semisimple complex Lie algebra. This basis is often used by physicists. The properties of the Cartan–Weyl basis are given by the following

THEOREM 3. *For each* $\alpha \in \Delta$ *we can select a vector* $E_\alpha \in L^\alpha$, *such that, for all* $\alpha, \beta \in \Delta$, *we have*

$$[H_i, E_\alpha] = \alpha(H_i)E_\alpha \quad \text{for } H_i \in H,$$

$$[E_\alpha, E_\beta] = \begin{cases} 0, & \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Delta, \\ H_\alpha, & \text{if } \alpha+\beta = 0, \\ N_{\alpha,\beta}E_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Delta, \end{cases} \tag{13}$$

*where the constants $N_{\alpha,\beta}$ satisfy*

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta}. \tag{14}$$

*For any such choice*

$$N^2_{\alpha,\beta} = \frac{q(1-p)}{2}(\alpha, \alpha), \tag{15}$$

*where the numbers $p$ and $q$ are defined by the series $\beta+k\alpha$ of th. 2.2°.* ▼

(For the proof cf. e.g. Helgason 1962, ch. III, § 5.)

## B. *Dynkin Diagrams*

We have seen, th. 1.4°, that to every root there corresponds a unique vector $H_\alpha$ in the Cartan subalgebra $H$. On the other hand the number of roots is, in general, larger than the dimension of the Cartan subalgebra; this may be clearly seen in example 1, $\mathrm{sl}(n, C)$, where $n^2-n$ roots are expressed in terms of $n-1$ basis vectors of $H$. Hence in general root vectors are linearly dependent. It is therefore natural to introduce a basis in the root space. One might expect that the problem of classification of all root systems $\Delta$ might be reduced to a simpler problem of classification of all nonequivalent systems of basis vectors in the root space. This is the main idea of Dynkin which led him to the concept of simple roots and to so-called *Dynkin diagrams.*

Let $H^*$ be a subalgebra of the Cartan subalgebra, defined in th. 2.4°, and let $X_1, X_2, \ldots, X_l$ be a basis in $H^*$. A vector $X \in H^*$ is said to be *positive* if its first coordinate which is different from zero is positive.

We call a positive root $X \in \Delta$ *simple* if it is impossible to represent it as the sum of two positive roots. The properties of a system $\Pi(L)$ of simple roots of a semisimple Lie algebra $L$ are described in the following theorem:

THEOREM 4. 1° *If $\alpha \in \Pi$, $\varphi \in \Pi$, then $\varphi-\alpha \notin \Pi$.*

2° *If $\alpha \in \Pi$, $\varphi \in \Pi$, $\alpha \neq \varphi$, then $-\dfrac{2(\varphi, \alpha)}{(\alpha, \alpha)}$ is a non-negative integer.*

3° *The $\Pi$-system is a linearly independent set and is a basis for the space $H^*$. An arbitrary root $\varphi \in \Delta$ has a representation in the form*

$$\varphi = \varepsilon \sum_{i=1}^{l} k_i \alpha_i, \tag{16}$$

*where $\varepsilon = \pm 1$, $k_i$ are non-negative integers.*

4° *If the positive root $\varphi$ is not simple, then $\varphi = \alpha+\psi$, $\alpha \in \Pi$, $\psi \in \Delta$, $0 < \psi < \varphi$*.*

PROOF: *ad* 1°. Assume $\varphi-\alpha = \psi \in \Delta$. Then by th. 2.1°, $-\psi \in \Delta$ and $\varphi = \alpha+\psi$, $\alpha = \varphi+(-\psi)$. Thus, since either $\psi > 0$ or $-\psi > 0$, then either $\varphi$ or $\alpha$ is not a simple root. Hence we have a contradiction.

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* $\psi < \varphi$ means that the first non-zero coordinate of $\varphi-\psi$ is positive.

*ad* 2°. By th. 2.2°

$$\frac{2(\varphi, \alpha)}{(\alpha, \alpha)} = -(p+q),$$

where $p, q$—integers and $p \leqslant q$. By 1°, $p = 0$. Therefore

$$\frac{2(\varphi, \alpha)}{(\alpha, \alpha)} = -q \leqslant 0.$$

*ad* 3°. Let $\lambda$ be a positive root. If $\lambda$ is simple, then $\lambda = \alpha_i$. If $\lambda$ is not simple, then $\lambda = \alpha+\beta$, where $\alpha$ and $\beta$ are positive. If $\alpha$ or $\beta$ or both are not simple, then we repeat this procedure. Finally we obtain the form (16) with $\varepsilon = +1$. If $\lambda$ is negative, then we apply our decomposition to the vector $-\lambda$ and get eq. (16) with $\varepsilon = -1$.

A set of positive vectors $x_1, x_2, \ldots, x_m$ of $R^m$ obeying the conditions

$$(x_i, x_k) \leqslant 0, \quad i \neq k, \tag{17}$$

is a linearly independent set. In fact, suppose that vectors $x_1, x_2, \ldots, x_m$ are linearly dependent and let $y_1, y_2, \ldots, y_n$ be a minimal linearly dependent subsystem. Then we would have

$$\sum_{i=1}^{n} a_i y_i = 0, \quad \text{where } a_i \neq 0,\ i = 1, 2, \ldots, n. \tag{18}$$

Let $u$ be the sum of all terms in (18) with positive coefficients, and $-v$ the sum of all terms with negative coefficients. Then eq. (18) becomes $u = v$, from which we get $(u, u) = (u, v)$. But $(u, u) > 0$, and $(u, v)$, by (17), is non-positive. Hence we have a contradiction. Therefore simple roots, which are positive, and satisfy the condition (17) are linearly independent. They constitute a basis of the space $H^*$ because of (15) and th. 2.4°.

*ad* 4°. If we add to the system $\Pi$ a positive root $\varphi \in \Pi$, then we obtain a linearly dependent system. Therefore at least one of the scalar products of the type (17) is positive, i.e. $(\varphi, \alpha_i) > 0$. By th. 2.2°, this inequality implies, for some simple root $\alpha_i$, $p \neq 0$ and consequently $\psi = \varphi - \alpha \in \Delta$. The inequality $\psi < 0$ is impossible, because otherwise the simple root $\alpha$ would be represented as the sum of two positive roots. ▼

EXAMPLE 2. We determine now the $\Pi$-system for the sl$(n, C)$-algebra. The $\Delta$-system was given by eq. (12). We choose $\tilde{H}_i$, $i = 1, 2, \ldots, n$, as the basis vectors in $H^*$, and define

$$\sum_{i=1}^{n} \lambda_i \tilde{H}_i > 0,$$

if the first non-zero component of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is positive. Then the roots

$$H_{\alpha_{ik}} = \tilde{H}_i - \tilde{H}_k, \quad i, k = 1, 2, \ldots, n,\ i \neq k,\ i < k,$$

are positive, and the roots

$$H_{\alpha_{i,i+1}} = \tilde{H}_i - \tilde{H}_{i+1} = \frac{1}{2n} H_i, \quad i = 1, 2, \ldots, n-1,$$

are simple where $H_i$ is given by eq. (6). Denoting the root $\alpha_{i,i+1}$ by the symbol $\alpha_i$ and using eq. 2(14) we find

$$(\alpha_i, \alpha_k) = (H_{\alpha_i}, H_{\alpha_k}) = \begin{cases} \dfrac{1}{n} & \text{for } i = k, \\ -\dfrac{1}{2n} & \text{for } |i-k| = 1, \\ 0 & \text{for } |i-k| > 1, \end{cases} \tag{19}$$

i.e. the angles $\langle \alpha_i, \alpha_k \rangle$ between the roots $\alpha_i$ and $\alpha_k$ are

$$\langle \alpha_i, \alpha_k \rangle = \begin{cases} 120°, & \text{if } |i-k| = 1, \\ 90°, & \text{if } |i-k| > 1. \blacktriangledown \end{cases} \tag{20}$$

The metric properties of the $\Pi$-system of a semisimple Lie algebra $L$ determine the $\Delta$-system. An inductive method for the construction of the $\Delta$-system is contained in the proof of the next theorem.

THEOREM 5. *The $\Delta(L)$-system of all roots of a given semisimple Lie algebra $L$ can be constructed from its $\Pi(L)$-system of simple roots.*

PROOF: According to th. 2.1°, we can restrict ourselves to the problem of the construction of positive roots only. Let $\beta$ be a positive root of $\Delta$, and $\beta = \sum_{i=1}^{n} k_i \alpha_i$ be its decomposition in terms of simple roots as in eq. (16). We call a root $\beta$ a *root of order s* if $\sum_{i=1}^{n} k_i = s$. Clearly the simple roots are all of order one. Suppose now that we have constructed all roots of order less than $s$. By th. 4.4°, roots of order $s$ have the form $\psi + \alpha$, where $\psi$ is a root of order $s-1$ and $\alpha \in \Pi$. We use the formula

$$q = -p - \frac{2(\psi, \alpha)}{(\alpha, \alpha)} \tag{21}$$

(th. 2.2°), if the vector $\psi + \alpha \in \Delta$. The vectors $\varphi = \psi + k\alpha$, $k = 0, -1, -2, \ldots$, are, by th. 4.3°, positive and of order less than $s$. Therefore, we can determine, by induction, whether or not they belong to the set $\Delta$, and we can find the smallest value $k_{\min} \equiv p$. Using formula (21), we find a number $q$. If $q > 0$ then the series $\varphi = \psi + j\alpha$, $j = 1, 2, \ldots, q$, contains the root $\psi + \alpha$. Otherwise the vector $\psi + \alpha$ is not a root. $\blacktriangledown$

The above theorem and th. 3 show that in fact the problem of a classification of all simple complex Lie algebras can be reduced to the problem of classification of all $\Pi$-systems of simple roots. According to th. 4 the latter problem can be reduced to a simpler combinatorial problem of the classification of all finite systems $\Gamma$ of vectors of $R^n$ satisfying the following conditions:

1° $\Gamma$ is a linearly independent system of vectors.

2° If $\alpha$, $\beta \in \Gamma$ then $\dfrac{2(\alpha,\beta)}{(\beta,\beta)}$ is a non-negative integer.

Clearly every $\Pi(L)$-system of simple roots is a $\Gamma$-system.

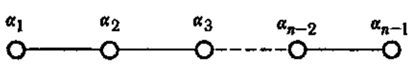
The problem of classification of $\Pi$-systems for simple Lie algebras can be simplified by introducing the device of the Dynkin diagrams. Let us observe first that according to th. 4.2°, if $\alpha, \beta \in \Pi$, $\alpha \neq \beta$, the quantity

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\cdot\frac{2(\alpha,\beta)}{(\beta,\beta)} = 4\cos^2\langle\alpha,\beta\rangle \tag{22}$$

is a non-negative integer; hence $4\cos^2\langle\alpha,\beta\rangle$ takes one of the values $0, 1, 2$ or $3$; consequently the corresponding angles are 90°, 120°, 135° and 150°, respectively.

We can represent graphically a $\Pi$-system (or $\Gamma$) of vectors $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ as a connected linear complex (or graph). The vertices $(\alpha_1), (\alpha_2), \ldots, (\alpha_n)$ are in one-to-one correspondence with the vectors $\alpha_i$ of the $\Pi$-system. Two different vertices of the complex are connected with a single, a double, or a triple line, when the two corresponding vectors span an angle of 120°, 135° or 150°, respectively. If all vectors $\alpha_i$ have the same lengths we denote the vertices by small open circles ○; if vectors $\alpha_i$ have two different lengths we denote the vertices corresponding to vectors with a smaller length by the full dots ●, and by small open circles otherwise.

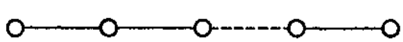
EXAMPLE 3. Let $L = \mathrm{sl}(n, C)$. We construct the Dynkin diagram for this Lie algebra. According to eq. (19) all simple roots $\alpha_i$ have the same length. Moreover, by virtue of (20), the simple roots $\alpha_i$ and $\alpha_{i+1}$ will be connected with a single line; any other pair $\alpha_i, \alpha_k$ $k \neq i+1$, of simple roots form an angle $\langle\alpha_i, \alpha_k\rangle = 90°$ and therefore is not connected. Consequently the Dynkin diagram for $\mathrm{sl}(n, C)$ ($\sim A_{n-1}$) has the form

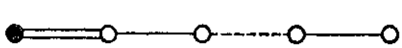
$A_{n-1}$. 

(23)

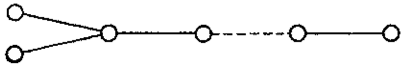
The following fundamental theorem gives a description of the Dynkin diagrams and the $\Pi$-systems of simple roots for all simple complex Lie algebras.

THEOREM 6. *The four infinite sequences of diagrams*

$A_n$:  (24)

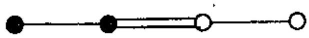
$B_n$:  (25)

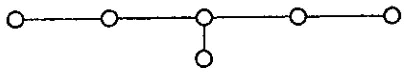
$C_n$:  (26)

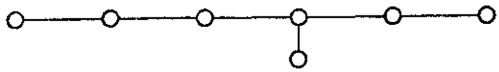
$D_n$:  (27)

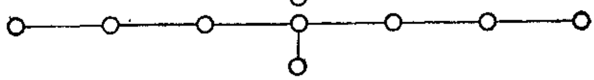
*and the five single diagrams*

$G_2$:  (28)

$F_4$:  (29)

$E_6$:  (30)

$E_7$:  (31)

$E_8$:  (32)

*constitute the set of all diagrams which can be associated with $\Pi$-systems. The corresponding $\Pi$-systems of simple roots are explicitly given by*

$$
\begin{aligned}
\Pi(A_n) &= \{h_{i+1}-h_i,\ i = 1, \dots, n\},\\
\Pi(B_n) &= \{h_1, h_{i+1}-h_i,\ i = 1, \dots, n-1\},\\
\Pi(C_n) &= \{2h_1, h_{i+1}-h_i,\ i = 1, \dots, n-1\},\\
\Pi(D_n) &= \{h_1+h_2,\ h_{i+1}-h_i,\ i = 1, 2, \dots, n-1\},\\
\Pi(G_2) &= \{h_2-h_1,\ h^{(3)}-3h_2\},\\
\Pi(F_4) &= \{h_3-h_2,\ h_2-h_1,\ h_1,\ \tfrac{1}{2}(h_4-h_1-h_2-h_3)\},\\
\Pi(E_6) &= \{h_{i+1}-h_i,\ i = 1, 2, \dots, 5,\ \tfrac{1}{2}\sqrt{2}\,h_7+\tfrac{1}{2}h^{(6)}-h_4-h_5-h_6\},\\
\Pi(E_7) &= \{h_{i+1}-h_i,\ i = 1, 2, \dots, 6,\ \tfrac{1}{2}h^{(8)}-h_4-h_5-h_6-h_7\},\\
\Pi(E_8) &= \{h_1+h_2,\ h_{i+1}-h_i,\ i = 1, 2, \dots 6,\ h_8-\tfrac{1}{2}h^{(8)}\}.
\end{aligned}
$$

*The vectors $h_i$ are orthogonal basis vectors of the corresponding Euclidean space and have the same but arbitrary length and $h^{(r)} = h_1+h_2+ \dots +h_r$.*

*A simple complex Lie algebra can be associated with each of the diagram* (24)–(32). ▼

(For the proof cf. Dynkin 1947, § 7.)

We observe that the diagrams $A_1$, $B_1$ and $C_1$ are identical. The same holds for the pair of algebras $B_2$ and $C_2$, and for the pair $A_3$ and $D_3$. All other diagrams are different. Thus we have the

COROLLARY. *The four infinite sequences of Lie algebras*

$$A_n,\ n \geqslant 1, \quad B_n,\ n \geqslant 2, \quad C_n,\ n \geqslant 3, \quad D_n,\ n \geqslant 4,$$

*and the five exceptional Lie algebras $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ constitute all the non-isomorphic simple complex Lie algebras.* ▼

The sequence $D_n$ does not contain the algebra $D_2$ because it is not simple ($D_2 \sim D_1 \oplus D_1$, direct sum of two ideals). Clearly $A_1 \sim B_1 \sim C_1$, $B_2 \sim C_2$, $A_3 \sim D_3$ on the basis of the identity of their Dynkin diagrams.

A problem of great practical interest is the reconstruction of a simple Lie algebra from its $\Pi$-system of simple roots. This problem can be solved along the following steps:

1° Reconstruct the $\Delta$-system from the $\Pi$-system (cf. th. 5).

2° Calculate (up to a sign) the structure constants $N_{\alpha,\beta}$ with the help of formula (15).

3° Determine the sign of $N_{\alpha,\beta}$.

The choice of the sign of $N_{\alpha,\beta}$ must be made in such a manner that the axioms 1(2) and 1(3) of a Lie algebra are satisfied.

When these steps are carried out, the commutation relations of an arbitrary simple complex Lie algebra are given by formulas (13). The dimensions of simple complex Lie algebras $A_n$, $B_n$, $C_n$ and $D_n$ can be calçulated from their defining matrix realization (cf. § 1, A) and are given by

| | $A_n$ | $B_n$ | $C_n$ | $D_n$ |
|---|---|---|---|---|
| Dimension | $n(n+2)$ | $n(2n+1)$ | $n(2n+1)$ | $n(2n-1)$ |

The dimensions of exceptional Lie algebra are: $G_2$: 14, $F_4$: 52, $E_6$: 78, $E_7$: 133, $E_8$: 248. These can be also calculated from their realizations (cf. § 5 and § 9, D).

## § 5. Classification of Simple, Real Lie Algebras

We have given in § 4 the classification of all complex simple Lie algebras. This classification provides also a natural starting point for the classification of all simple real Lie algebras. This is because one can relate to every simple complex Lie algebra a sequence of real simple Lie algebras introduced in § 1. A by means of the following two processes:

A. Selection of all non-isomorphic real forms $L^r$ of a given complex simple Lie algebra $L$;

B. Construction of a real Lie algebra $L^R$ associated with a given complex simple Lie algebra $L$.

These two processes give us, as we shall see, all the simple real Lie algebras.

First note that every real form $L^r$ of a complex simple Lie algebra $L$ is simple. In fact, a real form $L^r$ is generated by a special basis of a given complex simple Lie algebra $L$, in which all structure constants are real. There is, however, no basis in the simple complex Lie algebra $L$, in which structure constants could satisfy the conditions 1(9). Hence, $L^r$ has no ideals and, therefore, is simple. Using the same arguments, we conclude that the Lie algebra $L^R$ associated with a simple, complex Lie algebra $L$ is also simple.

The converse of the above statement is not true: the complex extension of a real simple Lie algebra may not be simple. For instance, the complex extension of the Lorentz Lie algebra o(3, 1) given by eq. 1(23a) is the complex Lie algebra $o(4, C) \sim D_2$ which is isomorphic to the direct sum $D_1 \oplus D_1$ of two ideals.

THEOREM 1. *All real simple Lie algebras are obtained by applying the processes* A *and* B *to all simple complex Lie algebras.*

PROOF: Let $L$ be an arbitrary simple real Lie algebra and $L^c$ its complex extension. In general, the algebra $L^c$ may not be simple. Accordingly, we distinguish two cases:

(i) $L^c$ is simple. In this case the original real simple Lie algebra $L$ is one of the real forms of the algebra $L^c$.

(ii) $L^c$ is not simple. By lemma 2.4, $L^c$ is, at any rate, semisimple, and can be decomposed by th. 3.6 into a direct sum of simple ideals. If $L_1 \neq \{0\}$ is a simple direct summand of $L^c$ and $\sigma$ is the conjugation of $L^c$ with respect to $L$, then $\sigma L_1$ is also a simple direct summand of $L^c$ because $[L_1, \sigma L_1] = 0$.

The original real Lie algebra $L$ consists of the elements invariant under $\sigma$, i.e., of the elements of the form $X+\sigma X$ with $X \in L_1$. The map $X \to X+\sigma X$ of $L_1$ into $L$ is a real isomorphism. For

$$X+Y \to (X+\sigma X)+(Y+\sigma Y) = (X+Y)+\sigma(X+Y),$$
$$\alpha X \to \alpha(X+\sigma X) = \alpha X+\sigma(\alpha X), \qquad \alpha \text{ real},$$
$$[X, Y] \to [X, Y]+\sigma[X, Y] = [X+\sigma X, Y+\sigma Y].$$

If $L^c$ would contain more ideals other than $L_1$ and $\sigma L_1$, then $L$ would be a direct sum of real ideals. Because $L$ is simple this is impossible, therefore $L^c = L_1 \dot{+} \sigma L_1$.

The last equality follows also from the relation $[L_1, \sigma L_1] = 0$. Consequently, the real simple Lie algebra $L$ is isomorphic to the complex simple Lie algebra $L_1$ considered as a real Lie algebra $L_1^R$ of twice the dimension. ▼

It follows from the proof of th. 1 that the complex extension $(L^R)^C$ of the real simple Lie algebra $L^R$ obtained by the process B is not simple while the complex extension $(L^r)^c$ of the real simple Lie algebra $L^r$ obtained by the process A is simple. Consequently the processes A and B provide disjoint classes of real simple Lie algebras. The process B associates to every complex simple Lie algebra $L$ a uniquely determined real simple Lie algebra $L^R$, whose structure constants can be obtained directly from the structure constants of algebra $L$. Hence the classification of all complex simple Lie algebras given by th. 4.6 provides simultaneously a classification of all real simple Lie algebras obtained by process B. In order to complete a classification of all simple Lie algebras it remains to classify the Lie algebras obtained by the process A. In the solution of this last problem a compact real form of a given complex simple Lie algebra plays an important role. We therefore show first that a compact real form of $L$ exists.

THEOREM 2. *Every semisimple complex Lie algebra has a real form which is compact.*

PROOF: Let $H_\alpha$, $E_\alpha$, $\alpha \in \Delta$, be the set of generators, which satisfy the commutation relations of th. 4.3. By items 5° and 3° of th. 4.1, we have $(E_\alpha, E_{-\alpha}) = 1$, and $(E_\alpha, E_\alpha) = 0$. Hence the vectors

$$U_\alpha = \mathrm{i}(E_\alpha + E_{-\alpha}),$$
$$V_\alpha = E_\alpha - E_{-\alpha},$$
$$\tilde{H}_\alpha = \mathrm{i}H_\alpha, \quad \alpha \in \Delta$$

satisfy

$$(U_\alpha, U_\alpha) = -2,$$
$$(V_\alpha, V_\alpha) = -2,$$
$$(U_\alpha, V_\alpha) = 0,$$
$$(\tilde{H}_\alpha, \tilde{H}_\alpha) = -(\alpha, \alpha) < 0.$$

Because $(E_\alpha, E_\beta) = 0$ for $(\alpha+\beta) \neq 0$, it follows that the Killing form is negative definite on the real linear subspace given by

$$L_k = \sum_{\alpha\in\Delta} R_\alpha \tilde{H}_\alpha + \sum_{\alpha\in\Delta} R_\alpha U_\alpha + \sum_{\alpha\in\Delta} R_\alpha V_\alpha, \qquad R_\alpha \text{ real numbers.}$$

If $X, Y \in L_k$, then by th. 4.3, the commutator $[X, Y]$ is expressed again in terms of elements $U_\alpha$, $V_\alpha$ and $\tilde{H}_\alpha$ with coefficients proportional to $N_{\alpha,\beta}$ or $(\alpha, \beta)$. Therefore because both $N_{\alpha,\beta}$ and $(\alpha, \beta)$ are real, and the Killing form is negative definite, the subspace $L_k$ is the real compact Lie algebra. Moreover we have

$$L = L_k \dot{+} \mathrm{i}L_k,$$

i.e. $L$ is the direct sum of the subalgebra $L_k$ and the vector space $\mathrm{i}L_k$. ▼

We give now the explicit construction of all simple real algebras, which admit a given simple complex algebra $L$ as their complex extension.

Let $L_k$ be the compact form of the complex simple algebra $L$, and let $X_1, X_2, \ldots, X_n$ be a basis of $L_k$. Clearly, the basis $\{X_i\}$ considered over $C$ provides also a basis in $L$.

Let $P$ be a linear transformation in $L$, which transforms the basis $X_i$, $i = 1, 2, \ldots, n$, into a new basis

$$Y_l = P_{kl} X_k, \quad l = 1, 2, \ldots, n, \tag{1}$$

in which the structure constants $c_{ij}{}^k$, $i, j, k = 1, 2, \ldots, n$, defined by the commutators

$$[Y_l, Y_s] = c_{ls}{}^k Y_k \tag{2}$$

are real. To each such basis there corresponds a real simple algebra $L^r$ spanned by generators $Y_i$, $i = 1, 2, \ldots, n$, with the commutation relations (2). The problem of the classification of all non-isomorphic real forms of a given simple complex Lie algebra $L$ is now reduced to the problem of finding all transformations of the form (1) in $L_k$ which lead to non-isomorphic real Lie algebras (2). This problem is solved by the following theorem:

THEOREM 3. *Let $L$ be a complex simple Lie algebra, $L_k$ its compact form, and $\Sigma$ the set of all involutive automorphisms of $L_k$. The linear transformations*

$$P = \sqrt{S} \equiv \frac{1-\mathrm{i}}{2} S + \frac{1+\mathrm{i}}{2} I, \qquad S \in \Sigma \tag{3}$$

*realize all non-isomorphic real forms of $L$. Two linear transformations $P_1$ and $P_2$ given by eq.* (3) *transform $L_k$ into two isomorphic real Lie algebras if and only if*

$$P_1 = AP_2 R, \tag{4}$$

*where $A$ is an automorphism of $L$ and $R$ is a real transformation in $L_k$.* ▼

(For the proof cf. Gantmacher 1939b, §§ 2 and 3.)

Two involutive automorphisms $S_1$ and $S_2$ will be called *equivalent*, if the corresponding transformations $P_1$ and $P_2$ satisfy eq. (4).

The following theorem gives a direct method of construction of all non-isomorphic real forms of a complex simple Lie algebra $L$.

THEOREM 4. *All non-isomorphic real forms of a given complex simple Lie algebra $L$ may be obtained in the following manner:*

1° *Find all nonequivalent involutive automorphisms $S$ of the compact form $L_k$ of $L$.*

2° *Choose a basis in $L_k$ such that the matrix $S$ is diagonal. Multiply those basis vectors of $L_k$ corresponding to eigenvalue $-1$ by* i, *and leave the remaining basis vectors unchanged. To the basis so obtained there corresponds a real simple Lie algebra $L^{(S)}$.*

PROOF: An involutive automorphism $S$ of a compact Lie algebra $L_k$ can be brought to the diagonal form with diagonal elements equal to $+1$ or $-1$. This is because the Killing form is negative definite and consequently the automorphism $S$ is a unitary operator, i.e. $S^* = S^{-1}$. From $S^2 = 1$ it follows that $S^{-1} = S$, or $S^* = S$, hence the automorphism $S$ can be represented as the difference of two projection operators

$$P^+ = \tfrac{1}{2}(1+S) \quad \text{and} \quad P^- = \tfrac{1}{2}(1-S).$$

Putting $L_k^+ = P^+ L_k$ and $L_k^- = P^- L_k$, we find that $L_k$ is a direct sum of orthogonal subspaces $L_k^+$ and $L_k^-$ and each element $X \in L_k^+ (L_k^-)$ is an eigenvector of $S$ with the eigenvalue $+1$ $(-1)$. According to eq. (3), the transformation $P = \sqrt{S}$ has the form

$$P = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \mathrm{i} & & \\ & 0 & & & & \mathrm{i} & \\ & & & & & & \ddots \\ & & & & & & & \mathrm{i} \end{bmatrix}. \tag{5}$$

Choosing any basis in $L_k^+$ and $L_k^-$ and applying the transformation (5), we obtain the basis in the real form $L^{(S)}$, specified by th. 4. If we then take all nonequivalent involutive automorphisms $S$ of $L_k$ and apply th. 3, we obtain all non-isomorphic simple real Lie algebras associated with the given simple complex Lie algebra $L$.

In other words, if

$$L_k = K \dot{+} P \tag{6}$$

is the decomposition of the algebra $L_k$ implied by the involutive automorphism $S$ (i.e., $S(X) = X$ for $X \in K$, and $S(Y) = -Y$ for $Y \in P$), then

$$L^{(S)} = K \dot{+} \mathrm{i}P \tag{7}$$

is the real form of the simple complex Lie algebra $L$ associated with the involutive automorphism $S$.▼

The Cartan metric tensor $g_{ik}^{(S)}$ in the real Lie algebra $L^{(S)}$ may be put equal to the matrix $S$. Indeed, because the Killing form for $L_k$ is definite, the Cartan metric tensor in $L_k$ may be taken in the form $g_{ik} = \delta_{ik}$. Hence,

$$g_{ik}^{(S)} = (\sqrt{S})_{il}(\sqrt{S})_{kn} g_{ln} = S_{ik}. \tag{8}$$

It follows, therefore, that two involutive automorphisms $S_1$ and $S_2$ of $L_k$ with different signatures lead to non-isomorphic simple real Lie algebras $L^{(S_1)}$ and $L^{(S_2)}$, respectively.

Example 1. Let $L = \mathrm{o}(3, C)$. The compact form $L_k = \mathrm{o}(3)$ of $L$ is defined by the following commutation relations

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2. \tag{9}$$

We have in the present case six involutive transformations in o(3):

$$S_{(1)} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}, \quad S_{(2)} = \begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix}, \quad S_{(3)} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix},$$

$$S_{(4)} = \begin{bmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{bmatrix}, \quad S_{(5)} = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & -1 \end{bmatrix}, \quad S_{(6)} = \begin{bmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 1 \end{bmatrix}.$$

The transformations $S_{(2)}$ and $S_{(3)}$ are not automorphisms of o(3), because they do not conserve the commutation relations (9). $\sqrt{S_{(1)}} = I$ and transforms o(3) onto o(3). The transformations $S_{(4)}$, $S_{(5)}$ and $S_{(6)}$ are involutive automorphisms of o(3). From eq. (3) we obtain

$$\sqrt{S_{(4)}} = \begin{bmatrix} 1 & & 0 \\ & \mathrm{i} & \\ 0 & & \mathrm{i} \end{bmatrix}, \quad \sqrt{S_{(5)}} = \begin{bmatrix} \mathrm{i} & & 0 \\ & 1 & \\ 0 & & \mathrm{i} \end{bmatrix}, \quad \sqrt{S_{(6)}} = \begin{bmatrix} \mathrm{i} & & 0 \\ & \mathrm{i} & \\ 0 & & 1 \end{bmatrix}. \tag{10}$$

We now show using (4) that authomorphisms $\sqrt{S_{(5)}}$ and $\sqrt{S_{(6)}}$ are equivalent to $\sqrt{S_{(4)}}$. In fact, the automorphisms (10) of o(3) are also automorphisms of o(3, $C$) in the basis (8). Furthermore the automorphisms of o(3, $C$) of the form

$$A_{(5)} = \sqrt{S_{(4)}}(\sqrt{S_{(5)}})^3 \quad \text{and} \quad A_{(6)} = \sqrt{S_{(4)}}(\sqrt{S_{(6)}})^3$$

transform both $\sqrt{S_{(5)}}$ and $\sqrt{S_{(6)}}$ into $\sqrt{S_{(4)}}$. Hence, by virtue of eq. (4), they provide the real forms $L^{(S_{(5)})}$ and $L^{(S_{(6)})}$ isomorphic to $L^{(S_{(4)})}$. Using now th. 4, we obtain ($Y_k = (S_{(4)})_{lk}X_l$)

$$[Y_1, Y_2] = Y_3, \quad [Y_2, Y_3] = -Y_1, \quad [Y_3, Y_1] = Y_2,$$

which is the Lie algebra o(2, 1) of the noncompact Lorentz group in three-dimensional space-time (see also exercice 2.1°). ▼

The problem of classification of all non-equivalent involutive automorphisms of compact simple Lie algebras can be solved by means of geometrical methods (cf. Cartan 1929), or algebraic methods (cf. Gantmacher 1939a,b or Hausner and Schwartz 1968, ch. III). In what follows we restrict ourselves to the enumeration of the concrete forms of the classical simple real Lie algebras implied by these automorphisms.

### I. *Real Forms of* sl($n$, $C$) ($\sim A_{n-1}$, $n > 1$)

This Lie algebra has the following real forms

(i) $L_k$ = su($n$) — the Lie algebra consisting of all skew-hermitian matrices $Z$ of order $n$ with $\mathrm{Tr}\,Z = 0$.

(ii) sl($n$, $R$) — the Lie algebra consisting of all real matrices $X$ of order $n$ with $\mathrm{Tr}X = 0$.

(iii) su($p$, $q$), $p+q = n$, $p \geqslant q$ — the Lie algebra of all matrices of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_2^* & Z_3 \end{bmatrix},$$

where $Z_1$, $Z_3$ are skew-hermitian of order $p$ and $q$, respectively, $\mathrm{Tr}\,Z_1 + \mathrm{Tr}\,Z_3 = 0$, $Z_2$ arbitrary.

(iv) su*($2n$) — the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix},$$

where $Z_1$, $Z_2$ complex matrices of order $n$, $\mathrm{Tr}\,Z_1 + \mathrm{Tr}\,\bar{Z}_1 = 0$.

### II. *Real Forms of* so($2n$, $C$) ($\sim D_n$, $n \geqslant 1$)

(i) $L_k$ = so($2n$) — the Lie algebra consisting of all real skew-symmetric matrices of order $2n$.

(ii) $so(p, q)$, $p+q = 2n$, $p \geqslant q$ — the Lie algebra of all real matrices of order $2n$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix},$$

where all $X_i$ real, $X_1$, $X_3$ skew-symmetric of order $p$ and $q$, respectively, and $X_2$ arbitrary.

(iii) $so^*(2n)$ — the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{bmatrix},$$

where $Z_1$, $Z_2$ complex matrices of order $n$, $Z_1$ skew-symmetric and $Z_2$ hermitian.

III. *Real Forms of* $so(2n+1, C)$ $(\sim B_n, n \geqslant 1)$

(i) $L_k = so(2n+1)$ — the Lie algebra of all real skew-symmetric matrices of order $2n+1$.

(ii) $so(p, q)$, $p+q = 2n+1$, $p \geqslant q$ — the Lie algebra of all real matrices of order $2n+1$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix},$$

where all $X_i$ real, $X_1$, $X_3$ skew-symmetric of order $p$ and $q$, respectively, and $X_2$ arbitrary.

IV. *Real Forms of* $sp(n, C)$ $(\sim C_n, n \geqslant 1)$

(i) $L_k = sp(n)$ — the Lie algebra of all skew-hermitian traceless matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^T \end{bmatrix},$$

where all $Z_i$ complex matrices of order $n$, $Z_2$ and $Z_3$ symmetric, (i.e. $sp(n) = sp(n, C) \cap su(2n)$).

(ii) $sp(n, R)$ — the Lie algebra of all real matrices of order $2n$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & -X_1^T \end{bmatrix},$$

where $X_1$, $X_2$, $X_3$ real matrices of order $n$, $X_2$, $X_3$ symmetric.

(iii) $\mathrm{sp}(p, q)$, $p+q = n$, $p \geqslant q$ — the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{12}^* & Z_{22} & Z_{14}^T & Z_{24} \\ -\bar{Z}_{13} & \bar{Z}_{14} & \bar{Z}_{11} & -\bar{Z}_{12} \\ Z_{14}^* & -\bar{Z}_{24} & -Z_{12}^T & \bar{Z}_{22} \end{bmatrix},$$

where $Z_{ij}$ complex matrices: $Z_{11}$ and $Z_{13}$ of order $p$, $Z_{12}$ and $Z_{14}$ with $p$ rows and $q$ columns, $Z_{11}$ and $Z_{22}$ skew-hermitian, $Z_{13}$ and $Z_{24}$ symmetric.

The list of global real Lie groups associated with these Lie algebras is given in ch. III, § 7.

There are important isomorphisms among the lowest members of the non-exceptional real simple Lie algebras. They are induced (except in one case) by the isomorphisms of corresponding complex simple algebras.

Table I shows all the known isomorphisms. We include also for convenience the isomorphisms induced by the fact that the complex Lie algebra $D_2$ is isomorphic to $A_1 \oplus A_1$.

Table I

| Isomorphisms of complex algebras | Isomorphisms of real forms |
|---|---|
| $A_1 \sim B_1 \sim C_1$ | $\mathrm{su}(2) \sim \mathrm{so}(3) \sim \mathrm{sp}(1)$ |
| | $\mathrm{sl}(2, R) \sim \mathrm{su}(1, 1) \sim \mathrm{so}(2, 1) \sim \mathrm{sp}(1, R)$ |
| $B_2 \sim C_2$ | $\mathrm{so}(5) \sim \mathrm{sp}(2)$ |
| | $\mathrm{so}(3, 2) \sim \mathrm{sp}(2, R)$ |
| | $\mathrm{so}(4, 1) \sim \mathrm{sp}(1, 1)$ |
| $D_2 \sim A_1 \oplus A_1$ | $\mathrm{so}(4) \sim \mathrm{so}(3) \oplus \mathrm{so}(3)$ |
| | $\mathrm{so}(2, 2) \sim \mathrm{sl}(2, R) \oplus \mathrm{sl}(2, R)$ |
| | $\mathrm{sl}(2C) \sim \mathrm{so}(3, 1)$ |
| | $\mathrm{so}^*(4) \sim \mathrm{sl}(2, R) \oplus \mathrm{su}(2)$ |
| $A_3 \sim D_3$ | $\mathrm{su}(4) \sim \mathrm{so}(6)$ |
| | $\mathrm{sl}(4, R) \sim \mathrm{so}(3, 3)$ |
| | $\mathrm{su}(2, 2) \sim \mathrm{so}(4, 2)$ |
| | $\mathrm{su}(3, 1) \sim \mathrm{so}^*(6)$ |
| | $\mathrm{su}^*(4) \sim \mathrm{so}(5, 1)$ |

There exists another isomorphism of the real forms which is not induced by the above isomorphisms of the complex simple Lie algebras. Namely

$$\mathrm{so}^*(8) \sim \mathrm{so}(6, 2)$$

(cf. Morita 1956).

For real forms of exceptional Lie algebras see e.g. Helgason 1962, p. 354, or Hausner and Schwartz 1968.

## § 6. The Gauss, Cartan and Iwasawa Decompositions

We have seen that by Levi–Malcev theorem, an arbitrary Lie algebra $M$ admits the decomposition

$$M = N \oplus L,$$

where $N$ is the radical of $M$ and $L$ is a semisimple Lie algebra. In order to better elucidate the structure and the properties of an arbitrary semisimple Lie algebra we give in this section three further decompositions of the Levi factor $L$. These decompositions play a fundamental role in the representation theory of the Lie algebras as well as of the corresponding Lie groups.

### A. *The Gauss Decomposition*

Let $L$ be a semisimple complex Lie algebra and let $\Delta$ be its system of non-zero roots. Let $\Delta^+$ denote the set of all positive roots and $L^+$ the linear hull of eigenvectors $E_\alpha$ defined by the equations

$$[H_i, E_\alpha] = \alpha(H_i)E_\alpha, \quad \alpha \in \Delta^+, \quad H_i \in H. \tag{1}$$

For $\alpha$, $\beta \in \Delta^+$ we have from 4(13)

$$[E_\alpha, E_\beta] = \begin{cases} 0, & \text{if } \alpha+\beta \neq 0,\ \alpha+\beta \notin \Delta, \\ N_{\alpha\beta}E_{\alpha+\beta}, & \text{if } \alpha+\beta \in \Delta. \end{cases} \tag{2}$$

If $\alpha$, $\beta \in \Delta^+$, then $\alpha+\beta$ is also in $\Delta^+$. Hence, by virtue of (2), the vector space $L^+$ forms a Lie algebra. Let $\Delta^-$ be the set of all negative roots and let $L^-$ be the linear hull of all eigenvectors of eq. (1) for $\alpha \in \Delta^-$. Using eq. (2) we conclude that $L^-$ forms a Lie algebra as well.

From eqs. (1) and (2) and commutativity of the Cartan subalgebra $H$, it follows that the direct sum $L^+ \dotplus H$ is a subalgebra of $L$. Moreover, eq. (1) implies that $L^+$ is an ideal of $L^+ \dotplus H$. Hence, the subalgebra $L^+ \dotplus H$ is, in fact, the semidirect sum $L^+ \oplus H$ of the ideal $L^+$ and the Cartan subalgebra $H$. Similarly, $L^- \oplus H$ is the semidirect sum of the ideal $L^-$ and $H$. Moreover, we have

THEOREM 1. *Let $L$ be a complex semisimple Lie algebra. Then*

1° *The subalgebras $L^+$ and $L^-$ are nilpotent.*

2° *The subalgebras $L^+ \oplus H$ and $L^- \oplus H$ are solvable.*

3° *$L$ admits the following decomposition*

$$L = L^+ \dotplus H \dotplus L^-. \tag{3}$$

PROOF: *ad* 1°. We first show that $L^+$ is nilpotent. Because $\Delta$ is a finite set, there exists a positive integer $N$ such that

$$\alpha_1 + \alpha_2 + \dots + \alpha_N \notin \Delta$$

for an arbitrary sequence $\alpha_1, \ldots, \alpha_N$ of positive roots. Consequently, by virtue of eq. (2), the multiple commutator

$$[\ldots[[[[E_{\alpha_1}, E_{\alpha_2}], E_{\alpha_3}], E_{\alpha_4}], \ldots, E_{\alpha_{N-1}}], E_{\alpha_N}] \sim E_{\alpha_1+\alpha_2+\ldots+\alpha_N} \tag{4}$$

is zero. Thus, there exists a positive integer $N$ such that the sequence of ideal $(L^+)_{(k)}$, $k = 0, 1, 2, \ldots, N$, given by eq. 2(3) terminates with $(L^+)_{(N)} = 0$. Consequently, $L^+$ is nilpotent. Similarly, one proves that $L^-$ is nilpotent.

*ad* 2°. By virtue of eqs. (1) and (2) and commutativity of $H$ we have

$$(L^+ \oplus H)^{(1)} \equiv [L^+ \oplus H, L^+ \oplus H] = L^+.$$

Because $L^+$ is nilpotent, there exists a positive integer $N$ such that $(L^+ \oplus H)^{(N)} = 0$. Hence, $L^+ \oplus H$ is solvable according to def. 2.1. Similarly one shows that $L^- \oplus H$ is solvable.

*ad* 3°. According to th. 4.1, we can write

$$L = H \dot{+} \sum_{\alpha \in \Delta} \dot{+} L^\alpha. \tag{5}$$

Combining, in eq. (5), the root subspaces $L^\alpha$ corresponding to positive and negative roots, respectively, we obtain

$$L = \sum_{\alpha \in \Delta^+} \dot{+} L^\alpha \dot{+} H \dot{+} \sum_{\alpha \in \Delta^-} \dot{+} L^\alpha,$$

which gives eq. (3). ▼

The decomposition (3) of a complex semisimple Lie algebra $L$ is called the *Gauss decomposition*. As an illustration we determine the explicit form of the Gauss decomposition for $\mathrm{sl}(n, C)$.

EXAMPLE 1. Let $L = \mathrm{sl}(n, C)$. The Cartan subalgebra $H$ is spanned by the basis vectors $H_i = e_{ii} - e_{i+1, i+1}$, $i = 1, 2, \ldots, n-1$ (cf. example 4.1). The vectors $e_{sk}$, $s, k = 1, 2, \ldots, n$, $s \neq k$, span the one-dimensional root subspaces $L^{\alpha_{sk}}$. Equation (1) takes in the present case the form

$$[H_i, e_{sk}] = \alpha_{sk}(H_i) e_{sk}.$$

We showed in example 4.2 that the roots $\alpha_{sk}$ for $s < k$ are positive and the roots $\alpha_{sk}$ for $s > k$ are negative. Consequently, for $\mathrm{sl}(n, C)$ we have

$$L^+ = \sum_{s<k} \dot{+} L^{\alpha_{sk}}, \quad L^- = \sum_{s>k} \dot{+} L^{\alpha_{sk}}. \tag{6}$$

The vectors $e_{sk}$ for $s < k$ have non-vanishing matrix elements above the main diagonal only. Hence, $L^+$ is the nilpotent subalgebra consisting of all upper triangular matrices. Similarly, $L^-$ is the nilpotent algebra consisting of all lower triangular matrices. Thus, the Gauss decomposition (3) of $\mathrm{sl}(n, C)$ is nothing but the decomposition of an arbitrary traceless matrix into the direct sum of upper triangular, diagonal and lower triangular traceless matrices. ▼

It is useful to extend the concept of the Gauss decomposition also to nonsemisimple Lie algebras. We say in general that a Lie algebra $L$ admits a Gauss decomposition (3) if the subalgebras $L^+$, $H$ and $L^-$ satisfy conditions 1° and 2° of th. 1. In particular the Gauss decomposition of $\mathrm{gl}(n, C)$ is of the form

$$\mathrm{gl}(n, C) = L^+ \dot{+} H \dot{+} L^-, \tag{7}$$

where $L^+$ and $L^-$ are nilpotent subalgebras given by eq. (6) and $H$ is the subalgebra of all diagonal complex matrices of order $n$.

## B. *The Cartan Decomposition*

The Cartan decomposition of a semisimple real Lie algebra $L$ is the direct sum of the maximal compact subalgebra and the vector space spanned by the remaining noncompact generators. Consider for example the Lorentz Lie algebra $o(3, 1)$. Denoting by $K_i$, $i = 1, 2, 3$, the generators of the compact $so(3)$ subalgebra and by $N_i$, $i = 1, 2, 3$, the generators of pure Lorentz transformations, we can write the commutation relations 1(23a) in the form

$$\begin{aligned} [K_i, K_j] &= \varepsilon_{ijl} K_l, \\ [K_i, N_j] &= \varepsilon_{ijl} N_l, \\ [N_i, N_j] &= -\varepsilon_{ijl} K_l. \end{aligned} \tag{8}$$

If $K$ denotes the compact subalgebra $o(3)$ and $N$ a vector space spanned by the generators $N_i$, the Lie algebra $L = o(3, 1)$ can be written in the form

$$L = K \dot{+} N.$$

Moreover, the commutation relations (8) become

$$[K, K] \subset K, \quad [K, N] \subset N, \quad [N, N] \subset K.$$

Similar decompositions are known for other familiar semisimple real Lie algebras, such as the de Sitter algebra $so(4, 1)$, the conformal algebra $so(4, 2)$, etc. This observation is in fact a general property:

THEOREM 2 (Cartan). *A semisimple real Lie algebra $L$ has a decomposition of the form*

$$L = K \dot{+} P, \tag{9}$$

*satisfying*

$$[K, K] \subset K, \quad [K, P] \subset P, \quad [P, P] \subset K \tag{10}$$

*and*

$$\begin{aligned} (X, X) &< 0 \quad \text{for } X \neq 0 \text{ in } K, \\ (Y, Y) &> 0 \quad \text{for } Y \neq 0 \text{ in } P. \end{aligned} \tag{11}$$

*If conditions* (10) *and* (11) *are satisfied, then $K$ is the maximal compact subalgebra of $L$.* ▼

(For the proof cf. Helgason 1962, ch. III, § 7.)

The decomposition of a semisimple real algebra specified by th. 2 is called the *Cartan decomposition.*

The form of the commutation relations given by (10) implies that the mapping

$$\theta(X) = X \qquad \text{for } X \in K, \tag{12}$$

$$\theta(Y) = -Y \quad \text{for } Y \in P, \tag{13}$$

is an involutive automorphism of the algebra $L$ (i.e. $\theta^2 = 1$). Conversely, for every involutive automorphism $\theta$ of a real semisimple Lie algebra there exists a basis in $L$ such that the formulas (12) and (13), and consequently commutation relations (10), hold. This follows immediately from the fact that every involutive automorphism of a real semisimple Lie algebra can be reduced to the diagonal form with diagonal elements equal to $+1$ or $-1$ (cf. the proof of th. 5.4).

The vector spaces $K$ and $P$ are orthogonal because

$$(X, Y) = \big(\theta(X), \theta(Y)\big) = -(X, Y) \quad \text{for every } X \in K \text{ and } Y \in P. \tag{14}$$

Hence $(X, Y) = 0$.

EXAMPLE 2. Let $L = \mathrm{sl}(n, R)$. Consider the subalgebra $\mathrm{so}(n)$ spanned by skew-symmetric matrices 1(14). The mapping

$$\theta : X \to -X^T \tag{15}$$

satisfies the condition

$$\theta(X) = X \quad \text{for } X \in \mathrm{so}(n) = K$$

and therefore is an admissible candidate for an automorphism $\theta$ of the Cartan decomposition. The vector space $P$ is defined by the condition (13)

$$\theta(Y) = -Y \Rightarrow Y^T = Y,$$

i.e. $P$ is the collection of all symmetric traceless matrices. Because the commutator of a skew-symmetric and a symmetric matrix is a symmetric matrix and the commutator of two symmetric matrices is an anti-symmetric matrix, the commutation relations (10) are satisfied. The Killing form for $\mathrm{sl}(n, R)$ was given in eq. 2(15), i.e.

$$(X, Y) = 2n\mathrm{Tr}(XY).$$

A skew-symmetric matrix, when brought to a diagonal form, has only pure imaginary non-vanishing diagonal elements. Hence if $D$ denotes the matrix, which diagonalizes a given skew-symmetric matrix, then

$$(X, X) = 2n\mathrm{Tr}(DXD^{-1}\, DXD^{-1}) < 0.$$

Similarly one verifies that if $Y$ is a symmetric matrix, then $(Y, Y) > 0$. Therefore conditions (11) are also satisfied. Hence $\mathrm{so}(n)$ is the maximal compact subalgebra of $\mathrm{sl}(n, R)$. Therefore, the Cartan decomposition $L = K \dot{+} P$ in the present case is just the well-known decomposition of a matrix into its skew-symmetric and traceless symmetric parts. ▼

The Cartan decomposition takes a particularly simple form for the real Lie algebra $L^R$ associated with a complex simple Lie algebra $L$. Namely, if $U$ is a compact form of $L$ (which by th. 5.2 always exists), then

$$L^R = U + \mathrm{i}U \tag{16}$$

is the Cartan decomposition of the algebra $L^R$. In fact, it is evident that commutations relations (10) are satisfied. One can show that conditions (11) are also satisfied: Let $(\cdot, \cdot)^R$ and $(\cdot, \cdot)$ be the Killing forms on $L^R$ and $L$, respectively. By lemma 2.4 we have

$$(X, Y)^R = 2\,\mathrm{Re}(X, Y) \quad \text{for } X, Y \in L^R.$$

Because the Killing form $(\cdot, \cdot)$ is negative definite on $U \times U$ and positive definite on $\mathrm{i}U \times \mathrm{i}U$, relations (11) of th. 2 follow. Hence, the decomposition (16) is the Cartan decomposition of the algebra $L^R$.

## C. *The Iwasawa Decomposition*

The third type of decomposition of semisimple real Lie algebras is based on the Cartan decomposition and the decomposition of a complex semisimple algebra on its root subspaces.

Let $L = K \dot{+} P$ be the Cartan decomposition of a semisimple real algebra $L$ and let $L^C$ be the complex extension of $L$. Let $\sigma$ and $\tau$ be the conjugations of the algebra $L^C$ and of the compact algebra $U = K \dot{+} \mathrm{i}P$, respectively, i.e.,

$$\begin{aligned} \sigma&: X + \mathrm{i}Y \to X - \mathrm{i}Y, \quad X, Y \in L, \\ \tau&: X + \mathrm{i}Y \to X - \mathrm{i}Y, \quad X, Y \in U. \end{aligned} \tag{17}$$

Clearly, by virtue of eq. 1(25), the transformation $\theta = \sigma\tau$ is an automorphism of $L^C$.

Let $\alpha$ be a root of the algebra $L^C$. The linear function $\alpha^\theta(X) \equiv \alpha(\theta X)$ on a Cartan subalgebra $H$ is also a root. For if

$$L^\alpha = \{Y \in L: [X, Y] = \alpha(X) Y \text{ for all } X \in H\}$$

is the root subspace of $L^C$ corresponding to the root $\alpha$, then $L^{\alpha^\theta} = \theta^{-1} L^\alpha$ is the root subspace of $L^C$ corresponding to $\alpha^\theta$. Let now

$$\begin{aligned} B_+ &= \{\alpha: \alpha \in \Delta^+, \ \alpha \neq \alpha^\theta\}, \quad N = \sum_{\alpha \in B_+} \dot{+} L^\alpha, \\ N_0 &= L \cap N, \quad S_0 = H_P \dot{+} N_0, \end{aligned} \tag{18}$$

where $H_P$ is a maximal abelian subalgebra of $P$. Then, we have

THEOREM 3. *The spaces $N$ and $N_0$ are nilpotent Lie algebras, $S_0$ is a solvable Lie algebra and*

$$L = K \dot{\oplus} H_p \dot{\oplus} N_0 . \blacktriangledown \tag{19}$$

(For the proof cf. Helgason 1962, ch. VII, § 3.)

The decomposition (19) of a real semisimple Lie algebra $L$ is called the *Iwasawa decomposition.*

EXAMPLE 3. Consider $L = \mathrm{sl}(n, R)$. Let us find first the form of the nilpotent algebras $N$ and $N_0$. From example 1 we have that the set $\Delta^+$ of positive roots of $L^C = \mathrm{sl}(n, C)$ consists of

$$\alpha_{ik} = \frac{1}{2n}(e_{ii} - e_{kk}), \qquad i < k,$$

where, from eq. 4(5),

$$\alpha_{ik}(X) = \lambda_i - \lambda_k \quad \text{for } H \ni X = \sum_{s=1}^{n} \lambda_s e_{ss}, \; \sum_{s=1}^{n} \lambda_s = 0.$$

We verify, using eq. (17), that

$$\sigma(X) = \sum \bar{\lambda}_s e_{ss}, \qquad \tau(X) = \sum -\bar{\lambda}_s e_{ss}.$$

Hence,

$$\theta(X) = \sigma\tau(X) = -X.$$

Consequently,

$$\alpha_{ik}^{\theta}(X) = \lambda_k - \lambda_i \neq \alpha_{ik}(X).$$

This implies that $B_+ = \Delta^+$. Therefore, the nilpotent algebra $N$ has the form

$$N = \sum_{\alpha_{ik} \in \Delta^+} \dot{+} L^{\alpha_{ik}} = \sum_{i<k} \dot{+} ((e_{ik}))_c,$$

where $((e_{ik}))_c$ are one-dimensional complex rays. The algebra $\mathrm{sl}(n, R)$ is spanned by the generators $e_{ik}$, $i \neq k$, $i, k = 1, 2, \ldots, n$, and the generators of a Cartan subalgebra $H_L = \{e_{ii} - e_{i+1,\,i+1},\ i = 1, 2, \ldots, n-1\}$. Therefore

$$N_0 = L \cap N = \sum_{i<k} \dot{+} ((e_{ik}))_R,$$

where $((e_{ik})_R$ are one-dimensional real rays. This algebra consists of all upper real triangular matrices with zeros on the diagonal. By example 2, the vector space $P$ consists of all symmetric matrices with trace zero. Because all elements of $H_L$ are symmetric matrices, we have $H_P = H_L \cap P = H_L$. Consequently, the solvable algebra $S_0$ is

$$S_0 = H_P \dot{+} \sum_{i<k} \dot{+} ((e_{ik}))_R,$$

i.e., it consists of all upper real triangular matrices with zero trace.

The Iwasawa decomposition (19) for $\mathrm{sl}(n, R)$ is just the decomposition of an arbitrary real traceless matrix into a sum of a skew-symmetric, a traceless diagonal, and a zero-diagonal upper triangular real matrix. ▼

## § 7. An Application. On Unification of the Poincaré Algebra and Internal Symmetry Algebras

In the physics of elementary particles that interact strongly, it seems to be an empirical fact that these particles and resonances can be grouped into multiplets which correspond to irreducible representations of some so-called *internal symmetry Lie algebras*, like multiplets of isospin algebra su(2), or multiplets of su(3)-algebra. The members of a given multiplet have the same parity and spin, but might have different masses. Thus the overall symmetry algebra cannot be a direct sum $P \oplus S$ of two ideals, the Poincaré algebra $P$ and an internal symmetry algebra $S$, because otherwise all masses within a multiplet would have to be the same. It is natural therefore to inquire if there exists a larger algebra $L$ which contains $P$ and $S$ as subalgebras in such a manner that at least one of the generators of $S$ does not commute with $P$, so that the (mass)$^2$ operator $P_\mu P^\mu$ is no longer an invariant of the larger algebra. Because the eigenvalues of the basis elements $H_i$ of a Cartan subalgebra $H$ of $S$ are used to label the states in multiplets and these so-called quantum numbers (like hypercharge or third component of isospin) are Poincaré invariant, it is natural to demand that $H$ commutes with $P$. We show here that there are severe restrictions on the form of such combined Lie algebras.

THEOREM 1. *Let $L$ be a Lie algebra which is spanned by the basis elements of the Poincaré algebra $P$ and by those of semisimple Lie algebra $S$. If $H$ is a Cartan subalgebra of $S$ and*

$$[P, H] = 0 \tag{1}$$

*then $L$ is a direct sum of ideals:*

$$L = P \oplus S. \tag{2}$$

PROOF: Let $X_\varrho$, $\varrho = 1, 2, \ldots, 10$, denote the basis elements of the Poincaré algebra satisfying the commutation relations

$$[X_\varrho, X_\sigma] = c_{\varrho\sigma}{}^{\tau} X_\tau, \tag{3}$$

and let $H_i$ and $E_\alpha$ be the basis elements of $S$ satisfying the Weyl canonical commutation relations 4(13). An arbitrary commutator $[E_\alpha, X_\varrho]$ can be written in the form

$$[E_\alpha, X_\varrho] = x_{\alpha\varrho}{}^{\beta} E_\beta + y_{\alpha\varrho}{}^{j} H_j + z_{\alpha\varrho}{}^{\tau} X_\tau, \tag{4}$$

where $x_{\alpha\varrho}{}^{\beta}$, $y_{\alpha\varrho}{}^{j}$ and $z_{\alpha\varrho}{}^{\tau}$ are expansion coefficients in the basis $E_\beta$, $H_j$, $X_\tau$ of the algebra $L$. The Jacobi identity

$$[[E_\alpha, X_\varrho], H_i] + [[X_\varrho, H_i], E_\alpha] + [[H_i, E_\alpha], X_\varrho] = 0, \tag{5}$$

by virtue of eqs. (1), (3), (4) and 4(13), takes the form:

$$\sum_\beta \big(x_{\alpha\varrho}{}^{\beta}\beta(H_i) - x_{\alpha\varrho}{}^{\beta}\alpha(H_i)\big)E_\beta - \sum_j y_{\alpha\varrho}{}^{j}\alpha(H_i)H_j - \sum_\tau z_{\alpha\varrho}{}^{\tau}\alpha(H_i)X_\tau = 0. \tag{6}$$

For every root $\alpha$, $\alpha(H_l) \neq 0$ for at least one basis element $H_l$ and $\alpha(H_l) \neq \beta(H_l)$ for at least one basis element $H_l$ if $\alpha \neq \beta$: hence

$$y_{\alpha\varrho}{}^{j} = 0, \quad z_{\alpha\varrho}{}^{\tau} = 0 \tag{7}$$

and

$$x_{\alpha\varrho}{}^{\beta} = \delta_{\alpha}{}^{\beta} x_{\alpha\varrho} \quad \text{(no summation).} \tag{8}$$

The Jacobi identity

$$[[X_\varrho, X_\sigma], E_\alpha] + [[X_\sigma, E\ ], X_\varrho] + [[E_\alpha, X_\varrho], X_\sigma] = 0 \tag{9}$$

yields then the equation

$$-\sum_\tau c_{\varrho\sigma}{}^{\tau} x_{\alpha\tau} E_\alpha - x_{\alpha\sigma} x_{\alpha\varrho} E_\alpha + x_{\alpha\varrho} x_{\alpha\sigma} E_\alpha = -\sum_\tau c_{\varrho\sigma}{}^{\tau} x_{\alpha\tau} E_\alpha = 0. \tag{10}$$

The commutation relations 1(23) for $P$ imply that for every $\tau$ we can choose $\varrho$ and $\sigma$ in such a manner that $c_{\varrho\sigma}{}^{\tau} = 0$ and $c_{\varrho\sigma}{}^{\tau'} = 0$ for $\tau' \neq \tau$. Hence $X_{\alpha\beta} = 0$ according to eq. (10). This in turn implies $[E_\alpha, X_\varrho] = 0$ by virtue of eqs. (7), (8) and (4). Consequently, $L$ is the direct sum $P \oplus S$ of two ideals. ▼

The structure th. 3.2 implies the following generalization of th. 1.

THEOREM 2. *Let $L$ be a Lie algebra which is spanned by the basis elements of the Poincaré algebra $P$, and the basis elements of an arbitrary compact Lie algebra $K$. Let $C$ be a maximal commutative subalgebra of $K$. If*

$$[P, C] = 0 \tag{11}$$

*then $L$ is the direct sum of ideals*:

$$L = P \oplus K. \tag{12}$$

PROOF: According to th. 3.2, a compact Lie algebra is a sum of ideals $N \oplus S$ where $N$ is the center of $K$ and $S$ is semisimple. Clearly $[P, N] = 0$ by virtue of (11). Let $H$ be a Cartan subalgebra of $S$. Because $S$ is semisimple and $[P, H] = 0$, th. 1 implies that $[P, S] = 0$. Hence $L = P \oplus N \oplus S = P \oplus K$—direct sum of ideals. ▼

These results do not exclude, in principle, a possibility of imbedding part of the Poincaré and other symmetry Lie algebras in some larger symmetry algebras. We shall come back to these problems in ch. 21, § 3.

## § 8. Contraction of Lie Algebras

Let $L$ be a Lie algebra. A contracted Lie algebra $L'$ of $L$ may be abstractly introduced as follows. (We shall later show that this operation has a physical implementation when certain physical parameters tend to zero or infinity.) Let $X_1, \ldots, X_r$ be a basis of $L$. For a subset $X_1, \ldots, X_\varrho$, $\varrho \leqslant r$, of basis elements, we define

$$Y_i \equiv \lambda^{-1} X_i, \quad i = 1, 2, \ldots, \varrho \leqslant r \tag{1}$$

and express the commutation relations in terms of $Y_i$:

$$\begin{aligned}
[Y_i, Y_j] &= c_{ij}{}^k \lambda^{-1} Y_k + \lambda^{-2} c_{ij}{}^m X_m, \\
[Y_i, X_m] &= c_{im}{}^k Y_k + c_{im}{}^n \lambda^{-1} X_n, \\
[X_m, X_n] &= c_{mn}{}^i \lambda Y_i + c_{mn}{}^s X_s,
\end{aligned} \tag{2}$$

$$i, j, k \leqslant \varrho, \quad \varrho < m, n, s \leqslant r.$$

Now we let $\lambda \to \infty$, and determine when the elements

$$Y_1, \ldots, Y_\varrho, X_{\varrho+1}, \ldots, X_r$$

would form again a Lie algebra, namely the contracted Lie algebra $L'_\varrho$. This is the case, if the condition

$$c_{mn}{}^i = 0, \quad i \leqslant \varrho, \quad \varrho < m, n \leqslant r \tag{3}$$

is satisfied. Clearly, if $\varrho = r$, $L'_r$ is the abelian Lie algebra

$$[Y_i, Y_j] = 0, \quad i, j = 1, \ldots, r.$$

There are many important and interesting non-trivial cases with $\varrho < r$.

EXAMPLE 1. The contraction of the de Sitter Lie algebras o(3, 2) or o(4, 1) to the Poincaré Lie algebra.

The de Sitter Lie algebras have basis elements $M_{ab} = - M_{ba}$, $a, b = 1, 2, \ldots, 5$, satisfying

$$[M_{ab}, M_{cd}] = -(g_{bc} M_{ad} - g_{ac} M_{bd} + g_{ad} M_{bc} - g_{bd} M_{ac}) \tag{4}$$

where

$$\begin{aligned}
g_{ab}&: - - - + + \quad \text{for } o(3, 2), \\
g_{ab}&: - - - + - \quad \text{for } o(4, 1).
\end{aligned}$$

Let now $M_{5\mu} = RP_\mu$ and $R \to \infty$. Then ($\mu, \nu, \varrho = 1, 2, 3, 4$)

$$\begin{aligned}
[P_\mu, P_\nu] &= \frac{1}{R^2}[M_{5\mu}, M_{5\nu}] = -\frac{1}{R^2} g_{55} M_{\mu\nu} \to 0, \\
[M_{\mu\nu}, P_\varrho] &= \frac{1}{R}[M_{\mu\nu}, M_{5\varrho}] = -\frac{1}{R}(g_{\mu\varrho} M_{\nu 5} - g_{\nu\varrho} M_{\mu 5}) = -g_{\nu\varrho} P_\mu - g_{\mu\varrho} P_\nu .
\end{aligned} \tag{5}$$

Defining now generators $\tilde{M}_{\mu\nu} = -M_{\mu\nu}$ we obtain for $M_{\mu\nu}$ and $P_\sigma$ the commutation relations of the Poincaré algebra (cf. eq. 1(23)).

EXAMPLE 2. The contraction of the Poincaré Lie algebra to the Galilean Lie algebra.

This problem has one abstract mathematical solution, and another distinct physical solution, as we shall see in ch. 13. For the formal solution, consider the elements of the Poincaré Lie algebra $P_\mu$, $M_{\mu\nu}$ with the commutation relations given by eq. 1(23). Let

$$M_{0i} \equiv cK_i, \quad c \to \infty .$$

Then

$$[M_{ij}, K_k] = \frac{1}{c}[M_{ij}, M_{0k}] = \frac{1}{c}(g_{jk}M_{0i} - g_{ik}M_{0j}) = g_{jk}K_i - g_{ik}K_j,$$

$$[K_i, K_j] = \frac{1}{c^2}[M_{0i}, M_{0j}] = \frac{1}{c^2}M_{ij} \to 0, \tag{6}$$

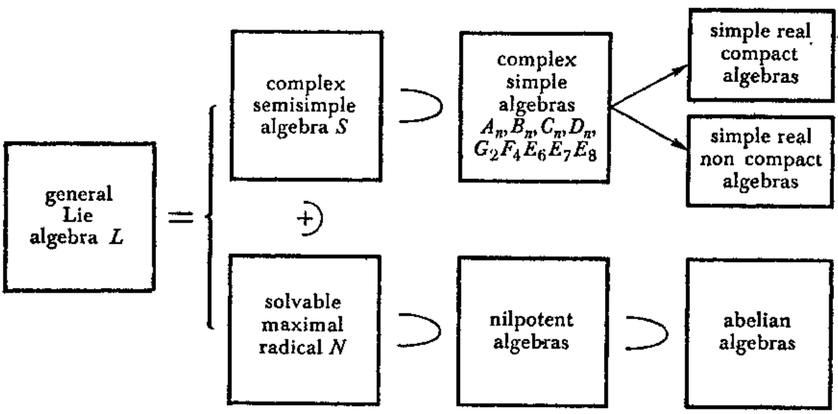
$$[K_i, P_\mu] = \frac{1}{c}[M_{0i}, P_\mu] \to 0.$$

The elements $M_{ij}$, $K_i$, $P_i$, $P_0$ form a basis of the Lie algebra of the Galilei group.

For the underlying physical motivation and significance of the contraction procedure we refer to ch. 13.

## § 9. Comments and Supplements

A) The following diagram describes the connections between various types of Lie algebras:




B) Figures 1 and 2 describe the set of all *real* simple Lie algebras corresponding to classical simple complex Lie algebras (obtained by both methods A and B of § 5).

C) The first important but incomplete results concerning the classification of complex simple Lie algebras were obtained by Killing 1888–1890. Killing's theory was completed and extended by Cartan in his thesis 1894. He introduced the roots as zeros of the characteristic polynomial $\det[\lambda I - \operatorname{ad} X]$. Later, H. Weyl 1925, 1926, I, II, III and 1935 and B. L. van der Waerden 1933 considerably simplified the theory using the Cartan subalgebra as the main tool in the classification problem. Here we follow the elegant method elaborated by Dynkin 1947; see also the method of H. Freudenthal 1958.

The problem of classification of real simple Lie algebras was also solved by Cartan 1914. Here we followed an algebraic derivation due to Gantmacher 1939a,b.

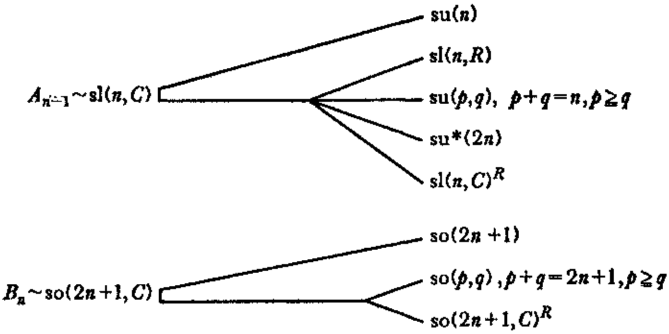



Fig. 1

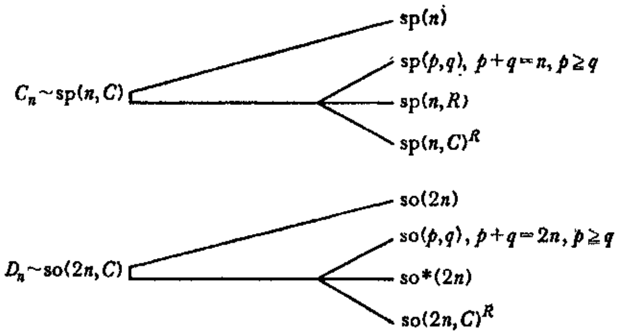



Fig. 2

An excellent readable exposition of the theory of semisimple Lie algebras is given in the monograph of Hausner and Schwartz 1968.

D) The matrix realization of classical Lie algebras $A_n$, $B_n$, $C_n$ and $D_n$ is relatively simple; in fact, it is given by the algebra of $n \times n$-matrices satisfying conditions of skew-hermiticity, symmetry, and tracelessness (§ 5). The concrete realization of exceptional Lie algebras is somewhat more complicated; in general, they have the following structure

$$L = S \dot{+} V \dot{+} V',$$

where $S$ is a simple complex Lie algebra, $V$ a complex vector space and $V'$ its conjugate space. The connection between various summands of $L$ is given by the relations

$$[S, V] \subset V, \quad [S, V'] \subset V', \quad [V, V] \subset V', \quad [V', V'] \subset V. \tag{7}$$

For example, for $L = G_2$ we have $S = \mathrm{sl}(3, C)$ and $V = C^3$ (for details cf. Hausner and Schwartz 1968, ch. II.4 and ch. III).

E) Th. 7.1 on space-time and internal symmetry algebras was first proved by McGlinn 1964 following suggestions to couple these two algebras, and in a general form presented here by Coester, Hamermesh and McGlinn 1964. See also the review by Hegerfeldt and Hennig 1968 for a collection of references.

F) The concept of group contractions and their representations was first introduced by Inönü and Wigner 1953.

## § 10. Exercises

§ 1.1. Show that the vector product $a \times b$, $a, b \in R^3$, satisfies the axioms of a Lie algebra.

§ 1.2. Show that the Poisson-brackets of classical mechanics

$$[f, g] \equiv \sum_{k=1}^{n} \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right), \quad f, g \in C,$$

where $R^{2n}$ is the phase-space, define a Lie algebra. Show the same for the Jacobi-brackets of vector-valued functions $[f, g] = g \cdot \nabla f - f \cdot \nabla g$.

§ 1.3. The *centralizer* $C$ of a Lie algebra $L$ consists of all elements $C$ such that $[C, X] = 0$ for all $X \in L$. Show that $C$ is a subalgebra.

§ 1.4. Find $L^C$, $(L^C)^R$ and $((L^C)^R)^C$ for the Lorentz Lie algebra $L = \mathrm{so}(3, 1)$.

§ 1.5. Find all Lie algebras whose basis elements are polynomials of $Q = x$ and $P = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$.

§ 1.6. Find the three-dimensional matrix representation of the nilpotent Heisenberg algebra $[P, Q] = -\mathrm{i}I$. (Cf. th. 2.2.)

§ 1.7. Find the second-order differential operators which, together with the Hamiltonian

$$H = \frac{1}{2m} P^2 + \frac{\omega^2}{2} Q^2,$$

form the Lie algebra su(1, 1), where $Q = x$, $P = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$.

§ 1.8. Let $L$ be any non-associative finite-dimensional algebra with a multiplication law $xy \in L$, $x \in L$, $y \in L$, and let $D$ be a derivation of $L$. Show that

$$\varphi_t \equiv \mathrm{e}^{tD} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D^\nu$$

satisfies

$$\varphi_t(xy) = (\varphi_t x)(\varphi_t y)$$

and

$$\varphi_t[x, y] = [\varphi_t x, \varphi_t y].$$

*Hint*: Use the Leibniz rule

$$D^n(xy)/n! = \sum_{j=0}^{n} \left(\frac{1}{j!} D^j x\right) \left(\frac{1}{(n-j)!} D^{n-j} y\right).$$

§ 1.9. Let $L$ be a Lie algebra with basis elements $X_1, \dots, X_r$. Let $L'$ be the vector space spanned by $X_1, \dots, X_r$ and a new element $Y$. In order that $L'$ be a Lie algebra (extension of $L$), there must be some restrictions on the coefficients $a^k{}_i$ in the equation

$$[Y, X_i] = a^k{}_i X_k.$$

Show that there is a trivial extension $L' = L \oplus \{Y\}$ (improper extension); and determine the proper nontrivial extension.

§ 1.10. Let $H$ be the Hamiltonian of a physical system in a Hilbert space $\mathscr{H}$ and let $\mathscr{A}$ be the set of all linear operators in $\mathscr{H}$ which commute with $H$. Show that $\mathscr{A}$ is a Lie algebra (in general, an infinite-dimensional Lie algebra).

§ 1.11. Show that the derivative algebra of the Lie algebra 1(23) of the Poincaré group is the 11-parameter Weyl algebra consisting of the Poincaré Lie algebra plus the generator $D$ of dilatations with

$$[D, P_\mu] = -P_\mu, \quad [D, M_{\mu\nu}] = 0.$$

§ 2.1. Classify all Lie algebras of dimension 2 and 3. (There are two Lie algebras of dimension 2, one abelian and one solvable given by $[e_1, e_2] = e_1$. The Lie algebras of dimension 3 are: (a) an abelian; (b) a nilpotent $[e_1, e_2] = 0$, $[e_2, e_3] = e_1$, $[e_3, e_1] = 0$; (c) $[e_1, e_2] = e_1$, $[e_1, e_3] = 0$, $[e_2, e_3] = 0$; (d) a class of solvable algebras $[e_1, e_2] = 0$, $[e_1, e_3] = \alpha e_1 + \beta e_2$, $[e_2, e_3] = \gamma e_1 + \delta e_2$, $\alpha\beta - \beta\gamma \neq 0$ (includes Euclidean algebras e(3) and e(2,1)); (e) two simple algebras so(3) and so(2, 1).)

More specifically, writing $[e_i, e_j] = f_k$ ($i, j, k$ cyclic), we have

| | I | II | III | IV | V | VI | VII | VIII | IX |
|---|---|---|---|---|---|---|---|---|---|
| $f_1$ | 0 | 0 | $e_1$ | $e_2$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ | $e_1$ |
| $f_2$ | 0 | 0 | 0 | $-e_1$ | $e_2$ | $-e_2$ | $e_1 + pe_2$ | $e_2$ | $-e_2$ |
| $f_3$ | 0 | $e_1$ | 0 | 0 | 0 | 0 | 0 | $e_3$ | $e_3$ |

§ 2.2. Show that the following Lie algebra is nilpotent: $[1, 2] = 5$, $[1, 3] = 6$, $[1, 4] = 7$, $[1, 5] = -8$, $[2, 3] = 8$, $[2, 4] = 6$, $[2, 6] = -7$, $[3, 4] = -5$, $[3, 5] = -7$, $[4, 6] = -8$, all other commutators zero.

§ 2.3. Show that any four-dimensional nilpotent algebra has a three-dimensional ideal.

§ 2.4. Find all solvable subalgebras of the Lie algebras of (a) the Lorentz group and (b) the Poincaré group.

§ 2.5. Evaluate the Killing form for the Poincaré group.

§ 2.6. The commutation relations for so$(p, q)$ Lie algebra are

$$[L_{ab}, L_{cd}] = -g_{bc}L_{ad} - g_{ad}L_{bc} + g_{ac}L_{bd} + g_{bd}L_{ac},$$

where $g_{ab}$ is the metric tensor. Show that the Cartan metric tensor

$$g_{ab,\alpha\beta} = (c_{ab})_{cd}{}^{ef}(c_{\alpha\beta})_{ef}{}^{cd}$$

has the simple form

$$g_{ab,\alpha\beta} = \mathrm{const}(g_{a\beta}g_{b\ } - g_{a\ }g_{b\beta}).$$

§ 2.7. Let $\psi(x)$ be a nonrelativistic quantum field at fixed $t$ satisfying the canonical commutation relations

$$[\psi(x), \psi^*(y)] = \mathrm{i}\delta^{(3)}(x-y),$$
$$[\psi(x), \psi(y)] = [\psi^*(x), \psi^*(y)] = 0.$$

Define the current

$$J_k(x) = \frac{1}{2\mathrm{i}}\psi^*(x)\overleftrightarrow{\partial}_k\psi(x).$$

Show that

$$[J_k(x), J_l(y)] = -\mathrm{i}\frac{\partial}{\partial x^l}[J_k(x)\delta^{(3)}(x-y)] + \mathrm{i}\frac{\partial}{\partial y^k}[J_l(y)\delta^{(3)}(x-y)].$$

Is $\{J_k(x)\}$ a Lie algebra?

§ 2.8. Set in the previous problem

$$J_k(n) \equiv \int_{-\pi}^{\pi} J_k(x)\mathrm{e}^{-\mathrm{i}n\cdot x}d^3x$$

Show that

$$[J_k(n), J_l(m)] = m_l J_k(m+n) - n_k J_l(m+n).$$

Find a finite-dimensional subalgebra of this Lie algebra.

§ 2.9. Let $\varrho(x) = \psi^*(x)\psi(x)$ be the charge density of the previous problems. Show that

$$[\varrho(x), \varrho(y)] = 0$$

and

$$[\varrho(x), J_k(y)] = \mathrm{i}\frac{\partial}{\partial x^k}\delta^{(3)}(x-y)\varrho(y).$$

Show that $\varrho(x)$ and $J_k(x)$ form an infinite-dimensional Lie algebra.

§ 3.1. Classify all ten-dimensional Lie algebras which contain a four-dimensional abelian algebra (i.e., classification of all space-time groups).

*Hint*: Use Levi–Malcev theorem.

§ 4.1. Determine the Weyl–Cartan form (cf. th. 4.3) of the Lie algebras su(3), su(2, 2), o$(p, q)$.

§ 4.2. The real Lie algebra so(4, 2) of dimension 15 is given by the following commutation relations of the basis elements:

$$\begin{aligned}
&M_{\mu\nu} \text{ and } P_\mu \text{ exactly as in eqs. 1(23)},\\
&[D, M_{\mu\nu}] = 0,\\
&[D, P_\mu] = -P_\mu,\\
&[K_\lambda, M_{\mu\nu}] = g_{\mu\lambda}K_\nu - g_{\nu\lambda}K_\mu,\\
&[K_\nu, P_\mu] = -2(g_{\mu\nu}D - M_{\mu\nu}),\\
&[D, K_\mu] = K_\mu,\\
&[K_\mu, K_\nu] = 0.
\end{aligned}$$

Show that although the maximal commutative subalgebra is four-dimensional, the Cartan subalgebra is of dimension 3.

§ 4.3. Consider the reflection $\Sigma_\alpha$ in the $l$-dimensional root-space by a plane perpendicular to a root $\alpha$. Show that the reflected root

$$\Sigma_\alpha\beta = \beta - \frac{2(\beta\alpha)}{(\alpha\alpha)}\alpha$$

is also a root and $\Sigma_\alpha\alpha = -\alpha$ (Weyl reflection).

§ 6.1. Find the Iwasawa decomposition for the Lorentz Lie algebra so(3, 1).

# Chapter 2

# Topological Groups

## § 1. Topological Spaces

In this section we describe the basic properties of topological spaces such as convergence of a sequence, continuity of mappings, compactness, connectedness, etc. We also give the corresponding notions in metric spaces, which are more intuitive.

*Notation*: The union $A \cup B$ of two sets $A$ and $B$ is the set of all points which belong to $A$ or $B$, i.e.:

$$A \cup B = \{x: x \in A \text{ or } x \in B\},$$

and the intersection $A \cap B$ is

$$A \cap B = \{x: x \in A \text{ and } x \in B\}.$$

The sets $A$ and $B$ are called *disjoint* if $A \cap B = \emptyset$, where $\emptyset$ is the empty set.

### A. *Topological Spaces*

DEFINITION 1. We say that a pair $\{X, \tau\}$, where $X$ is an arbitrary set and $\tau$ a collection of subsets $\tau_i \subset X$, is a *topological space* if $\tau$ satisfies the following conditions:

1° $\emptyset \in \tau$, $X \in \tau$.

2° If $U_1 \in \tau$ and $U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

3° If $U_s \in \tau$ for each $s \in S$, where $S$ is an arbitrary index set, then $\bigcup_{s \in S} U_s \in \tau$.

Every subset $U \subset X$ belonging to the collection $\tau$ is called an *open set* and the collection $\tau$ 'a *topology*' in the set $X$. We also use the expressions: $\tau$ defines a topology on $X$; $X$ is provided with a topology $\tau$. It follows from property 2° that the intersection of an arbitrary finite number of open sets is an open set.

A *neighborhood* of an element $x \in X$ is an arbitrary set which contains an *open neighborhood*, i.e. an open set containing $x$.

Clearly, a set that has more than one element can be provided with different topologies (cf. exercice 1).

EXAMPLE 1. Let $X$ be an arbitrary set and $\tau$ the family of all of its subsets. Clearly, $\{X, \tau\}$ is a topological space; one says that $X$ has a *discrete topology*, and $\{X, \tau\}$ is a discrete (topological) space. If $\tau$ consists of $X$ and the empty subset $\emptyset$ of $X$, the resultant topology is called the *coarsest topology* on $X$.

EXAMPLE 2. Consider the real line $R$ and the family $\tau$ of all sets $U \subset R$ obeying the condition that for each $x \in U$ there exists an $\varepsilon > 0$ such that the interval $(x-\varepsilon, x+\varepsilon) \subset U$. The family $\tau$ satisfies conditions 1°, 2° and 3° of def. 1 for open sets and generates, what is called, the *natural topology* of the real line. The family $\tau$ consists of all open intervals with rational end points, their finite intersections and arbitrary unions. ▼

A vector space $X$ over a field $K$ with a topology $\tau$ on $X$ is called the *topological vector space* if the maps $(x, y) \to x+y$ of $X \times X$ into $X$ and $(\lambda, x) \to \lambda x$ of $K \times X$ into $X$ are continuous in the topology $\tau$.

## B. *Convergence and Continuity*

The usual definitions of a convergent sequence and of a continuous function are reformulated in the language of open sets as follows:

DEFINITION 2. 1° A sequence $\{x_n\}$, $x_n \in X$, *converges to a limit* $x \in X$ if, for each open set $U$ containing $x$, there is an integer $N$ such that for $n \geqslant N$, $x_n \in U$.

2° A mapping $f\colon X \to Y$ from a topological space $\{X, \tau\}$ into a topological space $\{Y, \tau'\}$ is *continuous*, if for each $U \in \tau'$, open in $Y$, the inverse image $f^{-1}(U)$ is open in $X$. (One can also similarly define the continuity at a point $x \in X$.)

A continuous one-to-one transformation of $X$ onto $Y$ is said to be a *homeomorphism** (also called a *topological mapping*) if $f^{-1}$ is continuous. In other words, $X$ and $Y$ are *homeomorphic* if every open set of $X$ has an open set of $Y$ as image and every open set of $Y$ is the image of an open set of $X$.

EXAMPLE 3. The transformation $[0, 1) \ni x \to y = \exp(2\pi i x)$ is continuous and one-to-one, but it is not a homeomorphism, because the inverse transformation is not continuous. (See exercise 1.5 for another example.)

## C. *Metric Spaces*

We show now that the general defs. 2.1° and 2.2° of convergence and continuity coincide with the usual definitions, if $X$ is a metric space. In this case the concept of distance allows us to give a precise formulation of 'nearness'

A metric space $(X, d)$ is a set $X$ on which a real two-point function $d(\cdot, \cdot)$, the *distance* between the two points, is defined satisfying the following conditions

$$\begin{aligned} &1^\circ\ d(x, y) \geqslant 0,\\ &2^\circ\ d(x, y) = 0, \text{ if and only if } x = y,\\ &3^\circ\ d(x, y) = d(y, x),\\ &4^\circ\ d(x, y) \leqslant d(x, z)+d(z, y) \text{ for all } x,\ y,\ z \text{ in } X. \end{aligned} \tag{1}$$

* Do not confuse this concept with the algebraic concept of *homomorphism*.

EXAMPLE 4. 1° The two-point function

$$d(x, y) = \Big(\sum^{n} |x_i - y_i|^2\Big)^{1/2} \tag{2}$$

in the $n$-dimensional Euclidean space $R^n$, and

2° the function

$$d(x, y) = \|x-y\| \equiv [(x-y, x-y)]^{1/2} \tag{3}$$

in a Hilbert space $H$ with scalar product $(\cdot, \cdot)$, define a metric in a finite and infinite-dimensional space, respectively. (See exercise 1.2 for other examples.) ▼

We say that a *metric* is defined on the space, and different metrics can be defined on the same underlying set.

Let $x \in (X, d)$ and let $r$ be a positive number. The sets

$$S(x, r) = \{y \in X\colon\ d(x, y) < r\}, \tag{4}$$

$$\bar{S}(x, r) = \{y \in X\colon\ d(x, y) \leqslant r\} \tag{5}$$

are called the *open* and *closed balls* of radius $r$ with center $x$. ((4) is also called the *r-neighborhood*).

DEFINITION 3. A set $V \subset X$ of a metric space $(X, d)$ is said to be *open* if it is a union of open balls.

In particular each ball is an open set, and one readily verifies that a collection $\tau$ of all open sets in a metric space $(X, d)$ satisfies all the axioms of def. 1 and, therefore, defines a topology in $X$.

If $x_n \to x$ according to def. 2.1° then, for every open ball $S(x\ \varepsilon)$ we have $d(x_n, x) < \varepsilon$ for $n \geqslant N$. Hence, $\lim x_n = x$ according to the Cauchy definition of convergence. Conversely, if $x_n \to x$ in the sense of Cauchy, then for every open ball $S(x, \varepsilon)$ there exists an $N$ such that for $n \geqslant N$, $x_n \in S(x, \varepsilon)$; this is also true for any open set $V$ in $(X, d)$ containing $x$ according to def. 3. Hence, def. 2.1° and the ordinary Cauchy definition of convergence coincide in metric spaces.

It is also evident that def. 2.2° of continuity in a topological space $(X, \tau)$ coincides with the ordinary Cauchy definition if $X$ is a metric space and $\tau$ is the topology implied by the metric $d$.

The defs. 2.1° and 2.2° of convergence and continuity are thus a direct generalization to an aribtrary topological space of ordinary Cauchy definitions in metric spaces. It is in fact necessary to free the notions of neighborhoods, convergence and continuity from the more restrictive concept of distance.

EXAMPLE 5. Let $H$ be a Hilbert space and let formula (3) define a distance in $H$. Let $\tau$ be a collection of open sets which are generated by balls (4) according to def. 3. The topology so obtained is called the *strong topology*, and the corresponding convergence — the *strong convergence.*

Let $\tau$ be a collection of all open sets which are unions of 'weak' open spheres of the form

$$S_W(x) = \{u \in H: |(g_k, u-x)| < \varepsilon, \text{ for } k = 1, 2, \dots, m\} \tag{6}$$

for all possible choices of a positive number $\varepsilon$, a positive integer $m$, and vectors $g_1, \dots, g_m$.

The topology so obtained is called the *weak topology* in $H$ and the corresponding convergence—the *weak convergence.*

Notice that a sequence $u_n$ converges weakly to zero if $|(g, u_n)| \to 0$ for all vectors $g$ in $H$; in particular the sequence $u_n = e_n$, $e_n$—basis vectors in $H$, converges weakly to zero, although it does not converge strongly. ▼

The notion of weak and the strong convergence in Hilbert space play a fundamental role in group representation theory.

Although the def. 2 of a convergent sequence and of a continuous mapping are valid in arbitrary topological spaces, they are not sufficiently restrictive in every topological space $\{X, \tau\}$. For example, in a discrete space $X$ every function $f: X \to Y$, where $Y$ is an arbitrary topological space, is continuous, and in the coarsest topology $\tau = \{\emptyset, X\}$ every sequence $\{x_n\}$, $x_n \in X$, is convergent to each point $x \in X$.

In order to define the limiting point of a convergent sequence uniquely, an *axiom of separation* (also called the *Hausdorff separation axiom*) is introduced:

Definition 4. We say that a topological space is a *Hausdorff space* (or *$T_2$-space*) if for every pair of distinct points $x_1$ and $x_2$ there exist neighborhoods $U_1$ and $U_2$ such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. ▼

The discrete spaces and the real line with the natural topology are Hausdorff spaces. However, the real line $R$ with the topology $\tau = \{\emptyset, R\}$ is not a Hausdorff space.

Proposition 1. *In Hausdorff spaces every convergent sequence has a unique limit.*

Proof: Let $x_n \to x_1$ and $x_n \to x_2$. Suppose that $x_1 \neq x_2$. On the basis of the separation axiom, we have neighborhoods $U_1$ of $x_1$ and $U_2$ of $x_2$ such that $U_1 \cap U_2 = \emptyset$. From the def. 2.1° of a convergent sequence, it follows that there exists an integer $N$, such that for $n \geqslant N$, $x_n \in U_1$. Therefore, $x_n \notin U_2$ for $n \geqslant N$ and consequently $x_n \nrightarrow x_2$. Thus, $x_1 = x_2$. ▼

The *complement* $A'$ of a set $A \subset X$ is the set

$$A' \equiv X \backslash A \equiv \{x: x \in X \text{ and } x \notin A\}.$$

The set $A$ is called *closed* in $X$ if its complement

$$A' = X \backslash A$$

is open.

It follows directly from the definition of a closed set and a topology $\tau$ that in every topological space, the empty set and the whole space are both simultaneously closed and open. In the discrete topology every subset is both open and closed.

*Remark*: One could use closed sets instead of open sets in the defs. 2.1° and 2.2° with the appropriate changes.

The *closure* $\bar{A}$ of a set $A \subset X$ is the intersection of all closed sets which contain $A$, i.e. the smallest closed set containing $A$.

A set $A \subset X$ is called *dense in* $X$ if $\bar{A} = X$. A simple example of a dense set is provided by the set of all rational numbers in $R$.

A topological space $X$ is *separable*, if there exists a denumerable set $A \subset X$ which is dense in $X$.

EXAMPLE 6. Let $X$ be the Hilbert space equipped with the weak or strong topology, and let $\{e_n\}^\infty$ be a denumerable basis in $X$. Because the linear hull $A = \left\{\sum_{k=1}^{\infty} c_k e_k\right\}$, $c_k$ rational, $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ is dense in $X$, $X$ is separable.

## D. *Induced Topology, Product Topology and Quotient Topology*

Let $X$ be a topological space and $S \subset X$ a subset. Consider in $S$ a collection $\tau_s$ of sets of the form $S \cap U$, where $U$ is an open subset of $X$. The pair $(S, \tau_s)$ satisfies the conditions 1°, 2° and 3° of def. 1. Indeed, condition 1° is satisfied, because $\emptyset = S \cap \emptyset$ and $S = S \cap X$. Using the relations

$$(S \cap U_1) \cap (S \cap U_2) = S \cap (U_1 \cap U_2),$$
$$\bigcup_{k \in K} (S \cap U_k) = S \cap \bigcup_{k \in K} U_k,$$

we see immediately that conditions 2° and 3° of def. 1 are also satisfied. Thus if we consider $\tau_s$ as a collection of open sets in $S$ we transform the set $S$ into the topological space $(S, \tau_s)$.

DEFINITION 5. The topological space $(S, \tau_s)$ is said to be a *subspace* of the space $\{X, \tau\}$ and the topology $\tau_s$ is called the *induced* (or *relative*) *topology* by the topology in $X$. ▼

Let $\{X_1, \tau_1\}$ and $\{X_2, \tau_2\}$ be two topological spaces and let

$$X = X_1 \times X_2 = \{x_1, x_2 \colon x_1 \in X_1 \text{ and } x_2 \in X_2\} \tag{7}$$

be the Cartesian product of $X_1$ and $X_2$. We define the *product topology* $\tau$ on $X$ by choosing as a basis for the topology $\tau$ the class of all sets of the form $V \times U$ where $V \in \tau_1$ and $U \in \tau_2$.

It is evident that one can define in this manner the product topology for any finite number of topological spaces. In particular, the topology on $R^1$ (or $C^1$) defines a product topology on $R^n$ (or $C^n$).

Let $\{X, \tau_X\}$ be a topological space with a topology $\tau_X$ and let $f$ be a function on $X$ with the range $Y$. The inverse function $f^{-1}$ from $Y$ to $X$ defined for $y \in Y$ by $f^{-1}(y) = \{x \in X: f(x) = y\}$ has the properties

$$f^{-1}\Big(\bigcup_i U_i\Big) = \bigcup_i f^{-1}(U_i), \qquad U_i \subset Y,$$

$$f^{-1}\Big(\bigcap_i U_i\Big) = \bigcap_i f^{-1}(U_i).$$

Hence the family $\tau_Y$ of *all* sets $U \subset Y$ for which $f^{-1}(U)$ is open in $X$ is the topology on the space $Y$. The topology $\tau_Y$ is the largest topology for $Y$ with the property that the function $f$ is continuous. The topology $\tau_Y$ is called the *quotient topology for* $Y$ (the quotient topology relative of $f$ and relative to the topology $\tau_X$ of $X$).

Let $(X, \tau_X)$ be a fixed topological vector space and $R$ an equivalence relation on $X$. Let $\pi$ be the natural projection of $X$ onto the family $X/R$ of equivalence classes. The quotient space is the family $X/R$ with the quotient topology $\tau_{X/R}$ (relative to the map $\pi$ and topology $\tau_X$ on $X$).

If $U \subset X/R$, then $\pi^{-1}(U) = \bigcup \{u: u \in U\}$. Hence $U$ is open (closed) relative to the quotient topology $\tau_{X/R}$ if and only if $\bigcup \{u: u \in U\}$ is open (respectively closed) in $X$.

## E. *Compactness*

The concept of compactness or local compactness of a group will play a fundamental role in representation theory. This notion is purely a topological one and can be defined in terms of open sets only. It is instructive, however, to define this notion first for metric spaces in the language of sequences.

DEFINITION 6. A metric space $X$ is said to be *compact* if from every sequence $p_1, p_2, \ldots$ of points of $X$ we can choose a subsequence, which is convergent to a point $p \in X$, i.e. there exists a sequence of indices

$$k_1 < k_2 < \ldots$$

and a point $p \in X$, such that

$$\lim_{n\to\infty} p_{k_n} = p.$$

EXAMPLE 7. Let $X$ be the closed interval $a \leqslant x \leqslant b$, $a, b < \infty$. The classical Bolzano–Weierstrass theorem states that from every infinite bounded point sequence we can select a convergent subsequence. Hence $X$ is compact. ▼

We can easily prove, using the Bolzano–Weierstrass theorem and the concept of coordinates, that a subset $Y \subset R^n$ is compact if and only if it is closed and bounded. (This is the classical Borel–Lebesgue theorem, also known as the Heine–Borel theorem.)

To give a definition of compactness in an arbitrary topological space we first express it in the language of open sets: in fact, we have*

THEOREM 2 (the Borel–Lebesgue theorem). *Every collection of open sets, whose union covers a compact metric space $X$, contains a finite subcollection, whose union covers $X$.* ▼

Consequently in a general topological space we assume

DEFINITION 7. A topological Hausdorff space $X$ is *compact* if every collection of open sets, whose union covers $X$, contains a finite subcollection, whose union covers $X$ (i.e. every open covering of $X$ contains a finite subcovering).

EXAMPLE 8. 1° A discrete space is compact then and only then, when it is finite. If it is infinite, then the cover $X = \bigcup_{x \in X} \{x_i\}$ does not contain a finite subcovering.

2° A sphere $S^n$, $n = 1, 2, \ldots < \infty$, is compact. In fact, a sphere $S^n$ is a closed bounded subspace of $R^{n+1}$. Hence, by the Bolzano–Weierstrass theorem it is compact. ▼

The compactness of topological Hausdorff spaces is invariant not only under homeomorphisms, but also under continuous transformations. In fact, we have

PROPOSITION 3. *Let $X$ be a compact space and $Y$ a Hausdorff space. If there exists a continuous transformation $f$ of the space $X$ onto the space $Y$, then the space $Y$ is also compact.*

PROOF: Take an arbitrary open cover $\{U_s\}_{s \in S}$ of the space $Y$. The sets $\{f^{-1}(U_s)\}_{s \in S}$ generate an open cover of the space $X$. Therefore there exists a finite number of indices $s_1, s_2, \ldots, s_k \in S$ such that

$$f^{-1}(U_{s_1}) \cup f^{-1}(U_{s_2}) \cup \ldots \cup f^{-1}(U_{s_k}) = X.$$

Taking the images of both sides in this equation we obtain

$$U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_k} = Y. \blacktriangledown$$

A subset $A \subset X$ is compact if it is compact as a topological space with the induced topology.

DEFINITION 8. A topological space is *locally compact* if each point has a compact neighborhood.

Clearly, every compact space is locally compact. Every discrete space is locally compact. The straight line $R$ is locally compact, because every point $x \in R$ has a neighborhood $N$ (e.g., $N = (x-\varepsilon, x+\varepsilon)$), whose closure, by Bolzano–Weierstrass theorem, is compact.

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* A family $\{U_\lambda\}$ of (open) subsets of $X$ is called an (open) *covering* of a subset $A \subset X$, if $A \subset \bigcup U_\lambda$. A covering is finite (or countable) if it consists of a finite (or countable) number of sets.

The following, fundamental theorem gives a simple characterization of a class of non-locally compact spaces.

THEOREM 4 (Gleason). *A locally convex topological vector space X is locally compact if and only if* $\dim X < \infty$.

In particular every infinite-dimensional Hilbert space is not locally compact. One can show this directly; in fact, for instance, the sequence $x_n = Re_n$, where $\{e_n\}$ is an orthonormal basis in $H$ and $R$ any positive number does not contain a convergent subsequence in the strong topology; hence any ball $S(x, R)$ does not have a compact closure.

## F. *Connectedness*

In the theory of topological groups and group representations it is important to know, intuitively speaking, the number of different 'pieces' of a given topological space. We introduce now a mathematical formalism, which describes this feature of topological spaces.

DEFINITION 9. A topological space $X$ is called *connected* if it is *not* the union of two non-empty, disjoint subsets $A$ and $B$. A subspace $Y \subset X$ is called *connected* if it is connected as a topological subspace with the relative topology.

EXAMPLE 9. Every discrete space $X$ containing more than one point is not connected, because it can be represented as a sum of two sets $A = \{x_0\}$, $x_0 \in X$ and $A' = X \setminus \{x_0\}$, which are non-empty, open and disjoint. ▼

The connectedness is also invariant not only under homeomorphisms, but also under continuous transformations:

PROPOSITION 5. *Let X be a connected space and f: X → Y a continuous transformation onto the space Y. Then Y is connected.*

PROOF: Suppose that $Y$ is not connected. Then, there exist two non-empty open sets $A$ and $B$ in $Y$ such that

$$A \cup B = f(X) = Y$$

and

$$A \cap B = \emptyset.$$

It follows from the definition of the inverse mapping that

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(Y) = X$$

and

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \emptyset.$$

The sets $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty. Moreover, the continuity of the mapping $f$ implies that the sets $f^{-1}(A)$ and $f^{-1}(B)$ are open. Hence, $X$ can be represented as the union of two non-empty, disjoint, open sets, which contradicts the connectedness of $X$. ▼

DEFINITION 10. A *component* of a point $x$ of a topological space $X$ is the union of all connected subspaces of the space $X$ containing the point $x$.

The closure of a connected space is connected; hence the component is closed.

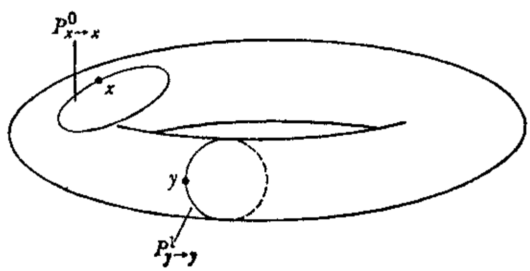
If all components of a space $X$ are one-point sets, then we call the space $X$ *totally disconnected.*

## G. *Simply-connected and Multi-connected Spaces*

Two points $x$ and $y$ of a space $X$ are connected by a path $P_{x \to y}$, if one can proceed continuously from $x$ to $y$ passing through the elements of a one-parameter subset $P$ of $X$. If $x$ coincides with $y$, one obtains either a closed path $P_{x \to x}$, or the null path $P$ at $x$. In other words, a *path* in $X$ is a continuous mapping of the closed interval $[0, 1]$ into $X$.

The two paths $P_{x \to y}$ and $Q_{x \to y}$ are said to be *homotopic* (or *deformable*, or *equivalent*), denoted by $P \simeq Q$, if there exists a continuous deformation of the path $Q_{x \to y}$ into the path $P_{x \to y}$ which leaves the end points unaltered.

EXAMPLE 10. Let $X$ be the surface of a torus. The closed path $P^0_{x \to x}$ which does not wrap the ring is homotopic to the null path. The closed path $P^1_{y \to y}$




which once wraps the ring is neither homotopic to $P^0_{x \to x}$, nor to the path which wraps twice the ring. ▼

A topological space is called *simply-connected* if every closed path is homotopic to the null path. The space in example 10 is not simply-connected.

Similarly, by making the stereographic projection, we conclude that the $n$-sphere $S^n$ is simply-connected for $n > 1$.

The definition of the homotopy of paths satisfies the following conditions:

$$P \simeq P \quad \text{(reflectivity)},$$
$$P \simeq Q \Rightarrow Q \simeq P \quad \text{(symmetry)},$$
$$P \simeq Q,\ Q \simeq T \Rightarrow P \simeq T \quad \text{(transitivity)}.$$

Hence, it is an equivalence relation: consequently all closed paths (at a point $x$) are classified into the so-called *homotopy classes.*

A topological space is said to be *n-connected* if it has $n$ homotopy classes at each point.

EXAMPLE 11. Let $X$ be the torus. It is evident that a closed path, which wraps the ring $k$-times is nonequivalent to a closed path, which wraps the ring $l$-times for $k \neq l$. Hence, the torus has infinite number of homotopy classes at each point. Consequently it is infinitely connected. ▼

## § 2. Topological Groups

We combine the concepts of an abstract group and a topological space specified on the same set $G$ into that of a topological group $G$. The consistency of this combination is provided by continuity.

DEFINITION 1. A *topological group* is a set $G$ such that:

1° $G$ is an abstract group,

2° $G$ is a topological space,

3° the function $g(x) = x^{-1}$, $x \in G$, is a continuous map from $G \to G$ and the function of $f(x, y) = x \cdot y$ is a continuous map from $G \times G \to G$.

EXAMPLE 1. Consider the Euclidean space $R^n$ as an abelian algebraic group and as a topological space with the product topology of $R^1$. The functions

$$g(x) = x^{-1} = -x, \quad \text{and} \quad f(x, y) = x+y$$

are continuous in this topology. Therefore, this abelian group is a topological group. ▼

The condition 3° expresses the compatibility between the algebraical and the topological operations on the set $G$. The following example illustrates that this compatibility condition is not automatically satisfied on a set, which is both an abstract group and a topological space.

EXAMPLE 2. Consider the cyclic group $C_3 = (1, x, x^2)$.* Let us define a topology $\tau$ by means of the following open sets

$$\emptyset, \ \{1\}, \ \{x\}, \ \{1, x\}, \ C_3. \tag{1}$$

The function $g(x) = x^{-1}$ transforms the element $x^2$ onto $g(x^2) = x$. Therefore the inverse function $g^{-1}$ transforms the open set $\{x\}$ onto the set $\{x^2\}$, which is not open. Therefore the cyclic group $C_3$ with the topology (1) is not a topological group. ▼

DEFINITION 2. Let $G$ be a topological group. A set $H \subset G$ is called a *topological subgroup* of $G$ if

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* An abstract group $G$ is called *cyclic* if every group element is a power of some group element $x$, i.e. $g_i = x^{p_i}$, $x \in G$.

1. $H$ is a subgroup of the abstract group $G$.
2. $H$ is a closed subset of the topological space $G$. ▼

Thus the fact that $G$ is not only an abstract group, but a topological group, imposes on the subgroup $H$ of $G$ a new condition, namely, that it be closed. The condition 2 could be replaced by other equivalent condition; in fact we have:

PROPOSITION 1. *A subgroup $H$ of a topological group $G$, that is an open subset of $G$ is also a closed subset.*

PROOF: A coset $X$ is the set of all elements $x$ of $G$ which satisfy the condition $x^{-1}x' \in H$, $x' \in X$ (i.e. $X = x'H$). This definition provides an equivalence relation so that the space $G$ decomposes onto disjoint cosets. Because $H$ is open, each coset is open. The complement of $H$ consists of the union of cosets of $H$ and is therefore open. Consequently, $H$ is closed. ▼

Note that not every abstract subgroup $H$ of the abstract group $G$ is a topological subgroup of $G$ considered as a topological group. For example, the abstract subgroup $H$ of the additive group $R$ of real numbers consisting of rational numbers is not a topological subgroup because it is not closed in $R$.

A topological subgroup $N$ of a topological group $G$ is called an *invariant subgroup* if for each $n \in N$ and $g \in G$ we have

$$g^{-1}ng \in N, \quad \text{i.e.} \quad g^{-1}Ng \subset N.$$

EXAMPLE 3. Let $\mathrm{GL}(n, R)$ be the group of all real non-singular $n \times n$-matrices under multiplication. We can parametrize an arbitrary group element $x = \{x_{ik}\}$ by the matrix elements $x_{ik} \in R$ so that $\mathrm{GL}(n, R)$ is a subset of $R^{n^2}$. Let us choose on $\mathrm{GL}(n, R)$ the induced topology of $R^{n^2}$. Because the matrix elements $z_{ik}$ of $z = xy$ are algebraic functions of $x_{il}$ and $y_{lk}$, the composition law in $G$ is continuous. Similarly, the matrix elements of $x^{-1}$ are rational, non-singular and, therefore, continuous functions of $x$. Hence, $\mathrm{GL}(n, R)$ equipped with the induced topology of $R^{n^2}$ is a topological group.

The subgroup $G_1 = \{\lambda I: \lambda \in R,\ \lambda \neq 0\}$ forms a one-parameter invariant topological subgroup of $\mathrm{GL}(n, R)$. Consider the subgroup $\mathrm{O}(n) = \{x \in \mathrm{GL}(n, R),\ (x^{-1})_{ik} = x_{ki}\}$. The mapping $f: x \to x^T x = e$ of $\mathrm{O}(n)$ is continuous; hence, the inverse image $f^{-1}(e) = \mathrm{O}(n)$ of the closed set $e$ is closed in $\mathrm{GL}(n, R)$; consequently $\mathrm{O}(n)$ is a topological subgroup of $\mathrm{GL}(n, R)$. It is called the *orthogonal group*. ▼

Similarly it follows that with the induced topology of $C^{n^2}$ the group $\mathrm{GL}(n, C)$ of $n \times n$ complex non-singular matrices is a topological group.

Different topological groups might be homeomorphic as topological spaces; for instance, the abelian group $G_1$ consisting of matrices

$$\begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}, \quad a, b \text{ real},$$

and the noncommutative group $G_1$ consisting of matrices

$$\begin{bmatrix} e^a & b \\ 0 & e^{-a} \end{bmatrix}, \quad a, b \text{ real},$$

are both homeomorphic to $R^2$ as topological spaces.

On the other hand one can obtain different topological groups by taking different topologies on the same abstract group. This leads to the following concept of *isomorphism* of topological groups.

DEFINITION 3. Two topological groups will be called *isomorphic* if there is one-to-one correspondence between their elements, which is a group isomorphism, *and* a space homeomorphism (i.e. preserves open sets).

The topological groups are examples of the so-called homogeneous spaces:

DEFINITION 4. A topological space $\{X, \tau\}$ is *homogeneous* if for any pair $x$, $y \in X$ there exists a homeomorphism $f$ of the space $\{X, \tau\}$ onto itself such that $f(x) = y$. ▼

Every topological group $G$ is homogeneous, because any two points $x, y \in G$ can be connected by the left translation

$$y = T_a^L x = ax, \qquad a = yx^{-1}, \tag{2}$$

which, by virtue of the uniqueness and the continuity of the group multiplication, is a homeomorphism of $G$.

The homogeneity of a topological group considerably simplifies the study of their local properties. It is sufficient to investigate the local properties of a topological group in the neighborhood of one point, for example, in the neighborhood of the unit element. Homogeneity will then ensure the validity of these properties at any other point.

We say that a topological group $G$ has a topological property $B$ (e.g., $G$ is compact, connected, separable, quotient group etc.) if $G$, considered as a topological space, has the property $B$.

The most important topological properties of $G$, with fundamental implications in the theory of group representations are compactness and connectedness (cf. § 1). We begin with a simple example to illustrate the concept of compactness of a group.

EXAMPLE 4. 1° Consider the orthogonal group $O(n)$. The real $n \times n$ matrix $x = \{x_{ik}\}$, $i, k = 1, 2, \ldots, n$, that corresponds to a group element $x \in O(n)$ satisfies the condition

$$x^T x = e. \tag{3}$$

It means that columns of a matrix $x \in O(n)$ can be considered as the orthonormal vectors of $R^n$. Therefore, matrix elements $x_{ik}$, $i, k = 1, 2, \ldots, n$, obey the condition

$$\sum_{i,k=1}^{n} x_{ik}^2 = n, \tag{4}$$

i.e. group elements of $O(n)$ can be represented as points of the sphere $S^{n^2-1}$ of radius $\sqrt{n}$. The collection of all points on the sphere which correspond to points of $O(n)$ is a closed set. Therefore, the group space of $O(n)$ is a closed, bounded subset of $R^{n^2}$, in the topology induced by that of $R^{n^2}$. Hence, by Bolzano–Weierstrass theorem the group $O(n)$ is compact.

2° One can prove in a similar manner that the unitary group $U(n)$ is compact. ▼

Although the underlying, topological space of a group $G$ in $R^n$ or $C^n$ might be bounded, the corresponding topological group might still be noncompact because $G$ is not closed. For instance, the one-parameter Lorentz group can be parametrized by the numbers $v/c$, where $v$ is the velocity of a reference system and $c$ the velocity of light; however, the Lorentz group is noncompact because the interval $-1 < v/c < 1$ in which the group is defined is not closed.

The group $GL(n, R)$ is noncompact, but it is locally compact; in fact, the map

$$\psi: x \to \det x, \qquad x \in GL(n, R) \tag{5}$$

is a continuous map of $GL(n, R)$ into $R$. Hence, the inverse image $\psi^{-1}(0)$ of the closed set $\{0\}$ in $R$ is closed in $R^{n^2}$. The complement of $\psi^{-1}(0)$ in $R^{n^2}$ is $GL(n, R)$. Therefore $GL(n, R)$ is an open set in $R^{n^2}$. Consequently, $GL(n, R)$ is noncompact.

It is, however, locally compact, because each point of the open set in $R^{n^2}$ has a compact neighborhood (cf. def. 1.8).

Similarly it follows that $GL(n, C)$ is locally compact.

Clearly, there exist topological groups which are noncompact and not locally compact. In fact, they play an important role in theoretical physics.

EXAMPLE 5. 1° Let $H$ be a Hilbert space with the strong topology $\tau$ defined by the norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$. Consider $H$ as an abelian topological group with respect to addition. By virtue of the Gleason theorem (§ 1, th. 4) this group is not locally compact.

2° A physical example of a non-locally compact group is provided by the group of gauge transformations in classical or quantum electrodynamics: Let $A_\mu(x)$ be the vector potential. The gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \varphi(x)$$

leave the field quantities $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ invariant.

Clearly these transformations form a group $G$ with respect to addition. Introducing any topology in $G$, we can convert the space of gauge functions into a topological vector space: this space is not locally compact by virtue of Gleason theorem. ▼

We shall now derive some properties of connected topological groups. Note first that if $G$ is connected, then the component of the identity coincides with $G$. On the other hand, if a component of unity contains only unity, then, due to homogeneity of topological groups all components of $G$ are one-point sets, i.e.

$G$ is totally disconnected. The set of rationals considered as an abelian topological group with the relative topology of the reals is an example of such a group.

PROPOSITION 2. *Let $G$ be a topological group, and $G_0$ the component of the identity* (def. 1.10). *Then $G_0$ is a closed invariant subgroup of $G$.*

PROOF: Let $C$ be a connected subset of $G$; then $xC$ and $Cx$, $x \in G$, are also connected, because left and right translations are homeomorphisms. Consequently, $xG_0x^{-1}$, $x \in G$, is a connected component which contains the identity $e$ of $G$. Thus $xG_0x^{-1}$ coincides with $G_0$ for every $x$ in $G$. We know that any component is closed. Hence, $G_0$ is an invariant topological subgroup of $G$. ▼

A subgroup $H$ of a topological group $G$ is called *central* if each element of $H$ commutes with every element of the whole group $G$.

PROPOSITION 3. *If $G$ is a connected topological group and $H$ is an invariant discrete subgroup, then $H$ is central.*

PROOF: Consider the map of $G$ into $H$

$$G \ni g \to gxg^{-1} \in H, \qquad x \in H.$$

This homeomorphism transforms a connected set onto a connected set. But connected sets in $H$ are only one-point sets $\{x\}$. Therefore the image of $G$, which contains $x$, must coincide with $\{x\}$. ▼

The following important example of a simply connected group will be used in this book throughout.

EXAMPLE 6. Let $G$ be the unitary unimodular group SU(2). A group element $g$ can be parametrized by

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{6}$$

Because

$$g^* = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix}, \quad g^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \tag{7}$$

the unitarity condition, $g^* = g^{-1}$, implies

$$d = \bar{a} \quad \text{and} \quad c = -\bar{b}.$$

Consequently, the unimodularity condition $ad-bc = 1$ can be written in the form

$$a\bar{a}+b\bar{b} = 1. \tag{8}$$

Setting $a = x+iy$ and $b = z+it$ we conclude that the group manifold of SU(2) coincides with the surface $x^2+y^2+z^2+t^2 = 1$ of three-dimensional sphere $S^3$. Since every closed path on $S^3$ can be contracted to a point, the group SU(2) is simply connected.

The subgroup $D$

$$D = \{I, -I\} \tag{9}$$

represents the discrete center of SU(2). The quotient group SU(2)/$D$ represents the doubly-connected rotation group SO(3). Thus the group manifold of SO(3) is obtained by identifying the antipodal points of the sphere $S^3$. ▼

*Uniform Continuity*

A complex function $\varphi(x)$ on $R^1$ is said to be *uniformly continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|\varphi(x)-\varphi(x+z)| < \varepsilon, \quad \text{whenever } |z| < \delta. \tag{10}$$

This notion has a natural generalization for complex functions defined on a group manifold.

A complex function $\varphi(x)$ defined on a topological group $G$ is said to be *left uniformly continuous* if for arbitrary $\varepsilon > 0$ there exists a neighborhood $V$ of the identity $e$ in $G$ such that for $x^{-1}y \in V$ we have

$$|\varphi(x)-\varphi(y)| < \varepsilon \quad (\text{or } |\varphi(x)-\varphi(xz)| < \varepsilon, \text{ whenever } z \in V). \tag{11}$$

Similarly we say that $\varphi(x)$ is *right uniformly continuous* if for arbitrary $\varepsilon > 0$ there exists a neighborhood $U$ of $e$ such that for $xy^{-1} \in U$ we have

$$|\varphi(x)-\varphi(y)| < \varepsilon \quad (\text{or } |\varphi(x)-\varphi(zx)| < \varepsilon \text{ whenever } z \in U). \tag{12}$$

A function which is both left and right uniformly continuous is said to be uniformly continuous.

PROPOSITION 4. *Let $G$ be a topological group and let $S$ be a compact subset of $G$. Then a continuous function $\varphi$ defined on $S$ is uniformly continuous on $S$.*

PROOF: Because $\varphi$ is continuous, for every $\varepsilon > 0$ there exists, for each point $y \in S$, a neighborhood $V_y$ of $e$ such that if $x \in S$ and $xy^{-1} \in V_y$, then $|\varphi(x)-\varphi(y)| < \varepsilon/2$. Let $W_y$ be a neighborhood of $e$ such that $W_y^2 \subset V_y$; the collection of open sets $W_y y$ covers $S$ and because $S$ is compact, we can select a finite covering. Let $\{W_{y_i} y_i\}_1^n$ be a finite collection of open sets which cover $S$, and let

$$V = \bigcap_{i=1}^{n} W_{y_i}.$$

Now, because the sets $W_{y_i} y_i$ cover $S$ for $x, y \in S$, $xy^{-1} \in V$, there exists a number $k$ such that $yy_k^{-1} \in W_{y_k}$ and therefore $|\varphi(y)-\varphi(y_k)| < \varepsilon/2$. Next

$$xy_k^{-1} = xy^{-1}yy_k^{-1} \in VW_{y_k} \subset W_{y_k}^2 \subset V_{y_k}$$

so that $|\varphi(x)-\varphi(y_k)| < \varepsilon/2$; hence we obtain

$$|\varphi(x)-\varphi(y)| < |\varphi(x)-\varphi(y_k)|+|\varphi(y)-\varphi(y_k)| < \varepsilon.$$

Consequently for $x \in S$, $\varphi(x)$ is uniformly right continuous. One shows similarly the uniform left continuity of $\varphi(x)$ on $S$. ▼

The concept of a left or a right continuity has a natural extension for a function $\varphi(x)$ on $G$ with values in a topological vector space $H$; for instance, if $H$ is a Hilbert space then $\varphi(x)$ is said to be left uniformly continuous if for arbitrary $\varepsilon > 0$ there exists a neighborhood $V$ of $e$ such that

$$\|\varphi(x)-\varphi(y)\|_H < \varepsilon, \quad \text{whenever } x^{-1}y \in V. \tag{13}$$

## § 3. The Haar Measure

In this section we introduce the important concept of an invariant measure and an invariant integration over a topological group $G$. Let $G$ be a locally compact group and let $C_0(G)$ and $C_0^+(G)$ denote the space of continuous and continuous non-negative functions on $G$ with a compact support respectively. A positive Radon measure is a positive linear form $\mu$ on $C_0(G)$ which is non-negative on $C_0^+(G)$, i.e.,

$$\mu(f) \geqslant 0 \quad \text{for } f \in C_0^+(G). \tag{1}$$

A positive Radon measure $\mu$, which is left-invariant, i.e.,

$$\mu(T_g^L f) = \mu(f), \quad \text{where } T_g^L f(x) = f(g^{-1}x),\ x, g \in G, \tag{2}$$

is called a *left Haar measure* (or a *left Haar integral*).

One defines similarly a right Haar measure $\lambda$ which satisfies the condition

$$\lambda(T_g^R f) = \lambda(f), \quad \text{where } T_g^R f(x) = f(xg),\ x, g \in G.$$

THEOREM 1. *Every locally compact group has a left Haar measure $\mu$. If $\nu$ is any other non-zero left Haar measure, then $\nu = c\mu$ for some positive number $c$.* ▼

(For the proof cf. Hewitt and Ross 1963, ch. IV, § 15.)

Let $G$ be a locally compact group with the multiplication law $xy$ and let $G^*$ be a new group with the same elements and same topology but with a new group multiplication law $x \times y$ defined by

$$x \times y = yx. \tag{3}$$

If $G^*$ has a left Haar measure $\mu$ given by th. 1 then $G$ has a right Haar measure: indeed, for $g^* \in G^*$, we have

$$(T_g^L \times f)(x) = f(g^{*-1} \times x) = f(xg^{-1}) = T_{g^{-1}}^R f(x),$$

hence

$$\lambda(T_g^R f) \equiv \mu(T_g^R f) = \mu(T_{g^{*-1}}^L f) = \mu(f) = \lambda(f), \tag{4}$$

i.e., $\lambda$ is a right Haar measure. Consequently the existence of a left Haar measure implies the existence of a right Haar measure (see also exercises 4 and 6). Therefore, by th. 1 every locally compact group has also a right Haar measure $\lambda$ defined up to a positive constant factor. The Haar measure, which is both left- and right-invariant is called the *invariant measure*.

By Riesz theorem we can associate with a measure $\mu(f)$ a set function $\mu(X)$ for a measurable set $X \subset G$, such that

$$\mu(f) = \int_G f(g)\mathrm{d}\mu(g). \tag{5}$$

The left-invariance (2) of the Haar measure implies then

$$\mu(gX) = \mu(X), \quad \text{or} \quad \mathrm{d}\mu(gx) = \mathrm{d}\mu(x), \tag{6}$$

for all $X \subset G$, $g, x \in G$.

EXAMPLE 1. Let $G$ be the group of all complex $2\times 2$-matrices with determinant one, i.e., $G = \mathrm{SL}(2, C)$. We shall construct explicitly an invariant Haar measure for $G$:

Every element of $G$ is a matrix $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, and may be identified with a point of $C^4$: The unimodular matrices form, in $C^4$, a second order surface $\alpha\delta - \beta\gamma = 1$. We relate with this surface a differential form $\mathrm{d}\omega$ defined by the formula

$$\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta = \mathrm{d}(\alpha\delta-\beta\gamma)\mathrm{d}\omega = J\mathrm{d}(\alpha\delta-\beta\gamma)\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta \tag{7}$$

where $J$ is the Jacobian for the transformation $(\alpha, \beta, \gamma, \delta) \to [(\alpha\delta-\beta\gamma), \beta, \gamma, \delta]$ from which we obtain the following expression for $\mathrm{d}\omega$

$$\mathrm{d}\omega(g) = \frac{1}{\delta}\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta. \tag{8}$$

Under the left translation $g \to g_0 g$ by an element $g_0 \in \mathrm{SL}(2, C)$ the form $\mathrm{d}\alpha\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta$, as well as the determinant $(\alpha\delta-\beta\gamma)$ of the matrix $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ are conserved. Consequently the form $\mathrm{d}\omega$ is also conserved. The same is true for the right translations $g \to gg_0$. Thus the positive form

$$\mathrm{d}\mu(g) = \mathrm{d}\omega\mathrm{d}\bar{\omega} = \frac{1}{|\delta|^2}\mathrm{d}\beta\mathrm{d}\gamma\mathrm{d}\delta\mathrm{d}\bar{\beta}\mathrm{d}\bar{\gamma}\mathrm{d}\bar{\delta} \tag{9}$$

satisfies

$$\mathrm{d}\mu(g_0 g) = \mathrm{d}\mu(gg_0) = \mathrm{d}\mu(g). \tag{10}$$

Therefore eq. (9) provides an invariant Haar measure on $\mathrm{SL}(2, C)$. By th. 1 any other Haar measure is then proportional to the measure (9).

Notice that because $g^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix}$, we have in addition from eq. (9):

$$\mathrm{d}\mu(g^{-1}) = \mathrm{d}\mu(g). \tag{11}$$

which expresses the inversion invariance of the Haar measure. ▼

For other examples see exercises 2 and 5.

*Inversion Invariance*

Let $\mu(\cdot)$ be a left Haar measure and let $\mu_g(f) \equiv \mu(T_g^R f)$. Because left and right translations commute we have

$$\mu_y(T_g^L f) = \mu(T_y^R T_g^L f) = \mu(T_g^L T_y^R f) = \mu_y(f).$$

Thus a linear positive invariant measure $\mu_y(f)$ is a Haar measure. By th. 1 we conclude that $\mu_y = \Delta(y)\mu$. Hence,

$$\mu(T_y^R f) = \Delta(y)\mu(f). \tag{12}$$

Because the map $G \ni y \to T_y^R f \in C_0(G)$ is continuous, the function $\Delta(y)$ is also continuous. Moreover, it satisfies the functional equation

$$\Delta(xy) = \Delta(x)\Delta(y). \tag{13}$$

Indeed,

$$\Delta(xy)\mu(f) = \mu(T_{xy}^R f) = \mu\big(T_x^R(T_y^R f)\big) = \Delta(x)\mu(T_y^R f) = \Delta(x)\Delta(y)\mu(f).$$

The function $\Delta(x)$ is called the *modular function for the group G*. If $\Delta(x) \equiv 1$, then, by eq. (12) the right and the left Haar measures for the group $G$ coincide. If this is the case the group is called *unimodular*.

Clearly, every abelian locally compact group is unimodular, because $T_{g^{-1}}^R = T_g^L$ in this case. Moreover, we have

PROPOSITION 2. *Every compact group is unimodular.*

PROOF: If $G$ is compact, then the function $f(x) \equiv 1$, $x \in G$, is in $C_0^+(G)$; hence, normalizing the left Haar measure by the condition $\mu(1) = 1$ we obtain

$$\Delta(y) = \Delta(y) \cdot \mu(1) = \mu(T_y^R 1) = \mu(1) = 1. \blacktriangledown$$

The unimodular Lie groups are described in ch. 3, § 10.D.

We now derive the fundamental inversion property of a left Haar measure.

PROPOSITION 3. *Let $\mu(\cdot)$ be a left Haar measure and let $\check{f}(x) \equiv f(x^{-1})$. Then,*

$$\mu(f) = \mu\left(\check{f}\,\frac{1}{\Delta}\right) \quad \textit{for every } f \in C_0^+(G). \tag{14}$$

PROOF: Let $\check{\mu}(f) \equiv \mu\left(\check{f}\,\frac{1}{\Delta}\right)$; then,

$$\check{\mu}(T_y^L f) = \mu\left((T_y^L f)^{\vee}\frac{1}{\Delta}\right) = \mu\left(T_{y^{-1}}^R \check{f}\,\frac{1}{\Delta}\right) = \Delta(y^{-1})\mu\left(T_{y^{-1}}^R \check{f}\,\frac{1}{T_{y^{-1}}^R \Delta}\right)$$
$$= \Delta(y^{-1})\Delta(y)\mu\left(\check{f}\,\frac{1}{\Delta}\right) = \check{\mu}(f).$$

Therefore, $\check{\mu}(\cdot)$ is a left Haar measure; consequently $\check{\mu}(f) = c\mu(f)$.

Next we show that $c = 1$; let $\xi$ be a positive number and let $U$ be a neighborhood of $e$ in $G$ such that $\left|\frac{1}{\Delta(x)} - 1\right| < \varepsilon$ for all $x \in U$. Let $h$ be a non-zero element in $C_0^+(G)$ such that $\check{h} = h$ and $h$ vanishes on the complement $U'$ of $U$. Then,

$$\left|h(x) - h(x)\frac{1}{\Delta(x)}\right| \leqslant \varepsilon h(x) \quad \text{for all } x \in G;$$

consequently

$$\left|\mu(h)-\mu\left(h\frac{1}{\Delta}\right)\right| \leqslant \varepsilon\mu(h).$$

This implies $|1-c| < \varepsilon$, i.e., $c = 1$. Thus $\mu(f) = \mu\left(\check{f}\frac{1}{\Delta}\right)$ for all $f \in C_0^+(G)$. ▼

Note that for a unimodular group $G$ we have $\mu(f) = \mu(\check{f})$, i.e.,

$$\int_G f(x)\,\mathrm{d}\mu(x) = \int_G f(x^{-1})\,\mathrm{d}\mu(x) = \int_G f(x)\,\mathrm{d}\mu(x^{-1}). \tag{15}$$

i.e. $\mathrm{d}\mu(x) = \mathrm{d}\mu(x^{-1})$.

In other words every invariant Haar measure is also invariant under the inversion; eq. (11) expresses explicitly this property of the Haar measure for SL(2, $C$).

## § 4. Comments and Supplements

### A. *Mackey Decomposition Theorem*

The following theorem gives an important decomposition of an arbitrary element of a topological group $G$.

THEOREM 1. *Let $G$ be a separable locally compact group and let $K$ be a closed subgroup of $G$. Then there exists a Borel set $S$ in $G$ such that every element $g \in G$ can be uniquely represented in the form*

$$g = ks, \quad k \in K,\ s \in S. ▼ \tag{1}$$

(For the proof cf. Mackey 1952, part I, lemma 1.1.)

The decomposition (1) plays a fundamental role in the theory of induced representations of topological groups (cf. ch. 16).

### B. *The Universal Covering Group*

The connection between global and local properties of topological groups is described by the following theorem:

THEOREM 2. *Let $\Gamma$ be the class of all arcwise-connected, locally connected, locally simply-connected topological groups, which are locally isomorphic with a certain topological group $G$. Then there exists in the class $\Gamma$, up to isomorphism, one and only one simply-connected group $\tilde{G}$. Any other group of the class $\Gamma$ is a quotient group $\tilde{G}/N$, where $N$ is a discrete normal subgroup.* ▼

(For the proof cf. Pontryagin 1966, ch. IX, sec. 51.)

The group $\tilde{G}$ is called the *universal covering group* of all groups in the class $\Gamma$.

Th. 2 plays a fundamental role in group representation theory because the connectedness of the group space is directly related to the single-valuedness of the representations of $G$.

### C. *Invariant Metric*

It is interesting that a topological group $G$ possesses not only an invariant measure but also an invariant metric. In fact, we have:

THEOREM 3 (the Birkhoff–Kakutani theorem). *Let $G$ be a topological group whose open sets at the identity $e$ have a countable basis.* Then there exists a distance function $d(\cdot, \cdot)$, which is right-invariant, i.e.*

$$d(xg, yg) = d(x, y) \quad \text{for all } x, y \text{ and } g \text{ in } G, \tag{2}$$

*and which induces on $G$ the original topology.* ▼

(For the proof cf. Montgomery and Zippin 1955, ch. I, § 22.)

### D. *Bibliographical Notes*

The axiomatic definition of a topological group in the form used today was first given by Polish mathematician F. Leja 1927. This subject became very popular in the early 1930's and was investigated by many prominent mathematicians such as D. van Dantzig, A. Haar, J. von Neumann, and others.

The notion of invariant integration on continuous groups had already been introduced in the nineteenth century by Hurwitz 1897. Later on, Weyl 1925–1926, I, II, III computed the invariant integral for O($n$) and U($n$); soon Peter and Weyl 1927 showed the existence of an invariant integral for any compact Lie group. The crucial achievement was Haar's 1933 result, who directly constructed a left-invariant integral for a locally compact group with a countable open basis. This result was surprising even for the best mathematicians like von Neumann, who did not believe in an existence of invariant integrals for such an extensive class of topological groups.

Haar's construction was extended to an arbitrary locally compact group by A. Weil 1936a, b, 1940.

An invariant measure exists also on some non-locally compact groups. In particular, the construction of an invariant measure for the complete metric groups was treated by Oxtoby 1946. For a more detailed discussion, see also the books by Hewitt and Ross 1963 and L. Nachbin 1965. Cf. also Stone 1966.

## § 5. Exercises

§ 1.1. Show that there exist nine topological spaces consisting of 3 elements no two of them homeomorphic.

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* A family $B(x)$, $x \in X$, of neighborhoods of a point $x$ having the property that for every open set $V$ containing $x$ there exists $U \in B(x)$ such that $x \in U \subset V$ is called a *basis* of the topological vector space $(X, \tau)$ in the point $x$.

§ 1.2. Show that the following two-point functions (besides eq. 1(2) and 1(3)) define a metric on the corresponding sets:

(a) On the set of $n$-tuples of real numbers

$$d(x, y) = \max|x_i - y_i|,$$

(b) $X$ = set of all continuous real functions on the closed interval $[0, 1]$,

$$d(x, y) = \left[\int_0^1 (x(t)-y(t))^2 \mathrm{d}t\right]^{1/2},$$

or

$$d(x, y) = \max_{0\leqslant t\leqslant 1} |x(t)-y(t)|.$$

(c) $X$ = arbitrary set,

$$d(x, y) = 1 \qquad \text{for } x \neq y,$$
$$d(x, x) = 0.$$

§ 1.3. The sequence $A_n$ of operators in a Hilbert space is said to *converge to A in uniform* (or *norm*) *topology* if

$$\lim_{n\to\infty} ||A_n - A|| = 0.$$

Show that the unitary one-parameter group of translations

$$U_t\colon\ u(x) \to u(x+t)$$

in $H = L^2(R^1)$ is continuous in strong topology but not in the uniform topology. Notice that because for any $t$, $t'$, $t \neq t'$

$$||U_t - U_{t'}|| = 2,$$

the *curve* $t \to U_t$ is *discrete* in uniform topology. This example illustrates the different nature of continuity in different topologies.

§ 1.4. Let SU(3) be the group of all $3\times 3$ unitary and unimodular matrices. Study the connectedness of $\mathrm{SU}(3)/Z_3$, where

$$Z_3 = \{I, \mathrm{e}^{\mathrm{i}2\pi/3}I, \mathrm{e}^{\mathrm{i}4\pi/3}I\}$$

is the center of SU(3). Generalize it to $\mathrm{SU}(n)/Z_n$.

§ 1.5. Let $(R, \tau_1)$ and $(R, \tau_2)$ be the real line $R$ equipped with a topology $\tau_1$ and $\tau_2$, respectively. Show that the one-to-one mapping $f\colon x \to y = x$ from $(R, \tau_1)$ into $(R, \tau_2)$ is continuous, iff $\tau_1$ is stronger than $\tau_2$.

§ 2.1. Let $G$ be the group of all linear transformations in $C^n$ which leaves the quadratic form

$$z_1\bar{z}_1 + \ldots + z_p\bar{z}_p - z_{p+1}\bar{z}_{p+1} - \ldots - z_n\bar{z}_n = 1$$

invariant.

Define a topology $\tau$ on $G$ such that $G$ becomes a topological group.

§ 2.2. Let $G$ be the group $O(n, 1)$ of all real transformations in $R^{n+1}$ which conserves the quadratic form:

$$x_0^2 - x_1^2 - x_2^2 - \dots - x_n^2.$$

Show that $O(2, 1)$ is infinitely many times connected.

§ 2.3. Show that $O(3, 1)$ consists of four components. Verify that the result is also true for the group $O(n, 1)$, $n \geqslant 3$.

§ 2.4. Let $X = R^n$ and let $G$ be the set of all one-to-one $C^\infty$ transformations: $f\colon R^n \to R^n$ such that the inverse transformation is also $C^\infty$. The group $G$ is the group of coordinate transformations (diffeomorphism group). For $f$ and $g$ in $G$ the functions

$$d(f, g)_n = \max_{0 \leqslant |m| \leqslant n} \sup_{x \in R^n} \left|(1+|x|^2)^n \left(f^{(m)}(x) - g(x)^{(m)}\right)\right|$$

where

$$f^{(m)}(x) = (D^m f)(x), \qquad D^m = D^{m_1} \dots D^{m_l}, \qquad D^{m_j} = \frac{\partial}{\partial x_{m_j}}, \qquad |m| = l$$

define metrics in $G$. Let $\tau_d$ denote the topology in $G$ defined by the metrics $d_n$. Show that the group operations are continuous relative to $\tau_d$ and therefore $(G, \tau_d)$ is a topological group.

§ 2.5. Let $H$ be a Hilbert space. Let $G$ be the group of all unitary operators in $H$. Define a topology $\tau$ on $G$, such that $G$ becomes a topological group.

§ 2.6. Consider the Schwartz space of functions $S$ on $R^n$ as an abelian group. Let $N = S \circledS G$ ($G$ = the diffeomorphism group in $R^n$, cf. exercise 2.3)) be the group defined by the following composition law

$$(s, g)(s', g') = (s + s' \circ g, g \circ g'),$$

where $s' \circ g$ and $g \circ g'$ denote the composition of the corresponding maps on $R^n$ (e.g. $s' \circ g = s'[g(x)]$).

Show that $N$ equipped with the product topology of Schwartz topology on $S$ and the topology $\tau_d$ on $G$ is a topological group. (Note: This is the global group associated with the current algebraic commutation relations; cf. ch. 1.10, exercise 2.9.)

§ 3.1. Let $\mu$ be the left Haar measure on $G$. We define a new measure

$$\check{\mu}(f) = \mu(\check{f}), \quad \text{where } \check{f}(x) = f(x^{-1}).$$

Show that $\check{\mu}$ is the right Haar measure on $G$.

§ 3.2. Consider the group $G$ of all matrices of the form

$$G \ni g = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}, \quad x, y \in R, \ x \neq 0.$$

Show that the left Haar measure has the form

$$d\mu(g) = \frac{dx\,dy}{x^2}.$$

§ 3.3. Show that the only translationally, invariant measure on the real line $R$ is proportional to the Lebesgue measure, i.e. $d\mu(x) = c\,dx$, $c = \text{const.}$

§ 3.4. Let $G$ be the set of all real non-singular $2\times 2$-matrices of the form

$$G \ni g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad (\text{i.e., } G = GL(2, R)).$$

Show that the left and the right invariant Haar measures on $G$ have the form

$$d\mu(g) = \frac{d\alpha\, d\beta\, d\gamma\, d\delta}{(\alpha\delta - \beta\gamma)^2}.$$

§ 3.5. Let $G = GL(n, R)$. Show that the Haar measure has the form

$$d\mu(x) = \frac{dx}{|\det X|^n},$$

where

$$dx = \prod_{i,j=1}^{n} dx_{ij}, \quad X = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \cdot & \cdot & \cdot \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \in G.$$

§ 3.6. Let $G$ be the group of all real $n\times n$ triangular matrices

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ & x_{22} & \cdots & x_{2n} \\ & & \ddots & \\ & 0 & & x_{nn} \end{bmatrix}.$$

Show that the left Haar measure on $G$ has the form

$$d\mu(x) = \frac{dx_{11}\, dx_{12} \dots dx_{n-1,n}\, dx_{nn}}{|x_{11}^n\, x_{22}^{n-1} \dots x_{n-1,n-1}^2\, x_{nn}|},$$

whereas the right Haar measure has the form

$$d\mu(x) = \frac{dx_{11}\, dx_{12} \dots dx_{n-1,n}\, dx_{nn}}{|x_{11}\, x_{22}^2 \dots x_{nn}^n|}.$$

# Chapter 3

# Lie Groups

## § 1. Differentiable Manifolds

In this section we introduce the concepts of differentiable (smooth) and analytic manifolds. Let $M$ be a Hausdorff space. A *chart* on $M$ is a pair $(U, \varphi)$, where $U$ is an open subset of $M$ and $\varphi$ is a homeomorphism of $U$ onto an open subset of $R^n$, $n$-dimensional (real) Euclidean space. The number $n$ is called the *dimension* of the chart, and $U$ the *domain* of the chart.

In other words, a chart is a local coordinate system in $M$ with respect to $\varphi$.

A Hausdorff space $M$ is said to be *locally Euclidean* if at each point $p \in M$ there exists a chart $(U, \varphi)$ on a neighborhood $U$ (called a *coordinate neighborhood*) of $p$ of dimension $n$. We then say that $(U, \varphi)$ is a *chart at p*. A Hausdorff space which is locally Euclidean at each point is called a *topological manifold* (of dimension equal to the dimension of the chart).

EXAMPLES. $R^n$, the sphere $S^n$, the projective spaces (real or complex), orthogonal group ($n \times n$ orthogonal matrices as a subspace of $R^{n^2}$) are topological manifolds. ▼

Let $S$ and $S'$ be open subsets of $R^n$ and $\psi$ a map of $S$ into $S'$. The map $\psi$ is said to be *differentiable* (or *smooth*) if the coordinates $y^j(\psi(p))$, $j = 1, 2, \dots, n$, are infinitely differentiable functions of the coordinates $x^i(p)$, $i = 1, 2, \dots, n$, $p \in S$. We shall write in this case $\psi \in$ class $C^\infty(S)$. The map $\psi$: $S \to S'$ is said to be *analytic* (or *class* $C^\omega$) if for each $p \in S$ there exists a neighborhood $U$ of $p$, such that for $q \in U$ every one of the coordinates $y^j(\psi(q))$, $j = 1, 2, \dots, n$, can be expressed as a convergent power series in $x^i(q) - x^i(p)$, $i = 1, 2, \dots, n$.

DEFINITION 1. A differentiable structure of dimension $n$ (also called an *atlas of class* $C^\infty$) on a Hausdorff space $M$ is a collection of charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ on $M$, such that the following conditions are satisfied:

1° $M = \bigcup_{\alpha \in A} U_\alpha$ (i.e., the domains of the charts cover $M$).

2° For each pair $\alpha, \beta \in A$ the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a differentiable map of $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$.* ▼

* The symbol $\varphi \circ \psi(z)$ means the composition of mappings $\psi$ followed by $\varphi$, i.e., $\varphi[\psi(z)]$.

A chart $(U_\alpha, \varphi_\alpha)$, $\alpha \in A$, defines a local coordinate system on the manifold $M$. The local coordinates of a point $p \in U_\alpha$ are the components of the function $\varphi_\alpha(p) \coloneqq (x^1(p), \dots, x^n(p))$.

Condition 2° means, in fact, that the transformation $\varphi_\beta \circ \varphi_\alpha^{-1}$ relating different coordinates introduced on the set $U_\alpha \cap U_\beta$ by the chart $(U_\alpha, \varphi_\alpha)$ and by the chart $(U_\beta, \varphi_\beta)$ is differentiable; it expresses the compatibility of overlapping coordinate systems.

We define an analytic structure on a Hausdorff space $M$ in a similar manner. We just replace the condition of differentiability of $\varphi_\beta \circ \varphi_\alpha^{-1}$ by the condition of analyticity of this map.

A differentiable (analytic) manifold of dimension $n$ is a Hausdorff space $M$ with a differentiable (analytic) structure of dimension $n$.

The simplest example of an analytic manifold is provided by the Euclidean space $R^n$. A chart $(U_\alpha, \varphi_\alpha$ lefined by an open set $U_\alpha = R^n$ and the homeomorphism $\varphi_\alpha$ which assigns to a point $p \in U_\alpha$ its Cartesian coordinates $\varphi_\alpha(p) = (x^1(p), x^2(p), \dots, x^n(p))$. We shall denote this analytic manifold by $R^n$.

EXAMPLE 1. Consider the two-dimensional unit sphere $S^2$ embedded in $R^3$ with the center at $(0, 0, 0)$. We shall introduce a collection of charts on $S^2$ by means of stereographic projections. Consider first the stereographic projection from the south pole $s$ with coordinates $s = (0, 0, -1)$ onto the plane through the equator. This is a homeomorphism of the punctured sphere (i.e., without the south pole) into the Euclidean plane $R^2$. If the Cartesian coordinates of a point $p \in S^2$ are $(x, y, z)$, then the Cartesian coordinates of the projection into the plane are $\left(\frac{x}{1+z}, \frac{y}{1+z}\right)$. Taking an arbitrary point $s' \in S^2$ as a 'new south pole' and utilizing the stereographic projection, we obtain a new local coordinate system. It can be readily verified that the coordinates of a point $q \in S^2\backslash\{s\} \cap S^2\backslash\{s'\}$ in the first and the second local coordinate system are related by an analytic transformation. Hence, the collection of local charts constructed by the stereographic projection from each point of $S^2$ satisfies the conditions 1° and 2° of def. 1, and defines the analytic structure on $S^2$. ▼

A complex analytic manifold of dimension $n$ is defined in an analogous manner. We replace $R^n$ by $C^n$ in the definition of a chart and we replace the condition 2° of def. 1 by the condition that the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ should be a holomorphic function of coordinates $z^i(p)$, $i = 1, 2, \dots, n$, of a point $p \in U_\alpha \cap U_\beta$.

In the following, by an analytic manifold, we mean an analytic real manifold.

A real-valued function $f$ on an analytic manifold $M$ is said to be *analytic at* $p \in M$ if there exists a chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$, such that $f \circ \varphi_\alpha^{-1}$ is an analytic function on the set $\varphi_\alpha(U_\alpha)$. The function $f$ is said to be *analytic* if it is analytic at each point $p \in M$. If we restrict the argument $p$ of a function $f$ to a subset

$N \subset M$, we obtain a real-valued function defined on $N$; we denote the restriction of $f$ to $N$ by $f|N$.

DEFINITION 2. Given two analytic manifolds $M$ and $N$, we say that $N$ is an *analytic submanifold* of $M$ if:

1° $N \subset M$ (set theoretically).

2° For any chart $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ with $\varphi_\alpha(p) = (x^1(p), x^2(p), \dots, x^n(p))$ the functions $x^i|N$ are analytic functions in $N$, and at each point $p \in N$ at which they are defined, we can select a subset $(x^{i_1}|N, x^{i_2}|N, \dots, x^{i_\nu}|N)$ which forms a chart at $p$. ▼

Simple examples of submanifolds are provided by the set $R^m$ in $R^n$, $m < n$, by great circles in $S^2$ (cf. example 1) and by $S^2$ in $R^3$.

Let $M$ and $N$ be two analytic manifolds of dimension $m$ and $n$ respectively. We shall now construct the product of these manifolds. Regarding $M$ and $N$ as Hausdorff spaces, we can form their topological product, which consists of all ordered pairs $(p, q)$, $p \in M$, $q \in N$.

The topology in $M \times N$ is defined as the product topology. The analytic structure on the manifold $M \times N$ is defined in a natural manner with the help of analytic structures on $M$ and $N$ as follows: Let $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ and $(V_\beta, \psi_\beta)_{\beta\in B}$ be collections of charts determining the analytic structure on $M$ and $N$, respectively. Denote by $\varphi_\alpha \times \psi_\beta$, $\alpha \in A$, $\beta \in B$, the mapping $(p, q) \to (\varphi_\alpha(p), \psi_\beta(q))$ of the product of open sets $U_\alpha \times V_\beta$ into $R^{m+n}$. Then the collection $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)$ of charts on the product $M \times N$ satisfies conditions 1° and 2° and defines the analytic structure on $M \times N$.

## A. *Tangent Spaces and Vector Fields*

Let $M$ be an analytic manifold of dimension $n$, $p$ a point of $M$, and $A(p)$ the class of functions analytic at $p$.

DEFINITION 3. The mapping $L$: $f(p) \to R$, $f \in A(p)$, is said to be a *tangent vector* at $p$ if the following conditions are satisfied:

1° $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$, $\alpha, \beta \in R$, $f$, $g \in A(p)$,

2° $L(fg) = L(f)g(p) + f(p)L(g)$,

i.e., $L$ is a linear functional and a derivation. ▼

If $L$ and $L'$ are tangent vectors, then $\alpha L + \beta L'$ is also a tangent vector. Hence the set of tangent vectors forms a linear vector space over $R$, the *tangent space*.

Let $(U, \varphi)$ be any chart at $p$. If a function $f$ is analytic at $p$, then the function $f^* = f \circ \varphi^{-1}$ is analytic in a neighborhood of $(x^1(p), \dots, x^n(p))$. We shall write for simplicity $\partial f/\partial x^i$ for

$$\left.\frac{\partial f^*}{\partial x^i}\right|_{x^i = x^i(p)} \tag{1}$$

THEOREM 1. *The mapping $L$: $f(p) \to R$, $f \in A(p)$ is a tangent vector at $p$ if and only if it is given by the formula*

$$Lf = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} L(x^i). \tag{2}$$

*The collection of all tangent vectors at $p$ forms an $n$-dimensional vector space.*

*The tangent vectors*

$$L_i(p)(f) = \left.\frac{\partial f^*}{\partial x^i}\right|_{x^i=x^i(p)}, \qquad i = 1, 2, \ldots, n, \tag{3}$$

*form a basis of the tangent space at $p$.* ▼

PROOF: If the action of $L$ is given by eq. (2) then $L$ is evidently a tangent vector. To prove that the action of any tangent vector is given by eq. (2), we note that if $f = \text{const}$, then $Lf = 0$. Expanding $f^*$ around $x(p)$

$$f^* = a_0 + a_1\big(x^1 - x^1(p)\big) + \ldots + a_n\big(x^n - x^n(p)\big) + \\ + \sum_{i,j=1}^{n} \big(x^i - x^i(p)\big)\big(x^j - x^j(p)\big) g_{ij} + \ldots,$$

where

$$a_i = \left.\frac{\partial f^*}{\partial x^i}\right|_{x^i=x^i(p)} \equiv \frac{\partial f}{\partial x^i} \quad \text{and} \quad g_{ij} \in A(p),$$

we obtain

$$Lf^* = a_1 L(x^1) + \ldots + a_n L(x^n), \tag{4}$$

which is equivalent to (2).

The set of tangent vectors $L_i(p)$ defined by

$$L_i(p)(f) = \left.\frac{\partial f^*}{\partial x^i}\right|_{x^i=x^i(p)}, \quad i = 1, 2, \ldots, n,$$

forms a basis of the tangent space at $p$. Indeed,

$$\sum_{i=1}^{n} \lambda_i L_i(p)(x^j) = \lambda_j,$$

hence $L_i(p)$ are linearly independent. Moreover, if $L(p)$ is any tangent vector, then by (2), $L(p)(x^j) = \sum_{i=1}^{n} L(x^i) L_i(p)(x^j)$, $j = 1, 2, \ldots, n$, i.e.,

$$L = \sum_{i=1}^{n} L(x^i) L_i(p). \tag{5}$$

Hence, the tangent space at $p$ is an $n$-dimensional vector space. ▼

DEFINITION 4. A *vector field* $X$ on an analytic manifold $M$ is a map which assigns to every point $p \in M$ a tangent vector $X(p)$ at $p$. ▼

A vector field $X$ is sometimes called an *infinitesimal transformation*. The vector field $X$ on $M$ is said to be *analytic at p* if $Xf$ is analytic at $p$ for an arbitrary function $f$ analytic at $p$ and it is said to be *analytic on M* if it is analytic at each $p \in M$. The simplest examples of analytic vector fields are provided by operators $L_i$ given by eq. (3). Indeed, let $(U, \varphi)$ be a chart at $p \in M$ and let $f$ be an analytic function at $p$. Then, expressing $f^* = f \circ \varphi^{-1}$ as a function $f^*(x^1, x^2, \ldots, x^n)$ of coordinates

$$\varphi(q) = \{x^1(q), x^2(q), \ldots, x^n(q)\}, \qquad q \in U$$

and setting

$$L_i(p)f = \left.\frac{\partial f^*}{\partial x^i}\right|_{x^i = x^i(p)}, \tag{6}$$

we obtain that the map $p \to L_i(p)$ is analytic.

If $A$ is a vector field defined on $U$, then, by virtue of eq. (3)

$$A(p) = \sum_{i=1}^{n} a^i(p) L_i(p), \tag{7}$$

where $a^i$, $i = 1, 2, \ldots, n$, are functions defined on $U$, given by the formula $a^i = Ax^i$. Conversely, if $a^i$, $i = 1, 2, \ldots, n$, are functions defined and analytic on $U$, then $A = \sum a^i L_i$ is an analytic vector field on $U$. At every $p \in U$ we have

$$(Af)(p) = \sum a^i(p) \left.\frac{\partial f^*}{\partial x^i}\right|_{x^i = x^i(p)}. \tag{8}$$

Hence, we may represent a vector field $A$ by the symbol $\sum_{i=1}^{n} a^{*i} \frac{\partial}{\partial x^i}$, where the functions $a^{*i} \equiv a^i \circ \varphi^{-1}$ are said to be the components of $A$ with respect to the coordinates $x^1, x^2, \ldots, x^n$.

It follows from def. 4 that if $X$ and $Y$ are vector fields, then $\alpha X + \beta Y$, $\alpha, \beta \in R$, is also a vector field. Hence, the collection of all vector fields forms a real vector space. This vector space is infinite-dimensional. This follows from the fact that we cannot introduce a finite set of basis functions for components $a^i(p)$ which appear in eq. (7).

By def. 4 a vector field $X$ may be viewed as a mapping $X$: $C^\omega(M) \to C^\omega(M)$. Hence the product $XY$ of two vector fields is well defined. However their product $XY$ is not in general a vector field. For example, if $M = R^n$, $A = \frac{\partial}{\partial x^1}$, $B = \frac{\partial}{\partial x^2}$, then $ABf = \frac{\partial^2}{\partial x^1 \partial x^2} f$, and the map of

$$f \to \left.\frac{\partial^2 f^*}{\partial x^1 \partial x^2}\right|_{x^i = x^i(p)}$$

is not a tangent vector to $R^n$. It is possible, however, to introduce some type of product in the vector space of analytic vector fields which transforms this space into a Lie algebra; namely, we may associate with analytic vector fields $A = \sum a^i L_i$ and $B = \sum b^i L_i$ an object $C$

$$C = AB - BA \equiv [A, B], \tag{9}$$

which is called the *Lie product* or the commutator of $A$ and $B$. In terms of a local coordinate system $\{x^1, x^2, \dots, x^n\}$ at $p$, we obtain

$$Af^* = \sum a^{*i}(x)\frac{\partial f^*}{\partial x^i}, \quad Bf^* = \sum b^{*i}(x)\frac{\partial f^*}{\partial x^i},$$

and

$$Cf^* = \sum_{i,j=1}^{n}\left(b^{*i}\frac{\partial a^{*j}}{\partial x^i} - a^{*i}\frac{\partial b^{*j}}{\partial x^i}\right)\frac{\partial f^*}{\partial x^j}. \tag{10}$$

Hence, if $A$ and $B$ are analytic vector fields defined on a manifold $M$, then their commutator $[A, B]$, by eq. (10), is also an analytic vector field defined on $M$.

The commutator operation (9) has the following properties

$$\begin{aligned}
&1^\circ\ [\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C], \qquad \alpha, \beta \in R,\\
&2^\circ\ [A, B] = -[B, A],\\
&3^\circ\ [A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0.
\end{aligned} \tag{11}$$

We see, therefore, that the collection of all analytic vector fields on an analytic manifold is an infinite-dimensional real Lie algebra.

## B. *Transformation of Vector Fields*

Let $M$ and $N$ be two differentiable $C^\infty$-manifolds and $\Omega$ a mapping of $M$ into $N$. A mapping $\Omega$ is *differentiable at* $p \in M$ if $f \circ \Omega \in C^\infty(p)$ for every $f \in C^\infty(\Omega(p))$. A mapping $\Omega$ is *differentiable* if it is differentiable at each $p \in M$. One defines similarly analytical mappings.

Let $\varphi$: $q \to (x^1(q), \dots, x^m(q))$ be a system of coordinates in a neighborhood $U$ of a point $p \in M$ and $\varphi'$: $r \to (y^1(r), \dots, y^n(r))$ a system of coordinates in a neighborhood $U'$ of a point $\Omega(p)$ in $N$ and let $\Omega(U) \subset U'$. A mapping $\varphi' \circ \Omega \circ \varphi^{-1}$ of $\varphi(U)$ into $\varphi'(U')$ is given by a system of $n$ functions

$$y^j = \omega^j(x^1, \dots, x^m), \quad 1 \leqslant j \leqslant n, \tag{12}$$

which represent the mapping $\Omega$ in terms of the coordinates.

We now derive the transformation properties of tangent vectors. Note first that if $L$ is a tangent vector at $p \in M$, then the linear mapping $L'$: $C^\infty(N) \to R$ given by $L'(g) = L(g \circ \Omega)$, is a tangent vector in the point $\Omega(p)$. We call the map $d\Omega_p$: $L \to L'$ the differential of the mapping $\Omega$ in a point $p$. By virtue of th. 1 the basis vectors in tangent spaces at a point $p$ and $\Omega(p)$ are given by formulas

$$e_i\colon f \to \left.\frac{\partial f^*}{\partial x^i}\right|_{\varphi(p)}, \quad 1 \leqslant i \leqslant m,\ f^* = f \circ \varphi^{-1}, \tag{13}$$

$$e_j\colon g \to \left.\frac{\partial g^*}{\partial y^j}\right|_{\varphi'(\Omega(p))}, \quad 1 \leqslant j \leqslant m,\ g^* = g \circ \varphi'^{-1}. \tag{14}$$

Hence, we obtain

$$\mathrm{d}\Omega_p(e_i)g = e_i(g \circ \Omega) = \left.\frac{\partial (g \circ \Omega)^*}{\partial x^i}\right|_{\varphi(p)} \tag{15}$$

Because $(g \circ \Omega)^*(x^1, \ldots, x^m) = g^*(y^1, \ldots, y^m)$, where $y^j = \omega^j(x^1, \ldots, x^m)$, then

$$\mathrm{d}\Omega_p(e_i) = \sum_{j=1}^{n} \left.\frac{\partial \omega^j}{\partial x^i}\right|_{\varphi(g)} e'_j. \tag{16}$$

We see, therefore, that if we represent the map $\mathrm{d}\Omega_p$ as a matrix using the basis $e_i$, $1 \leqslant i \leqslant m$ and $e'_j$, $1 \leqslant j \leqslant n$, we obtain the well-known Jacobi matrix of the system (12).

Vector fields $X$ and $Y$ on manifolds $M$ and $N$ are said to be $\Omega$-related if

$$\mathrm{d}\Omega_p X(p) = Y\big(\Omega(p)\big) \quad \text{for all } p \in M. \tag{17}$$

PROPOSITION 2. *Let $X_i$ and $Y_i$, $i = 1, 2$, be $\Omega$-related. Then,*

$$\mathrm{d}\Omega[X_1, X_2] = [Y_1, Y_2]. \tag{18}$$

PROOF: The formula (17) can be written in the form

$$(Yf) \circ \Omega = X(f \circ \Omega) \qquad \text{for all } f \in C^\infty(N). \tag{19}$$

Hence,

$$Y_1(Y_2 f) \circ \Omega = X_1(Y_2 f \circ \Omega) = X_1\big(X_2(f \circ \Omega)\big). \tag{20}$$

Changing indices 1 and 2 and subtracting expressions (20) we obtain eq. (18). ▼

## § 2. Lie Groups

Having discussed the general properties of analytic manifolds we are now in a position to define Lie groups.

DEFINITION 1. An abstract group $G$ is said to be a *Lie group* if

1. $G$ is an analytic manifold.

2 The mapping $(x, y) \to xy^{-1}$ of the product manifold $G \times G$ into $G$ is analytic. ▼

The condition (2) is equivalent to the following two conditions:

2′. The mapping $x \to x^{-1}$ of $G$ into $G$ is analytic.

2″. The mapping $(x, y) \to x \cdot y$ of $G \times G$ into $G$ is analytic. Indeed, in condition 2, we can set $x = e$, and see that $y^{-1}$ is analytic in $y$ and, hence, $xy = x(y^{-1})^{-1}$

is analytic in both $x$ and $y$. Conversely, if $2'$ and $2''$ are satisfied, then $(x, y) \to (x, y^{-1})$ is an analytic mapping of $G \times G$ into itself and, therefore, the mapping $(x, y) \to (x, y^{-1}) \to xy^{-1}$ is analytic so that condition 2 holds. Note that by condition $2''$ the left translation $T_x^L y = xy$, and the right translation $T_x^R y = yx$ are both analytic mappings.

Any Lie group is a topological group with respect to the topology induced by its analytic structure. Indeed, a manifold is a Hausdorff space and the analytic mapping $(x, y) \to xy^{-1}$ is continuous. Hence, by def. 2 (2.1) a Lie group is a topological group.

Furthermore any Lie group is locally compact. This follows from the fact that a manifold is locally Euclidean and an Euclidean space $R^n$ is locally compact.

A simple example of a Lie group is the additive group $R^n$ associated with the manifold $R^n$. The mapping $(x, y) \to xy^{-1} \equiv x - y$, in this case, is evidently analytic. The next example will play an important role in the following considerations.

EXAMPLE 1. Let $G = \mathrm{GL}(n, R)$ (cf. example 2.2.3). Consider the matrix elements $x^{ij}$, $i, j = 1, 2, \ldots, n$, of an element $x = \{x^{ij}\} \in \mathrm{GL}(n, R)$ as the set of coordinates of a point in $R^{n^2}$. Because the map

$$\psi\colon\ x \to \det x,$$

is a continuous map of $R^{n^2}$ into $R$, the set $\psi^{-1}(0)$ is closed in $R^{n^2}$. Therefore, its complement $(\psi^{-1}(0))'$ in $\mathrm{GL}(n, R)$ is an open subset of $R^{n^2}$ which is an open analytic submanifold of $R^{n^2}$. The coordinates $z^{ij}$ of the element $z = xy^{-1}$ can be expressed as rational functions of $x^{ls}$ and $y^{tu}$, and the denominators of these rational functions are different from zero on $\mathrm{GL}(n, R)$. Hence, the map $(x, y) \to xy^{-1}$ is analytic and, consequently, $\mathrm{GL}(n, R)$ is a Lie group. ▼

Let $(U_e, \varphi)$ be a chart at the identity $e$ of a Lie group $G$. We denote by $x^i(p)$ $i = 1, 2, \ldots, n$, the coordinates of a point $p \in U_e$ determined by the homeomorphism $\varphi(p) = (x^1(p), x^2(p), \ldots, x^n(p)) \in R^n$. It follows from the condition $2''$ that for every neighborhood $U_e$ of $e$ and for every open set $V \times W$ of $G \times G$ such that* $VW \subset U_e$, the functions $f^i$, $i = 1, 2, \ldots, n$, defined by

$$(xy)^i = f^i(x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n) \equiv f^i(x, y), \qquad x \in V,\ y \in W \tag{1}$$

are analytic functions of their arguments. The functions $f^i(x, y)$ are called the *composition functions of* $G$. They satisfy the obvious relations:

$$f^i(x, e) = x^i, \quad f^i(e, y) = y^i, \tag{2}$$

$$\left.\frac{\partial f^i}{\partial x^j}\right|_{(e,e)} = \left.\frac{\partial f^i}{\partial y^j}\right|_{(e,e)} = \delta^j_i. \tag{3}$$

* The product $VW$ denotes the subset of $G$ consisting of all elements $vw$, $v \in V$, $w \in W$.

It follows from the continuity of group multiplication that in locally Euclidean topological groups the composition functions are always continuous.

There arises the natural question as to when a locally Euclidean topological group is a Lie group. This problem was raised by Hilbert in 1900 and is known as Hilbert's Fifth Problem. The following theorem gives the solution of this problem:

THEOREM 1. *A locally Euclidean topological group is isomorphic to a Lie group.* ▼ (For the proof cf. Montgomery and Zippin 1956, ch. IV, § 4. 10.)

Th. 1 asserts in particular that the existence of continuous composition functions in a locally Euclidean topological group implies the existence (in some proper coordinate system) of analytic composition functions.

An interesting class of topological groups, which are not Lie groups, is provided by the class of infinite-dimensional topological groups, which often occur in classical and quantum physics. For example, the abelian group of gauge transformations of classical electrodynamics

$$A_\mu \to A_\mu + \partial_\mu \varphi \tag{4}$$

where $\varphi$ is a scalar gauge function, is not a Lie group since it is not locally Euclidean (cf. example 2.2.5).

We remark that the def. 1 specifies in fact a real Lie group. In the following, by a Lie group, we shall mean always a real Lie group, unless otherwise stated. The complex Lie group is defined as follows:

DEFINITION 2. An abstract group is said to be a *complex Lie group* if

1° $G$ is a complex analytic manifold.

2° The mapping $(x, y) \to xy^{-1}$ of the product manifold $G \times G$ into $G$ is holomorphic.

EXAMPLE 2. Let $G = \mathrm{GL}(n, C)$. The matrix elements $x^{ij} \in C$, $i, j = 1, 2, \ldots, n$, of a matrix $x = \{x^{ij}\} \in \mathrm{GL}(n, C)$ can be considered as coordinates of a point in $C^{n^2}$. Because the set $X = \{x\colon \det x = 0\}$ is closed in $C^{n^2}$ (cf. example 1), the group space of $\mathrm{GL}(n, C)$ is an open subset of $C^{n^2}$ and, therefore, is a complex analytic submanifold of $\mathscr{C}^{n^2}$.* Similarly as in example 1, we verify that the coordinates $z^{ij}$, $i, j = 1, 2, \ldots, n$, of an element $z = xy^{-1}$ are holomorphic functions of the coordinates $x^{ls}$ and $y^{tu}$, $l, s, t, u = 1, 2, \ldots, n$. ▼

Any complex Lie group of complex dimension $n$ can be considered as a real Lie group of $2n$ real dimensions. In fact, a complex analytic manifold of complex dimension $n$ can be considered as a real one of $2n$ real dimensions and the holomorphic mapping $(x, y) \to xy^{-1}$ becomes an analytic mapping, when considered on this real $4n$-dimensional manifold.

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* $\mathscr{C}^n$ is the complex analytic manifold determined by $C^n$ and the Cartesian coordinates.

Note that there exist genuine real Lie groups which are defined by complex matrices, e.g., the groups $SU(2n)$, $n = 1, 2, \ldots$, but they cannot be considered as complex groups because they are odd dimensional.

DEFINITION 3. Let $G$ be a Lie group. A subset $H \subset G$ is said to be an *analytic subgroup* of $G$ if

1° $H$ is a subgroup of $G$.

2° $H$ is an analytic submanifold of $G$. ▼

One could expect that an analytic subgroup is itself a Lie group. Indeed, we have:

PROPOSITION 2. *Any analytic subgroup $H$ of a Lie group $G$ is a Lie group.*

PROOF: Let $a, b \in H$. Then, $ab \in H$ and there exists a local coordinate system $(U, \varphi)$, $\varphi(z) = (z^1, z^2, \ldots, z^n)$, at $ab$ in $G$, such that $z^i|H$, $i = 1, 2, \ldots, \nu$, $\nu = \dim H$, form a local coordinate system at $ab$ in $H$. The element $xy \in G$ is near to $ab$, when $x$ and $y$ are near $a$ and $b$, respectively. This remains true, when $x$ and $y$ are restricted to $H$. Thus, the map $(x, y) \to xy$ restricted to $H \times H$ is analytic. We can show similarly that the map $x \to x^{-1}$ restricted to $H$ is also analytic. Hence, the analytic subgroup $H$ of a Lie group $G$ is a Lie group. ▼

Because analytic subgroups of a Lie group $G$ are themselves Lie groups, they are usually called Lie subgroups of the Lie group $G$.

An analytic homomorphism $t \to x(t)$ of $R$ into a Lie group is said to be a *one-parameter subgroup* of $G$.

EXAMPLE 3. Consider the subgroup $GL(m, R)$, $m < n$, of $GL(n, R)$. The collection of elements of $GL(m, R)$ is a subset of $GL(n, R)$, which, by the results of example 1, is an analytic submanifold of $GL(n, R)$. Therefore, the conditions 1° and 2° of def. 3 are satisfied and $GL(m, R)$, $m < n$, is a Lie subgroup of $GL(n, R)$. ▼

The following theorem gives a convenient criterion for a locally compact topological group to be a Lie group.

THEOREM 3. *A locally compact topological group $G$ is a Lie group if it can be mapped into* $GL(n, R)$ *by a continuous one-to-one homomorphism.* ▼

(For the proof cf. Montgomery and Zippin 1956, ch. II, § 16.)

For example, the groups $O(n)$, $U(n)$, (which is a subgroup of $O(2n)$) and $Sp(n)$ (which is a subgroup of $O(4n)$) are Lie groups.

## A. *The Structure Constants*

Let $G$ be a Lie group and $(V_e, \varphi)$ a chart at $e$, with $\varphi(e) = 0$. Consider the Taylor expansion of the composition functions (1) at the point $x^i = y^i = 0$. Using eqs. (1), (2) and (3) we obtain

$$f^i = x^i + y^i + a_{jk}{}^i x^j y^k + b_{jkl}{}^i x^j x^k y^l + d_{jkl}{}^i x^j y^k y^l + r_4, \tag{5}$$

where

$$a_{jk}{}^{i} = \left.\frac{\partial f^i}{\partial x^j \partial y^k}\right|_{00}, \dots \tag{6}$$

The numbers

$$c_{jk}{}^{i} = a_{jk}{}^{i} - a_{kj}{}^{i} \tag{7}$$

are called the *structure constants*. Under a change of the coordinate system

$$x^i \to x^{i'} = x^{i'}(x^k).$$

The structure 'constants' are subjected to the following transformation:

$$c_{j'k'}{}^{i'} = \left.\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\right|_0 c_{jk}{}^{i}.$$

Therefore, $c_{jk}{}^{i}$ is a tensor with one contravariant and two covariant indices. It also follows from formula (7) that

$$\begin{aligned} &1.\ c_{lk}{}^{i} \text{ are real numbers.}\\ &2.\ \text{For commutative groups } c_{lk}{}^{i} = 0.\\ &3.\ c_{lk}{}^{i} = -c_{kl}{}^{i}.\\ &4.\ c_{ls}{}^{p} c_{jk}{}^{s} + c_{js}{}^{p} c_{kl}{}^{s} + c_{ks}{}^{p} c_{lj}{}^{s} = 0. \end{aligned} \tag{8}$$

The last identity is a consequence of the associativity of the group multiplication. To prove it, we first calculate, using expansion (5), the coordinates of the elements $w = x(yz)$, and then $w' = (xy)z$. Comparing third order terms we obtain the last identity.

EXAMPLE 5. As an illustration we shall find the structure constants for GL$(n, R)$. The composition functions in the present case are (cf. example 1):

$$z^{ij} = f^{ij}(x, y) = x^{ik} y^{kj}.$$

Hence, using definitions (5) and (6) we obtain

$$a_{sm,kr}{}^{ij} = \left.\frac{\partial f^{ij}(x, y)}{\partial x^{sm} \partial y^{kr}}\right|_{\substack{x=e\\ y=e}} = \delta^i_s \delta_{mk} \delta^j_r,$$

and

$$c_{sm,kr}{}^{ij} = \delta^i_s \delta_{mk} \delta^j_r - \delta^i_k \delta_{rk} \delta^j_m. \tag{9}$$

Notice that the structure constants (9) for GL$(n, R)$ coincide with those for the Lie algebra gl$(n, R)$ (cf. 1.1 (13)).

## § 3. The Lie Algebra of a Lie Group

Starting from the concept of Lie groups we shall now establish contact with the theory of Lie algebras discussed in ch. 1 by introducing the concept of the Lie algebra of a Lie group $G$.

Let $T(e)$ be the algebra of differentiable functions of class $C^1$ defined in a neighborhood of $e$, and let $x(t)$, $a \leqslant t \leqslant b$, be a curve representing the homomorphism of class $C^1$ of $[a, b]$ into $G$, such that $x(0) = e$. The vector tangent to the curve $x(t)$ at $e$ is the map $A$: $T(e) \to R$ defined by:

$$Af = \left.\frac{\mathrm{d}f(x(t))}{\mathrm{d}t}\right|_{t=0} \tag{1}$$

In a local coordinate system $\{x^1, x^2, \ldots, x^n\}$ at $e$ we have

$$Af = \left.\frac{\mathrm{d}f(x(t))}{\mathrm{d}t}\right|_{t=0} = \sum_{j=1}^{n} \left.\frac{\partial f}{\partial x^j}\right|_{x^j = x^j(e)} \left.\frac{\mathrm{d}x^j}{\mathrm{d}t}\right|_{t=0} = \sum_{j=1}^{n} a^j L_j(e) f, \tag{2}$$

where

$$L_j(e) f = \left.\frac{\partial f}{\partial x^j}\right|_{x^j = x^j(e)},$$

and the numbers

$$a^j = \left.\frac{dx^j(t)}{dt}\right|_{t=0}, \quad j = 1, 2, \ldots, n, \tag{3}$$

are the components of the vector $A$ (cf. eq. 1(7)). Clearly, a vector tangent to a curve $x(t)$ at $e$, according to def. 1.3, is a tangent vector at $e$. Moreover, every tangent vector at $e$ can be considered as a vector tangent to a curve. Indeed, if

$$A = \sum_{j=1}^{n} a^j L_j(e)$$

is any tangent vector at $e$, the tangent vector to the curve

$$x^i(t) = x^i(e) + a^i t$$

is precisely $\sum a^j L_j(e) = A$.

According to th. 1.1 we can represent the tangent vector (2) by its components, i.e., we set $A = (a^1, a^2, \ldots, a^n)$. We know by th. 1.1 that the tangent space at $e$ is an $n$-dimensional vector space. We convert this vector space into a Lie algebra by setting

$$c^i = [A, B]^i = c_{jk}{}^i a^j b^k, \tag{4}$$

where the structure constants $c_{jk}{}^i$ are given by eq. 2(7). Indeed, from eq. (4) and 2(8)3 and (8)4 it follows that

$$[\alpha A + \beta B, C] = \alpha[A, C] + \beta[B, C], \tag{5}$$

$$[A, B] = -[B, A], \tag{6}$$

and

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{7}$$

The Lie algebra so obtained is said to be the *Lie algebra of the Lie group G*.

If we represent an element $A$ of a Lie algebra, in some basis of the vector space $L$, as $A = a^l X_l$, then, from eqs. (4) and (5), we obtain

$$C = c_{lk}{}^i a^l b^k X_i = [a^l X_l, b^k X_k] = a^l b^k [X_l, X_k],$$

i.e.,

$$[X_l, X_k] = c_{lk}{}^i X_i. \tag{7'}$$

EXAMPLE 1. Consider the group $\mathrm{GL}(n, R)$. The one-parameter subgroups may be written in the form

$$[g^{(ik)}(t)]^{ls} = \delta^{ls} + \delta^{li}\delta^{sk}x^{ik}(t) \quad \text{(no summation over } i, k, l, s = 1, 2, \ldots, n) \tag{8}$$

for 'off diagonal' subgroups, and in the form

$$[g^{(ii)}(t)]^{ls} = \begin{cases} \delta^{ls}, & l \neq i \\ x^{ii}(t), & l = i,\ s = i, \end{cases} \tag{9}$$

for diagonal ones. Here the index $(i, k)$, $i, k = 1, 2, \ldots, n$, enumerates successive subgroups and the indices $l$, $s$; $l$, $s = 1, 2, \ldots, n$, enumerate matrix elements of a group element $g^{(ik)}(t)$. The tangent vector $A^{(ik)}$ to the curve (8)–(9) has the following components

$$(A^{(ik)})^{ls} = a^{ik}\delta^{li}\delta^{sk}, \quad i, k, l = 1, 2, \ldots, n, \tag{10}$$

where

$$a^{ik} = \left.\frac{\mathrm{d}x^{ik}(t)}{\mathrm{d}t}\right|_{t=0}.$$

Eq. (10) implies that the basis vectors of $\mathrm{gl}(n, R)$ are given by $n \times n$-matrices $e_{ik}$, $i, k = 1, 2, \ldots, n$, of the form

$$(e_{ik})^{ls} = \delta^{li}\delta^{sk}. \tag{11}$$

The commutation relations for the basis elements $e_{ik}$ follow from eq. (4)

$$[e_{sm}, e_{kr}] = c_{sm,kr}{}^{ij} e_{ij}. \tag{12}$$

Using expression 2(9) for the structure constants of $\mathrm{GL}(n, R)$ we obtain

$$[e_{sm}, e_{kr}] = \delta_{mk} e_{sr} - \delta_{rs} e_{km} \tag{13}$$

(cf. example 1.1.2).

Note that the Lie product (13) for the tangent vectors $e_{sm}$ and $e_{kr}$ coincides with the commutator $[e_{sm}, e_{kr}] = e_{sm}e_{kr} - e_{kr}e_{sm}$ of the corresponding matrices.

## A. *Transformation Groups*

The Lie algebras occur in theoretical physics in most cases as the Lie algebras of transformation groups.

DEFINITION 1. A Lie group $G$ is a said to be a (*right*) *Lie transformation group* of a differentiable manifold $M$ if to each pair $(p, x)$, $p \in M$, $x \in G$ there corresponds an element $q \in M$ denoted by $px$, such that

1. The map $(p, x) \to px$ of $M \times G$ onto $M$ is differentiable.
2. $pe = p$ for all $p \in M$.
3. $(px_1)x_2 = p(x_1 x_2)$ for all $p \in M$ and $x_1, x_2 \in G$.

Similarly one defines a left Lie transformation group $(x, p) \to xp$ of a manifold $M$.

The group $G$ is said to be *effective* on $M$; if $x = e$ is the only element of $G$ which satisfies $px = p$ or $xp = p$ respectively for all $p \in M$.

We find now a general expression for the generators of the one-parameter transformation groups.

Let $(U, \varphi)$ be a chart at $e$ and let $(V, \psi)$ be a chart at a point $p \in M$. Condition 1 states that the coordinates $q^i$ of $q = px$ are analytic functions

$$q^i = \Phi^i(p^s, x^k), \qquad i = 1, 2, \ldots, m, \tag{14}$$

of coordinates $p^s$, $s = 1, 2, \ldots, m$, of $p \in M$ and $x^k$, $k = 1, 2, \ldots, n$ of $x \in G$.

Let $\psi$ be an analytic function on $M$ and let $x^i = e^i + \lambda^i \delta t$ be the coordinates of an element $x$ in an infinitesimally small neighborhood of $e$. Using the Taylor expansion for the function $T_x^R \psi(p) = \psi(px)$ at $p$ we obtain

$$\psi(q) = \psi(p) + \lambda^i \delta t \frac{\partial \psi(q)}{\partial q^k} \frac{\partial q^k}{\partial x^i}\bigg|_{x=e} + \varepsilon[(\delta t)^2]. \tag{15}$$

We denote by $f^k{}_i$ the derivatives $\dfrac{\partial q^k}{\partial x^i}\bigg|_{x=e}$ of the composition function (14) and evaluate the change of a function $\psi(p)$ due to an infinitesimal right translation

$$\delta\psi = \psi(q) - \psi(p) = \lambda^i f^k{}_i \frac{\partial \psi(q)}{\partial q^k}\bigg|_{q=p} \delta t, \tag{16}$$

where we have neglected powers of $\delta t$ beyond the first. We see, therefore, that the operators

$$X_i = f^k{}_i \frac{\partial}{\partial q_k}, \qquad i = 1, 2, \ldots, n = \dim G, \tag{17}$$

play the role of the generators of one-parameter right translations.

LEMMA 1. *The functions* $f^k{}_i(q) = \dfrac{\partial q^k}{\partial x^i}\bigg|_e$ *satisfy the following equation*

$$\frac{\partial f^i{}_j(q)}{\partial q^a} f^a_k(q) - \frac{\partial f^i{}_k(q)}{\partial q^a} f^a{}_j(q) = c_{jk}{}^b f^i{}_b(q), \tag{18}$$

*where* $c_{jk}{}^b$ *are the structure constants for* $G$. ▼

The proof easily follows from proposition 1.1 and the corresponding definitions and we omit it.

By virtue of eq. 1(9) we have

$$[X, Y] = XY - YX. \tag{19}$$

Hence by eq. (18) we obtain

$$[X_k, X_j] = \left(f_k^a \frac{\partial f^i{}_j(q)}{\partial q^a} - f_j^a \frac{\partial f^i{}_k(q)}{\partial q^a}\right)\frac{\partial}{\partial q^i} = c_{kj}{}^b f^i_b \frac{\partial}{\partial q_i} = c_{kj}{}^b X_b, \tag{20}$$

i.e., the set of generators (17) is closed under the Lie multiplication (19).

Notice that if $M = G$, then eq. (17) provides an expression for generators of the right translations

$$T^R_x \psi(y) = \psi(yx) \quad \text{on } G$$

(cf. exercise 3.1).

One can similarly define generators of left translations on $M$ and $G$.

EXAMPLE 2. The group $\mathrm{GL}(n, R)$ can be considered as an effective transformation group on $R^n$. We find the Lie algebra of $\mathrm{GL}(n, R)$ corresponding to this realization. Formula (14) takes, in the present case, the form

$$q^i = x^{ik} p^k. \tag{21}$$

Hence,

$$f^i{}_{st}(p) = \left.\frac{\partial x^{ik} p^k}{\partial x^{st}}\right|_{x=e} = \delta^{is} p^t \tag{22}$$

and the generators (17) of one-parameter subgroups are

$$X_{st} = f^i{}_{st}(p)\frac{\partial}{\partial p^i} = \delta^{is} p^t \frac{\partial}{\partial p^i} = p^t \frac{\partial}{\partial p^s}. \tag{23}$$

They satisfy the commutation relations

$$[X_{sm}, X_{kr}] = \delta_{mk} X_{sr} - \delta_{rs} X_{km} \tag{24}$$

(cf. eq. (13)). ▼

## B. *Correspondence between Lie Groups and Lie Algebras*

The following theorem establishes a close correspondence between the structures of Lie groups and Lie algebras:

THEOREM 2. *Get $G$ be a Lie group, $L$ its Lie algebra and $H$ a Lie subgroup of $G$. Denote by $N$ the set of all tangent vectors to differentiable curves at $e$ in $H$. Then,*

1. *$N$ is a subalgebra of $L$; it is the Lie algebra of the Lie subgroup $H$.*
2. *If $H$ is an invariant subgroup, then $N$ is an ideal of $L$.*
3. *If $H$ is a central invariant subgroup, then $N$ is a central ideal.*

PROOF: By def. 1 a Lie subgroup is at the same time an analytic submanifold. Hence, a subset $N$ is a subspace of $L$. Let $A$ and $B$ be tangent vectors to the curves $x(t)$ and $y(t)$ in $H$. The curve $g(t) = x(t)y(t)x^{-1}(t)y^{-1}(t)$ as well as the curve $g(\sqrt{s})$, $t = \sqrt{s}$ lie in $H$ because $H$ is a subgroup. It is easy to verify that the tangent vector $C$ to the curve $q(\sqrt{s})$ is equal to $[A, B]$ given by eq. (4). Hence, the tangent space $N$ is a linear subspace of $L$ which is closed under the Lie multiplication, i.e., it is a Lie subalgebra of $L$.

Let now $H$ be a normal subgroup of $G$. Denote by $x(t)$ an arbitrary curve in $G$ with the tangent vector $A$, and by $y(t)$ a curve in $H$ with the tangent vector $B$. Then, the curve $x(t)y(t)x(t)^{-1}$ lies in $H$ and, therefore, the curve $q(t) = x(t)y(t)x^{-1}(t)y^{-1}(t)$ as well as the curve $q(\sqrt{s})$, $t = \sqrt{s}$ lie in $H$. The vector $C = [A, B]$, $A \in L$, $B \in N$ is tangent to the curve $q(\sqrt{s})$. Hence, $C$ belongs to $N$; consequently, $N$ is an ideal of $L$.

In the special case, when $H$ is a central invariant subgroup, the curve $q(t)$ reduces to the point $e$ and therefore the tangent vector $C$ is zero, i.e., $N$ is a central ideal of $L$. ▼

We see that a Lie group determines a Lie algebra up to isomorphism. The following theorem gives an answer to the inverse question: to what extent does a Lie algebra $L$ determine a Lie group?

THEOREM 3. *Every subalgebra of a Lie algebra L of a Lie group G is a Lie algebra of precisely one connected Lie subgroup of G. Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.* ▼

(For the proof cf. Helgason 1962, ch. II, § 1 and § 2.)

## C. *Lie Groups with Isomorphic Lie Algebras*

We can also give a relation among global Lie groups which have isomorphic Lie algebras. Indeed, let $\Gamma$ be the class of all connected Lie groups having isomorphic Lie algebras. Then, by th. 3, any two members of the class $\Gamma$ are locally isomorphic. Moreover, by virtue of th. 2.4.2 there exists in the class $\Gamma$, up to an isomorphism, one and only one simply connected group $\tilde{G}$, the *universal covering group* of the class $\Gamma$. Any group of the class $\Gamma$ is a quotient group $\tilde{G}/N$, where $N$ is a discrete central invariant subgroup.

Let us note that the members of the class $\Gamma$, although locally isomorphic, may be totally different globally. The simplest example is the rotation group SO(2) and the translation group $T^1$. These groups are locally isomorphic, because their Lie algebras are isomorphic. However, as global groups, they are completely different; namely, SO(2) is a compact and infinitely connected group, while $T^1$ is noncompact and simply connected by th. 2.4.2. There exists between these groups the relation

$$\mathrm{SO}(2) = T^1/N, \tag{25}$$

where the discrete central invariant subgroup $N$ is the subgroup of integers.

Another example: SU(2) is simply connected, SO(3) is doubly connected, they have isomorphic Lie algebras, and $\mathrm{SO}(3) = \mathrm{SU}(2)/Z_2$, where $Z_2$ is the discrete center $Z_2 = (e, -e)$ of SU(2).

D. *Adjoint Group*

Consider now a Lie group $G$ and denote by $L$ its Lie algebra. The map

$$\psi_x(y) = xyx^{-1}, \tag{26}$$

for a fixed $x \in G$ defines an automorphism of $G$.

Automorphisms of the form (26) are called *inner automorphisms of* $G$. Any other automorphism is called an *outer automorphism* of $G$. To every automorphism of $G$ there corresponds an automorphism of the Lie algebra $L$. We can calculate the explicit form of the induced automorphism $l_x$ of $L$ by introducing a coordinate system on $G$. Because

$$y'_x = \psi_x(y) = (xyx^{-1}y^{-1})y$$

then using the coordinates of the product $xyx^{-1}y^{-1}$ we obtain

$$y'^i_x = c_{lk}{}^i x^l y^k + y^i + \varepsilon = (c_{lk}{}^i x^l + \delta^i_k) y^k + \varepsilon^i, \tag{27}$$

where $\varepsilon^i$ is of the third order of smallness with respect to coordinates of $x$ and $y$. The explicit form of the automorphism $l_x$ of $L$ can be calculated from the definition of the tangent vector to the one-parameter subgroup $y^i(t)$. Differentiating both sides of (27) we get, for $t = 0$,

$$a'^i = c_{lk}{}^i x^l a^k, \tag{28}$$

where

$$a^i = \mathrm{d}y^i/\mathrm{d}t|_{t=0}$$

are the coordinates of a vector $A = a_i X^i \in L$ in a basis $X_i$ of $L$.

The automorphism $l_x$ of the Lie algebra $L$ has then the form

$$(l_x)_k{}^i = c_{lk}{}^i x^l, \tag{29}$$

where $x^l$, $l = 1, 2, \dots, n$, are coordinates of an element $x \in G$. The map $h\colon x \to l_x$ is the homomorphism of $G$ into the group $G_A$ of all automorphisms of $L$. Obviously the kernel of this homomorphism is the center of $G$. The automorphisms (29) of the algebra $L$ induced by the inner automorphisms (26) of $G$ are called *inner automorphisms of* $L$. All other automorphisms of $L$ are called *outer automorphisms*. The group $G_a$ of all inner automorphisms (29) of $L$ is called the *adjoint group*. We show now that the Lie algebra of the adjoint group $G_a$ is the *adjoint algebra* $L_a$. In fact, taking a one-parameter subgroup $l_{x(t)}$ we find by eq. (29) that the coordinates of the generator

$$P = \mathrm{d}l_{x(t)}/\mathrm{d}t|_{t=0}$$

are

$$P_k{}^i = c_{lk}{}^i b^l, \qquad \text{where} \qquad b^l = \mathrm{d}x^l/\mathrm{d}t. \tag{30}$$

Putting $B = \sum b^i X_i$, we find

$$P = \operatorname{ad} B, \quad \text{or} \quad P(A) = \operatorname{ad} B(A) = [B, A], \tag{31}$$

i.e., $P \in L_a$. The dimension of the Lie algebra $L_a$ of tangent vectors of inner automorphisms is equal to the dimension of the adjoint algebra $L_a$. Hence $\tilde{L}_a$ is identical to $L_a$. For this reason $L_a$ is also called the *algebra of inner derivations.*

## E. *Left- and Right- Invariant Lie Algebras*

Let $G$ be a Lie group. Let $\Omega_{g_0}$ be the mapping of $G$ onto $G$ given by the left translation $\Omega_{g_0}$: $g \to g_0 g$. It follows from the def. 2.1 of a Lie group that $\Omega_{g_0}$ is an analytic isomorphism of $G$ onto $G$. Let $\mathrm{d}\Omega_{g_0}$ denote the differential of $\Omega_{g_0}$ defined in 1.B. It follows from 1.B that $\mathrm{d}\Omega_{g_0}$ transforms the tangent space $L_e$ at the unity $e$ into a tangent space $L_{g_0}$ at $g_0$.

DEFINITION 2. A vector field $X = \{X_1, g \in G\}$ on $G$ is said to be *left-invariant* if for any $g, g' \in G$ we have

$$\mathrm{d}\Omega_{g'g^{-1}} X_g = X_{g'}. \tag{32}$$

The collection of all left-invariant vector fields on $G$ forms a Lie algebra. Indeed if $X$ and $Y$ are any two left-invariant vector fields, then evidently $\alpha X + \beta Y$ is also left-invariant; moreover, by virtue of proposition 1.2, we have

$$\mathrm{d}\Omega_{g'g^{-1}}([X, Y_g]) = [\mathrm{d}\Omega_{g'g^{-1}}(X), \mathrm{d}\Omega_{g'g^{-1}}(Y)] = [X, Y]_{g'}, \tag{33}$$

i.e., $[X, Y]$ is also left-invariant. The left-invariant Lie algebra is generated by right translations and is realized by virtue of th. 1.1 by first order differential operators. We denote this Lie algebra by $L^R$.

PROPOSITION 4. *Every left-invariant vector field is analytic.*

PROOF: Let $V_1$ be a neighborhood of an arbitrary point $g_0 \in G$ and let $\{t_1, \ldots, t_n\}$ be a coordinate system on $G$ at $g_0$. There exists a neighborhood $V_2$ of $g_0$ such that the condition $g, h \in V_2$ implies $g g_0 h^{-1} \in V_1$. For $g \in G$ by virtue of eqs. (1), 1(17), and 1(19), we have

$$X_g t_i = (\mathrm{d}\Omega_{gg_0^{-1}}^{-1} X_{g_0}) t_i = X_{g_0}(t_i \circ \Omega_{gg_0^{-1}}^{-1}). \tag{34}$$

Now the coordinates $t_i'(g, h) \equiv t_i(g g_0^{-1} h)$ are analytic on $V_2 \times V_2$; hence $t_i'(g, h) = f_i(t_1(g), \ldots, t_n(g);\ t_1(h), \ldots, t_n(h))$, where the functions $f_i(y_1, \ldots, y_n;\ z_1, \ldots, z_n)$ are analytic in all their $2n$ arguments in the neighborhood of the set of values $y_k = t_k(g_0)$, $z_k = t_k(g_0)$, $k = 1, 2, \ldots, n$. We obtain

$$X_g t_i = (X_{g_0} t_j) \left(\frac{\partial f_i}{\partial z_j}\right)\bigg|_{g, g_0}, \tag{35}$$

where the indices $g, g_0$ mean that the partial derivatives are taken for $y_k = t_k(g)$, $z_k = t_k(g_0)$. Now the quantities $X_{g_0} t_j$ are constant by virtue of th. 1.1 and $(\partial f_i / \partial z_j)|_{g, g_0}$ considered as a function of $g$ is analytic at $g_0$. Hence, the functions

$X_g t_i$ are analytic at $g_0$ and consequently the vector field $X = \{X_g, g \in G\}$ is also analytic. ▼

Proposition 4 implies that the left-invariant Lie algebra $L^R$ of $G$ consists of analytic vector fields on $G$.

Similarly, we can also introduce right-invariant vector fields and show that they are analytic and form a right-invariant Lie algebra $L^L$.

PROPOSITION 5. *The left and the right-invariant Lie algebras are isomorphic. This isomorphism is analytic.*

PROOF: Let $\mathscr{I}$ be the map $g \to g^{-1}$ of $G$ onto itself. Clearly $\mathscr{I}$ is an analytic isomorphism by def. 2.1. Let $X$ be any left-invariant vector field and let $Y_g \equiv \mathrm{d}\mathscr{I} X_{g^{-1}}$. Then $Y$ is right-invariant. Indeed denoting right infinitesimal translations on $G$ by $d\Sigma$, we have

$$\mathrm{d}\Sigma_g Y_e = \mathrm{d}\Sigma_g(\mathrm{d}\mathscr{I} X_e) = \mathrm{d}(\Sigma_g \circ \mathscr{I}) X_e. \tag{36}$$

Because $(\Sigma_g \circ \mathscr{I})$ maps $g_0$ into $g_0^{-1} g = (g^{-1} g_0)^{-1}$, we have $\Sigma_g \circ \mathscr{I} = \mathscr{I} \circ \Omega_{g^{-1}}$. Hence

$$\mathrm{d}\Sigma_g Y_e = \mathrm{d}\mathscr{I}(\mathrm{d}\Omega_{g^{-1}} X_e) = \mathrm{d}\mathscr{I}(X_{g^{-1}}) = Y_g. \tag{37}$$

Finally, for arbitrary $g_0$ we have

$$\mathrm{d}\Sigma_{g_0 g^{-1}} Y_g = \mathrm{d}\Sigma_{g_0 g^{-1}}(\mathrm{d}\Sigma_g Y_e) = \mathrm{d}(\Sigma_{g_0 g^{-1}} \circ \Sigma_g) Y_e = \mathrm{d}\Sigma_{g_0} Y_e = Y_{g_0}, \tag{38}$$

i.e., $Y$ is right-invariant. Because the map $\mathscr{I}$ is analytic the isomorphism $\mathrm{d}\mathscr{I}$ is also analytic. ▼

Th. 1.1 implies that a right-invariant Lie algebra can be represented by means of first-order differential operators; i.e., for $\tilde{X} \in L^L$, we have

$$\tilde{X}_g = a^k(g(t)) \frac{\partial}{\partial t^k}, \tag{39}$$

where $t^k(g)$ are coordinates of an element $g \in G$.

Similarly an element $\tilde{\tilde{Y}}$ in $L^R$ associated with the right translations (that is, in the left-invariant Lie algebra) has the form

$$\tilde{\tilde{Y}}_g = b^k(g(t)) \frac{\partial}{\partial t^k}. \tag{40}$$

By virtue of proposition 1, all functions $a^k(g)$ and $b^k(g)$, $k = 1, 2, \ldots, \dim G$, are analytic on $G$. Moreover, by virtue of proposition 2, the function $b^k(g)$ can be expressed in terms of $a^k(g)$, or, vice versa, by means of an analytic transformation determined by $\mathrm{d}\mathscr{I}$.

### F. *Identities in Lie Algebras*

Let $L$ be a real Lie algebra, and $G$ the corresponding connected and simply connected real Lie group. In this section we derive two identities in the Lie algebra $L$ involving elements, their transforms under inner automorphisms defined by elements of $G$, and derivatives of local coordinates of the second kind relative to some parameter.

As it is well known, the exponential $e^x$, for $x$ in some open neighbourhood $V$ of the origin in $L$, will realize some open neighbourhood of the identity in $G$. If $x_1, \ldots, x_r$ is a basis of $L$, $V$ can be chosen small enough so that for any $x \in V$ we shall have $e^x = e^{t_1 x_1} \ldots e^{t_r x_r}$, the coordinates of the second kind, $e^x \to (t_1, \ldots, t_r)$ being a local chart in $G$ over some neighbourhood $W$ of the identity, contained in $e^V$. Furthermore, we can suppose $W$ to be convex, namely that if $e^y$ and $e^x e^y$ belong to $W$, then $e^{tx}e^y \in W$, when $0 \leqslant t \leqslant 1$; this follows from the fact that (cf. Helgason 1962, p. 34 and 92–94) the translates $t \to e^{tx}e^y$ of the one-parameter groups are the geodesics of the Cartan–Schouten connection.

Thus if $e^x \in W$ and $0 \leqslant t \leqslant 1$, we have $e^{tx} = e^{t_1 x_1} \ldots e^{t_r x_r}$, where the coordinates $t_i$ are analytic in $t$.

To simplify the notations, we suppose (using e.g. Ado theorem) that the Lie algebra $L$ is realized faithfully as a matrix algebra. Thus, some neighbourhood of the identity in $G$, containing $W$ (and all the products of elements of $W$ needed below), will be realized as a matrix group neighborhood.

Therefore, from the identities

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{tx} = xe^{tx} = e^{tx}x = \left(\frac{\mathrm{d}t_1}{\mathrm{d}t}x_1 + \ldots + \frac{\mathrm{d}t_r}{\mathrm{d}t}e^{t_1x_1} \ldots e^{t_{r-1}x_{r-1}}x_r e^{-t_{r-1}x_{r-1}} \ldots e^{-t_1x_1}\right)e^{tx}$$
$$= e^{tx}\left(e^{-t_rx_r} \ldots e^{-t_2x_2}x_1\frac{\mathrm{d}t_1}{\mathrm{d}t}e^{t_2x_2} \ldots e^{t_rx_r} + \ldots + x_r\frac{\mathrm{d}t_r}{\mathrm{d}t}\right),$$

we find

$$\begin{aligned} x &= \frac{\mathrm{d}t_1}{\mathrm{d}t}x_1 + \ldots + \frac{\mathrm{d}t_r}{\mathrm{d}t}\mathrm{Int}(t_1x_1) \ldots \mathrm{Int}(t_{r-1}x_{r-1})x_r, \\ x &= \mathrm{Int}(-t_rx_r) \ldots \mathrm{Int}(-t_2x_2)x_1\frac{\mathrm{d}t_1}{\mathrm{d}t} + \ldots + x_r\frac{\mathrm{d}t_r}{\mathrm{d}t}, \end{aligned} \tag{41}$$

where $\mathrm{Int}(tx)$ denotes the inner automorphism $\mathrm{Ad}(e^{tx})$ of $L$ defined by $y \to e^{tx}ye^{-tx}$ in any realization.

Moreover, if $x, y \in L$ and $e^y$, $e^xe^y \in W$, then, as we have seen, for $0 \leqslant t \leqslant 1$, $e^{tx}e^y \in W$. We can then write:

$$e^{tx}e^y = e^{\alpha_1x_1} \ldots e^{\alpha_rx_r}, \qquad e^y = e^{\beta_1x_1} \ldots e^{\beta_rx_r}$$

and, from the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((e^{tx}e^y)e^{-y}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(e^{\alpha_1x_1} \ldots e^{\alpha_rx_r}e^{-\beta_rx_r} \ldots e^{-\beta_1x_1})$$
$$= \left(x_1\frac{\mathrm{d}\alpha_1}{\mathrm{d}t} + \ldots + e^{\alpha_1x_1} \ldots e^{\alpha_{r-1}x_{r-1}}x_r\frac{\mathrm{d}\alpha_r}{\mathrm{d}t}e^{-\alpha_{r-1}x_{r-1}} \ldots e^{-\alpha_1x_1}\right)e^{tx}$$

we derive the relation

$$x = \frac{d\alpha_1}{dt} x_1 + \dots + \frac{d\alpha_r}{dt} \mathrm{Int}(\alpha_1 x_1) \dots \mathrm{Int}(\alpha_{r-1} x_{r-1}) x_r . \tag{42}$$

In addition, as is well known, we have for all $x, y \in L$ and $t \in R$,

$$e^{tx} y e^{-tx} = \mathrm{Int}(tx) y = \sum_{n=0}^{\infty} \frac{1}{n!} (\mathrm{ad}(tx))^n y . \tag{43}$$

## § 4. The Direct and Semidirect Products

By means of direct and semidirect products of groups we can construct new groups from given ones and reduce the investigation of some complicated groups to simpler subgroups.

### A. *The Direct Product*

DEFINITION 1. Let $G_1$ and $G_2$ be abstract groups, then the *direct product* $G_1 \otimes G_2$ is the group of all ordered pairs $(g_1, g_2)$, $g_1 \in G_1$, $g_2 \in G_2$, with the multiplication law

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2'). \tag{1}$$

The unit element of $G_1 \times G_2$ is the element $e = (e_1, e_2)$ and the inverse element of $(g_1, g_2)$ is $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-2})$. ▼

The subgroup $\tilde{G}_1$ of $G_1 \times G_2$ consisting of all pairs of the form $(g_1, e_2)$ is an invariant subgroup of $G_1 \times G_2$ isomorphic to $G_1$. In fact, for any $(g_1, e_2)$ and $g_1', g_2') \in G_1 \times G_2$ we have

$$(g_1', g_2')^{-1}(g_1, e_2)(g_1', g_2') = (g_1'^{-1} g_1 g_1', e_2) \in \tilde{G}_1 .$$

This isomorphism is given by the mapping $\varphi$: $g_1 \to (g_1, e_2)$. Similarly $\tilde{G}_2 = \{(e_1, g_2)\}$ is an invariant subgroup of $G_1 \times G_2$ isomorphic to $G_2$ and the following properties of $\tilde{G}_1$ and $\tilde{G}_2$ hold

$$\tilde{G}_1 \cdot \tilde{G}_2 = G_1 \times G_2 , \tag{2}$$

$$\tilde{G}_1 \cap \tilde{G}_2 = (e_1, e_2). \tag{3}$$

Conversely, if a group $G$ contains two invariant subgroups $G_1$ and $G_2$ satisfying

$$G_1 \cdot G_2 = G , \tag{4}$$

$$G_1 \cap G_2 = e , \tag{5}$$

then $G$ decomposes onto a direct product of $G_1$ and $G_2$. It is easy to verify, using the invariance of $G_1$ and $G_2$ in $G$ and eq. (5), that every $g_1 \in G_1$ is commutative with every $g_2 \in G_2$, clearly, by (4) every element $g$ can be represented uniquely as a product $g = g_1 g_2$, $g_1 \in G_1$, $g_2 \in G_2$.

If $G_1$ and $G_2$ are topological groups, then the direct product $G_1 \times G_2$ can also be considered as a topological group. The topology on $G_1 \times G_2$ is the product topology. If $G_1$ and $G_2$ are Lie groups, then $G_1 \times G_2$ can be considered to be a Lie group, whose analytic manifold is the product of analytic manifolds of $G_1$ and $G_2$, respectively.

### B. *The Semidirect Product*

Let $G$ be an abstract group, $G_A$ the group of all automorphisms of $G$, $G_{\tilde{A}}$ a subgroup of $G_A$ and $\Lambda(g)$ the image of $g \in G$ under the analytic automorphism $\Lambda \in G_{\tilde{A}}$ (cf. 1.1.B).

DEFINITION 2. The semidirect product $G \otimes G_{\tilde{A}}$ of $G$ and $G_{\tilde{A}}$ is the group of all ordered pairs $(g, \Lambda)$ with the group multiplication defined by

$$(g, \Lambda)(g', \Lambda') = \big(g\Lambda(g'), \Lambda\Lambda'\big). \tag{6}$$

The unit element of $G \otimes G_{\tilde{A}}$ is $e = (e, I)$ and the inverse element of the pair $(g, \Lambda)$ is the pair

$$(g, \Lambda)^{-1} = \big(\Lambda^{-1}(g^{-1}), \Lambda^{-1}\big). \tag{7}$$

The topology in $G \otimes G_{\tilde{A}}$ is the product topology of the product of spaces $G$ and $G_{\tilde{A}}$. ▼

Semidirect products play a fundamental role in the theory of Lie groups and in representation theory. In fact, as we shall show, an arbitrary Lie group is locally isomorphic to a semidirect product of groups (cf. § 5). Moreover all irreducible unitary representations of an important class of semidirect products of groups can be obtained as induced representations (cf. ch. 17).

THEOREM 1. *The semidirect product* $G \otimes G_{\tilde{A}}$ *with* $G = \{(g, I)\}$ *and* $G_{\tilde{A}} = \{(e, \Lambda)\}$ *has the following properties*

1. *$G$ is a normal subgroup of $G \otimes G_{\tilde{A}}$.*
2. *$G \otimes G_{\tilde{A}}/G$ is isomorphic to $G_{\tilde{A}}$.*
3. *$G \otimes G_{\tilde{A}} = G \cdot G_{\tilde{A}}$ and $G \cap G_{\tilde{A}} = (e, I)$.*

PROOF: *ad* 1. Let $(g', I) \in G$ and $(g, \Lambda) \in G \otimes G_{\tilde{A}}$. We have

$$\begin{aligned}(g, \Lambda)(g', I)(g, \Lambda)^{-1} &= \big(g\Lambda(g'), \Lambda\big)\big(\Lambda^{-1}(g^{-1}), \Lambda^{-1}\big)\\ &= \big(g\Lambda(g')g^{-1}, I\big) \in G,\end{aligned}$$

i.e., $G$ is an invariant subgroup in $G \otimes G_{\tilde{A}}$.

*ad* 2. Note that the set $(G, \Lambda)$ for a fixed $\Lambda \in G_{\tilde{A}}$ represents a coset of $G \otimes G_{\tilde{A}}/G$. Hence the map $\varphi$: $(G, \Lambda) \to (e, \Lambda)$ is the isomorphic map of $G \otimes G_{\tilde{A}}/G$ onto $G_{\tilde{A}}$.

Property 3 follows from the definition of $G$ and $G_{\tilde{A}}$ and eq. (6). ▼

EXAMPLE 1. Let $a$ be a four-vector and $\Lambda$ a homogeneous Lorentz transformation of the four-dimensional Minkowski space. A Poincaré transformation $L = (a, \Lambda)$ is defined by

$$\tilde{x}_\mu = (Lx)_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu. \tag{8}$$

The product $LL' = (a, \Lambda)(a', \Lambda')$ gives

$$\tilde{\tilde{x}}_\varrho = (L\tilde{x})_\varrho = \Lambda_\varrho{}^\mu \Lambda'_\mu{}^\nu x_\nu + \Lambda^\mu{}_\varrho a'_\mu + a_\varrho. \tag{9}$$

Thus the product $(a, \Lambda)(a', \Lambda')$ can be represented by the transformation

$$(a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda\Lambda'), \tag{10}$$

i.e., the composition law for Poincaré transformations is the same as eq. (6) for semidirect products. Therefore the Poincaré group is the semidirect product $T^4 \circledS \mathrm{SO}(3, 1)$ of the four-dimensional translation grup $T^4$ and the homogeneous Lorentz group SO(3, 1). The group SO(3, 1) acts on $T^4$ as a group of automorphisms. Note that $T^4$ is an invariant subgroup of $T^4 \circledS \mathrm{SO}(3, 1)$ and $T^4 \circledS \mathrm{SO}(3, 1)/T^4$ is isomorphic to SO(3, 1).

Eq. (10) can be written as a matrix product if we let

$$(a, \Lambda) = \begin{bmatrix} \Lambda & a \\ 0 & 1 \end{bmatrix}. \tag{11}$$

EXAMPLE 2. Let $R^3$ be the Euclidean space, $R$ a rotation, $\boldsymbol{v}$ and $\boldsymbol{a}$ 3-vectors, and $b$ a real number. A Galileo transformation $g = (b, \boldsymbol{a}, \boldsymbol{v}, R)$ is defined by

$$\begin{aligned} \boldsymbol{x}' &= R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}, \\ t' &= t + b. \end{aligned} \tag{12}$$

This definition implies the composition law for the Galileo group

$$(b', \boldsymbol{a}', \boldsymbol{v}', R')(b, \boldsymbol{a}, \boldsymbol{v}, R) = (b' + b, \boldsymbol{a}' + R'\boldsymbol{a} + b\boldsymbol{v}', \boldsymbol{v}' + R'\boldsymbol{v}, R'R), \tag{13}$$

which is again of the semi-product type and can be written as a matrix product with $(b, \boldsymbol{a}, \boldsymbol{v}, R)$ represented by

$$\begin{bmatrix} R & \boldsymbol{v} & \boldsymbol{a} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}. \tag{14}$$

The inverse of element $g$ is $(b, \boldsymbol{a}, \boldsymbol{v}, R)^{-1} = \big(-b, R^{-1}(\boldsymbol{a} - b\boldsymbol{v}), -R^{-1}\boldsymbol{v}, R^{-1}\big)$. These formulas are obtained from the corresponding Poincaré transformations of the previous example by the limiting procedure ($x_0 \equiv ct$)

$$\begin{aligned} &\Lambda_{0i} \to 0, \ c \to \infty \quad \text{such that } \Lambda_{0i} c \to v_i, \\ \text{but} \\ &a_0 \to \infty, \ c \to \infty \quad \text{such that } a_0/c \to b. \end{aligned} \tag{15}$$

To the limiting procedure (15) there corresponds in the Lie algebra the contraction introduced in ch. 1, § 8, example 2.

## § 5. Levi–Malcev Decomposition

### A. *Solvable, Nilpotent, Simple and Semisimple Lie Groups*

Let $G$ be an abstract group. We associate with each pair of elements $x, y \in G$ an element $q = xyx^{-1}y^{-1}$, which is called the *commutator* of $x$ and $y$. The set $Q$ of all elements of $g \in G$, which can be represented in the form $g = q_1 q_2 \dots q_m$, where each $q_i$ is a commutator of two elements $x_i, y_i \in G$, is called the *commutant* of $G$. The commutant $Q$ is an invariant subgroup in $G$. Indeed, the product of $q = q_1 q_2 \dots q_n$ and $q' = q_1' q_2' \dots q_m'$ is an element of $Q$, and the inverse element $q^{-1} = q_n^{-1}, \dots, q_1^{-1}$ to $q$ is also an element of $Q$. Moreover if $g \in G$ and $q = q_1 q_2 \dots q_n \in Q$, then

$$g^{-1}qg = \prod_{i=1}^{n} gq_i g^{-1} = \prod_{i=1}^{n} gx_i g^{-1} gy_i g^{-1} gx_i^{-1} g^{-1} gy_i^{-1} g^{-1} \in Q,$$

i.e., $Q$ is a normal subgroup of $G$. However in general $Q$ is not a topological subgroup (cf. def. 2.2.2).

Taking the closure of $Q$ in the topology of $G$, we obtain a normal topological subgroup.

The quotient group $G/Q$ is an abelian group. In fact if $x, y \in G$ and $X = xQ$, $Y = yQ$ are any two elements of $G/Q$, then

$$XYX^{-1}Y^{-1} = xQyQx^{-1}Qy^{-1}Q = xyx^{-1}y^{-1}Q = qQ = Q \simeq e \in G/Q,$$

i.e., $G/Q$ is commutative.

Consider now the chain of commutants

$$G = Q_0 \supset Q_1 \supset \dots \supset Q_{n-1} \supset Q_n \supset \dots, \tag{1}$$

where each $Q_n$ is the commutant of $Q_{n-1}$. If for some $m$ we have $Q_m = \{e\}$, then the group $G$ is said to be *solvable*. If $H$ is a subgroup of a solvable group $G$, then the $n$th commutant of $H$, $(Q_H)_n \subset Q_n$. Hence every subgroup of a solvable group is solvable. A solvable group always has a commutative invariant subgroup. Indeed, if $Q_m = \{e\}$, but $Q_{m-1} \neq \{e\}$, then for any pair $x, y \in Q_{m-1}$ we have $xyx^{-1}y^{-1} = e$, i.e., $xy = yx$.

EXAMPLE 1. Let $G$ be the group of motions of the Euclidean plane $R^2$. Every element $g \in G$ can be represented in the form $g = (a, \Lambda)$, where $a$ is an element of the translation group $T^2$ and $\Lambda$ is an element of the rotation group SO(2). The group multiplication is (cf. example 4.1).

$$(a, \Lambda)(a', \Lambda') = (a + \Lambda a', \Lambda\Lambda'), \tag{2}$$

i.e., the group of motions of $R^2$ is the semidirect product $T^2 \circledS \mathrm{SO}(2)$.

The commutator of $x = (a, \Lambda)$ and $y = (a', \Lambda')$ is the element

$$q = (a, \Lambda)(a', \Lambda')(a, \Lambda)^{-1}(a', \Lambda')^{-1} = (a + \Lambda a' - a' - \Lambda' a, I) \in T^2,$$

by eqs. 4(7) and (2). Hence we have $Q_1 = T_2$, $Q_2 = (0, I) = e$, i.e., the group $T^2 \circledS \mathrm{SO}(2)$ is solvable. ▼

THEOREM 1. *Every solvable connected Lie group can be represented as the product*

$$G = T_1 T_2 \ldots T_m$$

*of one-parameter subgroups $T_i$, where the sets*

$$G_k = T_{k+1} T_{k+2} \ldots T_m$$

*for arbitrary $k$, $1 \leqslant k < m$ are normal subgroups in $G$.*

The proof follows directly from th. 1.2.2 and we leave it as an exercise for the reader.

Let $K$ be the set of all elements generated by commutators

$$q = xyx^{-1}y^{-1}, \quad x \in Q, \ y \in G.$$

We readily verify as in the previous case that $K$ is an invariant subgroup of $G$. Consider the sequence of invariant subgroups

$$G = K_0 \supset K_1 \supset K_2 \supset \ldots \supset K_{n-1} \supset K_n \supset \ldots, \tag{3}$$

where each $K_{n+1}$ is a subgroup of $G$ generated by commutators $q = xyx^{-1}y^{-1}$, $x \in K_n$, $y \in G$. If for some $m$, $K_m = \{e\}$, then the group $G$ is said to be *nilpotent*. It follows from the definition that any subgroup of a nilpotent group is nilpotent. Moreover since $K_n \supset Q_n$, $n = 1, 2, \ldots$, then every nilpotent group is solvable.

Every nilpotent group has a nontrivial center. In fact, if $K_m = \{e\}$, but $K_{m-1} \neq \{e\}$, then for any $x \in K_{m-1}$ and $y \in G$: $xyx^{-1}y^{-1} = e$, i.e., $xy = yx$.

A Lie group is said to be nilpotent if it is nilpotent as an abstract group.

A Lie group is said to be *simple* if it has no proper, connected invariant Lie subgroup. We emphasize that a simple Lie group, in contrast to a simple finite group, might contain a *discrete* invariant subgroup in $G$. For example, the group $\mathrm{SU}(n)$ has the discrete cyclic invariant subgroup of order $n$, $Z_n$, generated by the element

$$g = \exp\left[\frac{2\pi i}{n}\right] e, \quad e \in \mathrm{SU}(n). \tag{4}$$

However, all groups $G_i = \mathrm{SU}(n)/Z_i$, where $Z_i$ is a subgroup of $Z_n$ will be considered according to the definition as simple Lie groups.

A Lie group is said to be *semisimple* if it contains no proper invariant connected abelian Lie subgroup.

## B. *Levi–Malcev Decomposition*

The solvable and semisimple groups form two disjoint classes. In fact, every solvable Lie group contains an invariant abelian subgroup, whereas a semisimple one does not. The theory of Lie groups might be reduced in a certain sense to an investigation of the properties of solvable and semisimple groups. In fact, we have

THEOREM 1 (Levi–Malcev theorem). *Every connected Lie group $G$ is locally isomorphic to the semidirect product*

$$N \circledS S, \tag{5}$$

*where $N$ is a connected maximal solvable invariant subgroup of $G$ and $S$ is a connected semisimple subgroup of $G$.*

PROOF: Let $L$ be the Lie algebra of $G$. By virtue of Levi–Malcev th. 1.3.5, $L$ is a semidirect sum $\tilde{N} \oplus\!\!\!\!\supset \tilde{S}$ of the radical $\tilde{N}$ and a semisimple Lie subalgebra $\tilde{S}$. Let $N \circledS S$ denote a connected Lie group, with the Lie algebra $\tilde{N} \oplus\!\!\!\!\supset \tilde{S}$, where $N$ and $S$ are connected subgroups corresponding to $\tilde{N}$ and $\tilde{S}$, respectively, by virtue of th. 3.3. Then $G$ and $N \circledS S$ are locally isomorphic according to th. 3.3.

## § 6. Gauss, Cartan, Iwasawa and Bruhat Global Decompositions

In ch. 1, § 6 we discussed the Gauss, Cartan and Iwasawa decompositions for Lie algebras. We give now the corresponding global decompositions for Lie groups.

### A. *Gauss Decomposition*

DEFINITION 1. A topological group $G$ admits a Gauss decomposition if $G$ contains subgroups $\mathfrak{Z}$, $D$ and $Z$ satisfying the conditions:

1° The sets $\mathfrak{Z}D$ and $DZ$ are solvable connected subgroups in $G$, whose commutants are $\mathfrak{Z}$ and $Z$ respectively.

2° The intersections $\mathfrak{Z} \cap DZ$ and $D \cap Z$ consist of the unit element only and the set $\mathfrak{Z}DZ$ is dense in $G$.

It follows from the first condition, that $D$ is an abelian subgroup and $\mathfrak{Z}$ and $Z$ are solvable and connected. The second condition means that almost every element $g \in G$ has the decomposition in the form

$$g = \zeta\delta z, \quad \zeta \in \mathfrak{Z},\ \delta \in D,\ z \in Z \tag{1}$$

and if such a decomposition exists, then it is unique. An element $g \in G$ is called *regular* if it admits the decomposition (1) and *singular* otherwise.

THEOREM 1. *Every connected semisimple complex Lie group $G$ admits a Gauss decomposition*

$$G = \overline{\mathfrak{Z}DZ}, \tag{2}$$

*where the abelian group $D$ is connected, and the groups $\mathfrak{Z}$ and $Z$ are simply connected and nilpotent: $\mathfrak{Z}D$ and $DZ$ are maximal connected solvable subgroups in $G$. The set of singular points (complementary to $\mathfrak{Z}DZ$) is closed and has a smaller* *dimension than $G$: the components $\zeta$, $\delta$ and $z$ of a regular point $g \in \mathfrak{Z}DZ$ are continuous functions of $g$.*

*Two arbitrary decompositions of this type are connected by an automorphism of $G$.* ▼

(For the proof cf. Želobenko 1963, § 5.)

We illustrate this theorem by the example of the group SL(2, $C$) which is the covering group of the homogeneous Lorentz group.

EXAMPLE 1. Let $G = \mathrm{SL}(2, C)$. Let $\mathfrak{Z}$, $D$ and $Z$ be the subgroups of SL(2, $C$), consisting, respectively, of matrices of the type

$$\zeta = \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}. \tag{3}$$

We denoted for simplicity the matrix and the corresponding complex number by the same letter. The group $D$ is isomorphic to the multiplicative group of complex numbers and each one of the groups $\mathfrak{Z}$ and $Z$ is isomorphic to the additive group of complex numbers. We verify, by comparing the product $\zeta\delta z$, and the matrix

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in \mathrm{SL}(2, C), \quad g_{11}g_{22} - g_{12}g_{21} = 1,$$

that every element $g \in \mathrm{SL}(2, C)$ for which $g_{22} \neq 0$ has the unique decomposition in the form

$$g = \zeta\delta z, \quad \zeta \in \mathfrak{Z},\ \delta \in D,\ z \in Z, \tag{4}$$

where the components $\zeta$, $\delta$ and $z$ are

$$\zeta = \frac{g_{12}}{g_{22}}, \quad \delta = g_{22}, \quad z = \frac{g_{21}}{g_{22}}. \tag{5}$$

Let $S = DZ$; then the commutants are

$$S^{(1)} = Z, \quad \text{and} \quad S^{(2)} = Z^{(1)} = \{e\}.$$

Hence $S$ is a solvable subgroup. Similarly, we verify that $K = \mathfrak{Z}D$ is solvable. Both subgroups are connected, because every element of $S$ or $K$ can be reached continuously from the unit element. The set of matrices of the form (4) is dense in SL(2, $C$), because its complement defined by the condition $g_{22} = 0$, has a smaller dimension than SL(2, $C$). Therefore the decomposition (4) represents the Gauss decomposition for SL(2, $C$). ▼

*Remark:* If we take the abelian subgroup $D$ to consist of diagonal matrices of the form

$$\delta = \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix} \tag{6}$$

then $\mathfrak{Z}DZ$ gives a Gauss decomposition of GL(2, $C$) according to def. 1.

THEOREM 2. *Let $G = \mathrm{SL}(n, C)$ and let $\mathfrak{Z}$, $D$ and $Z$ be the subgroups of $G$ whose elements are of the form*

$$\xi = \begin{bmatrix} 1 & \zeta_{12} & \zeta_{13} & \cdots & \zeta_{1n} \\ & 1 & \zeta_{23} & \cdots & \zeta_{2n} \\ & & \ddots & & \vdots \\ & 0 & & & \zeta_{n-1,n} \\ & & & & 1 \end{bmatrix}, \qquad \delta = \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & 0 \\ & & \ddots & \\ 0 & & & \delta_n \end{bmatrix},$$

$$z = \begin{bmatrix} 1 & & & \\ z_{21} & 1 & & 0 \\ & & \ddots & \\ z_{n1} & z_{n2} & \cdots & 1 \end{bmatrix}. \tag{7}$$

*Then the Gauss decomposition for* $\mathrm{SL}(n, C)$ *can be written either in the form*

$$G = \overline{\mathfrak{Z}DZ} \tag{8}$$

*or*

$$G = \overline{ZD\mathfrak{Z}}. \tag{9}$$

PROOF: We first show that for almost every element $g \in \mathrm{SL}(n, C)$ there exists an element $z \in Z$ such that $gz \in K$ where $K$ is a subgroup of all upper triangular matrices with determinant one. In fact, it follows from the definition of $K$ that all matrix elements of $gz$ under the main diagonal should be equal to zero. Moreover $z_{pq} = 0$ for $p < q$ and $z_{pp} = 1$. Consequently the condition $gz \in K$ is equivalent to the set of linear equations

$$\sum_{s=q}^{n} g_{ps} z_{sq} = 0 \quad \text{for } p > q,$$

i.e.,

$$\sum_{s=q+1}^{n} g_{ps} z_{sq} = -g_{pq}, \quad p = q+1, \ldots, n. \tag{10}$$

For fixed $q$ the determinant of eqs. (10) coincides with the minor $g_{p+1}$ of the form

$$g_s = \begin{vmatrix} g_{ss} & \cdots & g_{sn} \\ \cdots & \cdots & \cdots \\ g_{ns} & \cdots & g_{nn} \end{vmatrix}, \quad s = p+1 = 2, 3, \ldots, n. \tag{11}$$

Hence if $g_{p+1}$ is not equal to zero, eqs. (10) have a solution relative to $z_{sq}$. Therefore almost all elements $g \in \mathrm{SL}(n, C)$ can be represented in the form $g = kz$. Now every element $k \in K$ can be uniquely represented in the form

$$k = \zeta\delta, \quad \zeta \in \mathfrak{Z} \text{ and } \delta \in D. \tag{12}$$

Indeed the equality (12) means that

$$k_{pq} = \zeta_{pq}\delta_q \quad \text{(no summation)}. \tag{13}$$

In particular for $p = q$, $\zeta_{pp} = 1$; hence

$$\delta_p = k_{pp}, \quad \zeta_{pq} = \frac{k_{pq}}{k_{pp}}. \tag{14}$$

Consequently every $g \in \mathrm{SL}(n, C)$ for which the minors (11) do not vanish has the decomposition in the form

$$g = \zeta\delta z, \quad \zeta \in \mathfrak{Z}, \ \delta \in D, \ z \in Z. \tag{15}$$

The set of singular points for which at least one of minors (11) does not vanish has a dimension smaller than that of $\mathrm{SL}(n, C)$; therefore its complementary set $\mathfrak{Z}DZ$ is dense in $G$. Thus the decomposition (8) follows. One verifies easily that the subgroups $\mathfrak{Z}$, $D$, $Z$, $\mathfrak{Z}D$ and $DZ$ have all properties stated in th. 1. Hence (8) gives the desired Gauss decomposition for $\mathrm{SL}(n, C)$.

The decomposition (9) is derived in a similar fashion. ▼

*Remark*: The explicit form of continuous functions $\zeta(g)$, $\delta(g)$ and $z(g)$ is given in exercise 11.6.1 and 6.2. ▼

The compact semisimple Lie groups do not admit a Gauss decomposition, because they do not possess solvable subgroups. However, for the noncompact semisimple real Lie groups there exist some analogues of the Gauss decomposition. Indeed we have

THEOREM 3. *Every connected semisimple real Lie group $G$ admits a decomposition*

$$G = \overline{\mathfrak{Z}DZ}$$

*where $D$ is the direct product*

$$D = A \otimes K$$

*of a simply-connected abelian group $A$ and a connected semisimple compact group $K$, whereas groups $\mathfrak{Z}$ and $Z$ are nilpotent and simply-connected.*

*The set of singular points (complementary to $\mathfrak{Z}DZ$) is closed and has a smaller dimension than $G$: in the decomposition of a regular point $g = \zeta\delta z$ all components $\zeta$, $\delta$ and $z$ are continuous functions of $g$.* ▼

(For the proof cf. Želobenko 1963, § 6.)

EXAMPLE 2. Let $G = U(p, q)$. Then $\mathfrak{Z}$ and $Z$ are intersections of $U(p, q)$ with subgroups $\mathfrak{Z}$ and $Z$ of $\mathrm{GL}(n, C)$, respectively. The abelian subgroup $A$ is the product of $p$-dimenstional toroid with the $p$-dimensional Euclidean space, whereas $K = U(q-p)$.

## B. *The Cartan Decomposition*

Let $L$ be a real semisimple Lie algebra and let

$$L = K \dot{+} P \tag{16}$$

be its Cartan decomposition (cf. th. 1.6.9). There exists a global version of this decomposition which is described by the following

THEOREM 4. *Let G be a connected semisimple Lie group with a finite center. The Lie algebra L of G has the Cartan decomposition* (16). *Let $\mathscr{K}$ be the connected subgroup of G, whose Lie algebra is K, and let $\mathscr{P}$ be the image of the vector space P under the exponential map. Then*

$$G = \overline{\mathscr{P}\mathscr{K}}. \tag{17}$$

(For the proof cf. Cartan 1929.)

EXAMPLE 3. Let $G = \mathrm{SL}(n, R)$. The Cartan decomposition of the Lie algebra $L$ of $G$ is the decomposition of an arbitrary traceless matrix onto skew-symmetric and traceless symmetric parts (cf. example 1.6.2). The connected subgroup $\mathscr{K}$ of $G$ whose Lie algebra is $K$, consists of orthogonal matrices. On the other hand the set $\mathscr{P}$ is the set of unimodular hermitian matrices. Hence the global decomposition (17) is, in the present case, the well-known polar decomposition of a unimodular matrix onto the product of its hermitian and orthogonal parts.

### C. *The Iwasawa Decomposition*

Let $L$ again be a real semisimple Lie algebra and let

$$L = K \dot{+} H_p \dot{+} N_0 \tag{18}$$

be its Iwasawa decomposition (cf. th. 1.6.3). The global version of the decomposition (18) is described in the following theorem.

THEOREM 5. *Let G be a connected group with the Lie algebra L and let $\mathscr{K}$, $\mathscr{A}_p$ and $\mathscr{N}$ be the connected subgroups of G corresponding to the subalgebras K, $H_p$ and $N_0$ respectively. Then*

$$G = \mathscr{K}\mathscr{A}_p\mathscr{N}, \tag{19}$$

*and every element $g \in G$ has a unique decomposition as a product of elements of $\mathscr{K}$, $\mathscr{A}_p$ and $\mathscr{N}$. The groups $\mathscr{A}_p$ and $\mathscr{N}$ are simply connected.* ▼

(For the proof cf. Helgason 1962, ch. VI, § 5.)

EXAMPLE 3. Let $L = \mathrm{sl}(n, R)$. The Iwasawa decomposition (18) for $\mathrm{sl}(n, R)$ consists on the decomposition of an arbitrary traceless matrix onto skew-symmetric, diagonal and upper triangular matrices with zeros on the diagonal (cf. example 1.6.3) Thus the group $\mathscr{K}$ is the orthogonal group $\mathrm{SO}(n)$, the group $\mathscr{A}_p$ is the abelian group and the group $\mathscr{N}$ is the nilpotent group consisting of upper triangular matrices with the one's on the main diagonal. Consequently the global Iwasawa decomposition (19) for the group $\mathrm{SL}(n, R)$ consists of the decomposition of an arbitrary unimodular matrix onto the product of orthogonal, diagonal and upper triangular matrices with ones along the diagonal. ▼

D. *The Bruhat Decomposition*

Let $\mathscr{G} = \mathscr{K}\mathscr{A}\mathscr{N}$ be the Iwasawa decomposition of a connected semisimple Lie group $\mathscr{G}$ with a finite center: let $\mathscr{M}$ be the centralizer of the Lie algebra $A$ of $\mathscr{A}$ in $\mathscr{K}$, i.e. $\mathscr{M} = \{k \in \mathscr{K}: \mathrm{Ad}_k X = X \text{ for each } X \text{ in } A\}$. Set $\mathscr{P} = \mathscr{M}\mathscr{A}\mathscr{N}$. Since both $\mathscr{A}$ and $\mathscr{M}$ normalize $\mathscr{N}$, $\mathscr{P}$ is a closed subgroup of $G$.

The subgroup $\mathscr{P}$ is called *minimal parabolic subgroup* of $\mathscr{G}$.

Let $L$ and $A$ denote the Lie algebras of $\mathscr{G}$ and $\mathscr{A}$, respectively, and let $W$ be the Weyl group of the pair $(L, A)$. Let $\mathscr{M}^*$ be the normalizer of $A$ in $\mathscr{K}$, i.e. $\mathscr{M}^* = \{k \in \mathscr{K}: \mathrm{Ad}_k A \subset A\}$. It is obvious that $\mathscr{M}$ is a normal subgroup of $\mathscr{M}^*$. The Weyl group $W$ can be identified with the quotient $\mathscr{M}^*/\mathscr{M}$.

Let $m^*_w$ be any element of $\mathscr{M}^*$ belonging to the coset associated with $w$. Denote by $\mathscr{P} w \mathscr{P}$ the double coset $\mathscr{P} m^*_w \mathscr{P}$. The following lemma gives the so-called *Bruhat decomposition* of $\mathscr{G}$.

LEMMA 5. *The mapping*

$$w \to \mathscr{P} w \mathscr{P}, \quad w \in W$$

*is a one-to-one mapping of $W$ onto the set of double cosets $\mathscr{P} x \mathscr{P}$, $x \in \mathscr{G}$, i.e.*

$$\mathscr{G} = \bigcup_{w \in W} \mathscr{P} w \mathscr{P} \quad \textit{(disjoint sum)}.$$

(For the proof cf. Bruhat 1956).

EXAMPLE 4. Let $\mathscr{G} = \mathrm{SL}(2, R)$. Then

$$\mathscr{K} = \left\{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}, \ \varphi \in (0, 2\pi)\right\}, \quad \mathscr{A} = \left\{\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}, \ a \in R^+\right\},$$

$$\mathscr{N} = \left\{\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \ c \in R\right\}.$$

One readily verifies that the centralizer $\mathscr{M}$ of $A$ in $\mathscr{K}$ consists of two elements $\mathscr{M} = \{e, -e\}$, $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The minimal parabolic subgroups $\mathscr{P}_x$ are

$$P_x = \left\{\pm x \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} x^{-1}, \ x \in \mathrm{SL}(2, R), \ a \in R^+, \ b \in R\right\}.$$

Using the definition of $\mathscr{M}^*$ one readily verifies that the condition $\mathrm{Ad}_k A \subset A$ implies that $\varphi = \frac{n}{2}\pi$. This implies that $\mathscr{M}^*$ is the three-element group

$$\mathscr{M}^* = \left\{e, -e, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\},$$

Consequently,

$$W = \mathscr{M}^*/\mathscr{M} = \left\{e, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}.$$

Therefore, $\mathrm{SL}(2, R)$ can be represented as the disjoint sum of two double cosets.

## § 7. Classification of Simple Lie Groups

The Killing–Cartan classification of simple Lie algebras, by virtue of th. 3.3, leads to the classification of the corresponding simple Lie groups. The explicit form of a simple Lie group corresponding to a simple complex or real Lie algebra $L$ can be easily obtained by exponentiating the explicit defining representation of $L$ given in ch. 1.5. For example, the Lie algebra $\mathrm{sl}(n, C)$ was realized as the set of all $n \times n$-complex traceless matrices. Hence the group $\mathrm{SL}(n, C)$ consists of all elements

$$x = \mathrm{e}^{X}, \qquad X \in \mathrm{sl}(n, C), \tag{1}$$

which by virtue of the identity $\det \mathrm{e}^{X} = \mathrm{e}^{\mathrm{Tr}\, X}$ is the set of all $n \times n$-unimodular matrices. Similarly, one calculates an explicit realization of all other simple Lie groups corresponding to the explicit realization in ch. 1.5 of associated Lie algebras which we now list:

A. Groups associated with algebras $A_{n-1}$:

$$\begin{tikzcd}[row sep=tiny]
 & \mathrm{SU}(n) \\
 & \mathrm{SL}(n, R) \\
\mathrm{SL}(n, C) \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \mathrm{SU}(p, q), \\
 & \mathrm{SU}^*(2n) \\
 & \mathrm{SL}(n, C)^R
\end{tikzcd} \qquad p+q = n,\ p \geqslant q. \tag{2}$$

(i) $\mathrm{SU}(p, q)$, $p+q = n$, $p \geqslant q$ is the group of all matrices in $\mathrm{SL}(n, C)$ which leaves the quadratic form in $C^n$

$$z_1 \bar{z}_1 + \dots + z_p \bar{z}_p - z_{p+1} \bar{z}_{p+1} - \dots - z_n \bar{z}_n \tag{3}$$

invariant. For $q = 0$, we obtain the unitary group $\mathrm{SU}(n)$ of unimodular matrices. The remaining groups $\mathrm{SU}(p, q)$, $q \neq 0$, may be called *pseudo-unitary groups.*

(ii) $\mathrm{SL}(n, R)$ is the group of all real matrices with determinant one.

(iii) $\mathrm{SU}^*(2n)$ (denoted also $Q_{2n}$) is the group of all matrices in $\mathrm{SL}(2n, C)$ which commute with the transformation $\sigma$ in $C^{2n}$ given by

$$\sigma\colon\ (z_1, \dots, z_{2n}) \to (\bar{z}_{n+1}, \dots, \bar{z}_{2n}, -\bar{z}_1, \dots, -\bar{z}_n). \tag{4}$$

(iv) $\mathrm{SL}(n, C)^R$ is the group $\mathrm{SL}(n, C)$ considered as a real Lie group.

B. Groups associated with algebras $B_n$:

$$\begin{tikzcd}[row sep=tiny]
 & \mathrm{SO}(2n+1) \\
\mathrm{SO}(2n+1, C) \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{SO}(p, q), \\
 & \mathrm{SO}(2n+1, C)^R
\end{tikzcd} \qquad p+q = 2n+1,\ p \geqslant q. \tag{5}$$

(i) $SO(2n+1, C)$ is the group of all matrices in $SL(2n+1, C)$ which conserve the quadratic form in $C^{2n+1}$:

$$z_1^2 + \ldots + z_{2n+1}^2. \tag{6}$$

(ii) $SO(p, q), p+q = 2n+1, p \geqslant q$, is the group of all matrices in $SL(2n+1, R)$ which conserve the quadratic form

$$x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{2n+1}^2. \tag{7}$$

For $q = 0$ we obtain the compact orthogonal group $SO(2n+1)$. The remaining groups $SO(p, q)$, $q > 0$, may be called pseudo-orthogonal.

(iii) $SO(2n+1, C)^R$ is the group $SO(2n+1, C)$ considered as a real Lie group.

C. Groups associated with algebras $C_n$:

$$\mathrm{Sp}(n, C) \begin{cases} \mathrm{Sp}(n) \\ \mathrm{Sp}(p, q) \\ \mathrm{Sp}(n, R), \qquad p+q = n,\ p \geqslant q. \\ \mathrm{Sp}(n, C)^R \end{cases} \tag{8}$$

(i) $\mathrm{Sp}(n, C)$ is the group of all matrices in $GL(2n, C)$ which conserve the exterior form in $C^{2n}$ *

$$z_1 z_{2n}' - z_{2n} z_1' + \ldots + z_n z_{n+1}' - z_{n+1} z_n'. \tag{9}$$

(ii) $\mathrm{Sp}(p, q)$ is the group of all matrices in $\mathrm{Sp}(n, C)$ which conserve the hermitian form in $C^{2n}$

$$z^T \eta_{pq} \bar{z}, \tag{10}$$

where

$$\eta_{pq} = \begin{bmatrix} -I_p & & & 0 \\ & I_q & & \\ & & -I_p & \\ 0 & & & I_q \end{bmatrix}. \tag{11}$$

For $q = 0$ we obtain the compact symplectic group $\mathrm{Sp}(n)$. It is evident from eq. (10) that

$$\mathrm{Sp}(n) = \mathrm{Sp}(n, C) \cap U(2n),$$

and

$$\mathrm{Sp}(p, q) = \mathrm{Sp}(n, C) \cap U(2p, 2q).$$

(iii) $\mathrm{Sp}(n, R)$ is the group of all matrices in $GL(2n, R)$ which conserve the exterior form in $R^{2n}$:

$$x_1 x_{2n}' - x_{2n} x_1' + \ldots + x_n x_{n+1}' - x_{n+1} x_n'. \tag{12}$$

(iv) $\mathrm{Sp}(n, C)^R$ is the group $\mathrm{Sp}(n, C)$ considered as a real Lie group.

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* The exterior form is $(x \wedge y)^{ij} = \frac{1}{2}(x^i y^j - x^j y^i)$.

D. Groups associated with algebras $D_n$:

$$\mathrm{SO}(2n, C) \begin{cases} \mathrm{SO}(2n) \\ \mathrm{SO}(p, q) \\ \mathrm{SO}^*(2n) \\ \mathrm{SO}(2n, C)^R \end{cases} \quad p+q = 2n,\ p \geqslant q. \tag{13}$$

(i) The definitions of $\mathrm{SO}(2n, C)$, $\mathrm{SO}(p, q)$, $p+q = 2n$, $p \geqslant q$, $\mathrm{SO}(2n, C)^R$ groups follow from definitions B(i), B(ii) and B(iii) by replacing index $2n+1$ by $2n$ in corresponding formulas.

(ii) The group $\mathrm{SO}^*(2n)$ is the group of all matrices in $\mathrm{SO}(2n, C)$ which conserve in $C^{2n}$ the skew-hermitian form

$$-z_1\bar{z}_{n+1}+z_{n+1}\bar{z}_1-z_2\bar{z}_{n+2}+z_{n+2}\bar{z}_2- \ldots -z_n\bar{z}_{2n}+z_{2n}\bar{z}_n. \tag{14}$$

E. *Connectedness of Classical Lie Groups*

We show in ch. 5 that if a Lie group $G$ is $n$-connected, then there are representations of $G$ which are $n$-valued. The following theorem gives a description of the connectedness-property of classical Lie groups.

THEOREM 1. (a) *The groups* $\mathrm{GL}(n, C)$, $\mathrm{SL}(n, C)$, $\mathrm{SL}(n, R)$, $\mathrm{SU}(p, q)$, $\mathrm{SU}^*(2n)$, $\mathrm{SU}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n, C)$, $\mathrm{SO}(n)$, $\mathrm{SO}^*(2n)$, $\mathrm{Sp}(n, C)$, $\mathrm{Sp}(n)$, $\mathrm{Sp}(n, R)$, $\mathrm{Sp}(p, q)$ *are all connected.*

(b) *The groups* $\mathrm{SL}(n, C)$ *and* $\mathrm{SU}(n)$ *are simply-connected.*

(c) *The groups* $\mathrm{GL}(n, R)$ *and* $\mathrm{SO}(p, q)$ $(0 < p < p+q)$ *have two connected components.* ▼

(For the proof cf. Helgason 1962, IX, § 4, and Želobenko 1962.)

The following table gives the description of the center $Z(G)$ of the universal covering group $G$ of the compact simple Lie groups:

Table 1

| $G$ | $Z(G)$ | $\dim G$ |
|---|---|---|
| $\mathrm{SU}(n)$ | $Z_n$ | $n^2-1$ |
| $\mathrm{SO}(2n+1)$ | $Z_2$ | $n(2n+1)$ |
| $\mathrm{Sp}(n)$ | $Z_2$ | $n(2n+1)$ |
| $\mathrm{SO}(2n)$ | $Z_4$ if $n =$ odd<br>$Z_2 \times Z_2$ if $n =$ even | $n(2n-1)$ |

## § 8. Structure of Compact Lie Groups

We show here the remarkable result that any compact Lie group is the direct product of its center and finite number of compact simple subgroups.

We defined in 1.2.D that a Lie algebra $L$ is compact if there exists in $L$ a positive definite quadratic form $(\cdot, \cdot)$ satisfying the condition

$$([X, Y], Z)+(Y, [X, Z]) = 0. \tag{1}$$

We now show

PROPOSITION 1. *A Lie algebra $L$ of a compact Lie group $G$ is compact.*

PROOF: Let $(X, X)$ be any positive definite quadratic form on $L$ (e.g., $(X, X) = \sum x_i^2$, where $x_i$ are the coordinates of $X$ in a basis).

Set $\varphi_g(X) = (l_g X, l_g X)$, where $l_g X$ denotes the action of the adjoint group in $L$ given by eq. 3.3(29). For fixed $g \in G$, $\varphi_g(X)$ considered as a function of a vector $X \in L$ is a positive definite quadratic form, while for fixed $X$, $\varphi_g(X)$ is a continuous positive function on $G$. Because $G$ is compact, the new bilinear form defined by

$$(X, X)' = \int_G \varphi_g(X) \mathrm{d}g$$

is a positive definite quadratic form on $L$. For an arbitrary $h \in G$ by virtue of invariance of the Haar measure, we have

$$(l_h X, l_h X)' = \int_G (l_{hg} X, l_{hg} X) \mathrm{d}g = \int_G (l_g X, l_g X) \mathrm{d}g = (X, X)', \tag{2}$$

i.e., $(\cdot, \cdot)'$ is invariant relative to the action of the adjoint group. Eq. (1) results by taking in eq. (2) one-parameter subgroups $h(t_i)$, $i = 1, 2, \ldots, \dim G$, of the adjoint group and differentiating. ▼

We now prove the main theorem.

THEOREM 2. *A compact connected Lie group $G$ is a direct product of its connected center $G_0$ and of its simple compact connected Lie subgroups.*

PROOF: Let $L$ be the Lie algebra of $G$. By virtue of proposition 1, $L$ is compact. Hence by th. 1.3.2 we conclude that

$$L = N \oplus S_1 \oplus S_2 \oplus \ldots \oplus S_n, \tag{3}$$

where $N$ is the center of $L$ and $S_k$, $k = 1, 2, \ldots, n$, are simple ideals of $L$. Consequently, by virtue of th. 3.3 we obtain

$$G = G_0 \times G_1 \times G_2 \times \ldots \times G_n, \tag{4}$$

where $G_0$ is the connected center of $G$ and $G_k$, $k = 1, 2, \ldots, n$, are simple connected Lie subgroups of $G$.

## § 9. Invariant Metric and Invariant Measure on Lie Groups

### *A. Invariant Metric*

We know by the Birkhoff–Kakutani theorem 2.4.3 that every Lie group admits a right invariant metric. We shall now explicitly construct this metric for an arbitrary matrix Lie group.

PROPOSITION 1. *Let $G$ be a matrix Lie group. Let* $\mathrm{d}g$ *be the matrix consisting of differentials of all matrix elements of $g \in G$ and let $w(g, \mathrm{d}g)$ be a differential form on $G$ given by the formula $w(g, \mathrm{d}g) = \mathrm{d}gg^{-1}$. Then*

$$\mathrm{d}s^2 = \mathrm{Tr}\, ww^T = \sum_{i,j} w_{ij}^2 \tag{1}$$

*is a right invariant metric on $G$.*

PROOF: The form $w(g, \mathrm{d}g)$ is right invariant on $G$. Indeed $w(gh, \mathrm{d}(gh)) = \mathrm{d}ghh^{-1}g^{-1} = w(g, \mathrm{d}g)$. Hence $\mathrm{d}s^2$ is right invariant and positive. ▼

Let now $G$ be a simple Lie group and let $c_{kl}{}^m$ be the structure constants of the Lie algebra $L$ of $G$. The map $g \to g_0 g g_0^{-1}$ induces the automorphism 3(29) of the Lie algebra $L$: consequently the structure constants and the Cartan metric tensor

$$g_{ij} = c_{il}{}^m c_{jm}{}^l \tag{2}$$

are two-sided invariant. Hence, the metric

$$\mathrm{d}s^2(t) = g_{ij}\,\mathrm{d}t^i \mathrm{d}t^j \tag{3}$$

is also two-sided invariant. If $G$ is compact, the metric tensor $g_{ij}$ is positive definite; otherwise, it is indefinite. Hence every simple Lie group is either a Riemannian or pseudo-Riemannian space.

Now if $q$ and $h$ are arbitrary elements of $G$ then we define a distance $d(g, h)$ by the formula

$$d(g, h) \equiv \inf_{\gamma} \int_{\gamma} \mathrm{d}s, \tag{4}$$

where inf is taken with respect to all continuous curves connecting $g$ and $h$. Clearly the distance (4) has the same invariance properties as the metric $\mathrm{d}s$.

EXAMPLE 1. Let $G = \mathrm{SL}(n, R)$. The invariant Cartan metric tensor for $\mathrm{SL}(n, R)$ has the form 1.2(14)

$$g_{sm,s'm'} = 2n\delta_{sm'}\delta_{ms'}. \tag{5}$$

By virtue of example 2.1 the coordinates $t^{ij}$ of the element

$$x = \{x^{ij}\}_{i,j=1}^n \in \mathrm{GL}(n, R) \tag{6}$$

are matrix elements $x^{ij}$. Hence by virtue of eqs. (1) and (6) the invariant metric has the form

$$\mathrm{d}s^2 = g_{sm,s'm'}\mathrm{d}x^{sm}\mathrm{d}x^{s'm'} = 2n\mathrm{Tr}(\mathrm{d}x)^2, \tag{7}$$

where $\mathrm{d}x$ is the matrix $[\mathrm{d}x^{ij}]$. This metric is two-sided invariant.

The distance between two arbitrary points is then obtained by inserting (7) into (4).

B. *Invariant Measure*

We have shown in ch. 3, § 3, the existence, on an arbitrary locally compact topological group $G$ of a left- or right-invariant Haar measure $d\mu(x)$. This implies in particular that all Lie groups possess left- or right-invariant Haar measures.

EXAMPLE 2. Let $G$ be the three-dimensional group of triangular matrices

$$g(\alpha, \beta, \gamma) = \begin{bmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \equiv (\alpha, \beta, \gamma), \qquad \alpha, \beta, \gamma \in R^1.$$

This group is called the *Weyl group*. The group space of $G$ is isomorphic with $R^3$. The multiplication law in $G$ is given by the following formula

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha+\alpha', \beta+\beta', \gamma+\gamma'+\alpha\beta').$$

Hence the Euclidean measure on $R^3$ given by

$$dg(\alpha, \beta, \gamma) = d\alpha d\beta d\gamma$$

is both left- and right-invariant. ▼

Other examples of invariant measures for the specific Lie groups are given in exercises.

## § 10. Comments and Supplements

A. *Exponential Mapping*

We discuss here the properties of one-parameter subgroups $g(t)$ in $G$ obtained by exponentiation of elements $X$ of the Lie algebra $L$ of $G$. We first elaborate this problem for the matrix Lie algebras.

Let $X$ be an arbitrary $n \times n$-matrix $X = [X_{ij}]$. Let $\mu \equiv \max_{ij} |X_{ij}|$. Then for the matrix elements $(X^k)_{ij}$ of the $k$th power, $X^k$ $(0 \leqslant k < \infty)$ of $X$, we have

$$|(X^k)_{ij}| \leqslant (n\mu)^k. \tag{1}$$

Indeed, eq. (1) is true for $k = 0$. Assuming that (1) is true for some integer $k \geqslant 0$, we obtain

$$|(X^{k+1})_{ij}| = |(X^k)_{il} X_{lj}| \leqslant n(n\mu)^k \mu = (n\mu)^{k+1}.$$

Hence by induction, eq. (1) is valid for an arbitrary $k$. Now set

$$\theta(X) = \exp X \equiv 1 + \frac{X}{1} + \frac{X^2}{2!} + \dots + \frac{X^k}{k!} + \dots \tag{2}$$

By virtue of eq. (1), for fixed $i, j$, every series

$$\sum \frac{1}{k!} (X^k)_{ij}$$

is majorized by the series $\sum \frac{1}{k!}\mu^k$; hence, it is absolutely convergent. Consequently for a matrix $X = \{X_{ij}\}$ satisfying the condition $|X_{ij}| < \infty$ the exponential $\exp X$ always exists. By virtue of Jacobi equality

$$\det \exp X = \exp \operatorname{Tr} X \tag{3}$$

we conclude that the exponential of any matrix is a regular matrix.

Now every Lie algebra, by virtue of Ado's theorem, has a faithful representation given by a finite-dimensional matrices. Let $X_1, \ldots, X_n$ be a basis in this matrix algebra. Then the map

$$(t_1X_1 + \ldots + t_nX_n) \to \exp(t_1X_1 + \ldots + t_nX_n) \tag{4}$$

provides a map of a neighborhood of 0 in $L$ into a neighborhood of the identity $e$ in $G$.

Clearly, by virtue of eq. (2), we have

$$\exp(t+s)X = \exp tX \exp tY \tag{5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp tX = X \exp tX.$$

Hence $X$ is a tangent vector to a curve $\exp tX$ at $t = 0$. Eq. (1) implies that the map $t \to \exp tX$ of $R$ into $G$ is analytic.

In some cases the map $X \to \exp X$, $X \in L$, covers the whole group $G$ (cf. exercises). However, in other cases this is not so. For instance a diagonal matrix in $GL(n, R)$ with negative matrix elements cannot be represented as an exponential of any real matrix.

In applications it is useful to have an abstract formulation of the concept of exponential maps. This is given by the following theorem.

THEOREM 1. *Let $G$ be a Lie group and $L$ its Lie algebra. Then*

(i) *For every $X \in L$ there exists a unique analytic homomorphism $\theta(t) \equiv \exp tX$ of $R$ into $G$ such that*

$$\exp(t+s)X = \exp tX \exp sX, \tag{6}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\exp tX\right|_{t=0} = X, \tag{7}$$

$$\exp 0X = I. \tag{8}$$

(ii) *For $X, Y \in L$, we have*

$$\exp tX \exp tY = \exp\left\{t(X+Y) + \frac{t^2}{2}[X, Y] + O(t^3)\right\}, \tag{9}$$

$$\exp(-tX)\exp(-tY)\exp tX \exp tY = \exp\{t^2[X, Y] + O(t^3)\}, \tag{10}$$

$$\exp tX \exp tY \exp(-tX) = \exp\{tY + t^2[X, Y] + O(t^3)\}. \tag{11}$$

*In each case $O(t^3)$ denotes a vector in $L$ with the following property: there exists an $\varepsilon > 0$ such that $t^{-3}O(t^3)$ is bounded and analytic for $|t| < \varepsilon$.*

(iii) *There exists an open neighborhood $N_0$ of 0 in $L$ and an open neighborhood $V_e$ of identity $e$ in $G$ such that the map* exp *is an analytic diffeomorphism of $N_0$ onto $V_e$.* ▼

(For the proof cf. Helgason 1962, ch. II, § 1.)

Let $X_1, \ldots, X_n$ be a basis in $L$. The mapping

$$\exp(t_1 X_1 + \ldots + t_n X_n) \to (t_1, \ldots, t_n) \tag{12}$$

of $V_e$ onto $N_0$ is a coordinate system on $V_e$, the so-called *canonical coordinate system.*

B. *Taylor's Expansion*

Let $G$ be a Lie group and $L$ its Lie algebra. Let $\tilde{X}$ and $\tilde{\tilde{X}}$ be left- and right-invariant vector fields, respectively, given by 3(39) and 3(40), corresponding to an element $X \in L$. Then we have

THEOREM 2. *Let $f$ be an analytic function on $G$. Then for $0 \leqslant t \leqslant 1$ we have*

$$f(g\exp tX) = \sum_{k=0}^{\infty} \frac{t^k}{k!} [\tilde{X}^k f](g), \tag{13}$$

$$f(\exp tXg) = \sum_{k=0}^{\infty} \frac{t^k}{k!} [\tilde{\tilde{X}} f](g). \tag{14}$$

(For the proof cf. Helgason 1962, ch. II, § 1.)

C. *Levi–Malcev Theorem for Groups*

We state an extended version of th. 1 of sec. 5.

THEOREM 3 (the Levi–Malcev theorem). *Let $G$ be a connected Lie group, $L = N \oplus S$, the Levi–Malcev decomposition of its algebra and $\mathscr{N}$ and $\mathscr{S}$ analytic subgroups associated with $N$ and $G$, respectively. Then*

$$G = \mathscr{N} \otimes \mathscr{S}, \tag{15}$$

*where $\mathscr{N}$ is the invariant subgroup in $G$ and $\mathscr{S}$ is the maximal semisimple, connected subgroup in $G$.*

*If $G$ is simply-connected, then the subgroups $\mathscr{N}$ and $\mathscr{S}$ are simply-connected and for any $g \in G$ the decomposition $g = ns$, where $n \in \mathscr{N}$ and $s \in \mathscr{S}$ is unique.*

(For the proof cf. Malcev 1942.)

D. *Unimodular Lie Groups*

It is important to know in applications if a given Lie group $G$ is unimodular (cf. 2.3). The following theorem gives the list of known unimodular Lie groups:

THEOREM 4. *The following Lie groups are unimodular:*

1° *Lie groups $G$ for which the set of values of modular functions $\{\Delta(x), x \in G\}$ is compact.*

2° *Semisimple Lie groups.*

3° *Connected nilpotent Lie groups.* ▼

(For the proof cf. Helgason 1962, ch. X, § 1.)

E. *Measures on Semi-Direct Product Lie Groups*

THEOREM 5. *Let $G = T \rtimes K$ and let $\mathrm{d}t$ and $\mathrm{d}k$ denote left-invariant Haar measures on $T$ and $K$, respectively. Then the left-invariant Haar measure on $G$ has the form*

$$\mathrm{d}g = \frac{\mathrm{d}t\,\mathrm{d}k}{\delta^T(k)} \tag{16}$$

*and the modular function $\Delta^G(g)$ on $G$ has the form*

$$\Delta^G(g) = \Delta^T(t)\Delta^K(k)/\delta^T(k), \tag{17}$$

*where the function $\delta^T(k)$ is a unique positive function satisfying*

$$\int_T f(k^{-1}(t))\mathrm{d}t = \delta^T(k)\int_T f(t)\mathrm{d}t, \quad f \in L(T, \mathrm{d}t). \blacktriangledown \tag{18}$$

(For the proof cf. Nachbin 1965, ch. II, § 7.)

Formula (17) implies that $G$ is unimodular if and only if $T$ is unimodular and $\delta^T(k) = \Delta^K(k)$.

F. *Bibliographical Comments*

The concept of a local Lie group was introduced by Sophus Lie as a tool for an analysis of the properties of partial differential equations (Engel and Lie 1893). The connection between local and global Lie groups was first clarified by E. Cartan 1926, who proved that every Lie algebra over $R$ is a Lie algebra of a Lie group. The first systematic presentation of the theory of Lie groups from a global point of view was given by Chevalley 1946.

## § 11. Exercises

§ 1.1. Let $T^1$ be the quotient space $R/Z$ where $Z$ is the set of integers. We endow $T^1$ with the natural topology of the quotient space. Show that if a coset $p \in T^1$

does not contain the number 1/4 or 3/4, the function $\sin 2\pi p$ can be used to define a system of coordinates (i.e., a chart $(U, \sin 2\pi p)$) at $p$. And if the coset does not contain 0 or 1/2, then the function $\cos 2\pi p$ can be used.

§ 2.1. Show that every element $g \in \mathrm{SU}(2)$ can be written in the form

$$g = u_0\sigma_0 + \mathrm{i}u_k\sigma_k \tag{1}$$

where $u_\mu \in R$ and satisfy the condition

$$u_0^2+u_1^2+u_2^2+u_3^2 = 1, \tag{2}$$

and $\sigma_0 = I$, $\sigma_k$ were given in ch. 1.1.A.

§ 2.2. Show that SU(1, 1) consists of those elements $g \in \mathrm{SL}(2, C)$ which satisfy the condition

$$g^*\sigma_3 g = \sigma_3. \tag{3}$$

§ 2.3. Show that elements $g \in \mathrm{SU}(1, 1)$ can be written in the form

$$g = v_0\sigma_0+v_1\sigma_1+v_2\sigma_2+\mathrm{i}v_3\sigma_3 \tag{4}$$

where $v_\mu \in R$ and

$$v_0^2-v_1^2-v_2^2+v_3^2 = 1. \tag{5}$$

§ 2.4. Show that the elements of SL(2, $R$) consist of those elements of SL(2, $C$) which satisfy the condition

$$g^*\sigma_2 g = \sigma_2. \tag{6}$$

and can be written in the form

$$g = w_0\sigma_0+w_1\sigma_1+\mathrm{i}w_2\sigma_2+w_3\sigma_3 \tag{7}$$

where $w_\mu \in R$.

§ 2.5. Show that the map

$$g' = \varrho^* g\varrho \tag{8}$$

where

$$\varrho = \exp[\mathrm{i}\pi\sigma_1/4] \tag{9}$$

maps SU(1, 1) onto SL(2, $R$).

§ 2.6. Show that every element $g \in \mathrm{SO}(3)$ may be represented in the form

$$g(\varphi, \vartheta, \psi) = R_z(\varphi)R_y(\vartheta)R_z(\psi) \tag{10}$$

where $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \vartheta \leqslant \pi$ and $0 \leqslant \psi \leqslant 2\pi$ and

$$R_z(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_y(\vartheta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{bmatrix}, \tag{11}$$

are the rotations around the $z$- and $y$-axis, respectively. Find the geometric meaning of the Euler angles $\varphi$, $\vartheta$ and $\psi$.

§ 2.7. Show that the group SU(2) is the two-fold universal covering group for SO(3).

*Hint.* Introduce in $R^3$ the coordinates of the stereographic projection

$$\xi = \frac{x}{\frac{1}{2}-z}, \qquad \eta = \frac{y}{\frac{1}{2}-z}$$

and the complex variable $\zeta = \xi + \mathrm{i}\eta$ and show that the rotation $g$ in $R^3$ implies the projective mapping

$$\zeta \to \zeta' = \frac{\alpha\zeta+\beta}{\gamma\zeta+\delta} \tag{12}$$

in $C$, where the matrix

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \tag{13}$$

is an element of the group SU(2).

§ 2.8. Show that the elements $g \in \mathrm{SO}(3)$ can be parametrized as $3\times 3$ real orthogonal matrices $R$ with matrix elements

$$R_{ij} = \cos\theta\,\delta_{ij} + (1-\cos\theta)n_i n_j - \sin\theta\,\varepsilon_{ijk} n_k,$$

$$i,j = 1,2,3, \qquad 0 \leqslant \theta \leqslant \pi, \qquad \sum_{i=1}^{3} n_i^2 = 1.$$

Show that the group space is the ball with radius $\pi$ (the origin and the surface of the ball being identified), or the sphere $S^3$ in four dimensions with antipodes identified.

§ 2.9. Show that every element of SU(2) can be written in the form

$$u = \exp\left(\mathrm{i}\frac{\mu}{2}\sigma_3\right)\exp\left(\mathrm{i}\frac{\xi}{2}\sigma_2\right)\exp\left(\mathrm{i}\frac{\nu}{2}\sigma_3\right),$$

$$0 \leqslant \mu < 2\pi, \qquad 0 \leqslant \xi \leqslant \pi, \qquad -2\pi < \nu < 2\pi.$$

Show that the Euler angles $\mu$, $\xi$ and $\nu$ are coordinates on $S^3$.

§ 2.10. Show in particular that to a rotation $g(\varphi, \vartheta, \psi)$ there corresponds the unitary matrix of the form

$$u = \pm\begin{bmatrix} \exp(\mathrm{i}\varphi/2) & 0 \\ 0 & \exp(-\mathrm{i}\varphi/2) \end{bmatrix}\begin{bmatrix} \cos\vartheta/2 & \mathrm{i}\sin\vartheta/2 \\ \mathrm{i}\sin\vartheta/2 & \cos\vartheta/2 \end{bmatrix}\begin{bmatrix} \exp(\mathrm{i}\psi/2) & 0 \\ 0 & \exp(-\mathrm{i}\psi/2) \end{bmatrix}$$

$$= \pm\begin{bmatrix} \cos(\vartheta/2)\exp\left(\mathrm{i}\dfrac{\varphi+\psi}{2}\right) & \mathrm{i}\sin(\vartheta/2)\exp\left(-\mathrm{i}\dfrac{\psi-\varphi}{2}\right) \\ \mathrm{i}\sin(\vartheta/2)\exp\left(\mathrm{i}\dfrac{\psi-\varphi}{2}\right) & \cos(\vartheta/2)\exp\left(\quad\dfrac{\varphi+\psi}{2}\right) \end{bmatrix}. \tag{14}$$

§ 2.11*. The canonical equations of motion of classical mechanics

$$\frac{\partial H}{\partial q_k} = -\dot{p}_k, \qquad \frac{\partial H}{\partial p_k} = \dot{q}_k, \qquad k = 1, 2, \ldots, N,$$

where $H$ is the Hamiltonian of the classical system, can be written in the phase-space $R^{2n}$ with coordinates

$$x_i = q_i, \qquad x_{n+i} = p_i, \qquad 1 \leqslant i \leqslant n,$$

in matrix form as $\frac{\partial H}{\partial x} = J\dot{x}$, where $J$ is the matrix

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

with $I = (n \times n)$-identity matrix.

(i) Find the maximal symmetry group of canonical equations.

(ii) Show that the evolution of canonical variables is given by the one-parameter group of symplectic transformations.

§ 3.1. Let $x \to T_x$ be a representation of a Lie group $G$ given by right translations on $H = L^2(G)$:

$$T_x^R \psi(y) = \psi(yx), \qquad \psi \in H.$$

Determine the form of the infinitesimal generators as first-order differential operators on $H$.

§ 3.2. Let $G = SO(p, q)$. Show that the generators of the Lie algebra so$(p, q)$ can be represented in $C^3(R^{p+q})$ in the form

$$\begin{aligned} L^i{}_j &= x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad i, j = 1, 2, \ldots, p \text{ or } i, j = p+1, \ldots, p+q, \\ B^i{}_j &= x^i \frac{\partial}{\partial x^j} + x^j \frac{\partial}{\partial x^i}, \quad i = 1, 2, \ldots, p, \; j = p+1, \ldots, p+q \end{aligned} \tag{15}$$

and satisfy the following commutation relations

$$\begin{aligned} [L_{ij}, L_{rs}] &= \delta_{is} L_{jr} + \delta_{jr} L_{is} - \delta_{ir} L_{js} - \delta_{js} L_{ir}, \\ [B_{ij}, B_{rs}] &= \delta_{ir} L_{js} + \delta_{is} L_{jr} + \delta_{jr} L_{is} + \delta_{js} L_{ir}, \\ [L_{ij}, B_{rs}] &= \delta_{jr} B_{is} + \delta_{js} B_{ir} - \delta_{ir} B_{js} - \delta_{is} B_{jr}. \end{aligned} \tag{16}$$

Show that the generators $L_{ij}$ form a basis of the maximal compact subalgebra so$(p)\oplus$so$(q)$.

§ 3.3. Let $G$ be a connected Lie group, $\{X_i\}_1^d$—a basis in the left-invariant Lie algebra $L^R$ of $G$ and $\{\tilde{X}_i\}_1^d$—a basis in the right-invariant Lie algebra $L^L$ of $G$. Let $\alpha = (\alpha_1, \ldots, \alpha_p)$, where $\alpha_j = 1, 2, \ldots, d$, represent a multi-index, and let

$$X_\alpha = X_{\alpha 1} \ldots X_{\alpha p}, \qquad \tilde{X}_\alpha = \tilde{X}_{\alpha 1} \ldots \tilde{X}_{\alpha p}.$$

Let $|\alpha|$ denote the order of the multi-index.

Show that

(i) $$X_\alpha = \sum_{|\beta| \leqslant |\alpha|} a_\alpha{}^\beta \tilde{X}_\beta.$$

(ii) An arbitrary first order differential operator $P$ with $C^\infty(G)$ coefficients can be written in one of the forms

$$\sum p_\alpha X_\alpha \quad \text{or} \quad \sum \tilde{p}_\alpha \tilde{X}_\alpha,$$

where $A = [a_{\alpha\beta}(x)]$ is an analytic matrix on $G$.

§ 3.4*. Let $u$ be an infinitely differentiable positive definite function on a Lie group $G$ and let $K$ be an element of the right-invariant enveloping algebra of $G$. Show that $(K^+Ku)(e) \geqslant 0$.

§ 5.1. If $R_1(\alpha)$ is a rotation around $x$-axis in $R^3$ by an angle $\alpha$, $R_2(\beta)$ a rotation around $y$-axis by an angle $\beta$, then the *commutant* $q = R_1(\alpha)R_2(\beta)R_1^{-1}(\alpha)R_2^{-1}(\beta)$ is, for infinitesimal angles, a rotation around $z$-axis by an angle $\alpha\beta$.

§ 5.2. Show that the Lie group $G_n$ associated with the Canonical Commutation Relations (CCR) in quantum mechanics given by eq. 1.1(41) has the following composition law

$$(\xi, \eta, s)(\xi', \eta', s') = \big(\xi+\xi', \eta+\eta', \exp(-i\eta\xi')ss'\big), \tag{17}$$

where $\xi$, $\eta \in R^n$ and $s \in S^1$ (one-dimensional sphere). Show that $G_n$ is nilpotent.

*Hint*. Set

$$G_n \ni g(\xi, \eta, s) = \exp[i(\xi Q + \eta P + sI)]$$

and use the Baker–Hausdorff formula.

§ 5.3. Show that the infinite-dimensional Lie group $G_\infty$ associated with CCR $[\varphi(x), \pi(y)] = i\delta(x-y)$ in quantum field theory has the form

$$G_\infty = H \dotplus H \dotplus S^1$$

where $H$ is a real infinite-dimensional Hilbert space.

§ 5.4. Show that the three-dimensional real group defined by the multiplication law

$$(x_1 y_1 z_1)(x_2 y_2 z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+x_1 y_2), \quad x, y, z \in R^1 \tag{17a}$$

is a nilpolent group. Show that the subgroup $Z = \{(0, 0, z)\}$ is the center of $G$ and the subgroup $N = \{(0, y, z)\}$ is normal in $G$.

Show that the group $G$ defined by (17a) is the semi-direct product

$$G = N \circledS S, \tag{17b}$$

where $S = \{(x, 0, 0)\}$ (cf. example 9(2)).

§ 6.1. Let $G = \mathrm{SL}(n, C)$ and let $\mathfrak{Z}$, $D$ and $Z$ be the subgroups given by th. 6.2. Show that the matrix elements of Gauss factors 6(7) for the decomposition 6(8) have the form

$$\zeta_{pq} = \frac{1}{g_q}\begin{bmatrix} p & q+1 & \dots & n \\ q & q+1 & \dots & n \end{bmatrix}, \quad p < q,$$
$$\delta_p = \frac{g_p}{g_{p+1}}, \quad z_{pq} = \frac{1}{g_p}\begin{bmatrix} p & p+1 & \dots & n \\ q & p+1 & \dots & n \end{bmatrix}, \quad p > q, \tag{18}$$

where $g_p$ is the minor given by eq. 6(11) and

$$\begin{bmatrix} p_1 & p_2 & \cdots & p_m \\ q_1 & q_2 & \cdots & q_m \end{bmatrix}$$

is the minor of the matrix obtained by deleting from the element $g \in G$ all rows except the rows $p_1, p_2, \dots, p_m$ and all columns except $q_1, q_2, \dots, q_n$.

§ 6.2. Show that the real subgroup of GL($n$, $C$) determined by the condition

$$s^{-1}gs = g^{*-1}, \quad s = \begin{bmatrix} 0 & 0 & \sigma \\ 0 & e & 0 \\ \sigma & 0 & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & & & 1 \\ & & \cdot^{\cdot^{1}} & \\ & 1^{\cdot} & & \\ 1 & & & 0 \end{bmatrix} \tag{19}$$

where $\sigma$—matrix of order $p$, $e$—unit matrix of order $n-2p$, is isomorphic to the $U(p, q)$ group. Show that the Gauss factor $D$ of $U(p, q)$ consists of all bloc-diagonal matrices of the form

$$\delta = \begin{bmatrix} \lambda & 0 & \\ & u & \\ 0 & & \bar{\lambda}^{-1} \end{bmatrix} \tag{20}$$

where $\lambda$—diagonal complex matrix of order $p$ and $u$—unitary matrix of order $n-2p$. Show that the remaining Gauss factors have the form $\mathfrak{Z}_0 = \mathfrak{Z} \cap U(p, q)$ and $Z_0 = Z \cap U(p, q)$.

§ 6.3. Let $G = \mathrm{GL}(n, C)$ and let $g = \zeta\delta z$ be the Gauss decomposition of an element $g$. Show that

$$\det\begin{bmatrix} g_{11} & \cdots & g_{1p} \\ \cdots & \cdots & \cdots \\ g_{p1} & \cdots & g_{pp} \end{bmatrix} = \delta_1\delta_2 \dots \delta_p \equiv \Delta_p. \tag{21}$$

§ 6.4. Using Cartan decomposition show that every Lorentz transformation $g \in \mathrm{SO}_0(3, 1)$ can be written as a product of a rotation and a pure Lorentz transformation (boost), i.e.

$$g = \exp(\mathrm{i}\alpha J)\exp(\mathrm{i}\beta N). \tag{22}$$

§ 6.5. Find the Iwasawa decomposition for the Lorentz group.

§ 6.6. *Polar decomposition.* Show that an arbitrary element $g \in \mathrm{GL}(n, C)$ admits the following unique decomposition

$$g = hu \tag{23}$$

where $h$ is a positive definite hermitian matrix and $u$ is a unitary matrix.

*Hint.* Take $p = gg^*$ and show that $p = h^2$ where $h$ is positive definite. Show that $u = h^{-1}q$ is unitary.

*Note*. Every nonsingular operator $X$ in a Hilbert space has the decomposition

$$X = HU, \tag{24}$$

where $H$ is a positive definite self-adjoint operator and $U$ is an isometric operator.

§ 6.7. *Gramm decomposition*. Show that an arbitrary element $g \in \mathrm{GL}(n, C)$ admits the following unique decomposition

$$g = z\varepsilon u \tag{25}$$

where $z$ is the element of lower triangular subgroup of $\mathrm{GL}(n, C)$, $\varepsilon$ is an element of the set $E$ of all positive definite diagonal matrices and $u$ is unitary.

§ 6.8. Show that with the same notation as in exercise 6, we have the decomposition

$$\mathrm{GL}(n, C) = \mathrm{U}E\mathrm{U}. \tag{26}$$

*Hint*. Set $g = hu$ and reduce $h$ to the diagonal form.

*Remark*: The decomposition (26) for an arbitrary semisimple Lie group $G$ takes the form

$$G = KAK, \tag{27}$$

where $K$ is the maximal compact subgroup of $G$ and $A$ is the factor in the Iwasawa decomposition $(G = NAK)$ (cf. Bruhat 1956).

§ 7.1. Show that the group $\mathrm{SL}(2, C)$ is a two-fold universal covering group of $\mathrm{SO}(3, 1)$ given by the formula

$$L_\mu{}^\nu = \tfrac{1}{2}\mathrm{Tr}(\sigma_\mu \Lambda \sigma^\nu \Lambda^*) \tag{28}$$

where

$$L \in \mathrm{SO}(3, 1), \quad A \in \mathrm{SL}(2, C), \quad \sigma_\mu = (I, \sigma) \quad \text{and} \quad \tilde{\sigma}_\mu = (I, -\sigma).$$

§ 7.2. Show the inverse formula

$$A = \pm N^{-1} L_{\mu\nu} \tilde{\sigma}^\mu \sigma^\nu, \qquad N^2 = L_{\mu\nu} L_{\gamma\delta} \mathrm{Tr}(\sigma^\gamma \tilde{\sigma}^\nu \sigma^\delta \tilde{\sigma}^\gamma). \tag{29}$$

Hence

$$\mathrm{SO}(3, 1) = \mathrm{SL}(2, C)/D,$$

where

$$D = \{I, -I\}.$$

*Hint*. Use the one-to-one correspondence between the hermitian $2\times 2$-matrices $X$ and the vectors in Minkowski space given by

$$R^4 \ni x \to X = x^0 I + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{bmatrix} x^0 + x^3 & x^1 - ix^3 \\ x^1 + ix^3 & x^0 - x^3 \end{bmatrix}$$

and the fact that $\mathrm{SL}(2, C)$ transformations $X' = \Lambda X \Lambda^*$ in the matrix space, induce the Lorentz transformations in $R^4$.

§ 7.3. Show that the element

$$g = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

of the covering group of $\mathrm{SO}_0(3, 1)$ cannot be written in the form (22).

§ 9.1 Let $d(x, y)$ be a left-invariant distance on a connected Lie group $G$. Set $\tau(x) = d(e, x)$ and $\nabla\tau(x) = \{X_1\tau, \ldots, X_n\tau\}$, where $\{X_i\}_1^n$ is a basis in the left-invariant Lie algebra $L$ of $G$. Show that

$$|\nabla\tau(x)| \leqslant |\nabla\tau(e)|.$$

§ 9.2. Let $\mu(\cdot)$ be a left-invariant measure on $G$. Show that there exists a constant $\lambda$ such that

$$\int_G \exp[-\lambda\tau(x)]\mathrm{d}\mu(x) < \infty.$$

9.3. Show that the coefficients $a_{\alpha\beta}(x)$ in the formula (i) of exercise 3.3 which connects the elements of the left- and the right-invariant enveloping algebras satisfy the inequality

$$|a_{\alpha\beta}(x)| \leqslant \exp[c + c\tau(x)],$$

where $c$ is a constant.

§ 9.4. Show that the invariant measure for the group SO(3), in terms of the two different parametrizations (exercises to § 2.6 and 2.8) are

$$\mathrm{d}g = \frac{1}{8\pi^2}\sin\vartheta\,\mathrm{d}\varphi\,\mathrm{d}\vartheta\,\mathrm{d}\psi, \tag{30}$$

$$\mathrm{d}g(\theta, \boldsymbol{n}) = \frac{1}{4\pi^2}\mathrm{d}\boldsymbol{n}\sin^2\frac{\omega}{2}\mathrm{d}\omega. \tag{31}$$

§ 9.5. Show that the invariant measure for $SU(2)$ is

$$\mathrm{d}u = \frac{1}{16\pi^2}\sin\xi\,\mathrm{d}\xi\,\mathrm{d}\mu\,\mathrm{d}\nu. \tag{32}$$

*Hint.* In

$$u = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

set

$$a = \exp[\mathrm{i}(\mu+\nu)/2]\cos\xi/2$$

and

$$b = \exp[\mathrm{i}/2(\mu-\nu)]\sin\xi/2.$$

§ 9.6. Show that

$$\mathrm{d}g(u) = \pi^{-2}\delta\Big(\sum_{j=0}^{3}u_j^2 - 1\Big)\prod_{i=0}^{3}\mathrm{d}u_i, \tag{33}$$

$$\mathrm{d}g(v) = \pi^{-2}\delta(\det v - 1)\prod_{i=0}^{3}\mathrm{d}v_i, \tag{34}$$

$$dg(w) = \pi^{-2}\delta(\det w - 1)\prod_{i=0}^{3} dw_i \tag{35}$$

are invariant measures on SU(2), SU(1, 1) and SL(2, $R$) with the parametrizations (1), (4) and (7), respectively.

§ 9.7. Let $G = T^2 \circledS SO(2)$. Show that in terms of the parameters $x_1$, $x_2$, $\alpha$ where $x = (x_1, x_2) \in T^2$ and $\alpha \in SO(2)$, $0 \leqslant \alpha < 2\pi$ the invariant measure has the form

$$d\mu(x, \alpha) = dx_1 dx_2 d\alpha. \tag{36}$$

§ 9.8. Show that the invariant measure on the Poincaré group $G = T^4 \circledS SL(2, C)$ has the form

$$dg = d^4 a\, d\hat{g}, \tag{37}$$

where $d^4a$ a is the Lebesgue measure on $R^4$ and $d\hat{g}$ is the invariant measure on SL(2, $C$) given by 2.3(9).

§ 9.9. Let $G = Z$, the lower triangular complex subgroup of GL($n$, $C$), whose elements are matrices of the form

$$z = \begin{bmatrix} 1 & & & \\ z_{21} & 1 & & 0 \\ z_{31} & z_{32} & 1 & \\ \cdots & \cdots & \cdots & \\ z_{n1} & \cdots & z_{n,n-1} & 1 \end{bmatrix}, \quad z_{kj} = x_{kj} + iy_{kj}, \; x_{kj}, y_{kj} \in R.$$

Show that the invariant Haar measure on $Z$ is the Euclidean measure in $C^{n(n-1)/2}$ given by

$$d\mu(z) = \prod_{\substack{k,j=1 \\ k<j}}^{N} dx_{kj} dy_{kj}. \tag{38}$$

*Hint*. Find a composition law in $Z$ as in example 9.1.

§ 10.1. Let $G = GL(n, C)$. Show that the exponential mapping $\exp X$, $X \in L$, covers the whole group $G$.

*Hint*. Use Jordan form of an arbitrary element $g \in GL(n, C)$.

§ 10.2. Show that $G = GL(n, R)$ cannot be covered by an exponential mapping.

*Hint*. Consider the diagonal matrix with all negative matrix elements.

# Chapter 4

# Homogeneous and Symmetric Spaces

## § 1. Homogeneous Spaces

Let $\Gamma$ be a topological space and $G$ a topological group. We say that $G$ is a *topological* (left) *transformation group* on $\Gamma$ if the following conditions are satisfied:

1° With each $g \in G$ there is associated a homeomorphism $\gamma \to g\gamma$ of $\Gamma$ onto $\Gamma$.

2° The identity element $e$ of $G$ is the identity homeomorphism of $\Gamma$.

3° The mapping $(g, \gamma) \to g\gamma$ of $G \times \Gamma$ into $\Gamma$ is continuous.

4° $(g_1g_2)\gamma = g_1(g_2\gamma)$ for $g_1, g_2 \in G$ and $\gamma \in \Gamma$.

The topological space $\Gamma$ on which $G$ acts is called a *G-space*.

We say that $G$ acts *transitively* on $\Gamma$ if for every pair of points $\gamma_1, \gamma_2 \in \Gamma$ there exists an element $g \in G$ such that $\gamma_2 = g\gamma_1$. If $e$ is the only element of $G$ which leaves each $\gamma \in \Gamma$ fixed, then it is said that $G$ acts *effectively* on $\Gamma$ and $G$ is called *effective*.

It follows from def. 2.2.4 that a Hausdorff space $\Gamma$ is homogeneous if $G$ acts on $\Gamma$ transtitively.

The subgroup of $G$ which leaves a point $\gamma \in \Gamma$ fixed is called the *stability* (stationary, isotropy, little) *group* of $\gamma$. If $H_\gamma$ is the stability group of $\gamma$, and $\gamma' = g\gamma$, then the stability group of the point $\gamma'$ is the group $H_{\gamma'} = gH_\gamma g^{-1}$. Hence, the stability groups of any two points of a homogeneous space $\Gamma$ are isomorphic.

An important realization of homogeneous spaces is provided by the quotient spaces $G/H$ as follows. Let $G$ be a topological group, $H$ a closed subgroup of $G$, and $G/H$ the collection of left cosets $xH$, $x \in G$. We define the topology on the space $G/H$ by means of the canonical projection $\pi$: $G \ni x \to xH \in G/H$; namely, we say that a set $X \subset G/H$ is open in $G/H$ if $\pi^{-1}(X)$ is open in $G$. It is easily verified that such a collection of open sets defines a Hausdorff topology on $G/H$. If we assign to each $g \in G$ the map $g$: $xH \to gxH$, then $G$ becomes a transitive topological transformation group acting on $G/H$ and consequently $G/H$ is a homogeneous space. We verify these statements by using the continuity of the group multiplication of $G$.

The group $G$ acts effectively on $G/H$ if and only if $H$ does not contain a normal subgroup $N$ of $G$. In fact, if $N \subset H$ is a normal subgroup of $G$, $n \in N$, and $x \in G$, then $x^{-1}nx = n' \in N$ and $nxH = xn'H = xH$, i.e., to every $n \neq e$ there corresponds the identity transformation; to prove the second part of the statement, note that by condition 2°, a set $N$ of elements $n \in G$, which satisfy the condition $nxH = xH$ for each $x \in G$, generates a subgroup of $G$. For each $x$, $g \in G$, $n \in N$ and $h \in H$ we have $(gng^{-1})xH = xH$, and $(hnh^{-1})H = H$. Hence, $N$ is a normal subgroup of $G$ contained in $H$.

This construction shows that every quotient space $G/H$ of a topological group $G$ over a closed subgroup $H$ is a homogeneous space. In particular, if $H = \{e\}$ we obtain that $G$ itself is a homogeneous space.

Analogously we denote the homogeneous space $\{Hg\}$ of *right* cosets by the symbol $H\backslash G$.

There arises an interesting question as to whether every homogeneous $G$-space can be represented in this form. This question is answered positively by the following

Theorem 1. *Let $G$ be a locally compact topological group with a countable basis acting transitively on a locally compact Hausdorff space $\Gamma$. Let $\gamma$ be any point of $\Gamma$ and $H$ the subgroup of $G$ which leaves $\gamma$ unchanged. Then,*

1° *$H$ is closed.*

2° *The map*

$$gH \to g\gamma$$

*is a homeomorphism of $G/H$ onto $\Gamma$.* ▼

(For the proof cf. Helgason 1962, ch. II, th. 3.2.)

The same is true for the homogeneous spaces of the right cosets $H\backslash G$.

Homogeneous spaces play an important role in the representation theory. We shall use them for the construction of induced representations of various groups (ch. 16 ff).

## § 2. Symmetric Spaces

In this section we consider a special class of homogeneous spaces whose fundamental group $G$ is a Lie group.

Let $G$ be a connected Lie group and let $\sigma$ be an involutive automorphism of $G$ (i.e., $\sigma^2 = 1$, $\sigma \neq 1$). Let $G_\sigma$ be a closed subgroup of $G$ consisting of all fixed points of $G$ under $\sigma$ and $G_\sigma^I$ the identity component of $G_\sigma$. Let $H$ be a closed subgroup such that $G_\sigma \supset H \supset G_\sigma^I$. We shall then say that $G/H$ is a *symmetric homogeneous space* (defined by $\sigma$). If we denote by $\sigma$ the involutive automorphism of the Lie algebra $L$ of $G$ induced by $\sigma$, then, by virtue of the considerations given in ch. 1, § 6, eqs. (9)–(13), we obtain

$$L = K \dotplus P, \tag{1}$$

where

$$K = \{X \in L:\ \sigma(X) = X\} \tag{2}$$

and coincides with the subalgebra corresponding to $H$, and

$$P = \{X \in L:\ \sigma(X) = -X\}. \tag{3}$$

We have obviously (cf. th. 1.6.2)

$$[K, K] \subset K, \quad [K, P] \subset P \quad \text{and} \quad [P, P] \subset K. \tag{4}$$

We call a two-point function $f(x, y)$ which satisfies the condition

$$f(gx, gy) = f(x, y), \quad x, y \in \Gamma,\ g \in G, \tag{5}$$

the *invariant* of the symmetric space.

The rank of a symmetric space $G/H$ is defined as the dimension of the maximal abelian subalgebra of $P$ in the decomposition (1). This notion is of great importance in representation theory, since it gives the number of algebraically independent invariant differential operators in the space $L^2(G/H)$ (cf. th. 15.1.1).

EXAMPLE 1. Let $G = \mathrm{SO}(n+1)$, and $\sigma$ be defined by the formula

$$\sigma(g) = SgS^{-1}, \quad g \in G, \tag{6}$$

where

$$S = \begin{bmatrix} -1 & 0 \\ 0 & I_n \end{bmatrix} \tag{7}$$

and $I_n$ is the unit matrix in $R^n$. We find that $G_\sigma = \mathrm{SO}(n)$ and is equal to its identity component, hence $H = \mathrm{SO}(n)$. The symmetric space $\Gamma = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ is homeomorphic with the $n$-dimensional sphere $S^n$. In fact, the group $G = \mathrm{SO}(n+1)$ acts transitively on the manifold $S^n$ given by the equation

$$(x^1)^2 + (x^2)^2 + \ldots + (x^{n+1})^2 = 1. \tag{8}$$

The transitivity of the sphere $S^n$ with respect to the group $\mathrm{SO}(n+1)$ follows from the fact that any real vector $x = (x^1, x^2, \ldots, x^{n+1})$ satisfying eq. (8) can be attained from the vector $e^1 = (1, 0, 0, \ldots, 0)$ by a rotation matrix $g(x)$, whose first column $g^i{}_1(x) = x^i$. Therefore, any two vectors $x', x'' \in S^n$ can be related to each other by the rotation matrix $g = g(x')g^{-1}(x'')$. The subgroup of $\mathrm{SO}(n+1)$ which leaves the point $x = e^1 \in S^n$ invariant is isomorphic to $H_\sigma = \mathrm{SO}(n)$. Hence, by th. 1.1 the map

$$gH \to ge^1 \tag{9}$$

is a homeomorphism of $\mathrm{SO}(n+1)/\mathrm{SO}(n)$ onto $S^n$.

The Cartan decomposition of the Lie algebra of $\mathrm{SO}(n+1)$ is given by

$$\mathrm{so}(n+1) = \mathrm{so}(n) \dot{+} P.$$

Because $P$ is spanned by the elements $M_{1,n+1}, \ldots, M_{n,n+1}$ it can be seen from the commutation relations that the maximal abelian subalgebra of $P$ is one-dimensional. Hence, the rank of $S^n$ is equal to one. ▼

THEOREM 1. *Every homogeneous space $G/H$, where $G$ is a Lie group and $H$ is a compact subgroup, admits an invariant metric.*

PROOF: Let $o$ be the point of $G/H$ represented by the coset $H$ and $\tilde{H}$ a group of linear transformations of the tangent space $T_0(G/H)$, induced by the elements of $H$. Because $H$ is compact, so is $\tilde{H}$ and, by a procedure similar to that in 3.8., eq. (2), there exists a positive definite inner product, say $g_0$, in $T_0(G/H)$ which is invariant under $\tilde{H}$. For each $\gamma \in G/H$ we take an element $x \in G$ such that $x(o) = \gamma$ and define an inner product $g_\gamma$ in $T_\gamma(G/H)$ by $g_\gamma(X, Y) = g_0(x^{-1}X, x^{-1}Y)$, $X, Y \in T_\gamma(G/H)$. The set $X$ of points $x \in G$ which transform $o \to \gamma$ generates a left coset of $G$ with respect to $H$. The metric $g_\gamma$ is independent of the choice of an element $x \in \tilde{X}$. In fact, if $x, y \in \tilde{X}$ (i.e., $x^{-1}y = h \in H$), then

$$g_0(y^{-1}X, y^{-1}Y) = g_0(h^{-1}x^{-1}X, h^{-1}x^{-1}X) = g_0(x^{-1}X, x^{-1}Y).$$

It is also readily verified that the Riemannian metric so obtained is invariant with respect to $G$. ▼

Let $G$ be a connected Lie group and $G/H$ a symmetric homogeneous space with a compact $H$. The space $G/H$ equipped with the $G$-invariant Riemannian metric given by th. 1 is called a *globally symmetric Riemannian space*. According to eqs. (1)–(4) we can associate, with every globally symmetric Riemannian space $G/H$, a pair $(L, \sigma)$ with the following properties:

(i) $L$ is a Lie algebra of $G$.

(ii) $\sigma$ is an involutive automorphism of $L$.

(iii) The set $K = \{X \in L\colon \sigma(X) = X\}$ is a compact subalgebra of $L$, and $K$, $P$ satisfy the commutation relation (4).

The pair $(L, \sigma)$ is called the *orthogonal symmetric Lie algebra*.

The association of an orthogonal symmetric algebra with a globally symmetric Riemannian space $\Gamma$ allows us to reduce the problem of the classification of these spaces to the problem of the classification of the orthogonal symmetric algebras.

A globally symmetric Riemannian space is said to be *irreducible* if the associated orthogonal symmetric Lie algebra satisfies the following conditions:

(i) $L$ is semisimple and $K$ contains no ideals $\neq \{o\}$ of $L$.

(ii) $K$ is a maximal proper subalgebra of $L$.

EXAMPLE 2. Let $H_n$ be the set of all positive definite hermitian matrices of order $n$ with determinant one. Consider motions in $H_n$ given by the formula

$$h \to h_g = ghg^* \in H_n, \tag{10}$$

where $g \in \mathrm{SL}(n, C)$. We verify that $\mathrm{SL}(n, C)$ acts transitively on $H_n$. The stability subgroup $H$ of the point $I_n \in H_n$ is, by eq. (10), the set of all matrices satisfying the condition $I_n = gg^*$, i.e., the subgroup $\mathrm{SU}(n)$. An involutive automorphism $\sigma$ of $\mathrm{SL}(n, C)$, which leaves every point of $\mathrm{SU}(n)$ fixed, is given by formula $\sigma(g)$

$= g^{*-1}$. Hence, $G/H$ is a globally symmetric Riemannian space which, by th. 1.1, is homeomorphic with $H_n$. We also verify that the orthogonal symmetric algebra $(\mathrm{sl}(n, C), \sigma)$ is irreducible. Consequently, the globally symmetric Riemannian space $\mathrm{SL}(n, C)/\mathrm{SU}(n)$ is also irreducible. ▼

Cartan showed that the problem of classification of globally symmetric Riemannian spaces can be reduced to the problem of the classification of irreducible ones and solved the latter problem (1926a, b; 1927a, b, c). In table I we list the compact and noncompact *irreducible* globally symmetric Riemannian spaces (type I and III in Cartan's classification), whose transformation group is a simple real connected classical Lie group.

Table 1

Irreducible Globally Symmetric Riemannian Spaces Whose Transformation Group Is a Simple Real Connected Lie Group

| Compact | Noncompact | Rank | Dimension |
|---|---|---|---|
| $\mathrm{SU}(n)/\mathrm{SO}(n)$ | $\mathrm{SL}(n, R)/\mathrm{SO}(n)$ | $n-1$ | $(n-1)(n+2)/2$ |
| $\mathrm{SU}(2n)/\mathrm{Sp}(n)$ | $\mathrm{SU}^*(2n)/\mathrm{Sp}(n)$ | $n-1$ | $(n-1)(2n+1)$ |
| $\mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}(p)\times \mathrm{U}(q))$ | $\mathrm{SU}(p, q)/\mathrm{S}(\mathrm{U}(p)\times \mathrm{U}(q))$ | $\min(p, q)$ | $2pq$ |
| $\mathrm{SO}(p+q)/\mathrm{SO}(p)\times \mathrm{SO}(q)$ | $\mathrm{SO}_0(p, q)/\mathrm{SO}(p)\times \mathrm{SO}(q)$ | $\min(p, q)$ | $pq$ |
| $\mathrm{SO}(2n)/U(n)$ | $\mathrm{SO}^*(2n)/U(n)$ | $[n/2]$ | $n(n-1)$ |
| $\mathrm{Sp}(n)/U(n)$ | $\mathrm{Sp}(n, R)/U(n)$ | $n$ | $n(n+1)$ |
| $\mathrm{Sp}(p+q)/\mathrm{Sp}(p)\times \mathrm{Sp}(q)$ | $\mathrm{Sp}(p, q)/\mathrm{Sp}(p)\times \mathrm{Sp}(q)$ | $\min(p, q)$ | $4pq$ |

All spaces in table I are simply connected. We did not include in table I symmetric spaces associated with the exceptional groups, because we shall not use them in the following.

In addition to the irreducible symmetric spaces listed, there exists still two other classes. The first one consists of irreducible globally symmetric Riemannian spaces which are the simple compact connected Lie groups (type II). The last class (type IV) contains irreducible globally symmetric Riemannian spaces which are cosets spaces $G/H$, where $G$ is a connected Lie group whose Lie algebra is $(L)^R$, the real form of a simple complex Lie algebra $L$, and $H$ is the maximal compact subgroup of $G$.

It should be also noted that isomorphisms of lower-dimensional complex and real simple Lie algebras (cf. ch. 1, § 5, table I) imply a series of coincidences of lower dimensional symmetric spaces; e.g.,

$$\mathrm{SU}(2, 2)/\mathrm{S}(\mathrm{U}(2)\times \mathrm{U}(2)) \sim \mathrm{SO}_0(4, 2)/\mathrm{SO}(4)\times \mathrm{SO}(2)$$

(corresponding to the isomorphism $\mathrm{su}(2, 2) \sim \mathrm{so}(4, 2)$). For the full list of coincidences cf. Helgason 1962, ch. IX.

In the next table we list symmetric spaces with noncompact stability groups

Table II. Symmetric Spaces $G/H$ with Noncompact Stability Group

| $H \backslash G$ | $SL(n, C)$ | $SL(n, R)$ | $SU(p, q)$ |
|---|---|---|---|
| | $SO(n, C)$ | $SL(p, R)\times SL(q, R)\times R^1$ | $SU(k, k+h)\times SU(p-k, n-k-h)\times U(1)$ |
| | $SL(n, R)$ | $SO(p, q)$ | $SO(p, q)$ |
| | $SL(p, C)\times SL(q, C)\times C^1$ | $Sp(n/2, R)$ | $Sp(p/2, q/2)$ |
| | $SU(p, q)$ | $Sp(n/2, C)\times R^1$ | $\left.\begin{matrix}SO^*(n)\\ Sp(n, R)\\ SL(n, C)\times R^1\end{matrix}\right\} p=q=n/2$ |
| | $Sp(n/2, C)$ | | |
| | $SU^*(n)$ | | |
| $H \backslash G$ | $SU^*(n)$ | $SO(n, C)$ | $SO(p, q)$ |
| | $SU^*(p)\times SU^*(q)\times R^1$ | $SO(p, C)\times SO(q, C)$<br>$p=1$ or $p>2, q>2$ | $SO(p, k+h)\times SO(p-k, n-k-h)$<br>$k+h>2, n-k-h>2$ |
| | $Sp(p/2, q/2)$ | $SO(n-2)\times C^1$ | $SO(p-2, q)\times U(1)$ |
| | $SO^*(n)$ | $SO(p, q)$ | $SO(p-1, q-1)\times R^1$ |
| | $SL(n, C)\times U(1)$ | $SL(n/2, C)\times C^1$ | $SU(p/2, q/2)\times U(1)$ |
| | | $SO^*(n)$ | $\left.\begin{matrix}SL(n/2, R)\times R^1\\ SO(n/2, C)\end{matrix}\right\} p=q=n/2$ |
| | | $SO^*(n/2, C)$ | $SO(n-2)\times U(1)\left\} \begin{matrix}p=2\\ q=n-2\end{matrix}\right.$ |
| | | $SU(p, q)\times U(1)$ | |
| $H \backslash G$ | $Sp(n, C)$ | $Sp(n, R)$ | $Sp(p, q)$ |
| | $SL(n, C)\times C^1$ | $Sp(p, R)\times Sp(q, R)$ | $Sp(k, k+h)\times Sp(p-k, n-k-h)$ |
| | $Sp(n, R)$ | $SU(p, q)\times U(1)$ | $SU(p, q)\times U(1)$ |
| | $Sp(p, C)\times Sp(q, C)$ | $SL(n, R)\times R^1$ | |
| | $Sp(p, q)$ | $Sp(n/2, C)$ | |
| | | $\left.\begin{matrix}SU^*(n)\times R^1\\ Sp(n/2, C)\end{matrix}\right\} p=q=n/2$ | |

*Note*: $p+q=n$, $R^1$ ($C^1$) additive group of real (complex) numbers. When $n/2, p/2, \ldots$ occur, $n$ and $p$ are even.

The full classification of symmetric spaces, including symmetric spaces associated with exceptional simple Lie groups was elaborated by Berger 1957.

## § 3. Invariant and Quasi-Invariant Measures on Homogeneous Spaces

Let $X$ be a homogeneous space with a transformation group which is a locally compact separable group $G$. We know from th. 1.1 that $X$ is isomorphic to the coset space $H\backslash G$ or $G/H$, where $H$ is the stability subgroup of a point $x_0 \in X$.

Let $S$ be a subset of $X = H\backslash G$. By the 'translate' of a subset $S$ by an element $g \in G$ we mean the set $Sg = \{xg\colon x \in S\}$. Let $\mathrm{d}\mu(x)$ be a positive measure on $X$; a measure $\mathrm{d}\mu_g(x) \equiv \mathrm{d}\mu(xg)$ will be, by definition, the measure given by the following formula

$$\mu_g(f) \equiv \int_X f(x)\mathrm{d}\mu(xg) = \int_X f(xg^{-1})\mathrm{d}\mu(x) \quad \text{for every } f \in C_0(X). \tag{1}$$

In other words, $\mu_g(S) \equiv \mu(Sg)$ for every Borel set $S$ in the space $X$.

We have seen that on every locally compact topological group there exists an invariant measure (cf. th. 2.3.1). The following example shows that on a homogeneous space an invariant measure might not exist.

Example 1. Let $G$ be the group of triangular real matrices of the form

$$g = \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{bmatrix}, \quad \alpha > 0 \tag{2}$$

and let

$$H = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \right\}.$$

Every element $g \in G$ may be represented in the form (Mackey decomposition, cf. th. 2.4.1)

$$g = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \gamma\alpha & 1 \end{bmatrix}. \tag{3}$$

Hence, every element of a right coset $Hg$ may be uniquely represented by a point $x = \gamma\alpha$ of the real line $R$. Consequently, $X = H\backslash G = R$. Because an element $x \in X$ corresponds to the group element $\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$, we obtain the action of $G$ in $X$ by the formula

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \alpha x + \gamma & \alpha^{-1} \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha^2 x + \alpha\gamma & 1 \end{bmatrix}, \tag{4}$$

or,

$$g\colon x \to \alpha^2 x + \alpha\gamma. \tag{5}$$

The invariant measure on $X$ relative to $G$ should be, in particular, invariant relative to translations $x \to x + \gamma$; hence, it should be proportional to the Lebesgue measure on $R$ (cf. exercise 2.3.3). Such a measure cannot, however, be invariant relative to homothetic transformations $x \to \alpha^2 x$; consequently, there is no measure $\mathrm{d}\mu(x)$ on $X$ invariant relative to $G$. ▼

The lack of invariant measures on homogeneous spaces leads to the concept of quasi-invariant measures.

DEFINITION 1. A positive measure $\mathrm{d}\mu(x)$ on $X$ is said to be *quasi-invariant* if the measure $\mathrm{d}\mu_g(x) \equiv \mathrm{d}\mu(xg)$ and $\mathrm{d}\mu(x)$ are equivalent for every $g \in G$.

*Remark*: Two positive measures $\mathrm{d}\mu_1$ and $\mathrm{d}\mu_2$ are said to be *equivalent* if they have the same sets of measure zero. According to the Radon–Nikodym theorem (app. A.5) there exists then a function $\varrho(x) \geqslant 0$ such that

$$\mathrm{d}\mu_1(x) = \varrho(x)\mathrm{d}\mu_2(x). \tag{6}$$

The function $\varrho(x) = \mathrm{d}\mu_1(x)/\mathrm{d}\mu_2(x)$ is called the *Radon–Nikodym derivative*. ▼

The following theorem describes the main properties of quasi-invariant measures on homogeneous spaces.

THEOREM 1. *Let $G$ be a locally compact separable group, $H$ a closed subgroup of $G$ and $X = H\backslash G$. Then*

1° *There exists a quasi-invariant measure on $X$ such that the Radon–Nikodym derivative $\mathrm{d}\mu_g(x)/\mathrm{d}\mu(x)$ is a continuous function on $G \times X$.*

2° *Any two quasi-invariant measures on $X$ are equivalent.*

3° *All quasi-invariant measures may be obtained in the following manner: Let $\varrho(g)$ be a strictly positive locally integrable Borel function satisfying*

$$\varrho(hg) = \frac{\Delta_H(h)}{\Delta_G(h)}\varrho(g) \quad \text{for all } h \in H, \tag{7}$$

*where $\Delta_H$ and $\Delta_G$ are modular functions for $H$ and $G$, respectively.*

*Then $\varrho$ is related to the quasi-invariant measure $\mu$ on $X$ by the formula*

$$\int_G f(g)\varrho(g)\mathrm{d}g = \int_X \mathrm{d}\mu(\mathring{g}) \int_H f(hg)\mathrm{d}h, \quad \mathring{g} \equiv Hg, \tag{8}$$

*for $f \in C_0(G)$. The measure $\mu$ satisfies the condition*

$$\mathrm{d}\mu(\mathring{g}g') = \frac{\varrho(\mathring{g}g')}{\varrho(g')}\mathrm{d}\mu(g'), \tag{9}$$

*and for a given $\varrho$, is determined uniquely up to a multiplicative constant.* ▼

(For the proof cf. Mackey 1952 and Loomis 1960.)

The following corollary provides a convenient criterion for the existence of an invariant measure on the space $X = H\backslash G$.

COROLLARY 1. *An invariant measure $\mu$ exists on a homogeneous space $X = H\backslash G$ if and only if $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$. This measure is unique up to a multiplicative constant and satisfies*

$$\int_G f(g)\mathrm{d}g = \int_X \mathrm{d}\mu(\mathring{g}) \int_H f(hg)\mathrm{d}h, \quad \mathring{g} = Hg. \tag{10}$$

PROOF: It is sufficient to take $\varrho(g) = 1$ for every $g$ in (8). ▼

In particular, if $G$ is unimodular and $H$ is also unimodular, then $\Delta_G(h) = \Delta_H(h) = 1$ and $X = H\backslash G$ possesses an invariant measure.

EXAMPLE 2. Let $G = SO(3, 1)$ and $H = SO(3)$. The homogeneous space $X = H\backslash G$ may be represented as a hyperboloid

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = \varrho^2 > 0 \tag{11}$$

(cf. exercise 1.2). Because $G$ and $H$ are unimodular, we have $\Delta_G(h) = \Delta_H(h) = 1$ for all $h \in H$. Consequently, $X$ possesses an invariant measure by virtue of corollary 1.

We now calculate the explicit form of the invariant measure $\mathrm{d}\mu$ on $X = H\backslash G$. We first relate with the second order surface (11) a differential form $\mathrm{d}\mu$ defined by the formula

$$\mathrm{d}x_0 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = \mathrm{d}(x_\varrho x^\varrho)\mathrm{d}\mu = \mathscr{I}\mathrm{d}(x_\varrho x^\varrho)\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3. \tag{12}$$

The Jacobian $\mathscr{I}$ for the transformation $(x_0, x_1, x_2, x_3) \to (x_\varrho x^\varrho, x_1, x_2, x_3)$ takes the form $\mathscr{I} = 1/2x_0$, where $x_0 = (\varrho^2 + x^2)^{1/2}$ (for the upper hyperboloid (11)). Hence, by virtue of eq. (12),

$$\mathrm{d}\mu(x) = \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3}{2x_0}. \tag{13}$$

The right translations $x \to xg$ does conserve the differential forms $\mathrm{d}x_0 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3$ and $\mathrm{d}(x_\varrho x^\varrho)$. Hence, the differential form (13) is invariant and gives an invariant measure on $X = H\backslash G$. This measure is unique up to a multiplicative constant by virtue of corollary 1.

In relativistic kinematics, the energy momentum four-vector of a particle of positive mass satisfies $p_0^2 - p^2 = m^2$, hence it is a point on the homogeneous space $H\backslash G$, and the invariant measure is therefore $\mathrm{d}^3p/2p_0$. ▼

We now give the so-called measure disintegration theorem. Let $X$ be a locally compact space, countable at infinity, $r$ an equivalence relation in $X$, $Y$ the quotient space $X/r$. Denote by $\pi$ the canonical map of $X$ onto $Y$. Let $\mu$ be a finite measure on $X$. We say that a set $E \subset Y$ is a Borel set iff $\pi^{-1}(Y)$ is a Borel set in $X$. This gives a Borel structure in $Y$ induced by the Borel structure in $X$. We define a measure $\tilde{\mu}$ on $Y$ by the formula

$$\tilde{\mu}(E) = \mu(\pi^{-1}E), \qquad E\text{—a Borel set in } Y. \tag{14}$$

If the Borel structure on $Y$ is separable (i.e., there exists a sequence of Borel sets at $Y$ which separates points of $Y$) then we have the following theorem:

THEOREM 2. *For every $y \in Y$ there exists a measure $\mu_y$ on $X$, with the support $\pi^{-1}(y)$ (i.e. $\mu_y\,(X - \pi^{-1}(y)) = 0$), such that for every function $f \in L^1(X, \mu)$ we have*

$$\int_X f(x)\mathrm{d}\mu(x) = \int_Y \mathrm{d}\tilde{\mu}(y) \int_X f(x)\mathrm{d}\mu_y(x). \tag{15}$$

(For the proof cf. Mackey 1952, § 11.)

## § 4. Comments and Supplements

A. The following theorem gives a useful measure disintegration theorem implied by the Iwasawa decomposition.

THEOREM 1. *Let $G$ be a connected semisimple Lie group and let $\mathscr{K}\mathscr{A}_p\mathscr{N}$ be its Iwasawa decomposition. Let* $\mathrm{d}k$, $\mathrm{d}a$ *and* $\mathrm{d}n$ *be the left-invariant measures on* $\mathscr{K}$, $\mathscr{A}_p$ *and* $\mathscr{N}$, *respectively. Then the left-invariant measure* $\mathrm{d}g$ *on $G$ can be normalized such that*

$$\int_G f(g)\,\mathrm{d}g = \int_{\mathscr{K}\times\mathscr{A}_p\times\mathscr{N}} f(kan)\exp[2\varrho(\log a)]\,\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n$$
$$= \int_{\mathscr{K}\times\mathscr{A}_p\times\mathscr{N}} f(kna)\,\mathrm{d}k\,\mathrm{d}n\,\mathrm{d}a, \tag{1}$$

*where* $\log a$ *denotes the unique element $X$ in the Lie algebra $H_p$ for which* $\exp X = a$ *and* $\varrho = \frac{1}{2}\sum_{\alpha\in B_+}\alpha$. ▼

(Cf. eq. 1.6(18).)

(For the proof cf. Helgason 1962, ch. X, § 1.)

B. *Historical Notes*.

É. Cartan after completing the classification of complex simple Lie groups (1894), real simple Lie groups (1914) began in 1925 an analysis of the properties of homogeneous spaces associated with simple Lie groups. In a series of impressive papers (1926–27), he succeeded in completing the classification of global irreducible symmetric Riemannian spaces. He also gave a geometric description of all of these spaces.

The classification of symmetric spaces with non-compact stability groups was given by Berger 1957.

The description of quasi-invariant measures on homogeneous spaces in the version presented by th. 3.1 was given by Loomis 1960.

## § 5. Exercises

§ 2.1. Let $G = \mathrm{U}(n)$ and $H = \mathrm{U}(n-1)$. Show that the space $X = G/H$ is symmetric and can be represented as the manifold in $C^n$ given by

$$z_1\bar{z}_1 + \ldots + z_n\bar{z}_n = 1. \tag{1}$$

§ 2.2. Let $G = \mathrm{SO}(p, q)$ and $H = \mathrm{SO}(p-1, q)$, show that the space $X = G/H$ is symmetric and can be represented as a hyperboloid given by

$$x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 = 1. \tag{2}$$

§ 2.3. Show that the stability group of the cone

$$x_0^2-x_1^2-x_2^2-x_3^2 = 0 \tag{3}$$

is the group $H = T^2 \circledS \mathring{S}O(2)$.

*Hint.* Use $2\times 2$-matrix description of the Minkowski space $M$ given by the correspondence

$$M \ni x \to X = x^{\mu}\sigma_{\mu}$$

(cf. exercise 3.7.3) and find the stability subgroup of the point $x = (1, 0, 0, 1)$ in this realization.

§ 2.4. Show that the symmetric space $X = SU(1, 1)/U(1)$ may be realized as the unit disc $D = \{z \in C\colon |z| < 1\}$. Show that the action of SU(1, 1) on $D$ is given by the projective transformations $\left(g = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix}, \ \alpha, \beta \in C, \ |\alpha|^2-|\beta|^2 = 1\right)$

$$g\colon z \to \frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}. \tag{4}$$

§ 2.5. Show that the Riemannian metric tensor on the space $D$ of the previous problem has the form

$$\begin{aligned} g_{ij} &= (1-|z|^2)^{-2}\delta_{ij}, \\ g^{ij} &= (1-|z|^2)^2\delta_{ij}, \end{aligned} \tag{5}$$

and that the volume element $d\mu$ on $D$ is given by

$$d\mu(z) = \sqrt{(\det g)}\, dx\, dy = [1-(x^2+y^2)]^{-2} dx\, dy. \tag{6}$$

§ 3.1. Show that the invariant metric tensor $g_{\alpha\beta}$ on the hyperboloid (2) has the form

$$g_{\alpha\beta}(t) = g_{ik}\frac{\partial x^i}{\partial t^{\alpha}}\frac{\partial x^k}{\partial t^{\beta}}, \tag{7}$$

where $\{x^i\}_{i=1}^{p+q}$ are the Cartesian coordinates on the Minkowski space $M^{p,q}$ in which the hyperboloid (2) is embedded and $\{t^{\alpha}\}_{\alpha=1}^{p+q-1}$ are any 'internal' coordinates on hyperboloid (e.g., spherical).

§ 3.2. Find a measure on the cone (3) invariant relative to SO(3, 1).

§ 3.3. Show that the SO$(p, q)$-invariant measure on the hyperboloid (2) has the form

$$d\mu(t) = (\det g)^{1/2} \prod_{\alpha=1}^{p+q-1} dt^{\alpha}$$

where $g_{\alpha\beta}(t)$ is given by eq. (7).

# Chapter 5

# Group Representations

## § 1. Basic Concepts

Let $G$ be a locally compact, separable, unimodular topological group and let $H$ be a separable complex Hilbert space.

DEFINITION 1. A map $x \to T_x$ of $G$ into the set $L(H)$ of linear bounded operators in $H$ is said to be a *representation* of $G$ in $H$ if the following conditions are satisfied:

$$T_{xy} = T_x T_y, \qquad T_e = I. \blacktriangledown \tag{1}$$

The condition $T_{xy} = T_x T_y$ means that the map $x \to T_x$ is a homomorphism of $G$ into a set of linear operators in $H$. The condition $T_e = I$ guarantees that the representation $x \to T_x$ is in terms of invertible operators; indeed

$$T_x T_{x^{-1}} = T_{x^{-1}} T_x = T_e = I.$$

Hence,

$$T_x^{-1} = T_{x^{-1}}.$$

In addition certain continuity conditions are also imposed on a representation $x \to T_x$ of $G$ in $H$. A representation is said to be *strongly continuous* if for all $u \in H$ the map $x \to T_x u$ is a continuous map of $G$ into $H$. This means that for any $x_0 \in G$

$$\|T_x u - T_{x_0} u\| \to 0, \qquad \text{as } x \to x_0 \tag{2}$$

for all vectors $u \in H$.

The condition (2) is equivalent to the following, apparently stronger, condition which is usually given in the definition of a representation of topological groups:

*The map $(u, x) \to T_x u$ of $H \times G$ into $H$ is continuous.*

PROOF: It is necessary to show that for $x \to x_0 \in G$ and $u \to u_0 \in H$

$$T_x u \to T_{x_0} u_0.$$

Let $K$ be a compact neighborhood of the point $x_0$ in $G$. For any $u \in H$ the set $S = \{T_x u: x \in K\}$, by condition (2), is a compact (and therefore bounded) subset of $H$. The Principle of Uniform Boundedness (cf. Appendix A.4) assures that there exists a constant $C_k$ such that

$$\|T_x\| \leqslant C_k \qquad \text{for all } x \in K. \tag{3}$$

We can assume, without loss of generality, that $x \in K$. Then,

$$\|T_x u - T_{x_0} u_0\| \leqslant \|T_x u - T_x u_0\| + \|T_x u_0 - T_{x_0} u_0\| \leqslant C_k \|u - u_0\| + \|T_x u_0 - T_{x_0} u_0\|.$$

The first term tends to zero since $u \to u_0$ and the second term tends to zero due to condition (2). Therefore for $x \to x_0 \in G$ and $u \to u_0 \in H$, $T_x u \to T_{x_0} u_0$, and the map $(u, x) \to T_x u$ of $H \times G$ into $H$ is continuous. ▼

A representation is said to be *bounded*, if $\sup\|T_x\| < \infty$. It follows from eq. (3) that a representation of any locally compact separable, topological group is bounded for any compact subset $K \subset G$. In particular a representation of a compact group is bounded.

A representation is said to be *unitary* if each $T_x$, $x \in G$, is a unitary operator in $H$, and *trivial* if $T_x = I$ for all $x$ in $G$.

We denote for brevity a representation $x \to T_x$ of $G$ by the symbol $T$. The space $H$ in which a representation $T$ acts is called the *carrier space* of $T$.

The def. 1 specifies in fact linear representations of $G$. In the following, by a representation of a topological, locally compact, separable group $G$ we shall mean a 'linear strongly continuous representation in a separable complex Hilbert space $H$', unless explicitly stated otherwise (e.g., nonlinear, weakly-continuous or discontinuous).

EXAMPLE 1. Let $G = R^1$, and $H$ be an arbitrary Hilbert space. Consider the maps

(i) $R^1 \ni x \to T_x = \exp[ipx] \cdot I$, $p \in R^1$,

and

(ii) $R^1 \ni x \to T'_x = \exp[px] \cdot I$, $p \in R^1$.

The conditions (1) and (2) for the maps (i) and (ii) are obviously satisfied. Hence, $T$ and $T'$ are representations of $R^1$. The representation $T'$ is not bounded on $R^1$, although it is bounded on every bounded subset of $R^1$.

EXAMPLE 2. Let $G$ be a topological transformation group which acts continuously on a locally compact measure space $S$ and leaves the measure invariant. Set $H = L^2(S, \mu)$. Let the map $x \to T_x$ be defined by means of the left translation, i.e.,

$$(T_x u)(s) = u(x^{-1}s), \quad u \in H,\ s \in S,\ x \in G. \tag{4}$$

Clearly, every $T_x$ is a linear operator. Moreover

$$[T_x(T_y u)](s) = (T_y u)(x^{-1}s) = u(y^{-1}x^{-1}s) = (T_{xy}u)(s),$$

i.e.,

$$T_x T_y = T_{xy} \quad \text{and} \quad T_e = 1.$$

Hence, the map (4) defines a representation of $G$ in $H$.

The invariance of the measure implies

$$(T_x u, T_x v) = \int u(x^{-1}s)\overline{v(x^{-1}s)}\,\mathrm{d}\mu(s) = (u, v),$$

i.e. $T_x$ is isometric, and because the domain $D_{T_x}$ of $T_x$ is equal to $H$, every $T_x$ is unitary.

If $u \in C_0(S)$,

$$\sup|u(x^{-1}s)-u(s)| \to 0,$$

as $x \to e$ by uniform continuity (cf. proposition 2.2.4). Moreover, there exists a fixed compact set $K \subset S$, supporting $u$ and $T_x u$, for $x$ sufficiently near $e$, and we have

$$\begin{aligned}||T_x u-u|| &= \Big[\int_S |u(x^{-1}s)-u(s)|^2 \mathrm{d}\mu(s)\Big]^{1/2} \\ &\leqslant \max_{s\in K}|u(x^{-1}s)-u(s)|\sqrt{\mu(K)} \to 0, \qquad \text{as } x \to e.\end{aligned}$$

The continuity property as $x \to y$ follows if one replaces $u$ by $T_y u$.

Now if $u \in L^2(S, \mu)$ and $\varepsilon > 0$, there exists a $v \in C_0(S)$ such that $||u- v|| < \varepsilon$ Then

$$||T_x u-u|| \leqslant ||T_x(u-v)||+||T_x v-v||+||v-u||$$

and the invariance of $\mu$ under $G$ implies

$$||T_x u-u|| \leqslant 2\varepsilon+||T_x v-v||.$$

Hence $||T_x u-u|| \leqslant 3\varepsilon$, if $x$ is sufficiently close to $e$. Consequently, the map (4) defines a strongly continuous unitary representation of $G$ in $L^2(S, \mu)$. ▼

If $S = G$ the representation (4) is called the *left regular representation.* If $S = G/K$, where $K$ is a closed subgroup of $G$, the representation (4) is called the *quasi-regular representation*. Clearly by the right translations

$$T_x u(y) = u(yx) \tag{5}$$

one can define similarly the *right regular representation* of $G$ in $L^2(G, \mu)$.

A representation $x \to T_x$ of $G$ in $H$ is said to be *weakly continuous,* if for arbitrary $u, v \in H$,

$$(T_x u, v) \to (T_{x_0} u, v), \qquad \text{as } x \to x_0. \tag{6}$$

The weak and the strong continuity for unitary representations are equivalent. Indeed, we have

PROPOSITION 1. *Let $T$ be a unitary representation of a group $G$ in a Hilbert space $H$. Then the following statements are equivalent.*

*1° $T$ is strongly continuous.*

*2° $T$ is weakly continuous.*

*3° The function $x \to (T_x u, u)$ is continuous at $e$ for all $u \in H$.*

PROOF: Clearly, 1° ⇒ 2° and 2° ⇒ 3°. Hence, it suffices to show that 3° ⇒ 1°. Indeed, for any $u \in H$ and $x, y \in G$ we have

$$\begin{aligned}||T_x u-T_y u||^2 &= (T_x u, T_x u)-(T_x u, T_y u)-(T_y u, T_x u)+(T_y u, T_y u) \\ &= 2(u, u)-2\operatorname{Re}(T_y u, T_x u) \leqslant 2|(u, u)-(T_y u, T_x u)| \\ &= 2|(u, u)-(T_{x^{-1}y} u, u)|.\end{aligned} \tag{7}$$

Consequently, if 3° is satisfied, $||T_x u - T_y u|| \to 0$, for $x \to y$, by virtue of eq. (7). Thus, 3° ⇒ 1°. ▼

*Remark:* For unitary representations the weak continuity also implies the strong left uniform continuity (cf. eq. 2.2.(13)). Indeed

$$||T_x u - T_y u|| = ||u - T_{x^{-1}y} u||.$$

Hence, for arbitrary $\varepsilon > 0$ there exists a neighborhood $V_\varepsilon$ of $e$ such that

$$||T_x u - T_y u|| < \varepsilon, \quad \text{whenever } x^{-1}y \in V_\varepsilon. \;\blacktriangledown$$

It is interesting that a unitary representation $x \to T_x$ of a group $G$ might not be continuous. This important fact is illustrated by the following

EXAMPLE 3. Let $G = R$ and let $\{\xi^\alpha\}$ be a Hamel basis* for $R$. Take $\xi^1 = 1$ and let $\xi^2$ be any other basis element, which from $r$-independence of basis elements, must be irrational. Clearly, any element $x \in G$ can be written as $x = \sum r_\alpha \xi^\alpha$. Consider the map

$$\varphi\colon\; x \to T_x = \exp(ir_2)I, \tag{8}$$

where $x \in G$ and $I$ is the unit operator in the carrier Hilbert space $H$. We have $T_{xy} = T_x T_y$ and $T_e = I$. Moreover, $T_x = I$, when $x$ is rational (because in this case $x = x\xi^1$) and $T_{\xi^2} = \exp i \cdot I$. Hence, the map (8) provides a discontinuous, unitary representation of $G$ in $H$. ▼

*Remark:* The representations of a topological group $G$ given by def. 1 do not exhaust all possible representations of $G$ which one encounters in theoretical physics or geometry. For instance, if a carrier space of a representation is a group manifold itself, then, the map $x \to T_x$ defined by

$$T_x^R y = yx, \quad \text{or} \quad T_x^L y = x^{-1}y, \tag{9}$$

satisfies the conditions (1), and due to continuity of a group multiplication, also satisfies continuity conditions. However, the map $x \to T_x$ is nonlinear even if $G$ is commutative, and therefore, there appears a new type of representation of $G$. A similar situation occurs if, for instance, $G$ acts on a curved homogeneous space $G/K$, or on a nonlinear differential manifold $M$. To distinguish these from linear representations they are often called *realizations*. ▼

Let $\varphi$ be the homomorphism $x \to T_x$ given in def. 1. The set $K$ of all elements of $G$, which satisfies the condition $\varphi(x) = I$, $x \in G$, is said to be the *kernel* of the homomorphism $\varphi$. If $x, y \in K$, then $xy \in K$ and $x^{-1} \in K$. Moreover, if $x \in K$ and $y \in G$, then $\varphi(yxy^{-1}) = T_y T_{y^{-1}} = I$, i.e., $yxy^{-1} \in K$. Therefore, $K$ is an invariant subgroup of $G$.

* The Hamel base in $R$ is an (uncountable) base of $R$, considered as a vector space over the field $Q$ of rational numbers.

A representation $x \to T_x$ of $G$ is said to be *faithful*, if the map $x \to T_x$ is one-to-one. In this case, $K = \{e\}$. If $K \neq \{e\}$, then all elements of a coset $xK$ for a fixed $x \in G$ are represented by the same operator and two different cosets by different operators. Hence, the homomorphism $\varphi$: $x \to T_x$, which provides a non-faithful representation of $G$, can be considered as a faithful representation of the quotient group $G/K$ given by the isomorphism $\varphi$: $xK \to T_x$.

Note that simple Lie groups without nontrivial discrete centers, e.g., $SU(n)/Z_n$, where

$$Z_n = \left\{\exp\left[\frac{2\pi i}{k}\right]e, \quad k = 1, 2, \ldots, n-1\right\},$$

have no invariant subgroups. Hence, all representations of simple Lie groups without nontrivial discrete centers are either faithful or trivial ones.

*The Matrix Form of Representations*

Let $\{e_i\}_1^N$, $N \leqslant \infty$, be an orthonormal basis in the carrier space $H$. An operator $T_x$, $x \in G$, transforms a basis element $e_j$ into $T_x e_j \in H$. The latter can be represented in the form

$$T_x e_j = D_{ij}(x) e_i, \quad j = 1, 2, \ldots, N. \tag{10}$$

Hence,

$$D_{ij}(x) = (T_x e_j, e_i), \quad i, j = 1, 2, \ldots, N. \tag{11}$$

Therefore, an operator $T_x$ can be represented in the basis $\{e_i\}_1^N$ by the finite, or infinite matrix $[D_{ij}(x)]$. It follows from proposition 1 that every matrix element (11) is a continuous function on $G$. The matrix of the operator $T_{xy}$ is the product of matrices (11), i.e.,

$$D_{ij}(xy) = D_{ik}(x) D_{kj}(y). \tag{12}$$

Indeed,

$$\begin{aligned} D_{ij}(xy) &= (T_{xy} e_j, e_i) = (T_x T_y e_j, e_i) = (T_y e_j, T_x^* e_i) \\ &= (T_y e_j, e_k)(e_k, T_x^* e_i) = (T_y e_j, e_k)(T_x e_k, e_i) \\ &= D_{ik}(x) D_{kj}(y). \end{aligned}$$

Clearly, the matrix form (11) of the operator $T_x$ depends upon the choice of a basis in the carrier space $H$. If $U$ is a unitary operator which maps $H$ onto itself, and if $h_i = Ue_i$, $i = 1, 2, \ldots, N$, then the basis $\{h_i\}_1^N$ is orthonormal and the matrix form of $T_x$ in the new basis is

$$U^{-1} D(x) U, \tag{13}$$

where the matrix elements $U_{ij}$ of the operator $U = \{U_{ij}\}$ are

$$U_{ij} = (Ue_j, e_i) = (h_j, e_i).$$

Indeed,

$$\begin{aligned} D_{ij}^{(h)}(x) &= (T_x Ue_j, Ue_i) = (T_x Ue_j, e_s)(e_s, Ue_i) \\ &= (Ue_j, T_x^* e_s)(e_s, Ue_i) = (Ue_j, e_p)(e_p, T_x^* e_s)(e_s, Ue_i) \\ &= U_{is}^{-1} D_{sp}(x) U_{pj}. \end{aligned}$$

A unitary operator $T_x$ is represented by a unitary matrix $[D_{ij}(x)]$ only if the basis $\{e_i\}_1^N$ is orthonormal.

Let $T$ be a matrix representation of a topological group $G$ in a Hilbert space $H$. Consider the maps

$$\begin{aligned} &1^\circ\ x \to \check{T}_x \equiv T_{x^{-1}}^T, \\ &2^\circ\ x \to \hat{T}_x = T_x^{*T} = \bar{T}_x, \\ &3^\circ\ x \to \tilde{T}_x = T_{x^{-1}}^*. \end{aligned} \tag{14}$$

It is easy to verify that any of the maps (14) defines a representation of $G$, for instance,

$$\tilde{T}_e = I, \quad \text{and} \quad \tilde{T}_{xy} = T_{(xy)^{-1}}^* = (T_{y^{-1}} T_{x^{-1}})^* = \tilde{T}_x \tilde{T}_y.$$

The continuity of the maps (14) follows from the fact that the operations '$T$' and '$*$' are continuous.

The representations (14)1°, (14)2° and (14)3° are said to be *contragradient*, *conjugate* and *conjugate-contragradient* to the representation $T$, respectively. For unitary representations contragradient and conjugate representations coincide.

## § 2. Equivalence of Representations

Let $x \to T_x$ be a representation of a topological group $G$ in a Hilbert space $H$. Let $S$ be a bounded isomorphism of $H$ onto a Hilbert space $H'$. Then the map $\varphi$: $x \to T'_x = ST_x S^{-1}$ defines a representation of $G$ in $H'$. Indeed

$$T'_{xy} = ST_x S^{-1} ST_y S^{-1} = T'_x T'_y, \quad T'_e = I, \tag{1}$$

and

$$\begin{aligned} \|T'_x u' - T'_y u'\| &= \|S(T_x S^{-1} u' - T_y S^{-1} u')\| \\ &\leqslant \|S\| \|T_x u - T_y u\| \to 0 \quad \text{as } x \to y. \end{aligned}$$

In this manner a whole class of new representations may be constructed, starting from a given representation $x \to T_x$ of $G$, acting in the same or isomorphic carrier spaces. These representations, however, are not essentially different. Hence, we collect them into one class of representations by means of the notion of equivalence of the representations.

DEFINITION 1. A representation $x \to T_x$ of a topological group $G$ in a Hilbert space $H$ is said to be *equivalent* to a representation $x \to T'_x$ in $H'$, if there exists a bounded isomorphism $S$ of $H$ onto $H'$ such that

$$ST_x = T'_x S \quad \text{for all } x \in G. \tag{2}$$

We shall write in this case $T_x \simeq T'_x$. This operation '$\simeq$' is reflexive, symmetric and transitive. It means

$$\begin{gathered} T \simeq T, \\ T \simeq T' \Rightarrow T' \simeq T, \\ T \simeq T' \text{ and } T' \simeq T'' \Rightarrow T \simeq T''. \end{gathered} \tag{3}$$

Hence, it is an equivalence relation. Therefore it partitions the set of all representations of $G$ into disjoint classes of equivalent representations.

Next we introduce the narrower concept of unitary equivalence of two representations in the Hilbert spaces $H$ and $H'$.

DEFINITION 2. Two representations: $x \to T_x$ in $H$, and $x \to T'_x$ in $H'$ are *unitarily equivalent* if there exists a unitary isomorphism $U$: $H \to H'$ such that $UT_x = T'_xU$ for every $x \in G$.

EXAMPLE 1. Let $T^L$ and $T^R$ be the left and the right regular representation of a group $G$. The involution $I$: $u(x) \to u(x^{-1})$ defines a unitary map of $H$ onto itself. We have

$$(IT^R_x u)(y) = (T^R_x u)(y^{-1}) = u(x^{-1}y) = (Iu)(y^{-1}x) = (T^L_y Iu)(x).$$

Hence

$$IT^R_x = T^L_x I.$$

PROPOSITION 1. *Two equivalent unitary representations are unitarily equivalent.*

PROOF: Taking the adjoint of both sides of eq. (2) we obtain

$$T_x S^* = S^* T'_x. \tag{4}$$

Hence, by eqs. (4) and (2) we have

$$SS^* T'_x = ST_x S^* = T'_x SS^*,$$

i.e., every $T'_x$ commutes with the positive hermitian operator $SS^*$ and, thereby, also with $A = \sqrt{(SS^*)}$. The operator $A^{-1}S$ is a unitary operator, which satisfies the condition (2). Indeed,

$$A^{-1}ST = A^{-1}T'S = T'A^{-1}S.$$

Hence, $T$ and $T'$ are unitarily equivalent. ▼

Two equivalent unitary representations: $x \to T_x$ in $H$, and $x \to T'_x$ in $H'$, can be described by the same matrices by a proper choice of bases in $H$ and $H'$. Indeed, let $S$ be an isomorphism of $H$ onto $H'$, such that $ST_x = T'_xS$, and let $\{e_i\}_1^N$, $N \leqslant \infty$, be a basis in $H$. Then, taking as a basis in $H'$ the set $\{e'_i = Se_i\}_1^N$, we obtain

$$T_x e_j = D_{ij}(x) e_i \tag{5}$$

and

$$D'_{ij}(x)e'_i = T'_x e'_j = T'_x Se_j = ST_x e_j = SD_{ij}(x)e_i = D_{ij}(x)e'_i,$$

i.e., both matrices coincide.

Let $T$ and $T'$ be representations of $G$ in $H$ and $H'$, respectively. A bounded operator $S$ from $H$ into $H'$ is said to be an *intertwining operator* for $T$ and $T'$, if $ST_x = T'_x S$ for every $x \in G$. The set of all intertwining operators forms a linear space, which we denote by $R(T, T')$. The proposition 1 can now be restated in the following form: Two unitary representations $T$ and $T'$ are equivalent if and only if there exists in $R(T, T')$ a unitary operator from $H$ onto $H'$

Note that for $T = T'$, $R(T, T)$ is an algebra.

## § 3. Irreducibility and Reducibility

Let $x \to T_x$ be a representation of a topological group $G$ in a Hilbert space $H$. A subspace or subset $H_1$ of $H$ is said to be *invariant* (with respect to $T$) if $u \in H_1$ implies $T_x u \in H_1$ for every $x \in G$.

Every representation has at least two invariant subspaces: the null-space $\{0\}$ and the whole space $H$. These invariant subspaces are said to be trivial. The non-trivial invariant subspaces or subsets will be called *proper*. We introduce now the concept of irreducibility, which plays a fundamental role in the representation theory.

DEFINITION 1 (Algebraic irreducibility). A representation $x \to T_x$ of a group $G$ in $H$ is said to be *algebraically irreducible*, if it has no proper invariant subsets in $H$.

DEFINITION 2 (Topological irreducibility). A representation $x \to T_x$ of a topological group $G$ in $H$ is said to be *topologically irreducible* if it has no proper *closed* invariant subspace.

Clearly, algebraic irreducibility implies topological irreducibility. A representation, which has proper invariant subspaces is said to be *reducible*. Unless otherwise stated, we use in the following the name irreducible (reducible) for topologically irreducible (reducible) representations.

At least two new representations can be associated with every topologically reducible representation. The first one is obtained by the restriction of every $T_x$ to the closed subspace $H_1$. This representation is called the *subrepresentation of* $T$ and is denoted by ${}^{H_1}T$. The second one can be realized in the quotient space $H/H_1$. Indeed, because $H_1$ is invariant, a coset $u+H_1$ is transformed by $T_x$ into $T_x u+H_1 \in H/H_1$, i.e., $H/H_1$ is also an invariant space.

Let $H$ be a Hilbert space, and $H_1$ a proper invariant subspace. The orthogonal complement $H_1^\perp$ of $H_1$ may not, in general, be an invariant subspace of $H$. However, if a representation $x \to T_x$ in $H$ is unitary, then this is true. Indeed, we have

PROPOSITION 1. *Let $x \to T_x$ be a unitary representation of any group $G$ in a Hilbert space $H$. Let $H_1$ be a subspace of $H$ and let $P_1$ be the projection operator in $H$, whose range is $H_1$. Then,* 1° *The orthogonal complement $H_1^{\perp}$ of $H_1$ is invariant if and only if $H_1$ is invariant.*

2° *$H_1$ is invariant if and only if $P_1T_x = T_xP_1$, for every $x \in G$.*

PROOF: *ad* 1°. Let $H_1$ be invariant. Then, for $u \in H_1$, $v \in H_1^{\perp}$ and any $T_x$ we have

$$(T_x v, u) = (v, T_x^* u) = (v, T_{x^{-1}} u) = 0$$

because $T_{x^{-1}} u \in H_1$. Hence $H_1^{\perp}$ is also invariant. The converse statement follows by exchanging the roles of $H_1$ and $H_1^{\perp}$.

*ad* 2°. Let $H_1$ be invariant and $u \in H$. Then, $T_xP_1u \in H_1$ for every $x \in G$ and $P_1T_xP_1u = T_xP_1u$; because $u$ is arbitrary, we have $P_1T_xP_1 = T_xP_1$. Taking the adjoint of both sides, we obtain $P_1T_x^*P_1 = P_1T_x^*$, or $P_1T_{x^{-1}}P_1 = P_1T_{x^{-1}}$. Setting $y = x^{-1}$, we have $P_1T_y = P_1T_yP_1 = T_yP_1$ for every $y \in G$. Conversely, if $P_1T_x = T_xP_1$ for every $x \in G$, then, for $u_1 \in H_1$, we get $T_xu_1 = T_xP_1u_1 = P_1T_xu_1 \in H_1$. Hence, $H_1$ is invariant.▼

The following example shows that the assumption of unitarity in the proposition 1 is essential, i.e. for nonunitary representations the orthogonal complement of an invariant subspace is, in general, not invariant. ▼

EXAMPLE 1. Let $G = R^1$, and let $H$ be the two-dimensional real Hilbert space with the scalar product $(u, v) = u_1v_1 + u_2v_2$. We represent $R^1$ in $H$ by the triangular nonunitary matrices

$$R \in x \to T_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \quad \text{i.e.,} \quad T_x \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 + xu_2 \\ u_2 \end{bmatrix}. \tag{1}$$

It follows from eq. (1) that the subspace $H_1$ consisting of vectors $u = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$ is invariant with respect to $T$, while the orthogonal complement $H_1^{\perp}$ consisting of vectors $u = \begin{bmatrix} 0 \\ u_1 \end{bmatrix}$ is not invariant. ▼

A Hilbert space $H$ is said to be the *direct sum* of its subspaces $H_1, H_2, \ldots$, i.e.,

$$H = H_1 \oplus H_2 \oplus \ldots = \sum_i \oplus H_i, \tag{2}$$

if the following conditions are satisfied:

1° $H_i \perp H_j$ for $i \neq j$.

2° Every element $u \in H$ decomposes into the convergent series

$$u = \sum_i u_i, \quad \text{where } u_i \in H_i.$$

DEFINITION 3. A representation $T$ of $G$ in a Hilbert space $H$ is said to be the *direct sum* of representations $T_i$ of $G$ in $H_i$ if $H_i$ are invariant subspaces of $H$, such that $H = \sum_i \oplus H_i$ and if each $T_i$ is a subrepresentation of $T$. ▼

We write in this case

$$T = \sum_i \oplus T_i. \tag{3}$$

A representation $T$ of $G$ in $H$ is said to be *fully* or *completely reducible* (or *discretely decomposable*) if it can be expressed as a direct sum of irreducible subrepresentations. Finite-dimensional representations which are reducible but not fully reducible are called *indecomposable* representations (e.g. example 1). If we take as basis vectors in $\sum_i \oplus H_i$, the basis vectors of orthogonal subspaces $H_i$, we see that the matrix representation of a completely reducible representation is of the form

$$D(x) = \begin{bmatrix} D^1(x) & 0 & \dots & 0 & \dots \\ 0 & D^2(x) & \dots & 0 & \dots \\ 0 & 0 & \dots & D^l(x) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}, \tag{4}$$

where each $D^l(x)$ is irreducible. Example 1 shows that a reducible representation need not be completely reducible. However, for unitary *finite-dimensional* representations, we have

COROLLARY 2. *A finite-dimensional unitary representation of any group is completely reducible.*

PROOF: If $H_l$ is a proper, invariant subspace of $H$, then, by proposition 1, $H_1^\perp$ is also invariant and $H = H_1 \oplus H_1^\perp$. If $H_1$ or $H_1^\perp$ contains a proper invariant subspace, then we use again proposition 1 until we obtain a decomposition into irreducible invariant subspaces of $H_1$, provided this procedure converges (hence finite-dimensional case). ▼

The following proposition is fundamental in the group representation theory:

PROPOSITION 3 (Schur's lemma). *Let $T$ and $T'$ be unitary, irreducible representations of $G$ in $H$ and $H'$, respectively. If $S$ is a bounded linear map of $H \to H'$ such that*

$$ST_x = T'_x S \quad \textit{for every } x \in G, \tag{5}$$

*then, either $S$ is an isomorphism of the Hilbert spaces $H$ and $H'$ (i.e., $T \simeq T'$), or $S = 0$.*

PROOF: The adjoint of eq. (5) gives $TS^* = S^*T'$. Hence, the positive definite, hermitian operator $V = S^*S$ commutes with $T$.

If $V = \int \lambda dE(\lambda)$ is the spectral decomposition of $V$, then $TE(\lambda) = E(\lambda)T$. Therefore every closed subspace $H(\lambda) = E(\lambda)H$ is invariant. Since $H$ is irreducible, $H(\lambda)$ coincides with $H$ or with the null-space $\{0\}$. This implies $V = \lambda I$. Similarly, one obtains $V' = SS^* = \lambda' I'$. Because $\lambda S = SS^*S = \lambda' S$, either $\lambda = \lambda'$ if $S \neq 0$, or $S = 0$ otherwise. In the first case setting $U = \lambda^{-1/2}S$ we obtain

$U^*U = I$ and $UU^* = I'$; hence $S$ is an isomorphism of $H$ and $H'$ and according to def. 2.1, we have $T \simeq T'$. ▼

The Schur's Lemma 3 implies the following criterion of irreducibility:

PROPOSITION 4 (Schur's lemma—unitary case). *A unitary representation $T$ of $G$ in $H$ is irreducible if and only if the only operators commuting with all the $T_x$ are scalar multiples of the identity.*

PROOF: If $ST_x = T_xS$, then $S^*T_x = T_xS^*$. Hence, the self-adjoint operators $S_1 = \frac{1}{2}(S+S^*)$ and $S_2 = \frac{1}{2i}(S-S^*)$ also commute with all the $T_x$. Therefore, $S = \lambda_1 I + \lambda_2 I = \lambda I$, by proposition 3. Conversely, if every operator $S$ commuting with $T$ has the form $\lambda I$, then the projection operator $P$ commuting with $T$ is either $I$ or 0. Hence, by proposition 1.2°, the only closed invariant subspaces are the null-space or the whole carrier space $H$. Consequently $T$ is irreducible. ▼

The result of the proposition 4 allows us to give a new definition of irreducibility:

DEFINITION 2′ A unitary representation $T$ of $G$ in $H$ is said to be *irreducible* if the only operators, which commute with all the $T_x$, are scalar multiples of the identity. This formulation of irreducibility is called *operator irreducibility* of $T$. ▼

The proposition 4 has the following analogon for finite-dimensional (unitary or not) representations.

PROPOSITION 5 (Schur's lemma—finite-dimensional case). *Let $T$ be an irreducible representation of $G$ in $H$,* $\dim H < \infty$. *The only operators which commute with all $T_x$ are scalar multiples of the identity.*

PROOF: Let

$$ST_x = T_xS \qquad \text{for all } x \in G, \tag{6}$$

and let $N = \{u \in H\colon \; Su = 0\}$. By virtue of (6) we have

$$\{0\} = T_xSN = ST_xN.$$

Therefore, $T_xN \subset N$, i.e., $N$ is invariant subspace of $H$. Because $T$ is irreducible $N = \{0\}$ or $H$. Hence, $S$ is either an isomorphism or $S = 0$. Now, let $S$ be any isomorphism which commutes with all $T_x$ and let $\lambda \neq 0$ be an eigenvalue of $S$· clearly, $(S-\lambda I)$ is not an isomorphism of $H$; hence $(S-\lambda I) = 0$. ▼

### *Reducible Representations*

There exists a useful classification of reducible representations according to properties of the center $\mathrm{CR}(T, T)$ of the algebra $R(T, T)$ of intertwining operators. We start with the case when $\mathrm{CR}(T, T)$ is minimal.

DEFINITION 4. A representation $T$ of $G$ is said to be a *factor representation* if the center of $R(T, T)$ contains only multiples of the identity. Representations of this type are called the *primary representations.*

Clearly, an irreducible representation is a factor representation. An interesting feature of factor representations is given by the following

PROPOSITION 6. *Let $T$ be a factor representation which contains an irreducible subrepresentation $V$. Then, there exists an integer $n$, $n = 1, 2, 3, \ldots$, such that $T \simeq nV \equiv V \oplus V \oplus V \oplus \ldots + \oplus V$ ($n$ terms).* ▼

(For the proof cf., e.g., Pozzi 1966, proposition 6.14.)

This result implies the following definition of the so-called type I factor representations.

DEFINITION 5. If a factor representation contains an irreducible subrepresentation, it is said to be of *type I*. A group $G$ is said to be of *type I* if it has only type I factor representations.

In this book we shall deal exclusively with factor representations of type I. The factor representations of type I appear most often in applications in the problem of the reduction of the tensor products of representations. We need, then, additional invariant operators (quantum numbers) to split out the factor $nT^i$ onto its irreducible parts. We deal with this problem in sec. 6 and ch. 18, § 2.

Another interesting class of representations is obtained if the center of $R(T, T)$ is as large as possible, i.e., if it coincides with the whole $R(T, T)$.

DEFINITION 6. A unitary representation $T$ is said to be *multiplicity-free*, if $R(T, T)$ is commutative.

Note that if $T$ is both a factor and multiplicity free, then $R(T, T) = \{\lambda I\}$, i.e., $T$ is irreducible by virtue of Schur's lemma.

If $T$ is multiplicity free and discretely decomposable, then

$$T = \sum_i \oplus T^i,$$

where all $T^i$ are mutually inequivalent and irreducible. This follows directly from the fact that $R(T, T)$ contains in this case the maximal number of mutually commuting invariant operators. Thus, in contradistinction to factor representations, in the case of the multiplicity-free representations the generators of $R(T, T)$ provide a labelling of representations.

## § 4. Cyclic Representations

It is useful, in the analysis of the properties of representations of a given group $G$, to decompose any representation $T$ into more elementary constituents. The cyclic representations may be used for this purpose.

A representation $T$ of $G$ in $H$ is said to be *cyclic* if there is a vector $v \in H$ (called a *cyclic vector* for $T$), such that the closure of the linear span of all $T_x v$ is $H$ itself. The following theorem allows us to restrict our attention, in the case of unitary representations, to cyclic representations only.

THEOREM 1. *Every unitary representation $T$ of $G$ in $H$ is a direct sum of cyclic subrepresentations.*

PROOF: Let $v_1 \in H$ be any non-zero vector and let $H_{v_1}$ be the closure of the linear span of all vectors $T_x v_1$, $x \in G$. The space $H_{v_1}$ is invariant relative to $T$. Indeed, let $\tilde{H}_{v_1}$ be the linear span of all vectors $T_x v_1$; then, for each $u \in H_{v_1}$ there exists a sequence $\{u_n\}$ of vectors $u_n \in \tilde{H}_{v_1}$, which converges to $u$. Clearly $T_x u_n \in \tilde{H}_{v_1}$. The continuity of each $T_x$ implies $T_x u_n \to T_x u$. Hence, the vector $T_x u \in H_{v_1}$ and consequently $H_{v_1}$ is invariant. Thus, the subrepresentation ${}^{H_{v_1}}T$ is cyclic and $v_1$ is the cyclic vector for it. If $H_{v_1} = H$, the proof is completed. Otherwise, choose any non-zero vector $v_2$ in $H_{v_1}^{\perp} = H - H_{v_1}$ and consider the closed linear span $H_{v_2}$, which is invariant relative to $T$ and orthogonal to $H_{v_1}$, and so forth.

Let $\tau$ denote the family of all collections $\{H_{v_i}\}$, each composed of a sequence of mutually orthogonal, invariant and cyclic subspaces and order the family $\tau$ by means of the inclusion relation $\subset$. Then $\tau$ is an ordered set to which Zorn's lemma (cf. app. A.1) applies, which assures the existence of a maximal collection $\{H_{v_i}\}_{\max}$. By the separability of $H$, there can be at most a countable number of subspaces in $\{H_{v_i}\}_{\max}$, and their direct sum, by the maximality of $\{H_{v_i}\}_{\max}$, must coincide with $H$. ▼

Using the th. 1, we can give now a convenient criterion for the irreducibility of a unitary representation.

PROPOSITION 2. *A unitary representation $T$ of $G$ in $H$ is irreducible if and only if every non-zero vector $u \in H$ is cyclic for $T$.*

PROOF: If $T$ is irreducible, then by the proof of th. 1, every non-zero vector is cyclic for $T$. In order to prove the converse statement, suppose that $H_1$ is a non-trivial invariant subspace of $H$ and choose a vector $0 \neq v_1 \in H_1$. Due to the invariance of $H_1$ we have $T_x v_1 \in H_1$, moreover the closure of the linear span of all $T_x v_1$, which by assumption is the whole $H$, is contained in $H_1$. Hence, we have a contradiction. Therefore $H$ does not contain a nontrivial invariant subspace and consequently $T$ is irreducible. ▼

We have, moreover, the following convenient criterion for the unitary equivalence of cyclic representations.

PROPOSITION 3. *Let $T$ and $T'$ be unitary cyclic representations of $G$ in $H$ and $H'$ with cyclic vectors $v \in H$ and $v' \in H'$, respectively.*

*If*

$$(T_x v, v)_H = (T'_x v', v')_{H'} \quad \textit{for every } x \in G, \tag{1}$$

*then $T$ is unitarily equivalent to $T'$.*

PROOF: Let $\tilde{H}$ (resp. $\tilde{H}'$) be the linear span of all vectors $T_x v$ (resp. $T_x v'$), which is dense in $H$ (resp. $H'$). Then any $u \in \tilde{H}$ is of the form

$$u = \sum_{i=1}^{n} \alpha_i T_{x_i} v. \tag{2}$$

We define a map $S$ by the formula

$$Su = \sum_{i=1}^{n} \alpha_i T'_{x_i} v'. \tag{3}$$

Then, by eqs. (3) and (1):

$$\begin{aligned} \|Su\|^2_{H'} &= \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (T'_{x_i} v', T'_{x_j} v')_{H'} \\ &= \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (T'_{x_j^{-1} x_i} v', v')_{H'} \\ &= \sum_{i,j=1}^{n} \alpha_i \bar{\alpha}_j (T_{x_j^{-1} x_i} v, v)_{H}. \\ &= \|u\|^2_{H}. \end{aligned}$$

Therefore, $S$ is a linear isometric (hence, continuous) map, such that $ST_x u = T'_x Su$ for all $u \in H$. Thus, $S$ can be uniquely extended to a unitary map $\bar{S}$ from $H$ onto $H'$, such that $\bar{S}T = T'\bar{S}$. Consequently $T$ is unitarily equivalent to $T'$. ▼

## § 5. Tensor Product of Representations

### A. *Tensor Product of Spaces and Operators*

DEFINITION 1. Let $\overset{1}{E}$ end $\overset{2}{E}$ be two vector spaces. Let $\overset{1}{E}\square\overset{2}{E}$ be a vector space whose elements are formal linear combinations

$$\sum c_{x,y}(x, y), \quad x \in \overset{1}{E},\ y \in \overset{2}{E},$$

with a finite number of coefficients $c_{x,y} \in C$ different from zero. Let $N$ denote the subspace of $\overset{1}{E}\square\overset{2}{E}$ spanned by all vectors of the form

$$(x, y_1+y_2)-(x, y_1)-(x, y_2), \quad (x_1+x_2, y)-(x_1, y)-(x_2, y),$$
$$(\lambda x, y)-\lambda(x, y), \quad (x, \lambda y)-\lambda(x, y).$$

The tensor product is then defined as the quotient space.

$$\overset{1}{E}\otimes\overset{2}{E} = \overset{1}{E}\square\overset{2}{E}/N. ▼$$

Let $\varphi_2$ be the restriction of the canonical map $\psi: \overset{1}{E}\square\overset{2}{E} \to \overset{1}{E}\otimes\overset{2}{E}$ to the Cartesian product space $\overset{1}{E}\times\overset{2}{E}$: then we set $\varphi[(x, y)] \equiv x\otimes y$. We have

$$\begin{aligned}
&\text{(i)}\ x\otimes(y_1+y_2) = x\otimes y_1+x\otimes y_2,\\
&\text{(ii)}\ (x_1+x_2)\otimes y = x_1\otimes y+x_2\otimes y,\\
&\text{(iii)}\ (\lambda x)\otimes y = x\otimes(\lambda y) = \lambda(x\otimes y).
\end{aligned} \tag{1}$$

Let $\{\overset{k}{e}_i\}_{i=1}^{\dim \overset{k}{E}}$, $k = 1, 2$, be bases in $\overset{1}{E}$ and $\overset{2}{E}$ respectively. Then the map $\varphi$ associates with each pair $(\overset{1}{e}_i, \overset{2}{e}_k)$ in $\overset{1}{E}\times\overset{2}{E}$ an element $\overset{1}{e}_i\otimes\overset{2}{e}_k$; for $x = x^i\overset{1}{e}_i$ and $y = y^k\overset{2}{e}_k$ where only finite number of coordinates is different from zero, we have

$$x\otimes y = x^i y^k \overset{1}{e}_i\otimes\overset{2}{e}_k. \tag{2}$$

If $\overset{1}{E}$ and $\overset{2}{E}$ are Hilbert spaces with scalar products $(\cdot\,,\cdot)_i$, $i = 1, 2$, then the scalar product in $\overset{1}{E}\otimes\overset{2}{E}$ can be defined by the formula

$$(x_1\otimes y_1, x_2\otimes y_2) \equiv (x_1, x_2)_1(y_1, y_2)_2. \tag{3}$$

If either $\overset{1}{E}$ or $\overset{2}{E}$ is finite-dimensional, then the space $\overset{1}{E}\otimes\overset{2}{E}$ equipped with the scalar product (3) is complete. If both $\overset{1}{E}$ and $\overset{2}{E}$ are infinite-dimensional, we complete $\overset{1}{E}\otimes\overset{2}{E}$ in the norm defined by (3) and denote it by the symbol $\overset{1}{E}\dot{\otimes}\overset{2}{E}$.

EXAMPLE 1. Let $\overset{i}{H} = L^2(\overset{i}{\Omega}, \overset{i}{\mu})$, where $\overset{i}{\Omega}$ are open subsets of $R^n$ and $\overset{i}{\mu}$ are measures on $\Omega_i$. Let $\overset{1}{\mu}\otimes\overset{2}{\mu}$ denote the product measure on $\overset{1}{\Omega}\times\overset{2}{\Omega}$. Then

$$L^2(\overset{1}{\Omega}, \overset{1}{\mu})\dot{\otimes}L^2(\overset{2}{\Omega}, \overset{2}{\mu}) = L^2(\overset{1}{\Omega}\times\overset{2}{\Omega}, \overset{1}{\mu}\otimes\overset{2}{\mu}).\blacktriangledown$$

If $A$ and $B$ are bounded operators in $\overset{1}{E}$ and $\overset{2}{E}$, respectively, then the tensor product $A\otimes B$ of $A$ and $B$ is defined in $\overset{1}{E}\dot{\otimes}\overset{2}{E}$ by the formula

$$(A\otimes B)(x\otimes y) \equiv Ax\otimes By. \tag{4}$$

It follows from eq. (4) that

$$(A\otimes B)(A'\otimes B') = AA'\otimes BB'. \tag{5}$$

### B. *Tensor Product of Representations*

Let $\overset{1}{G}$ and $\overset{2}{G}$ be topological groups. Let $\overset{1}{g} \to T^1_{\overset{1}{g}}$ and $\overset{2}{g} \to T^2_{\overset{2}{g}}$ be representations of $\overset{1}{G}$ and $\overset{2}{G}$ acting in $\overset{1}{E}$ and $\overset{2}{E}$, respectively. We define in $\overset{1}{E}\otimes\overset{2}{E}$ the operator function

$$\overset{1}{G}\times\overset{2}{G} \ni (\overset{1}{g}, \overset{2}{g}) \to T^1_{\overset{1}{g}}\otimes T^2_{\overset{2}{g}}. \tag{6}$$

We have

$$\overset{1}{T}{}^1_e \otimes \overset{2}{T}{}^2_e = \overset{1}{I} \otimes \overset{2}{I}, \tag{7}$$

and by virtue of (5)

$$(\overset{1}{T}{}^1_g \otimes \overset{2}{T}{}^2_g)(\overset{1}{T}{}^1_{g_0} \otimes \overset{2}{T}{}^2_{g_0}) = (\overset{1}{T}{}^1_g \overset{1}{T}{}^1_{g_0} \otimes \overset{2}{T}{}^2_g \overset{2}{T}{}^2_{g_0}) = \overset{1}{T}{}^{11}_{gg_0} \otimes \overset{2}{T}{}^{22}_{gg_0}. \tag{8}$$

Hence the map (6) provides a representation of the direct product group $G_1 \times G_2$ in the tensor product space $\overset{1}{E} \otimes \overset{2}{E}$.

DEFINITION 2. The representation (6) of $\overset{1}{G} \times \overset{2}{G}$ in the space $\overset{1}{E} \otimes \overset{2}{E}$ is called the *outer tensor product representation*. If $\overset{1}{G} = \overset{2}{G}$ and $\overset{1}{g} = \overset{2}{g}$ then the prepresentation (6) is called the *inner tensor product* (or *Kronecker product*) *representation.*

Let $\overset{1}{E}$ and $\overset{2}{E}$ be Hilbert spaces and let $H = \overset{1}{E} \dot{\otimes} \overset{2}{E}$ be the Hilbert space tensor product. It follows then from eqs. (6) and (4) that if the representations $\overset{1}{T}$ and $\overset{2}{T}$ of $\overset{1}{G}$ and $\overset{2}{G}$, respectively, are continuous, then $\overset{1}{T} \otimes \overset{2}{T}$ is also continuous.

Let $\overset{1}{e_i} \otimes \overset{2}{e_k}$ be the basis in the Hilbert space $\overset{1}{E} \dot{\otimes} \overset{2}{E}$. Then by virtue of eqs. (4) and (3), the matrix elements of the tensor product representation $\overset{1}{T} \otimes \overset{2}{T}$ have the form

$$\begin{aligned} D_{ik,jm}(\overset{1}{g}, \overset{2}{g}) &= ((\overset{1}{T}{}^1_g \otimes \overset{2}{T}{}^2_g)(\overset{1}{e_j} \otimes \overset{2}{e_m}), \overset{1}{e_i} \otimes \overset{2}{e_k}) \\ &= (\overset{1}{T}{}^1_{\overset{1}{g}} \overset{1}{e_j}, \overset{1}{e_i})_1 (\overset{2}{T}{}^2_{\overset{2}{g}} \overset{2}{e_m}, \overset{2}{e_k})_2 = \overset{1}{D}_{ij}(\overset{1}{g}) \overset{2}{D}_{km}(\overset{2}{g}). \end{aligned} \tag{9}$$

EXAMPLE 1. Let $\overset{1}{Q}$ be a relativistic particle with mass $\overset{1}{m}$. Its wave function $\overset{1}{\psi}(\overset{1}{p})$ in the momentum space is an element of the space $\overset{1}{E} = L^2(\overset{1}{\Omega}, \overset{1}{\mu})$, where $\overset{1}{\Omega}$ is the mass hyperboloid $\overset{1}{p}{}^2 = \overset{1}{m}{}^2$, and $d\overset{1}{\mu} = \frac{d^3\overset{1}{p}}{\overset{1}{p}_0}$. The Poincaré group $\Pi = T^4 \rtimes SO(3,1)$ has the unitary continuous representation in $\overset{1}{E}$ given by the formula

$$\overset{1}{T}_{\{a,\Lambda\}} \overset{1}{\psi}(\overset{1}{p}) = \exp(ipa)\psi(\Lambda^{-1}\overset{1}{p}). \tag{10}$$

Let $\overset{2}{Q}$ be a second relativistic particle with mass $\overset{2}{m}$. The wave function $\psi(\overset{1}{p}, \overset{2}{p})$ of two-particle systems is an element of the tensor product space $L^2(\overset{1}{\Omega}, \overset{1}{\mu}) \dot{\otimes} L^2(\overset{2}{\Omega}, \overset{2}{\mu})$. One readily verifies using the def. 1 that

$$L^2(\overset{1}{\Omega}, \overset{1}{\mu}) \dot{\otimes} L^2(\overset{2}{\Omega}, \overset{2}{\mu}) = L^2(\overset{1}{\Omega} \times \overset{2}{\Omega}, \overset{1}{\mu} \otimes \overset{2}{\mu}), \tag{11}$$

where $\overset{1}{\mu}\otimes\overset{2}{\mu}$ denotes the product of measures $\overset{1}{\mu}$ and $\overset{2}{\mu}$. The tensor product representation $\overset{1}{T}_g\otimes\overset{2}{T}_g$ of the Poincaré group in the tensor product space (11), by virtue of eq. (4), is given by the formula

$$(\overset{1}{T}_{\{a,\Lambda\}}\otimes\overset{2}{T}_{\{a,\Lambda\}})\psi(\overset{1}{p},\overset{2}{p}) = \exp[\mathrm{i}(\overset{1}{p}+\overset{2}{p})a]\psi(\Lambda^{-1}\overset{1}{p},\Lambda^{-1}\overset{2}{p}).$$

Evidently this representation is unitary and continuous in the tensor product space (11). ▼

Let us note that even if the representations $\overset{1}{T}$ and $\overset{2}{T}$ of $G$ are irreducible, the inner tensor product $\overset{1}{T}\otimes\overset{2}{T}$ is in general highly reducible, i.e.,

$$\overset{1}{T}_g\otimes\overset{2}{T}_g \cong \oplus m_\lambda \overset{\lambda}{T}_g \tag{12}$$

where $\overset{\lambda}{T}$ are irreducible representations of $G$ and $m_\lambda$ is the multiplicity of $\overset{\lambda}{T}$ in the tensor product $\overset{1}{T}\otimes\overset{2}{T}$.

The determination of the multiplicities $m_\lambda$ of $\overset{\lambda}{T}$ in the tensor product $\overset{1}{T}\otimes\overset{2}{T}$ is the 'Clebsch–Gordan series' problem. It is one of the most difficult problems in group representation theory, whose solution is known only for several groups and certain types of representations. Even for such important groups like the Lorentz group, this problem is not yet completely solved.

Let $C$ be an operator in $\overset{1}{E}\otimes\overset{2}{E}$ which reduces the inner tensor product $\overset{1}{T}\otimes\overset{2}{T}$ of $G$ to a block diagonal form, i.e.,

$$C(\overset{1}{T}\otimes\overset{2}{T})C^{-1} = \oplus m_\lambda \overset{\lambda}{T}, \qquad C(\overset{1}{E}\otimes\overset{2}{E}) = \oplus m_\lambda \overset{\lambda}{E}. \tag{13}$$

The matrix elements of the operator $C$ are called the 'Clebsch–Gordan coefficients'. They allow one to express the basis elements $\overset{\lambda}{e}_l$ of the carrier space $\overset{\lambda}{E}$ of an irreducible representation $\overset{\lambda}{T}$ in terms of the tensor basis $\overset{1}{e}_i\otimes\overset{2}{e}_k$. The Clebsch–Gordan coefficients play for physical symmetry groups (like rotation, Lorentz or Poincaré groups) a fundamental role in particle physics.

## § 6. Direct Integral Decomposition of Unitary Representations

Let $g \to T_g$ be a unitary representation of a physical symmetry group in a Hilbert space $H$. In applications in most cases the representation $T$ is reducible. However, only the irreducible components $T(\lambda)$ of $T$ have a more direct physical meaning. Hence it is of fundamental importance to have a formalism which provides the description of $T$ in terms of its irreducible components.

In general the decomposition of a given reducible unitary representation onto a direct *sum* of irreducible representations is impossible, and one must use the concept of direct integral of representations and the direct integral of the corresponding carrier spaces. We illustrate this on a simple example.

Let $G$ be the translation group of the real line $R$ and let $g \to T_g$ be a unitary representation of $G$ in the Hilbert space $H = L^2(R)$ given by

$$T_g u(x) = u(x+g). \tag{1}$$

By virtue of Schur's lemma every irreducible representation of $G$ is one-dimensional (cf. proposition 6.1). Hence representation (1) is reducible. Suppose that $H_1$ is a one-dimensional invariant subspace in $H$. Then for every $u_1$ in $H_1$ we have

$$T_g u_1(x) = u_1(x+g) = \lambda_1(g) u_1(x).$$

Hence $u_1(x)$ must be an exponential function. But the only exponential function which is in $L^2(R^1)$ is $u_1 = 0$. Consequently, $H_1 = \{0\}$. Thus $H$ does not contain one-dimensional invariant subspaces. However, if we pass to the direct integral of Hilbert spaces we find explicit one-dimensional spaces in which irreducible representations of $G$ are realized. Indeed, let $A = \mathrm{i}\dfrac{\mathrm{d}}{\mathrm{d}x}$ be a self-adjoint operator in $H$. Using the spectral theorem we know that $A$ induces a decomposition of $H$ onto the direct integral (cf. app. B.3)

$$H \leftrightarrow \hat{H} = \int_{\Lambda} H(\lambda)\,\mathrm{d}\mu(\lambda), \tag{2}$$

where $\Lambda$ is the spectrum of $A$, $H(\lambda)$ are the one-dimensional Hilbert spaces and $\mathrm{d}\mu(\lambda)$ is the spectral measure associated with $A$. Each element $u$ of $\hat{H}$ is a vector-function $u = \{u(\lambda),\ u(\lambda) \in H(\lambda)\}$. In the present case the connection between the elements $u(\lambda)$ of $H(\lambda)$ and $u(x) \in H$ are given by the ordinary Fourier transform:

$$u(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\mathrm{i}\lambda x) u(x)\,\mathrm{d}x.$$

We obtain the transformation law of the elements $u(\lambda)$ in $H(\lambda)$ by taking the Fourier transform of $T_g u(x)$. We have

$$(T_g u)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(\mathrm{i}\lambda x)(T_g u)(x)\mathrm{d}x = \exp(-\mathrm{i}\lambda g) u(\lambda).$$

Hence every Hilbert space $H(\lambda)$ in the direct integral (2) is an invariant space of $T_g(\lambda)$. Thus the decomposition (2) of the carrier space $H$ induced by the operator $A$ implies the decomposition

$$T_g \leftrightarrow \hat{T}_g = \int T_g(\lambda)\,\mathrm{d}\mu(\lambda) \tag{3}$$

of $T_g$ onto the direct integral of irreducible representations. Notice that the operator $A$ (or more precisely, its spectral projections $E(\lambda)$) is in the commutant

$T'$ of the representation $T$. Thus the decompositions (2) and (3) may be considered to be the direct integral decomposition implied by the abelian '*'-algebra $T'$.

We now give a general definition of a *direct integral of representations*. Let $(\Lambda, \mu)$ be a Borel space with a measure $\mu$ and let

$$\hat{H} = \int_{\Lambda} H(\lambda) \mathrm{d}\mu(\lambda)$$

be a direct integral of Hilbert space (cf. app. B, § 3). Suppose that for each $\lambda \in \Lambda$ an operator $T(\lambda)$ on $H(\lambda)$ is defined. We say that the operator field $\lambda \to T(\lambda)$ is *integrable* iff the following conditions are satisfied:

(i) The $\{T(\lambda)\}$ are uniformly bounded, i.e. there exists a number $M$ such that

$$\|T(\lambda)\|_{H(\lambda)} \leqslant M \quad \text{for any } \lambda \in \Lambda.$$

(ii) Given any $u, v \in H$, the complex valued function $\lambda \to \big(T(\lambda)u(\lambda), v(\lambda)\big)_\lambda$ is $\mu$-measurable.

Then it is possible to define an operator $T$ on $\hat{H}$ by setting

$$Tu \equiv \int_{\Lambda} T(\lambda)u(\lambda)\mathrm{d}\mu(\lambda) \quad \text{whenever } u = \int_{\Lambda} u(\lambda)\mathrm{d}\mu(\lambda). \tag{4}$$

Conditions (i) and (ii) assure that $T$ is a bounded operator on $\hat{H}$, i.e. $Tu \in \hat{H}$ for $u \in \hat{H}$ and $\|T\|_{\hat{H}} \leqslant M$.

Let $G$ be a group and $\Lambda$ a Borel space with a measure $\mu$. Suppose that for every $\lambda \in \Lambda$ a unitary representation $T(\lambda)$ of $G$ is given in $H(\lambda)$: we say that the representation field $\lambda \to T(\lambda)$ is *integrable* iff for any $g \in G$ the operator field $\lambda \to T_g(\lambda)$ is integrable. Since $\|T_g(\lambda)\| = 1$ a representation field $T_g(\lambda)$ is integrable iff the function $\big(T_g(\lambda)u(\lambda), v(\lambda)\big)_\lambda$ is $\mu$-integrable.

For any integrable representation field we can define an operator

$$T_g = \int_{\Lambda} T_g(\lambda)\mathrm{d}\mu(\lambda) \tag{5}$$

in $\hat{H}$. It is evident from definition that

1° $T_e = I$,

2° $T_{g_1 g_2} = T_{g_1} T_{g_2}$,

3° $T_g^* = T_g^{-1}$.

Hence the map $g \to T_g$ in $H$ provides a unitary representation of $G$ in $H$.

The importance of the concept of a direct integral of representations follows from the following

THEOREM 0. *Every representation $T$ of a separable locally compact group $G$ is a direct integral of irreducible representations*

$$T = \int_{\Lambda} T(\lambda)\mathrm{d}\mu(\lambda), \tag{6}$$

*where $(\Lambda, \mu)$ is some measure space and the $T(\lambda)$ are irreducible.*

*If $G$ is of type I then the decomposition in* (6) *is essentially unique.* ▼

(For the proof cf. Mackey (1955), ch. 1, § 4.)

We shall now elaborate the general formalism of the decomposition of a unitary representation $g \to T_g$ of $G$ onto irreducible components. This formalism is based on a theorem of von Neumann on diagonal and decomposable operators (cf. app. B.3).

The basic steps of the decomposition of a reducible unitary representation $T_g$ onto irreducible components are the following:

(i) Consider a '$*$'-algebra $T$ of operators generated by $T_g$

$$T = \left\{ \sum_{i=1}^{n} c_i T_{x_i}:\ c_i \in C \right\}.$$

(ii) Find an abelian '$*$'-subalgebra $\mathscr{A}$ in the commutant $T'$ of $T$.

(iii) Apply the theorem of von Neumann and obtain a decomposition of $H$ onto a direct integral of Hilbert spaces:

$$H \leftrightarrow \hat{H} = \int H(\lambda) \mathrm{d}\mu(\lambda) \tag{7}$$

implied by the algebra $\mathscr{A}$.

Because $T_g$ is in $\mathscr{A}'$, $T_g$ is a decomposable operator. Hence

$$T_g \leftrightarrow \hat{T}_g = \int T_g(\lambda) \mathrm{d}\mu(\lambda). \tag{8}$$

Clearly if $\mathscr{A}_1$ and $\mathscr{A}_2$ are two abelian '$*$'-algebras in $T'$, and $\mathscr{A}_1 \subset \mathscr{A}_2$, then the decomposition $\int_{\Lambda_2} \overset{2}{T}(\lambda_2) \mathrm{d}\mu(\lambda_2)$ implied by $\mathscr{A}_2$ is a refinement of the decomposition $\int_{\Lambda_1} \overset{1}{T}(\lambda_1) \mathrm{d}\mu(\lambda_1)$ implied by $\mathscr{A}_1$. One may expect a most effective decomposition in the case when $\mathscr{A}$ is an abelian maximal '$*$'-algebra in $T'$. This is the content of the following fundamental theorem.

THEOREM 1 (Mautner). *Let $G$ be a separable locally compact group. Let $g \to T_g$ be a continuous unitary representation of $G$ in a Hilbert space $H$. Let $\mathscr{A}$ be an abelian '$*$'-algebra in the commutant $T'$ of $T$. Then*

(i) *there exists a direct integral decomposition of $H$ and $T$ given by eqs.* (4) *and* (5), *respectively,*

(ii) *$T(\lambda)$ are $\mu$-a.a. irreducible in $H(\lambda)$ if and only if $\mathscr{A}$ is maximal in $T'$.*

SKETCH OF A PROOF: The decomposition (7) and (8) of $H$ and $T$ onto direct integral (5) is a direct consequence of von Neumann theorem (cf. app. B.3). We now show that maximality of $\mathscr{A}$ implies essentially the irreducibility of $T(\lambda)$.

Let $\Lambda_0 \subset \Lambda$ be a set which has a positive measure $\mu(\Lambda_0) > 0$. For each $\lambda_0 \in \Lambda_0$, let $B(\lambda_0) \neq I(\lambda_0)$ be a bounded operator in $H(\lambda)$ which commutes with each $T_g(\lambda_0)$, $g \in G$, and put $B(\lambda) = 0$ for $\lambda \in \Lambda - \Lambda_0$. One may show that there exists a measurable operator field $B(\lambda)$. Set $B \equiv \int_{\Lambda_0} B(\lambda)\mathrm{d}\mu(\lambda)$. Because $B$ is not diagonal in $H = \int_{\Lambda} \hat{H}(\lambda)\mathrm{d}\mu(\lambda)$, $B \notin \mathscr{A}$, but since $B$ is decomposable $B \in \mathscr{A}'$. Thus the algebra $\mathscr{A} \cup \mathscr{B}$ is a commutative subalgebra in $T'$ and $\mathscr{A} \cup \mathscr{B}$ contains properly $\mathscr{A}$. Thus $\mathscr{A}$ is not maximal in $T'$. Consequently, the maximality of $\mathscr{A}$ implies the irreducibility of $T(\lambda)$ for $\mu$-a.a. $\lambda$.

In order to show the converse, let

$$T = \int_{\Lambda} T(\lambda)\mathrm{d}\mu(\lambda)$$

where for $\mu$-.a.a. $\lambda$ the representations $T(\lambda)$ are irreducible in $\tilde{H}(\lambda)$. If the commutative algebra $\mathscr{A}$ is not maximal, then there exists an orthogonal nontrivial projection $E \in (T' - \mathscr{A})$ commuting with $\mathscr{A}$. Consequently,

$$E = \int_{\Lambda_0} E(\lambda)\mathrm{d}\mu(\lambda),$$

where $E(\lambda)$ is a non-zero projection in $H(\lambda)$ for $\lambda$ in a certain set $\Lambda_0$ with $\mu(\Lambda_0) > 0$. Now, because $E \in T'$, we have

$$E(\lambda)T(\lambda) = T(\lambda)E(\lambda).$$

Because for $\mu$-a.a. $\lambda, T(\lambda)$ is irreducible, $E(\lambda) = I$ and we obtain a contradiction. Hence $\mathscr{A}$ must be maximal. ▼

(For the complete proof cf., e.g., Mackey 1955.)

The Mautner theorem plays an important role in group representation theory, and in its applications.

The following fundamental theorem which is a direct consequence of the theorem of Mautner shows that a topological group has always nontrivial irreducible representations.

THEOREM 2 (the Gel'fand–Raikov theorem). *Let $G$ be a separable topological group. Then for every two elements $g_1, g_2 \in G$, $g_1 \neq g_2$, there exists an irreducible representation $g \to T_g$ of $G$ such that $T_{g_1} \neq T_{g_2}$.*

PROOF: Let $T^L$ be a left-regular representation of $G$ in $L^2(G)$. Because $T^L$ is faithful, $T^L_{g_1} \neq T^L_{g_2}$. Let

$$T^L_g = \int_{\Lambda} T_g(\lambda)\mathrm{d}\mu(\lambda)$$

be a direct integral decomposition of $T^L$. If $T_{g_1}(\lambda) = T_{g_2}(\lambda)$ for $\mu$-a.a. $\lambda$, $\lambda \in \Lambda$, then $T^L_{g_1}$ would be equal to $T^L_{g_2}$ which is a contradiction. ▼

The Gel'fand–Raikov theorem was proved (by technique of positive definite functions) at the beginnings of the development of group representation theory in 1943. It presents one of the most important results in representation theory. In case of abelian groups, th. 2 gives:

COROLLARY. *Let $G$ be abelian separable topological group. Then for each $g_1 \neq g_2$ there exists a character $\chi(g)$ such that $\chi(g_1) \neq \chi(g_2)$.*

It should be stressed that the selection of a maximal commuting algebra $\mathscr{A}$ in $T'$ is nonunique. An explicit example of a selection of different, unitarily non-equivalent, sets of commuting operators in $T'$ is given in eq. 9.6(11).

There is a general feeling among physicists that a set of invariant operators of a group $G$ (and consequently also $T'$) is commutative; the following is a counterexample.

EXAMPLE 1. Let $K$ be a closed subgroup of a Lie group $G$ such that $X = G/K = \{gK, g \in G\}$ possesses an invariant measure $\mu$. Let $N(K)$ be the normalizer of $K$ in $G$, i.e., the set of all $n \in G$ such that $nKn^{-1} \subset K$. Let $H = L^2(X, \mu)$ and let $g \to T_g$ be a unitary representation of $G$ in $H$ given by

$$T_g u(x) = u(g^{-1}x). \tag{9}$$

Let $T_n^R$, $n \in N(K)$, be an operator in $H$ defined by

$$T_n^R u(gK) = u(gKn) = u(gnK).$$

Then

$$\begin{aligned}(T_n^R T_{g_0})u(gK) &= (T_{g_0})u(gKn) = u(g_0^{-1}gKn)\\ &= (T_{g_0}T_n^R u)(gK).\end{aligned} \tag{10}$$

Hence the right translations by elements $n \in N(K)$ are well defined in $X$ and commute with all $T_{g_0}$, $g_0 \in G$. Thus every right translation $T_n^R$, $n \in N(K)$, is in the commutant $T'$. Hence if $N(K)/K$ is a noncommutative subgroup, then $T'$ is non-abelian. ▼

One may expect that the decomposition (8) would be simplest in the case $T'$ is abelian. The representations with this property are called *multiplicity-free*. In this case, we can take $\mathscr{A} = T'$ and obtain an essentially unique decomposition of $T$ onto irreducible components. The following theorem shows that for type I groups we have essentially a unique decomposition of the representation $T$ onto its irreducible components:

THEOREM 3. *Let $G$ be a separable, topological type I group. Let $g \to T_g$ be a unitary representation of $G$ in a Hilbert space $H$, and let $\hat{G}$ be the set of equivalence classes of irreducible representations. Then there exists a standard Borel measure $\hat{\mu}$ on $\hat{G}$ and a function $\hat{n}(\lambda)$ on $\hat{G}$ such that*

$$H \leftrightarrow H = \int_{\hat{G}} H(\lambda)\hat{n}(\lambda)\,d\hat{\mu}(\lambda)$$

*and*

$$T \leftrightarrow \hat{T} = \int_{\hat{G}} T(\lambda)\hat{n}(\lambda)\,d\hat{\mu}(\lambda). ▼$$

(For the proof cf. Maurin 1968, ch. V, § 2.)

## § 7. Comments and Supplements

### A. *Some Generalizations of Continuous Unitary Representations*

So far we have discussed the properties of unitary, continuous linear representa tions of a topological group $G$. One could ask about the extensions of the theory if some of these conditions imposed on the representatives $T_x$ of $G$ are relaxed. Firstly, one can construct homomorphisms of $G$ into $L(H)$ for which $T_e \neq I$; indeed, if $G = R^1$, for example, then the map

$$R^1 \ni x \to T_x = \begin{bmatrix} \exp(ix) & 0 \\ 0 & 0 \end{bmatrix}, \quad 0 = \text{zero operator in a subspace of } H,$$

satisfies the condition $T_{xy} = T_x T_y$ in any Hilbert space, but $T_e \neq I$. The following proposition shows, however, that we practically lose nothing by imposing the condition $T_e = I$. Indeed, we have

PROPOSITION 1. *Let $x \to T_x$ be a homomorphism of $G$ into $L(H)$. Then $H$ is the direct sum $H_1 \oplus H_0$ of invariant subspaces and $T$ has the form*

$$T_x = \begin{bmatrix} \overset{1}{T}_x & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

*where $\overset{1}{T}$ is the representation of $G$ in $H_1$.* ▼

PROOF: Set $H_1 = \{u \in H\colon T_e u = u\}$ and $H_0 = \{u \in H\colon T_e u = 0\}$. Clearly, $H_1 \cap H_0 = \{0\}$. For arbitrary $u$ in $H$ we have $u = T_e u + (u - T_e u)$, where $T_e(T_e u) = T_e u$ and $T_e(u - T_e u) = 0$. Hence, $H$ is the direct sum $H_1 \oplus H_0$. It is evident that $H_1$ and $H_0$ are closed invariant subspaces of $H$. For $u$ in $H_0$ implies $T_x u = T_x T_e u = 0$. Thus, $T_x H_0 = \{0\}$, and the map $x \to T_x$ takes the form (1). ▼

Secondly, we might drop the continuity condition. In order to see what happens in this case we introduce the notion of the so-called *measurable representations*. Let $\mu$ be the Haar measure on $G$ and let $T$ be a unitary representation of $G$ in a separable Hilbert space $H$. $T$ is said to be *$\mu$-measurable* if the function $x \to (T_x u, v)$ is $\mu$-measurable for all $u, v$ in $H$. We have

PROPOSITION 2. *A unitary representation $T$ is continuous iff it is $\mu$-measurable.* ▼

(For the proof cf. Hewitt and Ross 1963, I (22.20b)).

This result shows that discontinuous representations must be nonmeasurable. Because for nonmeasurable functions we have only existence theorems, we do not expect explicit constructive realizations of discontinuous representations (cf. example 1.3). Hence, their physical meaning is doubtful.* One could obtain, however, measurable discontinuous representations if one would admit non-separable Hilbert spaces; for example, the von Neumann infinite tensor product of Hilbert spaces $\prod_i \otimes H_i$.

An interesting characterization of unitary representations in nonseparable Hilbert spaces is given by the following

PROPOSITION 3. *Let $T$ be a unitary, $\mu$-measurable representation of a locally compact group $G$ on a nonseparable Hilbert space $H$. Let $H_s$ be the subspace of all vectors $u$ in $H$ such that for all $\varphi$ in $L^1(G)$ and all $v$ in $H$ we have*

$$\int \varphi(x)(T_x u, v)\,\mathrm{d}\mu(x) = 0.$$

*Then,*

1° *$H$ is the direct sum of two invariant subspaces: $H = H_c \oplus H_s$.*

2° *The representation ${}^{H_c}T$ is continuous. The representation ${}^{H_s}T$ is singular in the sense that the map $x \to (T_x u, u)$ is equivalent to zero for all $u$ in $H_s$.***

3° *If $H$ is separable, then the subspace $H_s$ is absent.* ▼

(For the prof cf. Segal and Kunze 1968.)

B. *Comments*

(i) In mathematics equivalent representations $T$ and $T'$ of a group $G$ are indistinguishable. However, in physics representations which are unitarily equivalent are not necessarily physically equivalent. Indeed, let $H$ be a Hamiltonian of a system of two interacting non-relativistic particles, or of a particle in a potential field. Suppose that for such an interaction, $H$ has a continuous spectrum only (i.e., absence of bound states). Then, there exists a unitary scattering operator $S$ such that

$$SH_0 = HS,$$

where $H_0$ is the free Hamiltonian. Hence, the time displacement operators $U_t = \exp(\mathrm{i}tH)$ and $U_t^0 = \exp(\mathrm{i}tH_0)$ are unitarily equivalent. However, they are not physically equivalent as they describe the time evolution of essentially different physical systems. The same conclusion holds if one considers equivalent representations of the Galilei group or the Poincaré group. The reason for this distinction lies in the fact that in physics we do not use abstract groups but groups whose generators are identified with physical observables. The same group can be used to describe different physical situations.

(ii) In § 3 we described type I factor representations. Other types of factor representations can be described by using the notion of *finite representations.* If $\infty T \simeq T$ we say that $T$ is *infinite.* If no one subrepresentation of $T$ is infinite we say that $T$ is *finite.* A factor representation $T$, which has a finite subrepresentation but no irreducible subrepresentation is said to be of type II.

* A noncontinuous three-dimensional representation of the rotation group SO(3) is obtained if we replace in the matrix elements the functions $\cos\varphi$ and $\sin\varphi$ by $\cos f(\varphi)$, $\sin f(\varphi)$ where $f(\varphi_1+\varphi_2) = f(\varphi_1)+f(\varphi_2)$, and $f(\varphi)$ discontinuous.

** A function $f(x)$ on $G$ (in our case $f(x) = (T_x u, u)$) is *equivalent* to zero, if for $\mu$-almost all $x \in G$. $f(x) = 0$

Finally if $T$ is not irreducible, but every proper subrepresentation of $T$ is equivalent to $T$, then $T$ is said to be the *factor representation* of type III. Such a $T$ is necessarily a factor representation and infinite.

A general representation $T$ of $G$ need not belong to any of the above three types. We have, however,

THEOREM 1. *Let $T$ be any unitary representation of $G$. Then, there exists uniquely determined projections $P_1$, $P_2$ and $P_3$ in the center of $R(T, T)$ such that*

(i) $P_1+P_2+P_3 = I$.

(ii) ${}^{P_1H}T$, ${}^{P_2H}T$ *and* ${}^{P_3H}T$ *are of type I, II and III, respectively.*

(iii) *For $i \neq j$ no subrepresentation of* ${}^{P_iH}T$ *is equivalent to a subrepresentation of* ${}^{P_jH}T$. ▼

This theorem shows that we can restrict our analysis to type I, II, and III factor representations.

In almost all applications we encounter type I representations only. The properties of type II and III factor representations are less intuitive. A relatively simple construction of factor II representations is given in Naimark's book (1970, § 38, p. 484). Explicit examples of factor representations of type III were recently also constructed (cf. Dixmier 1969).

It is interesting that in relativistic quantum field theory we probably cannot avoid the use of type III factors. In fact, Araki and Woods have recently shown that the representation of the canonical commutation relations of a scalar relativistic quantum field leads to type III factors (cf. Araki and Woods 1966).

The beautiful exposition of the theory of factors and its applications to group representation theory is presented in Dixmier's book (1969).

## § 8. Exercises

§ 1.1. Show that the matrix elements of irreducible representations of SO(3) group have the form

$$D^J_{MM'}(\varphi, \vartheta, \psi) = \exp(-\mathrm{i}M\varphi)d^J_{MM'}(\vartheta)\exp(-\mathrm{i}M\psi), \tag{1}$$

where $\varphi \in [0, 2\pi)$, $\vartheta \in [0, \pi)$, $\psi \in [0, 2\pi]$, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, M, M' = -J, -J+1, \ldots, J-1, J$ and

$$d^J_{MM'}(\vartheta) = \left(\frac{1+\cos\vartheta}{2}\right)^M P^{0,2M}_{J-M}(\cos\vartheta). \tag{2}$$

Here $P^{\alpha,\beta}_{\gamma}(x)$ it the Jacobi polynomial.

§ 5.1. Show that the 'Clebsch–Gordan series' for SO(3) has the following form:

$$T^{J_1}\otimes T^{J_2} = \sum_{J=|J_1-J_2|}^{|J_1+J_2|} T^J. \tag{3}$$

# Chapter 6

# Representations of Commutative Groups

We begin the analysis of the representation theory of locally compact groups with the commutative groups. The commutativity of the group multiplication implies a considerable simplification of representation theory. This does not make, however, the theory trivial, because in most cases we have to use direct integrals to describe the properties of the representations.

## § 1. Irreducible Representations and Characters

We first show the simple but fundamental property of irreducible representations of abelian groups. Unless stated otherwise we consider locally compact abelian group.

PROPOSITION 1. *Any irreducible unitary representation of an abelian group $G$ in complex space is one-dimensional.*

PROOF: For every $x \in G$ and a fixed $y \in G$ we have

$$T_x T_y = T_{xy} = T_{yx} = T_y T_x. \tag{1}$$

Hence, by proposition 5.3.4

$$T_y = \alpha(y) I, \quad \alpha(y) \in C. \tag{2}$$

Consequently any one-dimensional subspace $H_1$ of the carrier space $H$ is invariant. But $T$ is irreducible; so $H_1$ must coincide with $H$. ▼

A *character* of an abelian locally compact group $G$ is any continuous function $\hat{x}$: $G \to C$, which satisfies

$$|\hat{x}(x)| = 1, \tag{3}$$

$$\hat{x}(x_1 x_2) = \hat{x}(x_1)\hat{x}(x_2). \tag{4}$$

It follows from eqs. (3) and (4) that $\hat{x}(e) = 1$ and $\hat{x}(x^{-1}) = \overline{\hat{x}(x)} = \hat{x}(x)^{-1}$.

Therefore, a character is a one-dimensional continuous unitary representation of $G$.

The *dual space* $\hat{G}$ of an arbitrary group $G$, is the set of equivalent classes of all continuous, irreducible unitary representations of $G$. According to proposition 1 for abelian groups $\hat{G}$ consists then of all characters of $G$. If $\hat{x}_1$ and $\hat{x}_2$ are in $\hat{G}$, then the function $x \to (\hat{x}_1 \hat{x}_2)(x) = \hat{x}_1(x)\hat{x}_2(x)$ satisfies the conditions

(i) $|(\hat{x}_1 \hat{x}_2)(x)| = 1,$ (5)

(ii) $\hat{x}_1\hat{x}_2(xy) = \hat{x}_1(xy)\hat{x}_2(xy) = (\hat{x}_1\hat{x}_2)(x)(\hat{x}_1\hat{x}_2)(y).$ (6)

Moreover, because $\hat{x}^{-1}(x) = \overline{\hat{x}(x)}$, $\hat{G}$ is also an abelian group.

EXAMPLE 1. 1° Consider $G = R^n$ as an additive vector group. Then, every character $\hat{x}(\cdot)$ has the form

$$\hat{x}(x) = \exp i(\hat{x}_1 x_1 + \dots + \hat{x}_n x_n) = \exp[i(\hat{x}\cdot x)], \quad \hat{x} \in R^n. \tag{7}$$

Thus, the character group $\hat{G}$ is isomorphic with $G$.

2° If $G$ is the multiplicative group of complex numbers of modulus one, $x = \exp(i\theta)$, then, every character has the form

$$\hat{x}(x) = \exp(in\theta), \quad n = \text{integers}, \tag{8}$$

Thus, in this case $\hat{G}$ is isomorphic with the additive group of integers. If $n$ is an arbitrary real number, then the character (8) is a multi-valued function on $G$. It is however a single-valued representation of the covering group $\tilde{G} = R^1$ of $G$. ▼

In these examples the character group $G$ is also a locally compact abelian topological group. One can show that this property holds for an arbitrary abelian locally compact group if we endow $\hat{G}$ with the topology of uniform convergence on compact sets (cf., e.g., Weil 1940, § 2g).

In order to take advantage of the symmetry between $G$ and $\hat{G}$ we introduce a more symmetric notation for characters by setting $\hat{x}(x) = \langle x, \hat{x}\rangle$. Then, eqs. (3)–(6) take the form:

$$|\langle x, \hat{x}\rangle| = 1, \tag{3'}$$

$$\langle x_1 x_2, \hat{x}\rangle = \langle x_1, \hat{x}\rangle\langle x_2, \hat{x}\rangle, \tag{4'}$$

$$\langle x, \hat{x}_1\hat{x}_2\rangle = \langle x, \hat{x}_1\rangle\langle x, \hat{x}_2\rangle. \tag{6'}$$

A complex character of an abelian locally compact group $G$ is a representation of $G$ in $C$.

## § 2. Stone and SNAG Theorems

We now derive a fundamental decomposition theorem for an arbitrary unitary representation of an abelian group.

THEOREM 1 (Stone, Naimark, Ambrose, Godement theorem). *Let $T$ be an unitary continuous representation of an abelian locally compact group $G$ in a Hilbert space $H$. Then, there exists on the character group $\hat{G}$ a spectral measure $E(\cdot)$ such that**

$$T_x = \int_{\hat{G}} \langle x, \hat{x}\rangle \mathrm{d}E(\hat{x}). \tag{1}$$

* See app. B.3, for properties of spectral measure $\mathrm{d}E$ and von Neumann spectral theory.

PROOF: Let $u \in H$. Then the function $x \to (T_x, u, u)$ is positive definite; hence, by the Bôchner theorem, there exists a finite regular Borel measure $\mu_{u,u}$ on $\hat{G}$ such that

$$(T_x u, u) = \int_{\hat{G}} \langle x, \hat{x} \rangle \, d\mu_{u,u}(\hat{x}),$$

and, in particular,

$$\int_{\hat{G}} d\mu_{u,u} = \mu_{u,u}(\hat{G}) = (u, u).$$

Using the polar decomposition, one can write $(T_x u, v)$ as a linear combination of terms like $(T_x u', u')$. Hence there exists a unique complex measure $\mu_{u,v}$ such that

$$(T_x u, v) = \int_{\hat{G}} \langle x, \hat{x} \rangle \, d\mu_{u,v}(\hat{x}).$$

Now fix any Borel set $\hat{B} \subset \hat{G}$; then $\mu_{u,v}(\hat{B})$ is a bilinear functional $F_{\hat{B}}(u, v)$ on $H$, which is hermitian since

$$F_{\hat{B}}(u, v) = \mu_{u,v}(\hat{B}) = \mu_{v,u}(\hat{B}) = F_{\hat{B}}(v, u)$$

and is bounded since

$$|F_{\hat{B}}(u, v)|^2 \leqslant \mu_{u,u}(\hat{B}) \mu_{v,v}(\hat{B}) \leqslant ||u||^2 ||v||^2. \tag{2}$$

Indeed, set, for any real $\lambda$,

$$u' \equiv u + \lambda \mu_{v,u}(\hat{B}) v$$

and then it follows that

$$0 \leqslant \mu_{u',u'}(\hat{B}) = \mu_{u,u}(\hat{B}) + 2\lambda |\mu_{u,v}(\hat{B})|^2 + \lambda^2 |\mu_{u,v}(\hat{B})|^2 \mu_{v,v}(\hat{B}),$$

which implies that

$$|\mu_{u,v}(\hat{B})|^4 - \mu_{u,u}(\hat{B}) \mu_{v,v}(\hat{B}) |\mu_{u,v}(\hat{B})|^2 \leqslant 0$$

from which eq. (2) follows.

Hence, by the Riesz theorem, for each Borel set $\hat{B} \subset \hat{G}$ there exists an operator $E(\hat{B})$ on $H$ such that for any $u, v \in H$ one has:

$$\big(E(\hat{B}) u, v\big) = F_{\hat{B}}(u, v) = \mu_{u,v}(\hat{B}). \tag{3}$$

It is obvious that $[E(\hat{B})]^* = E(\hat{B})$; moreover, a simple calculation shows that

$$\big(E(\hat{B}) T_x u, v\big) = \int_{\hat{B}} \langle x, \hat{x} \rangle \big(E(d\hat{x}) u, v\big).$$

Therefore, if one sets, for every Borel set $\hat{B}_1$, $Q(\hat{B}_1) \equiv E(\hat{B} \cap \hat{B}_1)$ one has

$$\int\limits_{\hat{B}} \langle x, \hat{x} \rangle (E(\mathrm{d}\hat{x})u, v) = \int\limits_{\hat{G}} \langle x, \hat{x} \rangle (Q(\mathrm{d}\hat{x})u, v)$$

$$= (E(\hat{B})T_x u, v) = \int\limits_{\hat{G}} \langle x, \hat{x} \rangle (E(\hat{B})E(\mathrm{d}\hat{x})u, v)$$

from which it is easy to conclude that

$$E(\hat{B} \cap \hat{B}_1) = E(\hat{B})E(\hat{B}_1).$$

One readily verifies that the operator function $\hat{B} \to E(\hat{B})$ satisfies all conditions imposed on the spectral measure (cf. § 3 of app. B). Hence by eq. (3) we have

$$(T_x u, v) = \int\limits_{\hat{G}} \langle x, \hat{x} \rangle \mathrm{d}\mu_{u,v}(\hat{x}) = \int\limits_{\hat{G}} \langle x, \hat{x} \rangle (E(\mathrm{d}\hat{x})u, v).$$

This equality implies the assertion of th. 1. ▼

In the special, but very important, case of the abelian vector groups we obtain

THEOREM 2 (Stone's theorem). *Consider $G = R^n$ as an additive vector group, and let $T$ be a unitary continuous representation of $G$ in a Hilbert space $H$. Then, there exists a unique set of mutually strongly commuting self-adjoint operators $Y_1, \ldots, Y_n$ such that*

$$T_x = \prod_{k=1}^{n} \exp(\mathrm{i}x_k Y_k). \tag{4}$$

PROOF: By virtue of example 1.1° the character group $\hat{G} = R^n$ and $\langle x, \hat{x} \rangle = \exp[\mathrm{i}(x\hat{x} + \ldots + x_n\hat{x}_n)]$. Hence by virtue of eq. (1) we have:

$$T_x = \int\limits_{R^n} \exp[\mathrm{i}(x_1\hat{x}_1 + \ldots + x_n\hat{x}_n)] \mathrm{d}E(\hat{x}). \tag{5}$$

Using now ths. 4.3, item 3°, and 4.2 of app. B one obtains

$$T_x = \prod_{k=1}^{n} \int\limits_{R^n} \exp[\mathrm{i}x_k\hat{x}_k] \mathrm{d}E(\hat{x}) = \prod \int\limits_{R^1} \exp[\mathrm{i}x_k\hat{x}_k] \mathrm{d}E(\hat{x}_k) = \prod \exp[\mathrm{i}x_k Y_k], \tag{6}$$

where

$$\mathrm{d}E(\hat{x}_k) = \int\limits_{R^{n-1}} \mathrm{d}E(\hat{x}) \quad \text{and} \quad Y_k = \int \hat{x}_k \mathrm{d}E(\hat{x}_k). \;▼ \tag{7}$$

EXAMPLE 1. Let $G = T^{3,1}$ be the translation group of the Minkowski space $M^4$ and let $x \to T_x$ be a unitary representation of $G$ in a Hilbert space $H$. The dual space $\hat{G}$ is identified in physics with the momentum space $P$ which is isomorphic to $M^4$. Hence, formula (4) can be written in the form

$$T_x = \int\limits_{P} \exp(\mathrm{i}xp) \mathrm{d}E(p), \quad xp = x^\mu p_\mu, \tag{8}$$

where $E(\cdot)$ is a spectral measure on momentum space. The commutative set of self-adjoint operators defined by eq. (7) are in this case

$$P_\mu = \int_P p_\mu \mathrm{d}E(p), \quad \mu = 0, 1, 2, 3, \tag{9}$$

and represent the energy-momentum four-vector.

## § 3. Comments and Supplements

### A. *Duality Theorem of Pontryagin*

We describe in this section a fundamental property of representations of abelian, locally compact groups.

Note first that the map $\hat{G} \ni \hat{x} \to \langle x, \hat{x} \rangle$ defines a continuous function on $\hat{G}$, which satisfies eqs. 1(3′) and 1(6′). Hence, every $x \in G$ defines a character $\hat{\hat{x}}$ of the group $\hat{G}$; consequently $G \subset \hat{\hat{G}}$, the set of all $\hat{\hat{x}}$. The following theorem asserts that there are no other characters on $\hat{G}$ besides those induced by the elements of $G$.

THEOREM 1. *The map $G \ni x \to \hat{\hat{x}} \in \hat{\hat{G}}$ is a topological isomorphism*

$$G \cong \hat{\hat{G}}. \blacktriangledown$$

(For the proof cf. Hewitt and Ross 1963, § 24.)

Example 1 provides two simplest illustrations of the Pontryagin duality.

### B. *Comments*

The SNAG theorem is usually presented in the operator form given by eq. 2(1). The following interesting form of this theorem based on Bochner theorem was recently given by Hewitt and Ross.

THEOREM 2. *Let $T$ be a continuous cyclic unitary representation of an abelian locally compact group $G$. There is a positive measure $\nu(\cdot)$ on $G$ such that $T$ is unitarily equivalent to the following representation*

$$U_x v(\hat{x}) = \hat{x}(x) v(\hat{x}) = \hat{\hat{x}}(\hat{x}) v(\hat{x}), \quad x \in G, \; v \in L^2(\hat{G}, \nu). \blacktriangledown \tag{1}$$

(For the proof cf. Hewitt and Ross 1970, § 33.8.)

The SNAG Theorem was originally proved by Stone for $G = R^n$ (1930, 1932). Later Naimark 1943, Ambrose 1944 and Godement 1944 gave various extensions of this theorem for an arbitrary abelian locally compact group. We follow here the presentation given by Maurin 1963, 1968, ch. VI.

C. The harmonic analysis on locally compact commutative groups will be considered in ch. 14, § 1

## D. *Indecomposable Representations*

We give the construction of indecomposable representations of vector groups, which are most important in applications. Let $G = R^n$. The simplest example of an indecomposable representation is given by the formula

$$R^n \ni x \to T_x = \exp(\mathrm{i}px)\begin{bmatrix} 1 & \gamma(px) \\ 0 & 1 \end{bmatrix},$$

where $px = p_k x^k$ is the scalar product in $R^n$ and $\gamma \in C$. Using the induction method one may find that an $n$-dimensional indecomposable representation of $R^n$ may be taken to be in the form:

$$R^n \ni x \to T_x = \exp(\mathrm{i}px) \times$$

$$\times \begin{bmatrix} 1 & \gamma_{n-1}(px) & \gamma_{n-2}\gamma_{n-1}(px)^2/2 \ldots & \frac{\gamma_2 \cdots \gamma_{n-1}}{(n-2)!}(px)^{n-2} & \frac{\gamma_1 \cdots \gamma_{n-1}}{(n-1)!}(px)^{n-1} \\ & 1 & \gamma_{n-2}(px) \quad \ldots & \frac{\gamma_2 \cdots \gamma_{n-2}}{(n-3)!}(px)^{n-3} & \frac{\gamma_1 \cdots \gamma_{n-2}}{(n-2)!}(px)^{n-2} \\ & & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ 0 & & 1 \quad \ldots & \gamma_2(px) & \frac{\gamma_1\gamma_2}{2}(px)^2 \\ & & & 1 & \gamma_1(px) \\ & & & & 1 \end{bmatrix}$$

These representations are nonunitary. They are used for a group-theoretical description of unstable particles (see ch. 17, § 4).

The indecomposable representations of other commutative groups may be constructed similarly with the help of triangular matrices: for instance the indecomposable representation of the multiplicative group of complex numbers may be taken to be in the form

$$C \ni z \to T_z = \begin{bmatrix} 1 & \ln z \\ 0 & 1 \end{bmatrix}$$

## § 4. Exercises

§ 1.1. Let $D_n = \{\delta = (\delta_1, \ldots, \delta_n), \delta_k \in C\}$ be the multiplicative group of complex numbers. Show that the map

$$\delta \to \chi_\delta = \prod_{s=1}^{n} |\delta_{ss}|^{m_s + i\varrho_s} \delta_{ss}^{-m_s} \tag{1}$$

where $m_s$ are integers and $\varrho_s$ are real numbers, is the character of $D_n$.

§ 1.2. Set $\varrho_s \in C$ in eq. (1). Show that in this case the map (1) gives the complex character of $D_n$.

§ 1.3. Construct a discontinuous irreducible unitary representation of $G = R^1$.

*Hint*: Use the Hamel basis.

§ 1.4. Let $G = N \otimes K$ where $N$ is commutative. Let $n \to U_n$ and $k \to V_k$ be representations of $N$ and $K$, respectively, in a carrier space $H$. What conditions $U$ must satisfy in order that the map $(n, k) \to U_n V_k$ be a representation of $G$?

§ 3.1.** Classify all finite-dimensional indecomposable representations of $G = R^1$

§ 3.2.*** Classify all indecomposable representations of $R^1$ in Hilbert space.

§ 3.3.** Classify all finite-dimensional indecomposable representations of $G = R^n$

Chapter 7

# Representations of Compact Groups

## § 1. Basic Properties of Representations of Compact Groups

The representation theory of compact groups forms a bridge between the relatively simple representation theory of finite groups and that of noncompact groups. Most of the theorems for the representations of finite groups have direct analogues for compact groups and these results in turn serve as the starting point for the representation theory of noncompact groups.

Everywhere in this section $G$ will denote a compact topological group and $dx$ an invariant measure on $G$, normalized to unity. And by a representation of $G$ we shall mean a strongly continuous representation in a Hilbert space $H$. Recall that by eq. 5.1(3), any representation of a compact, topological group is bounded.

We first show that in the case of representations of compact topological groups we can restrict ourselves without loss of generality to an analysis of unitary representations only.

THEOREM 1. *Let $T$ be an arbitrary representation of a compact group $G$ in $H$. There exists in $H$ a new scalar product defining a norm equivalent to the initial one, relative to which the map $x \to T_x$ defines a unitary representation of $G$.*

PROOF: Let $(\cdot, \cdot)$ be the initial scalar product in $H$. We define the new scalar product by

$$(u, v)' \equiv \int_G (T_x u, T_x v) dx. \tag{1}$$

It is easily verified that $(\cdot, \cdot)'$ is a scalar product in $H$. In particular

$$(u, u)' = \int_G (T_x u, T_x u) dx = 0 \Rightarrow u = 0.$$

Indeed, $(T_x u, T_x u)$ is zero almost everywhere; if $x \in G$ is such that $T_x u = 0$, then $T_x^{-1} T_x u = u = 0$.

For every $T_y$, we have then

$$(T_y u, T_y v)' = \int_G (T_{yx} u, T_{yx} v) dx = \int_G (T_z u, T_z v) dz = (u, v)'. \tag{2}$$

Hence, every $T_y$, $y \in G$, is isometric and $D_{T_y} = H$. Thus, every $T_y$, $y \in G$, is unitary.

To show the continuity of the representation in the topology induced by the scalar product (1) we first prove the equivalence of norms; note that

$$\begin{aligned}||u||'^2 &= (u, u)' = \int_G (T_x u, T_x u)\mathrm{d}x \leqslant (\sup_{x\in G}||T_x||)^2 \int_G (u, u)\mathrm{d}x \\ &= N^2(u, u) = N^2||u||^2,\end{aligned}$$

where we set $N = \sup_{x\in G}||T_x||$. Conversely, from the inequality

$$||u||^2 = (T_{x^{-1}}T_x u, T_{x^{-1}}T_x u) \leqslant (\sup_{x\in G}||T_x||)^2(T_x u, T_x u) = N^2||T_x u||^2$$

it follows that

$$||u||^2 = \int_G (u, u)\mathrm{d}x \leqslant N^2 \int (T_x u, T_x u)\mathrm{d}x = N^2(u, u)' = N^2||u||'^2.$$

Hence,

$$N^{-1}||u|| \leqslant ||u||' \leqslant N||u||, \tag{3}$$

i.e., the norms $||\cdot||$ and $||\cdot||'$ are equivalent. The equivalent norms $||\cdot||$ and $||\cdot||'$ define equivalent strong topologies $\tau$ and $\tau'$, respectively, on $H$. Hence the map $x \to T_x u$ of $G$ into $H$ is continuous relative to $\tau'$ and, consequently, the map $x \to T_x$ is the unitary continuous representation of $G$ in $H$. ▼

The next important result for compact groups shows that every irreducible unitary representation is finite-dimensional. We first prove the following useful lemma.

LEMMA 2. *Let $T$ be a unitary representation of $G$ and let $u$ be any fixed vector in the carrier space $H$. Then, the Weyl operator $K_u$ defined for all $v \in H$ by the formula*

$$K_u v = \int_G (v, T_x u) T_x u \mathrm{d}x \tag{4}$$

*has the following properties*

1° *$K_u$ is bounded.*

2° *$K_u T_x = T_x K_u$ for every $x \in G$ and $u \in H$.* ▼

PROOF: *ad* 1°.

$$||K_u v|| \leqslant \int_G |(v, T_x u)|\,||T_x u||\mathrm{d}x \leqslant \int_G ||v||\,||T_x u||\,||u||\mathrm{d}x = ||u||^2\,||v||.$$

*ad* 2°. For every $y \in G$ and $v \in H$ we have

$$\begin{aligned}T_y K_u v &= \int_G (v, T_x u) T_{yx} u \mathrm{d}x = \int_G (T_y v, T_{yx} u) T_{yx} u \mathrm{d}x \\ &= \int_G (T_y v, T_x u) T_x u \mathrm{d}x = K_u T_y v,\end{aligned}$$

i.e., $T_y K_u = K_u T_y$. ▼

We come now to the fundamental theorem.

THEOREM 3. *Every irreducible unitary representation $T$ of $G$ is finite-dimensional.*

PROOF: We have by lemma 2: $K_u T_y = T_y K_u$ for every $y \in G$ and $u \in H$. Hence by proposition 5.3.4, $K_u = \alpha(u)I$ and consequently

$$(K_u v, v) = \int_G (v, T_x u)(T_x u, v)\mathrm{d}x = \alpha(u)(v, v).$$

Hence

$$\int_G |(T_x u, v)|^2 \mathrm{d}x = \alpha(u)||v||^2. \tag{5}$$

By interchanging the roles of $u$ and $v$ in (5) and using the equality

$$\int_G f(x^{-1})\mathrm{d}x = \int_G f(x)\mathrm{d}x,$$

we get

$$\alpha(v)||u||^2 = \int_G |(T_x v, u)|^2 \mathrm{d}x = \int_G |(u, T_x v)|^2 \mathrm{d}x$$
$$= \int_G |(T_{x^{-1}} u, v)|^2 \mathrm{d}x = \int_G |(T_x u, v)|^2 \mathrm{d}x = \alpha(u)||v||^2.$$

Hence, $\alpha(u) = c||u||^2$ for all $u \in H$, where $c$ is a constant. Setting $v = u$ and $||u|| = 1$ in eq. (5) we obtain in particular that

$$\int_G |(T_x u, u)|^2 \mathrm{d}x = \alpha(u)||u||^2 = c||u||^4 = c. \tag{6}$$

Hence, $c > 0$, because the non-negative continuous function $x \to |(T_x u, u)|$ assumes the value $||u|| = 1$ at $x = e$. Now we prove the essential part of the theorem. Let $\{e_i\}_1^n$ be any set of orthonormal vectors in $H$. Setting $u = e_k$ and $v = e_1$ in eq. (5) we obtain

$$\int_G |(T_x e_k, e_1)|^2 \mathrm{d}x = \alpha(e_k)||e_1||^2 = c, \quad k = 1, 2, \ldots, n.$$

Hence, using orthonormality of the vectors $T_x e_k$, $k = 1, 2, \ldots, n$ and the Parseval inequality, we obtain

$$nc = \sum_{k=1}^{n} \int_G |(T_x e_k, e_1)|^2 \mathrm{d}x = \int_G \sum_{k=1}^{n} |(T_x e_k, e_1)|^2 \mathrm{d}x \leqslant \int_G ||e_1||^2 \mathrm{d}x = 1. \tag{7}$$

Eq. (7) shows that the dimension of the carrier space $H$ cannot exceed $1/c$ and hence it is finite. ▼

We have seen, by corollary 5.3.2, that a finite-dimensional unitary representation of any group is completely reducible. This result, in the case of compact topological groups, can be sharpened to the following one:

THEOREM 4. *Every unitary representation $T$ of $G$ is a direct sum of irreducible finite-dimensional unitary subrepresentations.*

PROOF: We first show that the operator $K_u$, which is defined by eq. (4) has the following properties:

1° $K_u^* = K_u$.

2° $K_u$ is a Hilbert–Schmidt operator.

3° Every eigenspace $H_i$ of $K_u$ is $T$-invariant.

4° $H = H_0 + \sum_i \oplus H_i$, $\dim H_i < \infty$, where $H_0$ is the 0-eigenspace of $K_u$, which may be infinite-dimensional.

*ad* 1°. For every $v, w \in H$ we have

$$(K_u v, w) = \int (v, T_x u)(T_x u, w)\mathrm{d}x = \int \overline{(w, T_x u)}\,(v, T_x u)\mathrm{d}x$$
$$= \left(v, \int (w, T_x u) T_x u \mathrm{d}x\right) = (v, K_u w).$$

Hence, $K_u^* = K_u$.

*ad* 2°. We recall that an operator $A$ is Hilbert–Schmidt if and only if for arbitrary basis $\{e_i\}$ in $H$ we have $\sum_i \|Ae_i\|^2 < \infty$. In our case we have

$$\sum_i \|K_u e_i\|^2 = \sum_i \int_G\int_G (e_i, T_x u)(T_x u, T_y u)(T_y u, e_i)\mathrm{d}x\mathrm{d}y$$
$$= \int_G\int_G \sum_i (e_i, T_x u)(T_x u, T_y u)(T_y u, e_i)\mathrm{d}x\mathrm{d}y$$
$$= \int_G\int_G (T_x u, T_y u)(T_y u, T_x u)\mathrm{d}x\mathrm{d}y < \infty.$$

The change of the order of summation and integration is justified by Lebesgue theorem and that

$$\sum_i |(e_i, T_x u)(T_y u, e_i)|^2 \leqslant \Big(\sum_i |(e_i, T_x u)|^2 \sum_k |(e_k, T_y u)|^2\Big)^{1/2} = \|u\|^2.$$

*ad* 4°. Follows from Rellich–Hilbert–Schmidt spectral theorem for compact operators (cf. app. B.3).

We now come to the proof of the main theorem. For all vectors $v \in H$, which are not orthogonal to $u$, we have $(K_u v, v) > 0$ (cf. eq. (6) and below). Therefore, the operator $K_u$ has at least one eigenvalue different from zero, i.e., the space $H \ominus H_0$ is not empty.

We have shown, therefore, that the space $H \ominus H_0 = \sum_i \oplus H_i$ is the direct orthogonal sum of finite-dimensional, invariant subspaces $H_i$. Using corollary 5.3.2 we can split each $H_i$ into the direct, orthogonal sum of irreducible, invariant subspaces. In order to split the space $H_0$ we consider the subrepresentation $T_x^0 \equiv {}^{H_0}T_x$ and construct for it the operator (4). Therefore, according to the properties 3° and 4° of $K_u$ and corollary 5.3.2, we conclude that $H_0$ contains also a non-trivial, finite-dimensional, minimal invariant subspace. It is evident from these considerations that the conclusion is true for any invariant space.

To complete the proof of th. 4 we show that the smallest subspace $M$ of $H$ containing all the minimal, mutually orthogonal, invariant subspaces is $H$ itself. Indeed, because $T$ is unitary and $M$ is invariant, then $M^\perp$ is also invariant. Hence, by the above conclusion $M^\perp$ contains a nontrivial, invariant, finite-dimensional subspace, which contradicts the definition of $M$. Therefore $M^\perp = 0$ and $M = H$. ▼

Next we derive the useful orthogonality relations for matrix elements of irreducible, unitary representations.

THEOREM 5. *Let $T^s$ and $T^{s'}$ be any two irreducible unitary representations of $G$ labelled by indices $s$ and $s'$, respectively. Then, their matrix elements satisfy the relations*

$$\int_G D^s_{ij}(x)\overline{D^{s'}_{mn}}(x)\,\mathrm{d}x = \begin{cases} 0, & \text{if } T^s \text{ and } T^{s'} \text{ are not equivalent},\\ \dfrac{1}{d_s}\delta_{im}\delta_{jn}, & \text{if } T^s \cong T^{s'}, \end{cases} \tag{9}$$

*where $d_s$ is the dimension of $T^s$.*

PROOF: Consider the operators

$$E_{ij} = \int_G T^s_x e_{ij} T^{s'}_{x^{-1}}\,\mathrm{d}x, \tag{10}$$

where $(e_{ij})^{mn} = \delta^m_i \delta^n_j$, $i, m = 1, 2, \ldots, d_s$, $j, n = 1, 2, \ldots, d_s$. For every $y \in G$ the operators (10) satisfy the relation

$$T^s_y E_{ij} = E_{ij} T^{s'}_y.$$

Indeed,

$$T^s_y E_{ij} = \int_G T^s_{yx} e_{ij} T^{s'}_{x^{-1}}\,\mathrm{d}x = \int_G T^s_{x'} e_{ij} T^{s'}_{x'^{-1}y}\,\mathrm{d}x' = E_{ij} T^{s'}_y. \tag{11}$$

Hence, if $T^s$ is not equivalent to $T^{s'}$, then, by Schur's lemma, we have $E_{ij} = 0$, or in matrix form,

$$\int_G D^s_{li}(x) D^{s'}_{jk}(x^{-1})\,\mathrm{d}x = \int_G D^s_{li}(x)\overline{D^{s'}_{kj}(x)}\,\mathrm{d}x = 0. \tag{12}$$

If $s = s'$, the operator given by (10) also satisfies the condition (11) and, therefore, by proposition 5.4.3, $E_{ij} = \lambda_{ij} I$. Hence, for $(l, i) \neq (k, j)$ the orthogonality relations (12) are still satisfied. If, however, $(l, i) = (k, i)$, then, by eq. (10) and $E_{ii} = \lambda_{ii} I$ (no summation) we obtain

$$(E_{ii})_{ll} = \int_G D^s_{li}(x) D^s_{il}(x^{-1})\,\mathrm{d}x = \int_G |D^s_{li}(x)|^2\,\mathrm{d}x = \lambda_{ii} \tag{13}$$

(no summation).

In order to calculate the constant $\lambda_{ii}$ we set $i = j$ in eq. (10) and take the trace of both sides. We obtain

$$\operatorname{Tr} E_{ii} = d_s \lambda_{ii} = \int_G \operatorname{Tr}(T_x^s e_{ii} T_{x^{-1}}^s) \mathrm{d}x = \operatorname{Tr} e_{ii} = 1$$

or, $\lambda_{ii} = 1/d_s$. This completes the proof of eq. (9). ▼

We now show that the right regular representation contains all irreducible representations. Indeed, we have

PROPOSITION 6. *Every irreducible unitary representation $T^s$ of $G$ is equivalent to a subrepresentation of the right regular representation.*

PROOF: Let $\{D_{jk}^s(x)\}$, $j, k = 1, 2, \ldots, d_s$, be a matrix form of $T^s$ and let $H^s$ be a subspace of $L^2(G)$ spanned by the orthonormal vectors $e_k^s = \sqrt{(d_s)} D_{ik}^s(x)$. The subrepresentation ${}^{H^s}T^R$ of the right regular representation $T^R$ is irreducible and is equivalent to $T^s$. In fact,

$$T_{x_0}^R e_k^s(x) = D_{ik}^s(x x_0) = D_{il}^s(x) D_{lk}^s(x_0) = D_{lk}^s(x_0) e_l^s(x). ▼$$

A *character* $\chi(x)$ of a finite-dimensional representation $T$ of $G$ is the trace of the operator $T_x$, i.e.,

$$\chi(x) = \operatorname{Tr} T_x = (T_x e_i, e_i) = D_{ii}(x). \tag{14}$$

The properties of the characters of irreducible unitary representations of $G$ are summarized as follows:

PROPOSITION 7.

$$\begin{aligned}
&1^\circ\ \chi(y^{-1}xy) = \chi(x),\\
&2^\circ\ \chi(x^{-1}) = \overline{\chi}(x),\\
&3^\circ\ \textit{If } T^s \simeq T^{s'}, \textit{ then } \chi^s = \chi^{s'},\\
&4^\circ \int_G \chi^s(x)\overline{\chi}^{s'}(x)\,\mathrm{d}x = \begin{cases} 0 & \textit{if } T^s \neq T^{s'}, \\ 1 & \textit{if } T^s \simeq T^{s'}. \end{cases}
\end{aligned} \tag{15}$$

PROOF: *ad* 1° $\chi(y_y^{-1}xy) = \operatorname{Tr}(T_{y^{-1}xy}) = \operatorname{Tr}(T_{y^{-1}} T_x T_y) = \operatorname{Tr} T_x = \chi(x)$.

*ad* 2° $\chi(x^{-1}) = \operatorname{Tr} T_{x^{-1}} = \operatorname{Tr} T_x^* = \overline{D}_{ii}(x) = \overline{\chi}(x)$.

*ad* 3° If $T \simeq T'$, then $T = S^{-1}T'S$ and $\operatorname{Tr} T_x = \operatorname{Tr} S^{-1} T_x' S = \operatorname{Tr} T_s'$.

*ad* 4° From eq. (9),

$$\int_G \chi^s(x)\overline{\chi}^{s'}(x)\,\mathrm{d}x = \int_G D_{ii}^s(x)\overline{D}_{jj}^{s'}(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } T^s \neq T^{s'}, \\ 1 & \text{if } T^s \simeq T^{s'}. \end{cases} ▼$$

Let $T$ be a finite dimensional representation of $G$. Then, by corollary 5.3.2 and eq. (14) we have

$$\chi(x) = m_i \chi_i(x), \tag{16}$$

where $m_i$ is the multiplicity with which an irreducible representation $T^i$, $i = 1, 2, \ldots, n$, of $G$ appears in the decomposition of $T$. Using (15)4° we obtain

$$m_i = \int_G \chi(x)\overline{\chi}_i(x)\,\mathrm{d}x \tag{17}$$

and

$$\sum_{i=1}^{n} m_i^2 = \int_G \chi(x)\bar{\chi}(x)\mathrm{d}x. \tag{18}$$

Eq. (17) shows that a character $\chi$ defines a finite-dimensional representation $T$ of $G$ up to an equivalence. Formula (18) can be used as a criterion for irreducibility of $T$; namely a representation $T$ is irreducible if and only if $\int_G \chi(x)\bar{\chi}(x)\mathrm{d}x = 1$. If $T$ is reducible, then $\int_G \chi(x)\bar{\chi}(x)\mathrm{d}x > 1$.

## § 2. Peter–Weyl and Weyl Approximation Theorems

In this section we prove several useful theorems which will allow us to extend the ordinary Fourier analysis on the real line to the harmonic analysis on compact groups. We begin with the celebrated Peter–Weyl theorem.

THEOREM 1 (Peter–Weyl). *Let $\hat{G} = \{T^s\}$ be the set of all irreducible non-equivalent unitary representations of G. The functions*

$$\sqrt{(d_s)}\, D^s_{jk}(x), \qquad s \in \hat{G},\ 1 \leqslant j,\ k \leqslant d_s, \tag{1}$$

*where $d_s$ is the dimension of $T^s$ and $D^s_{jk}(x)$ are the matrix elements of $T^s$, form a complete orthonormal system in $L^2(G)$.*

PROOF: Let $L$ be the linear closed subspace of $L^2(G)$ spanned by all functions (1) and let $L^\perp$ be the orthogonal complement of $L$. The space $L$ is invariant under the right translations $T^R$ (cf. the proof of proposition 1.6). Hence, by proposition 5.3.1, $L^\perp$ is also an invariant subspace relative to $T^R$. Let $0 \neq v \in L^\perp$ and set

$$u(x) = \int (T^R_x v(y))\bar{v}(y)\mathrm{d}y. \tag{2}$$

This is a continuous function on $G$ belonging to $L^\perp$: indeed by (2) we have

$$\int u(x)\overline{D^s_{jk}}(x)\mathrm{d}x = \sum_l \int v(x)\overline{D^s_{jl}}(x)\mathrm{d}x \overline{\int v(y)\overline{D^s_{kl}}(y)\mathrm{d}y} = 0.$$

Moreover $u(e) = ||v||^2 > 0$. Set now

$$w(x) = u(x) + \overline{u(x^{-1})} \tag{3}$$

and consider the operator

$$A\psi(x) = \int w(xy^{-1})\psi(y)\mathrm{d}y. \tag{4}$$

By virtue of (3) and because invariant measure on $G$ is finite, $A$ is a self-adjoint compact operator. Because $w \neq 0$ there exists an eigenvalue $\lambda$, $\lambda \neq 0$ and a finite-dimensional eigensubspace $H(\lambda)$ of $A$. Let $\psi_\lambda(x)$ be an eigenfunction of $A$. By virtue of (4) we have

$$\int \psi_\lambda(x)\overline{D^s_{ij}}(x)\,\mathrm{d}x = \frac{1}{\lambda}\int (A\psi_\lambda)(x)\overline{D^s_{ij}}(x)\,\mathrm{d}x$$

$$= \frac{1}{\lambda}\sum_k \int w(x')\overline{D}^s_{ik}(x')\,\mathrm{d}x' \int \psi_\lambda(x)\overline{D^s_{kj}}(x)\,\mathrm{d}x = 0.$$

Hence $\psi_\lambda(x) \in L^\perp$. The operator $A$ is $T^R$-invariant; indeed $(T^R_z A\psi)(x) = \int w(xzy^{-1})\psi(y)\mathrm{d}y = \int w(xy^{-1})\psi(zy)\mathrm{d}y = (AT^R_z\psi)(x)$. Consequently $H(\lambda)$ is also invariant with respect to $T^R$ and the formula $T_x\psi_\lambda(y) = \psi_\lambda(yx)$ defines the representation of $G$ in $H(\lambda)$. This representation is fully reducible. Let $\{e^s_k\}_1^{d_s}$ be a basis of an irreducible subspace $H^s(\lambda)$ of $H(\lambda)$, given by eigenfunction of $A$. We have

$$T^R_x e^s_k(y) = e^s_k(yx) = D^s_{jk}(x)e^s_j(y).$$

Since all eigenfunctions of $A$ are continuous, we have (no summation over $s$)

$$e^s_k(x) = D^s_{jk}(x)e^s_j(e),$$

i.e., $e^s_k \in L$. Hence $H(\lambda) = \{0\}$ contrary to the previous conclusion: consequently $L^\perp = 0$ by (3) and (2). ▼

COROLLARY. *Let $u(x) \in L^2(G)$. Then*

$$u(x) = \sum_{s\in\hat{G}} \sum_{j,k=1}^{d_s} c^s_{jk}D^s_{jk}(x), \tag{4}$$

*and*

$$\int_G |u(x)|^2\mathrm{d}x = \sum_{s\in\hat{G}} d_s \sum_{j,k=1}^{d_s} |c^s_{jk}|^2, \tag{5}$$

*where*

$$c^s_{jk} = d_s\int_G u(x)\overline{D^s_{jk}(x)}\,\mathrm{d}x \tag{6}$$

*and the convergence in eq.* (4) *is understood in the sense of norm on $L^2(G)$.*

PROOF: Formula (4) follows from the completeness of functions (1). To show (5), set $u(x) = u_N(x)+\varepsilon_N(x)$, where

$$u_N(x) = \sum_{s=1}^{N}\sum_{j,k=1}^{d_s} c^s_{jk}D^s_{jk}(x).$$

Clearly, $\|\varepsilon_N(x)\| \to 0$ for $N \to \infty$. Then, by orthogonality relations 1(9), one obtains

$$\int_G |u(x)|^2\mathrm{d}x = \sum_{s=1}^{N} d_s^{-1}\sum_{j,k=1}^{d_s}|c^s_{jk}|^2 + (u_N, \varepsilon_N) + (\varepsilon_N, u_N) + (\varepsilon_N, \varepsilon_N).$$

Using the Schwartz inequality, one obtains

$$\Big|\int_G |u(x)|^2 \mathrm{d}x - \sum_{s=1}^{N} \sum_{j,k=1}^{d_s} |c^s_{jk}|^2\Big| \leqslant (2\|u_N\| + \|\varepsilon_N\|)\|\varepsilon_N\| \underset{N\to\infty}{\to} 0. \blacktriangledown$$

The equality (5) is called the *Parseval equality*.

Due to proposition 1.6 we know that every irreducible, unitary representation of $G$ is a subrepresentation of the right regular representation. Moreover, we also know from th. 1.4 that the regular representation is a direct sum of irreducible, unitary (hence, finite-dimensional) representations. The following theorem completes the description of the structure of the regular representation in terms of its irreducible components.

THEOREM 2. *Every irreducible unitary representation $T^s$ of $G$ occurs in the decomposition of the regular representation with a multiplicity equal to the dimension of $T^s$. The orthonormal vectors*

$$Y^s_{(j)k}(x) = \sqrt{d_s}\, D^s_{jk}(x), \qquad s \in \hat{G} \text{ and } j,k = 1,2,\dots,d_s \tag{7}$$

*for fixed $j$ and fixed $s$ span the invariant irreducible subspaces of the right regular representation and the orthonormal vectors*:

$$\tilde{Y}^s_{k(j)}(x) = \sqrt{d_s}\, \overline{D^s_{kj}}(x), \qquad s \in \hat{G} \text{ and } j,k = 1,2,\dots,d_s \tag{8}$$

*for fixed $j$ and fixed $s$ span the invariant, irreducible subspaces of the left regular representation.*

PROOF: Denote by $H^s_{(j)}$, $s \in \hat{G}$ and $j$ fixed, the invariant irreducible subspaces of $H = L^2(G)$ spanned by the orthonormal vectors (7). Let $u(x) \in L^2(G)$. According to formula (4), we have

$$u(x) = \sum_{s\in\hat{G}} \sum_{j,k=1}^{d_s} c^s_{jk} D^s_{jk}(x) = \sum_{s\in\hat{G}} \sum_{j=1}^{d_s} u^s_{(j)}(x), \tag{9}$$

where $u^s_{(j)} \in H^s_{(j)}$. Because $H^s_{(j)} \perp H^{s'}_{(j')}$, if $(s,j) \neq (s',j')$, the decomposition (9) is unique. Hence,

$$H = \sum_{s\in\hat{G}} \sum_{j=1}^{d_s} \oplus H^s_{(j)}. \tag{10}$$

Moreover,

$$T^R_{x_0} u(x) = \sum_{s\in\hat{G}} \sum_{j=1}^{d_s} {}^{H^s_{(j)}}T^R_{x_0} u^s_{(j)}(x), \tag{11}$$

where ${}^{H^s_{(j)}}T^R$ are irreducible unitary subrepresentations of $T^R$ in $H^s_{(j)}$ given by formula

$${}^{H^s_{(j)}}T^R_{x_0} Y^s_{(j)k}(x) = D^s_{lk}(x_0)\, Y^s_{(j)l}(x). \tag{12}$$

Thus, for every $j = 1, 2, \ldots, d_s$, ${}^{H^s_{(j)}}T^R \simeq T^s$ and, therefore, by def. 5.3.3.,

$$T^R = \sum_{s \in \hat{G}} \oplus\, d_s T^s. \tag{13}$$

The same result can be proved for the left regular representation if we take the functions (8) as basis vectors of irreducible subspaces. ▼

The Peter–Weyl theorem can be considerably sharpened. For continuous functions on $G$ instead of the approximation in the norm of $L^2$-space, we can obtain a uniform approximation. This is the content of the following Weyl approximation theorem.

THEOREM 3. *Let $f$ be a continuous function on $G$. For every $\varepsilon > 0$ there exists a linear combination*

$$\sum_{s=1}^{N_\varepsilon} \sum_{j,k} c^s_{jk} D^s_{jk}(x)$$

*of the matrix elements of irreducible unitary representations such that*

$$\left| f(x) - \sum_{s=1}^{N_\varepsilon} \sum_{j,k=1}^{d_s} c^s_{jk} D^s_{jk}(x) \right| \leqslant \varepsilon \quad \textit{for all } x \in G. \tag{14}$$

PROOF: We shall prove th. 3 using the method of the so-called 'smeared-out' operators. This method is very useful in the solution of many problems in the representation theory. Let $\varphi \in C(G)$, $f \in L^2(G)$ and let $L_\varphi$ be the operator

$$L_\varphi f(x) = \int_G \varphi(y) T^L_y f(x) \mathrm{d}y = \int_G \varphi(y) f(y^{-1}x) \mathrm{d}y. \tag{15}$$

The operation (15) is also denoted by $\varphi * f$ and called the *convolution* of $\varphi$ and $f$. The operator $L_\varphi$ has the following properties

1° It is a continuous map from $L^2(G)$ into $C(G)$.

2° If $H_N$ is the set of finite linear combinations

$$\sum_{s=1}^{N} \sum_{j,k} c^s_{jk} D^s_{jk}(x),$$

where $D^s_{jk}(x)$ are matrix elements of irreducible, unitary representation of $G$, then,

$$L_\varphi(H_N) \subset H_N. \tag{16}$$

Indeed, using the Cauchy inequality, we have

$$|L_\varphi f(x)|^2 = \left| \int_G \varphi(y) f(y^{-1}x) \mathrm{d}y \right|^2 \leqslant \int_G |\varphi(y)|^2 \mathrm{d}y \int_G |f(y^{-1}x)|^2 \mathrm{d}y.$$

Let $\|f\|_{C(G)} \equiv \sup_{x \in G} |f(x)|$, then

$$\|L_\varphi f\|_{C(G)} \leqslant c\|f\|_{L^2}, \quad \text{where } c = \|\varphi\|_{L^2}. \tag{17}$$

Moreover, if $f_N(x) \in H_N$, we find

$$\begin{aligned} L_\varphi f_N(x) &= \sum_{s=1}^{N} \sum_{jk} c^s_{jk} \int_G \varphi(y) D^s_{jk}(y^{-1}x)\,\mathrm{d}y \\ &= \sum_{s=1}^{N} \sum_{jkp} c^s_{jk} D^s_{pk}(x) \int_G \varphi(y) D^s_{jp}(y^{-1})\,\mathrm{d}y \\ &= \sum_{s=1}^{N} \sum_{kp} \tilde{c}^s_{pk} D^s_{pk}(x) \in H_N, \quad \text{where } \tilde{c}^s_{pk} = \sum_j c^s_{pj}(\varphi) c^s_{jk}(f). \end{aligned}$$

To prove the main theorem let $f$ be any element of $C(G)$. Because any continuous function on $G$, by proposition 2.2.4., is uniformly continuous, there exists a neighborhood $V_\varepsilon$ of unity such that

$$|f(x_1)-f(x_2)| < \varepsilon, \quad \text{whenever } x_1 x_2^{-1} \in V_\varepsilon. \tag{18}$$

Let $\varphi_\varepsilon \in C(G)$ be a non-negative function, which differs from zero only on $V_\varepsilon$ and satisfies $\int_G \varphi_\varepsilon(x)\,\mathrm{d}x = 1$. Then, for the smeared out operator $L_{\varphi_\varepsilon}$ we obtain

$$\|L_{\varphi_\varepsilon} f - f\|_{C(G)} < \varepsilon. \tag{19}$$

Indeed, by eq. (18),

$$\begin{aligned} \|L_{\varphi_\varepsilon} f - f\|_{C(G)} &= \sup_{x \in G} \left| \int_G \varphi_\varepsilon(y) [f(y^{-1}x) - f(x)]\,\mathrm{d}y \right. \\ &= \sup_{x \in G} \int_{V_\varepsilon} \varphi_\varepsilon(x) |f(x^{-1}y) - f(y)|\,\mathrm{d}x < \varepsilon. \end{aligned}$$

We know by th. 1 that every element $f \in C(G)$ can be arbitrarily approximated in the norm of $L^2(G)$ by elements of $H_N$, i.e., in particular,

$$\|f - f_N\|_{L^2} \leqslant \|\varphi_\varepsilon\|_{L^2}^{-1} \cdot \varepsilon.$$

Thus, according to eqs. (19) and (17), we have

$$\begin{aligned} |f(x) - L_{\varphi_\varepsilon} f_N(x)| &\leqslant |f(x) - L_{\varphi_\varepsilon} f(x)| + |L_{\varphi_\varepsilon}(f(x) - f_N(x))| \\ &< \varepsilon + \|f - f_N\|_{L^2} \|\varphi_\varepsilon\|_{L^2} < 2\varepsilon. \end{aligned}$$

Because the function $L_{\varphi_\varepsilon} f_N(x)$, $f_N \in H_N$, is an element of $H_N$, the proof of th. 3 is completed. ▼

Note that from eqs. (4) and (6) we obtain the relation

$$\sum_{s=1}^{\infty} \sum_{j,k=1}^{d_s} d_s \bar{D}^s_{jk}(x') D^s_{jk}(x) = \delta(x - x'). \tag{20}$$

This is an alternate form of the completeness relation of $D^s_{jk}$-functions, which is very useful in calculations.

## § 3. Projection Operators and Irreducible Representations

We consider in this section the properties of the projection operators associated with the irreducible representations of compact groups. The technique of projection operators is extremely useful, elegant and effective in the solution of various practical problems in representation theory and quantum physics.

Let $D^s_{pq}(x)$ be the matrix elements of an irreducible representation $T^s$ and define the operators

$$P^s_{pq} \equiv d_s \int_G \bar{D}^s_{pq}(x) T_x \mathrm{d}x, \tag{1}$$

where

$d_s$: the dimension of an irreducible representation $T^s$,

$dx$: the invariant Haar measure on $G$,

$x \to T_x$: a unitary representation of $G$ in the carrier space $H$.

Because $D^s_{pq}(x)$ and $T_x$ are continuous functions on $G$, and because $G$ is compact, the operator integral (1) is well defined, (cf. app. B.2). In particular, all operators $P^s_{pq}$ are bounded. Indeed,

$$||P^s_{pq}u|| \leqslant d_s \int |\bar{D}^s_{pq}(x)| \, ||T_x u|| \mathrm{d}x \leqslant d_s \sup_{x\in G} |D^s_{pq}(x)| \, ||u||.$$

Hence,

$$||P^s_{pq}|| \leqslant d_s \sup_{x\in G} |D^s_{pq}(x)|.$$

PROPOSITION 1. *The operators $P^s_{pq}$ have the following properties*

1° $(P^s_{pq})^* = P^s_{qp}$, (2)

2° $P^s_{pq} P^{s'}_{p'q'} = \delta^{ss'} \delta_{qp'} P_{pq'}$. (3)

PROOF: *ad* 1°. Because for every bounded operator $A$ we have $||A^*|| = ||A||$, the map $A \to A^*$ is continuous in weak operator topology. Hence, in eq. (1) we can interchange the adjoint operation and integration, i.e.,

$$(P^s_{pq})^* = d_s \int_G D^s_{pq}(x) T^*_x \mathrm{d}x = d_s \int_G D^s_{pq}(x^{-1}) T_x \mathrm{d}(x^{-1}) = d_s \int \bar{D}^s_{qp}(x) T_x \mathrm{d}x = P^s_{qp},$$

where we used $T^*_x = T^{-1}_x = T_{x^{-1}}$, as well as the invariance of the Haar measure.

*ad* 2°. From (1)

$$P^s_{pq} P^{s'}_{p'q'} = d_s d_{s'} \int \bar{D}^s_{pq}(x) \bar{D}^{s'}_{p'q'}(x') T_x T_{x'} \mathrm{d}x \mathrm{d}x'.$$

Using the group property $T_x T_{x'} = T_{xx'}$ and the relation

$$D^s_{pq}(x) = D^s_{pq}(\tilde{x}x'^{-1}) = D^s_{pr}(\tilde{x}) D^s_{rq}(x'^{-1}) = D^s_{pr}(\tilde{x}) \bar{D}^s_{qr}(x'),$$

where $xx' = \tilde{x}$, as well as the orthonormality relations 1(9), we obtain

$$P^s_{pq} P^{s'}_{p'q'} = d_s d_{s'} \int D^s_{qr}(x') \bar{D}^{s'}_{p'q'}(x') \mathrm{d}x' \int \bar{D}^s_{pr}(\tilde{x}) T_{\tilde{x}} \mathrm{d}\tilde{x} = \delta^{ss'} \delta_{qp'} P^s_{pq'}. \; \blacktriangledown$$

COROLLARY. *The operators $P^s_p \equiv P^s_{pp}$ are projection operators, i.e.,*

$$(P^s_p)^* = P^s_p, \quad P^s_p P^{s'}_{p'} = \delta^{ss'}\delta_{pp'}P^s_p. \;\blacktriangledown \tag{4}$$

The operators $P^s_{pq}$ have simple transformation properties with respect to the action of the group $G$. Indeed, we have

PROPOSITION 2. *Let $P^s_{pq}$ be given by formula* (1). *Then* (*no summation over* $s$)

$$T_x P^s_{pq} = D^s_{rp}(x) P^s_{rq}, \tag{5}$$

$$P^s_{pq} T_x = D^s_{qr}(x) P^s_{pr}. \tag{6}$$

PROOF: Because $T_x$ is continuous, it can be brought under the integral sign in (1). Using the group properties of $D^s_{pq}$ functions, one obtains

$$\begin{aligned} T_x P^s_{pq} &= d_s \int \bar{D}^s_{pq}(x') T_{xx'} \mathrm{d}x' = d_s \int \bar{D}^s_{pq}(x^{-1}\tilde{x}) T_{\tilde{x}} \mathrm{d}\tilde{x} \\ &= d_s D^s_{rp}(x) \int \bar{D}^s_{rq}(\tilde{x}) T_{\tilde{x}} \mathrm{d}\tilde{x} = D^s_{rp}(x) P^s_{rq}. \end{aligned}$$

Formula (6) is proved in a similar manner. ▼

Note that the vectors

$$|s;p\rangle = P^s_{pq} u, \quad q \text{ fixed}, \; u \in H,$$

by virtue of eq. (5), transform as the basis vectors of the carrier space $H^s$ of the irreducible representation $T^s$. This fact is the starting point of most applications of the projection operators $P^s_{pq}$ (cf. § 4.A and § 4.B).

Note also that by eqs. (5) and (6) we have

$$T_x P^s_{pq} T_x^{-1} = D^s_{rp}(x) \bar{D}^s_{tq}(x) P^s_{rt}. \tag{7}$$

Formula (7) means that $P^s_{pq}$ transforms as a tensor operator corresponding to the tensor product of a basis vector $e^s_p$ and an adjoint vector to $e^s_q$ (i.e., as the product $|s;p\rangle\langle s;q|$ in Dirac's notation).

There are also useful projection operators associated with the characters

$$\chi^s(x) = \sum_{p=1}^{d_s} D^s_{pp}(x).$$

They are defined in the following manner:

$$P^s = d_s \int_G \chi^s(x) T_x \mathrm{d}x. \tag{8}$$

PROPOSITION 3. *The operators $P^s$ have the following properties*

$$(P^s)^* = P^s, \tag{9}$$

$$P^s P^{s'} = \delta_{ss'} P^s, \tag{10}$$

$$T_x P^s = P^s T_x. \tag{11}$$

PROOF: Because $P^s = \sum P^s_p$, eqs. (9)–(10) follow directly from propositions 1 and 2, and eq. (10) follows from (8) and the fact that $\chi(x) = \chi(yxy^{-1})$.

We prove still another useful result

PROPOSITION 4. *Let $T$ be a unitary representation of $G$ in $H$. Then*

$$\sum_s P^s = I, \tag{12}$$

*and*

$$\sum_{s,p} P^s_p = I. \tag{13}$$

PROOF: Let $\{e^s_r\}$ be a basis in $H$. Then by virtue of (8) we have

$$\sum_{s'} P^{s'} e^s_r = \sum_{s'} d_{s'} \int \mathrm{d}x \sum_p \overline{D}^{s'}_{pp}(x) D^s_{mr}(x) e^s_m = e^s_r. \tag{14}$$

This implies eq. (12). Eq. (13) follows from the definition of $P^s$ and eq. (12).

EXAMPLE. Let $G = \mathrm{SO}(3)$. If we describe the rotations in terms of Euler angles $\varphi$, $\vartheta$ and $\psi$,

$$0 \leqslant \varphi < 2\pi, \quad 0 \leqslant \vartheta < \pi, \quad 0 \leqslant \psi < 2\pi \tag{15}$$

then, by virtue of exercise 5.8.1.1 and 3.11(30) we have

$$\begin{gathered} d_J = 2J+1, \quad \mathrm{d}x = (8\pi^2)^{-1} \sin\vartheta \,\mathrm{d}\varphi \,\mathrm{d}\vartheta \,\mathrm{d}\psi, \\ D^J_{M,M}(\varphi, \vartheta, \psi) = \left(\frac{1+\cos\vartheta}{2}\right)^M P^{0;2M}_{J-M}(\cos\vartheta) \exp[-\mathrm{i}M(\varphi+\psi)], \\ \chi(\varphi, \vartheta, \psi) = \sum_{M=-J}^{J} D^J_{MM}(\varphi, \vartheta, \psi), \end{gathered} \tag{16}$$

$$T_{x(\varphi,\vartheta,\psi)} = \exp(-\mathrm{i}\varphi J_z) \exp(-\mathrm{i}\vartheta J_y) \exp(-\mathrm{i}\psi J_z).$$

Therefore, the projection operators $P^J_M$ and $P^J$ are given by

$$P^J_M = \frac{2J+1}{8\pi^2} \int \overline{D}^J_{M,M}(\varphi, \vartheta, \psi) T_{x(\varphi,\vartheta,\psi)} \sin\vartheta \,\mathrm{d}\varphi \,\mathrm{d}\vartheta \,\mathrm{d}\psi, \tag{17}$$

$$P^J = \frac{2J+1}{8\pi^2} \int \overline{\chi}^J(\varphi, \vartheta, \psi) T_{x(\varphi,\vartheta,\psi)} \sin\vartheta \,\mathrm{d}\varphi \,\mathrm{d}\vartheta \,\mathrm{d}\psi. \; \blacktriangledown \tag{18}$$

## § 4. Applications

### A. *Decomposition of a Factor Representation onto Irreducible Representations*

In many problems an irreducible representation $T^s$ of $G$ appears several times in the carrier space $H$. It is then required, in many applications, to decompose a factor representation $n_s T^s$ (no summation) onto its irreducible components and to construct explicitly the corresponding orthogonal carrier spaces. We solve these problems using the projection operators $P^s_{pq}$. As we noted the vectors

$$|s;p\rangle \equiv P^s_{pq} u, \quad q = \text{fixed}, \; u \in H, \tag{1}$$

transform as basis vectors $e_p^s$ of the carrier space $H^s$ of the irreducible representation $T^s$, i.e.

$$T_x|s;p\rangle = D^s_{rp}(x)P^s_{rq}u = D^s_{rp}(x)|s;r\rangle. \tag{2}$$

This relation provides us with a simple and elegant method of an explicit construction of the orthogonal basis vectors $|s;p\rangle$ of the irreducible carrier space $H^s$, from the vectors of the Hilbert space $H$, in which the reducible unitary representation $T$ of $G$ is realized. Consider first the case, when every irreducible representation $T^s$ appears in the decomposition of $T$ only once. Let $P^s_{pq}u \neq 0$, $p = 1, 2, \ldots, d_s$, $d_s$ = dimension of the irreducible representation $T^s$, for some $u \in H$ and for fixed $s$ and $q$. Then, the vectors

$$|s;p\rangle \equiv \frac{1}{N} P^s_{pq}u, \quad p = 1, 2, \ldots, d_s, \ s \text{ and } q \text{ fixed}, \tag{3}$$

where $N^2 = (u, P^s_{qq}u)$, form an orthonormal set of vectors. In fact,

$$\begin{aligned}\langle s;p'|s;p\rangle &= \frac{1}{N^2}(P^s_{p'q}u, P^s_{pq}u) = \frac{1}{N^2}\left(u, (P^s_{p'q})^* P^s_{pq}u\right)\\ &= \frac{1}{N^2}(u, P^s_{qp'}P^s_{pq}u) = \delta_{p'p}\frac{1}{N^2}(u, P^s_{qq}u) = \delta_{p'p}.\end{aligned} \tag{4}$$

According to eq. (2), the closed linear hull $H^s$ of orthonormal vectors (3) forms the carrier space in which the irreducible unitary representation $T^s = \{D^s_{ij}\}$ is realized.

It is remarkable that this method works even when a reducible unitary representation $T$ contains an irreducible representation $T^s$ several, say $(n_s)$, times. Indeed, let $T = \sum_s \oplus n_s T^s$ be a decomposition of $T$ onto factor representations $n_s T^s$, and let $H = \sum_s \oplus n_s H^s$ be the corresponding decomposition of the carrier space $H$. Moreover, let $H^s_q = P^s_q H$, where $s$ and $q$ are arbitrary, but fixed. Then, we have

PROPOSITION 1. *If $u \neq 0$ is an arbitrary vector from $H^s_q$, $s$ and $q$ fixed, then there exists only one subspace $H^s_{(u)}$ containing this vector, which is the carrier space of the irreducible representation $T^s = \{D^s_{ij}\}$.*

*If $u \neq 0$ and $v \neq 0$ are orthogonal vectors from $H^s_q$ then the spaces $H^s_{(u)}$ and $H^s_{(v)}$ are the orthogonal carrier spaces of the irreducible representation $T^s = \{D^s_{ij}\}$.*

PROOF: Let $0 \neq u \in H^s_q$ and let $||u|| = 1$. The vectors

$$|s;p\rangle = P^s_{pq}u, \quad p = 1, 2, \ldots, d_s, \ s \text{ and } q \text{ fixed}, \tag{5}$$

constitute the orthonormal set of vectors. According to eq. (2), the closed linear hull $H^s_{(u)}$ of all vectors (5) constitutes the carrier space of the irreducible representation $T^s = \{D^s_{ij}\}$. The vector $|s;q\rangle = P^s_{qq}u = u$ cannot belong to two different irreducible subspaces $H^s_{(u)}$ and $\hat{H}^s_{(u)}$ because the intersection $L = H^s_{(u)} \cap \hat{H}^s_{(u)}$ is an invariant subspace, which satisfies the following conditions:

$$L \subset H^s_{(u)}, \quad L \subset \hat{H}^s_{(u)}, \quad L \neq H^s_{(u)}, \quad L \neq \hat{H}^s_{(u)}.$$

Consequently, because of the irreducibility of $H^s_{(u)}$ and $\hat{H}^s_{(u)}$, this invariant subspace is empty, i.e., $L = 0$.

If $u \neq 0$ and $v \neq 0$ are orthogonal vectors from $H^s_q$, then the spaces $H^s_{(u)}$ and $H^s_{(v)}$ are the orthogonal carrier spaces of the same irreducible representation $T^s = \{D^s_{ij}\}$. In fact,

$$\begin{aligned}(P^s_{p'q}v, P^s_{p''q}u) &= (v, (P^s_{p'q})^* P^s_{p''q}u) = (v, P^s_{qp'} P^s_{p''q}u) \\ &= \delta_{p'p''}(v, P^\lambda_{qq}u) = \delta_{p'p''}(v, u) = 0. \blacktriangledown\end{aligned}$$

Thus, by taking successive orthogonal vectors from the space $H^s_q$, we obtain as many orthogonal carrier spaces of the same irreducible representation $T^s = \{D^s_{ij}\}$ as the dimension of the subspace $H^s_q$. Clearly, $\dim H^s_q = n_s$. Thus, proposition 1 provides, in the general case, a systematic method of separation of irreducible carrier subspaces $H^s_{(u_i)}$, $i = 1, 2, \ldots, n_s$, from the reducible space $H$ along the following steps:

1° Find the subspace $H^s_q = P^s_q H$ ($s$ fixed, $q$ arbitrary but fixed),

2° Select in an arbitrary manner in the subspace $H^s_q$ an orthogonal base $u_1, u_2, \ldots, u_{n_s}$, $n_s = \dim H^s_q$,

3° Apply successively the formula (5) to each of the vectors $u_i$, $i = 1, 2, \ldots, n_s$, to find irreducible subspaces $H^s_{(u_i)}$ containing these vectors.

According to proposition 1 the corresponding irreducible subspaces $H^s_{(u_1)}, H^s_{(u_2)}, \ldots, H^s_{(u_{n_s})}$, will be mutually orthogonal. The collection of all of them provides the effective decomposition of the reducible subspace $P^s H = n_s H^s$ into irreducible components $H^s_{(u_i)}$, $i = 1, 2, \ldots, n_s$ in which the same irreducible unitary representation $T^s = \{D^s_{ij}\}$ is realized. Applying successively this method for all $s$ we obtain the effective decomposition of $T$ into irreducible components $T^s$.

## B. *The Coupling Coefficients ('Clebsch–Gordan Coefficients')*

Let $T^{s_1}$ and $T^{s_2}$ be two irreducible representations of $G$ in Hilbert spaces $H^{s_1}$ and $H^{s_2}$, respectively. Let $|s_i p_i\rangle$ be the orthonormal basis vectors in $H^{s_i}$, $i = 1, 2$. Suppose first that $G$ is *simply reducible*, i.e., that in the *tensor product* definition of the two irreducible representations the multiplicity of a given representation is at most one. In the tensor product space $H = H^{s_1} \otimes H^{s_2}$, we can construct two sets of orthogonal basis vectors. The first consists of the Kroenecker product of the original basis vectors

$$|s_1 p_1, s_2 p_2\rangle = |s_1 p_1\rangle |s_2 p_2\rangle, \quad p_1 = 1, 2, \ldots, d_{s_1}, \; p_2 = 1, 2, \ldots, d_{s_2}, \tag{6}$$

while the second one contains the basis vectors

$$|sps_1 s_2\rangle \tag{7}$$

which span an irreducible carrier space $H^s$ contained in the tensor product space $H^{s_1}\otimes H^{s_2} = \sum_s \oplus H^s$. According to eq. (3) the basis vectors (7) can be obtained from the basis vectors (6) by means of the formula

$$|s_1 s_2 sp\rangle = (N^{ss_1s_2}_{p'p_1'p_2'})^{-1} d_s \int_G \bar{D}^s_{pp'}(x) T_x |s_1 p_1' s_2 p_2'\rangle \mathrm{d}x, \quad p', p_1', p_2' \text{ fixed}, \tag{8}$$

where $N^{ss_1s_2}_{p'p_1'p_2'}$ is a normalization constant. The operator $T_x$ acts on the basis vectors (6) by means of the formula

$$T_x|s_1 p_1 s_2 p_2\rangle = D^{s_1}_{p_1'p_1}(x) D^{s_2}_{p_2'p_2}(x)|s_1 p_1' s_2 p_2'\rangle . \tag{9}$$

The so-called 'Clebsch–Gordan coefficients' are the matrix elements of the unitary operator (called the *transition matrix*) connecting the basis vectors (6) and (7) and are given by (from (8) and (9))

$$\langle s_1 p_1 s_2 p_2|sps_1 s_2\rangle = (N^{ss_1s_2}_{p'p_1'p_2'})^{-1} d_s \int_G \bar{D}^s_{pp'}(x) D^{s_1}_{p_1p_1'}(x) D^{s_2}_{p_2p_2'}(x) \mathrm{d}x. \tag{10}$$

In order to find the normalization constant $N^{ss_1s_2}_{p'p_1'p_2'}$ we calculate the square of basis vector (8)

$$\begin{aligned}
1 &= (N^{ss_1s_2}_{p'p_1'p_2'})^{-2}\langle s_1 p_1' s_2 p_2'|P^{s*}_{pp'} P^s_{pp'}|s_1 p_1' s_2 p_2'\rangle \\
&= (N^{ss_1s_2}_{p'p_1'p_2'})^{-2}\langle s_1 p_1' s_2 p_2'|P^s_{p'p'}|sp_1' s_2 p_2'\rangle \\
&= (N^{ss_1s_2}_{p'p_1'p_2'})^{-2} d_s \int_G \mathrm{d}x D^s_{p'p'}(x) \bar{D}^{s_1}_{p_1'p_2'}(x) \bar{D}^{s_2}_{p_1'p_2'}(x) \\
&= (N^{ss_1s_2}_{p'p_1'p_2'})^{-1} \overline{\langle s_1 p_1' s_2 p_2'|s_1 s_2 sp'\rangle} .
\end{aligned}$$

Hence

$$N^{ss_1s_2}_{pp_1'p_2'} = \overline{\langle s_1 p_1' s_2 p_2'|s_1 s_2 sp'\rangle} = \langle s_1 s_2 sp'|s_1 p_1' s_2 p_2'\rangle . \tag{11}$$

We see, therefore, that the normalization constant is itself another Clebsch–Gordan coefficient. Because the indices $p'$, $p_1'$ and $p_2'$ are arbitrary, we can select C–G coefficient (11) as simple as possible.

The C–G coefficient (10) are not determined uniquely. Indeed, if we multiply the basis vectors (6) by constant phase factors $\varphi_s$, $|\varphi_s| = 1$, we obtain again a complete orthonormal system. Hence, the C–G coefficients are determined up to a phase factor. We can use this arbitrariness to set one of the C–G coefficients to be non-negative. Then the normalization constant in (10) is determined uniquely. Indeed, setting in eq. (10) $p = p'$, $p_1 = p_1'$, and $p_2 = p_2'$ one obtains, by virtue of eq. (11),

$$N^{ss_1s_2}_{p'p_1p'_2} = \left[\int_G \bar{D}^s_{p'p'}(x)\, D^{s_1}_{p'_1p'_1}(x)\, D^{s_2}_{p_2p_2}(x)\, \mathrm{d}x\right]^{1/2}. \tag{12}$$

Thus, the knowledge of the matrix elements $D^s_{pq}$ of irreducible representations allows us to determine completely the C–G coefficients for simple reducible groups.

If the given group $G$ is not simply reducible we first split out a factor representation $n_s T^s$ in the tensor product space onto irreducible representations using the technique of the subsection $A$. Having constructed the basis vectors (7) in the carrier spaces $H^s_{(u_1)}, H^s_{(u_2)}, \ldots, H^s_{(u_{n_s})}$, we proceed as above.

The formulas (10) and (12) constitute the basis for the explicit determination of the C–G coefficients for compact groups of physical interest, such as SO(3), SO(4), SU(3), etc. Because the matrix elements $D^s_{pq}(x)$ can be expressed in terms of the products of special functions (cf. ch. 14) the problem of the determination of C–G coefficients reduces to the problem of integration of the product of three special functions over a finite region.

### C. *A Physical Application: Distribution of Isotopic-Spin States*

The strongly interacting particles can be assigned certain internal quantum numbers, in addition to mass, total angular momentum (spin) and parity. The internal quantum numbers distinguish particles with the same spin and parity, and they are conserved in strong interactions. In particular we assign to each such particle an isotopic spin $t$, associated with an (internal) SU(2)-symmetry group. Let $I_1$, $I_2$ and $I_3$ be the generators of this SU(2)-group. The state of the particle is then characterized by $|t, \theta, x\rangle$, where $I^2 = I_1^2+I_2^2+I_3^2$ and $I_3$ have the eigenvalues $t(t+1)$ and $\theta$, respectively, and $x$ stands for the remaining set of quantum numbers. For a collection of free particles we take, as far as the isotopic spin quantum number is concerned, the tensor product space $|t_1\theta_1, t_2\theta_2, \ldots\rangle$. The total isotopic spin of the systems is defined by the vector sum $I = \sum_i t_i$. In a strong collision process the total isotopic spin of the initial particles is equal to the total isotopic spin of the final particles. In other words the $S$-matrix expressing the transition probability amplitude from the initial state to the final state commutes with the representation $g \to T_g$ of the isotopic spin group SU(2) in the Hilbert space of state vectors. (See ch. 13 for more details on quantum mechanical invariance properties.)

Consider the process of scattering of two particles in the reaction

$$A+B \to C_1+C_2+ \ldots +C_n,$$

where the particle $C_i$, $i = 1, 2, \ldots, n$, has the isospin $t_i$ and the third isospin component $\theta_i$. The probability that a certain final state with values $\theta_1, \theta_2, \ldots, \theta_n$ will be found for a given total isospin $I$ and its third component $I_3$ is

$$P^I_{\theta_1, \dots, \theta_n} = \Big[\sum_\alpha |\langle t_1\theta_1, \dots, t_n\theta_n|II_3\alpha\rangle|^2\Big]\Big[\sum_{(\theta_i)}\sum_\alpha |\langle t_1\theta_1, \dots, t_n\theta_n|II_3\alpha\rangle|^2\Big]^{-1}. \quad (13)$$

Here $\alpha$ stands for all additional isotopic quantum numbers required to describe the $n$-particle state completely in the isospin space. We have assumed that each state $\alpha$ for fixed $I$, $I_3$ occurs with equal probability. Using the technique of projection operators we can easily find the final expression for the probability (13). In fact the quantity

$$\sum_\alpha |\langle t_1\theta_1, \dots, t_n\theta_n|II_3\alpha\rangle|^2 \quad (14)$$

is nothing but the square of the projection of $|t_1\theta_1, \dots, t_n\theta_n\rangle$ onto the subspace in the tensor product space, spanned by the basis vectors $|I, I_3, \alpha\rangle$ with given $I, I_3$. Therefore, if one could compute directly the length of this projection one could dispense with the lengthy computation of the components and the determination of the complete set of quantum numbers $\alpha$. In general, more than one $(2I+1)$-dimensional subspace of a representation $T^I$ may occur in the tensor product of one-particle representations $\prod_{i=1}^n \otimes T^{t_i}$. Let us denote the direct sum of these subspaces by $\Gamma^{(I)}$. Then the projection on $\Gamma^{(I)}$ is given by the formula 3(18) where now

$$P^{\Gamma^{(I)}} = \frac{2I+1}{8\pi^2}\int \bar{\chi}^{(I)}(\varphi, \vartheta, \psi)\, T_{g(\varphi, \vartheta, \psi)} \sin\vartheta\, d\varphi\, d\vartheta\, d\psi, \quad (15)$$

$$T_g = \prod_{k=1}^n T_g^{(k)}, \qquad T_g^{(k)} = \exp[-i\varphi(t_k)_z]\exp[-i\vartheta(t_k)_y]\exp[-i\psi(t_k)_z]$$

or in the matrix form

$$[T_{g(\varphi, \vartheta, \psi)}]_{\theta_1'\theta_1, \dots, \theta_n'\theta_n} = \prod_{k=1}^n D^{t_k}_{\theta_k'\theta_k}(\varphi, \vartheta, \psi).$$

The square of the projection of $|t_1\theta_1, \dots, t_n\theta_n\rangle$ onto $\Gamma^{(I)}$ is

$$|P^{\Gamma^{(I)}}|t_1\theta_1, \dots, t_n\theta_n\rangle|^2 = \langle t_1\theta_1, \dots, t_n\theta_n|P^{\Gamma^{(I)}}|t_1\theta_1, \dots, t_n\theta_n\rangle,$$

where we made use of formula 3(10). Consequently

$$P^I_{\theta_1, \dots, \theta_n} = \frac{2I+1}{8\pi^2}\int \bar{\chi}^{(I)}(\varphi, \vartheta, \psi) \prod_{k=1}^n D^{(t_k)}_{\theta_k\theta_k}(\varphi, \vartheta, \psi) \sin\vartheta\, d\varphi\, d\vartheta\, d\psi. \quad (16)$$

Using the representation of $D^J_{MM}(g)$ in the form (cf. exercise 5.8.1.1)

$$D^J_{M,M}(\varphi, \vartheta, \psi) = \left(\frac{1+\cos\vartheta}{2}\right)^M P^{0,2M}_{J-M}(\cos\vartheta)\exp[-iM(\varphi+\psi)],$$

where $P^{0,2M}_{J-M}(x)$ is the Jacobi polynomial, and using the expression 3(16) for $\chi^{(I)}$ we get, after an integration over $\varphi$ and $\psi$,

$$P^{(I)}_{\theta_1,\ldots,\theta_n} = (2I+1)2^{-2I_3-1}\int_{-1}^{+1} dx(1+x)^{2I_3}P^{0,2I_3}_{I-I_3}(x)\prod_{l=1}^{n} P^{0,2\theta_l}_{t_l-\theta_l}(x). \tag{17}$$

In the derivation of the last formula the condition $I_3 \geqslant 0$ was assumed. This is no restriction, however, because one can prove easily that

$$P^I_{\theta_1,\ldots,\theta_n} = P^I_{-\theta_1,\ldots,-\theta_n}.$$

The integrand in eq. (17) is a polynomial of degree $N = I+\sum_{l=1}^{n} t_l-2I_3$. If we represent this polynomial by

$$\sum_{j=0}^{N} a_j x^j$$

then

$$P^I_{\theta_1,\ldots,\theta_n} = (2I+1)2^{-2I_3}\sum_{j=0}^{[N/2]'}\frac{a_{2j}}{2j+1}. \tag{18}$$

From eqs. (17) or (18) we obtain the final expression for $P^I_{\theta_1,\ldots,\theta_n}$ in various special cases:

(i) Case of $n$ pions (e.g., $p+\bar{p}\to n_+\pi^+ +n_0\pi^0+n_-\pi^-$)

$$P^I_{n_+,n_0,n_-} = (2I+1)2^{-2n_+-I+I_3}(-1)^{I-I_3}\times$$
$$\times\sum_{\nu=0}^{I-I_3}\binom{I-I_3}{\nu}\binom{I+I_3}{I-I_3-\nu}(-1)^\nu[A(2n_+ +\nu, n_0, I-I_3-\nu)+$$
$$+(-1)^{n_0}A(I-I_3-\nu, n_0, 2n_+ +\nu)],$$

where

$$A(a,b,c)\equiv\sum_{p=0}^{a}\binom{a}{p}\frac{(p+b)!c!}{(p+b+c+1)!}.$$

(ii) Case of one nucleon and $n$ pions

$$P^I_{m_1;n_+,n_0,n_-} = (J+1)2^{-(n_++n_-+J+1)}(-1)^{J-M}\times$$
$$\times\sum_{\nu=0}^{J-M}(-1)^\nu\binom{J-M}{\nu}\binom{J+M+2}{J-M-\nu}[A(2n_++1+\nu, n_0, J-M-\nu)+$$
$$+(-1)^{n_0}A(J-M-\nu, n_0, 2n_++1+\nu)],$$

where $J = I-1/2$, $M = I_3-1/2$, and $m_1$ is the third component of isospin of nucleon.

Note that the permutations which shuffle particles of the same kind and charge among themselves only do not lead to distinguishable isospin states. Therefore, in order to find the probability of a certain charge distribution, one has to multiply the coefficient $P^I_{\theta_1, \dots, \theta_n}$ with the number of permutations of the $\theta_1, \dots, \theta_n$, which result in a reordering of the numbers $\theta_1, \dots, \theta_n$ only. For example the weight of the charge distribution of $n$ pions, without consideration of their momenta is given by

$$\tilde{P}^I_{n_+, n_0, n_-} = \frac{n!}{n_+! n_0! n_-!} P^I_{n_+, n_0, n_-}.$$

## § 5. Representations of Finite Groups

In this section we treat the properties of finite groups and their representations. Finite groups have many important applications in quantum physics, especially in atomic, molecular and solid state physics. For this reason, and in order to be able to discuss the representations of the symmetric group $S_N$, we give a concise discussion of the representations of finite groups.

Every finite group is compact. Hence, all theorems of this chapter remain true for finite groups. It is only necessary to replace in all formulas the integral over the group manifolds $\int dx$ by the sum over group elements.

### A. *The Symmetric Group $S_N$*

The symmetric group $S_N$, the group of permutations of $N$ objects, of order $N!$, is fundamental in the study of finite groups, in the applications as well as in the representation theory of continuous groups. We have namely:

THEOREM 1 (Cayley). *Every finite group $G$ of order $N$ is isomorphic to a subgroup of $S_N$.*

PROOF: Consider the permutation of the elements of $G$ defined by left-multiplication by an element $x$

$$_x\pi = \downarrow \begin{pmatrix} x_1 & x_2 & \dots x_N \\ xx_1 & xx_2 & \dots xx_N \end{pmatrix}. \tag{1}$$

We denote the permutation of a set $(x_1 \dots x_N)$ into the set $(xx_1 \dots xx_N)$, i.e., an element of $S_N$, by an arrow as above. All permutations $\{_x\pi, x \in G\}$ form a group. The map $f\colon x \to {_x\pi}$ is one-to-one, and

$$f\colon xy \to {_x\pi}\,{_y\pi} = {_{xy}\pi}. \blacktriangledown$$

Clearly, one can also define a permutation by a right-multiplication.

The elements of $S_N$ can be generated from the simpler elements called cycles and transpositions. A *cycle* is a permutation in which some $r \leqslant N$ objects are permuted among themselves in a cyclic way. For example,

$$\downarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 1 & 6 & 7 & 5 & 8 & 9 & 10 \end{pmatrix} = (1\ 2\ 3\ 4)(5\ 6\ 7)(8)(9)(10), \tag{2}$$

where we have introduced the standard notation for cycles. A cycle can be written in any order: $(1\ 2\ 3\ 4) = (2\ 3\ 4\ 1) = (3\ 4\ 1\ 2) = (4\ 1\ 2\ 3)$; the product of disjoint cycles is commutative: $(1\ 2\ 3\ 4)(5\ 6\ 7) = (5\ 6\ 7)(1\ 2\ 3\ 4)$; and one-cycles may be omitted in the decomposition (2). A cycle of two symbols is called a *transposition.* Any cycle can be written as a product of transpositions: $(1\ 2\ 3\ 4) = (1\ 2)(1\ 3)(1\ 4)$ (operations from left to right). Continuing this process we have

THEOREM 2. 1° *Every permutation may be represented by a product of disjoint cycles (unique up to an ordering of factors).*

2° *Every permutation may be represented by a product of transpositions of adjacent symbols; the number of transposition in any decomposition is either always even or odd for a given $x \in S_N$.* ▼

Two group elements $x_1$ and $x_2$ are *conjugate* if there exists another group element $y$ such that $x_1 = yx_2y^{-1}$. This relationship is 1° reflexive: $x_1 = ex_1e^{-1}$, 2° symmetric: $x_2 = y^{-1}x_1y$, and 3° transitive: if $x_1 = yx_2y^{-1}$ and $x_2 = zx_3z^{-1}$, then $x_1 = (yz)x_3(yz)^{-1}$. Hence the group can be divided into *classes of conjugate group elements.* The identity element is a class itself. Elements in a class have a lot of things in common.

In $S_N$, if $x_1 = (1\ 5\ 3\ 6\ 7\ 4\ 2)(8\ 10)$, for example, and $y = \downarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ i_1 & i_2 & i_3 & i_4 & \dots & & \end{pmatrix}$, then $x_2 = yx_1y^{-1} = (i_1\ i_5\ i_3\ i_6\ i_7\ i_4\ i_2)(i_8\ i_{10})$. Thus, all elements in a class have the same cycle structure. The cycle lengths themselves are characterized by the partitions of $N$, hence the number of classes in $S_N$ is equal to the number of partitions of $N$.

Because the cycles commute, we can order them from large to small. Thus a cycle structure (a partition) is given by the set of numbers $\lambda_i$ satisfying

$$N = \lambda_1 + \lambda_2 + \dots + \lambda_k, \qquad \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \dots \geqslant \lambda_k \geqslant 0, \tag{3}$$

where $k$ is arbitrary. Alternatively, the partitions may be characterized by the set of non-negative integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ such that

$$N = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots + N\alpha_N, \qquad \alpha_i \geqslant 0. \tag{4}$$

Here $\alpha_k$ is the number of cycles of length $k$. Clearly $\alpha_1 \leqslant N$, $\alpha_N \leqslant 1$, etc.

There is no general formula for the number of partitions, excepting infinite series (e.g. Rademacher 1937) but there are tables (cf. Gupta 1958). We shall, however, answer the question as to the number of elements in each class of $S_N$.

Let $U$ be a subgroup of $G$. The elements of the form $\{xu|\ u \in U\} = xU$ form a coset. Clearly, two cosets $xU$ and $yU$ are either identical or have no elements in common. Every element $x$ of $G$ is in some coset, namely in the coset $xU$. All cosets have the same order equal to the order of $U$. Thus the group is divided into $\nu$ disjoint cosets, where

$$[G:U] = \nu \equiv \text{index of } U \text{ in } G = \frac{\text{order of } G}{\text{order of } U}. \tag{5}$$

Thus the order of a subgroup $U$ (or of a coset $xU$) is a factor of the order of $G$ (Lagrange's theorem).

The divisibility of the order of $G$ also holds with respect to classes. We have namely

THEOREM 3. *The order of a class of conjugate elements is a factor of the order of the group $G$.*

PROOF: We define a subgroup $U_x$ called *centralizer of* $x$ in $G$

$$U_x = \{y \mid yxy^{-1} = x\}. \tag{6}$$

We want to know the number of distinct conjugate elements to $x$. Two elements $uxu^{-1}$ and $vxv^{-1}$ are identical if and only if $u$ and $v$ belong to the same left coset of $U_x$. Hence the number of distinct elements conjugate to $x$ is equal to the number of cosets of $U_x$, or to the index of $U_x$ which is a factor of the order of $G$ by the previous theorem.

EXAMPLE 1. As an example we compute the order $h_\alpha$ of the class of $S_N$ defined by eq. (4). By eq. (5) it is sufficient to evaluate the order of $U_x$ defined by eq. (6). A permutation $x$ of cycle structure $\alpha = (\alpha_1, \ldots, \alpha_N)$ is left-invariant in the form $yxy^{-1} = x$ by $\prod_j \alpha_j! j^{\alpha_j}$ permutations, because a cycle of length $j$ remains unchanged by $j$ cyclic permutations and furthermore $\alpha_j$-cycles can be permuted among themselves. Thus

$$h_\alpha = N!/[\alpha_1! 1^{\alpha_1} \alpha_2! 2^{\alpha_2} \ldots \alpha_N! N^{\alpha_N}] \tag{7}$$

and $\sum_\alpha h_\alpha = N!$. ▼

A normal subgroup $N$ of $G$ consists entirely of classes.

EXAMPLE 2. For $N \neq 4$, the *alternating* group $A_N$, the subgroup of $S_N$ consisting of even permutations, is the only proper normal subgroup of $S_N$; its index is 2. For $N \neq 4$, $A_N$ is *simple*, i.e., contains no proper normal subgroup. For $N = 4$, $A_4$ contains the proper normal subgroup $V_4$ (the Klein four-group). (Note: $A_4$ is also known as $T$, the *tetrahedral group*, the group of rotations and reflections which leave a regular tetrahedron invariant.)

## B. *Properties of Representations of Finite Groups*

The general definitions of ch. 5 naturally apply here, except that the concept of continuity is not needed. The criteria in recognizing equivalent and irreducible representations are embedded again in two Schur's lemmas. We now list some useful results in the language of finite groups:

1° Every representation of a finite group $G$ is equivalent to a unitary representation (th. 1.1).

2° Every irreducible representation is finite-dimensional (th. 1.3).

3° Theorem of Maschke: Every reducible representation of finite groups is completely reducible, i.e., is a direct sum of irreducible representations (th. 1.4).

4° Let $T^s$ and $T^{s'}$ be two *irreducible* representations of $G$. Then the corresponding matrix elements in an orthonormal basis satisfy

$$\sum_{x\in G} D^{(s)^{-1}}_{ji}(x) D^{(s')}_{mn}(x) = \delta^{ss'}\delta_{im}\delta_{jn}\frac{h}{d_s}, \tag{8}$$

where $d_s$ is the dimension of $D^s$, $h$ is the order of the group and $s$ labels different irreducible representations. If $D^s$ is unitary, then $\overline{D^{s^{-1}}_{ji}(x)} = D^s_{ij}(x)$ (th. 1.5).

5° If $d_1, d_2, \ldots, d_k$ are the dimensions of the irreducible representations, then $h = d_1^2+d_2^2+ \ldots +d_k^2$.

6° The number of distinct irreducible representations is equal to the number of conjugate classes.

Thus, the symmetric group $S_N$ can have as many irreducible representations as there are partitions of $N$.

Because an arbitrary representation is a direct sum of irreducible finite-dimensional representations, an arbitrary character is given by

$$\chi(x) = \sum_s \lambda_s \chi^{(s)}(x), \qquad x \in G, \tag{9}$$

where $\chi^{(s)}$, $s = 1, 2, \ldots$, are the so-called *primitive* or *simple characters* of irreducible representations satisfying (8). Hence the general or *compound character* (9) satisfies

$$\sum_{x\in G} \bar{\chi}(x)\chi(x) = h\sum_s \lambda_s^2 \geqslant h. \tag{10}$$

Eq. (10) is a criterion of reducibility; a criterion of irreducibility is

$$(\chi, \chi) = h. \tag{11}$$

Consider now the right regular representation of $G$ defined in th. 1. The character of this representation is

$$\chi^{\text{reg}}(x) = \begin{cases} h, & x = e, \\ 0, & x \neq e. \end{cases} \tag{12}$$

Thus

$$\sum_{x\in G} \bar{\chi}^{\text{reg}}_{(x)}\chi^{\text{reg}}_{(x)} = h^2. \tag{13}$$

Consequently the regular representation is reducible in the form (9) and the coefficients $\lambda_s$ satisfy

$$\sum_s (\lambda_s^{\text{reg}})^2 = h. \tag{14}$$

On the other hand, from

$$\chi^{\text{reg}}(x) = \sum_s \lambda_s^{\text{reg}} \chi^{(s)}(x)$$

for $x = e$, we directly obtain

$$h = \sum_s \lambda_s^{\text{reg}} l_s. \tag{15}$$

From (14) and (15) we have $l_s = \lambda_s^{\text{reg}}$ and $h = \sum_s l_s^2$, thus

7° The dimension of an irreducible representation is equal to the number of times it is contained in the regular representation (th. 2.2). Every irreducible representation occurs in the regular representation (th. 1.6) and

$$h = \sum_{s=1}^{f} l_s^2. \tag{16}$$

### C. *Representations of $S_N$*

We begin with the regular representation of $S_N$ which is particularly suited to many physical situations. Consider functions of $N$ objects, $f(1\ 2 \dots N)$, for example, the wave function of a system of $N$ particles; each argument $1, 2, \dots, i, \dots$ stands for the set of quantum numbers of the $i$th particle. Let $x(1\ 2 \dots N)$ be a permutation of the $N$ objects, $x \in S_N$. The $N!$ functions

$$f\{x(1\ 2 \dots N)\} = f_x(1\ 2 \dots N), \qquad x \in S_N, \tag{17}$$

form a basis of the regular representation of dimension $h = N!$ In this basis a group element $x$ is represented by

$$xf_{x_i} = \sum_{j=1}^{N!} D(x)_{ji} f_{x_j}, \tag{18}$$

where

$$D(x)_{ji} = \delta_{jk}, \qquad \text{if } xx_i = x_k. \tag{19}$$

The regular representation is reducible. For the completely symmetric and the completely antisymmetric functions, denoted by $f(1\ 2 \dots N)$ and $f\{1\ 2 \dots N\}$ respectively, for example, form each a one-dimensional invariant subspace under the transformations (1). In fact, we know from the general theorems of the previous section that the regular representation must contain every irreducible representation of dimension $l_s$, $l_s$ times. Because the number of irreducible representations (equal to the number of conjugate classes) is given by the number of partitions of $N$, an irreducible representation corresponding to a partition

$$(\lambda_1, \dots, \lambda_k), \quad \sum \lambda_i = N, \quad \lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_k \geqslant 0,$$

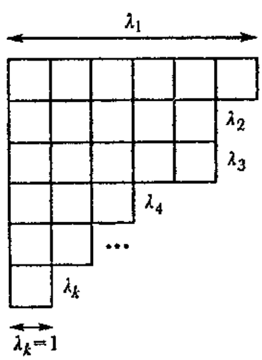
defines a set of functions of a definite symmetry type

$$f\{[1\,2\,\ldots\,\lambda_1][\lambda_1+1,\,\ldots,\,\lambda_1+\lambda_2],\,\ldots\} \tag{20}$$

which are completely symmetric in the first $\lambda_1$ variables, completely symmetric with respect to the next $\lambda_2$ variables, etc. The number of linearly independent functions of a definite symmetry type is equal to the dimension of the corresponding irreducible representation. Functions of a given symmetry type transform among themselves.

The above discussion characterizes in an intuitive way all the irreducible representations of $S_N$, but it is necessary to make the matters much more explicit and precise.

First of all, a partition of $N$ can be shown diagrammatically by the following 'Young frame', for example $(\lambda_1, \ldots, \lambda_\kappa)$ is




These are $N$ boxes and $N!$ ways of distributing the $N$ numbers $1, 2, \ldots, N$ into these boxes. A Young frame with the numbers written in it is called a '*Young tableau*' (Frobenius–Young tableau). Thus there are $N!$ tableaux corresponding to any frame.* In terms of our functions (4), the $N!$ functions corresponding to $N!$ tableaux are not linearly independent. The number of linearly independent functions are given by the number of standard tableaux.

A *standard tableaux* is one in which the integers $1, 2, \ldots, N$ are distributed in increasing order in every row from left to right, and in every column from top to bottom in the Young frame of the figure above.

It is then a combinatorial problem to evaluate the number of standard tableaux $l_{(\lambda)}$ corresponding to partition $(\lambda)$. We give three such formulas:

1)
$$l_{(\lambda)} = \frac{N!}{\prod_{i=1}^{k} n_i!} \prod_{i<k} (n_i - n_k), \tag{21}$$

* 'Frame' and 'tableau' are also called 'table' and 'diagrams', respectively.

where (see figure)

$$\begin{aligned} n_1 &= \lambda_1+(k-1), \\ n_2 &= \lambda_2+(k-2), \\ &\cdots\cdots\cdots\cdots \\ n_k &= \lambda_k. \end{aligned}$$

It can be verified that

$$\sum_{(\lambda)} l_{(\lambda)}^2 = N!.$$

2) Formula of W. Feit 1953

$$l_{(\lambda)} = N! \det[1/(\lambda_i-i+j)!], \qquad i,j = 1,2,\dots,N. \tag{22}$$

The determinant in (22) is that of an $N\times N$-matrix $A$ with matrix elements $a_{ij} = 1/(\lambda_i-i+j)!$. Note that

$$\frac{1}{x!} = 0 \quad \text{if } x<0, \quad \text{and} \quad 1/0! = 1.$$

3) Formula of J. S. Frame, G. de B. Robinson and R. M. Thrall 1954

$$l_{(\lambda)} = N! \Big/ \prod_{i,j} h_{ij}, \tag{23}$$

$$h_{ij} = 1+\lambda_i+\bar{\lambda}_j-(i+j).$$

Here $\bar{\lambda}_j$ = number of boxes in column $j$.

We shall see that to each frame corresponds an irreducible representation of $S_N$ of dimension $l_{(\lambda)}$.

In order to discuss the irreducible representations of the group $S_N$, we consider the group algebra $\mathscr{A}$ as the regular representation and decompose $\mathscr{A}$ into its irreducible parts by projection operators (idempotents).

Consider a tableau $T$. Let

$H(T)$ = the set of horizontal permutations, that is, the set of permutations $p \in S_N$ which permute the numbers in each row of $T$ but do not move any number from one row to another,

$V(T)$ = the set of vertical permutations, that is, permutations $q \in S_N$ which permute the numbers in each column of $T$ but do not move any number from one column to another.

$H(T)$ and $V(T)$ are subgroups of $S_N$, and clearly, they have only the identity element in common: $H(T) \cap V(T) = I$.

PROPOSITION. *The quantities defined, for each tableau, by*

$$e(T) = \sum_{\substack{p\in H(T)\\ q\in V(T)}} \varepsilon_q pq \in \mathscr{A},$$

$$\varepsilon_q = \begin{cases} +1 & \textit{for even permutations,} \\ -1 & \textit{for odd permutations} \end{cases}$$

*are essentially idempotent (that is a scalar multiple of an idempotent element in it;* $e(T)^2 = \dfrac{N!}{l_{(\lambda)}} e(T)$*) and provide us with the decomposition of* $\mathscr{A}$ *into irreducible subspaces.*

PROOF: Clearly $e(T) \neq 0$ in $\mathscr{A}$. Furthermore, we have the relations

$$p_1 e(T) = \sum \varepsilon_q p_1 pq = \sum \varepsilon_q p' q = e(T),$$

for $p_1 \in H(T)$, and similarly $e(T)q_1 = \varepsilon_{q_1} e(T)$ for $q_1 \in V(T)$. We form the ideal $\mathscr{A}e(T)$. We shall show that all these left ideals are minimal, that ideals coming from different tableaux but the same frame, are equivalent (isomorphic), and that ideals coming from different frames are nonequivalent.

Consider two tableaux $T$ and $T'$ of a given frame. We write $T' = gT$, where $g \in S_N$ is the permutation which changes the digits of $T$ into those of $T'$, that is, if the number $\alpha$ is in the position $(i,j)$ of $T$, then $g\alpha$ is in the position $(i,j)$ of $T' = gT$. For these two tableaux, $e(T)$, $e(T')$, $H(T)$, $H(T')$, are related by

$$\begin{aligned} e(T') &= ge(T)g^{-1} = e(gT), \\ H(T') &= gH(T)g^{-1} = H(gT), \\ V(T') &= gV(T)g^{-1} = V(gT). \end{aligned}$$

For, if $p \in H(T)$, then $p$ permutes the rows of $T$, and $gpg^{-1}$ permutes the rows of $T'$, and so on. Let us look now at the corresponding ideals $\mathscr{A}e(T)$ and $\mathscr{A}e(T')$:

$$\mathscr{A}e(T') = \mathscr{A}ge(T)g^{-1} = \mathscr{A}e(T)g^{-1}.$$

The two ideals are related to each other by a right multiplication, hence they are equivalent.

Consider now two different frames associated with the partitions $(n_1 n_2 \ldots n_\varkappa)$ and $(n'_1\ n'_2\ \ldots\ n'_n)$, respectively We write $(n_1\, n_2\ \cdots\ n_k) > (n'_1\ n'_2\ \ldots\ n'_n)$ meaning that at the first position where the arrays differ, $n_i > n'_i$. For two tableaux from two such frames we have

$$e(T_1)e'(T_2) = 0.$$

To show this we notice that there must exist two symbols $\alpha$ and $\beta$ which are collinear somewhere in $T$ and co-columnar somewhere in $T'$ otherwise one can show by suitable permutations that $n_1 = n'_1$, $n_2 = n'_2, \ldots$ Let $h$ be the permutation changing $\alpha$ and $\beta$. Then $h \in H(T_1)$ and $h \in V(T_2)$. Hence

$$e(T_2)e(T_1) = e(T'_2)hhe(T_1) = -e(T'_2)e(T_1),$$

where we used the property $e(T)q = \varepsilon_q q$. Hence

$$e(T_1)e'(T_2) = 0.$$

Next we show the idempotent character of $e(T)$. For any number $\alpha$, we get $p\alpha e(T)q = \varepsilon_q \alpha e(T)$. Conversely, let $a \in \mathscr{A}$ be such that $paq = \varepsilon_q a$, for all $p \in H(T)$ and $q \in V(T)$. Then there exists a number $\alpha$ such that $a = \alpha e(T)$. To see this, let $a = \sum \alpha(x)x$, $x \in S_N$. Then

$$a = \varepsilon_q p^{-1} a q^{-1} = \varepsilon_q \sum_x \alpha(x)(p^{-1}xq^{-1}) = \varepsilon_q \sum_y \alpha(pyq)y.$$

Thus

$$\alpha(y) = \varepsilon_q \alpha(pyq) \quad \text{for } p \in H(T),\ q \in V(T).$$

Setting $y = 1$ we have $\alpha(1) = \varepsilon_q \alpha(pq)$. To complete the proof we must show that $\alpha(x) = 0$, if $x$ is not of the form $pq$, $p \in H(T)$, $q \in V(T)$. This is indeed the case: if $x$ is not of the form $pq$, there must exist symbols collinear in $T$ and cocolumnar in $T' = xT$. Let $h$ be the permutation of $\alpha$ and $\beta$, then $h \in H(T)$ and $h \in V(xT)$, and so $h = xqx^{-1}$ for some $q \in V(T)$ and

$$\alpha(x) = \varepsilon_{q^{-1}} \alpha(hyq^{-1}) = \varepsilon_{q^{-1}} \alpha(x) = -\alpha(x).$$

Therefore $\alpha(x) = 0$, if $x$ is not of the form $pq$. Now consider

$$pe(T)^2 q = pe(T)e(T)q = \varepsilon_q e(T)^2.$$

By the preceding result, we have then $e(T)^2 = \alpha e(T)$. To evaluate $\alpha$ we consider the map $T(a) = ae(T)$, $a \in \mathscr{A}$, and the matrix form of $T$ in the basis consisting of group elements $x_1 = 1, x_2, x_3, \ldots, x_N$. Then if

$$e(T) = \alpha_1 x_1 + \alpha_2 x_2 + \ldots$$

we have

$$x_1 e(T) = \alpha_1 x_1 + \alpha_2 (x_1 x_2) + \ldots,$$

$$x_2 e(T) = \alpha_1 (x_2 x_1) + \alpha_2 x_2 + \ldots,$$

so that $\mathrm{Tr}(T) = \alpha_1 N!$. Furthermore $\alpha_1 = 1$, since $x_1 = 1$ occurs with coefficient 1 in $e(D)$. Consider now a second basis $(y_1, \ldots, y_l, \ldots, y_N)$, such that $(y_1, \ldots, y_l)$ is a basis for the ideal $J = \mathscr{A}e(T)$. Now $a_1 e(T) = \alpha a_1$ for $a_1 \in J$, and so

$y_1 e(T) = \alpha y_1,$

$y_l e(T) = \alpha y_l,$

$y_{l+i} e(T) =$ the first $l$ elements non-zero, the remaining elements zero since $y_{l+i} e(T) \in \mathscr{A}e(T)$.

Now trace $(T) = \alpha l$. Since trace is an invariant

$$\alpha l = N!, \qquad \alpha = N!/l.$$

Thus, the quantity $u = \dfrac{l}{N!} e(T)$ is truly idempotent.

Finally we show that the ideal $\mathscr{A}e(T)$ is minimal. It suffices to show that $e(T)\mathscr{A}e(T)$ is a numerical multiple of $e(T)$. Now

$$pe(T)\mathscr{A}e(T)q = e(T)\mathscr{A}e(T)\varepsilon_q, \quad p \in H(T),\ q \in V(T),$$

and by a previous lemma $e(T)\mathscr{A}e(T)$ is a multiple of identity.

If two tableaux $T_1$ and $T_2$ belong to different frames, we know that $e(T_1)e(T_2) = 0$. Hence for $x \in S_N$, $e(T_2)xe(T_1) = e(T_2)e(xT_1)x = 0$. Hence the two ideals $\mathscr{A}e(T_1)$ and $\mathscr{A}e(T_2)$ are inequivalent. ▼

To summarize we have shown that each tableau $T$ corresponding to a Young frame $(n_1\ n_2\ \ldots\ n_k)$ defines an essentially idempotent element $e(T) = \sum_{p,q} \varepsilon_q pq$, such that $\mathscr{A}e(T)$ is a minimal left ideal of the group algebra $\mathscr{A}$ of $S_N$ and thus an irreducible component of the regular representation. Further, ideals coming from different tableaux with the same frame are isomorphic, but ideals from different frames are not.

Unfortunately, there is no general algorithm which gives the minimal ideals of any finite group algebra as was the case for $S_N$. This seems to be an unsolved problem.

## § 6. Comments and Supplements

(i) In the proofs of most of the theorems in this chapter, we used explicitly the finiteness of the group volume $V = \int_G \mathrm{d}x$. Hence, we cannot expect that these theorems can be directly extended to noncompact groups for which $V = \infty$. However, a generalization of some of the theorems (in particular the Peter–Weyl theorem) to noncompact groups is possible (cf. ch. 14, § 2).

(ii) The functions $D^s_{pq}(x)$ play a special role in the theory of representations of compact groups and in its applications. Unfortunately these functions are explicitly known only in few cases: SO(3) (cf. exercise 5.8.1.1), SO(4) (cf. exercise 7.1.2), U(3) (cf. Chacón and Moshinski 1966). Gel'fand and Graev derived recursive formulas for $D$ functions for $U(n)$. See also the work of Leznov and Fedoseev 1971.

(iii) One can show that for simple Lie groups the functions $D^s_{pq}(x)$ are eigenfunctions of a maximal set of commuting operators in the enveloping algebra. (Cf. ch. 14, § 2 for a general proof of this statement for compact and noncompact groups.) This property is the starting point in the explicit calculation of $D^s_{pq}(x)$ for specific groups.

(iv) The th. 1.3 was first proved by Gurevich 1943. Here we followed the elegant proof given by Nachbin 1961 (cf. also Koosis 1956). In the proof of th. 1.4 we followed Auslander 1961. The projection operators for finite and compact groups were extensively used by Wigner 1959. He first showed the effectiveness of this technique in the solution of a number of problems in quantum mechanics.

It is interesting that the theory of projection operators can also be extended to noncompact groups (cf. ch. 14, § 5).

The first explicit calculation of C–G coefficients is due to Wigner, who derived formula 4(10) for SO(3) group.

In the calculation of the weights of isospin states we followed the work of Cerulus 1961. He also calculated the other special cases of eq. 4(18). More general formulas summing over all possible final states have been discussed recently (cf. P. Rotelli and L. G. Suttorp 1972).

(v) In table 7.I we give all finite groups up to order 15. Referring to this table:

1) $A_n$ = cyclic group of order $n$. If $n$ is a prime number there is only one group, namely the cyclic group.

2) If $p$ and $q$ are relatively prime to each other, then $Z_{pq} \sim Z_p \times Z_q$ (isomorphic to the direct product).

3) $D_n$ = dihedral groups of order $2n$ (group of transformations which map a regular $n$-polygon into itself, consisting of $n$ rotations by an angle $2\pi r/n$, $r = 0, 1, 2, \ldots, n-1$, and the reflection of the plane plus a rotation by $2\pi r/n$). $D_n$ may be generated by two elements $x$ and $y$ satisfying

$$x^2 = e, \quad y^n = e, \quad (xy)^2 = e.$$

4) $\langle 2, 2, m\rangle$ = dicyclic group of order $4m$. It is generated by two elements $x, y$ satisfying*

$$x^4 = e, \quad x^2 = y^m \quad \text{and} \quad yx = xy^{-1}.$$

Table I. All Finite Groups up to Order 15

| Order | Groups |
|---|---|
| 1 | $Z_1$ |
| 2 | $Z_2 \sim S_2$ |
| 3 | $Z_3$ |
| 4 | $Z_4$, $Z_2 \times Z_2 \sim D_2$ |
| 5 | $Z_5$ |
| 6 | $Z_6 \sim Z_2 \times Z_3$, $S_3 \sim D_3$ |
| 7 | $Z_7$ |
| 8 | $Z_8$, $D_4$, $Z_4 \times Z_2$, $Q$, $Z_2 \times Z_2 \times Z_2$ |
| 9 | $Z_9$, $Z_3 \times Z_3$ |
| 10 | $Z_{10} \sim Z_2 \times Z_5$, $D_5$ |
| 11 | $Z_{11}$ |
| 12 | $Z_{12} \sim Z_3 \times Z_4$, $Z_2 \times Z_6 \sim Z_2 \times Z_2 \times Z_3$, $D_6 \sim Z_2 \times D_3$ |
| | $A_4$, $\langle 2, 2, 3\rangle$ |
| 13 | $Z_{13}$ |
| 14 | $Z_{14} \sim Z_2 \times Z_6$, $D_7$ |
| 15 | $Z_{15} \sim Z_3 \times Z_5$ |

* H. S. M. Coxeter and W. O. J. Moser 1965 give large classes of groups generated by relations of this type.

(vi) We discussed in this chapter the strongly continuous representations only. It turns out however that compact groups have interesting non-continuous representations. We give two theorems which nicely illustrate this problem.

THEOREM 1. *A unitary representation of a connected compact semisimple Lie group in a finite-dimensional Hilbert space is neccessarily continuous.*

(For the proof cf. Van der Waerden 1933).

Clearly, by virtue of the structure of compact groups discussed in ch. 3.8 this theorem is false for nonsemisimple groups.

THEOREM 2. *Let G be a locally compact topological group whose every irreducible unitary representation in a Hilbert space is continuous. Then G is discrete.*

(For the proof see Bichteler 1968).

Theorem 2 implies that every connected compact semisimple Lie group must admit a noncontinuous infinite-dimensional irreducible unitary representation in a Hilbert space. It is interesting that in case of the rotation group such representations have an important physical meaning.

## § 7. Exercises

§ 1.1. Show that the matrix elements of irreducible representations of SO(4) have the form

$$D^{J_1,J_2}_{M_1M_1',M_2M_2'}(\varphi,\vartheta,\psi,\alpha,\beta,\gamma) = D^{J_1}_{M_1M_1'}(\varphi,\vartheta,\psi)D^{J_2}_{M_2M_2'}(\alpha,\beta,\gamma)$$

where the functions $D^J_{MM'}$ are given by eq. 5.8 (1).

*Hint*. Use isomorphism so(4) $\sim$ so(3) $\oplus$ so(3) given in table 1.5.1.

§ 3.1. Let $T^{\lambda_1}$ and $T^{\lambda_2}$ be irreducible representations of SO(3). Derive the following relation for matrix elements:

$$D^{\lambda_1}_{m_1m_1'}(g)\,D^{\lambda_2}_{m_2m_2'}(g) = \sum_{\lambda=|\lambda_1-\lambda_2|}^{\lambda_1+\lambda_2} \langle\lambda_1 m_1 \lambda_2 m_2|\lambda_1\lambda_2\lambda m\rangle D^{\lambda}_{mm'}(g) \times$$
$$\times \langle\lambda_1\lambda_2\lambda m'|\lambda_1 m_1'\lambda_2 m_2'\rangle.$$

*Hint*. Find the matrix elements of operator equality 5.8(3).

§ 4.1. Show that the Clebsch–Gordan series (5.8 (3) for SO(3)) implies the following relation for the matrix elements

$$D^{J_1}_{M_1M_1'}(g)D^{J_2}_{M_2M_2'}(g) = \sum_{J=|J_1-J_2|}^{J_1+J_2} \langle J_1M_1J_2M_2|J_1J_2JM\rangle \times$$
$$\times D^{J}_{M_1+M_2,M_1'+M_2'}(g)\langle J_1J_2JM'|J_1M_2'J_2M_2'\rangle.$$

*Hint*. Use eq. 5.8(3) and the completeness relation for $|J_1J_2JM\rangle$ states.

§ 5.1. Show that $S_N$ can be generated from two elements $x = (1\,2)$ and $y = (1\,2\,\ldots\,N)$.

*Hint*. Any permutation can be written as products of cycles. Any cycle is of the form

$$(i_1 i_2 \dots i_p) = (i_1 i_2)(i_2 i_3) \dots (i_{p-1} i_p).$$

Any transposition is of the form

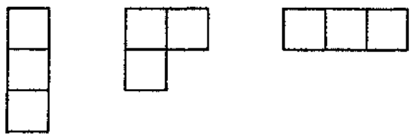
$$(i, j+1) = (j, j+1)(ij)(j, j+1)^{-1}.$$

Then $y^n x (y^n)^{-1}$ gives all transpositions of the form $(j, j+1)$.

§ 5.2. Find the multiplication table and a 2-dimensional and a 3-dimensional representation of $D_3 \sim S_3$. Find the normal subgroups and the conjugate classes.

§ 5.3. Show that the Pauli matrices $\pm I$, $\pm\sigma_1$, $\pm\sigma_2$, $\pm\sigma_3$ constitute a realization of the quaternion group $Q \sim \langle 2, 2, 2\rangle$ (see table 7.I).

§ 5.4. Consider $S_3$. There are three Young frames



The corresponding essential idempotents are

$$\begin{aligned} e_1 &= \sum_{x \in G} x, \\ e_2^{(1)} &= I + (1\,2) - (1\,3) - (1\,2), \\ e_2^{(2)} &= I + (1\,3) - (1\,2) - (1\,3\,2), \\ e_3 &= \sum_x \varepsilon(x) x. \end{aligned}$$

Discuss the properties of the idempotents, the minimal left ideals and two-sided ideals that they generate, and the corresponding irreducible representations.

§ 5.5. If $x$, $y$, $z$ are three identical objects show that $\frac{1}{\sqrt{6}}(x+y+z)$ and $\begin{pmatrix} \frac{1}{\sqrt{6}}(2x-y-z) \\ \frac{1}{\sqrt{6}}(y-z) \end{pmatrix}$ transform according to the one and 2-dimensional representation, respectively, of the permutation group $S_3$.

§ 5.6. Show that the Dirac matrices defined by

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, 3.$$

generate (with respect to matrix multiplication) a finite group of order $h = 32$; that it has 17 classes, hence 17 irreducible representations, 16 of them of dimension 1 and one of dimension 4.

# Chapter 8

# Finite-Dimensional Representations of Lie Groups

In this chapter we present the theory of finite-dimensional, irreducible representations of an arbitrary, connected Lie group in global form. The global approach brings considerable simplifications to the theory as compared to the infinitesimal, Cartan–Weyl approach; it provides the classification of finite-dimensional, irreducible representations in terms of the highest weights, and at the same time, a simple canonical realization of the carrier space in terms of polynomials in some complex variables. This in turn makes the solution of various practical problems possible, such as the reduction of a representation of the given group to a subgroup, the decomposition of the tensor product, the explicit calculation of the 'Clebsch–Gordan coefficients', and so on.

In § 1 we discuss the general properties of the representations of solvable and semisimple Lie groups. In particular, we derive the global forms of the celebrated Lie and Weyl theorems.

In § 2 we elaborate the techniques of induced, finite-dimensional representations of Lie groups and prove the main theorem that every finite-dimensional irreducible representation of a Lie group $G$, which admits a Gauss decomposition, is the representation induced by a one-dimensional representation of a certain subgroup.

In §§ 3–6 we develop the global representation theory of complex and real classical Lie groups.

Finally, in § 7 the classification of finite-dimensional irreducible representations of arbitrary connected Lie groups is discussed. The method is based on the use of the Levi–Malcev decomposition of $G$ and the properties of irreducible representations of solvable and semisimple groups.

In this chapter, because we are dealing exclusively with the finite-dimensional representations, we shall often omit, for simplicity, the term 'finite-dimensional'.

## § 1. General Properties of Representations of Solvable and Semisimple Lie Groups

The representation theory of Lie groups is based on the existence, for every complex or real Lie group, of a characteristic solvable connected subgroup. The explicit form of this subgroup is determined by the Levi–Malcev, Gauss or Iwasawa decompositions. We first prove the fundamental theorem of Lie on the representations of solvable groups. This theorem is the key for the classification of irreducible, finite-dimensional representations of arbitrary Lie groups. We give the global version of Lie's theorem which is convenient in the theory of induced representations.

## A. *Representation Theory of Solvable Groups*

Let $N$ be a solvable topological group. Let $Q(N)$ be its commutator subgroup i.e. the closure in the topology of $G$ of the set generated by the elements of the form $xyx^{-1}y^{-1}$. Set $Q_i(N) = Q\big(Q_{i-1}(N)\big)$. Because $N$ is solvable, we have $Q_p(N) = \{e\}$, for some $p$; the smallest $p$ is called the *height* of $N$. One easily sees that if $N$ is a connected solvable group then $Q(N)$ is also connected and solvable.

THEOREM 1 (Lie). *Every finite-dimensional irreducible representation of a connected topological, solvable group $N$ in a complex carrier space is one-dimensional.*

PROOF: We prove the theorem by induction. If $N$ has the height one (i.e. $N$ is abelian), the theorem follows from Schur's lemma 5.3.5. Assume that $N$ has height $p$ and the theorem is proved for groups of height $p-1$. Let $n \to T_n$ be a finite-dimensional, irreducible representation of $N$ in a vector space $H$. The subgroup $Z = Q(N)$ is of height $p-1$. Hence the representation $z \to T_z$ of the subgroup $Z = Q(N)$ contains a one-dimensional representation of $Z$. Consequently, we can find a complex character $z \to \chi(z)$ and a non-zero vector $u_\chi$ in $H$ such that

$$T_z u_\chi = \chi(z) u_\chi, \tag{1}$$

for every $z$ in $Z$. Denote by $\Phi$ the set of characters $\chi$ of $Z$ such that (1) has a non-zero solution $u_\chi$ in $H$. Clearly, $\Phi$ is a finite set. Because $Z$ is invariant in $N$, we can define, for every character $\chi$ of $Z$ and every $n \in N$, a new character $\chi_n$ given by the formula

$$\chi_n(z) \equiv \chi(n^{-1}zn). \tag{2}$$

Equation (1) implies

$$T_z T_n u_\chi = T_n T_n^{-1} T_z T_n u_\chi = \chi_n(z) T_n u_\chi. \tag{3}$$

Consequently, $\chi \in \Phi$ implies $\chi_n \in \Phi$ for every $n \in N$. Let us now introduce the following topology in $\Phi$: $\chi \to \chi'$ if $\chi(z) \to \chi'(z)$ for every $z$. In this topology $\Phi$ is a discrete space. The continuity of group multiplication implies that for a given $\chi$ the character $\chi_n$ depends continuously on $n \in N$. On the other hand, the connectedness of $N$ implies that for every $\chi$ in $\Phi$ the set of all $\chi_n$ is connected. Because this set is also finite we obtain $\chi_n = \chi$ for every $\chi \in \Phi$ and every $n \in N$. Thus we conclude that if $\chi \in \Phi$, the vectors $u_\chi$, which satisfy eq. (1), span an invariant subspace of $H$ under $T_n$. Because $H$ is irreducible under $T_n$ we conclude that for every $z \in Z$,

$$T_z = \chi(z) \cdot I. \tag{4}$$

Incidentally, $\Phi$ contains one character of $Z$ only.

Let $n_0$ be an arbitrary element of $N$ and let $c$ be any root of the polynomial $x \to \det(T_{n_0} - xI)$. Because $T_{n_0}$ is nonsingular, $c$ cannot be zero. Hence, there exists a non-zero $u_0$ in $H$ such that

$$T_{n_0} u_0 = c u_0 . \tag{5}$$

Because $n_0 n\, n_0^{-1} n^{-1} \in Z$, eq. (4) implies

$$T_{n_0} T_n = \chi(n_0 n n_0^{-1} n^{-1}) T_n T_{n_0}.$$

From this and eq. (5), we obtain

$$T_{n_0} T_n u_0 = c\chi(n_0 n n_0^{-1} n^{-1}) T_n u_0 .$$

Hence, for every $n \in N$, $T_n u_0$ is an eigenvector of $T_{n_0}$. The corresponding eigenvalue $c \cdot \chi(n_0 n n_0^{-1} n^{-1})$ depends continuously on $n$ and has only finite number of possible values. Hence, the connectedness of $N$ implies that $\chi$ does not depend on $n$. Setting $n = n_0$ we obtain $\chi(n_0 n n_0^{-1} n^{-1}) = 1$. Thus

$$T_{n_0} T_n u_0 = c T_n u_0 .$$

Therefore the linear subspace $\{u_0 \in H;\ T_{n_0} u_0 = c u_0\}$ is invariant under all $T_n$. Consequently it must coincide with $H$. This, in turn, implies by Schur's lemma that $T_{n_0}$ reduces to the scalar $cI$ for every $n_0 \in N$. Consequently, because of irreducibility of $T_{n_0}$, we obtain the assertion of Lie's theorem. ▾

COROLLARY 1. *In any representation space of a connected solvable group $N$, there is a non-zero vector and a non-zero continuous multiplicative function $\chi(n)$ such that*

$$T_n u_\chi = \chi(n) u_\chi \quad \textit{for all } n \in N.$$

PROOF: It is sufficient to pick up an irreducible subspace and apply Lie's theorem. ▾

COROLLARY 2. *Every representation of a connected, solvable group $N$ can be reduced to the triangular form*

$$T_n = \begin{bmatrix} \chi^1(n) & & & 0 \\ & \chi^2(n) & & \\ & & \ddots & \\ & * & & \chi^N(n) \end{bmatrix}. \tag{6}$$

PROOF: The existence of an eigenvector for all $T_n$ is equivalent to reducibility of the matrices $T_n$, i.e.,

$$T_n = \left[\begin{array}{c:c} \tilde{T}_n & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hdashline * \ldots * & \chi^N(n) \end{array}\right].$$

The matrices $\tilde{T}_n$ again form a representation of $N$. Hence, successive application of this form yields (6). ▼

Let us note that many important groups considered in physics are solvable: for instance, the Poincaré group $\Pi_2 = T^{(1,1)} \circledS \mathrm{SO}(1, 1)$ in two-dimensional space-time and the Heisenberg group associated with commutation relations

$$[X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0, \quad (\text{or } [a, a^*] = 1),$$

are solvable groups of height 2. Indeed, for instance, the group multiplication in $\Pi_2$ implies $Q_1(\Pi_2) = T^{(1,1)}$, $Q_2(\Pi_2) = \{e\}$. Because these groups are connected we have by Lie's theorem

COROLLARY 3. *Every finite-dimensional, irreducible representation of the Poincaré group $\Pi_2$ and the Heisenberg group is one-dimensional.* ▼

Thus the groups of motion of one or two-dimensional Minkowskian and Euclidean space-times must be solvable.

## B. *Representation Theory of Semisimple Lie Groups*

A connected simple Lie group has only the trivial one-dimensional representation, $g \to I$. Indeed, a connected simple group can have only two kinds of invariant subgroups: $G_1$ = discrete center of $G$ (or a subgroup of it), and $G_2 = G$. If the homomorphism $g \to T_g$ has $G_1$ as a kernel, then $T_g$ is a faithful representation of $G/G_1$. Consequently, it cannot be one-dimensional, because $G/G_1$ is noncommutative: if the kernel is $G_2$, then $T_g$ is the identity.

Because semisimple connected Lie groups are direct products of invariant, simple, connected subgroups, these groups also have only the trivial one-dimensional representation.

Next we prove an interesting property of the representations of simple Lie groups:

THEOREM 2. *A connected, simple, noncompact Lie group $G$ admits no finite-dimensional, unitary representations beside the trivial one.*

PROOF: Suppose first that the kernel of the mapping $g \to T_g$ consists of the identity only. Then, the homomorphism $g \to T_g$ is faithful. Consequently, the group $\{T_g\}$ is isomorphic to $G$. Moreover, $\{T_g\}$ is connected by continuity of the map $g \to T_g$. Hence, if $G$ admits a finite, say, $n$-dimensional, unitary representation, then $\{T_g\}$ is a connected and simple subgroup of $U(n)$ which by th. 3.10 is closed. Consequently, $G$ is compact.

Let, now, $Z_G$ be the center of $G$. Clearly, $Z_G$ is discrete. Let $\tilde{Z}_G$ be a subgroup of $Z_G$ which is the kernel of the homomorphism $g \to T_g$. $G$ is locally isomorphic to $G/Z_G$. By noncompactness of $G$ the Killing forms of both $G$ and $G/\tilde{Z}_G$ are not definite. Hence $G/\tilde{Z}_G$ is also noncompact and satisfies the assumption of the first part of the proof. ▼

This theorem has important consequences in quantum theory. Because a representation $g \to T_g$ of a physical symmetry group $G$ must conserve the probability (scalar product), $T$ must be unitary. (See ch. 14). On the other hand, many physical symmetry groups such as the Lorentz group SO(3, 1), or de Sitter group SO(4, 1), are simple and noncompact. Hence, we have to use infinite-dimensional representations for the description of states of the underlying physical objects.

*Remark*: Th. 2 is in general not true for *semisimple*, noncompact Lie groups. For instance, the semisimple, connected, noncompact Lie group

$$G = \mathrm{SO}(3, 1) \times \mathrm{SU}(3)$$

has a unitary finite-dimensional representation:

$$(g_1, g_2) \to I \cdot T_{g_2}, \quad g_1 \in \mathrm{SO}(3, 1), \quad g_2 \in \mathrm{SU}(3). \blacktriangledown$$

However, th. 2 implies the following corollary for semisimple groups.

COROLLARY 4. *A connected, semisimple, noncompact Lie group cannot admit faithful unitary finite-dimensional representations.*

PROOF follows from the decomposition of $G$ onto simple factors and from th. 2. ▼

Let now $G$ be a complex Lie group and let $t_1, t_2, \ldots, t_n$ be local (complex) coordinates in $G$. We distinguish the following classes of representations of $G$.

DEFINITION 1. A representation $g \to T_g$ of a complex group $G$ is said to be *complex-analytic* if it depends analytically on the parameters $t_1, \ldots, t_n$, *complex-antianalytic* if it depends analytically on $\bar{t}_1, \ldots, \bar{t}_n$ and *real-analytic*, if it depends analytically on parameters $\mathrm{Re}\, t_1, \mathrm{Im}\, t_1, \ldots, \mathrm{Re}\, t_n, \mathrm{Im}\, t_n$ (or $t_1, \ldots, t_n, \bar{t}_1, \ldots, \bar{t}_n$).

EXAMPLE 1. Let $G$ be a complex matrix group. Then, the representation $g \to g$ is analytic, $g \to \bar{g}$ is antianalytic and $g \to g \otimes \bar{g}$ is real.

If $g \to T_g$ is the complex-analytic irreducible representation of $G$ in $H$, its restriction to a subgroup $N$ might in general be reducible. However, if $N$ is a real form of $G$ (i.e., complex extension of $N$ coincides with $G$), then we have

THEOREM 3. *Let $T_G$ be a complex-analytic representation of $G$ and $T_N$ the restriction of $T_G$ to a real form $N$ of $G$. Then, $T_G$ is irreducible (fully reducible), if and only if $T_N$ is irreducible (fully reducible).*

PROOF: By assumption, every matrix element is an analytic function of the complex parameters $t_1, \ldots, t_n$ in $G$. If some matrix element is zero on $G$, then it is in particular zero on $N$: conversely by virtue of uniqueness of analytic continuation if a matrix element is zero on $N$ then it is zero also on $G$. This implies the assertion of th. 3. ▼

Using the 'Weyl unitary trick' (the construction of representations of real forms $G_R$ of a given complex Lie group $G_C$ by the restriction of the representations of $G_C$ to $G_R$) we obtain the global representations of real semisimple groups such as SL$(n, R)$, SU$(n)$, SU$(p, q)$ and so on, from the representations of the complex Lie group GL$(n, C)$. Because the structure of complex groups is simpler than the real groups, one obtains in this fashion a considerable simplification in the representation theory of semisimple Lie groups.

We now prove the fundamental Weyl theorem about full reducibility of representations of semisimple Lie groups

Theorem 4 (Weyl). *Let $G$ be a connected semisimple Lie group and let $g \to T_g$ be any finite-dimensional representation of $G$ in a carrier space $H$. Then,*

$$H = H_1 \oplus H_2 \oplus \ldots \oplus H_n, \tag{7}$$

*where each $H_i$ is invariant.*

PROOF: The proof consists of the reduction of the problem to the complete reducibility in the case of compact groups. Let $L$ be a Lie algebra of $G$, $L^c$ its complex extension and $L^c{}_k$ a maximal compact subalgebra of $L^c$. We know by th. 1.5.2 that $L^c{}_k$ is also a real form of $L^c$, i.e., complex extension of $L^c{}_k$ coincides with $L^c$. The representation $T$ of $G$ induces the representation $L \ni X \to T(X)$ of $L$ in $H$ by means of linear (matrix) transformations. Because $L^c = L + iL$, the representation $T(X)$ of $L$ provides a representation $T^c$ of $L^c$ and also a representation $T^c{}_k$ of $L^c{}_k$. By th. 3, a Lie algebra of linear transformations is completely reducible if and only if its complexification is completely reducible. Consequently, we can reduce the problem of proving the complete reducibility of $T(X)$ to that (via $T^c$) of $T^c{}_k$. Let $G_k$ be a compact Lie group associated with $L^c{}_k$. Then, by th. 7.1.4 we know that every representation of $G_k$ is completely reducible. Thus $T^c{}_k$ of $L^c{}_k$ and consequently also $T(X)$ must be completely reducible. Now exponentiating the representation $T(X)$ of $L$ to the global representation $T$ of $G$ we obtain the desired complete reducibility of $T$. ▼

The Weyl theorem states in fact that every representation of a semisimple Lie group $G$ is built out of irreducible ones. Hence, the problem of classification of finite-dimensional representations of semisimple Lie groups reduces to the problem of classification of all irreducible representations. This problem we solve in §§ 3, 4 and 5.

The generalization of the Weyl theorem to arbitrary connected Lie groups is given in § 7 of this chapter.

## § 2. Induced Representations of Lie Groups

We have seen in proposition 7.6 that every finite-dimensional irreducible representation of a compact group occurs in the regular representation. We now show that every continuous irreducible representation $g \to T_g$ of an arbitrary topological group $G$ can be imbedded in the regular representation realized in the space $C(G)$. Indeed, let $H$ be the carrier space of $T$ and let $\hat{H}$ be the dual space. Take a fixed $0 \neq v \in \hat{H}$ and set

$$f_u(g) = \langle T_g u, v \rangle, \qquad u \in H.$$

The set of functions so obtained forms a linear subspace $\tilde{H} \subset C(G)$. The mapping $V: H \to \tilde{H}$ is one-to-one because the inverse image of the zero of $H$ is an invariant subspace which cannot be different from 0 because $H$ is irreducible. Under $V$, the function $f_u(gg_0)$ corresponds to the vector $T_{g_0}u$, i.e., $G$ is represented in $\tilde{H}$ by the right translations $T^R_{g_0}$. We can select in the space $\tilde{H}$ a basis consisting of the functions

$$e_i(g) = D_{ii}(g), \qquad i = 1, 2, \dots, \dim H, \tag{1}$$

where $D_{ij}(g)$ are the matrix elements of the representation $T_g$. Then

$$T^R_{g_0} e_i(g) = e_i(gg_0) = D_{ii}(gg_0) = D_{ik}(g) D_{ki}(g_0) = D_{ki}(g_0) e_k(g).$$

Thus the space $\tilde{H}$ spanned by the continuous functions $e_i(g)$, $i = 1, 2, \dots, n = \dim H$, in $G$ can be taken to be the carrier space of the given irreducible representation $g \to T_g$ of $G$.

We now give a method of construction of induced finite-dimensional representations of complex, classical Lie groups based on a generalization of this idea. We shall start with the construction of the carrier space.

### *Representations of G Induced* by a Representation L of a Subgroup K*

Let $K$ be a closed subgroup of $G$ and let $k \to L_k$ be a finite-dimensional representation of $K$ in a Hilbert space $H$. Consider a linear space $\tilde{H}^L$ of functions $u$ with domain in $G$, range in $H$, satisfying the following conditions:

$$\begin{aligned} &1^\circ \text{ The scalar product } (u(g), v)_H \text{ is continuous in } G \text{ for arbitrary } v \in H, \\ &2^\circ\ u(kg) = L_k u(g) \text{ for all } k \in K. \end{aligned} \tag{2}$$

Define

$$T^L_{g_0} u(g) = u(gg_0). \tag{3}$$

* Induced representations, specially infinite-dimensional ones, will be discussed in more detail in ch. 16 ff.

The function $u(gg_0)$ satisfies conditions 1° and 2° and therefore belongs to $\tilde{H}^L$ We have, moreover,

$$(T^L_{g_1}T^L_{g_2}u)(g) = u(gg_1g_2) = T^L_{g_1g_2}u(g).$$

Consequently,

$$T^L_{g_1}T^L_{g_2} = T^L_{g_1g_2} \quad \text{and} \quad T^L_e = I.$$

By condition (2)1° the map $g \to T^L_g$ is continuous. Hence, the map $g \to T^L_g$ defines a continuous representation of $G$, in general infinite-dimensional.

The map $g \to T^L_g$ is called the *representation of $G$ induced by the representation $L$ of $K$.*

The realization of an induced representation $g \to T^L_g$ of $G$ by means of the right regular representation obliterates the individuality of a given representation. Therefore, we give another realization of $T^L$ in the linear space $H^L(Z)$ of functions in $Z = K\backslash G$.

Let $G$ be a classical Lie group, which admits the Gauss decomposition of the form

$$G = \overline{\mathfrak{Z}DZ}. \tag{4}$$

where $D$ is the abelian closed subgroup of $G$, $\mathfrak{Z}D$ and $DZ$ are solvable, connected subgroups in $G$, whose commutator subgroups are $\mathfrak{Z}$ and $Z$, respectively, and

$$\mathfrak{Z} \cap DZ = \{e\}, \quad D \cap Z = \{e\}.$$

Let $K = \mathfrak{Z}D$ and let $k \to L_k$ be a one-dimensional representation of $K$. Because $\mathfrak{Z}$ is the commutator subgroup of $K$ the representation $k \to L_k$ is trivial on $\mathfrak{Z}$, i.e., $L_\zeta = I$. Consequently, the map

$$\mathfrak{Z}D \ni k = \zeta\delta \to IL_\delta \tag{5}$$

defines, in fact, the one-dimensional representation of $D$. If $\tilde{H}^L$ is the space of functions satisfying conditions (2) with $L_\delta$ given by (5), then it follows from eqs. (4) and (5) that for $u(g) \in \tilde{H}^L$, we have

$$u(g) = u(\zeta\delta z) = L_{\zeta\delta}u(z) = L_\delta u(z), \quad z \in Z. \tag{6}$$

Because $L_\delta$ is fixed, we can replace each function $u(g) \in \tilde{H}^L$ by its contraction $u(z)$ defined on the domain $Z$ and consider instead of the linear space $\tilde{H}^L$ on $G$ the corresponding linear space $\tilde{H}^L(Z)$ of functions with the domain $Z$. The map $\tilde{H}^L \to \tilde{H}^L(Z)$ is one-to-one. In fact, the inverse image of zero in $\tilde{H}^L(Z)$ is zero in $\tilde{H}^L$. And, if $u(z) \equiv 0$, then $u(g) = 0$ for a regular point $g$ in $G$, which admits the decomposition $g = kz$. On the other hand $u(g)$ is continuous and $\mathfrak{Z}DZ$ is dense in $G$. Consequently, $u(g) = 0$. We now find the representation $g \to T^L_g$ in this realization of the carrier space:

LEMMA 1. *The action of operators $T^L_g$ in the space $\tilde{H}^L(Z)$ is given by the formula*

$$T^L_{g_0}u(z) = L_{\tilde{\delta}}\, u(z_{\tilde{g}}), \tag{7}$$

*where* $\tilde{\delta}$ *and* $z_{\tilde{g}}$ *are determined from the Gauss decomposition of the element* $\tilde{g} = zg_0 = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$.

PROOF: Writing

$$gg_0 = kzg_0 = k\tilde{\zeta}\tilde{\delta}z_{\tilde{g}} \tag{8}$$

we obtain from eqs. (6) and (5) that the vector $T^L_{g_0}u(z)$ in $\tilde{H}^L(Z)$ corresponding to the vector $(T^L_{g_0}u)(g) = u(gg_0)$ in $\tilde{H}^L$ has the form

$$L_{\tilde{\delta}}^{-1}u(gg_0) = L_{\tilde{\delta}}^{-1}L_k L_{\tilde{\zeta}\,\tilde{\delta}}\,u(z_{\tilde{g}}) = L_{\tilde{\delta}}\,u(z_{\tilde{g}}) \tag{9}$$

by (5) and (6). Hence eq. (7) follows. ▼

*Remark*: Strictly speaking the vectors $u(z)$ as well as the representation in $\tilde{H}^L(Z)$ should be denoted by different symbols, say $\tilde{u}(z)$ and $\tilde{T}^L_g$. For simplicity we have used the same symbol, as there will be no confusion.

It should be also stressed that the representation $T^L$ in $\tilde{H}^L(Z)$ might be reducible and infinite-dimensional. An irreducible subspace of $\tilde{H}^L(Z)$ which contains the function $u_0(z) \equiv 1$ we shall denote by $H^L(Z)$. The restriction of the representation $T^L$ to $H^L(Z)$ we shall denote, for the sake of simplicity, also by the symbol $T^L$.

Equation (7) implies further

$$T^L_{z_0}u(z) = u(zz_0) \qquad \text{for all } z_0 \in Z. \tag{10}$$

$$T^L_\delta u(z) = L_\delta u(\delta^{-1}z\delta) \qquad \text{for all } \delta \in D. \tag{11}$$

Clearly, the one-dimensional representations $L$ of $D$ (and therefore also of $K$) are given by the characters. If

$$D \ni \delta = \begin{bmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{bmatrix}, \qquad \delta_i \neq 0, \tag{12}$$

then, the most general complex-analytic character $\delta \to L_\delta$ has the form

$$L^\delta = \delta_1^{m_1}\delta_2^{m_2}\ldots\delta_n^{m_n}, \tag{13}$$

where $m_i$, $i = 1, \ldots, n$, are integers. The most general complex antianalytic character has the form:

$$L_\delta = \bar{\delta}_1^{m_1}\bar{\delta}_2^{m_2}\ldots\bar{\delta}_n^{m_n}$$

where $m_i$ are integers.

A character $L$ which determines an induced irreducible finite-dimensional representation $T^L$ of $G$ is said to be *inductive relative* to the group $G$. We shall prove later that only certain characters $L$ of $D$ can be inductive relative to $G$.

The following theorem constitutes the main result in the theory of finite-dimensional representations of Lie groups.

THEOREM 2. *Let G be a Lie group which admits a Gauss decomposition* $G = \overline{\mathfrak{Z}DZ}$. *Then, every irreducible, finite-dimensional representation of G is a representation*

*$T^L$ induced in the space $H^L$ by a uniquely defined character $\delta \to L_\delta$ of the subgroup $D$. Two irreducible representations $T^{L_1}$ and $T^{L_2}$ are equivalent if and only if $L^1 = L^2$.*

PROOF: Let $T$ be an irreducible representation of $G$ and let $T_K$ denote its restriction on the solvable connected subgroup $K = \mathfrak{Z}D$. We know by corollary 2 to Lie's theorem that all operators $T_k$, $k \in K$, can be simultaneously reduced to the triangular form, i.e.,

$$T_k = \begin{bmatrix} L_k^1 & & & \\ & L_k^2 & & 0 \\ & * & \ddots & \\ & & & L_k^r \end{bmatrix}, \tag{14}$$

where $k \to L_k^i$ are characters of the group $K$. Now, every character of $K$ is trivial on the commutator group $\mathfrak{Z}$ of $K$. Hence

$$L_k^i = L_{\zeta\delta}^i = L^i{}_\delta .$$

$L_k^i$ is in fact a character of $D$. The carrier space $H$ of the irreducible representation $T$ can be spanned by vectors

$$e_i(g) = D_{1i}(g), \tag{15}$$

where $D_{ji}(g)$ are matrix elements of $T_g$ (cf. eq. (1)). By virtue of eq. (14) we have

$$e_i(kg) = D_{1s}(k)D_{si}(g) = L_k^1 e_i(g).$$

Therefore, every element $u(g)$ in $\tilde{H}^L$ is a continuous function on $G$, which satisfies the condition

$$u(kg) = L_k^1 u(g).$$

Thus, the conditions 1° and 2° of eq. (2) are satisfied. Consequently, $T$ can be realized as the representation $T^{L^1}$ of $G$ induced by the one-dimensional representation $k \to L_k^1$ of the subgroup $K$. The action of $T_g^{L^1}$ in the carrier space $\tilde{H}^{L^1}$ is given by the right translation (3) and in the carrier space $\tilde{H}^L(Z)$ by the formula (7).

Applying corollary 1 to Lie's theorem for the solvable subgroup $N = DZ$ we conclude that the space $\tilde{H}^L(Z)$ contains a common eigenvector $u_0(z)$ for all operators $T_n$, $n \in N$. Clearly, this vector is invariant under the action of the commutator group $Z$ of $N$, i.e., $T_{z_0}u_0(z) = u_0(z)$ for all $z_0$ in $Z$. Because, by eq. (10), the subgroup $Z$ acts in $\tilde{H}^L(Z)$ by the right translation, the fixed eigenvector of all $T_z$ can be only a constant, e.g., $u_0(z) \equiv 1$. Hence $\tilde{H}^L$ coincides with $H^L(Z)$ and, by virtue of eq. (11), we obtain

$$T_\delta^{L^1} u_0(z) = L_\delta^1 u_0(z). \tag{16}$$

Consequently, the inductive character $L^1$ is uniquely defined.

If $L^1 = L^2$, then, obviously $T^{L^1}$ and $T^{L^2}$ are equivalent. Conversely, if $T^{L^1}$ and $T^{L^2}$ are equivalent, then there exists an operator $V$ such that $VT^{L^1}V^{-1} = T^{L^2}$ and $H_2 = VH_1$. Hence, by virtue of (16), we obtain $VT_\delta^{L^1}V^{-1}(Vu_0)(z) = L^1{}_\delta(Vu_0)(z)$: this implies $L^1 = L^2$. ▼

The final part of the proof gives the following important result:

COROLLARY 1. *There is one and only one (apart from normalization) invariant vector $u_0(z)$ of the subgroup $Z$ in the carrier space $H^L$ of every irreducible, finite-dimensional representation of $G$, i.e.,*

$$T_z^L u_0 = u_0 \quad \textit{for all } z \in Z. \tag{17}$$

*This invariant vector satisfies, moreover, the condition*

$$T_\delta^L u_0 = L_\delta u_0 \quad \textit{for all } \delta \in D, \tag{18}$$

*and can be properly normalized such that*

$$u_0(z) = 1. \blacktriangledown \tag{19}$$

The character $\delta \to L_\delta$ is said to be the *integral highest weight* of the irreducible representation $T^L$. Because $T^L = \exp(\sum_k H_k \alpha_k)$, where $H_k$ are the generators of the representation of the subgroup $D$, and $\alpha_n$ are the parameters in the Lie algebra of $D$, we can pass to the infinitesimal transformations and obtain by eqs. (18) and (13)

$$H_k u_0 = m_k u_0, \quad k = 1, 2, \ldots, n. \tag{20}$$

The vector $m = (m_1, m_2, \ldots, m_n)$ is called the *highest weight* of the representation $T^L$ and the vector $u_0$ the *highest vector*. By th. 2, $m$ is uniquely determined and in turn determines the irreducible representation $T^L$. The highest vector $u_0$ corresponding to $m$ will be also denoted by $u_m$.

COROLLARY 2. *If the carrier space $H$ of a representation $T$ contains only one invariant vector of the subgroup $Z$, then $T$ is irreducible.*

PROOF: Every $T$ of $G$ is completely reducible by the Weyl theorem. Hence, it can be reduced to a bloc diagonal form 5.3(4) of irreducible representations $D^i$, $i = 1, 2, \ldots, N$. Repeating the construction of th. 2 for each bloc $D^i$ we find $N$ invariant vectors of the subgroup $Z$. Hence, if $N = 1$, $T$ must be irreducible. $\blacktriangledown$

Eq. (19), the Gauss decomposition and the composition law for the operators $T_g^L$ imply the following useful result:

COROLLARY 3. *The carrier space $H^L(Z)$ of the irreducible representation $T^L$ is spanned by vectors*

$$u_g(z) = L_\delta u_0 = L_{\tilde{\delta}}, \tag{21}$$

*where $g$ ranges over $G$. The Gauss factor $\tilde{\delta}$ of the element $zg$ is a continuous function of $z$ and $g$. These functions satisfy the relation*

$$L_{\tilde{\delta}(z, g_1 g_2)} = L_{\tilde{\delta}(z, g_1)} L_{\tilde{\delta}(z g_1, g_2)}. \blacktriangledown \tag{22}$$

Eqs. (7) and (21) imply that if $L$ is an analytic (antianalytic) representation then the representation $T_g^L$ is also analytic (antianalytic).

The corollaries 2 and 3 give the following procedure for the decomposition of a reducible representation $T$ of $G$:

1° Find in the carrier space $H$ a maximal subspace $H_0$ which is fixed under the subgroup $Z$.

2° Select a normalized basis $u_0^{(i)}$ in $H_0$. Then, the carrier space $H^{(i)}$ of the irreducible representation $T^i$ is spanned by the vectors

$$u_g^{(i)}(z) = T_g u_0^{(i)} = L^{(i)}\mathfrak{z}\, u_0^{(i)} = L^i\mathfrak{z}\,.$$

The action of $T^{(i)}$ in the carrier space $H^{(i)}$ is given by (7). ▼

The following proposition describes the structure of the carrier space $H^L(Z)$ of the irreducible representation $T^L$. For simplicity we assume that $Z$ is a connected nilpotent group (it is just this case that we shall need later).

PROPOSITION 3. *The carrier space $H^L(Z)$ of an irreducible representation $T^L(Z)$ consists of functions $u(z)$, which are polynomials in the matrix elements $z_{pq}$ of an element $z \in Z$.*

PROOF: By virtue of corollary 2 to th. 1.1 every representation $n \to T_n$ of the solvable connected group $N = DZ$ can be written in the triangular form. Hence, the representations $T_z$ of the commutator subgroup $Z$ can be written in the triangular form with $L$'s on the main diagonal. This means that the Lie algebra $A$ of $Z$ generated by matrices $X_{pq}, p > q$, is mapped into an algebra of nilpotent matrices (i.e., for $X \in A$, $X^m = 0$ for some integer $m$). Therefore, the matrix elements of matrices $T_z$ ($= \exp \sum_{p>q} z_{pq}X_{pq}$) are polynomials in the matrix elements $z_{pq}$ of element $z \in Z$. On the other hand by eqs. (15) and (14) we have

$$e_l(g) = D_{ll}(g) = D_{ll}(kz) = D_{ls}(k)D_{sl}(z) = L_k^1 D_{ll}(z) = L_\delta^1 e_l(z)\,.$$

Because the map $H^L(G) \to H^L(Z)$ is one-to-one the matrix elements $e_l(z)$ of $T_z$ span the space $H(Z)$. ▼

The following proposition is useful in the determination of all characters $\delta \to L_\delta$ of $D$, which are inductive relative to $G$.

PROPOSITION 4. *Suppose that the Gauss decomposition of $G$ induces a Gauss decomposition of a subgroup $G_0$ of $G$:*

$$G_0 = \mathfrak{Z}_0 D_0 Z_0,$$

*where $\mathfrak{Z}_0$, $D_0$ and $Z_0$ are intersections of $G_0$ with the subgroups $\mathfrak{Z}$, $D$ and $Z$ of $G$, respectively: Let $L^0{}_{\delta_0}$ be the restriction of the character $L_\delta$ of $D$ to the subgroup $D_0$. If the character $L$ is inductive relative to $G$, then the character $L^0$ is inductive relative to the subgroup $G_0$.*

PROOF: Because the character $L$ of $D$ is inductive, the linear hull of functions $u_g(z) = L\tilde{\mathfrak{z}}$, $\tilde{\delta} = \tilde{\delta}(z, g)$, consists of polynomials in $z_{pq}$ and has a finite dimension by proposition 3. This is also true for functions

$$u_{g_0}(z_0) = L\mathfrak{z}_0, \quad \tilde{\delta}_0 = \tilde{\delta}(z_0, g_0), \quad z_0 \in Z_0,\ g_0 \in G_0\,.$$

Clearly, the functions $u_{g_0}(z_0)$ are continuous on $Z_0 \times G_0$. Consequently, some representation of $G_0$ is realized in the linear envelope $H_0$ of these functions.

The vector $u_0(z_0) = 1$ is the unique vector in $H_0$ which is fixed for the subgroup $Z_0$. Hence, the representation of $G_0$ in $H_0$ is irreducible by corollary 3. Consequently, the character $L^0$ of $D_0$ is inductive. ▼

The method of induced representations has a number of advantages when compared with the infinitesimal Cartan–Weyl method. It provides the classification of irreducible representations in terms of the highest weights and at the same time, it gives a natural realization of the carrier space as the linear space $H^L(Z)$ of polynomials over the standard subgroup $Z$. This is very useful in the solution of various practical problems.

We now consider the explicit construction of the operators $T_g^L$ and the carrier space $H^L(Z)$ for the group SL(2, $C$), the covering group of the Lorentz group SO(3, 1).

EXAMPLE 1. Let $G = \mathrm{SL}(2, C)$. The Gauss factors $\mathfrak{Z}$, $D$ and $Z$ in this case are given by (cf. eq. 3.6 (3)):

$$\mathfrak{Z} = \left\{\begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}\right\}, \quad D = \left\{\begin{bmatrix} \delta^{-1} & 0 \\ 0 & \delta \end{bmatrix}\right\}, \quad Z = \left\{\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}\right\},$$

where $\zeta$, $\delta$ and $z$ are in $C^1$. If $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}(2, C)$, then the factors $\tilde{\delta}$ and $z_{\tilde{g}}$ of the element $\tilde{g} = zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$ have the form

$$\tilde{\delta} = \beta z + \delta, \qquad z_{\tilde{g}} = \frac{\alpha z + \gamma}{\beta z + \delta}.$$

An arbitrary complex analytic character of $D$, by eq. (13), is given by

$$\delta \to L_\delta = \delta^m, \tag{23}$$

where $m$ is an integer, which we determine below. According to th. 2 and eq. (7), every irreducible representation $g \to T_g^L$ induced by the one-dimensional representation (23) of $D$ is given by the formula

$$T_g^L u(z) = L_{\tilde{\delta}} u(z_{\tilde{g}}) = (\beta z + \delta)^m u\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{24}$$

It remains only to determine the number $m$. By corollary 3, the carrier space $H^L(Z)$ is spanned by the vectors

$$u_g(z) = L_{\tilde{g}} = (\beta z + \delta)^m \tag{25}$$

where $\beta$ and $\delta$ take all admissible values for the elements $g$ in SL(2, $C$). Thus, in particular the space $H^L(Z)$ contains all translations

$$u_i(z) = (z + \delta_i)^m.$$

Because $H^L(Z)$ is finite-dimensional, there exists a number $r \geqslant 1$ such that an arbitrary set of $r+1$ functions is linearly dependent. Consequently the determinant

$$\Delta(z) = \begin{vmatrix} u_1(z) & u_2(z) & \dots & u_{r+1}(z) \\ u_1'(z) & u_2'(z) & \dots & u_{r+1}'(z) \\ \dots & \dots & \dots & \dots \\ u_1^{(r)}(z) & u_2^{(r)}(z) & \dots & u_{r+1}^{(r)}(z) \end{vmatrix}$$

is identically zero. Here

$$u_1^{(s)}(z) \equiv \frac{\partial^s}{\partial z^s} u_1(z)$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right),$$

Setting $z = 0$, we obtain

$$\Delta(0) = \begin{vmatrix} \delta_1^m & \delta_2^m & \dots & \delta_{r+1}^m \\ m\delta_1^{m-1} & m\delta_2^{m-1} & \dots & m\delta_{r+1}^{m-1} \\ \dots & \dots & \dots & \dots \end{vmatrix}$$
$$= m^r(m-1)^{r-1} \dots (m-r+1)[\delta_1 \delta_2 \dots \delta_{r+1}]^{m-r} \cdot W(\delta),$$

where

$$W(\delta) = \prod_{i<j} (\delta_i - \delta_j).$$

One can always choose the numbers $\delta_i \neq 0$ and different between themselves. Hence if $m$ is not equal to one of the integers $0, 1, \dots, r-1$, then $\Delta(0) \neq 0$. Consequently, only non-negative integers $m = 0, 1, 2, \dots$ give the inductive complex analytic characters $L$ of the form (23). For integer $m \geqslant 0$, according to eq. (25), the carrier space $H^L$ contains all monomials $1, z, z^2, \dots, z^m$ and is spanned by them. To summarize:

THEOREM 5. *Every complex analytic irreducible representation of* SL(2, $C$) *determines and is determined by an integer $m \geqslant 0$. It is realized by formula* (24) *in the space $H^m(Z)$ of all polynomials of degree not greater than $m$.* ▼

We have also complex 'antianalytic' irreducible representations $T^{\bar{L}}$ induced by the character

$$\bar{L}_\delta = \bar{\delta}^n. \tag{26}$$

By the same considerations $n$ must again be a non-negative integer. The representation $g \to T_g^{\bar{L}}$ is given in the space $H^n(\bar{Z})$ of polynomials in the variables $1, \bar{z}, \bar{z}^2, \dots, \bar{z}^n$ by the formula

$$T_g^{\bar{L}} u(z) = \overline{(\beta z + \delta)^n} u\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{27}$$

Finally, if we take a real analytic character $\delta \to L_\delta = \delta^m\bar{\delta}^n$ we obtain the real analytic representations of SL(2, $C$). Thus, every irreducible, finite-dimensional representation of SL(2, $C$) is determined by a pair $(m, n)$ of non-negative integers. It is given in the space of all polynomials $u(z, \bar{z})$ of degree not greater than $m$ relative to $z$ and not greater than $n$ relative to $\bar{z}$, by the formula

$$T_g^L u(z,\bar{z}) = (\beta z+\delta)^m\overline{(\beta z+\delta)}^n u\left(\frac{\alpha z+\gamma}{\beta z+\delta}, \frac{\bar{\alpha}\bar{z}+\bar{\gamma}}{\bar{\beta}\bar{z}+\bar{\delta}}\right). \tag{28}$$

In other words, every real, analytic, irreducible representation of SL(2, $C$) is a tensor product of the form

$$T^{L_1}\otimes\overline{T^{L_2}}, \tag{29}$$

where $T^{L_1}$ and $T^{L_2}$ are complex-analytic, irreducible representations of $G$ and $\overline{T^L}$ denotes the representation conjugate to $T^L$.

The formula (24) restricted to the subgroup SU(2) provides an irreducible, unitary representation of SU(2) (cf. exercise 9.2.1). In fact, using the 'Weyl unitary trick' we conclude that every irreducible representation of SU(2) is the restriction of an irreducible, complex-analytic representation of SL(2, $C$) to the subgroup SU(2).

## § 3. The Representations of GL($n$, $C$), GL($n$, $R$), $U(p, q)$, $U(n)$, SL($n$, $C$), SL($n$, $R$), SU($p$, $q$), and SU($n$)

Th. 2.2 reduces the problem of the classification of all irreducible representations of a group $G$ to the problem of the enumeration of all inductive highest weights. In this section we solve this problem for the full linear group GL($n$, $C$), for SL($n$, $C$) and for their real forms.

### A. *The Representation of* GL($n$, $C$)

The Gauss decomposition of GL($n$, $C$), $G = \overline{\mathfrak{Z}DZ}$ is given by

$$\zeta = \begin{bmatrix} 1 & \zeta_{12} & \cdots & \zeta_{1n} \\ & 1 & \cdots & \vdots \\ & & \ddots & \vdots \\ & & & \zeta_{n-1,n} \\ 0 & & & 1 \end{bmatrix}, \quad \delta = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix},$$

$$z = \begin{bmatrix} 1 & & & & \\ z_2 & 1 & & 0 & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ z_{n1}, & z_{n2} & \cdots & z_{n,n-1} & 1 \end{bmatrix}. \tag{1}$$

A complex, analytic character $L$ of $D$ has the form

$$\delta \to L_\delta = \delta_1^{m_1}\delta_2^{m_2} \ldots \delta_n^{m_n}, \tag{2}$$

where $m_i$ are integers.

THEOREM 1. *Every complex analytic irreducible representation of* GL($n$, $C$) *determines and is in turn determined by the highest weight* $m = (m_1, m_2, \ldots, m_n)$, *whose components are integers satisfying the conditions*

$$m_1 \geqslant m_2 \geqslant m_3 \ldots \geqslant m_n. \tag{3}$$

*The carrier space* $H^L(Z)$ *consists of polynomials of the matrix elements* $z_{pq}$ *of the elements* $z \in Z$.

*The representation* $T^L$ *is realized in* $H^L(Z)$ *by means of the formula*

$$T_g^L u(z) = L_{\tilde{\delta}} u(z_{\tilde{g}}), \tag{4}$$

*where* $\tilde{\delta}$ *and* $z_{\tilde{g}}$ *are factors of the Gauss decomposition of the element* $\tilde{g} \equiv zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$.

PROOF: According to th. 2.2 every complex analytic irreducible representation of GL($n$, $C$) is the representation $T^L$ induced by the complex analytic character $\delta \to L_\delta$ of $D$. Hence $m_i$ are integers. Proposition 2.3 assures that the representation $T^L$ can be realized in the space $H^L(Z)$ of polynomials $u(z)$ of matrix elements $z_{pq}$ of $z \in Z$. To classify all irreducible representations, it is necessary to find all admissible highest weights.

Let $L^0$ be the restriction of $L$ to the subgroup $D_0$ consisting of all matrices of the form

$$\delta_0 = \begin{bmatrix} \lambda & & & & \\ & \lambda^{-1} & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}. \tag{5}$$

The subgroup $G_0$ of all matrices of the form

$$g = \begin{bmatrix} \alpha & \beta & & & \\ \gamma & \delta & & 0 & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{bmatrix}, \qquad \alpha\delta - \beta\gamma = 1 \tag{6}$$

is isomorphic to SL(2, $C$). Hence, by proposition 2.4 we conclude that the character

$$L^0{}_{\delta_0} = \lambda^{m_1 - m_2}$$

is inductive relative to SL(2, $C$). Consequently, $m_1 - m_2$ is a non-negative integer by th. 2.5.

Moving along the main diagonal in (6), we conclude similarly that $m_2-m_3, \ldots, m_{n-1}-m_n$ are also non-negative integers. This proves the necessity of condition (3).

Conversely, when conditions (3) are satisfied the functions

$$u_g(z) = L_{\tilde{\delta}}, \qquad \tilde{\delta} = \tilde{\delta}(z, g) \tag{7}$$

are polynomials of the matrix elements $z_{pq}$ of $z$, whose degrees are uniformly bounded with respect to $g \in G$. Indeed, the character (2) can be written in the form

$$L_\delta = \Delta_1^{f_1} \Delta_2^{f_2} \ldots \Delta_n^{f_n}, \tag{8}$$

where

$$\Delta_p = \delta_1 \delta_2 \ldots \delta_p$$

and

$$f_p = m_p - m_{p+1}, \quad p = 1, 2, \ldots, n, \ m_{n+1} = 0.$$

According to exercise 3.11.6.2, $\Delta_p$ is equal to the main diagonal minor of $g$ of the order $p$. Consequently, $L_{\tilde{\delta}}$ is the polynomial in the matrix elements of $z_{\tilde{g}}$, where $\tilde{\delta}$ and $z_{\tilde{g}}$ are factors in the Gauss decomposition of the element $\tilde{g} \equiv zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$. Therefore, the linear hull $H^L$ of vectors (8) is finite-dimensional, and the representation $T^L$ of $G$ in $H^L$ is given by eq. (4). The vector $u_0(z) \equiv 1$ is the only element of $H^L$, which is fixed relative to the subgroup $Z$. Hence, by corollary 2 to th. 2.2 the representation $T^L$ is irreducible. This proves that the condition (3) is also sufficient.

These considerations and th. 2.2 imply that an irreducible representation $T^L$ can be realized in the space $H^L(Z)$ of polynomials on $Z$ by means of the formula (4). ▼

*Remark* 1: We can write the formula (4) in a more explicit form convenient for further calculations. Indeed, by virtue of eqs. (4) and (8)

$$\begin{aligned} T_g^L u(z) &= L_{\tilde{\delta}} u(z_{\tilde{g}}) = \Delta_1^{f_1}(\tilde{\delta}) \Delta_2^{f_2}(\tilde{\delta}) \ldots \Delta_n^{f_n}(\tilde{\delta}) u(z_{\tilde{g}}) \\ &= (zg)_{11}^{m_1-m_2} \begin{vmatrix} (zg)_{11} & (zg)_{12} \\ (zg)_{21} & (zg)_{22} \end{vmatrix}^{m_2-m_3} \ldots (\det g)^{m_n} u(z_{\tilde{g}}). \end{aligned} \tag{9}$$

Here we used the fact that $\det(zg) = \det g$ for the last minor.

*Remark* 2: One often uses the symbol $f = [f_1, f_2, \ldots, f_n]$, $f_k = m_k - m_{k+1}$, $m_{n+1} = 0$, for labelling an irreducible representation associated with the highest weight $m = (m_1, m_2, \ldots, m_n)$. In the following we use the curly bracket for the highest weight, and the square bracket for the symbol $f$. ▼

The antianalytic representations of GL($n$, $C$) are realized in the space of polynomials of variables $\bar{z}_{pq}$ of $\bar{z} \in Z$. These representations are induced by the characters $\delta \to \overline{L_\delta}$, where $L_\delta$ is the complex-analytic integral highest weight.

An arbitrary real analytic irreducible finite-dimensional representation of

GL($n$, $C$) is induced by a character $L_1\bar{L}_2$. According to eq. (4) and eq. 2(21) it can be written as the tensor product

$$T^{L_1} \otimes T^{\bar{L}_2} \tag{10}$$

of complex-analytic and complex-antianalytic irreducible representations. (Cf. example 2.1, eq. 2(29).)

The group GL($n$, $C$) has in particular a series of one-dimensional representations. These representations are of the form $\eta^p\bar{\eta}^q$, where

$$\eta(g) = \det g.$$

One can associate with these representations the infinitely many valued and indecomposable representations

$$g \to \eta^r(g)\begin{bmatrix} 1 & \log|\eta| \\ 0 & 1 \end{bmatrix}, \tag{11}$$

where $r$ is a complex number.

B. *Representations of* SL($n$, $C$)

Let $T^L$ be an irreducible representation of GL($n$, $C$) in the space $H^L(Z)$. The formula (9) implies that every irreducible representation $T^L$ is of the form

$$T_g{}^L = (\det g)^{m_n} \tilde{T}_g{}^L \tag{12}$$

where $g \to (\det g)^{m_n}$ is the one-dimensional representation of GL($n$, $C$) and $\tilde{T}_g^L$ is again a representation of GL($n$, $C$) which is also irreducible. The representation $g \to \tilde{T}_g^L$ restricted to SL($n$, $C$) provides an irreducible representation of SL($n$, $C$) in the carrier space $H^L(Z)$. Equation (12) implies that two irreducible representations of GL($n$, $C$) give the same representation of SL($n$, $C$) if and only if they differ by some power of $\det g$. This fact, by virtue of (9) allows us to normalize the integral highest weight $m = (m_1, \ldots, m_n)$ in such a fashion that $m_n = 0$. By a direct application of th. 2.2 to SL($n$, $C$) we see that every complex analytic irreducible representation of SL($n$, $C$) is a restriction of a representation of GL($n$, $C$). Consequently, we have

THEOREM 2. *Every irreducible, finite-dimensional representation* $\tilde{T}^L$ *of* SL($n$, $C$) *determines and is in turn determined by the highest weight* $m = (m_1, m_2, \ldots, m_{n-1})$, *whose components are integers satisfying the condition*

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{n-1} \geqslant 0. \tag{13}$$

*All of these representations are realized in a space* $H^L(Z)$ *of polynomials by means of formula* (4). ▼

Clearly, one has also the corresponding complex-antianalytic and real analytic representations of SL($n$, $C$). Because SL($n$, $C$) is simply-connected, every irreducible representation is single-valued.

## C. *The Representations of* $\mathrm{GL}_+(n, R)$ *and* $\mathrm{SL}(n, R)$

The group $\mathrm{GL}(n, R)$ has two connected components by virtue of th. 3.7.1. The complex extension of the group $\mathrm{GL}_+(n, R)$, whose elements satisfy the condition $\det g > 0$, coincides with $\mathrm{GL}(n, C)$. Hence we restrict our attention to $\mathrm{GL}_+(n, R)$. Using the th. 1.3 and th. 1 we conclude that every analytic irreducible, finite-dimensional representation of $\mathrm{GL}_+(n, R)$ determines and is determined by the highest weight $m = (m_1, m_2, \ldots, m_n)$, whose components are integers satisfyings the conditions

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n. \tag{14}$$

Let $\mathfrak{Z}_R D_R Z_R$ be the Gauss decomposition for $\mathrm{GL}_+(n, R)$. Then, the irreducible representation $T^L$ induced by the one-dimensional representation $L$ of $D_R$ is realized by the formula

$$T^L_g u(z) = L_{\tilde{\delta}} u(z_{\tilde{g}}), \tag{15}$$

where $\tilde{\delta} \in D_R$ and $z_{\tilde{g}} \in Z_R$ are determined from the decomposition $\tilde{g} \equiv zg = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$.

Using similar arguments as in case of $\mathrm{SL}(n, C)$ we obtain that every irreducible representation of $\mathrm{SL}(n, R)$ determines and is determined by the highest weight $m = (m_1, m_2, \ldots, m_{n-1})$, whose components are integers satisfying the condition (13). The action of $T^L$ in the carrier space $H^L(Z_R)$ is given by eq. (4).

## D. *The Representations of* $U(p, q)$, $p+q = n$, $U(n)$, $\mathrm{SU}(p, q)$, $p+q = n$, $\mathrm{SU}(n)$, *and* $Q_{2n}$

These groups are also real forms of $\mathrm{GL}(n, C)$ and $\mathrm{SL}(n, C)$, respectively. Hence, by the same arguments as in subsection C, every irreducible representation of $U(p, q)$, $p+q = n$, and of $U(n)$ is characterized by the highest weight $m = (m_1, m_2, \ldots, m_n)$ whose components satisfy the conditions (3). Similarly, irreducible representations of $\mathrm{SU}(p, q)$, $p+q = n$, $\mathrm{SU}(n)$ and $Q_{2n}$ are characterized by the highest weight $m = (m_1, m_2, \ldots, m_{n-1})$, whose components are integers satisfying the condition (13).

## § 4. The Representations of the Symplectic Groups $\mathrm{Sp}(n, C)$, $\mathrm{Sp}(n, R)$ and $\mathrm{Sp}(n)$

The symplectic group $\mathrm{Sp}(n, C)$ can be realized as the set of all linear transformations of the $n$-dimensional complex vector space ($n$ even $= 2\nu$), which conserve the skew-symmetric form

$$[x, y] = x_1 y_n + x_2 y_{n-1} + \ldots + x_\nu y_{\nu+1} - x_{\nu+1} y_\nu - \ldots - x_n y_1. \tag{1}$$

Thus, $g \in \mathrm{Sp}(n, C)$, if and only if

$$\sigma^{-1} g \sigma = (g^T)^{-1}, \quad \text{where } \sigma = \begin{bmatrix} 0 & -S \\ S & 0 \end{bmatrix} \tag{2}$$

and $S$ is the $\nu$-by-$\nu$ matrix given by

$$S = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

In particular, $\mathrm{Sp}(2, C)$ is isomorphic to $\mathrm{SL}(2, C)$.

The Gauss decomposition of $\mathrm{GL}(n, C)$ induces the Gauss decomposition of $\mathrm{Sp}(n, C)$:

$$\mathrm{Sp}(n, C) = \overline{\mathfrak{Z}_S D_S Z_S}, \tag{3}$$

where $\mathfrak{Z}_S$, $D_S$ and $Z_S$ are intersections of $\mathrm{Sp}(n, C)$ and the corresponding subgroups of $\mathrm{GL}(n, C)$.

It follows from eq. (2) that

$$\delta = \begin{bmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{bmatrix}$$

is an element of $D_S$ if any two of the numbers

$$\delta_1, \delta_2, \dots, \delta_\nu, \delta_{\nu+1}, \dots, \delta_{n-1}, \delta_n \tag{4}$$

in symmetric positions with respect to the center are mutually inverse; i.e., $\delta_n = \delta_1^{-1}$, $\delta_{n-1} = \delta_2^{-1}$, etc. Taking $\delta_1, \dots, \delta_\nu$ as independent parameters of $\delta \in D_S$ we see that every complex analytic character of $D_S$ has the form

$$\delta \to L_\delta = \delta_1^{m_1} \delta_2^{m_2} \dots \delta_\nu^{m_\nu}. \tag{5}$$

We have then

THEOREM 1. *Every complex-analytic, irreducible representation $T^L$ of* $\mathrm{Sp}(n, C)$ *determines and is in turn determined by the highest weight $m = (m_1, m_2, \dots, m_\nu)$, whose components are integers satisfying the condition*

$$m_1 \geqslant m_2 \geqslant \dots \geqslant m_\nu \geqslant 0. \tag{6}$$

*The carrier space $H^L(Z_S)$ of $T^L$ consists of polynomials of the matrix elements $z_{pq}$ of elements $z \in Z_S$. The representation $T^L$ is realized in $H^L(Z_S)$ by means of the formula*

$$T_g^L u(z) = L_{\tilde{\delta}} u(z_{\tilde{g}}), \tag{7}$$

*where $\tilde{\delta}$ and $z_{\tilde{g}}$ are factors of the Gauss decomposition* (3) *of the element $\tilde{g} \equiv zg = \tilde{\zeta} \tilde{\delta} z_{\tilde{g}}$.*

PROOF: Let $T^L$ be an irreducible representation of $\mathrm{Sp}(n, C)$ induced by the character $\delta \to L_\delta$ of $D_S$ given by eq. (5). Let $G_1, G_2, \dots, G_\nu$ be a sequence of subgroups of $\mathrm{Sp}(n, C)$ isomorphic to $\mathrm{SL}(2, C)$. In particular, $G_1$ consists of all linear transformations of the form

$$\begin{bmatrix} \alpha & \beta & & & & & \\ \gamma & \delta & & & 0 & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & 0 & & & \alpha & -\beta \\ & & & & & -\gamma & \delta \end{bmatrix}, \quad \alpha\delta-\beta\gamma = 1. \tag{8}$$

The subgroups $G_2, \ldots, G_{\nu-1}$ are obtained similarly by moving along the main diagonal. The subgroup $G_\nu$ consists of all unimodular transformations, which change the coordinates $x_\nu$ and $x_{\nu+1}$ only. Repeating now the arguments in the proof of th. 3.1, we conclude that all numbers $m_1-m_2, \ldots, m_{\nu-1}-m_\nu, m_\nu$ are non-negative integers. This proves the necessity of condition (6).

Suppose now that $m = (m_1, \ldots, m_\nu)$ satisfies condition (6). Then the highest weight $m = (m_1, \ldots, m_\nu, 0, \ldots, 0)$ of the subgroup $D$ of GL$(n, C)$ is inductive relative to GL$(n, C)$ by th. 3.1. Consequently the character (5) is inductive relative to Sp$(n, C)$ by proposition 2.4. This proves that condition (6) is also sufficient.

It follows from the proposition 2.3 and eq. 3.(7) that the carrier space $H^L(Z_S)$ consists of polynomials in $Z_S$. The formula (7) results then from lemma 2.1. ▼

The properties of the complex-antianalytic and real analytic irreducible representations of Sp$(n, C)$ are analogous to the corresponding representations of SL$(n, C)$. Because Sp$(n, C)$ is simply connected, all irreducible representations $T^L$ of Sp$(n, C)$ are single-valued.

Using the th. 1.3 and the th. 1 we conclude that every analytic irreducible representations of the real symplectic group Sp$(n, R)$ determines and is determined by the highest weight $m = (m_1, m_2, \ldots, m_\nu)$, whose components are integers satisfying the condition (6). The same is true for the compact, symplectic groups Sp$(n)$ = Sp$(n, C) \cap U(2n)$.

## § 5. The Representations of Orthogonal Groups SO$(n, C)$, SO$(p, q)$, SO*$(n)$, and SO$(n)$

The defining representation of the orthogonal group SO$(n, C)$ is the set of all linear unimodular transformations, which preserve the quadratic form

$$z_1^2+z_2^2+ \ldots +z_n^2.$$

However, for our purposes it is more convenient to realize SO$(n, C)$ as the group of unimodular, linear transformations which conserve the form

$$z_1 z_n+z_2 z_{n-1}+ \ldots +z_n z_1. \tag{1}$$

Over the field of complex numbers both forms coincide. Every element $g \in$ SO$(n, C)$ satisfies the condition

$$\sigma^{-1} g \sigma = (g^T)^{-1}, \quad \text{where } \sigma = \begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix} \tag{2}$$

and

$$S = \begin{bmatrix} 0 & \dots & & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & \dots & & 0 \\ 1 & 0 & \dots & & 0 \end{bmatrix}.$$

In the realization (1) the Gauss decomposition of $\mathrm{GL}(n, C)$ induces the corresponding Gauss decomposition of $\mathrm{SO}(n, C)$, i.e.,

$$\mathrm{SO}(n, C) = \overline{\mathfrak{Z}_0 D_0 Z_0} \tag{3}$$

where $\mathfrak{Z}_0$, $D_0$ and $Z_0$ are intersections of $\mathrm{SO}(n, C)$ with the subgroups $\mathfrak{Z}$, $D$ and $Z$ respectively of $\mathrm{GL}(n, C)$.

The group $\mathrm{SO}(2n+1, C)$ is doubly connected and $\mathrm{SO}(2n, C)$ is fourfold connected (cf. ch. 3.7.E). Hence, one might expect that $\mathrm{SO}(n, C)$ has also multivalued irreducible representations. This is the special feature of the representation theory of orthogonal groups. For example, the four-dimensional Dirac equation can be viewed as a direct consequence of the existence of additional spinor representations of the orthogonal group $\mathrm{SO}(4, C)$.

The full orthogonal group $O(n, C)$ consists of two connected components $O^+(n, C)$ and $O^-(n, C)$, whose elements satisfy the condition $\det g = \pm 1$, respectively. Hence, starting from an arbitrary element $o$ in $O^-$, one can obtain all other elements of $O^-$ by applying the left or the right translation by $g \in O^+$. For the element $o$ one can take the matrix $o = -e$ in case when $n$ is odd, or the $2\nu \times 2\nu$ matrix

$$O = \left[\begin{array}{ccccc:cc:ccccc}
1 & & & & & & & & & & & \\
 & \ddots & & & & & & & & & & \\
 & & 1 & & & & & & & 0 & & \\
 & & & \ddots & & & & & & & & \\
 & & & & 1 & & & & & & & \\
\hdashline
 & & & & & 0 & 1 & & & & & \\
 & & & & & 1 & 0 & & & & & \\
\hdashline
 & & & & & & & 1 & & & & \\
 & & & & & & & & \ddots & & & \\
 & & 0 & & & & & & & 1 & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & 1
\end{array}\right] \tag{4}$$

in case when $n$ is even, $n = 2\nu$. The matrix (4) corresponds to the transposition of the coordinates $z_\nu$ and $z_{\nu+1}$. In both cases we have $o^2 = e$.

The group $\mathrm{SO}(n, C) = O^+(n, C)$ is clearly the normal subgroup in $O(n, C)$. This means, in particular, that the map

$$g \to \check{g} \equiv ogo^{-1} \tag{5}$$

leaves the subgroup $\mathrm{SO}(n, C)$ invariant. We shall call the outer automorphism (5) of $\mathrm{SO}(n, C)$ the mirror automorphism, and the corresponding transformation $o$ the mirror reflection. One readily verifies, using the explicit form of the subgroups $\mathfrak{Z}_0$, $D_0$ and $Z_0$ that the mirror automorphism leaves the subgroups $\mathfrak{Z}_0$, $D_0$ and $Z_0$ invariant. Consequently the Gauss decomposition $g = \zeta\delta z$ goes over into the corresponding Gauss decomposition of the element $\check{g}$.

The matrix $\delta \in D$ conserves the form (1) if in the series $\delta_1, \delta_2, \ldots, \delta_\nu, \delta_{\nu+1}, \ldots, \delta_{n-1}, \delta_n$ the symmetrically situated elements are mutually inverse (i.e., $\delta_1 = \delta_n^{-1}$, $\delta_2 = \delta_{n-1}^{-1}$, etc.). Hence, in case of even $n$ ($n = 2\nu$) as well as odd ($n = 2\nu+1$) we have only the numbers $\delta_1, \delta_2, \ldots, \delta_\nu$ as the independent matrix elements of $\delta \in D_0$. Consequently, every one-dimensional complex analytic representation $L$ of $D_0$ has the form

$$\delta \to L_\delta = \delta_1^{m_1}\delta_2^{m_2} \ldots \delta_\nu^{m_\nu}. \tag{6}$$

LEMMA 1. *If $T$ is an irreducible representation of* $\mathrm{SO}(2\nu, C)$ *corresponding to the highest weight $m = (m_1, m_2, \ldots, m_\nu)$, then, the mirror-conjugated representation $\check{T}_g = T_{\check{g}}$ corresponds to the highest weight*

$$\check{m} = (m_1, \ldots, m_{\nu-1}, -m_\nu). \tag{7}$$

*If $n = 2\nu+1$, then every irreducible representation is mirror-self-conjugated.*

PROOF: Because the map $g \to \check{g}$ conserves the structure of $\mathfrak{Z}_0$, $D_0$, $Z_0$ it is sufficient to find the image of the character $L_\delta$. If $n$ is even, $n = 2\nu$, then all parameters $\delta_i$ remain unchanged except the parameter $\delta_\nu$, which goes into $\delta_\nu^{-1}$. If $n$ is odd, then $o = -e$ and $\delta$ is unchanged. ▼

The following theorem gives the classification of all complex analytic, irreducible representations of $\mathrm{SO}(n, C)$.

THEOREM 2. *The group* $\mathrm{SO}(n, C)$ *has two series of complex analytic irreducible representations. Every representation of the first series determines and is in turn determined by a highest weight $m = (m_1, m_2, \ldots, m_\nu)$ whose components $m_i$ are integers and satisfy the conditions*

$$\begin{aligned} &1^\circ \textit{ for } n = 2\nu:\ m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant |m_\nu|, \\ &2^\circ \textit{ for } n = 2\nu+1:\ m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant m_\nu \geqslant 0. \end{aligned} \tag{8}$$

*Every representation of the second series determines and is determined by a highest weight $m = (m_1, m_2, \ldots, m_\nu)$ whose component $m_i$ are half-odd integers and also satisfy the conditions* (8).

PROOF: *The case $n = 2\nu$*. Let $G_0$ be the subgroup of $SO(2\nu, C)$ consisting of all matrices of the form

$$\begin{bmatrix} g & 0 \\ 0 & \hat{g} \end{bmatrix}, \quad g \in SL(\nu, C), \tag{9}$$

where $\hat{g} = S^{-1}(g^T)^{-1}S$. Clearly, $G_0$ is isomorphic with $SL(\nu, C)$. If the character $L^m$ of $D_0$ is inductive relative to $SO(2\nu, C)$, then its restriction to $G_0$ is inductive relative to $G_0$. Hence, by virtue of eq. 3(13) we get

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant m_\nu, \tag{10}$$

where $m_i$ are integers. Repeating these arguments for mirror-conjugated representation we find

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant -m_\nu. \tag{11}$$

Moreover, by eq. 3(13), we see that both of the two numbers

$$m_i - m_\nu \quad \text{and} \quad m_i + m_\nu$$

are integers. Consequently, all components $m_1, \ldots, m_\nu$ should be simultaneously either all integers, or all half-odd integers. This proves the necessity of condition (8) 1°.

Suppose now that the condition (8) 1° is satisfied for some weight $m = (m_1, m_2, \ldots, m_\nu)$, where all $m_i$ are simultaneously integers or half-odd integers. Because a miror-conjugated character can be inductive only simultaneously with a given character, we can suppose that $m_\nu \geqslant 0$. Then, the character of the subgroup $D(2\nu)$ of $GL(2\nu, C)$ defined by the weight $m = (m_1, \ldots, m_\nu, 0, \ldots, 0)$ is inductive relative to $GL(2\nu, C)$ by th. 3.1. Consequently, by virtue of proposition 2.4, its restriction on $SO(2\nu, C)$ is inductive relative to this group.

*The case $n = 2\nu+1$*. The proof runs similar to the case of even $n$. The last condition in eq. (8) 2° ($m_\nu \geqslant 0$) follows from the consideration of the subgroup $SO(3, C)$, whose elements consist of rotations transforming coordinates $x_\nu$, $x_{\nu+1}$ and $x_{\nu+2}$ only. Indeed, since $SO(3, C)$ is locally isomorphic to $SL(2, C)$ we obtain $m_\nu \geqslant 0$ by th. 3.2. ▼

The irreducible representations of $SO(n, C)$ associated with the highest weights with integer components are the tensor representations. The remaining representations are called spinor representations.

The lowest spinor representations play an essential role in physics. In the case of $SO(2\nu+1, C)$ the lowest spinor representation is determined by the highest weight

$$m_+ = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}). \tag{12}$$

In the case of $SO(2\nu, C)$ there are two lowest spinor representations, namely $m_+$ (eq. (12)), and

$$m_- = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}). \tag{13}$$

These representations are mirror-conjugated. The linear objects transforming according to representations $T^{L^m+}$ and $T^{L^m-}$ are called *spinors of the first* and the *second kind*, respectively.

The group $SO(n, C)$ has also complex-antianalytic and real-analytic irreducible representations. Their properties are analogous to those of the corresponding representations of $SL(n, C)$.

Using the th. 3.1 we conclude that th. 2 is also true for the connected components of the real forms of $SO(n, C)$, i.e., $SO(p, q)$, $p+q = n$, $SO^*(n)$, $n = 2\nu$, and $SO(n)$.

## § 6. The Fundamental Representations

Let $G$ be a Lie group which admits a Gauss decomposition $G = \overline{\mathfrak{Z}DZ}.$ As we showed in § 2 every irreducible representation $T^{L^m}$ of $G$ is an induced representation, namely induced by the one-dimensional representation

$$\delta \to L_\delta^m = \delta_1^{m_1}\delta_2^{m_2} \dots \delta_n^{m_n} \tag{1}$$

of the subgroup $D$ of $G$. The components of the highest weight $m = (m_1, m_2, \dots, m_n)$ satisfy certain restrictions which we have stated for each class of the classical groups.

DEFINITION 1. An irreducible representation $T^{L^m}$ is called the *Young product* of irreducible representations $T^{L^{m'}}$ and $T^{L^{m''}}$ if

$$L_\delta^m = L_\delta^{m'} L_\delta^{m''}. \blacktriangledown \tag{2}$$

Eq. (2) implies by virtue of eq. 2 (21) that the carrier space $H^{L^m}(Z)$ is the linear envelope of the products of polynomials $p'(z)p''(z)$, where $p'(z) \in H^{L^{m'}}(Z)$ and $p''(z) \in H^{L^{m''}}(Z)$. Note the difference between this Young product and the tensor product $T^{L^{m'}} \otimes T^{L^{m''}}$, whose carrier space is spanned by the products $p'(z')p''(z'')$.

Using the concept of Young product, we can express an arbitrary irreducible representation $T^{L^m}$ in terms of a set of simplest representations, called the *fundamental representations*. In the case of $GL(n, C)$ we can take the following representations to be the fundamental ones (we assume for simplicity that $m_n$ is integer):

$$\begin{aligned}
\overset{1}{m} &= (1, 0, \dots, 0),\\
\overset{2}{m} &= (1, 1, 0, \dots, 0),\\
&\dots\dots\dots\\
\overset{k}{m} &= (\underbrace{1, 1, \dots, 1}_{k}, 0, \dots, 0),\\
&\dots\dots\dots\\
\overset{n}{m} &= (\underbrace{1, 1, \dots, 1}_{n}).
\end{aligned} \tag{3}$$

It is evident from eq. (1) that every other irreducible representation of $GL(n, C)$ is the Young product of representations of the type (3).

In the case of $SL(n, C)$ and $Sp(2n, C)$ the fundamental weights coincide with the first $n-1$ fundamental weights of $GL(n, C)$. In the case of $SO(2\nu+1)$ the fundamental highest weights are:

$$\begin{aligned}
\overset{1}{m} &= (1, 0, 0, \dots, 0), \\
\overset{2}{m} &= (1, 1, 0, \dots, 0), \\
&\cdots\cdots\cdots \\
\overset{\nu-1}{m} &= (1, 1, \dots, 1, 0), \\
\overset{\nu}{m} &= (\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{2}).
\end{aligned} \tag{4}$$

The last weight corresponds to the spinor representation. Finally, in the case of $SO(2\nu)$ we have two spinor representations and the fundamental weights have the form

$$\begin{aligned}
\overset{1}{m} &= (1, 0, \dots, 0), \\
\overset{2}{m} &= (1, 1, 0, \dots, 0), \\
&\cdots\cdots\cdots \\
\overset{\nu-2}{m} &= (1, 1, \dots, 1, 0, 0), \\
\overset{\nu-1}{m} &= (\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{2}), \\
\overset{\nu}{m} &= (\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, -\tfrac{1}{2}).
\end{aligned} \tag{5}$$

If we use the conditions imposed on the components of highest weights corresponding to an inductive character, we obtain

THEOREM 1 (the Cartan theorem). *For every simple, classical Lie group $G$ of rank $n$ there exists $n$ fundamental weights $\overset{i}{m}$, $i = 1, 2, \dots, n$, such that every highest weight $m = (m_1, m_2, \dots, m_n)$ corresponding to the irreducible representation $T^{L^m}$ of $G$ is given by the linear combination*

$$m = \sum_{i=1}^{n} f_i \overset{i}{m}$$

*with non-negative integral coefficients*

$$f_i = m_i - m_{i+1}, \quad i = 1, 2, \dots, n, \; m_{n+1} = 0. \tag{6}$$ ▼

In the current literature the symbol $D^N[f_1, f_2, \dots, f_n]$ is used for labelling an irreducible representation of dimension $N$ with the highest weight $m = f_1 \overset{1}{m} + f_2 \overset{2}{m} + \dots + f_n \overset{n}{m}$ expressed in terms of the fundamental weights. This symbol should be distinguished from the symbol $D^N(m_1, m_2, \dots, m_n)$, where

$m_i$ are the components of the highest weight, which define the inductive character $L^m = \delta_1^{m_1}\delta_2^{m_2} \ldots \delta_n^{m_n}$.

Clearly, by virtue of formula (1) and def. 1, the representation $T^{L^m}$ corresponding to a highest weight $m = (m_1, m_2, \ldots, m_n)$ is the Young product of the fundamental representations $T^{\overset{i}{L^m}}$ each taken $f_i$ times.

## § 7. Representations of Arbitrary Lie Groups

We know, by Levi–Malcev theorem, that an arbitrary simply-connected Lie group $G$ can be represented as a semidirect product

$$G = R \circledS G_S, \tag{1}$$

where $R$ is the maximal, simply-connected, solvable, normal subgroup in $G$ and $G_S$ is a semisimple, simply-connected subgroup of $G$. The subgroup $R$ is called the *radical* of $G$. This property of Lie groups allows us to develop a representation theory for arbitrary Lie groups analogous to that of semisimple groups. We first show an interesting property of the restriction $T_R$ of an irreducible representation $T$ of $G$ to the radical $R$.

THEOREM 1. *Let $g \to T_g$ be an irreducible representation of a simply-connected Lie group $G$ and let $N$ be an arbitrary connected solvable normal subgroup of $G$. Then, for all $n \in N$, we have*

$$T_n = \chi(n)I, \tag{2}$$

*where $\chi$ is the character of $N$ satisfying the condition*

$$\chi(g^{-1}ng) = \chi(n) \quad \textit{for all } g \in G. \tag{3}$$

PROOF: By Lie's theorem there exists in the carrier space $H$ of $T$ a non-zero vector $u_\chi$ and a character $\chi$ of $N$ such that

$$T_n u_\chi = \chi(n)u_\chi \quad \text{for all } n \in N.$$

Because $N$ is a normal subgroup of $G$, we have

$$T_n T_g u_\chi = T_g T_{g^{-1}} T_n T_g u_\chi = \chi(g^{-1}ng) T_g u_\chi, \tag{4}$$

i.e., $T_g u_\chi$ is also an eigenvector of $T_n$, for all $n \in N$. This implies that the restriction $T_N$ of $T_G$ contains together with a character $\chi(n)$ all characters

$$\chi_g(n) \equiv \chi(g^{-1}ng).$$

Repeating now the arguments of the proof of Lie's Theorem (below eq. 1 (3)) we obtain that

$$\chi(g^{-1}ng) = \chi(n) \quad \text{for all } g \in G. \tag{5}$$

Because $T_G$ is irreducible, the invariant linear hull of all vectors $T_g u_\chi$ coincides with the whole space $H$. Hence, $T_n = \chi(n)I$, by eqs. (4) and (5). ▼

As a corollary to th. 1, we obtain

THEOREM 2. *Every irreducible representation $T$ of a simply-connected Lie group $G$ is of the form*

$$T = \chi \otimes \tilde{T},$$

*where $\chi(g)$ is a character of $G$ which is equal to identity for $g \in G_S$, and satisfies the condition* (3), *and $\tilde{T}$ is an irreducible representation of the semisimple group $G_S$.*

PROOF: The restriction $T_R$ of $T_G$ to $R$, by th. 1, is given by the character $\chi(r)$ $r \in R$. By virtue of Levi–Malcev theorem every $g \in G$ can be represented in the form $g = \{r, g_s\}$, $r \in R$, $g_s \in G_s$. Setting $\chi(g) = \chi(\{r, g_s\}) = \chi(r)$ and using eq. (3), we can extend the character $\chi$ onto $G$. The representation of $G$ defined by $\tilde{T} \equiv \chi^{-1} \otimes T$ of $G$ is trivial on $R$. Hence, $\tilde{T}$ provides a representation of the factor group $G/R \simeq G_S$. Consequently, $T = \chi \otimes \tilde{T}$. Clearly, the irreducibility of $T$ implies the irreducibility of $\tilde{T}$. ▼

Thus, the problem of the classification of irreducible representations of an arbitrary, simply-connected Lie group is reduced to the problem of the classification of characters of solvable groups and the classification of irreducible representations of semisimple Lie groups. Furthermore, the representation theory of semisimple Lie groups allows us to give the explicit realization of the operators $T_g$ and the carrier space $H$ of an irreducible representation $T$ of $G$. In fact, if the Levi factor $G_S$ of $G$ is complex, then every irreducible representation of a simply-connected Lie group $G$ is realized by means of the formula

$$U_{\{r, g_s\}} u(z) = \chi(r) L_{\tilde{\delta}} u(z_{\tilde{g}}), \quad g = \{r, g_s\} \in G = R \circledS G_S, \tag{6}$$

where $\chi$ is a character of the radical $R$ satisfying eq. (3), $\delta \to L_\delta$ is the character of the Gauss factor $D$ in the Gauss decomposition $G_S = \overline{\mathfrak{Z} DZ}$ and $\tilde{\delta}$ and $z_{\tilde{g}}$ are determined by the decomposition $\tilde{g} = zg_s = \tilde{\zeta}\tilde{\delta}z_{\tilde{g}}$ of the element $zg_s$. The elements $u(z)$ of the carrier space $H$ are polynomials in the matrix elements $z_{pq}$ of elements $z \in Z$.

If the Levi factor $G_S$ of $G$ is a real semisimple Lie group, then the representation $T_{\{r, g_s\}}$ of $G$ can also be realized by eq. (6); the subgroup $Z$ represents in this case the Gauss factor of the complex extension $(G_S)^c$ of $G_S$.

The following example shows that the condition (3) imposes severe restrictions on the class of admissible characters, and consequently on the class of finite-dimensional irreducible representations of a simply-connected Lie group $G$.

EXAMPLE 1. Let $G = T^4 \circledS \mathrm{SL}(2, C)$, i.e., the Poincaré group. An arbitrary character of $T^4$ has the form

$$\chi(a) = \exp ipa, \quad pa = p_\mu a^\mu, \quad p_\mu \in C^1. \tag{7}$$

From the group multiplication law, we find for ($g = \{b, \Lambda\}$, $a = \{a, I\}$)

$$g^{-1}ag = \{-\Lambda^{-1}b, \Lambda^{-1}\}\{a, I\}\{b, \Lambda\} = \{\Lambda^{-1}a, I\}.$$

Hence, eq. (3) implies

$$\chi(\Lambda^{-1}a) = \chi(a) \quad \text{for all } \Lambda \text{ in } \mathrm{SL}(2, C).$$

Hence $\Lambda p = p$ for all $\Lambda$, by virtue of eq. (7). This is only possible if $p$ is the zero vector. Hence, $\chi(a) = I$.

Consequently, all irreducible, finite-dimensional representations of the Poincaré group are the irreducible finite-dimensional representations of SL(2, $C$) lifted to the Poincaré group by means of eq. (6). ▼

The th. 2 has various interesting consequences. In particular, we have

PROPOSITION 3. *A simply-connected Lie group $G$ admits a non-trivial irreducible unitary representation $T$ with $1 < \dim T < \infty$, if and only if the Levi factor $G_S$ contains a non-trivial compact normal subgroup.*

PROOF: By th. 2, $T = \chi \otimes \tilde{T}$, where $\tilde{T}$ is an irreducible representation of a connected, semisimple subgroup $G_S$ of $G$. Hence, if $G_S$ contains a non-trivial, compact normal subgroup $K$, then choosing $\chi$ to be unitary and $\tilde{T}$ to be an irreducible unitary representation of $K$, we can extend $\tilde{T}$ to $G_S$ and we obtain a unitary irreducible representation $T$ of $G$ with $1 < \dim T < \infty$. If, however, $G_S$ does not contain a compact normal subgroup, then it is a direct product of simple, connected, noncompact groups. Hence, by th. 1.2, $G_S$ does not admit a non-trivial, finite-dimensional unitary representation. ▼

It is interesting that the Weyl theorem can be generalized to arbitrary simply-connected Lie groups:

THEOREM 4. *A representation $T_G$ of a simply-connected Lie group $G$ is fully reducible if and only if its restriction $T_R$ to the radical $R$ is fully reducible.*

PROOF: It is sufficient to consider the case when an invariant subspace $H_1$ of the carrier space $H$ and the factor space $H/H_1$ are irreducible. If the restriction $T_R$ of $T_G$ to the radical $R$ is fully reducible, then there exists a space $H_2 \subset H$, the complement of $H_1$, which is invariant relative to $R$. It follows from th. 1 that the action of the radical $R$ in $H_1$ and $H_2$ reduces to the multiplication by characters $\chi_1$ and $\chi_2$ of $R$, respectively. If $\chi_1 \neq \chi_2$, then $H_2$ is the maximal subspace in $H$, on which the action of the radical $R$ is given by the multiplication by $\chi_2$; the considerations used in the proof of th. 1 show that $H_2$ is invariant also relative to the whole $G$. Hence, $T$ is fully reducible in $H$.

If $\chi_1 = \chi_2 = \chi$ then, setting $\chi(g) = \chi(\{r, g_s\}) = \chi(r)$ and using eq. (3), we can extend the character $\chi$ to the whole group $G$. Taking now the tensor product $\chi^{-1}(g) \otimes T_g$, we obtain the representation of $G$, which is trivial on the radical. Thus, it gives the representation of the semisimple group $G_S \simeq G/R$, which is fully reducible by Weyl's theorem. ▼

## § 8. Further Results and Comments

We discuss now briefly a series of further results which are important for applications.

### A. *Reduction of a Representation to a Subgroup*

In many physical problems the following question arises: what irreducible representations of the subgroup $G_0$ of a group $G$ occur if an irreducible representation $T$ of $G$ is restricted to the subgroup $G_0$?

We first give the main result for $\mathrm{GL}(n, C)$.

THEOREM 1. *An irreducible representatiom of* $\mathrm{GL}(n, C)$ *determined by the highest weight* $m = (m_1, m_2, \ldots, m_n)$, *restricted to the subgroup* $G_0 \simeq \mathrm{GL}(n-1, C)$ *contains all irreducible representations of* $G_0$ *with highest weights* $l = (l_1, \ldots, l_{n-1})$ *for which the following conditions are satisfied*

$$m_1 \geqslant l_1 \geqslant m_2 \geqslant l_2 \geqslant m_3 \geqslant \ldots \geqslant m_{n-1} \geqslant l_{n-1} \geqslant m_n. \tag{1}$$

*Every irreducible component occurs with multiplicity one.* ▼

(For the proof cf. e.g. Želobenko 1962, § 13.)

By virtue of th. 3.1 this theorem is also true for all real forms of $\mathrm{GL}(n, C)$ and in particular for the reduction of irreducible representations of the unitary group $U(n)$ with respect to $U(n-1)$.

For orthogonal groups we have a similar result.

THEOREM 2. *An irreducible representation of* $\mathrm{SO}(2\nu+1, C)$ *determined by the highest weight* $m = (m_1, m_2, \ldots, m_\nu)$ *with integer (half-integer) components restricted to the subgroup* $G_0 \simeq \mathrm{SO}(2\nu, C)$ *contains all irreducible representations of* $G_0$ *with highest weights* $q = (q_1, q_2, \ldots, q_\nu)$ *for which the following conditions are satisfied*

$$m_1 \geqslant q_1 \geqslant m_2 \geqslant q_2 \geqslant \ldots \geqslant m_\nu \geqslant q_\nu \geqslant -m_\nu. \tag{2}$$

*The components* $q_i$ *are simultaneously all integers (if* $m_i$ *are integers) or all half-odd integers (if* $m_i$ *are half-odd integers). Every irreducible representation occurs with multiplicity one.*

*Similarly, the restriction of the irreducible representations of* $\mathrm{SO}(2\nu)$ *determined by the highest weight* $m = (m_1, m_2, \ldots, m_\nu)$ *with integral (or half-odd integral) components contains all irreducible representations of the subgroup* $G_0 \simeq \mathrm{SO}(2\nu-1)$ *with the highest weights* $p = (p_1, p_2, \ldots, p_{\nu-1})$ *for which*

$$m_1 \geqslant p_1 \geqslant m_2 \geqslant p_2 \ldots \geqslant m_{\nu-1} \geqslant p_{\nu-1} \geqslant |m_\nu|.$$

*The components* $p_i$ *are simultaneously all integers (all half-odd integers) together with* $m_i$. *Every irreducible representation occurs with multiplicity one.* ▼

(For the proof cf. Želobenko 1962, § 13.)

Clearly, all statements of th. 2 hold for real forms of $\mathrm{SO}(n, C)$ and in particular for orthogonal real groups $\mathrm{SO}(n)$ and $\mathrm{SO}(p, q)$, $p+q = n$.

Analogous, but more complicated results, hold for symplectic groups.

The proofs of these theorems can be given by various methods. In particular, an elementary proof can be given by means of Young diagrams (cf. Hamermesh 1962, ch. 10).

The technique of induced representations used by Želobenko 1962 allows one not only to prove ths. 1 and 2 in an elegant way, but also to construct the carrier spaces in which irreducible representations of the corresponding subgroups are realized.

The problem of reduction of representations of $SU(m+n)$ with respect to $SU(m)\times SU(n)$ was considered by Hagen and Macfarlane 1966. Some special cases of reduction of $SU(m+n)$ with respect to $SU(m)\times SU(n)$ and $SU(n)$ with respect to $SO(n)$ were considered by Želobenko 1970, ch. XVIII. These problems were also treated by Whippman 1965.

## B. *Weight Diagrams*

Let $T^{L^m}$ be an irreducible representation of a semisimple Lie group $G$ corresponding to the highest weight $m$ and let $u_m(z)\equiv 1$ be the highest vector in the carrier space $H^m(Z)$ of $T^{L^m}$. Denoting the generators of the subgroup $Z(\mathfrak{Z})$ by $E_\alpha(E_{-\alpha})$ and the generators of the subgroup $D$ by $H_i$ and using the Cartan–Weyl commutation relations, one obtains

$$H_i E_{\pm\alpha} u_m = \big(E_{\pm\alpha} H_i \pm \alpha(H_i) E_{\pm\alpha}\big) u_m = \big(m_i \pm \alpha(H_i)\big) E_{\pm\alpha} u_m. \tag{3}$$

The eigenvectors of $H_i$ are called the *weight vectors* and the eigenvalues are the components of a *weight*. Hence, the vectors $E_{\pm\alpha} u_m$ are, together with $u_m$, formally also the weight vectors in the carrier space $H^m$. However, the action of the subgroup $Z$ in $H^m$ implies (cf. eq. 2(17))

$$T^{L^m}_{z_0} u_m(z) = u_m(zz_0) = u_m, \tag{4}$$

or, infinitesimally,

$$E_\alpha u_m = 0. \tag{5}$$

Hence by eq. (3), there cannot be a weight vector in $H^m$ with the weight $m' = \big(m_1+\alpha(H_1), \ldots, m_n+\alpha(H_n)\big)$.* This explains the names 'highest weight' and 'highest vector' for $m$ and $u_m$, respectively.

Let

$$v = E_{-\alpha^{(1)}} E_{-\alpha^{(2)}} \ldots E_{-\alpha^{(s-1)}} E_{-\alpha^{(s)}} u_m, \qquad s = 1, 2, \ldots, \tag{6}$$

where the generators $E_{-\alpha}$ and $E_\alpha$ are in an arbitrary order. Then, eq. (3) implies

$$H_i v = \big(m_i - \alpha^{(1)}(H_i) - \alpha^{(2)}(H_i) + \ldots + \alpha^{(s-1)}(H_i) - \alpha^{(s)}(H_i)\big) v, \tag{7}$$

i.e., every non-zero vector (6) is a weight vector. Now, the highest vector $u_m$ is cyclic for the representation $T^{L^m}$, by corollary 3 to th. 2.2. Hence, the weight vectors (6) span the carrier space $H^m$ of $T^{L^m}$.

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* We say that a weight $m'$ is *higher* than $m$ if the first non-vanishing component of the vector $m'-m$ is positive.

By virtue of eq. (7), an arbitrary weight has the form

$$m-k_1\alpha_1-k_2\alpha_2-\ldots-k_n\alpha_n, \tag{8}$$

where $k_i$ are non-negative integers and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are simple roots and $n$ is the dimension of the Cartan subalgebra. We can restrict ourselves, in eq. (8), to simple roots only, due to the fact that every positive root is a sum of simple roots with non-negative coefficients. Because the dimension of $H^m$ is finite, the number of different weights is finite.

It is convenient to associate with every weight (8) a point of an $n$-dimensional vector space $R^n$. The diagram in $R^n$ corresponding to the collection of all weights is called the *weight diagram of a given representation*.

Eq. (7) implies, in particular, that all generators $H_i$ of the Cartan subgroup $D$ are diagonal in the space $H^m$. In physical applications, if $G$ is a symmetry group of some physical system, the generators $H_i$ are simultaneously diagonalizable, hence observables. For instance, in the case of the SU(3)-symmetry in particle physics, the generator $H_1$ can be taken to be the third component of isospin, and the generator $H_2$ to be the hypercharge. Hence, the weights give the values of measurable quantities.

It is crucial for applications to determine all weights associated with a given highest weight and their multiplicities. This problem was solved by Freudenthal (cf. Freudenthal and De Vries 1969) and Kostant 1959.

In order to state the Freudenthal theorem, it is necessary to introduce a scalar product for roots and weights. Notice first, that roots and weights are elements of the dual space $H^*$ of the Cartan algebra $H$. On the other hand, by formula 1.4(3), for every $\lambda \in H^*$, there is a uniquely determined element $H_\lambda \in H$ such that

$$\lambda(X) = (H_\lambda, X) \quad \text{for all } X \in H, \tag{9}$$

where $(\cdot, \cdot)$ is the Killing form of the Lie algebra $L$ of $G$. When $\lambda, \mu \in H^*$, the scalar product $(\lambda, \mu)$ can be defined by

$$(\lambda, \mu) = (H_\lambda, H_\mu). \tag{10}$$

Because $G$ is semisimple, by th. 1.4.1, the restriction of the Killing form on $H$, and consequently, the scalar product (10) are non-degenerate. The Freudenthal formula expresses the multiplicity $n_M$ of a weight $M$ in terms of multiplicities of weights $M+k\alpha$, $\alpha > 0$. Explicitly, we have

THEOREM 3. *The multiplicity $n_M$ of a weight $M$ in the weight diagram associated with a highest weight $m$ is given by the recursion formula*

$$[(m+r, m+r)-(M+r, M+r)]n_M = 2\sum_{k=1}^{\infty}\sum_{\alpha>0} n_{M+k\alpha}(M+k\alpha, \alpha), \tag{11}$$

*where*

$$r = \frac{1}{2}\sum_{\alpha>0}\alpha. \blacktriangledown$$

(For the proof cf. Freudenthal and De Vries 1969.)

The formula (11) gives an effective method of calculating the multiplicity $n_M$ of a weight $M$ starting with $n_m = 1$. Clearly, by virtue of eq. (8) the summation over $k$ in eq. (11) is finite.

We now introduce the Weyl group in order to give the Kostant formula. Let $\mu$ be a vector from the weight (or root) space and let $\alpha$ be a root. Set

$$\mu' \equiv S_\alpha(\mu) \equiv \mu - \frac{2(\mu, \alpha)}{(\alpha, \alpha)}\alpha. \tag{12}$$

Because $S_\alpha(\alpha) = -\alpha$, and for $\mu \perp \alpha$, $S_\alpha(\mu) = \mu$ the map $\alpha \to S_\alpha$ is a reflection with respect to the hyperplane perpendicular to the vector $\alpha$. Clearly, $S_\alpha^2 = I$ and $S_\alpha$ is an orthogonal transformation, i.e.,

$$\big(S_\alpha(\mu_1), S_\alpha(\mu_2)\big) = (\mu_1, \mu_2). \tag{13}$$

The linear group generated by the transformations $S_{\alpha_i}$, $\alpha_i$ = simple roots, is called the *Weyl group W*. If $s \in W$ is represented by an even product of reflections relative to hyperplanes perpendicular to roots, then $\det S = 1$. Otherwise $\det S = -1$.

THEOREM 4. *Let $P(R)$, $R \in H^*$, be a partition function, which is equal to the number of solutions $(k_\alpha, k_\beta, \ldots, k_\zeta)$ of the equation*

$$R = \sum_{\alpha>0} k_\alpha \alpha, \tag{14}$$

*where $\alpha$'s are positive roots and $k_\alpha$'s are non-negative integers. Then, the multiplicity $n_M$ of a weight $M$ associated with a highest weight $m$ is given by the formula*

$$n_M = \sum_{S\in W} (\det S) P[S(m+r)-(M+r)]. \blacktriangledown \tag{15}$$

(For the proof cf. Kostant 1959a.)

The Kostant formula is useful rather in theoretical considerations. It is tedious in practical calculations because there is no effective method of computing the partition function $P(R)$.

In the case of the algebras of rank 2 and the algebra $A_3$, the explicit expressions for the function $P(R)$ were found by Janski 1963.

The other formulas for multiplicities of weights, sometimes more convenient for applications, were found by Klimyk 1966 and 1967a, b.

C. *Decomposition of the Tensor Product*

The problem of the decomposition of a tensor product $T \otimes T'$ of irreducible representations $T$ and $T'$ of a topological group $G$ is called the *'Clebsch–Gordan series' problem.*

First, we consider the case of $\mathrm{GL}(n, C)$. We define, on representations

$$L^m = \delta_1^{m_1} \delta_2^{m_2} \dots \delta_n^{m_n},$$

of the subgroup $D$, the following operators $\hat{\delta}_k$, $k = 1, 2, \dots, n$,

$$\hat{\delta}_k L^m \equiv \begin{cases} \delta_1^{m_1} \delta_2^{m_2} \dots \delta_k^{m_k+1} \dots \delta_n^{m_n}, & \text{if } m_{k-1} > m_k, \\ 0, & \text{if } m_{k-1} = m_k. \end{cases} \tag{16}$$

Note that the operators $\hat{\delta}_k$ are noncommutative. The following theorem gives the general formula for the decomposition of the tensor product $T^{L^m} \otimes T^{L^{m'}}$ of irreducible representations in terms of the operators $\hat{\delta}_k$:

THEOREM 5. *Let $T^{L^m}$ and $T^{L^{m'}}$ be irreducible representations of* $\mathrm{GL}(n, C)$ *induced by the representations $L^m$ and $L^{m'}$ of the subgroup $D$, respectively. Then the tensor product $T^{L^m} \otimes T^{L^{m'}}$ reduces to*

$$\sum_{m''} \oplus T^{L^{m''}}$$

*where $L^{m''}$ are summands in the expansion of the following determinant*

$$\Gamma_{m_1 m_2 \dots m_n} L^{m'} \equiv \begin{vmatrix} \Gamma_{m_1} & \Gamma_{m_1+1} & \dots & \Gamma_{m_1+(n-1)} \\ \Gamma_{m_2-1} & \Gamma_{m_2} & \dots & \Gamma_{m_2+(n-2)} \\ \dots & \dots & \dots & \dots \\ \Gamma_{m_n-(n-1)} & \Gamma_{m_n-(n-2)} & \dots & \Gamma_{m_n} \end{vmatrix} \tag{17}$$

*where*

$$\Gamma_m = \sum_{\nu_1+\nu_2+\dots+\nu_n=m} \hat{\delta}_1^{\nu_1} \hat{\delta}_2^{\nu_2} \dots \hat{\delta}_n^{\nu_n}, \qquad \Gamma_m = 0 \quad \textit{for } m < 0. \blacktriangledown$$

(For the proof cf. Želobenko 1963, th. 12.)

In the following, for simplicity, we shall use the symbol $m \otimes m'$ for the tensor product $T^{L^m} \otimes T^{L^{m'}}$.

As examples, we consider two important classes of tensor products of representations of $\mathrm{GL}(n, C)$.

1° Multiplication of the vector $\overset{(1)}{m}$ by a tensor

$$m' = (m'_1, m'_2, \dots, m'_n), \quad \min(m'_i - m'_j) \geqslant 1.$$

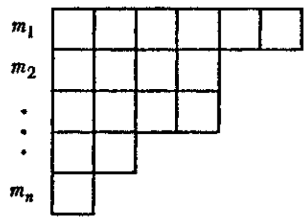
The formula (17) gives immediately

$$\Gamma_{1, 0, \dots, 0} L^{m'} = \hat{\delta}_1 L^{m'} + \hat{\delta}_2 L^{m'} + \dots + \hat{\delta}_n L^{m'}. \tag{18}$$

We will now give a useful graphical representation of these results. First note that to each highest weight

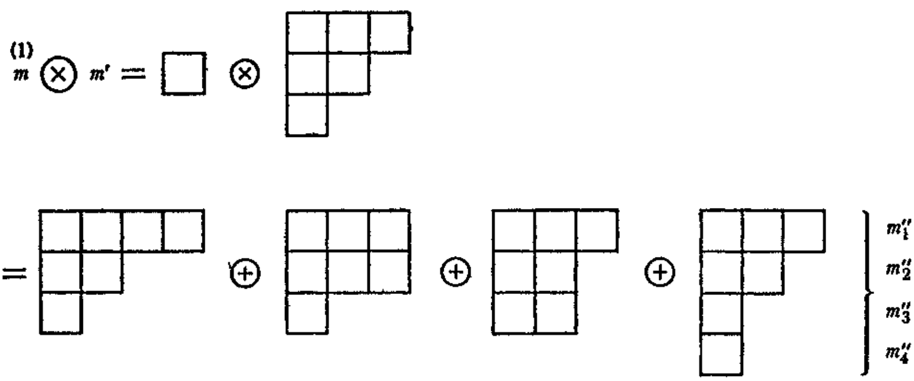
$$m = (m_1, \dots, m_n), \qquad m_1 \geqslant m_2 \geqslant \dots \geqslant m_n, \tag{19}$$

we can associate the following diagram




where the first row contains $m_1$ boxes, the second $m_2$ boxes, etc. It follows from (19) that the lengths of successive rows are non-increasing and the number of rows is at most $n$. Such diagrams are called admissible. They are in one-to-one correspondence with the highest weights of irreducible representations. These diagrams are precisely the Young frames characterizing the irreducible representations of the permutation group (cf. ch. 7.5.C).

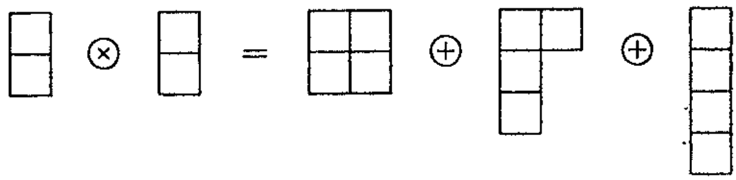
Using the Young diagrams, we can illustrate eq. (18) graphically as follows, e.g., for $m' = (3, 2, 1, 0)$, one obtains




2° Tensor product of two polyvectors: let $\overset{(l)}{m}$ and $\overset{(k)}{m}$ denote the highest weight associated with polyvectors. Then, th. 5 gives immediately

$$\Gamma_{(\underbrace{11 1\dots1}_{l}\ 000\dots0)} \overset{(k)}{L^m} = \overset{(l)}{L^m}\overset{(k)}{L^m} + \overset{(l+1)}{L^m}\overset{(k-1)}{L^m} + \dots + \overset{(l+k)}{L^m} L^0. \tag{20}$$

This result can also be illustrated graphically by means of Young frames, e.g.,

$\overset{(2)}{m} \otimes \overset{(2)}{m} =$ 

These two examples suggest that there should exist a graphical method of the decomposition of an arbitrary tensor product $m \otimes m'$. Indeed, by virtue of eq. (16), and the fact that a component $m_i$ of $m$ is represented as the length of the $i$th row in the Young frame, the formula (17) yields the admissible Young frames of the irreducible representations occurring in the decomposition. The general rule can be stated in the following manner:

We take one Young frame corresponding to the highest weight $m$ as fixed, and label the rows of the second frame as follows, for example

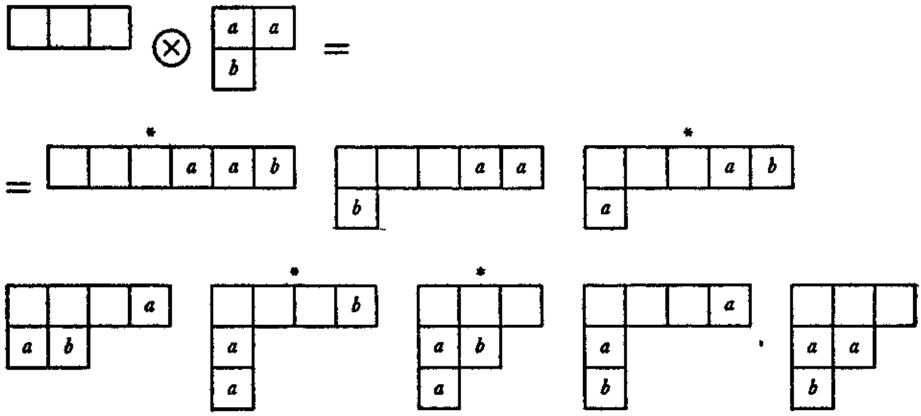
| | | | |
|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $b$ | |
| $c$ | $c$ | | |

We add now the boxes of the first row labelled '$a$' to the first frame in all possible ways so as to obtain again admissible frames. To all the frames thus obtained we add next the boxes of the second row, then the third row, etc., in each step requiring that the resulting frames are admissible (i.e., $m_k'' \geqslant m_{k-1}''$).

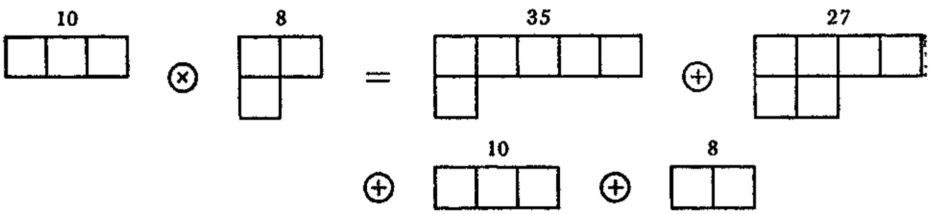
From this set of frames we eliminate those frames which contain equal labels appearing in a column. Next we drop the columns of length $n$ from the tableau. Finally, we order the boxes of the frames: We start with the first row and take the boxes in the order from right to left. Then we run through the second row from right to left and so on. This ordered sequence contains boxes with labels and empty boxes. If we cut this sequence at any point, the number of labels $b$ must not exceed the number of $a$'s, the number of $c$'s must not exceed the number of $b$'s, etc., counted from the start until the cut.

The resulting frames, which differ in form or in the places of the labels, correspond to distinct irreducible representations contained in the tensor product.

EXAMPLE 1. Let $G = \mathrm{SU}(3)$. Consider the tensor product of the representations $m = (3, 0)$ and $m' = (2, 1)$. The rule of adding of boxes of the second frame corresponding to $m'$ to the first frame corresponding to $m$ gives the following set of frames:




The frames with '*' are forbidden by our rules. Hence, we obtain




We have put over the frames the dimension of the resulting irreducible representations obtained from the Weyl formula (29) below (for the detailed proof of these rules of decomposition of tensor product $m \otimes m'$, see Itzykson and Nauenberg 1966 or Boerner 1963). Clearly, th. 1, by virtue of the th. 3.1, is valid for $GL(n, R)$, $U(n)$, $SL(n, C)$, $SL(n, R)$ and $SU(n)$.

A general formula for the decomposition of the tensor product of irreducible representations of an arbitrary semisimple Lie group $G$ was obtained by Kostant and Steinberg.

THEOREM 6. *Let $m$ and $m'$ be two irreducible representations of a semisimple Lie group. Then, the multiplicity $n_{m''}$ of the irreducible representation $m''$ in the tensor product $m \otimes m'$ is given by the formula*

$$n_{m''} = \sum_{S,T \in W} \det(ST) \cdot P\{S(m+r)+T(m'+r)-(m''+2r)\}, \tag{21}$$

*where $W$ is the Weyl group of $G$, $r = \frac{1}{2}\sum_{\alpha>0} \alpha$ and the partition function $P(R)$ is the same as defined in th. 4.* ▼

(For the proof cf. Steinberg 1961.)

The use of formula (21) is tedious even for low-dimensional Lie algebras. Fortunately, special computer programs has been elaborated by Pajas and tables of multiplicities for most important groups are published (Pajas 1967).

Some variants of formulae (21) for multiplicities were derived by Straumann 1965 and Klimyk 1966. There exists also an interesting graphical method elaborated by Speiser 1964. For more recent work we refer to Gruber 1968.

The problem of the decomposition of the tensor product $m \otimes m'$ onto irreducibles is complete if we can give a method of separation of the carrier space $H^{m''}$ in which the irreducible representation $m''$ is realized. To solve this problem, it is sufficient to express the basis vectors $e_k''$, $k = 1, 2, \ldots, \dim H^{m''}$, of $H^{m''}$ in terms of the basis vectors $e_i e_j'$ of the tensor product space $H^m \otimes H^{m'}$, i.e.,

$$e_k'' = c_k{}^{ij} e_i e_j'. \tag{22}$$

The coefficients $c_k{}^{ij}$ are called the *Clebsch–Gordan coefficients*. Clearly, they depend on the bases in the space $H^m$, $H^{m'}$ and $H^{m''}$, respectively. Unfortunately, we know the explicit form of the Clebsch–Gordan coefficients only in few cases: for SU(2) (cf., e.g., Edmonds 1957), SL(2, $C$) (cf., e.g., Gel'fand *et al.* 1958) and SU($n$), e.g., De Swart 1963 for SU(3) and Shelepin 1967 for SU($n$).

Ths. 5 and 6 were derived by means of algebraic methods. They can also be obtained by means of global methods. Indeed, one could use the tensor product theorem for global induced unitary irreducible representations of compact groups (cf. ch. 18, § 2). Then, using th. 3.1, one would extend these results for all noncompact complex and real semisimple groups associated with a given compact group. In all known cases this method is very elegant and effective.

## D. *Characters and Dimensions of Representations*

The *character of a representation* $T$ of $G$ is defined by the formula

$$\chi(\delta) = \operatorname{Tr} T_\delta, \qquad \delta = \text{Gauss factor of } g \text{ in } g = \zeta\delta z. \tag{23}$$

This fundamental concept was introduced by Weyl. The character (23) does not have the multiplicative property $\chi(\delta+\delta') = \chi(\delta)\chi(\delta')$, valid for the characters of abelian groups. Weyl has derived a general formula for the characters of irreducible representations of all simple Lie groups.

THEOREM 7. *Let $T^{L^m}$ be an irreducible representation of $G$ determined by the highest weight $m = \sum_i f_i \overset{i}{m}$. Let $k = r+m$, where $r = \frac{1}{2}\sum_{\alpha>0} \alpha$ and the sum is taken over positive roots. Set*

$$\xi(m) = \sum_{\dots} \det S \exp[\mathrm{i}(Sk)\delta], \tag{24}$$

*where $W$ is the Weyl group (defined by eqs.* (12) *and* (13)) *and* $a\delta = a_1\delta_1 + a_2\delta_2 + \ldots + a_n\delta_n$.* *Then,*

$$\chi^m(\delta) = \frac{\xi(m)}{\xi(0)}. \blacktriangledown \tag{25}$$

(For the proof cf. Weyl 1925.)

The formula (25) gives simpler expressions in the case of classical groups. In particular, for GL($n$, $C$), we have (cf. Weyl 1939, ch. VII.6)

$$\chi^m(\delta) = \frac{d(l_1, l_2, \ldots, l_n)}{d(n-1, n-2, \ldots, 0)}, \tag{26}$$

where $l_i = m_i + n - i$ and $d(l_1, l_2, \ldots, l_n)$ is the determinant

$$\begin{vmatrix} \delta_1^{l_1} & \delta_1^{l_2} & \ldots & \delta_1^{l_n} \\ \delta_2^{l_1} & \delta_2^{l_2} & \ldots & \delta_2^{l_n} \\ \ldots & \ldots & \ldots & \ldots \\ \delta_n^{l_1} & \delta_n^{l_2} & \ldots & \delta_n^{l_n} \end{vmatrix}. \tag{27}$$

Clearly, the formula (26) provides also a character for all real forms of GL($n$, $C$) and in particular for $U(n)$. In these cases $T_\delta = \exp(i\delta_j H_j)$ where $H_j$ are generators of Cartan subalgebra.

The explicit forms of the characters for the symplectic and orthogonal groups were also given by Weyl 1939, ch. VI, 8 and 9 respectively.

The formula (25) for characters can be used to calculate the dimension $N^m$ of an irreducible representation $T^{L^m}$ of $G$.

Indeed, because

$$N^m = \chi^m(e), \tag{28}$$

one obtains the dimension of $T^{L^m}$ by the limiting procedure $\delta \to e$. This gives

THEOREM 8.

$$N^m = \frac{\prod\limits_{\alpha>0} (\alpha, r+m)}{\prod\limits_{\alpha>0} (\alpha, r)}, \tag{29}$$

*where the multiplication is taken over all positive roots.*

(For the proof cf. Weyl 1934.)

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* $Sm = m - \frac{2m \cdot r}{r \cdot r} r$ is the reflection of the weight vector $m$ through a hyperplane with the normal $r$.

The formula (29) in case of GL($n$, $C$) (and in particular for $U(n)$) becomes

?

[illegible] ?

$$N^m = \frac{\prod_{i<j}(l_i - l_j)}{\prod_{i<j}(l_i^0 - l_j^0)}, \tag{30}$$

where $l_j = m_j + n - j$ and $l_j^0 = m_j - j$.

### E. *Comments*

The proof of Lie's theorem in the form given here was elaborated by Godement 1956, appendix. The properties of the representations of semisimple Lie groups were investigated by Cartan 1914 and Weyl (cf., e.g., Weyl 1934–35 or 1939). The possibility of using Lie's theorem as a tool for a global classification of irreducible representations of semisimple Lie groups, was demonstrated by Godement 1956, appendix. This approach was used and extended to arbitrary Lie groups by Želobenko 1962, 1963. Here we followed the approach of Želobenko.

Note that every irreducible representation of a semisimple Lie group $G$ is induced by the character of the subgroup $D$. We show in ch. 19 that infinite-dimensional representations of simple complex Lie groups are also induced by a complex character $L$ of $D$.

## § 9. Exercises

§ 1.1. Show that the Heisenberg algebra

$$[Q_i, P_j] = \delta_{ij} I, \quad i,j = 1, 2, \dots, n, \tag{1}$$

has no finite-dimensional irreducible representations.

*Hint*: Use the fact that $\mathrm{Tr}[A, B] = 0$ for matrices.

§ 1.2. Construct the three-dimensional representation of the Lie algebra:

$$[Q, P] = Z, \quad [Q, Z] = 0, \quad [P, Z] = 0 \tag{2}$$

§ 2.1. Let $G = \mathrm{SU}(2)$ and let $H^J$, $J = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$, be the space of all homogeneous monomials of degree $2J$

$$f(z_1, z_2) = \sum_{M=-J}^{J} a_M z_1^{J-M} z_2^{J+M}, \quad a_M \in C. \tag{1}$$

Set

$$u(z) = f(z, 1), \quad g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha = \bar{\delta}, \quad \gamma = -\bar{\beta} \tag{2}$$

and show that the transformation

$$T_g^J f(z_1, z_2) = f(\alpha z_1 + \gamma z_2, \beta z_1 + \delta z_2) \tag{3}$$

implies the transformation

$$(T_g^J u)(z) = (\beta z + \delta)^{2J} u\left(\frac{\alpha z + \gamma}{\beta z + \delta}\right). \tag{4}$$

which are spinor representations of SU(2) of weight $J$.

§ 2.2. Show that there exist a scalar product in $H^J$ with respect to which the vectors

$$\psi_M^J(z) = [(J-M)!(J+M)!]^{-\frac{1}{2}} z^{J-M} \equiv |JM\rangle \tag{5}$$

are orthonormal

*Hint.* Use th. 7.1.1.

§ 2.3. Show that the representation (4) is irreducible in $H^J$, and that every irreducible representation of SU(2) is equivalent to $T^J$.

§ 2.4. Show that on the space of functions $u(z)$ of problem 1.1 the generators of SU(2) are given by the differential operators

$$\begin{aligned} J_+ u &= -\frac{\mathrm{d}}{\mathrm{d}z} u, \\ J_- u &= z^2 \frac{\mathrm{d}}{\mathrm{d}z} u - 2Jzu, \\ J_3 u &= -z \frac{\mathrm{d}}{\mathrm{d}z} u + Ju. \end{aligned} \tag{6}$$

§ 2.5. Let $G = \mathrm{SL}(2, C)$ and $H$ be the space of polynomials $p(z, \bar{z})$ of degree $\leqslant m$ in $z$ and of degree $\leqslant n$ in $\bar{z}$. The irreducible spinor and tensor representations of $G$ of dimension $(m+1)(n+1)$ in $H$ can be written as

$$\begin{aligned} T(g)p(z, \bar{z}) &= (\beta z + \gamma)^m (\bar{\beta}\bar{z} + \bar{\delta})^n p\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \frac{\bar{\alpha}\bar{z} + \bar{\gamma}}{\bar{\beta}\bar{z} + \bar{\delta}}\right), \\ g &= \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \alpha\delta - \beta\gamma = 1. \end{aligned} \tag{7}$$

The equivalence of this form to two-component spinors with dotted and undotted indices is obtained if we represent $p(z, \bar{z})$ as follows:

$$p(z, z) = \sum_{A_i, \dot{B}_i = 0}^{1} \psi_{A_1 \dots A_m \dot{B} \dots \dot{B}_n} z^{(A_1 + \dots + A_m)} \bar{z}^{(\dot{B}_1 + \dots + \dot{B}_n)} \tag{8}$$

where the $(m+1)(n+1)$-dimensional 2-component spinor $\psi$ is symmetric in the indices $A_k$, and symmetric in the indices $\dot{B}_k$. Show that under $T(g)$ the $\psi$'s transform as

$$\psi'_{A_1 \dots A_m \dot{B}_1 \dots \dot{B}_n} = g_{A_1 C_1} \cdots g_{A_m C_m} \bar{g}_{\dot{B}_1 \dot{D}_1} \cdots \bar{g}_{\dot{B}_n \dot{D}_n} \psi_{C_1 \dots C_m \dot{D}_1 \dots \dot{D}_n} \tag{9}$$

where

$$g_{11} = \alpha, \quad g_{12} = \beta, \quad g_{21} = \gamma, \quad g_{22} = \delta.$$

*Remark*: The irreducible representations of SL(2, $C$) given by (7) are denoted by $D^{(m,n)}$.

§ 2.6. Let $\sigma_k$ be the Pauli matrices (cf. example 1.1.1). Show that the matrices

$$J_k = \tfrac{1}{2}\sigma_k, \quad N_k = \tfrac{1}{2}\mathrm{i}\sigma_k, \quad k = 1, 2, 3 \tag{10}$$

are generators of the representation $D^{(1/2,0)}$ of the Lorentz group. Show that the matrices

$$J_k = \tfrac{1}{2}\sigma_k, \quad N_k = -\tfrac{1}{2}\mathrm{i}\sigma_k \tag{11}$$

are generators of the representation $D^{(0,1/2)}$ of the Lorentz group.

§ 2.7. Let $\{\gamma_\mu\}_0^3$ be the set of Dirac matrices given by

$$\gamma_0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \gamma_k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}. \tag{12}$$

Show that the matrices

$$M_{\mu\nu} = \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{13}$$

are generators of the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of the Lorentz group in block-diagonal form

$$D(\Lambda) = \begin{bmatrix} D^{(1/2,0)}(\Lambda) & 0 \\ 0 & D^{(0,1/2)}(\Lambda) \end{bmatrix}. \tag{14}$$

§ 2.8. Show that in the so-called Dirac representation of $\gamma$ matrices given by

$$\gamma_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix} \tag{15}$$

the generators (13) have the form

$$J_k = \varepsilon_{klm} M_{lm} = \frac{\mathrm{i}}{2}\begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}, \quad N_k = M_{0k} = \frac{1}{2}\begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix}.$$

§ 3.1. Show that $SU(3)/Z_3$ has self-conjugate representations of dimensions $1, 8, 27, 64, \ldots, n^3, \ldots$ only. Show in particular that the fundamental representations of dimension 3 of SU(3) (so-called *quark representations*) are not representations of $SU(3)/Z_3$.

§ 7.1. Classify the finite-dimensional representations of the Euclidean group $T^3 \otimes SO(3)$

§ 8.1. Let $H^m$ be the carrier space of the irreducible representation of $u(n)$ characterized by the highest weight $m = (m_{1n}, m_{2n}, \ldots, m_{nn})$. Show that every vector in $H^m$ may be represented by a pattern given by

$$m = \begin{vmatrix} m_{1n} \cdots\cdots\cdots\cdots m_{nn} \\ m_{1,n-1} \cdots m_{n-1,n-1} \\ \cdots\cdots\cdots \\ m_{12} \;\; m_{22} \\ m_{11} \end{vmatrix},$$

where $m_{ij}$ satisfy the condition

$$m_{ij} \geqslant m_{i,j-1} \geqslant m_{i+1,j}, \quad i = 1, 2, \ldots, n-1, \; j = 2, 3, \ldots, n.$$

*Hint*: Use the th. 8.8.1.

§ 8.2. Let $H^2$ be the carrier space of the irreducible representation of $u(2)$ characterized by the highest weight $m = (m_{12}\, m_{22})$. Show that the representation of the generators $A_i^j$, $i, j = 1, 2$, is given by the formula

$$A_{kk}m = (r_k - r_{k-1})m \;\; (k = 1, 2), \quad A_{21}m = a_1^1(m)_1^1, \quad A_{12}m = b_1^1(m)m_1^{\hat{1}},$$

where $r_0 = 0$, $r_k = \sum_{j=1}^{k} m_{jk}$ $(k = 1, 2)$, and

$$a_1^1(m) = \left[\frac{\prod_{i=1}^{2}(l_{i2} - l_{11} + 1)}{(l_{21} - l_{11} + 1)(l_{21} - l_{11})}\right]^{1/2},$$

$$b_1^1(m) = \left[-\frac{\prod_{i=1}^{2}(l_{i2} - l_{11})}{(l_{21} - l_{11})(l_{21} - l_{11} - 1)}\right]^{1/2},$$

with

$$l_{ik} = m_{ik} - i, \quad m_1^1 = \begin{vmatrix} m_{12} & m_{22} \\ \multicolumn{2}{c}{m_{11}+1} \end{vmatrix}, \quad m_1^{\hat{1}} = \begin{vmatrix} m_{12} & m_{22} \\ \multicolumn{2}{c}{m_{11}-1} \end{vmatrix}.$$

§ 8.3. Using the graphical method show that the tensor product of a fundamental representation $T^{\overset{p}{m}}$, $\overset{p}{m} = (\underbrace{1, \ldots, 1}_{p}, 0, \ldots, 0)$, with an arbitrary representation $T^m$, $m = (m_1, \ldots, m_n)$ of $U(n)$ has the following decomposition:

$$T^{\overset{p}{m}} \otimes T^m = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n} \oplus T^{m' = (m_1, \ldots, m_{i_1}+1, \ldots, m_{i_p}+1, \ldots, m_n)}.$$

§ 8.4. Show that the irreducible representations of a semisimple Lie group $G$ form a semigroup with respect to multiplication of highest weights.

§ 8.5. Find unitary irreducible infinite-dimensional discontinuous representations of SU$(n)$.

*Hint.* Take the discontinuous representations of the abelian subgroup $D$ of SL$(n, C)$ given e.g. by example 5.1.3, induce them to SL$(n, C)$ and then restrict them to obtain representations of SU$(n)$.

# Chapter 9

# Tensor Operators, Enveloping Algebras and Enveloping Fields

Tensor operators $\{T_a\}$ and tensor-field operators $\{T_a(x)\}$ associated with group representations play a fundamental role in quantum theory. Physical quantities like the angular momentum $J$, the energy-momentum four-vector $P_\mu$, spin, fields and currents are identified with objects of this kind. The description of most physical phenomena in quantum theory reduces therefore to an analysis of the properties of certain tensor operators.

In sec. 1 we describe the basic properties of tensor operators and derive the Wigner–Eckart theorem.

In sec. 2 we discuss the basic properties of the enveloping algebra, which is in principle an algebra of the tensor operators. The properties of the invariant operators which are, in fact, the simplest tensor operators are given in sec. 3.

If a given group $G$ is a symmetry group of some physical system, then the spectra of the invariant operators associated with $G$ determine the observable quantum numbers of the physical system. Therefore, from the point of view of physical applications we are deeply interested to find explicitly:

(i) The set $\{C_p\}$ of independent invariant operators which generates the ring of the invariant operators in the enveloping algebra $E$ of the Lie algebra $L$ of $G$.

(ii) The spectra of these independent invariant operators $C_p$.

In sec. 4 we give an explicit solution to problems (i) and (ii) for all classical simple Lie algebras.

Finally, in sec. 5 we discuss the important concept of the enveloping field of a Lie algebra and in particular the famous Gel'fand–Kirillov theorem on the generators of the enveloping field.

## § 1. The Tensor Operators

In the quantum theory of atomic and nuclear spectroscopy a set of operators $\{T^J_m\}$, $m = -J, -J+1, \ldots, J-1, J$, appear which transform under the rotation group SO(3) as the spherical harmonics $Y^J_m(\vartheta, \varphi)$ (or, as the state vectors), i.e.,*

* We use the definition of tensor operators due to Wigner 1959.

$$U_g^{-1} T_m^J U_g = D_{mm'}^J(g) T_{m'}^J. \tag{1}$$

This has lead to the introduction, in mathematical physics, of the concept of tensor operators. We now give a general definition. Because we shall deal with non-compact groups, we shall distinguish between contravariant $\{T^a\}$ and covariant $\{T_a\}$ tensor operators.

DEFINITION 1. Let $g \to D(g)$ be a finite-dimensional representation of a group $G$ in a vector space $V$ and let $\{D^a{}_b\}$ be its matrix form in a basis $\{e_a\}_1^{\dim V}$ of $V$. Let $g \to U_g$ be a unitary representation of $G$ in a Hilbert space $H$.

A set $\{T^a\}$, $a = 1, 2, \ldots, \dim D$, of operators is said to be a *contravariant tensor operator* if

$$U_g^{-1} T^a U_g = D^a{}_b(g) T^b. \blacktriangledown \tag{2}$$

Thus a contravariant tensor operator $\{T^a\}$ in $H$ transforms as a contravariant vector with respect to the representation $g \to D(g)$ in $V$.

The corresponding definition of the tensor operator on the level of Lie algebra is obtained if we insert the representation of the generators

$$D(X) \equiv \frac{\mathrm{d}}{\mathrm{d}\Theta} D(\exp \Theta X)_{\Theta=0}, \quad \text{and} \quad \mathrm{i}U(X) \equiv \frac{\mathrm{d}}{\mathrm{d}\Theta} U_{\exp \Theta X}\big|_{\Theta=0} \tag{3}$$

into formula (2). We find

$$[U(X), T^a] = \mathrm{i} D^a{}_b(X) T^b, \quad X \in L. \tag{4}$$

*Remark* 1: We shall assume that the generators $U(X)$, $X \in L$, have a common dense invariant domain $D \subset H$. The construction of such domains is given in 11.1, as well as the precise definitions of generators (3). We shall assume that $D$ is a domain for operators $T^a$. We shall concentrate in this chapter on algebraic properties of tensor operators so the concrete form of $D$ is irrelevant.

In general, the definitions (2) and (4) of a tensor operator are equivalent if the representation of the Lie algebra $L$ of $G$ in (4) can be integrated to a global representation $U_g$ of $G$. And it is instructive to show also that (4) implies (2). The method used here is useful in many practical calculations. Let the global representation $g \to U_g$ be given by

$$U_g = \exp\left(\mathrm{i} s_\varrho U(X_\varrho)\right),$$

where we have denoted the representations of the generators by $U(X_\varrho)$. Consider a 'one-parameter' subgroup of the global representation $U_{g(\lambda)} = \exp\left(\mathrm{i}\lambda s_\varrho U(X_\varrho)\right)$. We set

$$T'^a(\lambda) = U_{g(\lambda)}^{-1} T^a U_{g(\lambda)} \tag{5}$$

and differentiate this equality with respect to $\lambda$

$$\frac{\mathrm{d}T'^a(\lambda)}{\mathrm{d}\lambda} = -\mathrm{i} s_\varrho \exp\left(-\mathrm{i}\lambda s_\varrho U(X_\varrho)\right)[U(X_\varrho), T^a] \exp\left(\mathrm{i}\lambda s_\varrho U(X_\varrho)\right).$$

Using eqs. (4) and (5), we obtain

$$\frac{\mathrm{d}T'^a}{\mathrm{d}\lambda} = s_\varrho D^a{}_b(X_\varrho)T'^a(\lambda).$$

This differential equation with the initial conditions $T'^a(0) = T^a$ has the following integral

$$T'^a(\lambda) = [\exp(\lambda s_\varrho D(X))]^a{}_b T'^b.$$

For $\lambda = 1$ we obtain the formula (2), by virtue of eq. (5).

The tensor operator $\{T^{(s)a}\}$ is said to be *irreducible*, if $g \to D^{(s)}(g)$ is irreducible. The simplest example of an irreducible tensor operator is provided by any invariant operator $C$ of $G$. In this case

$$U_g^{-1}CU_g = C,$$

i.e., $g \to D(g) \equiv 1$. A less trivial example is the following:

EXAMPLE 1. Let $G$ be the rotation group in $R^n$, $H = L^2(R^n)$ and $(U_g\psi)(x) = \psi(g^{-1}x)$. Let $T^\mu = \hat{x}^\mu$ be the coordinate operator $(\hat{x}^\mu\psi)(x) = x^\mu\psi(x)$. Then

$$\begin{aligned}(U_g^{-1}\hat{x}^\mu U_g\psi)(x) &= (\hat{x}^\mu U_g\psi)(gx)\\ &= g^\mu{}_\nu x^\nu(U_g\psi)(gx) = g^\mu{}_\nu x^\nu\psi(x)\\ &= g^\mu{}_\nu\hat{x}^\nu\psi(x).\end{aligned}$$

Hence,

$$U_g^{-1}\hat{x}^\mu U_g = g^\mu{}_\nu\hat{x}^\nu, \tag{6}$$

i.e., the set $\{\hat{x}^\mu\}$ is a contravariant tensor operator. ▼

DEFINITION 2. A set $\{T_a\}$, $a = 1, 2, \ldots, \dim D$, of operators is said to be a *covariant tensor operator* if it transforms according to the representation $\check{D}(g) = D^T(g^{-1})$ contragradient relative to $D(g)$, i.e.,

$$U_g^{-1}T_aU_g = D_a^{Tb}(g^{-1})T_b \equiv D^b{}_a(g^{-1})T_b. \text{▼} \tag{7}$$

On the level of Lie algebra, eqs. (7) and (3) give

$$[U(X), T_a] = -\mathrm{i}D^b{}_a(X)T_b. \tag{8}$$

Remark 1 also applies to def. 2.

If $g \to D(g)$ is a unitary representation of $G$, then the space $V$ has the metric tensor $g^{ab} = \delta^{ab}$. Consequently, we may set $D^a{}_b(g) = D_{ab}(g)$ in all formulas. We see then that the standard definition of tensor operator $T^j_m$ for SO(3) given by eq. (1) correspond to def. 1 of a contravariant tensor operator.

Let $L$ be an arbitrary Lie algebra with a basis $X_a$ and let $X_a \to U(X_a)$ be a representation of $L$ by self-adjoint operators in $H$. Then, the set $\{T_a\} = \{U(X_a)\}$, $a = 1, 2, \ldots, \dim L$, represents a covariant tensor operator for $L$. Indeed, by virtue of commutation relations in $L$ we have for this special tensor operator

$$[U(X_b), T_a] = \mathrm{i}c_{ba}{}^cT_c. \tag{9}$$

The set of matrices $D(X_a) = -C_a \equiv \|-c_{ab}{}^c\|$ provides a representation of $L$ by virtue of Jacobi identity 1.1(7). Hence, the condition (8) is satisfied. Thus, the set $\{U(X_a)\}$ is a covariant tensor operator.

For arbitrary tensor operators $\{Q_a\}$ and $\{T^a\}$ the operator $C = Q_a T^a$ is invariant: indeed

$$U_g^{-1} C U_g = D^b{}_a(g^{-1}) D^a{}_{b'}(g) Q_b T^{b'} = D^b{}_{b'}(g^{-1}g) Q_b T^{b'} = \delta^b{}_{b'} Q_b T^{b'} = C, \quad (10)$$

or,

$$[C, U(X_a)] = 0. \quad (10')$$

This provides a convenient method of construction of invariants of a Lie algebra, which we shall frequently use in what follows.

DEFINITION 3. A set $\{T^{\mu_1\mu_2\cdots\mu_r}\}$ is said to be *contravariant tensor operator of the rank r if*

$$U_g^{-1} T^{\mu_1\mu_2\cdots\mu_r} U_g = D^{\mu_1}{}_{\nu_1}(g) D^{\mu_2}{}_{\nu_2}(g) \ldots D^{\mu_r}{}_{\nu_r}(g) T^{\nu_1\nu_2\cdots\nu_r}. \quad (11)$$

One defines analogously the covariant tensor operators $\{T_{\mu_1\mu_2\ldots\mu_r}\}$ and *mixed tensor operators* $\{T^{\nu_1\nu_2\cdots\nu_s}_{\mu_1\mu_2\cdots\mu_r}\}$.

*Remark* 2: Not every set of operators $\{T_a\}$, which carries a tensor index $a$ represents a tensor operator. An important counter-example is provided by generators of the Poincaré group; indeed, under the Poincaré group the generators $P_\mu$ transform in the following manner

$$\begin{aligned}[U^{-1}{}_{(a\Lambda)} P_\mu U_{(a\Lambda)}\psi](p) &= (P_\mu U_{(a\Lambda)}) \exp(-ipa)\psi(\Lambda p) \\ &= (\Lambda)_\mu{}^\nu p_\nu \exp(ipa)(U_{(a\Lambda)})\psi(\Lambda p) \\ &= \Lambda_\mu{}^\nu P_\nu \psi(p) = (\Lambda^{-1})^\nu{}_\mu P_\nu \psi(p),\end{aligned}$$

i.e.,

$$U^{-1}{}_{(a\Lambda)} P_\mu U_{(a\Lambda)} = (\Lambda^{-1})^\nu{}_\mu P_\nu. \quad (11')$$

Thus, $P_\mu$ transform according to the contragradient representation of the Poincaré group given by

$$(a, \Lambda) \to (0, \Lambda)^{-1^T} = (0, \Lambda^{-1^T}),$$

i.e., $P_\mu$ is covariant tensor operator, according to def. 2. On the other hand the commutation relations of generators $M_{\mu\nu}$ with, for instance, $P_\mu$ gives

$$[P_\sigma, M_{\mu\nu}] = i(g_{\mu\sigma} P_\nu - g_{\nu\sigma} P_\mu).$$

Therefore, the set $\{M_{\mu\nu}\}$ alone cannot form a tensor operator for the Poincaré group $\Pi$ according to defs. (4) and (8). However the set $\{P_\mu, M_{\mu\nu}\}$ forms a tensor operator according to eq. (9).

DEFINITION 4. A covariant tensor $\{g_{\mu_1\ldots\mu_p}\}$ in $V$ is said to be *invariant* if it satisfies

$$D^{\nu_1}{}_{\mu_1}(g) \ldots D^{\nu_p}{}_{\mu_p}(g) g_{\nu_1\ldots\nu_p} = g_{\mu_1\cdots\mu_p}. \quad (12)$$

The Kronecker symbol $\delta_{ij}$ and the Levi–Civita symbol $\varepsilon_{ijk}$ are the only invariant tensors in $R^3$ with respect to SO(3).

The following theorem describes some important properties of tensor operators.

THEOREM 1. 1° *The contraction* $\{T^{\mu_1\dots\mu_s\alpha_1\dots\alpha_p}_{\mu_1\dots\mu_s\beta_1\dots\beta_q}\}$ *of a tensor operator* $\{T^{\nu_1\dots\nu_s\alpha_1\dots\alpha_p}_{\mu_1\dots\mu_s\beta_1\dots\beta_q}\}$ *is again a tensor operator.*

2° *If* $\{T^{\mu_1\dots\mu_p}\}$ *is a tensor operator which transforms according to the tensor product* $D\otimes\dots\otimes D$ *of representations* $g\to D(g)$, *and* $g_{\mu_1\dots\mu_p}$ *is a covariant invariant tensor relative to contragradient representation* $\hat{D}(g) = D^T(g^{-1})$, *then the operator*

$$T = g_{\mu_1\dots\mu_p}T^{\mu_1\dots\mu_p}$$

*is an invariant of* $G$, *i.e.,*

$$U_g^{-1}TU_g = T.$$

PROOF: *ad* 1°. This assertion follows from eq. (10).

*ad* 2°. Using eq. (11) and eq. (12) we obtain

$$U_g^{-1}TU_g = g_{\mu_1\dots\mu_p}D^{\mu_1}{}_{\nu_1}(g)\dots D^{\mu_p}{}_{\nu_p}(g)T^{\nu_1\dots\nu_p} = g_{\nu_1\dots\nu_p}T^{\nu_1\dots\nu_p} = T. \blacktriangledown$$

If $\{\overset{1}{T}{}^a\}$ and $\{\overset{2}{T}{}^a\}$, $a = 1, 2, \dots, \dim D$, are two contravariant tensor operators, which satisfy eq. (2), then,

$$T^a = \overset{1}{T}{}^a + \overset{2}{T}{}^a$$

is also a contravariant tensor operator of the same kind according to def. 1.

If $\{\overset{1}{T}{}^a\}$ and $\{\overset{2}{T}{}^a\}$ are two contravariant tensor operators which transform according to representations $\overset{1}{D}$ and $\overset{2}{D}$, respectively, then the set $\{T^{ab} = \overset{1}{T}{}^a\overset{2}{T}{}^b\}$ defines a contravariant tensor operator, which transforms as

$$U_g^{-1}T^{ab}U_g = D^{ab}{}_{a'b'}T^{a'b'}, \tag{11''}$$

where

$$D^{ab}{}_{a'b'}(g) = D^a{}_{a'}(g)D^b{}_{b'}(g).$$

The tensor operator $\{T^{ab}\}$ is called the *tensor product* of the tensor operators $\{\overset{1}{T}_a\}$ and $\{\overset{2}{T}_b\}$.

One can form a new irreducible tensor operator from the tensor product of two irreducible tensor operators $\{T^{\lambda_1}_{m_1}\}$ and $\{T^{\lambda_2}_{m_2}\}$. We show this construction for the case when $G$ is a *simply reducible compact group*.* Because every representation of a compact group is equivalent to a unitary representation we shall use only lower indices.

* A compact group is said to be *simply reducible* if in the decomposition of the tensor product of any two irreducible representations every irreducible component appears at most once.

Let $\{|\lambda_1; m_1\rangle\}$ be a basis in an irreducible space $H^{\lambda_1}$, $\{|\lambda_2; m_2\rangle\}$ a basis in an irreducible space $H^{\lambda_2}$ and $\{|\lambda_1\,\lambda_2\,\lambda m\rangle\}$ an orthonormal basis in the irreducible subspace $H^{\lambda}$ of $H^{\lambda_1}\otimes H^{\lambda_2}$. Set

$$T^{\lambda}_{m} = \sum_{m_1 m_2} \langle \lambda_1\,\lambda_2\,\lambda m|\lambda_1\,m_1\,\lambda_2\,m_2\rangle T^{\lambda_1}_{m_1} T^{\lambda_2}_{m_2}, \tag{13}$$

where

$$|\lambda_1\,m_1\,\lambda_2\,m_2\rangle \equiv |\lambda_1\,m_1\rangle|\lambda_2\,m_2\rangle. \tag{14}$$

Using eq. (1) for $T^{\lambda_1}_{m_1}$ and $T^{\lambda_2}_{m_2}$, we have

$$U_g^{-1}\,T^{\lambda}_{m}\,U_g = \sum \langle \lambda_1\,\lambda_2\,\lambda m|\,\lambda_1\,m_1\,\lambda_2\,m_2\rangle\, D^{\lambda_1}_{m_1 m_1'}(g)\cdot D^{\lambda_2}_{m_2 m_2'}(g)\, T^{\lambda_1}_{m_1'} T^{\lambda_2}_{m_2'}. \tag{15}$$

According to exercise 7.7.3.1.

$$D^{\lambda_1}_{m_1 m_1'}(g)\, D^{\lambda_2}_{m_2 m_2'}(g)$$

$$= \sum_{\tilde{\lambda}=|\lambda_1-\lambda_2|}^{\lambda_1+\lambda_2} \langle \lambda_1\,m_1\,\lambda_2\,m_2|\,\lambda_1\,\lambda_2\,\tilde{\lambda}\tilde{m}\rangle\, D^{\tilde{\lambda}}_{\tilde{m}\tilde{m}_1}(g)\langle \lambda_1\,\lambda_2\,\tilde{\lambda}\tilde{m}_1|\lambda_1\,m_1'\,\lambda_2\,m_2'\rangle \tag{16}$$

Inserting this expression in eq. (15), and using completeness and orthogonality relations for vectors $|\lambda_1\,m_1\,\lambda_2\,m_2\rangle$ and $|\lambda_1\,\lambda_2\,\lambda m\rangle$ respectively one obtains

$$U_g^{-1}\,T^{\lambda}_{m}\,U_g = D^{\lambda}_{mm'}(g)\,T^{\lambda}_{m'}.$$

Hence, the object (13) is an irreducible tensor operator.

The following theorem describes the fundamental property of an irreducible tensor operator, very useful in applications:

THEOREM 2 (the Wigner–Eckart theorem). *Let $U^{\lambda_1}_g$ and $U^{\lambda_2}_g$ be irreducible unitary representations of a simple reducible compact group $G$ in the Hilbert spaces $H^{\lambda_1}$ and $H^{\lambda_2}$, respectively. Let $\{|\lambda_1\,m_1\rangle\}$ and $\{|\lambda_2\,m_2\rangle\}$ be orthogonal sets of basis vectors in $H^{\lambda_1}$ and $H^{\lambda_2}$. Let $\{T^{\lambda}_{m}\}$ be an irreducible tensor operator. Then,*

$$\langle \lambda_2\,m_2|T^{\lambda}_{m}|\lambda_1\,m_1\rangle = \langle \lambda\lambda_1\,\lambda_2\,m_2|\lambda m\lambda_1\,m_1\rangle\, T(\lambda,\,\lambda_1,\,\lambda_2), \tag{17}$$

*where $\langle \lambda\lambda_1\,\lambda_2\,m_2|\lambda m\lambda_1\,m_1\rangle$ is the Clebsch–Gordan coefficient. $T(\lambda,\,\lambda_1,\,\lambda_2)$ is the so-called reduced matrix element of the tensor operator $\{T^{\lambda}_{m}\}$ given by*

$$T(\lambda, \lambda_1, \lambda_2) = \frac{1}{d_{\lambda_2}} \sum_{n_1, n_2} \langle \lambda n\,\lambda_1 n_1\,|\lambda\lambda_1\lambda_2 n_2\rangle\langle \lambda_2 n_2\,|T^{\lambda}_{n}|\,\lambda_1\,n_1\rangle \tag{18}$$

*and $d_{\lambda_2}$ is the dimension of $T^{\lambda_2}$.*

PROOF: By virtue of eq. (7) we have:

$$\langle \lambda_2\,m_2|T^{\lambda}_{m}|\lambda_1\,m_1\rangle = \sum_{n} D^{\lambda}_{nm}(g)\langle \lambda_2\,m_2|U_g^{-1}T^{\lambda}_{n}U_g|\,\lambda_1\,m_1\rangle, \tag{19}$$

using the equality $U_g|\,\lambda m\rangle = D^{\lambda}_{nm}(g)|\,\lambda n\rangle$ one obtains

$$\langle \lambda_2\,m_2|\,T^{\lambda}_{m}|\,\lambda_1\,m_1\rangle = \sum_{n, n_1, n_2} \bar{D}^{\lambda_2}_{n_2 m_2}(g)\, D^{\lambda}_{nm}(g)\, D^{\lambda_1}_{n_1 m_1}(g)\langle \lambda_2\,n_2|\,T^{\lambda}_{n}|\,\lambda_1\,n_1\rangle. \tag{20}$$

Integrating now over the group space $G$ and using the relation 7.4(10) and 7.4(11) one obtains

$$\begin{aligned}&\langle \lambda_2 m_2 | T^{\lambda}_m | \lambda_1 m_1 \rangle \\ &= \langle \lambda\lambda_1 \lambda_2 m_2 | \lambda m \lambda_1 m_1 \rangle d_{\lambda_2}^{-1} \sum_{n, n_1, n_2} \langle \lambda n \lambda_1 n_1 | \lambda\lambda_1 \lambda_2 n_2 \rangle \langle \lambda_2 n_2 | T^{\lambda}_n | \lambda_1 n_1 \rangle . \end{aligned} \tag{21}$$

This gives the assertion of the theorem. ▼

*Remark*: If $G$ is not simply reducible there is a complication due to the fact that in the tensor product $U^{\lambda_1} \otimes U^{\lambda_2}$ the irreducible representation $U^{\lambda}$ may occur more than once, i.e.,

$$U^{\lambda_1}_{\cdot} \otimes U^{\lambda_2} = \sum_{\lambda} \oplus c_{\lambda_2} U^{\lambda_2}, \qquad c_{\lambda_2} \geqslant 1 .$$

In this case one should split out the factor representation $c_\lambda U_\lambda$ onto irreducible components using formalism of sec. 7.4.A and proceed as above. However contrary to the common belief even in case of $U(n)$ groups, $n > 3$, one meets considerable difficulties with a derivation of Wigner–Eckart theorem. (Cf. Holman and Biedenharn 1971.)

In many applications we are interested in the ratios of the matrix elements (17) of a tensor operator $\{T^{\lambda}_m\}$: in such cases for fixed invariant numbers $\lambda$, $\lambda_1$ and $\lambda_2$ eq. (17) gives

$$\frac{\langle \lambda_2 m_2 | T^{\lambda}_m | \lambda_1 m_1 \rangle}{\langle \lambda_2 m'_2 | T^{\lambda}_{m'} | \lambda_1 m'_1 \rangle} = \frac{\langle \lambda\lambda_1 \lambda_2 m_2 | \lambda m \lambda_1 m_1 \rangle}{\langle \lambda\lambda_1 \lambda_2 m'_2 | \lambda m' \lambda_1 m'_1 \rangle}, \tag{22}$$

i.e., the problem is reduced to a calculation of the ratios of C–G coefficients only.

Notice also that for fixed invariant numbers $\lambda$, $\lambda_1$, $\lambda_2$ eq. (17) allows us to calculate an arbitrary matrix element of $T^{\lambda}_m$ from the knowledge of any particular matrix element.

More general objects are defined when a tensor operator $\{T_a\}$ depends on coordinates $x$; such objects are frequently encountered in quantum field theory and are called the *tensor field operators*. The transformation properties of the tensor field operators are defined by the formula

$$U_g^{-1} T^{\mu}(x) U_g = D^{\mu}{}_{\nu}(g) T^{\nu}(g^{-1}x). \tag{23}$$

In particular, if $G$ is the Poincaré group, $\{x\} = M$ is the Minkowski space and $T(x)$ is a scalar field operator, the formula (23) gives

$$U^{-1}_{(a\Lambda)} T(x) U_{(a\Lambda)} = T\big(\Lambda^{-1}(x-a)\big). \tag{24}$$

An example of tensor field operators is provided by the currents $\{j^{\mu}{}_k(x)\}$ which transform according to the direct product $G \otimes \Pi$ of an internal symmetry group $G$ (like SU(2) or SU(3)) and the Poincaré group $\Pi$. The transformation properties of $\{j^{\mu}{}_k(x)\}$ are

$$U_g^{-1} j^{\mu}{}_k(x) U_g = D_{k'k}(g^{-1}) j^{\mu}{}_{k'}(x), \qquad g \in G, \tag{25}$$

and

$$U^{-1}_{(a\Lambda)} j^{\mu}{}_k(x) U_{(a\Lambda)} = \big((\Lambda)^{\mu}{}_{\nu} j^{\nu}{}_k\big(\Lambda^{-1}(x-a)\big), \qquad (a, \Lambda) \in \Pi \tag{26}$$

## § 2. The Enveloping Algebra

Let $L$ be a Lie algebra over $K = R$ or $C$. Let $\tau$ be the (free) tensor algebra over $L$ considered as a vector space, i.e.,

$$\tau = \bigoplus_{r=0}^{\infty} \tau^r = K \oplus L \oplus (L \otimes L) \oplus (L \otimes L \otimes L) \oplus \ldots \tag{1}$$

The vector space $\tau$ is an associative algebra with the abstract multiplication law given by the tensor product $\otimes$. Let $J$ be the two-sided ideal in $\tau$ generated by elements of the form

$$X \otimes Y - Y \otimes X - [X, Y], \qquad \text{where } X, Y \in L. \tag{2}$$

Then, the quotient algebra $E = \tau/J$ is called the *universal enveloping algebra* of the Lie algebra $L$. Clearly, the enveloping algebra is associative.

Let $\pi$: $Z \to Z + J \equiv \tilde{Z}$, $\tilde{Z} \in E$, denote the canonical map (i.e. natural homomorphism) of $\tau$ onto $E$. Clearly, $\pi(Z_1 \otimes Z_2) = \pi(Z_1)\pi(Z_2) = \tilde{Z}_1 \tilde{Z}_2$.* The vector subspace of $E$ spanned by all elements $X_{i_1} X_{i_2} \ldots X_{i_r}$, $X_{i_k} \in L$, of order $r$ will be denoted by $E^r$. The element

$$([\pi(X), \pi(Y)] - \pi[X, Y]), \qquad X, Y \in L \tag{3}$$

is the image of an element (2) under canonical map and is, therefore, zero. Clearly the canonical map $\pi$ of the algebra $\tau$ onto $E$ induces a linear map of $L$ into $E$.

Let $X \to T(X)$ be a representation of a Lie algebra in a vector space $H$. The formula

$$T(X_{i_1} \otimes X_{i_2} \otimes \ldots \otimes X_{i_r}) = T(X_{i_1}) T(X_{i_2}) \ldots T(X_{i_r}) \tag{4}$$

defines uniquely a representation $\tilde{T}$ of the associative algebra $E$ in $H$.** Clearly, $\tilde{T}(X) = T(X)$ for $X$ in $L$. Moreover,

$$\tilde{T}(X \otimes Y - Y \otimes X - [X, Y]) = T(X)T(Y) - T(Y)T(X) - T([X, Y]) = 0.$$

Thus, every representation $T$ of $L$ can be extended to the representation $\tilde{T}$ of the universal enveloping algebra $E$ of $L$.

### *Bases in the Enveloping Algebra*

We shall construct two convenient bases in $E$. Notice first that if $X_1, \ldots, X_n$ is a basis in $L$ then monomials

$$\tilde{X}_{i_1} \tilde{X}_{i_2} \ldots \tilde{X}_{i_r}, \qquad \tilde{X}_{i_k} = \pi(X_{i_k}), \tag{5}$$

---

* For simplicity we shall omit the symbol of multiplication in $E$.

** The construction of the dense, invariant domain for operators (4) is given in 11.2.

span the space $E^r$. Using the relation $[\tilde{X}_1, \tilde{X}_2] = c_{ik}{}^l \tilde{X}_l$ for basis elements of $L$ in $E$ we can reduce (5) to the standard monomial form

$$e_{j_1 j_2 \dots j_r} = \tilde{X}_{j_1} \tilde{X}_{j_2} \dots \tilde{X}_{j_r}, \quad \text{where } j_1 \leqslant j_2 \leqslant \dots \leqslant j_r, \tag{6}$$

at the expense of introducing elements (5) of $E^{r-1}$. Then, the elements (5) of $E^{r-1}$ in turn can be reduced to the standard form (6). Thus, instead of $n^r$ elements, which span $E^r$, we obtain

$$\frac{(n+r-1)!}{(n-1)!\,r!} \tag{7}$$

elements which span $E^r$. For example, the space $R^3$ associated with the Lie algebra su(2) contains 10 basis elements of the form (6) instead of 27 of the form (5). Taking the collection of vectors (6) in $E^0, E^1, E^2, \dots$ we obtain a set $\{e_{i_1 i_2 \dots i_r};\ i_1 \leqslant i_2 \leqslant \dots \leqslant i_r,\ r = 0, 1, 2, \dots\}$ which span $E$. We leave as an exercise for the reader the demonstration that elements (6) are linearly independent. The basis (6) in $E$ is called the *Poincaré–Birkhoff–Witt basis*. The elements (6) can also be written in the form $e_{i_1 i_2 \dots i_r} = \tilde{X}_1^{k_1} \tilde{X}_2^{k_2} \dots \tilde{X}_n^{k_n}$ where $k_1 + k_2 + \dots + k_n = r$.

In the applications it is convenient to use the following symmetric basis in $E$.

PROPOSITION 1. *Let $L$ be a Lie algebra with a basis $X_1, \dots, X_r$. The elements*

$$e_{\{i_1 i_2 \dots i_r\}} \equiv \frac{1}{r!} \sum_{\sigma} \tilde{X}_{i_{\sigma(1)}} \dots \tilde{X}_{i_{\sigma(r)}}, \quad r = 0, 1, \dots, \tag{8}$$

*where $i_k = 1, 2, \dots, \dim L$, and $\sigma$ runs over all permutations of the set $(1, 2, \dots, r)$, form a basis in the universal enveloping algebra $E$ of $L$.*

PROOF: An element

$$\tilde{X}_{i_1} \dots \tilde{X}_{i_r} - \tilde{X}_{i_{\sigma(1)}} \dots \tilde{X}_{i_{\sigma(r)}} \tag{9}$$

of $E^r$ may be expressed in terms of the elements of $E^{r-1}$ if use is made of the commutation relations $\tilde{X}_i \tilde{X}_k = \tilde{X}_k \tilde{X}_i + c_{ik}{}^l \tilde{X}_l$. If we sum eq. (9) over all permutations $\sigma$, we obtain

$$\tilde{X}_{i_1} \dots \tilde{X}_{i_r} = e_{\{i_1 \dots i_r\}} + \text{terms in } E^{r-1}.$$

Repeating now this procedure for terms in $E^{r-1}$, $E^{r-2}$, etc., we find

$$\tilde{X}_{i_1} \dots \tilde{X}_{i_r} = \sum_{\substack{i_1, i_2, \dots, i_k = 1 \\ 0 \leqslant k \leqslant r}}^{\dim L} c^{i_1 \dots i_k} e_{\{i_1 \dots i_k\}}. \tag{10}$$

The map $\sigma$: $e_{i_1 i_2 \dots i_r} \leftrightarrow e_{\{i_1 i_2 \dots i_r\}}$ establishes a one to one correspondence of basis elements (6) and symmetric elements (8). Hence elements (8) form also a basis in $E$. ▼

The *center* $Z$ of the universal enveloping algebra $E$ is the set of all elements $C$ in $E$ which satisfy

$$[C, \tilde{X}] = 0 \quad \text{for all } \tilde{X} \text{ in } L. \tag{11}$$

One of the main problems in the representation theory is to find the center $Z$ and to determine the spectrum of the elements of $Z$ in irreducible carrier spaces. For semisimple Lie algebras this problem is solved explicitly in secs. 4 and 6A.

## § 3. The Invariant Operators

We describe in this section the general properties of invariant operators of arbitrary Lie algebras.

According to eq. 2(6) any element of the enveloping algebra $E$ of a given Lie algebra $L$ can be expressed as the sum of elements of the form*

$$g^{i_1 \dots i_s} X_{i_1} \dots X_{i_s}, \qquad s = 0, 1, \dots \tag{1}$$

The elements $X_{i_1} \dots X_{i_s}$ are tensor operators relative to the adjoint representation (or adjoint group). Therefore, the problem of construction of invariants of $E$ reduces, by th. 1.1.2, to the problem of finding appropriate invariant tensors $g^{i_1 \dots i_s}$. Indeed, we have

THEOREM 1 (Gel'fand). *In order that an element $P$*

$$P = cI + \sum_i g^i X_i + \sum_{i,k} g^{ik} X_i X_k + \sum_{i,k,j} g^{ikj} X_i X_k X_j + \dots \tag{2}$$

*of the enveloping algebra $E$ belong to the centre $Z$ of $E$ it is sufficient that the coefficients*

$$g^i, g^{ik}, g^{ikj}, \dots \tag{3}$$

*are invariant tensors for the adjoint group $G_A$. If, in addition, $P$ is written in the form in which the coefficients $g^{ik}$, $g^{ikj}$, ... are symmetric, then this condition is also necessary.*

PROOF: Denote by $P_r$ an element $g^{i_1 \dots i_r} X_{i_1} \dots X_{i_r}$. Then by virtue of the invariance of the tensor $g^{i_1 \dots i_r}$ we obtain

$$\mathrm{Ad}\, g P_r = g P_r g^{-1} = g^{i_1 \dots i_r} \prod_{k=1}^{r} \mathrm{Ad}\, g X_{i_k} = g^{i_1 \dots i_r} \prod_{k=1}^{r} (\mathrm{Ad}\, g)^{j_l}{}_{i_k} X_{j_l} = P_r .$$

Taking $g = \exp(tX_l)$ and passing to the infinitesimal form with the above equality we obtain

$$[P_r, X_l] = 0 .$$

This implies the first part of th. 1. Now let tensors (3) be symmetric; then by virtue of eq. (2.8) every element $P_r$ in (2) can be written in the form

$$P_r = \sum_{i_1, \dots, i_1} g^{i_1 \dots i_r} e_{\{i_1 \dots i_r\}} .$$

* From now on, we denote for simplicity the product $\tilde{X}_{i_1} \dots \tilde{X}_{i_s}$ in $E$ by $X_{i_1} \dots X_{i_s}$.

If $P \in Z$ then $P_r \in Z$ and $[P_r, X_l] = 0$. This, for arbitrary $g = \exp(tX_l)$, implies

$$gP_r g^{-1} = \operatorname{Ad} g P_r = P_r.$$

Hence

$$g^{i_1 \dots i_r} \prod_{k=1}^{r} (\operatorname{Ad} g)^{j_k}{}_{i_k} e_{\{j_1 \dots j_r\}} = g^{i_1 \dots i_r} e_{\{i \dots i_r\}}.$$

Consequently every tensor (3) must be invariant of $G_A$. ▼

The problem of the explicit construction of invariant operators for semisimple Lie algebras was first considered by Casimir 1931. Using the Cartan metric tensor $g_{ik}$ in $L$ he constructed the second order operator

$$C_2 = g^{ik} X_i X_k \tag{4}$$

which, due to 1.2(18) satisfies

$$[C_2, X_l] = g^{ik}[X_1, X_l]X_k + g^{ik} X_i [X_k, X_l] = (c_{kls} + c_{slk}) X^s X^k = 0.$$

In order to understand the invariance property of the operator (4) from the point of view of th. 1, note that according to eq. 1.2(8) the tensor $g_{ik}$ can be written in the form

$$g_{ik} = \operatorname{Tr} \hat{X}_i \hat{X}_k, \tag{5}$$

where $\hat{X}_i = -C_i \equiv \{-c_{il}{}^s\}$ is the adjoint representation of the Lie algebra $L$ determined by the structure constants. The elements of the adjoint group transform each $\hat{X}_i$ into $g\hat{X}_i g^{-1}$ and therefore leave the tensor (5) invariant. This implies that the operator (4) is an invariant for $L$ by virtue of th. 1.

It becomes straightforward now to construct higher order invariant operators; in fact, the tensor

$$g_{i_1 i_2 \dots i_p} = \operatorname{Tr} \hat{X}_{i_1} \hat{X}_{i_2} \dots \hat{X}_{i_p} = c_{i_1 l_1}{}^{l_2} c_{i_2 l_2}{}^{l_3} \dots c_{i_{p-1} l_{p-1}}{}^{l_p} c_{i_p l_p}{}^{l_1} \tag{6}$$

is an invariant of the adjoint group by the same arguments. Hence, the operators

$$C_p = g_{i_1 i_2 \dots i_p} X^{i_1} X^{i_2} \dots X^{i_p}, \quad p = 2, 3, \dots \tag{7}$$

are invariants of the enveloping algebra $E$ by virtue of th. 1.

For the construction of invariant tensors (6) we may take finite-dimensional representations $X \to V(X)$ of a given Lie algebra. Indeed. if

$$g_{i_1 i_2 \dots i_p} = \operatorname{Tr}\big(V(X_{i_1}) V(X_{i_2}) \dots V(X_{i_p})\big), \tag{8}$$

then, the adjoint transformation $X = gXg^{-1}$ implies that

$$\begin{aligned} g'_{i_1 i_2 \dots i_p} &= \operatorname{Tr}\big(V(gX_{i_1}g^{-1}) \dots V(gX_{i_p}g^{-1})\big) \\ &= \operatorname{Tr}\big(V_g V(X_{i_1}) V_g^{-1} \dots V_g V(X_{i_p}) V_g^{-1}\big) \\ &= \operatorname{Tr}\big(V(X_{i_1}) \dots V(X_{i_p})\big) = g_{i_1 i_2 \dots i_p}. \end{aligned} \tag{9}$$

i.e., the tensor (8) is invariant. We shall use this fact for the construction of independent invariant operators for arbitrary semisimple Lie groups. In the applications it is crucial to know the minimal number of invariant operators which generate the center $Z$ of the enveloping algebra $E$. The following theorem gives the solution of this problem for semisimple Lie algebras.

THEOREM 2. *For every semisimple Lie algebra L of rank n there exists a set of n invariant polynomials of generators $X_i$, whose eigenvalues characterize the finite-dimensional irreducible representations.*

(For the proof cf., e.g., Chevalley 1955.)

The problem of finding the explicit form of the spectra of the invariant operators is of fundamental importance, in particular, in applications. This problem can be solved easily for the second order invariant operators of semisimple Lie algebras. In the standard Cartan–Weyl basis of $L$ the operator $C_2$, eq. (4), has the form:

$$C_2 = \sum_{i=1}^{l} g^{ik} H_i H_k + \sum_{\alpha} E_\alpha E_{-\alpha}. \tag{10}$$

When this operator acts on the highest weight vector $u_m$ of an irreducible representation, one obtains, because of the condition $E_\alpha u_m = 0$ for positive roots (eq. 8.8(5)),

$$C_2 u_m = \Big\{ g^{ik} m_i m_k + \sum_{\alpha>0} [E_\alpha, E_{-\alpha}] \Big\} u_m = \Big[ m^2 + \sum_{\alpha>0} (\alpha, m) \Big] u_m. \tag{11}$$

We know that every irreducible representation is characterized by the components of the highest weight vector $m = (m_1, m_2, \ldots, m_n)$. By virtue of Schur's lemma every invariant operator in the carrier space of an irreducible representation is proportional to the identity, i.e., $C_i = \lambda_i I$; the number $\lambda_i$ is a function of the components of the highest weight $m = (m_1, m_2, \ldots, m_n)$ and represents a spectrum of the Casimir operator $C_i$, i.e.,

$$C_2(m) = m^2 + 2rm, \tag{12}$$

where

$$r = \frac{1}{2} \sum_{\alpha>0} \alpha \tag{13}$$

and summation runs over the positive roots only. With $k = m+r$ we have then

$$C_2(m) = k^2 - r^2. \tag{14}$$

In this form the eigenvalue of the Casimir operator $C_2$ is invariant relative to the action of the Weyl group $k \to Sk$. This fact holds for arbitrary invariant operators. Indeed, we have

THEOREM 3 ($S$-theorem). *Let $C$ be any invariant operator and $H^m$ the carrier space of an irreducible representation of $L^m$ determined by the highest weight $m$. Then,* *the eigenvalue $C(m)$ of $C$ expressed in terms of $k = m+r$ is invariant under the transformation of the Weyl group, i.e.,*

$$C'(Sk) = C'(k) \quad \textit{for all } S \in W, \tag{15}$$

*where*

$$C'(k) = C(k-r). \tag{16}$$

PROOF: The character $\chi^m(g) = D^m_{pp}(g)$ is an eigenfunction for an arbitrary invariant operator. Because the trace of the matrix $D^m$ is invariant under the similarity transformation $D^m(g) \to D^m(g')D^m(g)D^m(g'^{-1})$, the character $\chi^m(g)$ of a semisimple Lie group is a function of the classes of conjugate elements and takes the Weyl form 8.8(25). It is evident from the Weyl formula that a character $\chi^m(\delta)$ is left-invariant under the transformation $S$: $k \to Sk$ except for a possible change of sign. Consequently the eigenvalue

$$C'\chi^m(\delta) = C'(k)\chi^m(\delta)$$

is invariant under the transformation of the Weyl group. ▼

This $S$-theorem is useful in the determination of the explicit form of the spectra of Casimir operators for semisimple Lie groups, and will be used in the following sections.

## § 4. Casimir Operators for Classical Lie Group

### A. *Casimir Operators and Their Spectra for $U(n)$*

Let us consider first the group $U(n)$. The $n \times n$ matrices $u \in U(n)$ obey the condition $u^*u = 1$. Therefore the $n^2$ generators $M_i{}^k$, $i, k = 1, 2, \ldots, n$ of one-parameter subgroups satisfy

$$(M_i{}^k)^* = M_i{}^k. \tag{1}$$

However, because the commutation relations of the generators $M_i{}^k$ are not in a symmetric form we usually pass to the Lie algebra $\mathrm{gl}(n, R)$, whose commutation relations are simply

$$[A_j{}^l, A_i{}^k] = \delta_i{}^l A_j{}^k - \delta_j{}^k A_i{}^l. \tag{2}$$

If the generators $A_i{}^k$ satisfy the condition $(A_i{}^k)^* = A_k{}^i$, then the $n^2$ independent hermitian generators obeying (1) are given by

$$\begin{aligned} M_k{}^k &= A_k{}^k, \quad k = 1, 2, \ldots, n, \\ M_k{}^l &= A_k{}^l + A_l{}^k, \quad k < l \leqslant n \\ M_l{}^k &= \mathrm{i}(A_k{}^l - A_l{}^k), \quad k < l \leqslant n. \end{aligned} \tag{3}$$

If an element $F$ of the enveloping algebra $E$ satisfies

$$[F, A_i{}^k] = 0 \quad \text{for all } i,\ k,$$

then, due to (3), it also satisfies

$$[F, M_i{}^k] = 0, \quad i, k = 1, 2, \ldots, n.$$

Therefore, the problem of invariant operators of $u(n)$ is reduced to that of $\mathrm{gl}(n, R)$. The latter problem can be solved easily using th. 3.1. Indeed, using 3(8) and the adjoint representation of $\mathrm{gl}(n, R)$

$$V(A_i^{\ j})_l^{\ s} = \delta_{il}\delta^{js}, \tag{4}$$

we obtain

$$\begin{aligned} g_{i_1}{}^{j_1}{}_{i_2}{}^{j_2}{}_{\dots i_p}{}^{j_p} &= \mathrm{Tr}\left(V(A_{i_1}{}^{j_1}) \dots V(A_{i_p}{}^{j_p})\right) \\ &= \delta_{i_1 l_1}\delta^{j_1 s_1}\delta_{i_2 s_1}\delta^{j_2 s_2} \dots \delta_{i_p s_{p-1}}\delta^{j_p l_1} \\ &= \delta_{i_1}{}^{j_p}\delta_{i_2}{}^{j_1}\delta_{i_3}{}^{j_2} \dots \delta_{i_p}{}^{j_{p-1}}. \end{aligned} \tag{5}$$

Therefore, by eq. 3(7) the invariant operators have the form

$$\begin{aligned} C_p &= g_{i_1}{}^{j_1}{}_{i_2}{}^{j_2}{}_{\dots i_p}{}^{j_p} A_{j_1}{}^{i_1} A_{j_2}{}^{i_2} \dots A_{j_p}{}^{i_p} \\ &= A_{i_2}{}^{i_1} A_{i_3}{}^{i_2} \dots A_{i_p}{}^{i_{p-1}} A_{i_1}{}^{i_p}, \quad p = 1, 2, \dots \end{aligned} \tag{6}$$

The next two theorems give the explicit form of the spectra of the invariant operators (6) in carrier spaces of irreducible representations. Note that the use of tensor operators considerably simplifies the proof of th. 1.

THEOREM 1. *Let $H^m$ be the carrier space of an irreducible representation of the group $U(n)$ determined by the highest weight $m = (m_1, \dots, m_n)$. Then, the spectra of the invariant operators* (6) *in $H^m$ have the form*

$$C_p(m_1, \dots, m_n) = \mathrm{Tr}\,(a^p E), \tag{7}$$

*where the matrix $a = \{a_{ij}\}$, $i, j = 1, 2, \dots, n$, is*

$$\begin{aligned} a_{ij} &= (m_i + n - i)\,\delta_{ij} - Q_{ij}, \\ Q_{ij} &= \begin{cases} 1 & \text{for } i < j, \\ 0 & \text{for } i \geqslant j; \end{cases} \end{aligned} \tag{8}$$

*$a^p$ is the $p$-th power of the matrix $a$ and $E$ is the matrix with all elements $E_{ij} = 1$.*

PROOF: In order to calculate the spectra of the invariant operators $C_p$, $p = 1, 2, \dots, n$, we use an idea which Racah has used in the calculation of the spectrum of the Casimir operator $C_2$ of an arbitrary simple Lie group (cf. eqs. 3 (10)–3 (12)). Let us recall first the connection between the generators $A_i^{\ k}$ and the Cartan–Weyl generators $H_i$, $E_\alpha$ (cf. 1.4 (10) and 1.4 (11))

$$\begin{aligned} H_i &= A_i^{\ i}, \quad i = 1, 2, \dots, n, \\ E_{(e_i - e_k)} &= A_i^{\ k}, \quad i \neq k, \end{aligned} \tag{9}$$

where $e_i = (0, 0, \dots, \underset{(i)}{1}, \dots, 0, 0)$, $i = 1, 2, \dots, n$, are the orthonormal vectors of $R^n$. Therefore, the generators $A_i^{\ k}$, $i > k$, are associated with positive roots of the algebra $u(n)$ and they play the role of the raising operators.

Consider now an irreducible representation of $u(n)$ which is determined by the highest weight $m = (m_1, \dots, m_n)$ and denote by $\psi_m$ the highest weight vector.

Because for $i > j$ the operators $A_i{}^j$ are the raising operators, we obtain

$$A_i{}^j \psi_m = 0, \quad i > j. \tag{10}$$

Let us rewrite now eq. (6) in the form

$$C_p = (T_{p-1})_j{}^i A_i{}^j, \quad \text{where } (T_q)_j{}^i \equiv A_{i_1}{}^i A_{i_2}{}^{i_1} \ldots A_j{}^{i_{q-1}}. \tag{11}$$

The operator $(T_{p-1})_j{}^i$ has the same transformation property with respect to $U(n)$ as $A_j{}^i$, so it represents a tensor operator. Consequently, by virtue of eqs. 1(9) and (2)

$$[A_j{}^i, (T_{p-1})_l{}^k] = \delta_j{}^k (T_{p-1})_l{}^i - \delta_l{}^i (T_{p-1})_j{}^k \tag{12}$$

and

$$(T_{p-1})_j{}^i \psi_m = 0 \quad \text{for } i < j. \tag{13}$$

From (10), (11) and (13) it follows that

$$\begin{aligned} C_p \psi_m &= \sum_{i=1}^{n} (T_{p-1})_i{}^i A_i{}^i \psi_m + \sum_{\substack{i,j \\ i>j}} [(T_{p-1})_i{}^j, A_j{}^i] \psi_m \\ &= \left\{ \sum_{i=1}^{n} (T_{p-1})_i{}^i A_i{}^i + \sum_{\substack{i,j \\ i>j}} (T_{p-1})_j{}^j - (T_{p-1})_i{}^i \right\} \psi_m \end{aligned} \tag{14}$$

(no summation convention in eq. (14)).

Using $A_i{}^i \psi_m = m_i \psi_m$ we find that

$$C_p \psi_m = \sum_{i=1}^{n} (m_i + n + 1 - 2i)(T_{p-1})_i{}^i \psi_m. \tag{15}$$

The quantity $(T_{p-1})_i{}^i \psi_m$ can be calculated recursively. Namely, using (11), (13) and (12) we get

$$(T_q)_i{}^i \psi_m = \sum_{j=1}^{n} (T_{q-1})_j{}^i A_i{}^j \psi_m = \sum_{j=1}^{n} a_{ij} (T_{q-1})_j{}^j \psi_m, \tag{16}$$

where the matrix $a_{ij}$ is

$$a_{ij} = (m_i + n - i)\, \delta_{ij} - Q_{ij}, \tag{17}$$

$$Q_{ij} = \begin{cases} 1 & \text{for } i < j, \\ 0 & \text{for } i \geqslant j. \end{cases}$$

Successively lowering the degree of $T_q$ by using (16) and the identity

$$\sum_{i=1}^{n} a_{ij} = m_j + n + 1 - 2j, \quad \sum_{j=1}^{n} a_{ij} = m_i \tag{18}$$

we get the result:

$$C_p(m_1, m_2, \ldots, m_n) = \sum_{i,j=1}^{n} (a^p)_{ij}. \tag{19}$$

Introducing the matrix $E$ with the matrix elements $E_{ij} = 1$ $i, j = 1, 2, \ldots, n$, we may write the formula (19) in the form

$$C_p(m_1, \ldots, m_n) = \mathrm{Tr}(a^p E). \blacktriangledown \tag{20}$$

The next theorem gives a convenient generating function for the spectrum of successive Casimir operators $C_p$, $p = 1, 2, \ldots$

THEOREM 2. *The function*

$$G(z) = z^{-1}(1 - \Pi(z)), \quad z \in C^1, \tag{21}$$

*where*

$$\Pi(z) = \prod_{i=1}^{n} \left(1 - \frac{z}{1-\lambda_i z}\right) \tag{22}$$

*and*

$$\lambda_i = m_i + n - 1, \tag{23}$$

*is a generating function for the spectrum of the Casimir operators, i.e.,*

$$G(z) = \sum_{p=0}^{\infty} C_p(m_1, \ldots, m_n) z^p. \tag{24}$$

PROOF: By elementary methods the triangular matrix $a = \{a_{ij}\}$ can be reduced to the diagonal form. Hence we can express $C_p(m_1, \ldots, m_n)$ by the eigenvalues $\lambda_i$ of the matrix $a$ in the form

$$C_p(m_1, \ldots, m_n) = \sum_{i=1}^{n} \lambda_i^p \prod_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{\lambda_i - \lambda_j - 1}{\lambda_i - \lambda_j}, \tag{25}$$

$$\lambda_i = m_i + n - i.$$

We can further simplify eq. (25) using the following integral representation

$$C_p(m_1, \ldots, m_n) = \frac{1}{2\pi i} \oint \lambda^p \prod_{i=1}^{n} \left(1 - \frac{1}{\lambda - \lambda_i}\right) d\lambda, \tag{26}$$

where the contour of integration encloses (in the positive direction) all poles $\lambda = \lambda_i$. With $\lambda = z^{-1}$ we have

$$C_p(m_1, \ldots, m_n) = \frac{1}{2\pi i} \oint \frac{dz}{z^{p+2}} \prod_{i=1}^{n} \left(1 - \frac{z}{1 - \lambda_i z}\right). \tag{27}$$

From (27) it follows that the function

$$\Pi(z) = \prod_{i=1}^{n} \left(1 - \frac{z}{1-\lambda_i z}\right) = 1 - C_0 z - C_1 z^2 - \ldots, \quad C_0 = n \tag{28}$$

has, at $|z| < 1/\lambda_i$, the eigenvalues of successive Casimir operators as its expansion coefficients. Therefore the function

$$G(z) = z^{-1}[1-\Pi(z)] \tag{29}$$

is the generating function for the Casimir operators, i.e.,

$$G(z) = \sum_{p=0}^{\infty} C_p z^p. \blacktriangledown \tag{30}$$

The generating function (21) is very convenient for the calculation of the eigenvalues of an arbitrary Casimir operator $C_p$. In order to illustrate its power, let us consider the following example.

EXAMPLE 1. Let us calculate the eigenvalues of the Casimir operators for the totally symmetric $(f, 0, 0, \ldots 0)$, or the totally antisymmetric $\{1^k\}$ representation of $u(n)$. Actually, both of these representations are special cases of the more general representation characterized by the highest weight $(\underbrace{f, f, \ldots, f}_{k \text{ times}}, 0, \ldots, 0)$, $k \leqslant n$. For this more general case, we have

$$\lambda_i = \begin{cases} f+n-i & \text{for } 1 \leqslant i \leqslant k, \\ n-i & \text{for } i > k. \end{cases} \tag{31}$$

From (22) and (31) it follows that

$$\Pi(z) = \frac{[1-(f-n)z][1-(n-k)z]}{1-(f+n-k)z}, \tag{32}$$

hence

$$G(z) = n + \frac{kfz}{1-(f+n-k)z}. \tag{33}$$

Using (30) we find

$$C_p(\underbrace{f, \ldots, f}_{k \text{ times}}, 0, \ldots, 0) = kf(f+n-k)^{p-1}. \tag{34}$$

If we put in this expression $k = 1$ we get the spectra of the operators $C_p$ for the totally symmetric representation $(f, 0, \ldots, 0)$ and if we put $f = 1$, $k$ arbitrary, $k \leqslant n$, we get the spectra of $C_p$ for the totally antisymmetric representations $\{1^k\}$, respectively. $\blacktriangledown$

The formula (6) gives an infinite number of invariant operators. It is not evident however that it provides all the generators of the center $Z$ of the enveloping algebra $E$ of $u(n)$.

The algebra $u(n)$ has $n$ independent Casimir operators, by virtue of th. 3.2. One would expect that the first $n$ Casimir operators $C_1, C_2, \ldots, C_n$ given by eq. (6) generate the center $Z$. Indeed, by an elementary computation one can show that:

$$\frac{\partial(C_1, C_2, \ldots, C_n)}{\partial(m_1, m_2, \ldots, m_n)} = n! \prod_{i<j} (\lambda_i - \lambda_j). \tag{35}$$

Therefore, for $i < j$ we have $(\lambda_i - \lambda_j) > 0$; the Jacobian (35) is positive. Hence, the invariant operators $C_1, C_2, \ldots, C_p$ are independent and their eigenvalues determine uniquely the irreducible representations of $U(n)$.

B. *Casimir Operators of* SU$(n)$

The Lie algebra su$(n)$ is generated by the operators $A_i^j$, $i \neq j$, and by $\tilde{A}_i^i$ of the form

$$\tilde{A}_i^i = A_i^i - \frac{1}{n} \sum_{l=1}^{n} A_l^l. \tag{36}$$

The action of $\tilde{A}_i^i$ on the highest vector $\psi_m$ is given by eq. (10)

$$\tilde{A}_i^i \psi_m = \left( m_i - \frac{1}{n} \sum_{l=1}^{n} m_l \right) \psi_m = \tilde{m}_i \psi_m .$$

Thus, applying the same argument used in the derivation of (19) we obtain

$$C_p^{(su)}(\tilde{m}_1, \ldots, \tilde{m}_n) = \sum_{i,j}^{n} (\tilde{a}^p)_{ij}, \tag{37}$$

where

$$\tilde{a} = a - \frac{m}{n} \cdot I, \qquad m = \sum_{i=1}^{n} m_i$$

and $I$ denotes the unit matrix.

Therefore we obtain the corresponding expressions for the spectrum of the invariant operators $C_p^{(su)}$ of SU$(n)$ by simply replacing in all formulas for $C_p^{(u)}$ the numbers $m_i$ by

$$\tilde{m}_i = m_i - \frac{1}{n} \sum_{s=1}^{n} m_s .$$

In particular, for the representations which are determined by the highest weight $(\underbrace{f, f, \ldots, f}_{k \text{ times}}, 0, \ldots, 0)$, we get

$$C_p^{(su)}(\underbrace{f, \ldots, f}_{k \text{ times}}, 0, \ldots, 0) = \frac{kf(n+f)(n-k)}{n(n+f-k)} \cdot \left\{ \left[ \frac{(f+n)(n-k)}{n} \right]^{p-1} - \left[ -\frac{kf}{n} \right]^{p-1} \right\}. \tag{38}$$

C. *Casimir Operators and Their Spectra for $O(n)$ and* Sp$(n)$

1. The orthogonal group $O(n)$ consists of all linear transformations of the $n$-dimensional Euclidean space $E^n$ which conserve the quadratic form

$$(\xi^1)^2 + (\xi^2)^2 + \ldots + (\xi^n)^2 = 1 . \tag{39}$$

The symplectic group $\mathrm{Sp}(n, C)$ is formed by all the transformations of $2n$-dimensional complex space $C^{2n}$, that preserve the bilinear form

$$[x, y] = \sum_{i,j=-n}^{n} h_{ij} x^i y^j = \sum_{i=1}^{n} (x^i y^{-i} - x^{-i} y^i), \tag{40}$$

where the metric tensor $h_{ij}$ is

$$h_{ij} = \varepsilon_i \delta_{i,-j}, \qquad \varepsilon_i = \begin{cases} 0 & \text{for } i = 0, \\ 1 & \text{for } i > 0, \\ -1 & \text{for } i < 0. \end{cases} \tag{41}$$

In what follows it is convenient to consider both groups together. Therefore we go over in the case of orthogonal group from the Cartesian coordinates $\xi^i$, $i = 1, 2, \ldots, n$ to the 'spherical coordinates' $x^i$, $i = \pm 1, \pm 2, \ldots, \pm \left[\frac{n}{2}\right]$ (and $x^0$ if $n$ is odd)

$$x_1 = \frac{\xi^1 + \mathrm{i}\xi^2}{\sqrt{2}}, \quad x^{-1} = \frac{\xi^1 - \mathrm{i}\xi^2}{\sqrt{2}}, \ldots, \quad \text{and } x^0 = \xi^n \text{ if } n \text{ is odd.}$$

The quadratic form (39) takes now the form analogous to (40)

$$(x, y) = \sum_{i,j=-n}^{n} g_{ij} x^i y^j, \qquad g_{ij} = \delta_{i,-j}.$$

The generators $X_j{}^i$ of one parameter subgroups obey the following commutation relations:*

$$[X_j{}^i, X_l{}^k] = \delta_j{}^k X_l{}^i - \delta_l{}^i X_j{}^k + \\ + \begin{cases} \delta_j{}^{-l} X_{-i}{}^k - \delta_{-i}{}^k X_l{}^{-j} & \text{for } O(n), \\ \varepsilon_i \varepsilon_j \delta_j{}^{-l} X_{-i}{}^k - \varepsilon_j \varepsilon_k \delta_{-i}{}^k X_l{}^{-j} & \text{for } \mathrm{Sp}(2n). \end{cases} \tag{42}$$

The equality

$$X_{ij} = -X_{ji}, \quad i, j = 1, 2, \ldots, n, \tag{43}$$

for generators of $O(n)$ in cartesian coordinates corresponds now the following equality

$$X_j{}^i = -X_{-i}{}^{-j}, \quad i, j = -n, \ldots, +n. \tag{44}$$

For generators of $\mathrm{Sp}(2n)$ we have correspondingly

$$X_j{}^i = -\varepsilon_i \varepsilon_j X_{-i}{}^{-j}, \quad i, j = -n, \ldots, +n. \tag{45}$$

It follows from the commutation relations (42) that the operators $X_i{}^i$ commute with one another and correspond to the generators $H_i$ of Cartan basis, while the

* We conserve here notation of Perelomov and Popov 1966 for generators $X_j{}^i$. However in other works these generators are denoted usually by the symbol $X^i{}_j$.

the generators $X_j{}^i$ with $i > j$ correspond to the generators $E_\alpha$ associated with the positive roots of the algebra. The commutation relations for a tensor operator $T_l{}^k$, which has the same transformation properties as $X_j{}^i$ are given by the formula:

$$[X_j{}^i, T_l{}^k] = \delta_j{}^k T_l{}^i - \delta_l{}^i T_j{}^k + \\ + \begin{cases} \delta_j{}^{-l} T_{-i}{}^k - \delta_{-i}{}^k T_l{}^{-j} & \text{for } O(n), \\ \varepsilon_i \varepsilon_j \delta_j{}^{-l} T_{-i}{}^k + \varepsilon_j \varepsilon_k \delta_{-i}{}^k T_l{}^{-j} & \text{for } \mathrm{Sp}(2n). \end{cases} \tag{46}$$

In particular

$$[X_j{}^i, T_i{}^j] = (1 \mp \delta_{-j}{}^i)(T_i{}^i - T_j{}^j), \tag{47}$$

where the sign $-(+)$ refers to the orthogonal (symplectic) group.

Utilizing Gel'fand theorem 3.1 we easily verify that the operators

$$C_p = \sum_{i_1, \dots, i_p} X_{i_2}{}^{i_1} X_{i_3}{}^{i_2} \dots X_{i_1}{}^{i_p}, \tag{48}$$

are the invariant operators for $O(n)$ or $\mathrm{Sp}(2n)$ since the corresponding polynomials 3(6) in a dual space are invariants of the adjoint group. We shall calculate the spectra of invariant operators by the same method as that of $U(n)$. We shall take advantage of the fact that in an irreducible representation space $H^m$, which is determined by the highest weight $m = (m_1, \dots, m_n)$ ($n$—rank of a group) the highest weight vector $\psi_m$ has the following properties

$$X_j{}^i \psi_m = 0 \quad \text{for } i > j$$

and (49)

$$X_i{}^i \psi_m = m_i \psi_m.$$

On the basis of eqs. (44) and (45) we get $m_{-i} = -m_i$. Representing now the invariant operator (48) in the form

$$C_p = \sum_{i,j} (T^{(p-1)})_j{}^i X_i{}^j,$$

where

$$(T^{(p-1)})_j{}^i = \sum_{i_1, \dots, i_{p-2}} X_{i_1}{}^i X_{i_2}{}^{i_1} \dots X_j{}^{i_{p-2}}, \tag{50}$$

and using (49) and (47), we get:

$$C_p \psi_m = \Big( \sum_i (T^{(p-1)})_i{}^i X_i{}^i + \sum_{i>j} [(T^{(p-1)})_j{}^i, X_i{}^j] \Big) \psi_m = \sum_{i=-n}^{n} (m_i + 2r_i)(T^{(p-1)})_i{}^i \psi_m, \tag{51}$$

where

$$r_i = \tfrac{1}{2} \sum_{j<i} (1 \mp \delta_{-j}{}^i) \tag{52}$$

The expression $(T^{(p-1)})_i{}^i$ can be calculated recursively

$$\begin{aligned}(T^{(q)})_i{}^i\psi_m &= \Big\{(T^{(q-1)})_i{}^iX_i{}^i + \sum_{i>j}[(T^{(q-1)})_j{}^i, X_i{}^j]\Big\}\psi_m \\ &= \Big\{m_i(T^{(q-1)})_i{}^i + \sum_{j<i}(1\mp\delta_{-j}{}^i)[(T^{(q-1)})_i{}^i - (T^{(q-1)})_j{}^j]\Big\}\psi_m \\ &= \sum_{j=-n}^{n} a_{ij}(T^{(q-1)})_j{}^j\psi_m\end{aligned}$$

where

$$\begin{gathered} a_{ij} = (l_i+\alpha)\,\delta_{ij} - \theta_{ji} + \tfrac{1}{2}\beta(1+\varepsilon_i)\,\delta_{i,-j}, \\ l_i = m_i + r_i, \qquad \theta_{ji} = \begin{cases} 1 & \text{for } j<i, \\ 0 & \text{for } j \geqslant i. \end{cases} \end{gathered} \tag{53}$$

The constants $\alpha$ and $\beta$ for various groups as well as the explicit expressions for $r_i$ are given in the table.

Table I

| Algebra | Group | Invariant form | $\alpha$ | $\beta$ | $r_i$ | Index '$i$' runs over values |
|---|---|---|---|---|---|---|
| $A_{n-1}$* | $SU(n)$ | $\sum_{i=1}^{n} \bar{x}_i\, \bar{y}_i$ | $\frac{n-1}{2}$ | 0 | $\frac{n+1}{2}-i$ | $1, 2, \ldots, n$ |
| $B_n$ | $O(2n+1)$ | $\sum_{i=-n}^{n} x^i y^{-i}$ | $n-\frac{1}{2}$ | 1 | $(n+\frac{1}{2})\varepsilon_i - i$ | $1, 2, \ldots, n, 0, -n, \ldots$ $\ldots, -2, -1$ |
| $C_n$ | $Sp(2n)$ | $\sum_{i=1}^{n}(x^iy^{-i}-x^{-i}y^i)$ | $n$ | $-1$ | $(n+1)\varepsilon_i - i$ | $1, 2, \ldots, n, -n, \ldots$ $\ldots, -2, -1$ |
| $D_n$ | $O(2n)$ | $\sum_{i=1}^{n}(x^iy^{-i}+x^{-i}y^i)$ | $n-1$ | 1 | $n\varepsilon_i - i$ | $1, 2, \ldots, n, -n, \ldots$ $\ldots, -2, -1$ |

Using (51) and (53) we obtain finally

$$C_p(m_1, m_2, \ldots, m_n) = \mathrm{Tr}(a^pE). \tag{54}$$

Therefore the problem of finding of spectra of invariant operators (48) is reduced to the elementary problem of finding of $p$th power of the known matrix $[a_{ij}]$.

Using this formula (54) we easily calculate that the spectra of the lowest order invariant operators of $O(2n)$, $O(2n+1)$ and $Sp(2n)$ are of the form

$$\begin{gathered} C_2 = 2S_2, \quad C_3 = 2[\alpha - \tfrac{1}{2}(\beta-1)]C_2, \\ C_4 = 2S_4 - (2\alpha\beta+\beta-1)S_2, \end{gathered} \tag{55}$$

* For $U(n)$ the values of $\alpha, \beta$ and $r_i$ are the same as for $SU(n)$.

where

$$S_k = \sum_{i=1}^{n} (l_i^k - r_i^k). \tag{56}$$

From (55) and (56) it follows that the spectrum of $C_2$ is

$$C_2 = 2(m^2+2rm). \tag{57}$$

This is (up to factor 2) Racah's result, given in eq. 3(12).

The important problem of finding the independent invariant operators which generate the centre of enveloping algebra, can be solved with the help of Racah's $S$-theorem, considered in subsec. A. This theorem asserts, in the language of the variables $l_1, \ldots, l_n$, that the spectra of invariant operators are invariant under the action of Weyl's $S$-group. In terms of the variables $l_1, \ldots, l_n$ any element $s \in S$ for $O(2n+1)$ or $\mathrm{Sp}(2n)$ can be represented as a permutation of the numbers $l_1, \ldots, l_n$ and as an arbitrary number of inversions

$$l_i \to -l_i, \quad l_j \to l_j, \quad j \neq i.$$

Therefore the spectra of invariant operators can be expressed in terms of symmetric polynomials of even order of the variables

$$\tilde{S}_k = \sum_{i=1}^{n} l_i^k$$

or the variables

$$S_k = \sum_{i=1}^{n} (l_i^k - r_i^k) \tag{58}$$

which are more convenient for practical calculations (e.g. for the identity representation $m = (0, \ldots, 0)$ from $S_k = 0$ it follows directly $C_p = 0$).

Consequently the invariant operators $C_p$ with odd $p$ are not independent and can be expressed in terms of $C_{2q}$ operators with $2q < p$. It can be shown by direct calculation that for $O(2n+1)$ and $\mathrm{Sp}(2n)$ the Jacobian

$$\frac{\partial(C_2, C_4, \ldots, C_{2n})}{\partial(m_1, m_2, \ldots, m_n)}$$

does not vanish.* Thus the set of operators $C_2, C_4, \ldots, C_{2n}$ generates the ring of invariant operators of $O(2n+1)$ and $\mathrm{Sp}(2n)$ groups. A somewhat different situation occurs for $O(2n)$ group. The spectra of invariant operators $C_{2i}$, $i = 1, 2, \ldots, n$, are furthermore invariant under the action of Weyl's group, which in the case of $O(2n)$ reduces to the permutations of numbers $l_i$, $i = 1, 2, \ldots, n$, and pair inversions

$$l_i \to -l_i, \quad l_j \to -l_j, \quad l_k \to l_k, \quad k \neq i, j. \tag{59}$$

However, as we have shown in 8.5 for this group there exist two nonequivalent

* For analogous calculation see : M. Micu, 1964, Construction of Invariants for Simple Lie Groups, *Nuclear Physics* 60, 353–362.

fundamental spinor representations $\Delta_+$ and $\Delta_-$, whose highest weights are of the form

$$m_+ = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}), \qquad m_- = (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, -\tfrac{1}{2}). \tag{60}$$

The spectra of invariant operators $C_{2i}$, $i = 1, \ldots, n$, are expressed in terms of sums $S_k$ with even $k$ and therefore they are not affected by the substitution $m_n \to -m_n$ (which induces $l_n \to l_n$ since $r_n = 0$). The same holds also for the pair of representations $\Delta_+^p \Delta_-^q$ and $\Delta_+^q \Delta_-^p$. Therefore the set of invariant operators $C_{2i}$ cannot characterize the various nonequivalent representations. Because for the spinor representations $m_i \geqslant 0$, $i = 2, 3, \ldots, n-1$, and only $m_n$ can take both positive and negative values, in order to establish a one-to-one correspondence between the components of the highest weight $m_i$ and spectra of invariant operators, it is sufficient to replace the operator, say, $C_{2n}$ by a new invariant operator which is affected by the substitution $m_n \to -m_n$. Such an invariant operator can be constructed with the help of the totally antisymmetric tensor $\varepsilon_{i_1 j_1 \ldots i_n j_n}$, all of whose non-zero components in the spherical coordinates, are defined by the condition $\varepsilon_{n, n-1, \ldots, -n+1, -n} = -1$. The invariant operator constructed with the help of the totally antisymmetric tensor has the form:

$$C_n' = \sum \varepsilon_{i_1 j_1 \ldots i_n j_n} X^{i_1 j_1} \ldots X^{i_n j_n} = \sum_{i_1 j_1 \ldots i_n j_n} \varepsilon^{i_1 \ldots i_n}{}_{j_1 \ldots j_n} X_{i_1}{}^{j_1} \ldots X_{i_n}{}^{j_n}. \tag{61}$$

Acting by $C_n'$ on the highest weight vector $\psi_m$ and using (49) we can express the eigenvalue of $C_n$ in terms of the eigenvalues $m_i$ of diagonal operators $X_i{}^i$. The leading term has the form

$$\sum_{i_1, \ldots, i_n} \varepsilon^{i_1 \ldots i_n}{}_{i_1 \ldots i_n} m_{i_1} \ldots m_{i_n} = (-1)^{n(n-1)/2} 2^n n! m_1 \ldots m_n.$$

Therefore passing to the variables $l_i$ we find that $C_n'$ is a polynomial of degree $n$ in the variables $l_1, \ldots, l_n$ with the leading term $(-1)^{n(n-1)/2} 2^n n!\, l_1 \cdot l_2 \ldots l_n$. The symmetry with respect to the Weyl $S$-group expressed in terms of $l_i$ of $O(2n)$ asserts that the spectrum of invariant operators should remain unchanged under any permutation of the variables $l_1, \ldots, l_n$ and under any 'pair inversions' (59). These conditions are satisfied by the following symmetric polynomials of degree less than or equal to $nl_1 \cdot l_2 \ldots l_n$ and $l_1^\alpha + l_2^\alpha + \ldots + l_n^\alpha$ for even $\alpha < n$. In order to find the final form of the spectrum of $C_n$ we shall utilize the fact that the operator (61) is a pseudoscalar operator in the extended $O(2n)$ group containing space reflections. We have shown that if a given irreducible representation $T_g$ of $\mathrm{SO}(2n)$ is characterized by the highest weight $m = (m_1, \ldots, m_{n-1}, m_n)$ then the mirror-conjugate representation

$$\check{T}_g = T_{ogo^{-1}}, \qquad g \in \mathrm{SO}(2n), \qquad o \text{ — reflection},$$

has the highest weight $m = (m_1, \ldots, m_{n-1}, -m_n)$ (see lemma 8.5.1). Since for $O(2n)$ group $r_n = 0$ we get $m_n = l_n$ and

$$C'_n(l_1, \ldots, l_{n-1}, -l_n) = -C'_n(l_1, \ldots, l_{n-1}, l_n). \tag{62}$$

Due to Racah's $S$-theorem $C'_n$ is a symmetric function of the numbers $l_1, \ldots, l_n$ and therefore eq. (62) is satisfied for any $l_i$, $i = 1, 2, \ldots, n$. Therefore the expression for the spectrum of $C'_n$ can contain only the term proportional to $l_1, l_2, \ldots, l_n$, i.e.,

$$C'_n(m_1, \ldots, m_n) = (-1)^{n(n-1)/2} \cdot 2^n n! \, l_1 \cdot l_2 \ldots l_n.$$

It can be verified by direct calculation that

$$\frac{\partial(C_2, C_4, \ldots, C_{2(n-1)}, C'_n)}{\partial(m_1, m_2, \ldots, m_n)} \neq 0$$

(for analogous calculations see footnote on p. 363).

## 2. Special Cases

Consider first the case of totally symmetric representations of $O(2n)$, $O(2n+1)$ or $\mathrm{Sp}(2n)$, which are characterized by the highest weight vector $m = (f, 0, \ldots, 0)$. Utilizing formula (54) we get:

$$\begin{aligned} C_p(f, 0, \ldots, 0) &= (f+2\alpha)^p + (-f)^p + (2\alpha+\beta-1) + \\ &+ (2\alpha-1)\left(1 + \frac{\beta+1}{2(\alpha-1)}\right)\left[\frac{(-f)^p - 1}{f+1} - \frac{(f+2\alpha)^p - 1}{f+2\alpha-1}\right] + \\ &+ \frac{\alpha(\beta+1)}{2(\alpha-1)} \frac{(f+2\alpha)^p - (-f)^p}{f+\alpha}. \end{aligned} \tag{63}$$

For lowest values of $p$ this formula simplifies

$$C_2 = 2f(f+2\alpha), \tag{64}$$

$$C_4 = 2f(f+2\alpha)[f^2 + 2\alpha f + 2\alpha^2 - \alpha\beta - \tfrac{1}{2}(\beta-1)]. \tag{65}$$

In the case of totally antisymmetric fundamental representations characterized by the highest weights $m = (\underbrace{1, 1, \ldots, 1}_{k \text{ times}}, 0, \ldots, 0)$, the spectrum of $C_p$ operators is of the form:

$$\begin{aligned} C_p(\{1^k\} &= -(2\alpha+2-k)^p - k^p + (-1)^p(2\alpha+\beta+3) + \\ &+ (2\alpha+3)\left(1 + \frac{\beta-1}{2(\alpha+2)}\right)\left[\frac{k^p - (-1)^p}{k+1} + \frac{(2\alpha+2-k)^p - (-1)^p}{2\alpha+3-k}\right] + \\ &+ \frac{(\alpha+1)(\beta-1)}{2(\alpha+2)} \cdot \frac{k^p - (2\alpha+2-k)^p}{\alpha+1-k}. \end{aligned} \tag{66}$$

For $p = 2, 4$ we obtain

$$C_2 = 2k(2\alpha+2-k), \tag{67}$$

$$C_4 = 2k(2\alpha+2-k)[k^2-2(\alpha+1)k+(\alpha+1)(2\alpha+2-\beta)+\tfrac{1}{2}(\beta+1)]. \tag{68}$$

As we have mentioned already, for $O(2n+1)$ and $O(2n)$ we have besides the fundamental tensor representations $\{1^k\}$ still the fundamental spinor representations, whose highest weights are:

$$m = \begin{cases} (\tfrac{1}{2}, \ldots, \tfrac{1}{2}) & \text{for } O(2n+1), \\ (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \pm\tfrac{1}{2}) & \text{for } O(2n). \end{cases}$$

The spectra of $C_p$ for these representations are:

$$C_p = \begin{cases} n[n^{p-1}-(-\tfrac{1}{2})^{p-1}] & \text{for } O(2n+1), \\ (n-\tfrac{1}{2})[(n-\tfrac{1}{2})^{p-1}-(-\tfrac{1}{2})^{p-1}] & \text{for } O(2n). \end{cases}$$

## § 5. The Enveloping Field

Certain physical observables are described by operators that are quotients of polynomials of the generators of a Lie algebra; for instance, the square of the relativistic spin operators has the form

$$S^2 = \frac{W_\mu W^\mu}{P_\mu P^\mu}, \quad \text{where } W_\mu = \tfrac{1}{2}\varepsilon_{\mu\alpha\beta\gamma}M^{\alpha\beta}P^\gamma, \tag{1}$$

where $P^\mu$, $M^{\alpha\beta}$ are the generators of the Poincaré group $\Pi$. The quantities of type (1) are not in the enveloping algebra of $\Pi$. The latter consists of polynomials in the generators only. Therefore, we introduce the concept of the *enveloping field* of a Lie algebra which incorporates in a natural way, quotients of polynomials of the generators of the Lie algebra. The enveloping field has other interesting properties: it depends only weakly on the original Lie algebra and can be generated by a Heisenberg algebra $p_1, q_1, \ldots, p_n, q_n$, and a certain number of commutative operators $C_1, \ldots, C_k$. This result is of importance for the theory of dynamical groups in particle physics.

### *Rings and Quotient Fields*

We begin with an analysis of certain properties of rings and fields. We recall that a *ring* $R$ is an abelian group with respect to addition and a (in general noncommutative) multiplicative semigroup with or without the unit element. An element $b$ in $R$ is said to be a *non-zero divisor* if there is no $c \in R$, $c \neq 0$, such that $cb = 0$ or $bc = 0$.

A ring $R$ is called a *left Noether ring* if every chain of (left) ideals of $R$

$$R_1 \Subset R_2 \Subset \ldots \tag{2}$$

terminates* (i.e., there exists an index $n$ such that $R_n = R_{n+1} = \ldots$).

* $R_1 \Subset R_2$ means '$R$ is a proper ideal of $R_2$'.

A ring $R$ is called a (*left*) *Ore ring* if for every $a, b \in R$, where $b$ is a non-zero divisor, there exist $a', b' \in R$, where $b'$ is a non-zero divisor, such that $b'a = a'b$.

We now introduce the important concept of quotients. Let $(a, b)$ and $(c, d)$ be two ordered pairs from $R \times R$ and let $a$ and $c$ be a non-zero divisors; we say that $(a, b)$ is equivalent to $(c, d)$ if there exist $x, y \in R$ such that

$$(xa, xb) = (yc, yd). \tag{3}$$

We can easily verify that all the axioms of equivalence are satisfied.

A *quotient* associated with an Ore ring $R$ is defined as the ordered pair $(a, b)$, $a, b \in R$, $a$ a non-zero divisor, equipped with the above equivalence relation. We shall denote the $(a, b)$-quotient by the symbol $a^{-1}b$. Similarly one defines the right quotient, denoted by $ab^{-1}$.

Further, we introduce in the set of all quotients the operations of addition, subtraction, division and multiplication in the following natural manner

$$a^{-1}b_1 \pm a^{-1}b_2 \equiv a^{-1}(b_1 \pm b_2), \tag{4}$$

$$(a^{-1}b_1)^{-1}(a^{-1}b_2) \equiv b_1{}^{-1}b_2, \tag{5}$$

$$(a_1{}^{-1}b_1)(a_2{}^{-1}b_2) \equiv (b_1{}^{-1}a_1)^{-1}(a_2{}^{-1}b_2). \tag{6}$$

We recall that a ring $R$ with the unit element $I$ is called a *field*, if for any $a \in R$, $a \neq 0$, there exists an $a^{-1}$ such that $aa^{-1} = a^{-1}a = I$. We see that the set of all quotients associated with an Ore ring without zero divisor and equipped with operations (4)–(6) forms a field which we shall call the *quotient field*.

EXAMPLE 1. Let $R$ be the ring of all even integers $R = \{\pm 2n,\ n = 0, 1, \ldots\}$. Every element $a \in R$, $a \neq 0$ is a non-zero divisor of $R$. If $a = 2n$, $b = 2m$, $m \neq 0$, then there exist $a'$ (equal, e.g., $2n$) and $b' \neq 0$ (equal, e.g., $2m$) such that $b'a = a'b$. Hence, $R$ is an Ore ring.

Let $(a, b) \equiv (2n, 2m)$, $a \neq 0$. By virtue of the condition (3) any pair $(c, d) \equiv (2n', 2m')$ equivalent to $(a, b)$ satisfies the condition

$$\frac{m'}{n'} = \frac{m}{n}. \tag{7}$$

Thus, the abstract quotient $a^{-1}b$ corresponds to the class of all pairs $(2n', 2m')$ of the form $(xa, xb)$, $x \in R$ which have the constant ratios $\dfrac{m'}{n'} = \dfrac{b}{a}$. Consequently, the abstract field of quotients defined by relations (4)–(6) corresponds in the present case to the field of all rational numbers. ▼

### *Heisenberg Ring and Field*

We now introduce the field associated with the Heisenberg algebra generated by $p$, $q$ and a Noether ring $A$.

Let $A$ be an arbitrary Noether ring over $R$ or $C$ without zero divisor. We denote by $R_n(A)$ an algebra over $A$ with $2n$ generators $p_1, \ldots, p_n, q_1, \ldots, q_n$ satisfying the commutation relations

$$[p_i, q_j] = \delta_{ij} I, \quad [p_i, p_j] = 0, \quad [q_i, q_j] = 0. \tag{8}$$

We first introduce a basis in $R_n(A)$.

PROPOSITION 1. *The algebra $R_n(A)$ is a free $A$-module and has a basis consisting of all monomials of the form*

$$(p_1)^{k_1} \ldots (p_n)^{k_n} (q_1)^{l_1} \ldots (q_n)^{l_n} \equiv p^{(k)} q^{(l)}, \tag{9}$$

*where $p_i^{(0)} = I_R$ and $q_i^{(0)} = I_R$.*

PROOF: We show first that monomials (9) generate the $A$-module $R_n(A)$. Let $[R_n(A)]_k$ denote the set of all elements in $R_n(A)$ which can be written as polynomials in $\{p, q\}$ with coefficients in $A$ and of degree $\leqslant k$. Clearly,

$$R_n(A) = \bigcup_k [R_n(A)]_k. \tag{10}$$

Assume that our assertion is true for $k < k_0$; by virtue of commutation relations (8) we can reduce any monomial of degree $k_0$ to a monomial of the form (9) (i.e., with all $p$'s on the left side) modulo $[R_n(A)]_{k_0-2}$. Because the statement is obviously true for $k = 0$ and $k = 1$ our assertion follows by the method of induction. Next we show that the monomials (9) are linearly independent over $A$. Let $x = \Sigma\, a_{kl} p^{(k)} q^{(l)} = 0$, where $a_{kl} \in A$, and at least one of the $a_{kl}$ be non-zero. Let us introduce a lexicographic ordering of $(k, l) = (k_1, \ldots, k_n, l_1, \ldots, l_n)$ and et $(k^0, l^0)$ be the greatest set (according to the lexicographic ordering) for which $a_{kl} \neq 0$. By a simple calculation one verifies that

$$\prod_{i=1}^{n} ad^{k_i^0}(-q_i)\, ad^{l_i^0} p_i x = \prod_{i=1}^{n} k_i! l_i! a_{k^0 l^0} \neq 0 \tag{11}$$

which contradicts the equality $x = 0$. Hence, $p^{(k)} q^{(l)}$ cannot be linearly dependent. ▼

PROPOSITION 2. *$R_n(A)$ is an Ore ring.*

PROOF: We introduce a filtration of $R_n(A)$ given by

$$[R_n(A)]_0 \subset [R_n(A)]_1 \subset [R_n(A)]_2 \subset \ldots, \tag{12}$$

where $R_n(A)_k$ are the same objects as those used in the proof of the proposition 1. Consider the *graded ring structure* given by

$$\mathrm{gr}^{(k)} R_n(A) \equiv [R_n(A)]_k / [R_n(A)]_{k-1} \tag{13}$$

and

$$\mathrm{gr}\, R_n(A) \equiv \sum_{k=0}^{\infty} \mathrm{gr}^{(k)} R_n(A).$$

By virtue of proposition 1 one obtains the one-to-one correspondence

$$\operatorname{gr} R_n(A) \leftrightarrow R[p, q], \tag{14}$$

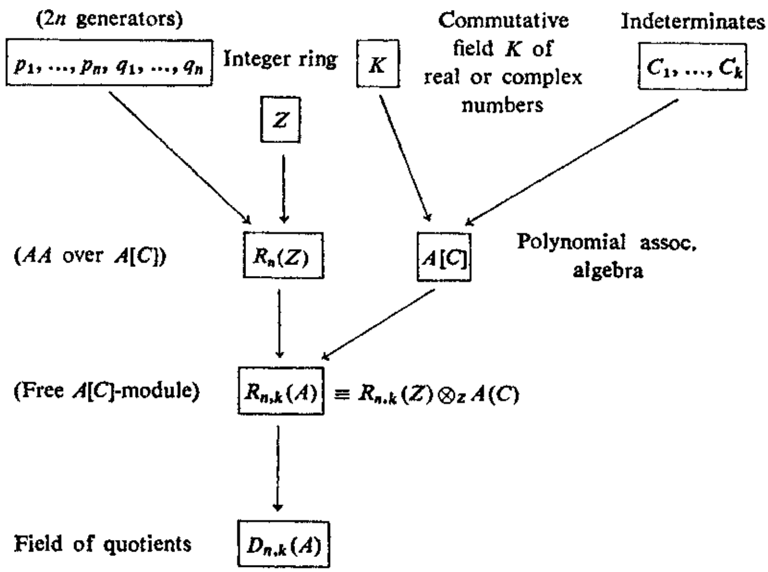
where $R[p, q]$ is the polynomial ring of $(p, q)$. Clearly, $R[p, q]$ is a Noether ring without zero divisors (because it is a polynomial ring). Consequently, $\operatorname{gr}R_n(A)$ is also a Noether ring. Hence, by virtue of propositions 1 and 2 of app. A.2 we conclude that $R_n(A)$ is an Ore ring without zero divisors. ▼

We now associate, by virtue of proposition 2, with the ring $R_n(A)$ a quotient field $D_n(A)$. In what follows the ground ring $A$ is a polynomial ring over a set of $k$ indeterminates $\{C_1, C_2, \ldots, C_k\} \equiv \hat{C}$ with coefficients in the set $K$ of real or complex numbers. Set

$$R_{n,k}(K) = R_n(A[\hat{C}]), \tag{15}$$

$$D_{n,k}(K) \equiv \text{field of quotients associated with } R_{n,k}(K). \tag{16}$$

The following picture illustrates the connection between various objects.




*Enveloping Field of a Lie Algebra*

Let $L$ be a Lie algebra over the set $K$ of real or complex numbers and let $E(L)$ be the enveloping algebra of $L$. We have

PROPOSITION 3. *$E(L)$ is an Ore ring.*

PROOF: Let us introduce the *filtration* of $E(L)$ given by

$$E^{(0)} \subset E^{(1)} \subset E^{(2)} \subset \ldots \tag{17}$$

where $E^{(k)}$ denotes the set of all elements in $E$ which can be written as polynomials in the generators of $L$ with coefficients in $K$ and of degree $\leqslant k$. Consider the corresponding *graded ring structure* defined by

$$\mathrm{gr}^k E \equiv E^{(k)}/E^{(k-1)} \tag{18}$$

and

$$\mathrm{gr} E \equiv \sum_{k=0}^{\infty} \mathrm{gr}^k E. \tag{19}$$

By virtue of proposition 1 and definitions (18) and (19) we have the one-to-one corespondence

$$\mathrm{gr} E \leftrightarrow R(X_1, \ldots, X_m), \tag{20}$$

where $R(X_1, \ldots, X_m)$ is the ring of all polynomials of $m$ variables, $m = \dim L$. Therefore, $\mathrm{gr} E$ is a Noether ring without zero-divisors (because it is a polynomial ring). By virtue of propositions 1 and 2 of app A.2 one concludes that $E$ is an Ore ring without zero divisors. ▼

The above property permits us to construct the field of quotients $D(L)$ from $E(L)$. This field we shall call the enveloping field of $L$ or simply the *Lie field.*

We see that the concept of the field of quotients allows us to consider in a natural manner the quotients of arbitrary elements of an enveloping algebra $E(L)$ of a given Lie algebra $L$. If $L$ is a Poincaré algebra the elements of the form (1) will be in the field $D(L)$ of $L$.

This is not the only advantage of the concept of a field $D(L)$ associated with $E(L)$. It is interesting that the properties of the field $D(L)$, in contradistinction to the properties of $E(L)$, depend only weakly on the original Lie algebra. We may roughly state this in the form:

The field $D(L)$ of a Lie algebra $L$ is isomorphic to a Heisenberg field $D_{n,k}(K)$.

We show this result first for the field $D(L)$ of the full linear algebra $\mathrm{gl}(n, C)$.

THEOREM 4. *The enveloping field $D(\mathrm{gl}(n, C))$ of the Lie algebra $\mathrm{gl}(n, C)$ is isomorphic to the Heisenberg field $D_{\frac{1}{2}n(n-1),n}(C)$.*

PROOF: We first show that if $L_n$ is the Lie algebra of all $n \times n$ matrices which have zeros in the last row, then

$$D(L_n) = D_{(n/2)(n-1),0}(C). \tag{21}$$

We show this by induction over $n$. Let $e_{ik}$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, n+1$, be a natural basis in $L_{n+1}$ given by 1.1(11). Set $q_i = e_{i,n+1}$, $p_i = e_{i,i} q_i^{-1}$, $\tilde{e}_{ik} = e_{ik} q_i^{-1} q_k$ (no summation).

Because $\det q_i = 0$, the quantity $q_i^{-1}$ is not defined as a matrix, but it is defined as a formal quantity whose rules of manipulation are conformed to the equivalence relation of quotients. By direct computation we verify that

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [p_i, q_j] = \delta_{ij} I, \tag{22}$$

$$[\tilde{e}_{ik}, q_j] = \delta_{kj} q_j, \qquad [\tilde{e}_{ik}, p_j] = -\delta_{kj} p_j \qquad \text{(no summation)} \tag{23}$$

Indeed, for instance

$$\begin{aligned}[q_i, q_j] = q_i q_j - q_j q_i &= e_{i,n+1} e_{j,n+1} - e_{j,n+1} e_{i,n+1} \\ &= \delta_{j,n+1} e_{i,n+1} - \delta_{i,n+1} e_{j,n+1} = 0.\end{aligned} \tag{24}$$

This relation implies also, by the definition of quotients, that

$$q_j q_i^{-1} = q_i^{-1} q_j, \tag{25}$$

i.e., $q_i^{-1}$ and $q_j$ commute.

Similarly we compute

$$\begin{aligned}[p_i, q_j] &= e_{ii} q_i^{-1} q_j - q_j e_{ii} q_i^{-1} = e_{ii} q_j q_i^{-1} - e_{j,n+1} e_{ii} q_i^{-1} \qquad \text{(no summation)} \\ &= e_{ii} e_{j,m+1} q_i^{-1} - \delta_{i,n+1} e_{ji} q_i^{-1} = \delta_{ij} e_{i,m+1} q_i^{-1} = \delta_{ij} q_i q_i^{-1} = \delta_{ij} I.\end{aligned}$$

Now, if the coefficients $c_{ik}$ of a matrix $c$ obey the condition

$$\sum_k c_{ik} = 0, \qquad i = 1, 2, \ldots, n, \tag{26}$$

then we see, by virtue of eq. (23), that the operator $\alpha(c) = c_{ik}\tilde{e}_{ik}$ commutes with all the operators $p_i$ and $q_j$. Let $\tilde{L}$ denote the set of all matrices $c$ of order $n$ whose entries satisfy the conditions (26). The set $\tilde{L}$ is isomorphic with $L_n$; indeed, if $l_{ij}$ are the matrix entries of an element $l \in L_n$, then the correspondence given by

$$c_{ij} = l_{ij}, \qquad i = 1, 2, \ldots, n-1, \; j = 1, 2, \ldots, n, \tag{27}$$

and

$$c_{nj} = -\sum_{i=1}^{n-1} l_{ij}, \qquad j = 1, 2, \ldots, n, \tag{28}$$

gives a one-to-one mapping of $\tilde{L}$ onto $L_n$.

With every matrix $c \in \tilde{L}$ we now associate an element $\alpha(c) \equiv c_{ik}\tilde{e}_{ik} \in D(L_{n+1})$ One readily verifies that

$$\alpha([c_1, c_2]) = [\alpha(c_1), \alpha(c_2)]. \tag{29}$$

It is clear that the Lie field $D(L_{n+1})$ is generated by elements of the form $\alpha(c)$, $c \in \tilde{L}$, and by the elements $p_1, \ldots, p_n, q_1, \ldots, q_n$. Hence, in order to conclude our assertion it is sufficient to show that the field generated by the elements $\alpha(c)$ is isomorphic to the field $D_{\frac{n}{2}(n-1),0}(L)$. Now $\tilde{L}$ is isomorphic to $L_n$; hence, eq. (21) for $L_{n+1}$ follows by induction.

The center of $E(\mathrm{gl}(n, C))$ is generated by $n$ elements of the form $C_1, C_2, \ldots, C_n$, $C_s = \mathrm{Tr}\, e^s$, $e \equiv \{e_{ik}\}$.

The elements of the last row enter linearly into the monomials $C_i$; consequently, $D(\mathrm{gl}(n, C))$ is generated by $D(L_n)$ and by the elements $C_1, C_2, \ldots, C_n$. Thus,

$$D(\mathrm{gl}(n, C)) = D_{(n/2)(n-1),n}(C). \blacktriangledown \tag{30}$$

COROLLARY. $D(\mathrm{sl}(n, C)) = D_{\frac{1}{2}n(n-1),\, n-1}(C)$.

PROOF: $D(\mathrm{sl}(n, C))$ is generated by $C_2, C_3, \ldots, C_n$ and $D(L_n)$. Hence eq. (30) follows from eq. (21). ▼

In order to generalize the basic result to an arbitrary semisimple Lie algebra $L$ it is convenient to use the concrete realization of $L$ as differential operators on the manifold $X = N\backslash G$, where $N$ is a nilpotent subgroup from the Iwasawa decomposition $G = KAN$. In this case there might exist invariant operators in the carrier space $H = L^2(X)$ which are not elements of the center of $E(L)$ of the Lie algebra $L$. Let $\tilde{E}(L)$ denote the extension of $E(L)$ by the ring of the invariant operators which are not generated by the elements of the center* $Z$ of $E(L)$. Then, we have

THEOREM 5. *Let $L$ be a semisimple algebra and let $D(L)$ be the enveloping field of the extended enveloping algebra $\tilde{E}(L)$ realized in the space $H = L^2(X)$, $X = N\backslash G$. Let $(t_\alpha, \tau_i)$, $\alpha = 1, 2, \ldots, n$, $n = \dim N$ and $i = 1, 2, \ldots, k = \operatorname{rank} G$, be coordinates in the space $X$. Then, the field $D(L)$ is isomorphic to the Heisenberg field $D_{n,k}(C)$ generated by operators*

$$p_\alpha = \frac{\partial}{\partial t_\alpha}, \quad q_\alpha = t_\alpha, \quad \textit{and} \quad C_i = \tau_i \frac{\partial}{\partial \tau_i} \quad (\textit{no summation}). \blacktriangledown \tag{31}$$

(For the proof cf. Gel'fand and Kirillov 1969.)

These authors also proved an analogous theorem for a class of nilpotent Lie algebras (cf. Gel'fand and Kirillov 1966).

These theorems show that there is a common underlying structure of the enveloping fields of all semisimple Lie algebras and this structure is isomorphic to the structure of Heisenberg fields.

The numbers $n$ and $k$ which appear in the Heisenberg field $D_{n,k}(C)$ have a definite meaning in the representation theory: namely, in the case of finite-dimensional representations $k$ represents the number of components of the highest weight $m = (m_1, \ldots, m_k)$ which characterizes the irreducible representation $T^{L^m}$ of $G$ (cf. ch. 8). The number $n$ represents the dimension of the domain $G_0\backslash G$ of functions of the carrier space $H(G_0\backslash G)$ of an irreducible representation $T^{L^m}$ of $G$ (cf. ch. 8). This interpretation of the numbers $k$ and $n$ is also valid in the theory of infinite-dimensional representations (cf. ch. 19).

The fact that an enveloping field of a Lie algebra is generated by the Heisenberg algebra $p_i$, $q_i$ and a set $C_1, \ldots, C_k$ of invariant operators is useful in quantum mechanics and in particle physics; in particular, it serves as a tool in the interpretation and analysis of the so-called dynamical groups (cf. ch. 13).

* Note that the center $Z$ of $E(L)$ contains invariant operators which are polynomials in the generators; $\tilde{E}(L)$ contains invariant operators which are rational functions of the generators. However, in general, there might be invariant operators which are more general functions of generators, e.g. pseudo-differential operators.

## § 6. Further Results and Comments

### A. *Casimir Operators and Their Spectra for Semisimple Lie Algebras*

We now extend th. 4.2 to other semisimple Lie algebras. There is no complete tensor calculus for arbitrary semisimple Lie algebras. Therefore we use eq. 3(7) for the Casimir operators

$$C_p = g_{\mu_1\mu_2\ldots\mu_p} X^{\mu_1}X^{\mu_2}\ldots X^{\mu_p}, \quad p = 2, 3, \ldots, \tag{1}$$

where

$$g_{\mu_1\mu_2\ldots\mu_p} = \mathrm{Tr}(\hat{X}_{\mu_1}\hat{X}_{\mu_2}\ldots\hat{X}_{\mu_p}) \tag{2}$$

and $\hat{X}_\mu$ is the representation of $X_\mu$ in an arbitrary irreducible representation of the given Lie algebra. For classical Lie algebras $A_n$, $B_n$, $C_n$ and $D_n$, we take for the representation $X_\mu \to \hat{X}_\mu$ the simplest fundamental representation $m = (1, 0, \ldots, 0)$. The following theorem gives the direct generalization of results of 4.2 for semisimple Lie algebras.

THEOREM 1. *Let $m = (m_1, \ldots, m_n)$ be the highest weight of an irreducible representation of anyone of the classical Lie algebras $A_n$, $B_n$, $C_n$, $D_n$, $n = 1, 2, \ldots$, and let $X_\mu \to \hat{X}_\mu$ be the simplest fundamental representation $m = (1, 0, \ldots, 0)$. The function*

$$G(z) = z^{-1}\left(1 + \frac{\beta z}{2-(2\alpha+1)z}\right)(1-\Pi(z)), \tag{3}$$

*where*

$$\Pi(z) = \prod_{(i)}\left(1 - \frac{z}{1-\lambda_i z}\right), \quad \lambda_i = l_i+\alpha,\ l_i = m_i+r_i,\ i>0,\ l_{-i} = -l_i, l_0 = 0, \tag{4}$$

*is a generating function for the spectrum of the Casimir operators* (1), *i.e.*,

$$G(z) = \sum_{p=0}^{\infty} C_p(m_1, \ldots, m_n) z^p. \tag{5}$$

*The parameters $\alpha$, $\beta$ and $r_i$ are given in table 1 for various Lie algebras.* ▼

(For the proof cf. Perelomov and Popov 1968.)

These authors have also calculated the form of the spectrum of Casimir operators for some exceptional Lie algebras.

### B. *Comments*

(i) The concept of a tensor operator was first introduced by Wigner 1931. The technique of tensor operators was successfully used by Racah, Eliott, Jahn and others in the theory of atomic and nuclear spectra (cf. the collection by Biedenharn and van Dam 1965 for references). More recently this technique has been used also in various problems of elementary particle theory and in particular in the current algebra approach (cf., e.g., Adler and Dashen 1968).

(ii) The various generalizations of the Wigner–Eckárt theorem were given by Sharp 1960, Stone 1961, Biedenharn 1963, Din 1963, Ginibre 1963, Moshinsky 1963, and Klimyk 1971.

(iii) The selection of the generators of the center $Z$ of the enveloping algebra $E$ is not unique; for instance, in the case of $u(n)$ one could take instead of the generators 4(6) the operators

$$C'_p = A_{i_1}{}^{i_2} A_{i_2}{}^{i_3} \dots A_{i_p}{}^{i_1} = \mathrm{Tr} A^p. \tag{6}$$

These are invariant operators of $u(n)$ by virtue of th. 1.1. Using the commutation relations 4(2) one may express the Casimir operators $C'_p$ in terms of $C_p, C_{p-1}, \dots, C_1$. For instance, in the case $p = 3$, one obtains

$$C'_3 = C_3 - nC_2 + (C_1)^2. \tag{7}$$

One often uses also the symmetrized Casimir operators of the form

$$C''_p = \frac{1}{p!} P(X_{i_2}{}^{i_1} X_{i_3}{}^{i_2} \dots X_{i_1}{}^{i_p}), \tag{8}$$

where the symbol $P$ denotes summation over $p!$ permutations of the generators in the bracket (cf., e.g., Gel'fand 1950, Berezin 1957). It seems, however, that the formula (1) for the Casimir operators is the most convenient one in the calculation of the spectra of invariant operators.

(iv) The invariant operators $C$ and $C'$ in $E$ always commute. This property is, however, not true for arbitrary invariant operators; for instance, the operators $r$ and $d/dr$, $r = |\vec{x}|$, are invariant operators of the rotation group acting in $H = L^2(R^3)$, but they do not commute.

(v) There exists two important results concerning the structure of the ring of invariant operators in the enveloping algebra in the space $H = L^2(X)$, where $X$ is a symmetric space.

GEL'FAND–CHEVALLEY THEOREM. *The number of generators of the ring of invariant operators in $L^2(X)$ is equal to the rank of symmetric space $X$.*

(For the proof see Helgason 1962, ch. 10.)

Thus in particular on symmetric spaces of rank one the center of enveloping algebra is generated by a single element. In this case we have

THEOREM 2. *The ring of invariant operators in $L^2(X)$, where $X$ is a symmetric space of rank one, is generated by the Laplace–Beltrami operator*

$$\Delta(x) = |\bar{g}|^{-1/2} \partial_\alpha g^{\alpha\beta}(x) |\bar{g}|^{1/2} \partial_\beta, \tag{9}$$

*where $g^{\alpha\beta}(x)$ is the left-invariant metric tensor on $X$ and*

$$\bar{g}(x) = \det[g_{\alpha\beta}(x)].$$

(For the proof cf. Helgason 1962, ch. 9.)

(vi) We now give the important so-called Commutativity Theorem of Segal concerning the structure of the algebra of invariant operators in $L^2(G, \mu)$. Let $G$ be a unimodular locally compact group with a Haar measure $\mu$ and let $g \to T_g^{\mathrm{L}}$ and $g \to T_g^{\mathrm{R}}$ be the left and the right regular representations of $G$ in $L^2(G, \mu)$.

Denote by $\mathscr{R}_{\mathrm{L}}$ (or $\mathscr{R}_{\mathrm{R}}$) the closure in the weak operator topology of the set of all linear combinations of the $T_g^{\mathrm{L}}$ or $T_g^{\mathrm{R}}$. Then we have

THEOREM 3. *If $G$ is unimodular then we have*

$$\mathscr{R}'_{\mathrm{L}} = \mathscr{R}_{\mathrm{R}}, \quad \mathscr{R}'_{\mathrm{R}} = \mathscr{R}_{\mathrm{L}}, \tag{10}$$

$$(\mathscr{R}_{\mathrm{L}} \cup \mathscr{R}_{\mathrm{R}})' = \mathscr{R}'_{\mathrm{L}} \cap \mathscr{R}'_{\mathrm{R}} = \mathscr{R}_{\mathrm{L}} \cap \mathscr{R}'_{\mathrm{L}} = \mathscr{R}_{\mathrm{R}} \cap \mathscr{R}'_{\mathrm{R}}. \tag{11}$$

(For the proof cf. Segal 1950 or Maurin 1968, ch. 6, § 7.)

## § 7. Exercises

§ 1.1. Let $G = \mathrm{SO}(3)$. Show that the hermitean adjoint of the tensor operator $Y_M^J$ given by

$$(Y_M^J)^* = (-1)^M Y_{-M}^J$$

is also a tensor operator.

§ 1.2. Prove the Wigner–Eckart theorem for arbitrary compact groups.

*Hint*: Decompose onto irreducible components that representation of $G$ with respect to which the product $T_m^\lambda U_{m_1}^{\lambda_1}$ transforms and use the method of proof of th. 2.

§ 2.1. Show that the enveloping algebra of the Heisenberg algebra $[P, Q] = I$ has the trivial center.

§ 3.1. Show that the center of the enveloping algebra of the Euclidean Lie algebra $T^n \circledS \mathrm{so}(n)$ is generated by $\{n/2\}$ elements.

§ 3.2. Find the generators of the center of the enveloping algebra of the semi-direct product $C^3 \circledS \mathrm{SU}(3)$.

§ 3.3. Show that the following operators

$$C_2 = \tfrac{1}{2} M_{\mu\nu} M^{\mu\nu} = \boldsymbol{J}^2 - \boldsymbol{N}^2,$$

$$C_2' = -\tfrac{1}{4} \varepsilon_{\alpha\beta\gamma\delta} M^{\alpha\beta} M^{\gamma\delta} = \boldsymbol{J} \cdot \boldsymbol{N},$$

where $\boldsymbol{J} = (M_{23}, M_{31}, M_{12})$ and $\boldsymbol{N} = (M_{01}, M_{02}, M_{03})$ generate the center of the enveloping algebra of the Lorentz group.

§ 5.1. Is the operator of helicity $\boldsymbol{J} \cdot \boldsymbol{p}/|\boldsymbol{p}|$ an element of the enveloping field of the Poincaré group?

§ 5.2.** Elaborate an extension of the concept of enveloping field of Euclidean Lie algebra which would contain invariant operators of the form $|r|$, $\dfrac{1}{|r|}$, $\dfrac{\mathrm{d}}{\mathrm{d}|r|}$, etc.

§ 5.3.* Let $g \to T_g$ be a quasi-regular representation of SO(3) in the space $H = L^2(X), X = \mathrm{SO}(3)/\mathrm{SO}(2)$. Show that there are no pseudodifferential invariant operators in $H$.

§ 5.4.* Find the generators of the center of the enveloping field $D(L)$ for semisimple algebras.

§ 5.5. Let $\Pi = T^4 \otimes SO(3,1)$, $N$ be the Iwasawa factor of $SO(3,1)$ and $G_0 = T^4 \otimes N$. Show that in the space $H = L^2(X)$, $X = \Pi/G_0$ there are more than two invariant differential operators of $\Pi$.

§ 6.1.** Find the spectra of the Casimir operators for the irreducible representations of the exceptional Lie algebras.

§ 6.2.** Find the generating function 6 (5) for the spectra of the Casimir operators for exceptional Lie algebras.

## Chapter 10

# The Explicit Construction of Finite-Dimensional Irreducible Representations

The method of induced representations presented in ch. 8 solves the problem of classification of all finite-dimensional irreducible representations of all simple Lie groups. However, in order to examine all the consequences for physical applications, we need to determine explicitly:

1° The set of independent invariant operators.

2° The complete set of commuting operators (CSCO) which we interpret as physical observables; the nature and the range of their spectra (cf. ch. 13).

3° The properties of the basis functions and the dimension of the carrier space that is identified with the space of physical states.

4° The properties of the decomposition of a representation of $G$ with respect to a subgroup $G_0$ of $G$.

5° The matrix elements of the operators $T_g$ or $T(X)$ of the representation of $G$ or Lie algebra $L$.

In this chapter we discuss four general methods of construction of irreducible representations for which some or all of the problems 1°–5° are explicitly solved.

### § 1. The Gel'fand–Zetlin Method

The first method of construction of irreducible representations which also provides solutions to the problems 1°–5° listed above is the so-called Gel'fand–Zetlin formalism. It can be applied to both compact and noncompact groups and seems to be especially suitable for applications in quantum physics. We describe this formalism in detail using the examples of the algebras $u(n)$ and $so(n)$.

A. *The Representations of* $u(n)$

The group $U(n)$ is defined as the transformation group in $C^n$ which conserves the hermitian form

$$z_1\bar{z}_1+z_2\bar{z}_2+\ldots+z_n\bar{z}_n = \text{const.} \tag{1}$$

Thus, for $u \in U(n)$, $u^*u = uu^* = 1$. Consequently the generators of the one-parameter subgroups obey the hermiticity condition*

$$M_{ik}^* = M_{ik}. \tag{2}$$

The set of $n^2$ generators (2) span the Lie algebra of $U(n)$. However, as we have noted in ch. 9, because the commutation relations of the $M_{ik}$ cannot be written in a symmetric way we start from the Lie algebra of the group $\mathrm{GL}(n, R)$ whose elements satisfy the commutation relations

$$[A_{ij}, A_{kl}] = \delta_{jk} A_{il} - \delta_{il} A_{kj}, \quad i, j = 1, 2, \ldots, n. \tag{3}$$

The Lie algebra (3) has an $n^2$-dimensional representation given by the Cartan–Weyl matrices

$$A_{ij} \to (e_{ij})_{lk} = \delta_{il} \delta_{jk}. \tag{4}$$

These matrices obey the condition

$$e_{ij}^* = e_{ji}. \tag{5}$$

Let us introduce the following $n^2$ operators

$$\begin{gathered} M_{kk} = A_{kk}, \\ M_{kl} = A_{kl} + A_{lk}, \quad \tilde{M}_{kl} = \mathrm{i}(A_{kl} - A_{lk}), \quad k < l \leqslant n. \end{gathered} \tag{6}$$

When $A_{ij} = e_{ij}$ the operators (6) are $n \times n$-matrices obeying the hermiticity condition (2). Thus they are generators of $U(n)$ and automatically fulfil the commutation relations of the algebra $u(n)$. Therefore, an arbitrary representation of the Lie algebra of $\mathrm{GL}(n, R)$ for which the condition (5) is fulfilled, induces a hermitian representation of $u(n)$ determined by eq. (6). But clearly this fact does not imply that the Lie algebras of $U(n)$ and $\mathrm{GL}(n, R)$ are isomorphic since both algebras are real and the transformation given by (6) is a complex linear substitution.

Next we construct the canonical basis of an arbitrary irreducible representation space of $u(n)$. The construction is based on the following two properties of irreducible representations of $u(n)$:

(i) A finite-dimensional irreducible representation of the Lie algebra $u(n)$ is uniquely determined by an $n$-dimensional vector $m_n = (m_{1n}, m_{2n}, \ldots, m_{nn})$ with integer components $m_{in}$ obeying the condition:

$$m_{1n} \geqslant m_{2n} \geqslant \ldots \geqslant m_{nn}. \tag{7}$$

This vector represents the highest weight of the representation. The space in which the irreducible representation, determined by the vector $m_n$, is realized is denoted by $H^{m_n}$. We shall assume that the Lie algebra $u(n-1)$ is imbedded in a natural manner in $u(n)$, i.e., it is spanned by the generators $A_{ij}$ $i, j = 1, 2, \ldots, n-1$ (cf. sec. 8.3.D).

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* In ch. 9 we used tensors of the form $M_i{}^j$ or $A_i{}^j$. This was convenient for the calculation of the invariants. In this section we shall use covariant tensors of the form $A_{ij}$, $A_{kl}$, etc.

(ii) In the decomposition of the irreducible finite-dimensional representations of $u(n)$ only those irreducible representations of $u(n-1)$ appear for which the components $m_{i,n-1}$ of the highest weight obey the condition

$$m_{in} \geqslant m_{i,n-1} \geqslant m_{i+1,n}, \quad i = 1, 2, \dots, n-1. \tag{8}$$

The multiplicity of every irreducible representation of $u(n-1)$ which appears in the decomposition of $u(n)$ is equal to one (cf. th. 8.8.1).

Let us consider the decreasing chain of algebras

$$u(n) \supset u(n-1) \supset \dots \supset u(2) \supset u(1) \tag{9}$$

and decompose the irreducible space $H^{m_n}$ into subspaces $H^{m_{n-1}}$ which are irreducible with respect to $u(n-1)$. Each of these subspaces decomposes further with respect to irreducible representations of $u(n-2)$, and so on till $u(1)$. Because the irreducible representations of $u(1)$ are one-dimensional, the intersection of the decreasing chain of subspaces

$$H^{m_n} \supset H^{m_{n-1}} \supset \dots \supset H^{m_2} \supset H^{m_1} \tag{10}$$

determines uniquely this one-dimensional subspace. The uniqueness follows from the result (ii). We denote the unit vector which spans this one-dimensional subspace by the so-called *Gel'fand-Zetlin pattern m*:

$$m = \begin{vmatrix} m_{1n} \quad m_{2n} \dots m_{n-1,n} \qquad\qquad m_{nn} \\ m_{1,n-1} \qquad m_{2,n-1} \cdots \qquad m_{n-1,n-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ m_{12} \qquad m_{22} \\ m_{11} \end{vmatrix}. \tag{11}$$

The first row of the pattern is determined by the components of the highest weight of an irreducible representation of $u(n)$. This row is fixed for a given irreducible representation of $u(n)$. In the following rows there are arbitrary integers which obey the following inequalities

$$m_{ij} \geqslant m_{i,j-1} \geqslant m_{i+1,j}, \quad \begin{array}{l} j = 2, 3, \dots, n, \\ i = 1, 2, \dots, n-1. \end{array} \tag{12}$$

These inequalities are reflected in the Gel'fand–Zetlin pattern by the fact that the numbers $m_{i,j-1}$ at the $(j-1)$st row are placed between the numbers $m_{ij}$ and $m_{i+1,j}$ at the $j$th row. For a definite $k$, $k = 1, 2, \dots, n-1$, the numbers $m_{ik}$, $1 \leqslant i \leqslant k$, represent the components of the highest weight $m_k$ of an irreducible representation of $u(k)$ which will appear in the decomposition of an irreducible representation of $u(n)$.

In order to determine a hermitian representation of the Lie algebra $u(n)$, eq. (6), it is sufficient to define the action of the operators $A_{ij}$, $i, j = 1, 2, \dots, n$, on the pattern $m$ and check that the commutation relations (3) and eq. (5) are fulfilled. We can restrict ourselves to the generators $A_{k,k}$, $A_{k,k-1}$ and $A_{k-1,k}$ because the action of the other generators can be obtained from the commutators of these generators: e.g., from eq. (3) we have

$$A_{k-2,k} = [A_{k-2,k-1}, A_{k-1,k}],$$

and generally,

$$\begin{aligned} A_{k,k-h} &= [A_{k,k-1}, A_{k-1,k-h}], \\ A_{k-h,k} &= [A_{k-h,k-1}, A_{k-1,k}], \end{aligned} \qquad h > 1. \tag{13}$$

In case of Lie algebra $u(2)$ the action of generators $A_{k,k}$, $A_{k,k-1}$ and $A_{k-1,k}$ is well known (cf. exercise 8.9.8.2). Guided by this case we define the action of the operators $A_{kk}$, $A_{k,k-1}$ and $A_{k-1,k}$ for an arbitrary $u(n)$ as follows:

$$A_{kk}m = (r_k - r_{k-1})m, \tag{14}$$

$$A_{k,k-1}m = \sum_{j=1}^{k-1} a^j_{k-1}(m) m^j_{k-1}, \tag{15}$$

$$A_{k-1,k}m = \sum_{j=1}^{k-1} b^j_{k-1}(m) \hat{m}^j_{k-1}, \tag{16}$$

where

$$r_0 = 0, \quad r_k = \sum_{j=1}^{k} m_{jk}, \quad k = 1, 2, \ldots, n, \tag{17}$$

$$a^j_{k-1}(m) = \left[ - \frac{\prod_{i=1}^{k} (l_{ik} - l_{j,k-1} + 1) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1})}{\prod_{i \neq j} (l_{i,k-1} - l_{j,k-1} + 1)(l_{i,k-1} - l_{j,k-1})} \right]^{1/2}, \tag{18}$$

$$b^j_{k-1}(m) = \left[ - \frac{\prod_{i=1}^{k} (l_{ik} - l_{j,k-1}) \prod_{i=1}^{k-2} (l_{i,k-2} - l_{j,k-1} - 1)}{\prod_{i \neq j} (l_{i,k-1} - l_{j,k-1})(l_{i,k-1} - l_{j,k-1} - 1)} \right]^{1/2} \tag{19}$$

and

$$l_{i,k} = m_{i,k} - i.$$

Here $m^j_{k-1}(\hat{m}^j_{k-1})$ represents the pattern obtained from $m$ by replacing the number $m_{j,k-1}$ in the $(k-1)$st row of $m$ by the number $m_{j,k-1}-1$ $(m_{j,k-1}+1)$. Note that formally there are patterns $m^j_{k-1}$ and $\hat{m}^j_{k-1}$ which do not obey the inequalities (12). However, such patterns do not arise in (15) or (16) because the coefficients $a^j_{k-1}$ $(b^j_{k-1})$ are different from zero only for the patterns obeying inequalities (12). Moreover, for admissible patterns the denominators of the coefficients $a^j_{k-1}$ and $b^j_{k-1}$ are not equal to zero and that the expressions under the square roots are non-negative. Therefore,

$$\bar{a}^j_{k-1} = a^j_{k-1} \quad \text{and} \quad \bar{b}^j_{k-1} = b^j_{k-1}. \tag{20}$$

We also have

$$a^j_{k-1}(m) = b^j_{k-1}(m^j_{k-1}), \quad b^j_{k-1}(m) = a^j_{k-1}(m^{\hat{j}}_{k-1}). \tag{21}$$

All these statements follow directly from eqs. (18), (19) and (12).

It should be remarked that for some non-admissible patterns the denominators of $a^j_{k-1}$ or $b^j_{k-1}$ can be equal to zero. However, in such cases the numerators are also zero and that the ratios, by definition, are equal to zero.

We show first that the operators $A_{kk}$ and $A_{k,k-1}$ satisfy the correct commutation relations, i.e.,

$$[A_{kk}, A_{k,k-1}] = A_{k,k-1}, \quad k = 2, 3, \dots, n. \tag{22}$$

Indeed, from (15) and (14) it follows that

$$X \equiv A_{kk}A_{k,k-1}m = \sum_{j=1}^{k-1} a^j_{k-1}(m)[r_k(m^j_{k-1}) - r_{k-1}(m^j_{k-1})]m^j_{k-1}.$$

Using the definition of the pattern $m^j_k$ and (17), we find

$$\begin{aligned} X &= [r_k(m) - r_{k-1}(m)]\sum_{j=1}^{k-1} a^j_{k-1}(m)m^j_{k-1} + \sum_{j=1}^{k-1} a^j_{k-1}(m)m^j_{k-1} \\ &= A_{k,k-1}A_{kk}m + A_{k,k-1}m \end{aligned}$$

so that

$$[A_{kk}, A_{kk-1}]m = A_{kk-1}m \tag{23}$$

which is precisely eq. (22), because of the arbitrariness of the pattern $m$. Similarly one shows that operators $A_{kk}$, $A_{k-1,k}$ and $A_{k,k-1}$ satisfy the commutation relations (3). Hence by virtue of (13) the operators $A_{ij}$, $i, j = 1, 2, \dots, n$, satisfy the commutation relations (3). Next we verify the hermiticity condition (5) imposed on the generators $A_{ij}$, which in turn insures that the generators (6) of $u(n)$ are hermitian, namely,

$$(n, A_{ij}m) = \overline{(n, A^*_{ji}m)}, \quad i, j = 1, 2, \dots, n, \tag{24}$$

for arbitrary patterns $n$ and $m$.

The generators $A_{kk}$, $k = 1, 2, \dots, n$, are hermitian, because they are diagonal in the representation space and their eigenvalues are real by eqs. (14)–(17). For the generators $A_{k,k-1}$ we have:

$$(n, A^*_{k,k-1}m) = \overline{(A^*_{k,k-1}m, n)} = \overline{(m, A_{k,k-1}n)} = \sum_{j=1}^{k-1} \overline{a^j_{k-1}(n)}\, \delta_{m,n^j_{k-1}}. \tag{25}$$

Because the $a^j_{k-1}(n)$ are real and obey (21) we obtain

$$\begin{aligned} (n, A^*_{k,k-1}m) &= \sum_{j=1}^{k-1} b^j_{k-1}(n^j_{k-1})\, \delta_{m,n^j_{k-1}} \\ &= \sum_{j=1}^{k-1} b^j_{k-1}(m)\, \delta_{n,m^{\hat{j}}_{k-1}} = (n, A_{k-1,k}m). \end{aligned} \tag{26}$$

The equality

$$\delta_{m,n^j_{k-1}} = \delta_{n,m^{\hat{j}}_{k-1}}$$

used here follows from the fact that $n$ coincides with $m^j_{k-1}$ if and only if $m$ coincides with $n^{\hat{j}}_{k-1}$.

Due to the arbitrariness of $n$ and $m$ in (25) and (26) we have

$$A^*_{k,k-1} = A_{k-1,k}, \quad A^*_{k-1,k} = A_{k,k-1}. \tag{27}$$

Applying the method of induction and utilizing the recurrence formulas (13), we obtain

$$\begin{aligned} A^*_{k,k-h} &= [A_{k,k-1}, A_{k-1,k-h}]^* = [A^*_{k-1,k-h}, A^*_{k,k-1}] \\ &= [A_{k-h,k-1}, A_{k-1,k}] = A_{k-h,k}. \end{aligned} \tag{28}$$

Therefore, the hermiticity condition (5) is fulfilled for an arbitrary $A_{ij}$, $i, j = 1, 2, \ldots, n$, in the representation space $H^{m_n}$. Consequently the generators (6) of $u(n)$ are represented in $H^{m_n}$ by hermitian operators.

The space $H^{m_n}$ in which the representation (14)–(19) is realized is by definition irreducible. An independent formal proof of irreducibility follows from the fact that the following pattern

$$m = \left|\begin{array}{ccccccccccc} m_{1n} & & m_{2n} & & m_{3n} & \cdots & m_{n-2,n} & & m_{n-1,n} & & m_{nn} \\ & m_{2n} & & m_{3n} & & \cdots\cdots & & m_{n-1,n} & & m_{nn} & \\ & & m_{3n} & & & \cdots\cdots\cdots & & & m_{nn} & & \\ & & & \cdot & & & & \cdot & & & \\ & & & & \cdot & & \cdot & & & & \\ & & & & & m_{n-1,n} \quad m_{nn} & & & & & \\ & & & & & m_{nn} & & & & & \end{array}\right|$$

is the only invariant vector of the subgroup $Z$ (i.e., $A_p{}^q m = 0$ for $p > q$) and from corollary 2 to th. 8.2.2.

EXAMPLE 1. The simplest Gel'fand–Zetlin pattern is obtained when all components of the highest weight are equal to each other:

$$m_{1m} = m_{2n} = \ldots = m_{nn} = m. \tag{29}$$

In this case, due to inequalities (12) all other entries in the pattern are also equal to $m$. Therefore, we get the one-dimensional representation of $U(n)$ which is of the type

$$U(n) \ni g \to T_g{}^{L^m} = (\det g)^m. \tag{30}$$

EXAMPLE 2. The Gel'fand–Zetlin pattern for the group $U(3)$. (This is an important internal higher symmetry group for fundamental particles; see ch. 13.) States spanning an irreducible representation space of $U(3)$ may be labelled by the eigenvalues of four commuting operators, namely $I^2$, $I_3$ which are associated with SU(2) subgroup and $Y$ and $B$. It turns out that Gel'fand–Zetlin patterns correspond exactly to the physical states labelled by these quantum numbers. Expressing the operators $I^2$, $I_3$ and $Y$ in terms of $A_i{}^k$ of the subgroup $U(2)$ and utilizing formulas (14), (15) and (16), we obtain

$$m = \left|\begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & I+\frac{1}{2}Y+B & & -I+\frac{1}{2}Y+B & \\ & & I_3+\frac{1}{2}Y+B & & \end{matrix}\right|,$$

where

$$m_{13}+m_{23}+m_{33} = 3B.$$

The quantity $B$ may be interpreted as the baryon number. We see that for a definite $I$ and $Y$ there are $2I+1$ patterns with different $I_3$. We may also express the components of the highest weight of $u(3)$ in physical terms: indeed let $Y_h$ be the highest possible values of $Y$ and $I_h$ be the corresponding (unique) value of $I$: then from eq. (8) we have

$$m_{13} = B+\tfrac{1}{2}Y_h+I_h, \quad m_{23} = B+\tfrac{1}{2}Y_h-I_h, \quad m_{33} = B-Y_h. \tag{31}$$

Also it follows from the inequalities (12) and formula (31) that irreducible representations with the same $Y_h$ and $I_h$ but different values of the baryon number $B$ have the same dimensions. ▼

We are now in a position to give the solutions of the problems listed in the introduction.

(1) *The dimension* of the carrier space $H^{m_n}$ of an irreducible representation is given by the Weyl formula (cf. eq. 8.8(30))

$$N = \frac{\prod_{i<j}(l_i-l_j)}{\prod_{i<j}(l_i^0-l_j^0)}, \tag{32}$$ ? n-[illegible]

where

$$l_j = m_{jn}+n-j, \quad l_j^0 = n-j.$$

(2) *The maximal set of commuting operators* consists of the invariant operators of the following chain of subalgebras

$$u(n) \supset u(n-1) \supset \ldots \supset u(2) \supset u(1), \tag{33}$$

i.e., it contains $(n^2+n)/2$ operators:

$$\begin{matrix} C_{1n}, \ C_{2n}, \ \ldots, \ C_{n-1,n}, \ C_{nn} \\ C_{1,n-1}, \ \ldots, \ C_{n-1,n-1} \\ \cdots\cdots\cdots\cdots \\ C_{12}, \ C_{22} \\ C_{11} \end{matrix} \tag{34}$$

Here

$$C_{pk} = A_{i_2}{}^{i_1}A_{i_3}{}^{i_2}\ldots A_{i_p}{}^{i_{p-1}}A_{i_1}{}^{i_p}, \quad p = 1, 2, \ldots, p \leqslant k,$$

where $A_i^j$ are generators of $\mathrm{gl}(k,R)$ and summation over repeated indices runs from 1 to $k$.

(3) *The eigenvalues* of any of these operators were explicitly expressed in terms of the highest weights in ch. 9, § 4. For example,

$$C_{1,k} = \sum_{i=1}^{k} m_{i,k}, \quad C_{2,k} = \sum_{i=1}^{k} m_{i,k}(m_{i,k}+n+1-2i), \tag{36}$$

and so forth.

(4) *The matrix elements of the generators* of the Lie algebra $u(n)$ follow from eqs. (14), (15), (16) and (6). For example, the matrix elements of the generators $A_{k,k-1}$ are

$$(m', A_{k,k-1}m) = \sum_{j=1}^{k=1} a_{k-1}^j(m)\,\delta_{m_{k-1}^j, m'}, \quad k = 2, 3, \dots, n-1. \tag{37}$$

We see that $A_{k,k-1}$ has non-vanishing matrix elements only between neighboring patterns $m$ and $m'$. Using the recursion formula (13) we can determine the action of any generator $A_{ik}$ on any pattern $m$. The explicit formulas are given in ch. 11, § 8, eqs. (15)–(24) for both the compact and the noncompact generators of the Lie algebra $u(p, q)$.

(5) Let $g \to \hat{T}_g$ be the conjugate representation to a group representation $g \to T_g$ of $u(n)$. Then by virtue of 5.1(14)2° we obtain for hermitian generators $\hat{M}_{ik}$

$$\hat{M}_{ik} = (-M_{ik})^T. \tag{38}$$

Thus for $U(n)$, we see using (6) that a representation is conjugate to a given representation if the generators $A_{ik}(= M_{ik} - \mathrm{i}\tilde{M}_{ik})$ obey the condition (38). It can be verified, using (37), and eqs. (14) and (19), that the irreducible representations determined by $\hat{m}_n = (\hat{m}_{1n}, \hat{m}_{2n}, \dots, \hat{m}_{nn})$ and $m_n = (m_{1n}, m_{2n}, \dots, m_{nn})$ are conjugate to each other if and only if

$$\hat{m}_{in} = -m_{n+1-i,n}, \quad i = 1, 2, \dots, n, \tag{39}$$

and a representation $m$ is self-conjugate if

$$m_{in} = -m_{n+1-i,n}, \quad i = 1, 2, \dots, n.$$

(6) The set of all representations determined by a highest weight $m_n = (m_{1n}, \dots, m_{nn})$ can be divided into equivalence classes of projectively equivalent representations, obtained as follows: We associate with a given representation determined by the highest weight $m_n$ the subclass of all irreducible representations determined by $\tilde{m}_n$ for which the components of the highest weights $\tilde{m}_{in}$ obey the condition

$$\tilde{m}_{in} - m_{in} = s, \tag{40}$$

where $s$ is an arbitrary integer. We can easily verify, on the basis of the Weyl formula (32) that the representations associated with any highest weight $\tilde{m}_n$

obeying condition (40) have the same dimension as the original representation associated with $m_n$. Moreover, on the basis of eqs. (14), (15) and (16), we verify that the matrix elements of any generator $A_i^j$ are related by

$$(\tilde{m}', A_i^j \tilde{m}) = (m', A_i^j m) + s\delta_i^j \delta_{m,m'}.$$

Therefore, the global representations $T^{L^{\tilde{m}_n}}$ and $T^{L^{m_n}}$ of $U(n)$ are related to each other. Indeed, from (42),

$$T^{L^{\tilde{m}_n}} = \exp is(\varphi_1 + \varphi_2 + \dots + \varphi_n) T^{L^{m_n}} = (\det \delta)^s T^{L^{m_n}} = (\det g)^s T^{L^{m_n}}$$

where $\exp(\varphi_k) = \delta_k$ and $\det \delta = \det g$ (cf. exercise 3.11.6.3). Consequently the representations $T^{L^{\tilde{m}_n}}$ and $T^{L^{m_n}}$ are projectively equivalent, and the set of irreducible representations of $U(n)$ can be divided into subclasses of projectively equivalent representation. Each subclass of projectively equivalent representations contains the infinite set of irreducible representations whose highest weights obey the condition (40). Any subclass of projectively equivalent irreducible representations of $U(n)$ becomes one irreducible representation of the group $SU(n)$.

B. *The Representations of* $o(n)$

The orthogonal group $O(n)$ is the set of all linear transformations $g$ of the $n$-dimensional Euclidean space $R^n$,

$$x_l' = g_{ls} x_s, \qquad s, l = 1, 2, \dots, n,$$

which conserve the quadratic form

$$x_1^2 + x_2^2 + \dots + x_n^2.$$

The group $O(n)$ contains $\frac{1}{2}n(n-1)$ different one-parameter subgroups, namely rotations in the planes $(x_i, x_k)$

$$g_{ik}(\vartheta) = \begin{array}{cc} \begin{array}{ccccccccccc} & & & & (i) & & (k) & & & & \\ \end{array} & \\ \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 & & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & & 0 & & & 0 \\ \cdots & & & & & & & & & \cdots \\ 0 & 0 & \dots & 1 & 0 & & 0 & & \dots & 0 \\ 0 & \cdots & & 0 & \cos\vartheta & \dots & \sin\vartheta & & \dots & 0 \\ 0 & \cdots & & 0 & 0 & \dots & 0 & & \dots & 0 \\ \cdots & & & & & & & & & \cdots \\ 0 & \cdots & & 0 & -\sin\vartheta & \dots & \cos\vartheta & 0 & \dots & 0 \\ 0 & \cdots & & & & & 0 & 1 & \dots & 0 \\ 0 & \cdots & & & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \\ \dots (i) \\ \\ \\ \dots (k) \\ \\ \\ \end{array} \end{array} . \tag{41}$$

In the representation $g \to T_g = g$, the generator $X_{ik}$ of the one parameter subgroup $g_{ik}$ is represented by a skew-symmetric $n \times n$-matrix with elements $(X_{ik})_{ik} = -(X_{ik})_{ki} = 1$ and zero for others. Therefore, the generators of $O(n)$

can be expressed in terms of the generators $e_{ik}$ (eq. (4)) of the group $\mathrm{GL}(n, R)$ as follows:

$$X_{ik} = e_{ik} - e_{ki}. \tag{42}$$

The commutation relations for the generators $X_{ik}$ can be obtained from those of $e_{ik}$:

$$[X_{ik}, X_{lm}] = \delta_{kl}X_{im} + \delta_{im}X_{kl} - \delta_{km}X_{il} - \delta_{il}X_{km}. \tag{43}$$

As in the case of $u(n)$ the construction of irreducible representations of $o(n)$ is based on the following two results:

(i) A finite-dimensional irreducible representation of the Lie algebra $o(n)$, $n = 2\nu$ or $n = 2\nu+1$, is uniquely determined by the highest weight $m = (m_1, m_2, \ldots, m_\nu)$ with integral or half-integral components obeying the condition

1° for $n = 2\nu$: $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant |m_\nu|$,

2° for $n = 2\nu+1$: $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant m_\nu \geqslant 0$

(cf. th. 8.5.2).

(ii) In the decomposition of an irreducible finite-dimensional representation of $o(n)$ every irreducible representation of $o(n-1)$ appear with multiplicity one: the components $p_i$ of the highest weight of these representations obey the conditions:

1° for $n = 2\nu$:

$$m_1 \geqslant p_1 \geqslant m_2 \geqslant p_2 \geqslant \ldots \geqslant m_{\nu-1} \geqslant p_{\nu-1} \geqslant |m_\nu|, \tag{44}$$

2° for $n = 2\nu+1$:

$$m_1 \geqslant q_1 \geqslant m_2 \geqslant q_2 \geqslant \ldots \geqslant m_\nu \geqslant q_\nu \geqslant -m_\nu$$

(cf. th. 8.8.2).

The construction of the irreducible representations of the Lie algebra (43) will be accomplished by the following steps:

(i) Construction of the set of orthonormal states associated with a given highest weight.

(ii) Determination of the action of the generators $X_{ik}$ on the basis states and verification of the commutation relations (43).

We denote the components of the highest weight by

$$m = (m_{1,2k+1}, m_{2,2k+1}, \ldots, m_{k+1,2k+1}), \tag{45}$$

when $n$ is even ($n = 2k+2$), and by

$$m = (m_{1,2k}, m_{2,2k}, \ldots, m_{k,2k}), \tag{46}$$

when $n$ is odd ($n = 2k+1$).

With a given highest weight we associate a Gel'fand–Zetlin pattern $m$. Repeating the arguments given for $U(n)$ and using the result (44), we conclude that the patterns are given by:

for $n$ even ($n = 2k+2$):

$$m = \left|\begin{array}{c} m_{1,2k+1} \quad m_{2,2k+1} \quad \cdots \quad m_{k,2k+1} \quad m_{k+1,2k+1} \\ m_{1,2k} \quad \cdots \quad m_{k,2k} \\ m_{1,2k-1} \quad \cdots \quad m_{k,2k-1} \\ m_{1,2k-2} \quad \cdots \quad m_{k-1,2k-2} \\ m_{1,2k-3} \quad \cdots \quad m_{k-1,2k-3} \\ \cdots\cdots\cdots\cdots \\ m_{14} \quad m_{24} \\ m_{13} \quad m_{23} \\ m_{12} \\ m_{11} \end{array}\right|, \tag{47}$$

for $n$ odd ($n = 2k+1$):

$$m = \left|\begin{array}{c} m_{1,2k} \quad m_{2,2k} \quad \cdots \quad m_{k-1,2k} \quad m_{k,2k} \\ m_{1,2k-1} \quad m_{2,2k-1} \quad \cdots \quad m_{k-1,2k-1} \quad m_{k,2k-1} \\ m_{1,2k-2} \qquad\qquad m_{k-1,2k-2} \\ m_{1,2k-3} \qquad . \qquad m_{k-1,2k-3} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ m_{14} \quad m_{24} \\ m_{13} \quad m_{23} \\ m_{12} \\ m_{11} \end{array}\right|. \tag{48}$$

The patterns (47) or (48) are determined by the upper row, which contains the fixed components of the highest weight of an irreducible representation. For $n = 2k+2$ the numbers $m_{ij}$ in the other rows obey the inequalities (cf. eq. (44))

$$\begin{gathered} m_{1,2k+1} \geqslant m_{1,2k} \geqslant m_{2,2k+1} \geqslant m_{2,2k} \geqslant \ldots m_{k,2k+1} \geqslant m_{k,2k} \geqslant |m_{k+1,2k+1}|, \\ m_{1,2k} \geqslant m_{1,2k-1} \geqslant m_{2,2k} \geqslant m_{2,2k-1} \geqslant \ldots \geqslant m_{k,2k-1} \geqslant -m_{k,2k}, \\ m_{1,2k-1} \geqslant m_{1,2k-2} \geqslant m_{2,2k-1} \geqslant \ldots \geqslant m_{k-1,2k-2} \geqslant |m_{k,2k-1}|, \end{gathered} \tag{49}$$

and so on, for an arbitrary row

$$\begin{gathered} m_{i,2p+1} \geqslant m_{i,2p} \geqslant m_{i+1,2p+1}, \quad i = 1, 2, \ldots, p-1, \\ m_{p,2p+1} \geqslant m_{p,2p} \geqslant |m_{p+1,2p+1}|, \\ m_{i,2p} \geqslant m_{i,2p-1} \geqslant m_{i+1,2p}, \quad i = 1, 2, \ldots, p-1, \\ m_{p,2p} \geqslant m_{p,2p-1} \geqslant -m_{p,2p}. \end{gathered}$$

The numbers $m_{ij}$ in the $j$th row represent the components of the highest weight of $o(j+1)$. The numbers $m_{ij}$ associated with a given pattern are simultaneously all integers or all half-integers. This is in contrast to the highest weight of the group $U(n)$ where all $m_{ij}$ were integers.

For $n = 2k+1$ we have

$$\begin{gathered} m_{1,2k} \geqslant m_{1,2k-1} \geqslant m_{2,2k} \geqslant m_{2,2k-1} \geqslant \cdots \geqslant m_{k,2k-1} \geqslant m_{k,2k}. \\ m_{1,2k-1} \geqslant m_{1,2k+2} \geqslant m_{2,2k-1} \geqslant \cdots \geqslant m_{k-1,2k-2} \geqslant |m_{2k-1,k}| \end{gathered}$$

and so on.

It follows from the commutation relations (43) that the action of the entire

Lie algebra $o(n)$ can be reproduced, if the action of generators $X_{2p+1,2p}$, $p = 1, 2, \ldots, [(n-1)/2]$ and $X_{2p+2,2p+1}$, $p = 0, 1, 2, \ldots, [(n-2)/2]$, is explicitly known. Now for $n = 3$ and 4 one may easily calculate directly the action of these generators: using these expressions one can deduce the action of generators $X_{2p+1,2p}$ and $X_{2p+2,2p+1}$ for an arbitrary $n$. Because these are straightforward algebraic calculations, we restrict ourselves for giving the final formulas only (for the complete derivation see Ottoson 1968).

Let $m_k^j(\hat{m}_k^j)$ be the pattern obtained from $m$ by replacing $m_{jk}$ by $m_{jk}-1(m_{jk}+1)$. Then the operators $X_{2p+1,2p}$ and $X_{2p+2,2p+1}$ are defined by the following relations:

$$X_{2p+1,2p}m = \sum_{j=1}^{p} A_{2p-1,j}(m)\hat{m}^j_{2p-1} - \sum_{j=1}^{p} A_{2p-1,j}(m^j_{2p-1})m^j_{2p-1},$$
$$p = 1, 2, \ldots, \left[\frac{n-1}{2}\right], \tag{50a}$$

and

$$X_{2p+2,2p+1}m = \sum_{j=1}^{p} B_{2p,j}(m)\hat{m}^j_{2p} - \sum_{j=1}^{p} B_{2p,j}(m^j_{2p})m^j_{2p} + \mathrm{i}C_{2p}(m)m. \tag{50b}$$

Using the notation

$$\begin{aligned} m_{p,2p-1} &= l_{p,2p-1}, & m_{p,2p}+1 &= l_{p,2p}, \\ m_{p-1,2p-1}+1 &= l_{p-1,2p-1}, & m_{p-1,2p}+2 &= l_{p-1,2p} \\ &\cdots\cdots\cdots\cdots & &\cdots\cdots\cdots\cdots \\ m_{1,2p-1}+p-1 &= l_{1,2p-1}, & m_{1,2p}+p &= l_{1,2p}, \end{aligned} \tag{51}$$

we define the coefficients $A$, $B$ and $C$ by the following formulas

$$\begin{aligned} A_{2p-1,j}(m) &= \frac{1}{2}\Big[\prod_{r=1}^{p-1}(l_{r,2p-2}-l_{j,2p-1}-1)(l_{r,2p-2}+l_{j,2p-1})\Big]^{1/2} \times \\ &\quad \times \Big[\prod_{r=1}^{p}(l_{r,2p}-l_{j,2p-1}-1)(l_{r,2p}+l_{j,2p-1})\Big]^{1/2} \times \\ &\quad \times \Big\{\prod_{r\neq j}(l^2_{r,2p-1}-l^2_{j,2p-1})[l^2_{r,2p-1}-(l_{j,2p-1}+1)^2]\Big\}^{-1/2}, \\ B_{2p,j}(m) &= \left[\frac{\prod_{r=1}^{p}(l^2_{r,2p-1}-l^2_{j,2p})\prod_{r=1}^{p+1}(l^2_{r,2p+1}-l^2_{j,2p})}{l^2_{j,2p}(4l^2_{j,2p}-1)\prod_{r\neq j}(l^2_{r,2p}-l^2_{j,2p})[(l_{r,2p}-1)^2-l^2_{j,2p}]}\right]^{1/2}, \end{aligned} \tag{52}$$

$$C_{2p} = \frac{\prod_{r=1}^{p} l_{r,2p-1}\prod_{r=1}^{p+1} l_{r,2p+1}}{\prod_{r=1}^{p} l_{r,2p}(l_{r,2p}-1)}.$$

From the commutation relations (43) and the expressions (50) for $X_{2p+1,2p}$ and $X_{2p+2,2p+1}$, we can then obtain the explicit form of any generator $X_{ik}$ of the Lie algebra $o(n)$. By straightforward calculation, as in the case of $u(n)$, we can check that the commutation relations for the generators $X_{ik}$ are fulfilled. Furthermore, if the Gel'fand–Zetlin patterns associated with a given weight $m$ are assumed to be orthonormal, then the generators $X_{ij}$ obey the hermiticity condition

$$X_{ij} = -X_{ji}. \tag{53}$$

EXAMPLE 1. The irreducible representations of the Lie algebra $o(4)$. The Gel'fand–Zetlin pattern in this case has the form

$$m = \begin{vmatrix} m_{13} \;\; m_{23} \\ m_{12} \\ m_{11} \end{vmatrix} \equiv \begin{vmatrix} m_1 \;\; m_2 \\ J \\ M \end{vmatrix}, \tag{54}$$

where the numbers $m_{13}$ and $m_{23}$ are the fixed components of the highest weight. It follows from the commutation relations (43) that in order to determine the action of any generator it is sufficient to define the action of $X_{21}$, $X_{32}$ and $X_{43}$. Using (50) we get

$$X_{21} m = \mathrm{i} M m,$$

$$X_{43} \begin{vmatrix} m_1 \;\; m_2 \\ J \\ M \end{vmatrix}$$

$$= \left[\frac{(J+M+1)(J-M+1)(m_1-J)(J-m_2+1)(J+m_2+1)(m_1+J+2)}{(2J+1)(2J+3)(J+1)^2}\right]^{1/2} \times$$

$$\times \begin{vmatrix} m_1 \;\; m_2 \\ J+1 \\ M \end{vmatrix} + \mathrm{i}M \frac{(m_1+1)m_2}{J(J+1)} \begin{vmatrix} m_1 \;\; m_2 \\ J \\ M \end{vmatrix} -$$

$$- \left[\frac{(J+M)(J-M)(m_1-J+1)(m_1+J+1)(J-m_2)(J+m_2)}{(2J+1)(2J-1)J^2}\right]^{1/2} \begin{vmatrix} m_1 \;\; m_2 \\ J-1 \\ M \end{vmatrix},$$

$$X_{32} \begin{vmatrix} m_1 \;\; m_2 \\ J \\ M \end{vmatrix} = \frac{1}{2}[(J-M)(J+M+1)]^{1/2} \begin{vmatrix} m_1 \;\; m_2 \\ J \\ M+1 \end{vmatrix} -$$

$$- \frac{1}{2}[(J-M+1)(J+M)]^{1/2} \begin{vmatrix} m_1 \;\; m_2 \\ J \\ M-1 \end{vmatrix},$$

where according to (49)

$$m_1 \geqslant J \geqslant |m_2|, \qquad J \geqslant M \geqslant -J$$

and $m_1$, $m_2$, $J$ and $M$ are simultaneously all integers or all half-integers. ▼

The carrier space $H^m$ of the representation (50) or $o(n)$ is spanned by the basis vectors (47) or (48), and is, by definition, irreducible. An independent formal proof using the method of $Z$-invariants can again be given, as in the case of $u(n)$.

The dimension of an irreducible representation determined by the components of the highest weight is given by Weyl's formula (see eq. 8.8(29)).

The maximal set of commuting operators in the representation space contains the following operators:

(i) $O(2k+2)$:

$$\begin{array}{ccccc} C_2(2k+2) & C_4(2k+2) & \cdots & C_{2k}(2k+2) & C_k'(2k+2) \\ C_2(2k+1) & C_4(2k+1) & \cdots\ C_{2(k-1)}(2k+1) & C_{2k}(2k+1) & \\ C_2(2k) & C_4(2k) & C_{2(k-1)}(2k) & C_k'(2k) & \\ & & \cdots\cdots\cdots\cdots & & \\ & & C_2(4)\ \ C_2'(4) & & \\ & & C_2(3) & & \\ & & X_{21} & & \end{array} \tag{55}$$

(ii) $O(2k+1)$:

$$\begin{array}{ccccc} C_2(2k+1) & C_4(2k+1) & \cdots & C_{2(k-1)}(2k+1) & C_{2k}(2k+1) \\ C_2(2k) & C_4(2k) & \cdots & C_{2(k-1)}(2k) & C_k'(2k) \\ & C_2(2k-1) & \cdots & C_{2(k-1)}(2k-1) & \\ & & \cdots\cdots\cdots\cdots & & \\ & & C_2(4)\ \ C_2'(4) & & \\ & & C_2(3) & & \\ & & X_{21} & & \end{array} \tag{56}$$

Here

$$C_{2i}(p) = \mathrm{Tr}\,X^{2i}(p) \tag{57}$$

and

$$C_l'(2l) = \varepsilon^{i_1j_1, i_2j_2, \ldots, i_lj_l} X_{i_1j_1} X_{i_2j_2} \ldots X_{i_lj_l}, \tag{58}$$

where $X^{2i}(p)$ denotes $2i$th power of the matrix $X(p) \equiv (X_{ik}(p))$ which is composed of the generators $X_{ik}(p)$ of the group $O(p)$ and $\varepsilon^{i_1j_1, i_2j_2, \ldots, i_lj_l}$ is the totally anti-symmetric Levi–Civita tensor (cf. 9.4.B). It should be noted that for the group $O(2k)$, in contradistinction to group $O(2k+1)$, the set of Casimir operators (57) does not provide a set of independent invariant operators and therefore the pseudoscalar operator (58) has to be included. The spectra of the operators (57) and (58) were given in ch. 9.4.B.

In general, if the components of the highest weight of $o(n)$ are not restricted by relations other than (44), we have in the representation space spanned by Gel'fand–Zetlin patterns

$$N = \frac{1}{2}\left[\frac{n(n-1)}{2} + \left\{\frac{n-1}{2}\right\}\right] \tag{59}$$

independent commuting operators, where $\frac{1}{2}n(n-1)$ and $\left\{\frac{n-1}{2}\right\}$ are the dimension and the rank of $O(n)$, respectively. However, if the components of the highest weight are not independent then some of the operators (57) or (58) become functions of others and the number of independent commuting operators is smaller. For example, if the highest weight is of the form

$$m(f, 0, 0, \ldots, 0), \tag{60}$$

then by virtue of the structure of the patterns (47) and eq. (56) only $n-1$ commuting operators

$$C_2(n), C_2(n-1), \ldots, C_2(3), X_{21}, \tag{61}$$

generate the ring of the commuting operators in the irreducible representation space of $o(n)$, which is defined by the highest weight (60).

Notice finally that because any highest weight (44) with all integer or all half-integer components gives rise, in the Gel'fand–Zetlin approach, to an irreducible representation, this formalism provides a description of all irreducible finite-dimensional representations of $o(n)$.

## § 2. The Tensor Method

Many physical laws such as the Maxwell or Einstein equations are most concisely stated in the language of tensors, objects transforming according to the finite-dimensional tensor representations of a physical symmetry group $G$.

In this section we elaborate the connection between the theory of tensor representations and the theory of induced representations. In particular, we give the classification of all irreducible tensor representations of groups $GL(n, C)$ and $SO(n, C)$. Clearly, using the th. 8.3.1 we also obtain a description of all irreducible tensor representations of the real compact and noncompact forms of these groups, e.g., $U(p, q)$ and $SO(p, q)$, $p+q = n$.

The tensor method does not solve directly the practical problems 1°–5° presented in the introduction. It gives, however, a beautiful connection between the representation theory of a group $G$ and the representation theory of the permutation group $S_n$. This connection cannot be seen by means of other methods.

### A. *Tensors*

Let a group $G$ be realized as a matrix group, i.e. $G \ni g \leftrightarrow \{g_i{}^k = D_i{}^k\}D_i{}^k$, $i, k = 1, 2, \ldots, N$. A quantity $T = \{T_{i_1 i_2 \ldots i_r}\}$, $T_{i_1 i_2 i_r} \in C$, $i_k = 1, 2, \ldots, N$, $k = 1, 2, \ldots, r$, is called a ***tensor of rank r relative to G*** if it has the following transformation law

$$T_{i_1 i_2 \ldots i_r} = D_{i_1}{}^{k_1} D_{i_2}{}^{k_2} \ldots D_{i_r}{}^{k_r} T_{k_1 k_2 \ldots k_r}. \tag{1}$$

If $\overset{1}{T}$ and $\overset{2}{T}$ are tensors of rank $r$ then $c_1\overset{1}{T}+c_2\overset{2}{T}$ is also a tensor of the same rank; hence the set of all tensors of a given rank forms a vector space $\overset{r}{H^T}$. It follows from (1) that $\overset{r}{H^T}$ is the carrier space of the tensor product $T_g\otimes T_g\otimes \ldots \otimes T_g$ ($r$ times). This representation is called *tensor representation.*

The following proposition gives a description of the space $\overset{r}{H^T}$.

PROPOSITION 1. *The components*

$$T_{i_1 i_2 \ldots i_r} = \overset{k_1}{x}_{i_1}\overset{k_2}{x}_{i_2} \ldots \overset{k_r}{x}_{i_r}, \qquad k_s = 1, 2, \ldots, N \tag{2}$$

*where* $\overset{k}{x}$, $k = 1, 2, \ldots, N$, *are linearly independent complex N-dimensional vectors, span the tensor space* $\overset{r}{H^T}$ *of all tensors T of rank r.*

PROOF: The components $T_{i_1 i_2 \ldots i_r}$ are points of $C^{N^r}$. Setting $\overset{k}{x} = e_k$, where $e_k$ are basis vectors of $C^N$, we obtain a basis in $C^{N^r}$. Hence the vectors (2) form a basis in $\overset{r}{H^T}$. ▼

The basic property of tensors, observed first by Weyl, is that the operation of permutations of indices $i_1, i_2, \ldots, i_r$ commutes with the action of $G$ in $H^T$. Indeed, if $s$ is an element of the symmetric group $S_r$, and

$$(sT)_{i_1 i_2 \ldots i_r} \equiv T_{s(i_1 i_2 \ldots i_r)} = T_{s(i_1)s(i_2)\ldots s(i_r)}, \tag{3}$$

then

$$\begin{aligned}
(sT')_{i_1 i_2 \ldots i_r} &= T'_{s(i_1)s(i_2)\ldots s(i_r)} \\
&= D_{s(i_1)}{}^{s(k_1)} D_{s(i_2)}{}^{s(k_2)} \ldots D_{s(i_r)}{}^{s(k_r)} T_{s(k_1)s(k_2)\ldots s(k_r)} \\
&= D_{s(i_1)}{}^{s(k_1)} D_{s(i_2)}{}^{s(k_2)} \ldots D_{s(i_r)}{}^{s(k_r)} (sT)_{k_1 k_2 \ldots k_r} \\
&= D_{i_1}{}^{k_1} D_{i_2}{}^{k_2} \ldots D_{i_2}{}^{k_2} \ldots D_{i_r}{}^{k_r} (sT)_{k_1 k_2 \ldots k_r} = (sT)'_{i_1 i_2 \ldots i_r}.
\end{aligned}$$

Consequently, those tensors $\{T_{i_1 i_2 \ldots i_r}\}$ whose components have a symmetry property corresponding to a given Young frame form an invariant subspace in the space $H^T$. Thus tensor representations are in general reducible.

A tensor $T = \{T_{i_1 i_2 \ldots i_r}\}$ is said to be *symmetric* if

$$(sT)_{i_1 i_2 \ldots i_r} \equiv T_{s(i_1 i_2 \ldots i_r)} = T_{i_1 i_2 \ldots i_r}$$

for every permutation $s \in S_r$ of indices.

A tensor is *skew-symmetric* if

$$sT = (-1)^{\delta_s} T, \qquad s \in S_r, \tag{4}$$

where $\delta_s$ is the parity of the permutation $s$. The skew-symmetric tensors with respect to all indices are also called *polyvectors.* Using the Young idempotent operators and eq. (2), we can obtain in some cases explicit compact formulas for components of a tensor with a given symmetry property corresponding to a Young frame. Indeed, for instance, the components of a symmetric tensor correspond to the Young frame

$$\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}\boxed{\phantom{x}}$$

and can be represented as elements of a tensor product of the vector $x$, i.e.,

$$T_{i_1 i_2 \dots i_r} = x_{i_1} x_{i_2} \dots x_{i_r} \equiv T_{\boxed{i_1}\boxed{i_2}\boxed{\cdots}\boxed{i_r}} \tag{5}$$

Antisymmetric tensors can be represented in terms of determinants; for instance, the quantities

$$e_{ij} = \begin{vmatrix} \overset{1}{x}_i & \overset{1}{x}_j \\ \overset{2}{x}_i & \overset{2}{x}_j \end{vmatrix} \equiv e_{\begin{smallmatrix}\boxed{i}\\ \boxed{j}\end{smallmatrix}}, \qquad \overset{i}{x} \in C^n, \ i, j = 1, \dots, n \tag{6}$$

represent the components of a skew-symmetric tensor of rank two associated with the Young frame $\begin{smallmatrix}\boxed{\phantom{x}}\\ \boxed{\phantom{x}}\end{smallmatrix}$. In general, the quantities

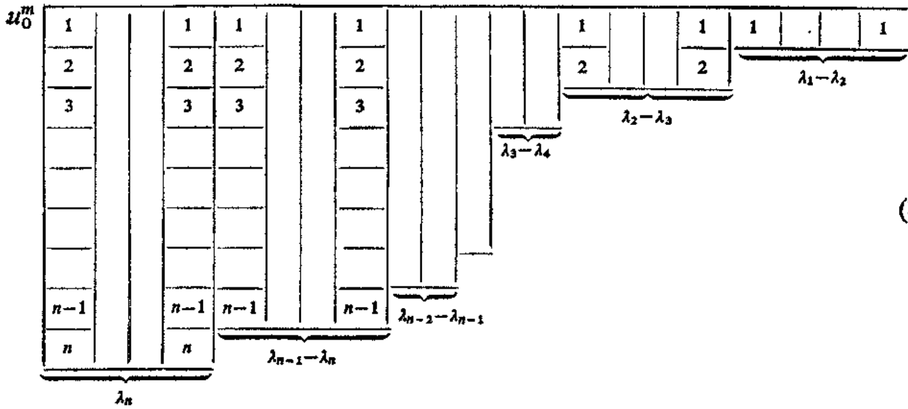
$$e_{i_1 i_2 \dots i_r} = \begin{vmatrix} \overset{1}{x}_{i_1} & \overset{1}{x}_{i_2} & \dots & \overset{1}{x}_{i_r} \\ \overset{2}{x}_{i_1} & \overset{2}{x}_{i_2} & \dots & \overset{2}{x}_{i_r} \\ \dots & \dots & \dots & \dots \\ \overset{r}{x}_{i_1} & \overset{r}{x}_{i_2} & \dots & \overset{r}{x}_{i_r} \end{vmatrix} e_{\begin{smallmatrix}\boxed{i_1}\\ \boxed{i_2}\\ \boxed{\vdots}\\ \boxed{i_r}\end{smallmatrix}}, \qquad i_1, i_2, \dots, i_r = 1, \dots, n \tag{7}$$

represent the components of a skew-symmetric tensor of rank $r$ determined by the Young partition $\overset{r}{\lambda} = (\underbrace{1, 1, \dots, 1}_{(r)}, 0, \dots, 0)$, $r = 1, 2, \dots, n$. The tensors (7) are *polyvectors*.

One can realize tensors (5) and (7) as functions on the $Z$ factor of the Gauss decomposition $G = \mathfrak{Z}DZ$. Indeed, let $G$ be e.g. $GL(n,C)$. Then $Z$ consists of all lower triangular complex matrices given by eq. 3.6 (7). The polyvectors (7) can be expressed in terms of minors of the elements $z \in Z$ in the form

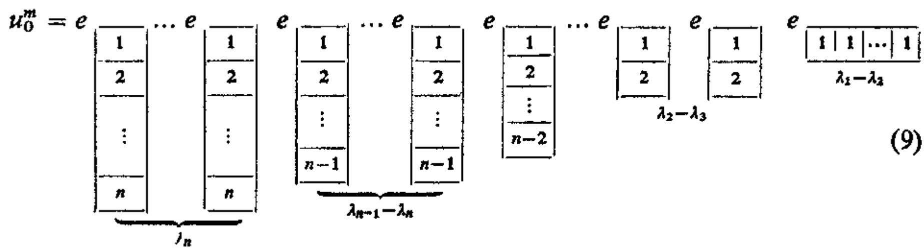
$$e_{\begin{smallmatrix}\boxed{i_1}\\ \boxed{i_2}\\ \boxed{\vdots}\\ \boxed{i_r}\end{smallmatrix}} = \begin{vmatrix} z_{i_1 1} & z_{i_1 2} & \dots & z_{i_1 r} \\ z_{i_2 1} & z_{i_2 2} & \dots & z_{i_2 r} \\ \dots & \dots & \dots & \dots \\ z_{i_r 1} & z_{i_r 2} & \dots & z_{i_r r} \end{vmatrix}, \qquad 1 < r \leqslant n.$$

Similarly one can show that symmetric tensors $T_{\boxed{i_1}\boxed{i_2}\boxed{\cdots}\boxed{i_r}}$ can be realized in the class of polynomials of elements of the first colum of the matrices $z \in Z$: $1, z_{21}, z_{31}, \dots, z_{n1}$ which are homogeneous of the degree $r$. Among the components of a tensor $T$ with a given Young symmetry $\lambda$ there exists a distinguished component given by the following Young tableau



(8)

associated wih a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. This component can be represented by the following product of polyvectors and a symmetric tensor



(9)

We arrange the indices of a tensor with a given Young symmetry $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1 + \lambda_2 + \ldots + \lambda_n = r$, in such a way that they correspond to the so-called *standard Young tableaux*. Accordingly, independent components of a tensor $T$ correspond to the standard tableaux which have non-decreasing indices from the left to the right and with increasing indices from the top to the bottom. For instance, for the tensor of order three, associated with the Young frame $\begin{array}{|c|c|}\hline \phantom{i} & \phantom{i} \\ \hline \phantom{i} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, we write $T_{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}$, where, according to the Young rule $i_1, i_2, i_3 = 1, 2, \ldots, n$ and $i_1 \leqslant i_2$, $i_1 < i_3$; for $n = 2$ we have therefore the following independent components $T_{\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}$ $T_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}}$

In particular, for the components of totally symmetric and totally antisymmetric tensors, according to this convention, we have then, respectively

$$T_{\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_r\\\hline\end{array}} \qquad \begin{gathered} i_1, i_2, \ldots, i_r = 1, 2, \ldots, n \\ \text{and } i_1 \leqslant i_2 \leqslant \ldots \leqslant i_r \end{gathered} \qquad T_{\begin{array}{|c|}\hline i_1\\\hline \vdots\\\hline i_r\\\hline\end{array}} \qquad \begin{gathered} i_1, i_2, \ldots, i_r = 1, 2, \ldots, n \\ \text{and } i_1 < i_2 < \ldots < i_r . \end{gathered} \tag{10}$$

B. *Tensor Representations of* GL$(n, C)$, SL$(n, C)$, GL$(n, R)$, SL$(n, R)$ $U(p, q)$, SU$(p, q)$, $U(n)$, SU$(n)$ *and* SU*$(n)$

In this section we shall identify an irreducible representation $T^{L^m}$ of GL$(n, C)$ with a certain tensor representation. We first associate a tensor representation to the representation $\overset{i}{T}{}^{L^m}$ with the fundamental weight $\overset{i}{m} = (\underbrace{1, 1, \ldots, 1}_{(i)}, 0, \ldots, 0)$ of GL$(n, C)$.

Let $x = (x_1, \ldots, x_n)$ be an element of the linear space $\overset{1}{H}{}^T = C^n$, in which GL$(n, C)$ is realized as $g \to T_g = D(g) = g$. Let $Z$ be the factor of the Gauss decomposition $G = \overline{\mathfrak{Z}DZ}$. Then the action (1) in $\overset{1}{H}{}^T$ of an element $z \in Z$ is given by the formula

$$zx \equiv \left\| \begin{matrix} 1 & & & & & \\ z_{21} & 1 & & 0 & & \\ z_{31} & z_{32} & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z_{n1} & z_{n2} & z_{n3} & \cdot & \cdot & 1 \end{matrix} \right\| \left\| \begin{matrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{matrix} \right\| = \left\| \begin{matrix} x_1 \\ z_{21}x_1 + x_2 \\ z_{31}x_1 + z_{32}x_2 + x_3 \\ \cdots\cdots\cdots\cdots \\ z_{n1}x_1 + z_{n2}x_2 + \ldots + x_n \end{matrix} \right\| . \tag{11}$$

We see, therefore, that element $T_{\begin{array}{|c|c|c|}\hline 1 & \cdots & 1\\\hline\end{array}} = x_1 \ldots x_1$, in space spanned by components $T_{\begin{array}{|c|c|c|}\hline i_1 & \cdots & i_r\\\hline\end{array}}$, is the only invariant vector of $Z$ defined up to a normalization constant. Hence, $T_{\begin{array}{|c|c|c|}\hline 1 & \cdots & 1\\\hline\end{array}}$ is the highest vector by virtue of corollary 1 to th. 8.2.2. For $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in D$ we obtain

$$T_\delta u_0^m = \delta^r u_0^m . \tag{12}$$

Hence $m = (r, 0, \ldots, 0)$. This implies in particular that the vector representation $T$: $g \to T_g = g$ corresponds to the representation $T^{L^m}$ associated with the fundamental highest weight $m = (0, 0, \ldots, 1)$.

The following theorem shows that all fundamental representations of GL$(n, C)$ can be realized as polyvector representations.

THEOREM 2. *The linear space $\overset{r}{H}{}^T$ of all polyvectors of rank $r$ is the carrier space of the irreducible representation $\overset{r}{T}{}^{L^m}$ of* GL$(n, C)$ *associated with the fundamental highest weight $\overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$.*

PROOF: We can take as the basis vectors in $\overset{r}{H^T}$ the polyvectors (7). Because for $\delta \in D$, $\delta x = (\delta_1 x_1, \delta_2 x_2, \ldots, \delta_n x_n)$, we have

$$(T_\delta e)_{i_1 i_2 \ldots i_r} = \delta_{i_1} \delta_{i_2} \ldots \delta_{i_r} e_{i_1 i_2 \ldots i_r}, \tag{13}$$

i.e., every basis vector $e_{i_1 i_2 \ldots i_r}$ is a weight vector. Using eqs. (11) and (7) we verify that the vector $e_{12\ldots r}$ is the only invariant of the subgroup $Z$. Hence by virtue of corollary 2 to th. 8.2.2, it is the highest vector of an irreducible representation. By virtue of (13) the corresponding integral highest weight is $\delta_1 \delta_2 \ldots \delta_r$; hence the irreducible representation $T^{L^m}$ in $\overset{r}{H^T}$ is determined by the highest weight $m = \overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$. ▼

Consequently, the representation (1) of $\mathrm{GL}(n, C)$ in the space of all polyvectors $\{e_{i_1 i_2 \ldots i_r}\}$ of rank $r$ corresponds to the representation $T^{L^{\overset{r}{m}}}$ determined by the fundamental weight $\overset{r}{m} = (\underbrace{1, 1, \ldots, 1}_{(r)}, 0, \ldots, 0)$.

Furthermore, by def. 8.6.1, and th. 8.6.1 we have

COROLLARY 1. *Every irreducible representation of* $\mathrm{GL}(n, C)$ *is a Young product of polyvector representations.* ▼

According to eq. (4), the linear space $\overset{r}{H^T}$ spanned by the components of tensor of a rank $r$ with a given Young symmetry, forms an invariant subspace relative to $G$. Thus, we expect a close connection between the induced representations $T^{L^m}$ and the tensor representations of $G$ realized in the space $\overset{r}{H^T}$. In fact, we have

THEOREM 3. *Let* $\overset{r}{H^T}$ *be the space of tensors of rank* $r$ *with respect to* $\mathrm{GL}(n, C)$ *which have the symmetry defined by the Young partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *with* $r = \lambda_1 + \lambda_2 + \ldots + \lambda_n$. *Then, the representation of* $\mathrm{GL}(n, C)$ *realized in* $\overset{r}{H^T}$ *is equivalent to the representation* $T^{L^m}$ *with the highest weight*

$$m = (\lambda_1, \lambda_2, \ldots, \lambda_n).$$

PROOF: Let $\overset{1}{x}, \overset{2}{x}, \ldots, \overset{r}{x}$ be a set of $r$ independent vectors from $C^n$. The tensor $\tilde{T}$ with components $\tilde{T}_{i_1 i_2 \ldots i_r} = \overset{1}{x}_{i_1} \overset{2}{x}_{i_2} \ldots \overset{r}{x}_{i_r}$ has no symmetry with respect to the symmetric group $S_r$. Hence, acting on $\tilde{T}_{i_1 i_2 \ldots i_r}$ by the Young idempotent operator $Y_\lambda$ which corresponds to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_n = r$, we obtain a non-zero tensor $T$ with this Young symmetry: hence $Y_\lambda \tilde{T} \in \overset{r}{H^T}$. We claim that the element $u_0^m$ of $\overset{r}{H^T}$ whose only non vanishing component is given by eq. (8) is invariant relative to the action of the subgroup $Z$ of $\mathrm{GL}(n, C)$.