If we put $G = R^n$ in th. 1, then the composition law (10) is the same as the composition law (9) for the Weyl group. Hence, the formula (11) provides an irreducible unitary representation of the Weyl group. It is evident that the generators of U_β have the same form as q_j given by eq. (2), and the generators of V_α have the same form as p_k. Thus every irreducible integrable representation of the canonical commutation relations is equal to the Schrödinger representation.

The set $\{V, U\}$, eq. (10), might be in general reducible. In this case one can show, however, that the carrier Hilbert space H can be decomposed into the orthogonal direct sum $\oplus H_s$ of subspaces, each invariant and irreducible relative to the set $\{V, U\}$ (cf. Mackey 1949, th. 1). Consequently, one concludes that any integrable representation of the canonical commutation relations (1) is unitarily equivalent to at most a countable sum of replicas of the Schrödinger representation. This shows, in fact, the equivalence of the Heisenberg and the Schrödinger formulations of quantum mechanics in the case of integrable representations.

§ 3. Comments and Supplements

A. The construction of the relativistic position operator and the proof of equivalence of the Schrödinger and the Heisenberg formulations in quantum mechanics given in this chapter were based in fact on the Imprimitivity Theorem. This once more demonstrates the importance and the power of this theorem. In fact, the exposition of quantum mechanics—nonrelativistic as well as relativistic—could be based on this theorem.

Historically, the equivalence of the Heisenberg and Schrödinger representations were proved by Lanczos 1925, Schrödinger 1926 and Pauli.

The concept of a relativistic position operator was first introduced in a fundamental paper by Newton and Wigner 1949. The derivation based on the Imprimitivity Theorem was given by Wightman 1962 and Mackey 1963.

The construction of position operators given in 1.B was elaborated by Lunn 1969. Position operators for massless particles were discussed by Bertrand 1972. The new and physically satisfactory theory of position operator for massless particles was presented in the beautiful paper by Angelopoulos, Bayen and Flato 1975.

The problem of representations of the canonical commutation relations was extensively investigated by Stone 1930 and v. Neumann 1931. The derivation of the equivalence of any irreducible representation to the Schrödinger representation based on the Imprimitivity Theorem was given by Mackey 1949. Here we follow essentially the Mackey derivation.

The Weyl group has apparently no direct meaning like the Galilei or the Poincaré group. Hence non-integrable or partially integrable representations of canonical commutation relations 2(1) might be also of some interest. An example of such representation (which is non-equivalent to the Schrödinger representation!) was constructed by Doebner and Melsheimer 1967. The physical meaning of this representation is, however, so far unclear. It would be very interesting to construct an explicit example of partially integrable (with respect to p_j) representation of canonical commutation relations and look for their physical meaning.

B. *Algebraic Definition of Position Operators*

We give now, for completeness two other definitions of position operators, which are frequently considered as more physical than the definition given in § 1. These definitions in general are not equivalent with definition 1.1. Given an irreducible representation of the generators P_μ and $M_{\mu\nu}$ of the Poincaré group Π for a massive particle we wish to define the position operators Q_j in the envelopping field of the Lie algebra of Π.

(1) From the physical assumptions of translational and rotational invariance, we have, as in eqs. 1 (8)–1 (9):

$$\begin{aligned} [Q^j, P^k] &= i\delta^{jk}, \\ [Q^j, J^k] &= i\varepsilon^{jkl}Q_l. \end{aligned} \tag{1}$$

Further, time translations and pure Lorentz transformations (boosts) give the conditions

$$\begin{aligned} [Q^j, P^0] &= iP^jP_0^{-1}, \\ [Q^j, M^{0k}] &= -i\delta^{jk}Q_0+iQ^jP^kP_0^{-1}, \end{aligned} \tag{2}$$

where $Q_0 = t$ is assumed to be a number in the carrier space. Notice that because of the spectral condition P_0^{-1} is well defined.

A form of Q_μ satisfying these requirements is

$$Q_\mu = tP_\mu P_0^{-1}+\frac{1}{m^2}M_{\mu\nu}P^\nu+\frac{1}{m^2}M^{\nu 0}P_\mu P_\nu P_0^{-1}. \tag{3}$$

Indeed, we have $Q_0 = t$, and in the rest frame $(P_0, \mathbf{0})$:

$$\mathring{Q}_i = \frac{1}{m}M_{i0},$$

which is the correct relation. We remark that Q_μ in (3) is not a four-vector.

However, contrary to the condition used in § 1, we have

$$[Q^j, Q^k] = \frac{1}{m^2}i\varepsilon^{jkl}S_l, \tag{5}$$

where S_l is the spin operator. For spinless particles the position operators commutes. But particles with spin cannot be localized better than their Compton wave-lengths. Note that for spin $J = 0$, the position operator (3) in momentum space is

$$Q_j = \mathrm{i}\frac{\partial}{\partial p^j} - \mathrm{i}\frac{p_i}{p_0^2} \tag{6}$$

whereas the Newton–Wigner hermitian position operator is

$$Q_j^{(NW)} = \mathrm{i}\partial/\partial p^j - \tfrac{1}{2}\mathrm{i}p_j/p_0^2. \tag{7}$$

(2) Often another position operator, namely a four-vector X_μ is introduced by the commutation relations

$$[X^\mu, P^\nu] = -\mathrm{i}g^{\mu\nu}. \tag{8}$$

In terms of X^μ and P^ν we can represent

$$M_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu + S_{\mu\nu}, \tag{9}$$

where $S_{\mu\nu}$ is the spin part of $M_{\mu\nu}$: $S_{\mu\nu}P^\nu = 0$. Clearly, X_0 plays a different role than Q_0 in (3).

If we introduce $D = Q_\mu P^\mu$, then we can easily derive from (9).

$$X_\mu = [\{D, P_\mu\} + \{M_{\mu\nu}, P^\nu\}]/2P^2. \tag{10}$$

In the 11-parameter Lie algebra P_μ, $M_{\mu\nu}$, D these operators have the commutation relations, in addition to (8)

$$\begin{aligned} [M^{\mu\nu}, X^\lambda] &= -\mathrm{i}(g^{\nu\lambda}X^\mu - g^{\mu\lambda}X^\nu), \\ [X^\mu, D] &= -\mathrm{i}X^\mu, \\ [X^\mu, X^\nu] &= -\mathrm{i}\varepsilon^{\mu\nu\lambda\sigma}P_\lambda W_\sigma/P^4, \end{aligned} \tag{11}$$

where $W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}M^{\nu\lambda}P^\sigma$. This position operator is formally covariant but does not satisfy the physical requirements (2).

§ 4. Exercises

§ 1.1. Show that the imprimitivity system $U_{\mathscr{E}^3}$, $E(\cdot)$ given by eq. 17.1(10) is not proper for the construction of position operators Q_i, $i = 1, 2, 3$.

§ 1.2.* Show that the position operator Q_k, $k = 1, 2, 3$, in the space H of positive energy solutions of the Dirac equation $(\gamma_\mu p^\mu - m)\psi(p) = 0$ has the following form

$$Q_k = \mathrm{i}\frac{\partial}{\partial p_k} + \mathrm{i}\frac{\gamma_k}{2p_0} - \frac{\mathrm{i}(\gamma p)p_k + (\Sigma \times p)_k p_0}{2p_0^2(p_0 + m)} - \frac{\mathrm{i}p_k}{p_0^2}. \tag{1}$$

Hint: Use the Induction-Reduction Theorem 18.1.1.

§ 1.3.*** Construct position operators for massless particle with arbitrary spin J.

Hint: Use Induction-Reduction Theorem 18.1.1 for the representation $U^{0,J}$ of Poincaré group and find the base $X = \Pi/K$ of spectral measure $E(S)$, $S \subset X$. Deduce from this what physical quantities of massless particle can be localized.

§ 1.4.*** Construct position operators for tachyons $m^2 < 0$ of arbitrary spin.

Hint: Use the same method as in the previous exercise.

§ 1.5. Show that for position operator (1) of a Dirac particle

$$\frac{\mathrm{d}Q_k}{\mathrm{d}t} = \mathrm{i}[H, Q_k] = \frac{p_k}{p_0}\,\frac{\gamma_0 m + \gamma_0 \gamma p}{p_0}, \tag{2}$$

which is equal to p_k/p_0 in H, whereas

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{i}\frac{\partial}{\partial p^k}\right) = \mathrm{i}\left[H, \mathrm{i}\frac{\partial}{\partial p^k}\right] = \gamma_0 \gamma_k, \tag{3}$$

which by virtue of the fact that $(\gamma_0 \gamma_k)^2 = I$ is equal to the velocity of the light.

§ 1.6.* Show that every Galilei invariant system with mass $m > 0$ is localizable.

§ 1.7.* Show that an elementary Galilean system with $m = 0$ described by an irreducible representation of the Galilei group is not localizable.

§ 1.8. Construct a position operator for a massive particle of arbitrary spin.

Hint: Use Induction-Reduction Theorem 18.1.1.

§ 1.9. Complete position operators Q_k, $k = 1, 2, 3$, given by 1(21) to a covariant position operator $Q = (Q_0, Q_1, Q_2, Q_3)$, $Q_0 = t$. Show that this operator associated with hyperplane H defined by the normal vector $n = (1, 0, 0, 0)$ transform covariantly in the following manner

$$Q_\mu(H) \xrightarrow[(a,\Lambda)]{} Q'_\mu(H') = \Lambda^\nu_\mu Q_\nu(H) + a_\mu$$

where

$$H = (n, \tau), \qquad Q_\mu n^\mu = \tau$$

and

$$H' = (n', \tau'), \qquad n'_\mu = \Lambda^\nu_\mu n_\nu, \qquad \tau' = \tau + n'_\mu a^\mu.$$

§ 1.10.* Show that the operator $Q'_\mu(H')$ is defined by the system of imprimitivity $(E'(\cdot), U'^{m,J})$, where

$$U'^{m,J}_{(a,\Lambda)}\psi(p) = \mathrm{e}^{\mathrm{i}pa} D^J(r')\psi(\Lambda^{-1}p)$$

$r' = \Lambda_{\bar\omega}^{-1} r \Lambda_\omega$, $\Lambda_{\bar\omega}^{-1}$ is the Lorentz transformation which takes H into H' and $E'(\cdot)$ is the spectral measure associated with U' as in proposition 1 based on $\mathrm{SO}(3)'\backslash T^3 \otimes \mathrm{SO}(3)' \sim H'$ where $\mathrm{SO}(3)' = \Lambda_{\bar\omega}^{-1}\mathrm{SO}(3)\Lambda_\omega$. ▼

§ 1.11. Show that the classical Poyting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ and the energy density $U = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{H}^2)$ are invariants of $E(2)$.

Hint: Introduce a complex vector $\boldsymbol{E} + \mathrm{i}\boldsymbol{H}$ and show that for $A \in \mathscr{E}^2$, $A = \begin{bmatrix} \alpha & \bar\alpha z \\ 0 & \bar\alpha \end{bmatrix}$, $|\alpha| = 1$, $z \in C$ one has $\boldsymbol{E} + \mathrm{i}\boldsymbol{H} \xrightarrow{A} \alpha^2(\boldsymbol{E} + \mathrm{i}\boldsymbol{H})$.

§ 2.1.** Construct simplest indecomposable representations of canonical commutation relations (2.1).

Hint: Use the indecomposable representations 6.3.D for an abelian subgroup of the Weyl group and induce them to the whole group.

§ 2.2.*** Find all nonintegrable representations of canonical commutation relations (2.1). Can they have a physical interpretation?

§ 2.3.*** Find all partially integrable representations (with respect to momenta P_j) of canonical commutation relations (2.1). Give a physical interpretation for these representations.

Chapter 21

Group Representations in Relativistic Quantum Theory

In this chapter we discuss a number of selected basic applications of group representations which are at the foundation of many approaches to relativistic quantum theory. We do not have a closed complete relativistic quantum theory in the same level of development as the nonrelativistic quantum mechanics. For this reason the group theoretical framework of relativistic theory plays a basic and guiding role in the establishment of models and theories for relativistic processes.

§ 1. Relativistic Wave Equations and Induced Representations

Wave equations for quantum systems are modelled after the wave equations of classical physics: electromagnetic waves, sound or water waves, etc., however, with a different interpretation of the wave function. In quantum physics the wave function represents a probability amplitude.

Relativistic wave equations provide an effective and practical way of implementing the induced representations of the Poincaré group. The solutions of the wave equations, the wave functions of the system, carries all the information about the spin and momenta of the system provided by the Poincaré group. In addition, the wave equation provides a conserved current density for the quantum system. And via the so-called minimal coupling to the electromagnetic field it also gives a very simple and natural covariant description of the interaction of the quantum system with the external electromagnetic field or radiation. These two last properties of the wave equations go much beyond the theory of induced representations. The real importance of wave equations lie in the covariant description of interactions.

The Klein–Gordon and Dirac equations are the best known examples of relativistic wave equations. But there are infinitely many other possible relativistic wave equations. In fact, we show in this section not only the relation of the wave equations to induced representations, but also give in a unified manner the wave equations corresponding to all induced representations of the Poincaré group.

We go even further and discuss the so-called infinite-component wave equations, which use representations of the Poincaré group induced from infinite-dimensional representations of the homogeneous Lorentz-group, or even of more general groups. These wave equations, we shall see, describe composite quantum systems with internal degrees of freedom.

A. *From Induced Representations to Wave Equations*

We start generally from a group G and its unitary representation U^L induced by a representation $k \to L_k$ of a closed subgroup K of G. We assume for simplicity that the space $X = K\backslash G$ has an invariant measure.

As we know from chs. 8 and 16, the Mackey 'wave functions' $f(g)$, $g \in G$, transform in the following manner

$$[U^L_{g_0} f](g) = f(gg_0), \quad g_0, g \in G, \tag{1}$$

and satisfy the subsidiary condition

$$f(kg) = L_k f(g), \quad k \in K, \tag{2}$$

where K is the closed inducing subgroup of G and $K \to L_k$ is a continuous unitary representation of K. The scalar product in the representation space H of L determines the scalar product for U^L (cf. 16.1(1.3°))

$$(f_1, f_2) = \int_X d\mu(\dot{g})(f_1, f_2)_H, \quad \dot{g} \in K\backslash G = X, \tag{3}$$

where

$$d\mu(\dot{g}g) = d\mu(\dot{g}), \quad \dot{g} \in X,\ g \in G.$$

The subsidiary condition (2) shows that the integrand in (3) depends only on $\dot{g} \in X$. In order not to carry along the subsidiary condition it is convenient to have it automatically built-in into the formalism. One method for this has already been discussed in detail, namely to write eq. (1) on the coset space $X = K\backslash G$ by introducing wave functions over the coset space. These are called the *Wigner-states* in the case of the Poincaré group.

A second method is to write covariant wave functions: Instead of inducing from the representation L of K, we start from a representation $\tilde{L}$ of G containing L as its restriction, reduce it with respect to K and then induce it to get another representation U^L of G. Let $f(g)$ satisfy eqs. (1) and (2) and define

$$h(g) \equiv \tilde{L}^{-1}_{g} f(g). \tag{4}$$

g

role k_g

Then it follows that

$$h(kg) = h(g), \quad k \in K,\ g \in G, \tag{5}$$

that is $h(g)$ depends only on cosets $\dot{g} = Kg$:

$$h(g) \equiv \psi(\dot{g}), \quad \dot{g} \in X = K\backslash G. \tag{6}$$

We have then by virtue of eqs. 16.1(14) and (12) ($\dot{g} \equiv x$, $\tilde{L}_g \equiv B_g$)

$$[U^L_{g_0} \psi](x) = \tilde{L}_{g_0} \psi(xg_0) \tag{7}$$

which is a simple 'covariant' transformation law, without subsidiary conditions; $\psi(x)$ is called, in the case of the Poincaré group, the *spinor wave functions.*

The scalar product (3) becomes

$$(\psi_1, \psi_2) = \int d\mu(x) (\tilde{L}_g \psi_1(x), \tilde{L}_g \psi_2(x))_H, \quad x = \dot{g}. \tag{8}$$

If the restriction L of $\tilde{L}$ to K is unitary, then the induced representation U^L of G is also unitary.

If we wish to obtain the particular representation U^{L^j} only, a complication of this method arises if the restriction of $\tilde{L}_g$ to K contains many representations of K other than L^j. Let $L = \sum \oplus L^j$ be the decomposition of the representation $\tilde{L}$ of G when restricted to K. Then by virtue of th. 16.2.1 the induced representation U^L of G have the form

$$U^L = \sum_j \oplus U^{L^j} \quad \text{and} \quad H^L = \sum_j \oplus H^{L^j}. \tag{9}$$

If we are interested in the specific representation U^{L^j} and the corresponding carrier space H^{L^j} we can eliminate the unwanted representations by imposing the subsidiary condition

$$\pi f(g) = f(g), \quad \text{or} \quad \pi f(e) = f(e), \quad f \in H^{L^j}, \tag{10}$$

where π is the projector for the representation $L^j \subset L$. The equivalence of the above relations follows from the fact that for arbitrary f in H^L, $f(g) = (U_g^L) f(e)$ by virtue of eq. (1). Because, by virtue of eqs. (6) and (4) $f(g) = \tilde{L}_g \psi(x)$ the eq. (10) gives

$$\tilde{L}_g^{-1} \pi \tilde{L}_g \psi(x) = \psi(x). \tag{11}$$

Note that the projector $\tilde{L}_g^{-1} \pi \tilde{L}_g$ depends only on the coset x. Indeed by definition of π we have

$$\tilde{L}_{kg}^{-1} \pi \tilde{L}_{kg} = \tilde{L}_g^{-1} L_k^{-1} \pi L_k \tilde{L}_g = \tilde{L}_g^{-1} \pi \tilde{L}_g.$$

Hence setting $\pi(x) = \tilde{L}_g^{-1} \pi \tilde{L}_g$, $x = \dot{g}$ we obtain

$$\pi(x) \psi(x) = \psi(x). \tag{12}$$

This is the general 'wave equation' for a function in the reducible carrier space H^L, which transforms according to the representations U^{L^j}. If only one representation of K occurs in L then $\pi = I$; hence $\pi(x) = I$ and we have no wave equation.

Note that if one uses left translations in eq. (1), we have to change $\tilde{L}$ into $\tilde{L}^{-1}$ in equations (4), (8) and (11) (cf. eq. 16.1(44) and (46)). Using eq. 16.1(45) we find that the transformation law for the wave function is

$$[\hat{U}_g \psi](x) = \tilde{L}_g^{-1} \psi(g^{-1}x)$$

(cf. eq. 17.2(41)).

B. *General Wave Equations for the Poincaré Group*

We shall now derive a general wave equation for a massive particle of arbitrary spin. In this case the stability subgroup is $K = T^4 \rtimes SU(2)$ (ch. 17, § 2) and it is convenient to realize the quotient space $X = G/K$ as the mass hyperboloid $p^2 = m^2, p = (p_0, p_1, p_2, p_3)$.

The irreducible representations of the Poincaré group were constructed in ch. 17, § 2. The wave function $\psi(p)$ corresponding to a massive particle with a spin j transforms in the spinor basis 17.2 (41) in the following manner

$$U^{mj}_{(a\Lambda)}\psi(p) = \exp[ipa]D^{(j,0)}(\Lambda)\psi(L_\Lambda^{-1}p). \tag{13}$$

The wave function $\psi(p)$ does not satisfy any wave equation besides the trivial one

$$(p^2 - m^2)\psi(p) = 0 \tag{14}$$

which expresses the mass irreducibility condition.

However by virtue of the fact that the parity operator transforms $D^{(j,0)}$ into $D^{(0,j)}$-representation (cf. 17, § 3) the wave function $P\psi$ will transform according to $D^{(0,j)}$-representation and therefore it will not be an element of the carrier space H^{mj}. Hence, if we want to construct a carrier space for a massive spin j particle which admits the parity operator we must start with a reducible representation $D(\Lambda)$ of SL(2, C), e.g. $D(\Lambda) = (D^{(j,0)} + D^{(0,j)})(\Lambda)$, restrict it to SU(2) and then induce to the Poincaré group. The elements $\psi(p)$ in this space will transform in the following manner

$$U^m_{(a,\Lambda)}\psi(p) = \exp(ipa)D(\Lambda)\psi(L_\Lambda^{-1}p). \tag{15}$$

However we have now twice as many components of the wave function $\psi(p)$ than we need for a description of the particle with the spin j. Hence we have to get rid of the redundant components. We now show that the condition which removes redundant components is just the wave equation.

It follows from the theory developed in ch. 17, § 2 that the wave function $\psi(p)$ defined on the single orbit $p^2 = m^2$ will transform with respect to the irreducible representation U^{mj} if and only if the wave function $\psi(\mathring{p})$, $\mathring{p} = (m, 0, 0, 0)$, in the rest frame will transform according to the single representation D^j of SU(2). The wave function $\psi(\mathring{p})$ which satisfies this condition is selected by the requirement

$$\pi\psi(\mathring{p}) = \psi(\mathring{p}), \tag{16}$$

where π is the projector for the representation $D^j \subset D$ of SU(2). Using this equation for the function $\varphi(\mathring{p}) = (U_{(0,\Lambda)}\psi)(\mathring{p})$ and utilizing the transformation law (15) we obtain

$$\pi D(\Lambda)\psi(p) = D(\Lambda)\psi(p), \quad p = L_\Lambda^{-1}\mathring{p},$$

or

$$\pi(p)\psi(p) = \psi(p), \tag{17}$$

where

$$\pi(p) = D^{-1}(\Lambda)\pi D(\Lambda). \tag{18}$$

Using the Mackey decomposition $\Lambda = \Lambda_p r$ of SL(2, C) (cf. 17.2 (32)), and the fact that the projection operator π commutes with the transformations $D(r)$, $r \in \mathrm{SU}(2)$, we obtain

$$\pi(p) = D^{-1}(\Lambda_p)\pi D(\Lambda_p). \tag{19}$$

Using again the Mackey decomposition of SL(2, C) we find

$$D^{-1}(\Lambda')\pi(p)D(\Lambda') = D^{-1}(\Lambda_{\Lambda'^{-1}p})\pi D(\Lambda_{\Lambda'^{-1}p}) = \pi(L_{\Lambda'}^{-1}p). \tag{20}$$

Hence $\pi(p)$ is a covariant matrix operator. Because $D(\Lambda)$ is finite-dimensional the rank of tensor coefficients in the powers of p in $\pi(p)$ is bounded. Consequently $\pi(p)$ is a covariant polynomial in p. If one makes the identification $p_\mu = \mathrm{i}\partial_\mu$ then eq. (17) becomes a covariant differential equation of finite order.

Let us note that every covariant relativistic wave equation for a massive particle with an arbitrary spin is a special case of eq. (17). Indeed to each pair $\{D(\Lambda), \pi\}$ there corresponds the unique covariant wave equation for the massive particle with spin j, and conversely to every covariant wave equation there corresponds a unique pair $\{D(\Lambda), \pi\}$. Hence eq. (17) represents the most general covariant relativistic wave equation.

We shall now find the form of the scalar product in the spinor basis. Using eq. (8) and the note at the end of subsec. A we find

$$(\psi_1, \psi_2) = \int (D^{-1}(\Lambda_p)\psi_1(p), D^{-1}(\Lambda_p)\psi_2(p))_H \frac{\mathrm{d}^3p}{p_0}. \tag{21}$$

We utilized in the last formula the fact that by 17.2(32) $\Lambda = \Lambda_p r$ and $(D^{-1}(r) \times \times \psi_1(p), D^{-1}(r)\psi_2(p))_H = (\psi_1(p), \psi_2(p))_H$.

Using now the relation $\Lambda_p \sigma \cdot \mathring{p} \Lambda_p^* = \sigma \cdot p$, $\sigma\mathring{p} = m$, we obtain

$$D^{-1*}(\Lambda_p)D^{-1}(\Lambda_p) = D((\Lambda_p\Lambda_p^*)^{-1}) = D\left(\left(\frac{\sigma\cdot p}{m}\right)^{-1}\right) = D\left(\frac{\sigma\cdot p}{m}\right). \tag{22}$$

Hence

$$(\psi_1, \psi_2) = \int \left(\psi_1(p), D\left(\frac{\sigma\cdot p}{m}\right)\psi_2(p)\right)_H \frac{\mathrm{d}^3p}{p_0}. \tag{23}$$

The formula (23) simplifies if the rest frame states $\psi(\mathring{p})$ are eigenstates of the parity operator η for $D(\Lambda)$, e.g.

$$\eta\psi(\mathring{p}) = \psi(\mathring{p}). \tag{24a}$$

Using this formula for $\varphi(\mathring{p}) = (U_{(0,\Lambda)}\psi)(\mathring{p}) = D(\Lambda)\psi(p)$ we obtain

$$\eta D(\Lambda)\psi(p) = D(\Lambda)\psi(p) \tag{24b}$$

for arbitrary $\psi(p)$ in the carrier space and arbitrary $\Lambda \in \mathrm{SL}(2, C)$. Replacing now $D(\Lambda_p)\psi_2(p)$ in eq. (21) by $\eta D(\Lambda_p)\psi_2(p)$ and using the formula 17.3(9) we obtain finally

$$(\psi_1, \psi_2) = \int (\psi_1(p), \eta\psi_2(p))_H \frac{d^3p}{p_0}. \tag{26}$$

This is the most convenient form of the scalar product which will be frequently used.

§ 2. Finite Component Relativistic Wave Equations*

We now derive all conventional relativistic wave equations for a massive particle with spin j by putting the particular forms of the representation $D(\Lambda)$ and projector π in the general wave equation 1 (17)

A. *Dirac Equation*

We want to derive a wave equation for a particle with a positive mass m and spin 1/2. The wave functions $\psi(p)$ from the carrier space $H^{m,1/2}$ of irreducible representations $U^{m,1/2}$ by virtue of eq. 17.2 (41) transform in the spinor basis in the following manner

$$U^{m,1/2}_{(a,\Lambda)}\psi(p) = \exp[ipa]D^{(1/2,0)}(\Lambda)\psi(L_\Lambda^{-1}p). \tag{1}$$

However the parity operator P cannot be defined in $H^{m,1/2}$ because $PD^{(1/2,0)}P^{-1} = D^{(0,1/2)}$ The minimal extension H^m of the space $H^{m,1/2}$ in which the parity operator is defined is the space of wave functions which transform according to $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation, i.e.

$$U^m_{(a,\Lambda)}\psi(p) = \exp[ipa]\,(D^{(1/2,0)} \oplus D^{(0,1/2)})\,(\Lambda)\psi(L_\Lambda^{-1}p). \tag{2}$$

However, we have now four components of the wave function, instead of two needed for the description of the particle with two spin projections. The projection operator π which removes the unwanted components has in the present case the form

$$\pi = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix} = \frac{1}{2}(\gamma_0 + I) \quad \text{with } \gamma_0 = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}. \tag{3}$$

Using the formulas 8.9(14)–(15) for the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ and γ_μ-matrices we readily verify that

$$(D^{(1/2,0)} \oplus D^{(0,1/2)})^{-1}(\Lambda)\gamma_\mu(D^{(1/2,0)} \oplus D^{(0,1/2)})(\Lambda) = (L_\Lambda^{-1})_\mu{}^\nu\gamma_\nu. \tag{4}$$

Hence by virtue of eq. 1(18) we obtain

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* In order to have a correspondence with notation used in the physical literature we have used exceptionally in this section a scalar product which is antilinear with respect to the first factor and linear with respect to the second one.

$$(D^{(1/2,0)}\oplus D^{(0,1/2)})^{-1}(\Lambda_p)\pi(D^{(1/2,0)}\oplus D^{(0,1/2)})(\Lambda_p) = \frac{1}{2m}(\gamma_\mu p^\mu + m), \tag{5}$$

where $p = L_\Lambda \mathring{p}$, $\mathring{p} = (m, 0, 0, 0)$.

Consequently, using the general wave equation 1(17), we obtain the equation

$$(\gamma_\mu p^\mu - m)\psi(p) = 0 \tag{6}$$

which is the Dirac equation. This example clearly shows that the subsidiary condition for the spin irreducibility represents the wave equation.

Using the explicit form of $D^{(1/2,0)}\oplus D^{(0,1/2)}$-representation given by eqs. 8.9(14) and formula 17.3(9) we conclude that in the case of $D^{(1/2,0)}\oplus D^{(0,1/2)}$-representation the parity operator η must satisfy the following conditions

$$\eta\gamma_k\eta = -\gamma_k \quad \text{and} \quad \eta\gamma_0\eta = \gamma_0. \tag{7}$$

This conditions are satisfied by $\eta = \gamma_0$. Hence the scalar product 1(26) takes the form

$$(\psi_1, \psi_2) = \int \frac{d_3 p}{p_0}(\psi_1(p), \gamma_0\psi_2(p))_H \doteq \int \bar{\psi}_{1\alpha}(p)\psi_\alpha(p)\frac{d^3p}{p_0}, \tag{8}$$

where

$$\bar{\psi}(p) = \psi^*(p)\gamma_0.$$

B. *Proca Equations*

We now want to derive the wave equation for a particle with a positive mass and spin 1. For reasons of relativistic covariance such a particle could be described by the three-component wave function $\tilde{\Phi}(p) = \{\tilde{\Phi}_k(p)\}_{k=1}^3$ (each component corresponding to a spin projection) which transforms in the following manner (cf. eq. 17.2(41))

$$U^{m,1}_{(a,\Lambda)}\tilde{\Phi}(p) = \exp(ipa)D^{(1,0)}(\Lambda)\tilde{\Phi}(L_\Lambda^{-1}p). \tag{9}$$

However the representation $D^{(1,0)}$ does not admit the parity operator p in the carrier space $H^{m,1}$. Hence we have either to double the space and consider the representation $D^{(1,0)}\oplus D^{(0,1)}$, or to start with the four-vector function $\Phi(p) = \{\Phi_\mu(p)\}_{\mu=0}^3$ which transforms with respect to the representation $D^{(1/2,1/2)}$. Because $D^{(1/2,1/2)}|_{SU(2)} \simeq D^1 + D^0$ we have in the latter case in addition to the spin 1 particle, another scalar particle. Because the $D^{(1/2,1/2)}$-representation gives the wave function with lesser number of components than $D^{(1,0)}\oplus D^{(0,1)}$ we shall use it for the description of spin 1 particles. The projector π onto the three-vector space can be written in the form

$$\pi = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \tfrac{1}{2}[\delta_{\mu\nu} - g_{\mu\nu}]. \tag{10}$$

The $D^{(1/2,1/2)}$-representation is the regular four-dimensional representation L of the Lorentz group given by eq. 17.2(2). Thus by virtue of eqs. (10) we obtain

$$\pi(p) = (D^{(1/2,\, 1/2)})^{-1}(\Lambda)\,\pi D^{(1/2,\, 1/2)}(\Lambda) = \frac{1}{2}\left[\frac{p_\mu p_\nu}{m^2} - g_{\mu\nu}\right].$$

Hence using eq. 1(17) we obtain

$$\frac{1}{2}\left(\frac{p_\nu p_\mu}{m^2} - g_{\mu\nu}\right)\Phi^\mu(p) = \Phi_\nu(p).$$

Multiplying both sides by p^ν we have

$$p^\nu \Phi_\nu(p) = 0 \tag{11}$$

which is the Proca equation. Clearly, since $p^2 = m^2$ each component $\Phi_\mu(p)$ satisfies also the Klein–Gordon equation

$$(p^2 - m^2)\Phi_\mu(p) = 0. \tag{12}$$

Because $D^{(1/2,1/2)}(\Lambda) = L_\Lambda$ the parity operator η for $D^{(1/2,1/2)}(\Lambda)$ satisfying eq. 17.3(9) is given by virtue of 17.2(5) by the metric tensor $g = ||g_{\mu\nu}||$. Hence by virtue of eq. 1(26) the scalar product has the form

$$(\Phi, \Phi) = \int \Phi^*(p) g \Phi(p) \frac{d^3p}{p_0} = \int \Phi^*_\mu(p)\Phi^\mu(p)\frac{d^3p}{p_0}. \tag{13}$$

The Proca equations (11) and (12) in coordinate space have the following form

$$\partial^\mu \Phi_\mu(x) = 0 \quad \text{and} \quad (\Box - m^2)\Phi_\mu(x) = 0. \tag{14}$$

These equations can be cast into the form of a set of first order equations. Indeed, setting

$$B_{\mu\nu} = \partial_\mu \Phi_\nu - \partial_\nu \Phi_\mu \tag{15}$$

we obtain

$$\partial^\mu B_{\mu\nu} - m^2 \Phi_\nu = 0. \tag{16}$$

One readily verifies by differentiation that original Proca equations (11) and (12) are equivalent to eqs. (15) and (16).

C. *Massive Tensor Fields Equations*

We start with the massive spin-2 field. For the description of this field we may use either $D^{(1,1)}$-representation or $D^{(2,0)} \oplus D^{(0,2)}$. We know that $D^{(1,1)}$-representation may be realized in the space of traceless symmetric tensor $\Phi_{\mu_1\mu_2}$ of order two. Because $D^{(1,1)}|_{SU(2)} \simeq D^2 \oplus D^1 \oplus D^0$ we have to cut down unwanted spin-1 and spin-0 components. Following the vector case we take the projector π in the form

$$\pi = \bigotimes_{r=1}^{2} \tfrac{1}{2}||\overset{r}{\delta} - \overset{r}{g}||. \tag{17}$$

Because the tensor $\Phi_{\mu_1\mu_2}$ transforms according to $L\otimes L$ representation, the projector $\pi(p)$ by virtue of 1(18) has the form

$$\pi(p) = \bigotimes_{r=1}^{2} \tfrac{1}{2}\left\| g^{\mu_r}{}_{\nu_r} - \frac{p^{\mu_r}p_{\nu_r}}{m^2}\right\|. \tag{18}$$

The wave function $\Phi_{\mu_1\mu_2}(p)$ in momentum space, by virtue of eq. 1(17) satisfies the condition

$$\pi(p)\Phi = \Phi. \tag{19}$$

Multiplying both sides by p^{μ} we obtain

$$p^{\mu}\Phi_{\mu,\mu_2}(p) = 0. \tag{20}$$

Solving this equation with respect to $\Phi_{0,\mu}$ we obtain

$$\Phi_{0,\mu}(p) = \frac{p_k\Phi_{k\mu}}{p_0}. \tag{21}$$

Hence the wave equations (20) allows us to express the component $\Phi_{0,0}(p)$ (spin-0 particle) and the components $\Phi_{0,k}(p)$ (spin-1 particle) in terms of five independent components $\Phi_{k,l}(p)$ corresponding to a spin-2 particle. Clearly by virtue of the mass condition we have also

$$(p^2-m^2)\Phi_{\mu_1,\mu_2}(p) = 0. \tag{22}$$

The procedure of this example can be applied directly to the representations $D^{(j,j)}$ of symmetric traceless tensors $\Phi_{\mu_1,\ldots,\mu_{2j}}$. The resulting wave equations have the form

$$p^{\mu}\Phi_{\mu_1,\mu_2,\ldots,\mu_{2j}}(p) = 0, \quad (p^2-m^2)\Phi_{\mu_1,\mu_2,\ldots,\mu_{2j}} = 0. \tag{23}$$

D. *Rarita–Schwinger Equations*

The example of the Dirac and tensor particles suggest to use for the description of massive particles with arbitrary half-integer spin the tensor product of the Dirac and the tensor representations of SL(2, C), i.e.,

$$D = (D^{(1/2,0)} \oplus D^{(0,1/2)}) \otimes D^{(j,j)}. \tag{24}$$

In this case the wave function $\psi(p)$ carries both a spinor index and symmetric traceless tensor indices: $\psi_{\alpha;\mu_1,\ldots,\mu_{2j}}(p)$. The projector π onto the highest spin $2j+\frac{1}{2}$ of the representation (24) is the tensor product of the Dirac (3) and the symmetric tensor (18) projectors, i.e.

$$\pi = \tfrac{1}{2}(\gamma_0+I)^2 \bigotimes_{r=1}^{2j} \tfrac{1}{2}(\overset{r}{I}-\overset{r}{g}). \tag{25}$$

Hence by virtue of eq. 1 (18), we obtain

$$\pi(p) = \frac{1}{2m}(\gamma p+m)\bigotimes_{r=1}^{2j}\left\| g^{\mu_r}{}_{\nu_r} - \frac{p^{\mu_r}p_{\nu_r}}{m^2}\right\|. \tag{26}$$

Multiplying both sides of eq. (26) by $(\gamma p-m)$ and p^{μ} respectively we obtain

$$(\gamma p-m)\psi_{\mu_1,\ldots,\mu_{2j}}(p) = 0, \tag{27}$$

$$p^{\mu}\psi_{\mu,\mu_2,\ldots,\mu_{2j}}(p) = 0. \tag{28}$$

We have also

$$(p^2-m^2)\,\psi_{\mu_1,\ldots,\mu_{2j}}\;p) = 0. \tag{29}$$

Eqs. (27)–(29) are called *Rarita–Schwinger equations*. We leave as an exercise for the reader to verify that the spin-tensor wave function $\psi_{\alpha;\mu_1,\ldots,\mu_{2j}}(p)$ satisfying eqs. (27)–(29) has $4j+1$ independent components.

E. *Bargmann–Wigner Equations*

We now want to derive the wave equation for a massive particle with arbitrary integer spin j. Clearly we can construct a particle with spin j from the tensor product of Dirac particles: the corresponding representation $D(\Lambda)$ of SL(2, C) has the form

$$D = \bigotimes_{r=1}^{j} (\overset{r}{D}{}^{(1/2,0)} \oplus \overset{r}{D}{}^{(0,\,1/2)}). \tag{30}$$

By virtue of eq. (3) the projector onto the highest spin has the form

$$\pi = \bigotimes_{r=1}^{j} \tfrac{1}{2}(\overset{r}{\gamma}_0+\overset{r}{I}). \tag{31}$$

Using eqs. 1(18) and (5) we find that the projector $\pi(p)$ has now the form

$$\pi(p) = \bigotimes_{r=1}^{2j} \frac{1}{2m}(\overset{r}{\gamma p}+m). \tag{32}$$

Multiplying the equation 1(17), i.e.,

$$\pi(p)\psi(p) = \psi(p) \tag{33}$$

by $\overset{i}{\gamma}p-m$, $i = 1, 2, \ldots, j$, and using the mass condition $p^2 = m^2$ one obtains a series of spinor equations

$$(\overset{i}{\gamma}p-m)_{\alpha_i\beta_i}\psi_{\beta_1,\ldots,\beta_i,\ldots,\beta_{2j}}(p) = 0, \quad i = 1, 2, \ldots, 2j. \tag{34}$$

These equations are the Bargmann–Wigner wave equations. They represent a direct generalization of the Dirac equation to particles with arbitrary integer spin.

F. 2(2j+1)-*Component Wave Equations*

In order to describe a massive particle with an arbitrary spin one may also use the representation $D = D^{(j,0)} \oplus D^{(0,j)}$ which is another direct generalization of the Dirac representation. We take the projector π in the form

$$\pi = \tfrac{1}{2}(\eta+I), \tag{35}$$

where similarly as in the Dirac case the operator η is the parity operator for the representation D, $\eta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, $\eta D^{(j,0)}\eta = D^{(0,j)}$. Using eqs. 1(17), 1(18) and (22), we obtain the following wave equation

$$\begin{aligned}\psi(p) &= \pi(p)\psi(p) = \tfrac{1}{2}D^{-1}(\Lambda)(\eta+I)D(\Lambda)\psi(p)\\ &= \tfrac{1}{2}(\eta D^*(\Lambda)D(\Lambda)+I)\psi(p) = \tfrac{1}{2}\left[\eta D\left(\frac{\sigma p}{m}\right)+I\right]\psi(p)\\ &= \frac{1}{2}\begin{bmatrix} I & D^{(j,0)}\left(\dfrac{\sigma p}{m}\right) \\ D^{(0,j)}\left(\dfrac{\sigma p}{m}\right) & I \end{bmatrix}\psi(p).\end{aligned} \tag{36}$$

The spinor $\psi(p)$ has the form

$$\psi(p) = \begin{bmatrix} \psi_{\alpha_1,\dots,\alpha_j}(p) \\ \psi_{\dot\beta_1,\dots,\dot\beta_j}(p) \end{bmatrix},$$

where the first row transforms according to $D^{(j,0)}$ representation and the second row according to $D^{(0,j)}$ representation. By convention the components of the spinor transforming according to $D^{(0,j)}$ representation are denoted by dotted indices. Using the equality

$$D^{(j,0)}\left(\frac{\sigma p}{m}\right) = \left(\frac{\sigma p}{m}\right)\otimes\dots\otimes\left(\frac{\sigma p}{m}\right) \quad (2j\text{-times}) \tag{37}$$

and

$$D^{(0,j)}\left(\frac{\sigma p}{m}\right) = \left(\frac{\tilde\sigma p}{m}\right)\otimes\dots\otimes\left(\frac{\tilde\sigma p}{m}\right) \quad (2j\text{-times}) \tag{38}$$

we can write eq. (36) in the spinorial form

$$\begin{aligned}(\sigma\cdot p)_{\alpha_1\dot\beta_1}(\sigma\cdot p)_{\alpha_2\dot\beta_2}\dots(\sigma p)_{\alpha_j\dot\beta_j}\psi^{\dot\beta_1\dots\dot\beta_j} &= m^{2j}\psi_{\alpha_1\alpha_2\dots\alpha_j},\\ (\tilde\sigma\cdot p)^{\dot\beta_1\alpha_1}(\tilde\sigma\cdot p)^{\dot\beta_2\alpha_2}\dots(\tilde\sigma p)^{\dot\beta_j\alpha_j}\psi_{\alpha_1,\dots\alpha_j} &= m^{2j}\psi^{\dot\beta_1\dots\dot\beta_j}.\end{aligned} \tag{39}$$

Projection operators of this type have been discussed by Joos 1962, Barut, Muzinich and Williams 1963, and Weinberg 1964.

G. *Wave Equations for Massless Particles*

If we use wave functions for a massless particle which transform covariantly under representations $D(\Lambda)$ of the Lorentz group according to eq. (15) we must project it onto the irreducible representations of the subgroup $\tilde E_2$ of SL(2, C)

which is the little group for massless particles. This different little group, as compared to massive particles, which is non-compact and non-semisimple has important consequences for the physics of massless particles. The finite dimensional representations of SL(2, C) reduced with respect to the subgroup $\tilde{E}(2)$ are indecomposable, hence nonunitary representations of the latter, unless it is one-dimensional. The structure of $\tilde{E}^2$ is $\tilde{E}^2 \simeq T^2 \rtimes U(1)$ and the restriction of the representation D of SL(2, C) to $\tilde{E}(2)$ is of the form $D(T^2)\exp(\mathrm{i}j\Phi)$ where $D(T^2)$ is in general indecomposable. In ch. 17, § 2.C, we have discussed the orbits of the representations of $\tilde{E}_2$. Thus, for a wave equation with a positive scalar product we must make sure that the representations $D(T^2)$ of T^2 are trivial, i.e.

$$D(T^2)\psi(\mathring{p}) = \psi(\mathring{p}), \quad \mathring{p} = (1, 0, 0, 1).$$

Let the generators of SL(2, C) be J_k and N_k, $k = 1, 2, 3$, then $J_1 - N_2$ and $J_2 + N_1$ are the commuting generators of T_2. Then for the trivial representation of T_2 the wave function must satisfy

$$(J_1 - N_2)\psi(\mathring{p}) = 0, \quad \text{and} \quad (J_2 + N_1)\psi(\mathring{p}) = 0. \tag{40}$$

Take the representation $D^{(j_1, j_2)}(A)$ of SL(2, C) in the SU(2)$\times$SU(2)-basis with $\boldsymbol{J}_1 = \boldsymbol{J} + \mathrm{i}\boldsymbol{N}$, $\boldsymbol{J}_2 = \boldsymbol{J} - \mathrm{i}\boldsymbol{N}$. The elementary calculation shows that equations (40) imply that $\psi(\mathring{p})$ is a highest weight vector for $\boldsymbol{J}_1$ and a lowest weight vector for $\boldsymbol{J}_2$, i.e.

$$J_1^{(+)}\psi(\mathring{p}) = 0, \quad J_2^{(-)}\psi(\mathring{p}) = 0, \tag{41}$$

where

$$J_k^{(\pm)} = (J_k)_1 + \mathrm{i}(J_k)_2, \quad k = 1, 2$$

or,

$$(J_1)_3\psi(\mathring{p}) = j_1\psi(\mathring{p}) \quad \text{and} \quad (J_2)_3\psi(\mathring{p}) = -j_2\psi(\mathring{p}). \tag{42}$$

In an arbitrary Lorentz frame, these equations take the form

$$(\boldsymbol{J}\cdot\boldsymbol{p})\psi(p) = p_0(j_1 - j_2)\psi(p) \tag{43a}$$

or, equivalently,

$$(\boldsymbol{N}\cdot\boldsymbol{p})\psi(p) = \mathrm{i}p_0(j_1 + j_2)\psi(p). \tag{43b}$$

The wave equations (43) can also be written as

$$W_0\psi(p) = p_0(j_1 - j_2)\psi(p), \tag{44}$$

where $W_\mu = \varepsilon_{\mu\lambda\sigma\nu}M^{\lambda\sigma}P^\nu$ is the spin operator. Thus eq. (44) indicates that $\psi(p)$ has only one component, namely, with helicity $\lambda = j_1 - j_2$, as we know from the representation theory generally. Parity invariance necessitates again the doubling of the space, so that massless particles, when parity is defined, have two states of polarization. The scalar product 1(23) reduces in this case to

$$(\psi_1, \psi_2) = \int \frac{\mathrm{d}^3p}{p_0}\psi_1^*(p)p_0^{-2(j_1+j_2)}\psi_2(p), \quad p_0 = |\boldsymbol{p}|. \tag{45}$$

EXAMPLE 1. $D(\Lambda) = D^{(1/2,0)}(\Lambda)$. Then $\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma}$, hence either one of eq. (43) gives

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})\psi(p) = p_0\psi(p) \tag{46}$$

which is the Weyl-equation for the neutrino.

EXAMPLE 2. Take $D(\Lambda) = (D^{(1,0)}\oplus D^{(0,1)})(\Lambda)$. Then $\psi = \psi_1\oplus\psi_2$, $\boldsymbol{J}^{(1,0)} = \boldsymbol{J}^{(0,1)}$ with $(J_a^{(1,0)})_{bc} = \mathrm{i}\varepsilon_{abc}$; hence $(\boldsymbol{J}^{(1,0)}\boldsymbol{p})_{bc} = \mathrm{i}\varepsilon_{abc}p_a = (\boldsymbol{J}^{(0,1)}\boldsymbol{p})_{bc}$ and from eq. (43a) we get

$$\boldsymbol{p}\times\psi_1(p) = -\mathrm{i}\omega\psi_1(p), \quad \boldsymbol{p}\times\psi_2(p) = \mathrm{i}\omega\psi_2(p).$$

Setting $\psi_1 = \boldsymbol{B}+\mathrm{i}\boldsymbol{E}$ and $\psi_2 = \boldsymbol{B}-\mathrm{i}\boldsymbol{E}$ we obtain finally the following equations in momentum space

$$\begin{aligned} \boldsymbol{p}\times\boldsymbol{E} &= \omega\boldsymbol{B}, \quad & \boldsymbol{p}\cdot\boldsymbol{B} &= 0, \\ \boldsymbol{p}\cdot\boldsymbol{E} &= 0, \quad & \boldsymbol{p}\times\boldsymbol{B} &= -\omega\boldsymbol{E} \end{aligned} \tag{47}$$

which are the free Maxwell equations.

H. *General Remarks*

(i) It is remarkable that all existing finite-dimensional relativistic wave equations are special cases of the general wave equation 1(17)

$$\pi(p)\psi(p) = \psi(p)$$

derived on the basis of the theory of induced representation. We have obtained particular wave equations by taking specific representations $D(\Lambda)$ of $\mathrm{SL}(2, C)$ and calculating the corresponding projection operators $\pi(p)$. These results show again the effectiveness and the elegance of the theory of induced representations.

(ii) In general the wave equation 1(17) represents the irreducibility condition for spin. The Klein–Gordon equation

$$(p^2-m^2)\psi(p) = 0 \tag{48}$$

satisfied by all wave functions represents the irreducibility condition for mass. In the case of the Dirac equation the mass irreducibility follows from the spin irreducibility. Indeed multiplying the Dirac equation

$$(\gamma_\mu p^\mu - m)\psi(p) = 0$$

by the operator $(\gamma_\mu p^\mu + m)$ we obtain eq. (48). However in the case of the Proca equation one cannot derive the mass irreducibility (48) from the spin irreducibility condition $p^\nu\Phi_\nu(p) = 0$. One can write however the wave equation in a form from which the mass and spin irreducibility follow: for instance the corresponding equation for a massive spin 1 particle has the form

$$(m^2 g_\nu{}^\mu + p_\nu p^\mu)\Phi_\mu(p) = p^2\Phi_\nu(p);$$

conversely one can write the Dirac equation in the form

$$(\gamma_\mu p^\mu - (p^2)^{\frac{1}{2}})\psi(p) = 0$$

from which the mass irreducibility does not follow. These two examples illustrate the fact that the mass and the spin irreducibility conditions are on the same footing and it is only a question of convenience if we represent them by a single or by two separate equations.

§ 3. Infinite Component Wave Equations

A. *Gel'fand–Yaglom Equations*

We have shown in sec. 1 that any representation of the Lorentz group whose restriction to SU(2) was reducible could be used for a construction of manifestly covariant wave equation. In sec. 2 we used finite-dimensional reducible representations of SL(2, *C*) to produce conventional finite component relativistic wave equations. One observes however in experiments that the elementary particles and resonances may be grouped into possible infinite families of particles. Fig. 1 shows an example of mass-spin relations for a multiplet of mesons (so-called ϱ and π Regge trajectories).

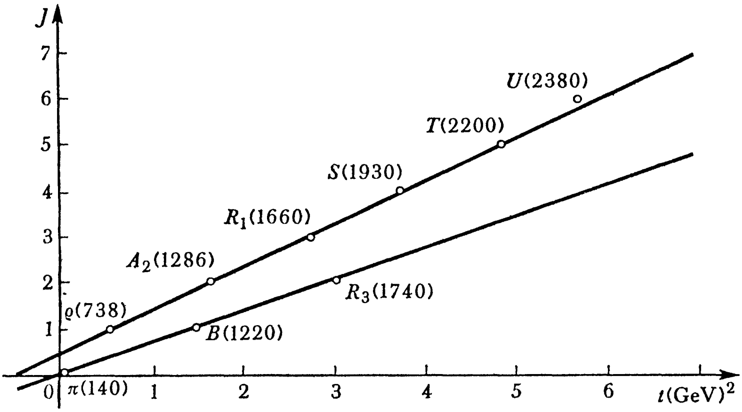


Fig. 1

It is therefore a highly attractive idea to consider infinite-component wave equations describing the properties of a whole family of particles. The simplest equation of this type would be a generalization of the Dirac equation

$$(\Gamma_\mu p^\mu - \varkappa)\psi(p) = 0, \tag{1}$$

where Γ_μ is a vector operator in the carrier space and $\varkappa$ is a scalar. However there arises the problem if and when in any infinite-dimensional space a vector operator exists which must satisfy the covariance condition

$$U_g^{-1}\Gamma_\mu U_g = \Lambda_\nu^{\ \mu}\Gamma_\nu \tag{2}$$

exists. This problem was studied by Gel'fand and Yaglom who found

THEOREM 1. *Let the irreducible representation of* SL(2, *C*) *be labelled by a pair of numbers* $[j_0, j_1]$, *where* j_0 *is the lowest spin in the representation and takes in-*

tegral or half-odd-integral values and j_1 is an arbitrary complex number (cf. ch. 19). *A four-vector operator Γ^μ exists in the direct sum $H = \bigoplus_s H^{[j_0^s, j_1^s]}$ of irreducible carrier spaces if for every irreducible component $H^{[j_0, j_1]}$ in H there exists an irreducible component $H^{[j_0', j_1']}$ in H whose invariant numbers are related to each other by*

$$\begin{aligned}[j_0', j_1'] &= [j_0+1, j_1] \\ &= [j_0-1, j_1] \\ &= [j_0, j_1+1] \\ &= [j_0, j_1-1]. \blacktriangledown\end{aligned} \tag{3}$$

PROOF: By virtue of eq. (2) we obtain

$$[\Gamma_\mu, M_{\lambda\varrho}] = \mathrm{i}(g_{\mu\lambda}\Gamma_\varrho - g_{\mu\varrho}\Gamma_\lambda), \tag{4}$$

where $M_{\lambda\varrho}$ are generators of SL(2, C). Setting $\boldsymbol{J} = (M_{32}, M_{13}, M_{21})$ and $\boldsymbol{N} = (M_{01}, M_{02}, M_{03})$, we obtain in particular

$$\mathrm{i}\Gamma_k = [\Gamma_0, N_k], \tag{5}$$

$$[\Gamma_0, J_k] = 0, \quad [\Gamma_3, N_3] = \mathrm{i}\Gamma_0 = -\mathrm{i}[[\Gamma_0, N_3], N_3]. \tag{6}$$

By virtue of eq. (5) it is sufficient to find Γ_0 in order to obtain $\{\Gamma_\mu\}_{\mu\neq 0}^3$. Hence it is sufficient to verify when the operator Γ_0 which is determined by eq. (6) exists. Let H be a reducible carrier space of representation $g \to U_g$ of SL(2, C) and let $\sum_\tau \oplus H^\tau$ be its decomposition onto irreducible subspaces H^τ, $\tau \equiv [j_0, j_1]$. Let $|\tau, JM\rangle$ be the canonical basis in H^τ. Let $[c_{JM,J'M'}^{\tau\tau'}]$ be the matrix elements of Γ_0 in this basis in the carrier space H. Then by virtue of eq. (6) we have

$$c_{JM,J'M'}^{\tau\tau'} = c_J^{\tau\tau'}\delta_{JJ'}\delta_{MM'}. \tag{7}$$

The action of the generator N_3 on the elements of the canonical basis is given in exercise 19.7.3.4. Taking now the matrix element of the equation $\Gamma_0 = [[\Gamma_0, N_3], N_3]$ in the canonical basis between basis elements $|\tau JM\rangle$ and $|\tau(J\pm 1)M\rangle$ we obtain six linear equations for three unknowns $c_J^{\tau\tau'}$, $c_{J-1}^{\tau\tau'}$ and $c_{J+1}^{\tau\tau'}$. We leave as an exercise for the reader to write down explicitly these equations. Solving the first three equations with respect to these unknowns and inserting the obtained expressions to the remaining three equations we readily verify that $c_J^{\tau\tau'}$ can be different from zero only then if $\tau(j_0 j_1)$ and $\tau'(j_1' j_0')$ are such that

$$[j_0', j_1'] = [j_0 \pm 1, j_1] \tag{8}$$

or

$$[j_0', j_1'] = [j_0, j_1 \pm 1]. \blacktriangledown \tag{8'}$$

Remark 1: Let us note that the condition (3) is satisfied also by finite dimensional representations. Indeed using the correspondence between indices $[j_0 j_1]$ and

(J_1, J_2) which characterize the finite-dimensional representation $D^{(J_1,J_2)}$ given by the formula

$$J_1 = \frac{j_0+j_1-1}{2}, \quad J_2 = \frac{j_1-j_0-1}{2}$$

we find that for instance the direct sums

$$D(\Lambda) = D^{(0,0)} \oplus D^{(1/2,1/2)},$$
$$D(\Lambda) = D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1/2,1/2)}$$

satisfy condition (3). Hence there are in these cases finite-dimensional Gel'fand–Yaglom equation of the type (1).

Remark 2: The eigenvalues of Casimir operators $C_2 = J^2 - N^2$ and $C_2' = JN$ for representation $[j_0, j_1]$ are $j_0^2 + j_1^2 - 1$ and $2_{ij_0j_1}$ (see exercise 19.7.3.1). The action of the parity operator is: P: $J \to J$ and $N \to -N$. Hence the parity transform of $[j_0 j_1]$ is $[j_0, -j_1]$. Consequently by eq. (3) Γ_μ exists on the direct sum of spaces of

(a) $[0, j_1]$ with $[1, j_1]$, $[0, j_1+1]$, $[0, j_1-1]$ (9)

and

$[j_0, 0]$ with $[j_0+1, 0]$, $[j_0-1, 0]$, $[j_0, 1]$, $[j_0, -1]$, (10)

(b) $[j_0, j_1]$ ($j_0 \neq 0$, $j_1 \neq 0$) with the set $[j_0', j_1']$, (11)

as in eq. (3) and with the same set corresponding to $[j_0', -j_1']$.

B. *Majorana Wave Equation*

The Gel'fand–Yaglom Theorem shows that in general one needs at least a pair of irreducible representations in order to be able to determine a vector operator Γ_μ in the carrier space H. However if we take the unitary irreducible representation of SL(2, C) in the form $[j_0, j_1] = [0, \frac{1}{2}]$ then by virtue of th. 1, $[j_0', j_1'] = [0, -\frac{1}{2}]$ is the second representation which together with $[0, \frac{1}{2}]$ defines the operator Γ_μ. However $[0, -\frac{1}{2}]$ is equivalent to $[0, \frac{1}{2}]$ by virtue of exercise 19.7.3.2. Hence, in this case the vector operator can be defined in the carrier space $H^{[0,1/2]}$ of the irreducible representation $[0, \frac{1}{2}]$. A similar situation holds for the representation $[\frac{1}{2}, 0]$. The corresponding wave equation

$$(\Gamma_\mu p^\mu - \varkappa)\psi(p) = 0 \tag{12}$$

associated with $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 0]$ representation is called the Majorana equation. These equations were introduced by Majorana in 1932 as a possibility for avoiding 'negative energy' states in Dirac theory, which caused a serious embarrassment at that time.

We find now the spectrum of the Majorana equation associated with $[\frac{1}{2}, 0]$ representation. Assuming that p is a time-like momentum and going to the rest frame $\mathring{p} = (m, 0, 0, 0)$ we obtain

$$(\Gamma_0 m - \varkappa)\psi(\mathring{p}) = 0. \tag{13}$$

To get the spectrum of Γ_0 we take a special realization of $[\frac{1}{2}, 0]$ representation. Let a_i and a_i^*, $i = 1, 2$, be the creation and annihilation operators. Then the vectors $|[\frac{1}{2}, 0]; JM\rangle$ in the space $H^{[1/2,0]}$ can be realized by means of the formula

$$|[\tfrac{1}{2}, 0]; JM\rangle = Na_1^{*J+M}a_2^{*J-M}|0\rangle, \tag{14}$$

where N is a normalization factor. The generators $\boldsymbol{J}$ and $\boldsymbol{N}$ of SL(2, C) have in the realization (3) the following form

$$\begin{aligned} \boldsymbol{J} &= \tfrac{1}{2}a^*\boldsymbol{\sigma}a, \\ \boldsymbol{N} &= \frac{i}{4}(a^*\boldsymbol{\sigma}Ca^* + aC\boldsymbol{\sigma}a), \end{aligned} \tag{15}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and C is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. By virtue of eq. (6) the operator Γ_0 must commute with $\boldsymbol{J}$ and satisfy the equality $[[\Gamma_0, N_3], N_3] = -\Gamma_0$. One readily verifies that a second order operator in a and a^* which satisfies these conditions has the form of the particle number operator

$$\Gamma_0 = \tfrac{1}{2}(a^*a+1). \tag{16}$$

Using the formula (5) one obtains

$$\boldsymbol{\Gamma} = -\frac{i}{4}(a^*\boldsymbol{\sigma}Ca - aC\boldsymbol{\sigma}a). \tag{17}$$

The action of Γ_0 on the states (14) gives

$$\Gamma_0|[\tfrac{1}{2}, 0]; JM\rangle = (J+\tfrac{1}{2})|[\tfrac{1}{2}, 0]; JM\rangle. \tag{18}$$

Using then eq. (13) we obtain the mass formula for the Majorana equation

$$m_J = \frac{\varkappa}{J+\frac{1}{2}}, \quad J = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \tag{19}$$

One obtains the same mass formula for Majorana equation associated with $[0, \frac{1}{2}]$ representation (with $J = 0, 1, 2, \ldots$, in this case).

It is interesting that the set of operators $\boldsymbol{J}$, $\boldsymbol{N}$ and Γ_μ closes to the Lie algebra so(2, 3). Consequently the carrier spaces $H^{[1/2,0]}$ or $H^{[0,1/2]}$ are at the same time the carrier spaces of irreducible representations of SO(2, 3).

The Majorana equations (12) have also solutions for space-like momenta $p^2 = m^2 < 0$. In this case in the rest frame $\mathring{p} = (0, 0, 0, m)$ the resulting equation has the form

$$(\Gamma_3 m - \varkappa)\psi(\mathring{p}) = 0. \tag{20}$$

Diagonalizing now Γ_3 we find a continuous mass spectrum in this case. This follows also directly from the observation that Γ_3 is a generator of a noncompact subgroup of SO(3, 2) and the fact that all such generators have continuous spectra.

Minimal Coupling

The importance of linear equations of the type of Dirac equation, 2(6), or Majorana eq. (12) lies in the fact that the behavior of the particles in an external electromagnetic field with potential $A_\mu(x)$ is described by the substitution

$$P_\mu \to P_\mu - eA_\mu(x), \tag{21}$$

where e is the electric charge. Hence we have the equation

$$(\Gamma_\mu p^\mu - e\Gamma_\mu A^\mu(x) - \varkappa)\psi = 0. \tag{22}$$

Compared with the free particle equation the second term $e\Gamma_\mu A^\mu(x)$ appears as the interaction term. This process obtained by the rule (21) is called the *minimal coupling*. Thus the operator $e\Gamma_\mu$ is the current operator of the quantum system. The potentials $A_\mu(x)$ themselves are in turn produced by currents. Consequently, in general, we have a system of coupled equations, (22) and

$$\Box A_\mu(x) = j_\mu(x). \tag{23}$$

For given external fields $A_\mu(x)$ one can however confine oneselve to eq. (22).

C. *Generalizations of Gel'fand–Yaglom Equations*

The mass spectrum (19) is rather unphysical: a decreasing mass with increasing spin. This form of the spectrum is typical for the general Gel'fand–Yaglom equation (1). Also the magnetic moment derived from eq. (23) turns out to have the wrong sign. Hence it is natural to look for a more general relativistic wave equation of the form

$$(\Gamma_\mu p^\mu - K)\psi(p) = 0, \tag{24}$$

where K is an invariant operator of the Lorentz group. For $K = \alpha p_\mu p^\mu + \varkappa$ one obtains

$$(\Gamma_\mu p^\mu - \alpha p_\mu p^\mu - \varkappa)\psi = 0. \tag{25}$$

This equation can also be solved exactly by going to the rest frame. Proceeding as in the case of the simple Majorana equation one obtains:

$$m_J = \frac{J+\frac{1}{2}}{2\alpha}\left(1 \pm \sqrt{1 - \frac{4\varkappa\alpha}{(J+\frac{1}{2})^2}}\right). \tag{26}$$

In particular, for $\varkappa = 0$, we have a linear spectrum in spin

$$m_J = \frac{1}{2\alpha}(J+\tfrac{1}{2}).$$

One obtains another interesting model of generalized equation (24) by taking the wave function $\psi(p)$ which transforms according to $(D^{(1/2,0)} \oplus D^{(0,1/2)}) \otimes U^{[j_0,0]}$ representation. Setting in this case $K = -(M_0 + M_1\sigma_{\mu\nu}M^{\mu\nu})$ where $M^{\mu\nu}$ are generators of $U^{[j_0,0]}$ representation and M_0 and M_1 are scalars we obtain the following wave equation

$$(\gamma_\mu p^\mu + M_0 + M_1 \sigma_{\mu\nu} M^{\mu\nu})\psi(p) = 0. \tag{27}$$

Passing to the rest frame and performing the similar analysis as in case of Majorana equations one obtains the following mass formula

$$\pm m_J = M_1(J+\tfrac{1}{2}) \pm \{(M_0 - M_1)^2 + M_1^2[J(J+1) - j_0(j_0+1) - \tfrac{3}{4}]\}. \tag{28}$$

Eq. (27) is called the Abers, Grodsky and Norton equation (1967).

D. *Applications of Infinite Component Wave Equations*

In sec. 1, we have embedded the inducing representation of the subgroup K of G into a representation $D(G)$ of G in order to have manifestly covariant wave equations. We can also embed $D(K)$ into a representation $D(\check{G})$ of a larger group $\check{G}$ containing G (hence K). Now the multiplicity of $D(K)$ in $D(\check{G})$ will be in general much larger. Physically these multiplicities will be identified with additional internal degrees of freedom of the system.

We now give an important example of this method. Let $\check{G} = \mathrm{SO}(4, 2)$. The Lie algebra of SO(4, 2) has a basis $L_{ab} = -L_{ba}$, $a, b = 0, 1, 2, \ldots 5$, containing the generators $M_{\mu\nu} = L_{\mu\nu}$; $\mu, \nu = 0, 1, 2, 3$, of SO(3, 1). With respect to $M_{\mu\nu}$, the elements $L_{\mu 5} = \Gamma_\mu$ are components of a four-vector operator, and $L_{45} = S$ is a scalar operator. Hence the most general Lorentz covariant wave equation linear in the elements of the Lie algebra is

$$(\Gamma_\mu p^\mu + \beta S + \gamma)\psi(p) = 0. \tag{29}$$

Eq. (29) can be solved again by transforming it into the rest frame $\mathring{p} = (p_0, 0, 0, 0)$:

$$(\Gamma_0 p^0 + \beta S + \gamma)\psi(0) = 0. \tag{30}$$

Here we can either diagonalize Γ_0 which has a discrete spectrum as the generator of the compact subgroup, or S with a continuous spectrum as the generator of the noncompact subgroup.

We choose for example the most degenerate discrete class of representations of SO(4, 2) discussed in ch. 15 and choose as a basis the eigenvectors of Γ_0, J^2, J_3 labelled by $|n, J, J_3\rangle$. In this representation the invariant operators of SO(4, 2) have the values

$$\begin{aligned} C_2 &= \tfrac{1}{2} L_{ab} L^{ab} = -3, \\ C_3 &= \varepsilon_{abcdef} L^{cd} L^{ef} L^{ab} = 0, \\ C_4 &= L_{ab} L^{bc} L_{cd} L^{da} = -12. \end{aligned} \tag{31}$$

Due to the fact that Γ_0, S and $L_{04} \equiv T$ generate an SU(1, 1) subgroup, we can solve (30) by defining $\tilde{\psi}(\mathring{p})$ by

$$\psi(\mathring{p}) \equiv \exp(\mathrm{i}\theta_n L_{04})\tilde{\psi}(\mathring{p}), \quad \Gamma_0|\tilde{\psi}_n(\mathring{p})\rangle = n|\tilde{\psi}_n(\mathring{p})\rangle \tag{32}$$

and choosing θ_n appropriately. Then

$$[(m^2-\beta^2)^{1/2}\Gamma_0+\gamma)\tilde{\psi}(\mathring{p}) = 0 \tag{33}$$

or

$$m^2 = (\beta^2-\gamma^2/n^2). \tag{34}$$

Consider the physically interesting case when β and γ are functions of the total mass m of the system. Clearly, in this case, the mass spectrum might be different from that given by eq. (34). Set

$$\beta = m\frac{\omega^2-m_1^2+(m-m_2)^2}{\omega^2+m_1^2-(m-m_2)^2}, \quad \gamma = \frac{-2\alpha\omega m(m+m_1-m_2)}{\omega^2+m_1^2-(m-m_2)^2}. \tag{35}$$

Solving (34) with respect to m one obtains

$$m = m_2+m_1\frac{1-\alpha^2/n^2}{1+\alpha^2/n^2}. \tag{36}$$

Expansion with respect to α^2/n^2 gives:

$$m = m_1+m_2-2m_1\alpha^2/n^2,$$

which is the mass spectrum of nonrelativistic hydrogen atom. We now show that choosing properly the representation of operators Γ_0 and S we obtain that eq. (30) with β and γ given by eq. (35) can be interpreted as the Klein–Gordon equation with scalar and vector potential of the type $1/r$ and equal coupling constants at both interactions. Indeed, take a representation of so(4, 2) algebra in terms of differential operators on $L^2(R^3)$ (cf. ch. 12, § 2). Generators of su(1, 1) subalgebra have in this case the form (cf. eq. 12.2(5))

$$\begin{aligned} \Gamma_0 &= \frac{1}{2\omega}(-r\nabla^2+\omega^2 r), \\ S &= \frac{1}{2\omega}(-r\nabla^2-\omega^2 r), \\ T &= -\mathrm{i}r\nabla-\mathrm{i}. \end{aligned} \tag{37}$$

Inserting this generators into eq. (30), defining $E_1 \equiv m-m_2$ and using expressions (35) for β and γ one obtains:

$$\left[\frac{1}{2\omega}(\omega^2+m_1^2-E_1^2)(-r\nabla^2+\omega^2 r)+\frac{1}{2\omega}(\omega^2-m_1^2+E_1^2)(-r\nabla^2-\omega^2 r)\right]\psi = 2\alpha\omega(E_1+m_1)\psi.$$

After elementary calculations one obtains

$$\left[-\nabla^2-\left(E_1+\frac{\alpha}{r}\right)^2+\left(m_1-\frac{\alpha}{r}\right)^2\right]\psi = 0. \tag{38}$$

This is the Klein–Gordon equation with the scalar potential $\varphi(r) = -\alpha/r$ and the vector potential $V(r) = -\alpha/r$. From eq. (36) we get the following spectrum for E_1:

$$E_1 = m_1\frac{1-\alpha^2/n^2}{1+\alpha^2/n^2}.$$

Expanding with respect to α^2/n^2 we get:

$$E_1 = m_1 - 2m_1\alpha^2/n^2.$$

This corresponds to the spectrum of the Schrödinger equation with the potential $U(r) = -2\alpha/r$: this fact is not surprising since in nonrelativistic approximation the scalar and the vector potential add together.

It is thus remarkable, that equations of the type (29) describe relativistically composite quantum systems, such as the H-atom. Notice that the labeling of the states $|nJJ_3\rangle$ agrees with the quantum numbers of the H-atom. With the covariant equations (29) we thus established contact to the dynamical group formalism of quantum mechanics discussed in ch. 12.

Another representation of the su(1, 1) algebra (53), namely

$$\begin{aligned} \Gamma_0' &= \tfrac{1}{2}\left[-r\nabla^2 + r + \frac{1}{r}(-\alpha^2 - \mathrm{i}\alpha\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}})\right], \\ S' &= \tfrac{1}{2}\left[-r\nabla^2 - r + \frac{1}{r}(-\alpha^2 - \mathrm{i}\alpha\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}})\right], \\ T' &= T = -\mathrm{i}r\nabla - \mathrm{i}, \quad \hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|, \end{aligned} \tag{40}$$

where $\boldsymbol{\alpha}$'s are the Dirac matrices, leads with a suitable choice of the parameters to

$$[\Gamma_0' + S' - (E^2 - m^2)(\Gamma_0' - S') - 2\alpha E]\psi = 0$$

or

$$\left[p^2 - (E^2 - m^2) - \frac{2\alpha E}{r} - \frac{1}{r^2}(\alpha^2 + \mathrm{i}\alpha\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}})\right]\psi = 0 \tag{41}$$

which is the second order Dirac equation for the Coulomb potential.

Finally using the representation given in eq. 12.2 (22) we obtain the equation for dyonium which is an atom of two dyons, particles having both electric and magnetic charges. The Lie algebra solving the dyonium is the same as in eqs. (35) or (40), except for the replacements

$$\begin{aligned} \boldsymbol{p} &\to \boldsymbol{\pi} = \boldsymbol{p} - \mu\boldsymbol{D}(\boldsymbol{r}), \\ \mathrm{i}\alpha\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}} &\to (\mu\boldsymbol{\sigma} + \mathrm{i}\alpha\boldsymbol{\alpha})\cdot\hat{\boldsymbol{r}}, \end{aligned} \tag{42}$$

where $\boldsymbol{D}(\boldsymbol{r})$ and μ are defined in lemma 12.2.5.

E. *The Dynamical Group Interpretation of Wave Equations*

In the previous sections, covariant wave equations have been interpreted as projections on certain subspaces of the reducible representations of the Poincaré group induced from those of SL(2, C) (cf. eq. 1(12)).

A second interpretation (which ties up with the discussion of ch. 13, § 2) is to specify the space of states in the rest frame of a system by an irreducible representation U of a dynamical group $\mathscr{G} \supset$ SL(2, C). Then states of momentum P_μ are obtained by a Lorentz transformation (i.e. a boost). This method is particularly useful for infinite-component wave equations, but is also applicable to finite-component equations.

EXAMPLE 1. Let $\mathscr{G} = O(4,2)$. Take U to be the 4-dimensional non-unitary representation in which the generators of $\mathscr{G}$ are given in terms of the 16 elements of the algebra of Dirac matrices as in exercise 13.6.4.1.

Because $\frac{1}{2}L_{56} = \gamma_0$ has eigenvalues $n = \pm 1$, taking the simplest mass relation $mn = \varkappa$, we can write

$$(m\gamma_0 - \varkappa)\psi(\mathring{p}) = 0, \tag{43}$$

where $\varkappa$ is a fixed constant.

Transforming this equation with the Lorentz transformation of parameter ξ

$$\begin{aligned} \psi(p) &= \exp(\mathrm{i}\xi N)\psi(p), \\ N &= \tfrac{1}{2}\gamma_0\gamma \end{aligned} \tag{44}$$

gives

$$(\gamma^\mu p_\mu - \varkappa)\psi(p) = 0$$

which is the Dirac Equation 2(6).

EXAMPLE 2. Let $\mathscr{G} = O(4,2)$. Let U be an infinite-dimensional unitary representation of $\mathscr{G}$ of the most degenerate series. Now $\frac{1}{2}L_{56} = \Gamma_0$ has the spectrum $n = \mu+1, \mu+2, \ldots$ The simplest mass relation linear in the group generators is

$$(M\Gamma_0 - \varkappa)\psi(\mathring{p}) = 0.$$

Corresponding to masses $M_n = \varkappa/n, n = \mu+1, \ldots$ Transforming this equation by $\exp(\mathrm{i}\xi N)$ (Lorentz transformations) we obtain

$$(\Gamma_\mu p_\mu - \varkappa)\psi(p) = 0$$

which is the Majorana equation 3(12). Or, taking the more general mass relation

$$(M\Gamma_0 + \alpha M^2 - \varkappa)\psi(\mathring{p}) = 0$$

we obtain the generalized Majorana Equation 3(25). ▼

As these examples show we have again an inducing process. The rest frame states $\psi(\mathring{p})$ transform according to a representation L of K in H: $\psi_k(\mathring{p}) = L_k\psi(\mathring{p})$, $k \in K$, and then we pass to the induced representations of the Poincaré group Π by (44).

The reduction of U with respect to SL(2, C) gives a reducible representation $\tilde{L}$ of the latter. Hence the spinorial wave functions $\psi(p)$ transform, according to a, in general reducible, representation $\tilde{L}$ of SL(2, C):

$$(U_{(a,\Lambda)}\psi)(p) = \exp(\mathrm{i}pa)\tilde{L}(\Lambda^{-1}p).$$

The general procedure is as follows:

(1) The choice of the dynamical group $\mathscr{G}$ and its representation U depends on the internal dynamics and degrees of freedom of the system (i.e. spin, or orbital and radial excitations). This is reflected in the number of quantum numbers and of states. Figs. 2 and 3 show two weight diagrams for $\mathscr{G} = O(4,2)$ where j and n are the eigenvalues of L^2 and $L_{56} = \Gamma_0$, respectively (cf. also ch. 12, Fig. 1).

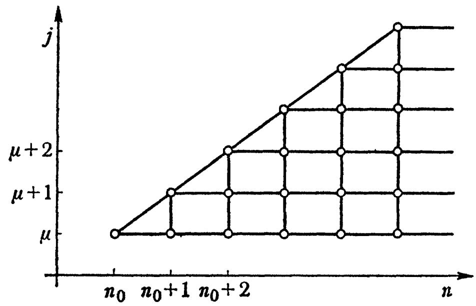


Fig. 2

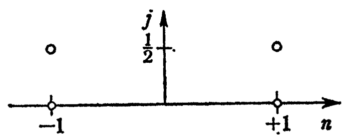


Fig. 3

(2) Let Γ_μ be a vector operator in U and S a scalar operator. Take any relation involving the quantities $P_0\Gamma^0$, $P_\mu P^\mu$, S, ... on the carrier space H of U:

$$f(\Gamma_0 P^0, P^2, S)\psi(\mathring{p}) = 0, \quad \psi(\mathring{p}) \in H \tag{45}$$

and transform this equation by the Lorentz transformation of the type (44)

$$f(\Gamma_\mu P^\mu, P^2, S)\psi(p) = 0, \tag{46}$$

which is a covariant equation incorporating the postulated mass relation (45).

The role of the dynamical group $\mathscr{G}$ in these considerations lies in the fact that it carries information about the internal dynamics of the system and dictates therefore which representation of SL(2, C) has to be chosen to construct a covariant wave equation. This latter choice was so far arbitrary in our discussion in § 1. Eqs. (45) and (46) can be generalized to spin dependent mass formulas.

F. *Physical Applications of Matrix Elements of Representations of Semisimple Non-Compact Lie Groups*

Let G be a Lie group and U_g a representation in H. Certain group elements can be parametrized as

$$\exp(\mathrm{i}\theta^k X_k),$$

where θ^k are the group parameters and X_k the generators of the Lie algebra of G. Let $u_m \in H$ be a basis in the carrier space of U_g. The matrix elements will be denoted by

$$D_{mn}(\theta) = \langle m|\exp(\mathrm{i}\theta\cdot X)|n\rangle \tag{47}$$

with respect to the scalar product in H. Clearly these functions generalize the functions D^j_{mn} of the group SU(2).

For the compact semisimple groups SO(n), the expressions for $D(\theta)$ has been given by Vilenkin 1968. For the noncompact case SO(2,1) these functions were first given by Bargmann 1947. Many other results are known for SO(3,1) and certain representations of SO(4,1), SO(4,2), ... in the physical literature.

Consider the wave equation of the type (12), (25) or (29).

The electromagnetic coupling of the system is described by the process of minimal coupling (cf. (21)).

The physical transition probability amplitude is given by the matrix elements of the current operator Γ_μ between the two states of momenta p_1 and p_2 (cf. 13).

$$A_\mu = \langle \psi_m(p_1)|\Gamma_\mu|\psi_n(p_2)\rangle .$$

Using Lorentz transformations we have

$$A_\mu = \langle \psi_m(\mathring{p})|\exp(-\mathrm{i}\xi\cdot N)\Gamma_\mu \exp(\mathrm{i}\xi\cdot N)|\psi_n(\mathring{p})\rangle .$$

These matrix elements are to be evaluated in general in an infinite-dimensional representation U of the dynamical group $\mathscr{G}$, SO(4, 2), for example. Thus transition amplitudes can be reduced to the matrix elements $D_{mn}(\theta)$ of the type (47). More general matrix elements appear in the physical applications (cf. Barut and Wilson 1976).

§ 4. Group Extensions and Applications

A. *Group Extension*

Group extension plays an important role in many parts of quantum theory. We discuss here the relation of group extension to cohomology.

Consider a following sequence of groups and homomorphisms

$$\to G_n \xrightarrow{f_n} G_{n+1} \xrightarrow{f_{n+1}} G_{n+2} \to . \tag{1}$$

Recall that if the kernel Ker$f_n = I$, f_n is called injective (then Image $f_n \cong G_n$), if Image $f_n = G_{n+1}$, then f_n is called surjective.

DEFINITION 1. The sequence (1) is an *exact sequence* if Image f_n = Kerf_{n+1}, for all n.

EXAMPLES:

$$(1°)\ 1 \to \mathrm{Ker}f \to G \xrightarrow{f} \mathrm{Im}f \to 1, \tag{2}$$

$$(2°)\ 1 \to C(G) \to G \xrightarrow{f} J(G) \to 1, \tag{3}$$

where $C(G)$ is the center of G and $J(G)$ is the group of non-trivial inner automorphisms. Clearly, $G/\mathrm{Ker}f \cong \mathrm{Im}f$ and $G/C(G) \cong J(G)$.

$$(3°)\ 1 \to J(G) \to \mathrm{Aut}\,G \to A(G) \to 1, \tag{4}$$

where $A(G)$ represents the automorphism classes of G modulo inner automorphisms, $A(G) \cong \mathrm{Aut}\ G/J(G)$.

DEFINITION 2. A group E is called an *extension of G by K* if it satisfies an exact sequence

$$1 \to K \to E \to G \to 1. \blacktriangledown \tag{5}$$

We have then that K is an invariant (normal) subgroup of E and $E/K = G$; thus E can be thought of as consisting of the cosets kG, $k \in K$ (written multiplicatively). An important mathematical problem is to find all groups E such that $E/K = G$, and K is an invariant subgroup of E.

$$\begin{tikzcd}
& 1 \arrow[d] & & & \\
& \mathrm{C}(K) \arrow[d] & & & \\
1 \arrow[r] & K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & J(K) \arrow[r] \arrow[d] & \mathrm{Aut}\,K \arrow[r] & A(K) \arrow[r] & 1 \\
& 1 & & &
\end{tikzcd} \tag{6}$$

Noting that elements from E induce automorphisms of K by $k \to x^{-1}kx$ for any $x \in E$ we get homomorphisms $E \to \mathrm{Aut}K$ and $G \to A(K)$. In general however the latter homomorphism does not come from a homomorphism $G \to \mathrm{Aut}K$.

B. *Extensions of the Poincaré Group*

The full relativistic invariance includes also the discrete symmetry operations of space-reflection Σ and time reflection Θ.

We introduce the two reflection operations Σ and Θ. Their identification with parity P and time reflection T, or their combinations, eg. PC, PT, ... depends on superselection rules. The full Lorentz group then consists of the cosets

$$L = (\Lambda, \Lambda\Sigma, \Lambda\Theta, \Lambda\Sigma\Theta), \tag{7}$$

where Λ is the restricted Lorenz group. L is the extension of Λ by the group of reflections. We again use the covering group $\mathrm{SL}(2, C)$ of Λ and write the group elements of the Poincaré group as

$$(a, \Lambda) \to (a \cdot \sigma, A), \qquad A \in \mathrm{SL}(2, C),$$

where $a \cdot \sigma = a^{\mu}\sigma_{\mu}$ (cf. ch. 17.2). We denote by

$$\begin{aligned}
U(a \cdot \sigma, A) &= \text{unitary representations of the restricted Poincaré group,} \\
U(\Sigma) &= \text{operators assigned to } \Sigma\text{, assumed to be unitary.} \\
U(\Theta), U(\Theta\Sigma) &= \text{operators assigned to } \Theta \text{ and } \Theta\Sigma\text{, assumed to be anti-unitary}
\end{aligned} \tag{8}$$

In ch. 13 we discussed the quantum mechanical unitary and antiunitary operators. The relation between the restricted groups, the full groups and the covering groups can be arranged into following sequences:

(i) The group of translations T^4, the restricted Poincaré group Π_0, the restricted Lorentz group Λ, the full Poincaré group Π, and the full Lorentz group L satisfy

$$\begin{tikzcd}
& & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & T_4 \arrow[r] & \Pi_0 \arrow[r] \arrow[d] & \Lambda \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & T_4 \arrow[r] & \Pi \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 1 \\
& & C_2 \oplus C_2 \arrow[d] & C_2 \oplus C_2 \arrow[d] & \\
& & 1 & 1 &
\end{tikzcd} \tag{9}$$

(ii) The quantum mechanical covering groups of (9) are

$$\begin{tikzcd}
& & 1 \arrow[d] & 1 \arrow[d] & \\
& & Z_2 \arrow[d] & Z_2 \arrow[d] & \\
1 \arrow[r] & T_4 \arrow[r] & \overline{\Pi}_0 \arrow[r] \arrow[d] & \mathrm{SL}(2, C) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & T_4 \arrow[r] & \Pi \arrow[r] \arrow[d] & \Lambda \arrow[r] \arrow[d] & 1 \\
& & 1 & 1 &
\end{tikzcd} \tag{10}$$

The group Z_2 here is the first homotopy group of Λ, i.e. $\mathrm{SL}(2, C)/Z_2 = \Lambda$.

Now we shall determine the representations of the extended group. By a choice of equivalent phase factors we can set $U(\Sigma) = I$ (cf. ch. 13). Let $\overset{*}{U}(\Theta)^2 = \varepsilon_\theta$ and $\overset{*}{U}(\Theta)^2 = \varepsilon_{\theta\Sigma}$. It follows from the associative law $\overset{*}{U}\overset{*}{U}{}^2 = \overset{*}{U}{}^2\overset{*}{U}$ that both ε's satisfy $\varepsilon^2 = \pm 1$. Correspondingly, we have four types of representations depending on $\varepsilon_\theta = \pm 1$, $\varepsilon_{\theta\Sigma} = +1$. $\overset{*}{U}(\Theta)$ and $\overset{*}{U}(\Theta\Sigma)$ are then determined up to factor $e^{i\alpha}$. Next, we determine the phases between the products of $U(\Sigma)$, $\overset{*}{U}(\Theta)$ and $\overset{*}{U}(\Theta\Sigma)$, and obtain from the group law the following multiplication table:

	$U(\Sigma)$	$\overset{*}{U}(\Theta)$	$\overset{*}{U}(\Theta\Sigma)$
$U(\Sigma)$	1	$\overset{*}{U}(\Theta\Sigma)$	$\overset{*}{U}(\Theta)$
$\overset{*}{U}(\Theta)$	$\varepsilon_\theta\varepsilon_{\theta\Sigma}\overset{*}{U}(\Theta\Sigma)$	ε_θ	$\varepsilon_\theta U(\Sigma)$
$\overset{*}{U}(\Theta\Sigma)$	$\varepsilon_\theta\varepsilon_{\theta\Sigma}\overset{*}{U}(\Theta)$	$\varepsilon_\theta U(\Sigma)$	$\varepsilon_{\theta\Sigma}$

(11)

This is a multiplication table of a finite group (cf. ch. 7).

The next problem would be to determine the phases between $U(\Sigma)$, $\overset{*}{U}(\Theta)$, $\overset{*}{U}(\Theta\Sigma)$ and the representations $U(a, \Lambda)$ of the restricted group. We write the group law as

$$U(\Sigma)\, U(a, \Lambda)\, U(\Sigma^{-1}) = \omega(a, \Lambda)\, U(\Sigma a, \Sigma\Lambda\Sigma^{-1}).$$

If one considers the product of two such transformations one obtains for the phase the equation

$$\omega\big(a + \Lambda(a)b, AB\big) = \omega(a, A)\omega(a, B). \tag{12}$$

Thus, ω is a one-dimensional representation of the Poincaré group Π. But Π has no invariant subgroup with an abelian factor group, hence $\omega = 1$.

Similarly, the phase ω' between $\overset{*}{U}(\Theta)$ and $\overset{*}{U}(\Theta\Sigma)$ must be one.

Consequently no new possibilities are introduced, and we have the four possible quantum mechanical full Poincaré groups corresponding to eq. (8) depending on the factors in the multiplication table,

$$U(a, L) = \big(U(a, \pm A),\ U(a, \pm A)\,U(\Sigma),\ U(a, \pm A)\overset{*}{U}(\Theta),\ U(a, \pm A)\overset{*}{U}(\Theta\Sigma)\big). \tag{13}$$

Note that, of course, the use of $\pm A \in \mathrm{SL}(2, C)$ in the quantum mechanical representation of $\Lambda \in \mathrm{SO}(3, 1)$ is already a group extension from the mathematical point of view. The Poincaré group can be further extended by other reflection operators, such as C = charge conjugation.

Let $\{|p, \sigma\rangle\}$ be the carrier-space vectors of the representation $U(a, \Lambda)$ of the restricted Poincaré group, where p^2 defines an orbit (cf. 17.2). Consider now the vectors $U(\Sigma)|\tilde{p}, \sigma\rangle$ where $\tilde{p}_\mu = (p_0, -\boldsymbol{p})$, and where we have identified Σ with the parity operator. Both vectors $|p, \xi\rangle$ and $U(\Sigma)|\tilde{p}, \sigma\rangle$ transform in the same way under the restricted Poincaré group. Hence we can define eigenstates of parity by

$$|p, \sigma, \pm\rangle = |p, \sigma\rangle \pm U(\Sigma)|p, \sigma\rangle.$$

If neither of these states vanishes, we have a new quantum number parity, with eigenvalues ± 1, hence a doubling of the representation space. Whether both of these spaces are observable depends on the existence of superselection rules (cf. ch. 13).

C. *Classification of Extensions*

DEFINITION 3. If K is abelian and $K \subset C$ the center of E, then E is called a *central extension*. We have then $C(K) = K$, $J(K) = 1$, $\mathrm{Aut}\,K = A(K)$.

Suppose we are given a homomorphism $\sigma\colon G \to \mathrm{Aut}\,K$; this is frequently described by saying that G acts via σ on K, or that K is a G-module (with respect to σ). We shall investigate the problem of describing group extensions

$$1 \to K \xrightarrow{f} E \xrightarrow{h} G \to 1$$

such that inner automorphisms of E restricted to K coincide with $\sigma \circ h$. Note that there always exists such an extension, namely the semidirect product $E = K \otimes G$ with the multiplication defined by

$$(\alpha, a)(\beta, b) = (\alpha + \sigma_a \beta, ab), \qquad \alpha, \beta \in K,\ a, b \in G, \tag{14}$$

is such extension; it is called a *trivial extension*. Observe that for trivial extensions

$$1 \to K \xrightarrow{f} E \xrightarrow{h} G \to 1$$

there exists a group homomorphism $D\colon G \to E$ such that $h \circ D$ = identity (D is called a *section* or *splitting*). We shall characterize nontrivial extensions by means of non necessarily homomorphic sections, i.e. extensions

$$1 \to K \xrightarrow{f} E \underset{\xleftarrow[D]{\hphantom{xxx}}}{\xrightarrow{\ \ h\ \ }} G \to 1,$$

where $D: G \to E$ satisfies $h \circ D =$ identity. We shall deal exclusively with the case of an abelian K. Consider $f(\alpha)D(a)$, $\alpha \in K$, $a \in G$. For fixed a, varying α we get fibers of the bundle E with base G (i.e. cosets of $f(K)$). For fixed α, varying a, we have sections. In particular, $f(1)D(G) \cong G$. $D(a)D(b)$ and $D(ab)$ are in the same coset (fiber) of $f(K)$, since they are both mapped under the homomorphism h into ab. Hence, they can only differ by a 'phase'

$$D(a)D(b) = \omega(a, b)D(ab). \tag{15}$$

The associativity of the group multiplication law implies

$$\omega(a, b)+\omega(ab, c)-\sigma_a\omega(b, c)-\omega(a, bc) = 0. \tag{16}$$

Such a map ω: $G \times G$ into K (identifying K with $f(K)$) is called a *factor system.* However we wish to disregard differences coming from replacing a map D by another D' such that $D'(a) = \varphi(a)D(a)$, where $\varphi(a) \in K$. D' leads to another factor system ω', such that

$$\omega'(a, b) = \omega(a, b)+\theta(a, b), \tag{17}$$

where $\theta(a, b)$ is given by

$$\theta(a, b) = \varphi(a)+\sigma_a\varphi(b)-\varphi(ab)$$

and is said to be a trivial factor system.

A normalized factor system by the definition satisfies

$$D(1) = 1, \quad \text{or} \quad \omega(a, 1) = \omega(1, b) = \omega(1, 1) = 0. \tag{18}$$

Conversely, given a homomorphism σ: $G \to \mathrm{Aut}K$ and a normalized factor system $\omega(a, b) \in K$, we can construct an extension E of G by K with elements (α, a), $\alpha \in K$, $a \in G$, with the composition law

$$(\alpha, a)(\beta, b) = \big(\alpha+\sigma_a\beta+\omega(a, b), ab\big) \tag{19}$$

where $\omega(a, b)$ satisfies eq. (18). The unit element of E is $(0, 1)$ and the inverse element of (α, a) is

$$(\alpha, a)^{-1} = \big(\sigma_{a^{-1}}\alpha\omega(a^{-1}, a), a^{-1}\big). \tag{20}$$

Two factor systems differing by a trivial factor system provide isomorphic or equivalent extensions. Furthermore, the inner automorphisms on E induces on K the automorphism σ because

$$(\alpha, a)(\alpha', 1)(\alpha, a)^{-1} = (\sigma_a\alpha', 1). \tag{21}$$

The mathematical problem now is to find all normalized factor systems up to a trivial one. The result can be expressed as

PROPOSITION 1. *The two-dimensional cohomology group $H^2(G, K)$ is exactly the group of all (equivalent classes of) those extensions E of G by K which realize the given group action σ of G on K.*

In order to explain this proposition we discuss now the relationship of group extensions to homology and cohomology.

'Homology' of a Lie algebra, or of a group (or of an associative algebra) goes back to the homology of topological spaces (i.e. the connectivity, cf. ch. 2). The homology groups quite generally are described in terms of chains and their boundaries. An n-dimensional chain is a formal linear combination of n-simplices in the space. All n-chains form a free abelian group C_n. The chain complex C is the sequence

$$C_0 \leftarrow C_1 \leftarrow \dots \leftarrow C_{n-1} \overset{\partial_n}{\leftarrow} C_n \overset{\partial_{n+1}}{\leftarrow} C_{n+1} \leftarrow \dots$$

where the boundary homomorphisms ∂ satisfy $\partial\partial = 0$. Its homology group in n-dimension is

$$H_n(C) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}. \tag{22}$$

Now we describe the corresponding cohomology groups. An n-dimensional cochain is a homomorphism f: $C_n \to K$ (abelian group), coboundary operator δ_n is the map δ: $\operatorname{Hom}(C_n, K) \to \operatorname{Hom}(C_{n+1}, K)$ defined (uniquely) by the condition

$$\delta_n f = f\partial.$$

Hence we get the cochain complex

$$\to C^n \equiv \operatorname{Hom}(C_n, K) \overset{\delta_n}{\to} C^{n+1} = \operatorname{Hom}(C_{n+1}, K) \to .$$

Kernels of δ are cocycles (those of ∂, cycles) and Image δ are coboundaries (those of ∂, are boundaries). Consequently, the n-dimensional cohomology of the complex C (or of the underlying space) with coefficients in K is

$$H^n(C, K) = \frac{\operatorname{Ker} \delta_n}{\operatorname{Im} \delta_{n-1}}. \tag{23}$$

Cohomology of groups deals with the action of a group G on an abelian group K:

$$\alpha \to \sigma_a \alpha, \qquad \alpha \in K, \qquad a \in G, \; \sigma: G \to \operatorname{Aut} K$$

and is defined, dimension by dimension, in terms of the following cocycles.

In dimension one, the cocycles are 'crossed homomorphisms' ϕ: $G \to K$ such that $\phi(ab) = \sigma_a \phi(b) + \phi(a)$, for all $a, b \in G$. All such homomorphisms form a group. The 'principal crossed homomorphisms', i.e. coboundaries are $k_\alpha a = \sigma_a \alpha - \alpha$, $a \in G$, for each α. Then we define

$$H^1(G, K) = \{\text{crossed homomorphisms}\}/\{\text{principal crossed homomorphisms}\}. \tag{24}$$

In dimension two: cocycles are defined to be the factor systems introduced in eq. (15), i.e. maps $\theta: G \times G \to K$ such that

$$\sigma_a \theta(b, c) + \theta(a, bc) = \theta(ab, c) + \theta(a, b). \tag{25}$$

All solutions of this equation form a group. The trivial factor systems which satisfy $\theta_\phi(a, b) = \sigma_a \phi(b) - \phi(ab) + \phi(a)$ are declared coboundaries and the two-dimensional cohomology group is defined by

$$H^2(G, K) = \frac{\{\theta: G \times G \to A\}}{\{\theta = \theta_\phi\}},$$

i.e. all solutions of the factor set equation (25) modulo trivial solutions. This proves proposition 1.

Generally we define $\alpha_n: G\times G\times \ldots \times G \to K$ and the sequence of abelian groups

$$C^n(G, K): \{\alpha_n(a_1 \ldots a_n), a_i \in G, \alpha_n \in K | \alpha_n = 0 \text{ if at least one } \alpha_i = 1\}$$

with the homomorphisms (boundary operator)

$$\begin{aligned}\delta_n[\alpha_n(a_1, \ldots, a_n)] &\equiv (\delta_n \alpha_n)(a_1, \ldots, a_{n+1}) \\ &\overset{\text{Def}}{\equiv} a_1\alpha_n(a_2, \ldots, a_{n+1}) + \sum_{k=1}^{n}(-1)^k \alpha_n(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_{n+1}) + \\ &\quad + (n-1)^{n+1}\alpha_n(a_1, \ldots, a_n),\end{aligned}$$

so that n-dimensional cohomology group $H^n(G, K)$ can be defined as above. Indeed, we have immediately $C^0(G, K) = K$, $(\delta_0 \alpha_0)a = a\alpha_0 - \alpha_0$. Let $\sigma_a K = aK = aC^0 \cdot \mathrm{Ker}\, \delta_0 = K^G$ = fixed point of K under G. In this case we have $B^0 = 0$, by definition, consequently $H^0(G, K) = K^G$.

If we look at δ_1 and δ_2 we find that

$$\delta_1[\alpha_1(a)] \equiv (\delta_1 \alpha_1)(a, b) = a\alpha_1(b) - \alpha_1(ab) + \alpha_1(a), \tag{26}$$

$$\delta_2[\alpha_2(a, b)] \equiv (\delta_2 \alpha_2)(a, b, c) = a\alpha_2(b, c) - \alpha_2(ab, c) + \alpha_2(a, bc) - \alpha_2(a, b), \tag{27}$$

so it coincides with the definitions given above.

As a special case the problem of phase representations of a Lie group G becomes simply a group extension of G by a one-dimensional abelian group K (cf. ch. 12), and we have

THEOREM 2 (Bargmann). *Let G be a connected and simply connected Lie group with a trivial second cohomology group $H^2(G, K)$, where K is a one-dimensional abelian group. Then any projective representation of G admits a lifting to a representation of G.*

(For the proof see V. Bargmann 1954, D. T. Simms 1971.)

D. *Examples: Some Further Applications of Group Extensions in Physics*

1) Extensions of U_1: $\{e^{i\theta}, -\pi \leqslant \theta \leqslant \pi\}$.

The only inner automorphism is the identity automorphism; and the only outer automorphism is $e^{i\theta} \to e^{-i\theta}$. If we represent U_1 by $e^{i\theta Q}$ (for example, Q is the charge operator) and the automorphisms by $\varrho e^{i\theta Q}\varrho^{-1} = e^{\pm i\theta Q}$, then

$[\varrho, Q] = 0$, $\varrho^2 = 1$ for inner automorphism,

$[\varrho, Q]_+ = 0$, $\varrho^2 = 1$,

or

$[\varrho, Q]_+ = 0$, $\varrho^2 = e^{i\pi Q} = (-1)^Q\}$ for outer automorphisms.

2) Extension of U_2.

Writing $U_2 = (U_1 \otimes \mathrm{SU}(2))/C_2$, where C_2 has two elements $(1, 1)$ and $(-1, -1)$, we interpret in strong interactions SU(2) with the isospin group, and U_1 with hypercharge Y. The commutation relations of isospin with the ordinary electric charge Q are

$$[I_3, Q] = 0, \quad [I_\pm, Q] = \pm I_\pm.$$

Hence defining $Y = Q - I_3$, we have

$$[I, Y] = 0,$$

thus the direct product $U_1 \otimes \mathrm{SU}(2)$. The representations of U_2 are $D^I(u)e^{i\alpha Y}$ with both elements of C_2 represented by identity. Thus $D^I(-1)e^{i\pi Y} = 1$, or

$$(-1)^{2I} = (-1)^Y.$$

This relation is satisfied empirically for all known particles.

Let now C be the charge conjugation operator which reverses the eigenvalues of Q. We can extend U_2 by C, $C^2 = I$, or more conveniently by $G = Ce^{i\pi I}$ $= G$-parity. We obtain $[G, I] = 0$. This extension corresponds to the automorphism $e^{i\alpha} \to e^{-i\alpha}$, hence we obtain (as in example 1)

$$GY + YG = 0.$$

The operator G is just the parity operator in isospin space, distinguishing axial or polar vectors, or tensor in the isospin space.

3) The physically meaningful representations of the Galilei group (in the space of solutions of the Schrödinger equation) are the projective unitary representations of its universal covering group (cf. ch. 13). Let $g = (b, \boldsymbol{a}, \boldsymbol{v}, R)$ be an element of the Galilei group. The projective representations satisfy

$$U(g')\,U(g) = \omega(g', g)\,U(g'g)$$

and the factor system $\omega(g', g)$ is explicitly given by

$$\omega(g', g) = \exp\left[i\frac{m}{2}\,(\boldsymbol{a}'(R'\boldsymbol{v}) - \boldsymbol{v}'(R\boldsymbol{a}) + b\boldsymbol{v}'(R'\boldsymbol{v})\right].$$

This procedure is equivalent to finding a central extension E of the Galilei group G by a one-dimensional abelian group $K = R$. This 11-parameter group E has the mass operator m as one of its invariants and leads to a superselection rule on mass, as we have noted in ch. 13.4. The exact sequence in this case is

$$1 \to R_1 \to E \to G \to 1.$$

Galilei group G does not satisfy Bargmann's criterion. The projective unitary representations of G are not induced by the unitary representations of G but of its extension E.

§ 5. Space-Time and Internal Symmetries

In ch. 1, § 7, we discussed possible unification of the space-time symmetries and internal symmetries of fundamental particles of physics on the algebraic level. Theorems 1 and 2, there give the limitations for attempts to combine these two types of symmetries within a larger finite Lie algebra. In this section we shall elaborate on the same problem on the group level. The following theorem, corollary and counterexamples describe the scope and results obtained. Finally we shall state how this problem is solved in practice from the physical point of view.

THEOREM (Jost). *Let G be a finite connected Lie group containing the Poincaré group as an analytic subgroup. Let U be a continuous unitary representation of G and P_μ the energy-momentum vector. Let the spectrum of P_μ be contained in $\{0\}\cup\cup V_+$, $V_+ =$ forward cone in the Minkowski space M^4. If the mass operator $M = (P_\mu P^\mu)^{1/2}$ has an isolated eigenvalue $m_1 > 0$, then the corresponding eigenspace H_1 is invariant under G.*

PROOF: Let the spectral resolution of continuous abelian group of translations be

$$e^{-ia^\mu P_\mu} = \int e^{-ia^\mu p_\mu} \mathrm{d}E(p). \tag{1}$$

If the mass operator $M = \int \sqrt{p^2}\,\mathrm{d}E(p)$ has isolated eigenvalues, e.g. $m_1 > 0$ and the rest, its spectrum consists of the mass hyperboloid $M_1 = \{p : p^0 > 0;\ p^2 = m_1^2\}$ and the rest M_2, which are O_M-separated: Two closed sets $M_\alpha \subset M^4$, $\alpha = 1, 2$, are said to be *O_M-separated*, if there are functions $h_\alpha \in O_M$ such that

(i) $0 \leqslant h_\alpha(p) \leqslant 1$,

(ii) $h_\alpha(p) = 1$ if $p \in M_\alpha$,

(iii) $\operatorname{supp} h_1 \cap \operatorname{supp} h_2 = \emptyset$.

Let

$$E_\alpha = \int_{M_\alpha} \mathrm{d}E(p) = \int_{M_\alpha} h_\alpha(p)\mathrm{d}E(p), \qquad \alpha = 1, 2. \tag{2}$$

The theorem is proved if we can show that

$$E_\alpha U_g = U_g E_\alpha \tag{3}$$

for every $g \in G$, for then $U_g \varphi_1 \in H_1$ together with $\varphi_1 \in H_1$. We must show that, because G is connected,

$$E_\alpha e^{itX} = e^{itX} E_\alpha, \qquad t \in R^1, \tag{4}$$

$$X = U(x), \quad x \in L, \text{ the Lie algebra of } G,$$

or,

$$E_1 X\psi = XE_1\psi,$$

$$\psi \in \Delta(X) = \text{the domain of definition of } X. \tag{5}$$

This last assertion follows from

LEMMA. *Let $h_\alpha \in O_M$ and bounded, $\alpha = 1, 2$, and let $\varphi \in D$ (Gårding domain). Then $h_\alpha(p)\varphi \in \Delta(X)$ for every X. Further if $\operatorname{supp} h_1 \cap \operatorname{supp} h_2 = \emptyset$ then*

$$h_1(p)Xh_2(p)\varphi = h_2(p)Xh_1(p)\varphi = 0. \blacktriangledown \tag{6}$$

For then using (2) we get

$$E_1 X E_2 \varphi = E_2 X E_1 \varphi = 0 \tag{7}$$

and, because $E_1+E_2 = I$, $E_1X\varphi = XE_1\varphi$. And if we have a sequence φ_k, $\varphi_k \in D$, for which $\varphi_k \to \psi$, then $X\varphi_k \to X\psi$, also $E_1X\varphi_k \to E_1X\psi$, hence $XE_1\varphi_k \to E_1X\psi$, Consequently, $E_1\psi \in \Delta(X)$ and $E_1X\psi = XE_1\psi$, or $E_1X \subset XE_1$.

Only the proof of the Lemma remains: For $\varphi \in D$ we have from $Adg: x \to gxg^{-1}$

$$U_g XU_{g^{-1}}\varphi = AdgX\varphi. \tag{8}$$

In particular

$$e^{ia^\mu P_\mu}Xe^{-ia^\mu P_\mu}\varphi = C(a)X\varphi \tag{9}$$

where $C(a)$ is a real polynomial in a_μ. (The last statement follows from the fact that $ad(a^\mu P_\mu)$ is nilpotent, i.e. there exists an N such that $[ad(a^\mu P_\mu)]^N = 0$.) Hence

$$Xe^{-ia^\mu P_\mu}\varphi = e^{-ia^\mu P_\mu}C(a)X\varphi. \tag{10}$$

We can apply this equation to finite sums $c_k e^{-ia^\mu_k p_\mu}$ and finally to the limits

$$X\int\tilde{f}(a)e^{-ia^\mu P_\mu}d^4a\varphi = \int\tilde{f}(a)C(a)e^{-ia^\mu P_\mu}d^4aX\varphi,$$

or, using (1) and $f(p) = \int\tilde{f}(a)e^{-ia^\mu P_\mu}d^4a$, $f \in S(R^4)$

$$X\int f(p)dE(p)\varphi = \int(C(\nabla)f)(p)dE(p)X\varphi,$$

where $\nabla = i\dfrac{\partial}{\partial p}$. Or in the operator form

$$Xf(p)\varphi = [C(\nabla)f](p)X\varphi, \quad f \in S(R^4),\ \varphi \in D.$$

If $g \in O_M$ and g is bounded with all its derivatives up to order $N-1$, then

$$Xg(p)\varphi = [C(\nabla)g](p)X\varphi. \tag{11}$$

The Gårding domain D is invariant under U_g: $XD \subset D$. Let

$$||P||^2 = P_0^2+P_1^2+P_2^2+P_3^2 = \int||p||^2dE(p).$$

For every $\varphi_1 \in D$ and $n \in Z_+$, $(1+||P||^2)^n\varphi_1 \in D$, hence

$$Xg(p)(1+||p||^2)^n\varphi_1 = [C(\nabla)g](p)X(1+||p||^2)^n\varphi_1.$$

Finally let $h \in O_M$. There exists $n \in Z_+$ such that

$$g = \frac{h}{(1+||p||^2)^n}$$

is bounded with all its derivatives up to order $N-1$. Hence

$$Xh(p)\varphi_1 = [C(\nabla)g](p)X(1+||p||^2)^n\varphi_1$$

and $\operatorname{supp}h = \operatorname{supp}g$.

Now for $h_\alpha \in O_M$, $\operatorname{supp}h_1 \cap \operatorname{supp}h_2 = \emptyset$, clearly $\operatorname{supp}h_1 \cap \operatorname{supp}C(\nabla)g_2 = \emptyset$. Then $h_1C(\nabla)g_2 = 0$, or

$$h_1(p)Xh_2(p)\varphi = (h_1C(\nabla)g_2)(p)X(1+||p||^2)^n\varphi = 0$$

and the lemma is proved. ▼

COROLLARY. *In an irreducible representation of G there can be no mass-splitting.*

In other words, in an irreducible representation an isolated mass value is the whole spectrum; or, the spectrum must be a connected set.

This still leaves reducible representations, or, non-integrable irreducible representations of the Lie algebra L of G, in which cases a discrete mass spectrum is possible.

EXAMPLE 1 (Flato, Sternheimer). This example is based on the simple observation that the Hamiltonian $H = p^2/2m$ of a non-relativistic free particle has a continuous spectrum when represented as a self-adjoint operator in $L^2(-\infty, \infty)$, but a discrete spectrum when represented as a self-adjoint operator in $L^2(a, b)$, in a box with suitable boundary conditions.

Consider now the su(2, 2)-algebra containing the Poincaré algebra Π (cf. ch. 13, 4(2)). Consider $H = L^2(Q)$, where Q is a 'box' in the Minkowski space, Q: $\{0 \leqslant x_\mu \leqslant a\}$, $x_\mu \in M^4$. The SU(2, 2) generators define hermitian operators in the domain of absolutely continuous functions, with L^2 derivatives, vanishing on the boundary ∂Q; and $-\partial_\mu^2$ is self-adjoint on $D(P_\mu^2)$ of $f \in L^2(Q)$ and $\partial_\mu f$ absolutely continuous with respect to x_μ, $\partial_\mu^2 f \in L^2(Q)$ and with periodic boundary conditions on f and $\partial_\mu f$. The mass operator $\square = -\partial_\mu \partial^\mu$ is defined on the common domain $\bigcap_{\mu=1}^{3} D(P_\mu^2)$, is essentially self-adjoint and has a spectral resolution

$$M^2 = \left(\frac{2\pi}{a}\right)^2 \sum_{m=-\infty}^{m=\infty} mE(m),$$

where $E(m)$ is the projection operator on the subspace H_m by $\exp\left[\left(i\frac{2\pi}{a}\right) n^\mu x_\mu\right]$. This representation is partially integrable on its translation subalgebra, but this domain of integrability does not coincide with the invariant domain of the entire su(2, 2). ▼

Finally, we reformulate th. 1 and corollary 2 of ch. 1, § 7 on the group level in a somewhat stronger form.

LEMMA 2. *Let G be any group and S and $\Pi T^4 \otimes$ SO (3, 1) $= T \oplus \Lambda$ (Poincaré group) subgroups of G such that any $g \in G$ has a unique decomposition into a product $g = s\pi$, $s \in S$, $\pi \in \Pi$. If there is one $\varrho \in \Pi$, $g \notin T$, such that $s^{-1}gs \in \Pi$ for all $s \in S$, then $G = S \otimes \Pi$.*

LEMMA 3. *Let S and Π be subgroups of G such that $G = S\Pi = \Pi S$ and $S \cap \Pi = \{1\}$. If there is one element $s \in S$ such that $\pi s \pi^{-1} \in S$ for all $\pi \in \Pi$, and if no proper invariant subgroup of S contain s, then $G = S \otimes \Pi$.*

(Proofs are straightforward, cf. Michel 1964.)

Solution of the Mass Spectrum Problem

From a physical point of view there are no compelling reasons to consider irreducible representations of finite Lie groups G containing Π. The origin of the idea of larger dynamical groups G goes back to the description of mass spectrum of composite systems, like H-atom, *at rest* by non-compact groups (cf. ch. 12). The relativistic generalization of this idea to *moving systems*, i.e. the inclusion of the momenta P_μ is also possible. An elegant solution of this problem is via the infinite-component wave equations (§ 3) which describe covariantly systems with internal degrees of freedom and provides a *discrete* mass spectrum. Here the irreducible representations of the dynamical group G is used again to generate the states of the system at rest, and the wave equation defines boosted states to a momentum p_μ. This turns out to be the proper relativistic generalization of the dynamical groups, and not a finite Lie group containing G. The algebraic structure of the infinite-component wave equations involves an infinite-parameter Lie algebra hence there is no contradiction with the above theorems.

§ 6. Comments and Supplements

Let us stress again that the general formalism presented in sec. 1 may be applied to the derivation of wave equations covariant with respect to an arbitrary topological group. In particular one may derive wave equations covariant with respect to SO(3), SO(4), SO(4, 2), $T^3 \circledS SO(3)$ or generalized Poincaré group $T^n \circledS SO(n-1, 1)$; in each case wave equations will represent irreducibility conditions with respect to invariant quantities characterizing irreducible representations of a given group like mass and spin irreducibility conditions in the case of the Poincaré group.

We now give for completeness a brief characterization of other conventional finite-component wave equations which were not discussed in sec. 2. They are all special cases of eq. 1 (17) or of Gel'fand–Yaglom equation 3 (1).

A. *Fierz–Pauli Equations*

These equations were postulated by Fierz 1939 and Fierz and Pauli 1939 in the form ($\partial_{\alpha\beta} \equiv (\sigma_\mu \partial^\mu)_{\alpha\beta}$)

$$\partial_{\alpha\dot{\beta}} \varphi^{\alpha,\alpha_1,\dots,\alpha_l}_{\dot{\beta}_1,\dots,\dot{\beta}_k}(x) = \mathrm{i}\varkappa \chi^{\alpha_1,\dots,\alpha_l}_{\dot{\beta},\dot{\beta}_1,\dots,\dot{\beta}_k}(x) \tag{1}$$

and

$$\partial^{\alpha\dot{\beta}} \chi^{\alpha_1,\dots,\alpha_l}_{\dot{\beta},\dot{\beta}_1,\dots,\dot{\beta}_k}(x) = \mathrm{i}\varkappa \varphi^{\alpha,\alpha_1,\dots,\alpha_l}_{\dot{\beta}_1,\dots,\dot{\beta}_k}(x). \tag{2}$$

Here both spinors are symmetric: hence they transform according to the following representation of SL(2, C)

$$D(\Lambda) = D^{[(l+1)/2,\, k/2]} \oplus D^{[l/2,\, (k+1)/2]}. \tag{3}$$

Consequently the highest spin is $j = \frac{1}{2}(l+k+1)$. Thus the Fierz–Pauli equation is characterized by the pair $(D(\Lambda), \pi)$ where $D(\Lambda)$ is given by eq. (3) and π is a projector onto the highest spin. The Fierz–Pauli equation does not admit a parity operator unless $l = k$. Cleary for $l = k = 0$ we obtain the Dirac equation.

B. *Duffin–Kemmer–Petiau Equations*

We noticed that the Gel'fand–Yaglom condition given by th. 3.1 may be also satisfied by a direct sum of finite-dimensional representations. In particular the representations

$$D(\Lambda) = D^{(0,0)} \oplus D^{(1/2,1/2)} \tag{4}$$

or

$$D(\Lambda) = D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1/2,1/2)} \tag{5}$$

satisfy the condition 3 (3).

The corresponding wave equation are

$$(\beta_\mu p^\mu + m)\psi(p) = 0 \tag{6}$$

where in the present case $\{\beta_\mu\}_{\mu=0}^{3}$ is a set of 5×5 or 10×10 matrices, respectively. The explicit form of these matrices can be found using eqs. 3 (6) and 3 (5). Eq. (6) represents the wave equation for spin zero and spin one particles, respectively and exhibit the spin irreducibility condition 1 (17). Eq. (6) with $D(\Lambda)$ given by eqs. (4) or (5) are called the *Duffin–Kemmer–Petiau equations*. They are equivalent to Proca equation: indeed setting

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \psi_9, \psi_{10})$$
$$= \left(-\frac{1}{m}B_{14}, -\frac{1}{m}B_{24}, -\frac{1}{m}B_{34}, -\frac{1}{m}B_{23}, -\frac{1}{m}B_{31}, -\frac{1}{m}B_{12}, \Phi_1, \Phi_2, \Phi_3, \Phi_4\right) \tag{7}$$

we find that Proca equations 2(15) and 2(16)

$$p_\mu \Phi_\nu - p_\nu \Phi_\mu = B_{\mu\nu} \quad \text{and} \quad p^\mu B_{\mu\nu} = m^2 \Phi_\nu \tag{8}$$

can be written in the form of eq. (6).

C. *Bhabba Equations*

Covariant wave equations like (6) in the case when $\psi(p)$ transforms according to a finite-dimensional representation $D(\Lambda)$ of SL(2, *C*) are often called *Bhabba type equations.* The Gel'fand–Yaglom Theorem gives a general method for construction of such equations. A non-trivial example is provided by the following representation $D(\Lambda)$ of SL(2, *C*)

$$D(\Lambda) = (D^{(1/2,0)} \oplus D^{(0,1/2)} \oplus D^{(1,1/2)} \oplus D^{(1/2,1)})(\Lambda) \tag{9}$$

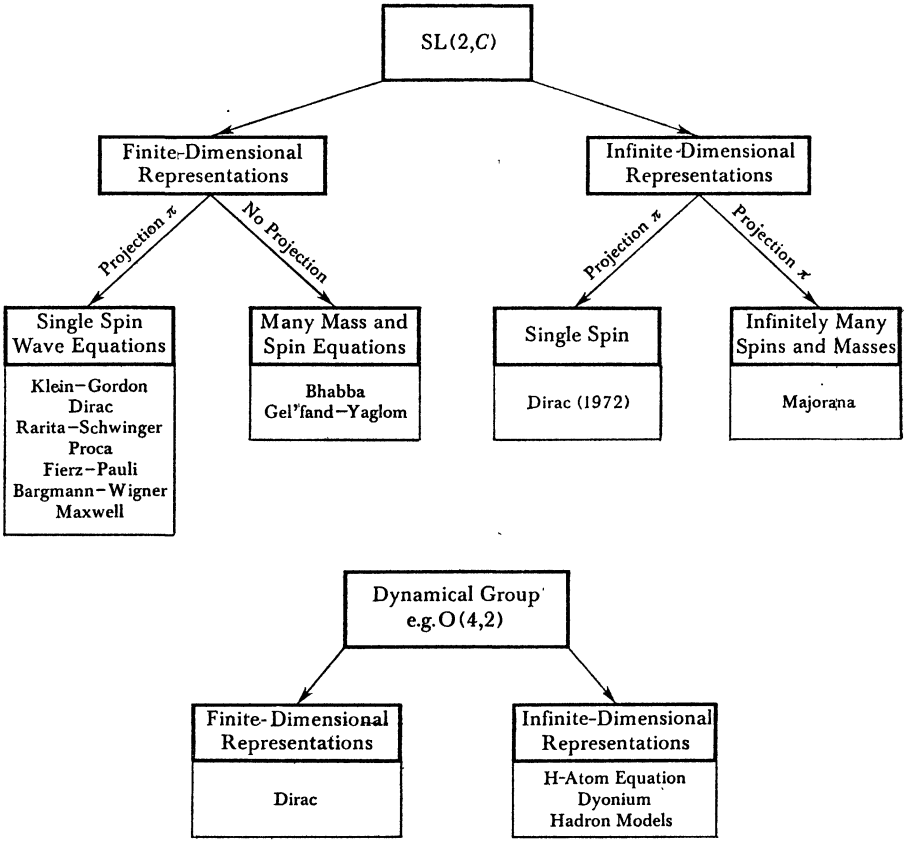
which satisfies conditions 3 (3). Clearly this equation describes particles with spin $\frac{3}{2}$ and $\frac{1}{2}$. Solving eq. 3 (6) for the representation (9) we obtain the matrix β_0. Passing to the rest frame and solving the eigenvalue equation $\beta_0 p_0 \psi = -m\psi$ one finds that the masses of corresponding particles are

$$m_{3/2} = m,\ m_{1/2}^{(1)} = \lambda_1^{-1},\ m_{1/2}^{(2)} = \lambda_2^{-1}, \tag{10}$$

where λ_1 and λ_2 are characteristic roots of the matrix β_0. Hence we have a mass spectrum as in the case of general Gel'fand–Yaglom equations 3 (1). This is a typical situation for general Bhabba-like equations, they contain a finite number of masses and spins.

Table 1 shows the group structure of different relativistic wave equations.

Table 1



D. *Applications of the Imaginary Mass ($m^2 < 0$) Representations of the Poincaré Group*

Although the representations of the Poincaré group with $m^2 < 0$ cannot be interpreted physically as isolated quantum systems with measurable mass, they occur in relativistic two-particle systems as follows.

Consider the scattering process of two particles a and b producing two same or other particles c and d:

$$a+b \rightarrow c+d. \tag{11}$$

The energy-momentum for vectors satisfy the conservation law

$$p_a+p_b = p_c+p_d. \tag{12}$$

For spinless particles the amplitudes for the scattering process depends only on the invariant products of the momenta which can be taken as, e.g.

$$s = (p_a+p_b)^2 \quad \text{and} \quad t = (p_a-p_c)^2. \tag{13}$$

For a fixed value of s the states of the two-particle system (a, b) belong to irreducible representations of the Poincaré group, the generators of which being $P^\mu = P_a^\mu + P_b^\mu$ and $J^{\mu\nu} = J_a^{\mu\nu} + J_b^{\mu\nu}$.

In the physical region of the process (11),

$$s > (m_a+m_b)^2 > 0, \tag{14}$$

and this is the usual representation of the Poincaré group in the tensor product space of the two particles.

On the other hand for each fixed value of t, we may consider another Poincaré group with generators

$$P'^\mu = P_a^\mu - P_c^\mu \quad \text{and} \quad J^{\mu\nu} = J_a^{\mu\nu} + J_c^{\mu\nu}, \tag{15}$$

so that we have, again in the physical region of the process (1) representations of the Poincaré group, but with $p'_\mu p'^\mu = (p_a^\mu - p_c^\mu)^2 < 0$. The significance of these remarks can be seen from the partial-wave expansions (i.e. harmonic analysis) of the scattering amplitude.

For the reaction (11) the scattering amplitude can be written as a matrix element

$$T^{S_cS_dS_aS_b}_{\lambda_c\lambda_d\lambda_a\lambda_b}(p) = \langle m_c S_c p_c \lambda_c; m_d S_d p_d \lambda_d | T | m_a S_a p_a \lambda_a, m_b S_b p_b \lambda_b \rangle, \tag{16}$$

where m, S, p, λ denote mass, spin, momentum and helicity of each particle. The relativistic invariance implies that these matrix elements transform according to the tensor products of representations of the Poincaré group to which the particles belong, denoted by $U(c)^*$, $U(d)^*$, $U(a)$ and $U(b)$ symbolically.

We can decompose the product $U(a) \otimes U(b)$ and similarly the product $U(c)^* \otimes \otimes U(d)^*$, into the direct integral of representations of the Poincaré group. We then obtain an expansion of the T-matrix (16) in terms of the two-particle states (a, b), or (c, d). Physically these are the states formed by the particles a and b, for example, resonances or compound states of a and b. The relevant Poincaré algebra then is (14). In particular, if we use in the expansion, a basis consisting of P^2, J^2, J_3 for (14), we obtain the so-called partial wave (or direct channel) expansion of the T-matrix, a sum over all possible total angular momenta of the intermediate states formed by a and b which then decay into the particles c and d.

We can also decompose the direct products

$$U(a)\, U(c)^* \quad \text{or} \quad U(a)\, U(d)^* \tag{17}$$

into the direct integrals of the representations of the Poincaré group. However, in this case, we see immediately that we must use the Poincaré group generated by (15). The little group in this case is $O(2, 1)$, so we are led to $m^2 = t < 0$— representations of the Poincaré group. Physically, the momenta $p_a - p_c$ in (15) correspond to the momentum of the 'exchange particle' (Fig. 1.B), and the expansion is called a crossed channel expansion of the T-matrix in terms of the possible two-particles states in the crossed channel.

These expansions gain a further significance due to the analytic continuation properties of the T-matrix elements in momenta, or, analytic properties of the representations of the Poincaré group. If we continue analytically p_c into $-p_c'$ and p_b into $-p_b'$, eq. (12) goes into

$$p_a+p_c' \to p_b'+p_d \tag{18}$$

and describes the physical momentum balance in the reaction

$$a+c \to b+d. \tag{19}$$

Looking at Fig. 1.B, we see that the 'exchanged' particles for the process (1) are just the two particle-states of the reaction (19). It is assumed in particle physics that a single analytic function describes both processes (11) and (19)—and others—in their respective physical region of momenta satisfying (12) or (18).

There exist a great many other forms of expansion of the T-matrix depending on which homogeneous or symmetric spaces we represent T.

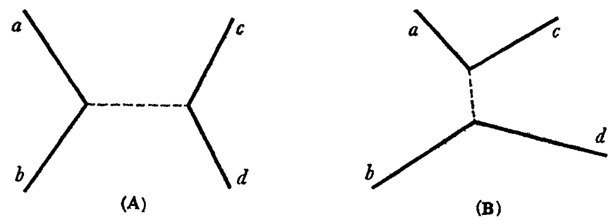


Fig. 1

E. *Bibliographical Remarks*

The relativistic wave equation for a massive spin $\frac{1}{2}$ particle was proposed by Dirac in 1928. At that time only tensor representations of the Lorentz group were known and the question arose if the Dirac equation can be covariant at all, as it does not transform according to any tensor representations. A clarification of this problem gave rise to a new class of representations of the Lorentz group, i.e., spinor representations. Next it was observed that Dirac equation admits negative energy solutions which represented a difficulty in the physical interpretation of the equation. In order to remove this difficulty Majorana proposed in 1932 the infinite-component wave equation which does not contain the negative energy states. It was however realized soon that the Majorana equation has also solutions $\psi(p)$ with $p^2 = m^2 \leqslant 0$.

It was later recognized that negative energy states of the Dirac equation have a natural interpretation in quantum field theory as antiparticles and in 1931 antiparticles were discovered experimentally. The wave equation for massless spin $\frac{1}{2}$ particle was first proposed by Weyl in 1932. This equation was discussed by

Pauli in 1933, but it was rejected because of its noninvariance under space inversions. When it was verified experimentally that parity is violated in reactions involving neutrinos, Landau 1957, Salam 1957 and Lee and Yang proposed the Weyl equation for the description of neutrino states.

The wave equation for a massive spin 1 particle was proposed by Proca in 1936 and by Petiau in his thesis also in 1936. The general aspects of these equations were discussed by Duffin 1938 and Kemmer 1939.

The general wave equations (1) and (2) for particles with arbitrary spin were discussed first by Fierz 1939 and by Fierz and Pauli 1939 in the language of spinors. The alternative formulation in the language of spin-tensors was given by Rarita and Schwinger 1941. The interesting description of tensor particles in terms of symmetric tensor wave functions was given by Bargmann and Wigner in 1946.

A relatively simple form of covariant wave equations for massive and massless particles with arbitrary spin was proposed by Joos 1962, Barut, Muzinich and Williams 1963, and Weinberg 1964.

The derivation of the finite-component relativistic wave equations presented in this chapter is based on the theory of induced representations and enables us to give a unified derivation of all finite-component covariant wave equations starting from the single equation 1(17) (cf. also Niederer and O'Raifeartaigh 1974).

All these equations concern massive particles with a specific mass. Bhabba proposed in 1945 an analogue of the Dirac equation for the description of a system with several masses and spins (see also 1949). A general theory of such equations was developed by Gel'fand and Yaglom in 1947, which included also infinite-component wave equations of Majorana type. Gel'fand–Yaglom equations provide in general a mass spectrum which is decreasing with increasing spin. Various other equations have been recently given and discussed by Nambu 1967, Fronsdal 1968, Abers, Grodsky and Norton 1967, Barut *et al.* 1967 and others. These equations modify the Gel'fand–Yaglom form in order to obtain an increasing mass spectrum.

A detailed discussion of Bhabba like equations and some other covariant wave equations is given in Umezawa's book 1956.

The application of infinite-component wave equations for the description of properties of realistic quantum systems was developed mainly by Barut and collaborators. Using this approach combined with SO(4, 2)-symmetry of quantum systems they succeeded to obtain not only the spectrum of hydrogen-like atom, but also they were able to take into account the recoil correction, to find forms factors and to describe other characteristic features of quantum systems (see Barut and Kleinert 1967 and Barut *et al.* 1968–75). The application of this approach for a description of the internal structure of hadrons was also successful (cf. Barut 1970).

§ 7. Exercises

§ 1.1. Can one derive the general Poincaré invariant wave equation 1(17) from the general wave equation 1(12).

Hint: The Poincaré group has no other finite-dimensional representations beside those of SL (2, C) lifted to Π (cf. example 8.7.1).

§ 2.1. Derive the wave equation for a spin 1 massive particle using the representation $D^{(1,0)} \oplus D^{(0,1)}$ of SL(2, C).

Hint: Follow the derivation of Dirac equation.

§ 2.2. 'Maxwell Group': consider the wave equation of a particle in a fixed given external field. For example the Schrödinger equation

$$\mathrm{i}\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m}(p-\mathrm{e}A)^2+\mathrm{e}A_0\right]\psi$$

or the Dirac equation

$$[\gamma^\mu(p_\mu-eA_\mu)-m]\psi = 0.$$

The symmetry groups of these equations are now only subgroups of the Galilei group, or Poincaré group, respectively. (Or a subgroup of the conformal group if either m is transformed, or $m = 0$.) Because of gauge invariance, this subgroup is given by those transformations which leave $F_{\mu\nu} = A_{\nu,\mu}-A_{\mu,\nu}$ invariant. In particular, determine the symmetry group in the case of constant $F_{\mu\nu}$ and study the induced projective representations of this group. (cf. H. Bacry, Ph. Combe and J. L. Richard 1970; J. L. Richard 1972; R. Schrader 1972). If the fields are also transformed, we regain the full invariance. Cf. also the following problem.

§ 3.1 Discuss the light-like solutions of the Majorana equation.

§ 4.1. Study the extension of the space-time group Π (Poincaré group or Galilei group) by the electromagnetic gauge group K, i.e. abelian group of all real-valued functions $\in C^\nu$ or C^∞, on the Minkowski space M^4 (cf. U. Cattaneo and A. Tanner 1974 and references therein).

§ 4.2. Show that the group of automorphisms of the Poincaré group Π is

$$\mathrm{Aut}\Pi = \Pi \circledS (C_2 \times C_2 \times D),$$

where D is the multiplicative group of real numbers $\varrho < 0$ occurring in $(a, \Lambda) \to (\varrho a, \Lambda)$, and $C_2 \times C_2$ is the group generated by space and time reflecting. What is the group of automorphisms of the Galilei group (compare exercise 1.10.1.11).

Appendix A

Algebra, Topology, Measure and Integration Theory

1. *Zorn's Lemma*

A relation R in a set X is called *reflexive* if xRx for $\forall x \in X$, *symmetric* if xRy implies yRx, *antisymmetric* if xRy and yRx implies $x = y$ for $\forall x, y \in R$, and *transitive* if xRy and yRz implies xRz for $\forall x, y, z \in X$.

A relation R on a set X is called an *equivalence relation* if it is reflexive, symmetric and transitive.

A relation R on a set X is called a *partial ordering* if it is reflexive, transitive and antisymmetric. If R is a partial ordering we write $x \prec y$ instead of xRy.

If for all x, y in X either $x \prec y$ or $y \prec x$, X is said to be *linearly ordered.*

An element $u \in X$ is called an *upper bound* for a subset $Y \subset X$ if $y \prec u$ for all $y \in Y$. An element $m \in X$ is called a maximal element of X if $m \prec x$ implies $x = m$.

ZORN'S LEMMA. *Every nonempty partially ordered set X with the property that every linearly ordered subset has an upper bound in X possesses a maximal element.*▼

2. *Rings*

PROPOSITION 1. *Let R be a ring. If grR is a (left) Noether ring without zero-divisors then R is also (left) Noether ring without zero-divisors.*

(For the proof cf. Chow (1969), proposition 13.)

PROPOSITION 2. *A (left) Noether ring without zero-divisors satisfies the (left) Ore condition.*

(For the proof cf. Chow 1969, proposition 14.)

3. *Semigroups*

A semigroup G is a nonempty set S and a mapping $(x, y) \to xy$ of $G \times G$ into G such that

$$x(yz) = (xy)z \quad \text{for all } x, y, z \text{ in } G.$$

Let X be any nonempty set. A *free* semigroup S is the set of all finite formal products

$$x_1 \cdot x_2 \dots x_n,$$

where $x_1, x_2, \ldots, x_n \in X$, $n \geqslant 1$ with the multiplication law

$$(x_1 \cdot x_2 \ldots x_a)(y_1 y_2 \ldots y_a) = x_1 \cdot x_2 \ldots x_n \cdot y_1 \cdot y_2 \ldots y_n.$$

4. *Principle of Uniform Boundedness*

A complete metric vector space over a field K is called an *F-space* if

(i) $d(x, y) = d(x-y, 0)$ for all $x, y \in X$,

(ii) the map $(\lambda, x) \to \lambda x$ of $K \times X$ into X is continuous with respect to λ for every x and continuous with respect to x for eve.y λ.

PRINCIPLE OF UNIFORM BOUNDEDNESS. *Let, for every element a of a set A, T_a be a continuous linear map of an F-space X into an F-space Y. If for every $x \in X$ the set $\{T_a x | a \in A\}$ is bounded then $\lim\limits_{x \to 0} T_a x = 0$ uniformly relative to $a \in A$.* ▼

(For the proof see Dunford and Schwartz 1958, ch. I.)

5. *Measures and Integration*

A Borel structure on a set X is a family B of subsets of X such that

(i) $\varnothing, X \in B$.

(ii) If $X_j \in B$, then $X - X_j$, $\bigcap\limits_{j=1}^{\infty} X_j$ and $\bigcup\limits_{j=1}^{\infty} X_j$ are in B.

The pair (X, B) is called a *Borel space*. Elements of B are called *Borel subsets* of X.

Every set X has at least one Borel structure. In fact if F is any family of subsets of a set X then there exists a smallest Borel structure for X which contains F.

If X and Y are Borel spaces then the map F: $X \to Y$ is called a *Borel map* if $f^{-1}(Y_i)$ is a Borel subset of X for every Borel subset Y_i of Y. A Borel map is called a *Borel isomorphism* if f is one-to-one and onto and if f^{-1} is a Borel map.

A Borel space is called *standard* if it is Borel isomorphic to the Borel space associated with Borel subsets of a complete separable metric space.

A Borel measure is a map μ: $B \to [0, \infty)$ such that

(i) $\mu(X_i) \geqslant 0$ for $X_i \in B$, $\mu(\varnothing) = 0$.

(ii) If $X_i \in X$ are disjoint elements of B then

$$\mu(\bigcup_i X_i) = \sum_i \mu(X_i).$$

(iii) There exists $X_k \in B$, $k = 1, 2, \ldots$ such that

$$\mu(X_j) < \infty \quad \text{and} \quad X = \bigcup_j X_j.$$

A Borel measure on X is called *standard* if there exists a Borel subset N such that $\mu(N) = 0$ and $X - N$ is a standard Borel space.

A real-valued nonnegative function on a Borel space X is called *measurable* if for any $a > 0$ the set $\{x\colon f(x) > a\}$ is measurable. A complex-valued function is measurable if it may be written in the form $f = f_1 - f_2 + i(f_3 - f_4)$ with f_i real, nonnegative and measurable. A function obtained from measurable functions by means of algebraic operations or passage to the limit is measurable.

The limit of a Lebesgue sum relative to the measure μ is called the *integral* $\int\limits_X f(x)\,d\mu(x)$ of f. If $\int\limits_X |f(x)|\,d\mu(x)$ is finite then f is called *integrable.*

Given a measure μ on a measure space X we can define a set of integrable functions on X and we can form $L^p(X, \mu)$ spaces as in the case when $X = R$.

A measure ν is said to be *absolutely continuous with* respect to the measure μ if and only if $\mu(A) = 0$ implies $\nu(A) = 0$. We have

RADON–NIKODYM THEOREM. *A measure ν is absolutely continuous with respect to μ if and only if there is a measurable function ϱ such that*

$$\nu(A) = \int \varrho(x)\,\chi_A\,d\mu(x)$$

for any measurable set A, where χ_A is the characteristic function of the set A. ▼

One often uses this theorem in the differential form

$$\frac{d\nu}{d\mu}(x) = \varrho(x).$$

6. *Lebesgue Theorems*

DOMINATED CONVERGENCE THORÈM. *If $\{u_n\}_1^\infty$ is a sequence of integrable functions, convergent almost everywhere to a function u and if $|u_n(x)| \leqslant v(x)$ almost everywhere, where v is integrable, then the function u is also integrable and*

$$\int\limits_X u(x)\,d\mu(x) = \lim_{n\to\infty} \int\limits_X u_n(x)\,d\mu(x). \;\blacktriangledown$$

The assumptions of the last theorem are satisfied in particular, when $\mu(X) < \infty$ and $|u_n(x)| \leqslant M$ almost everywhere, and M is a constant. In this case we obtain

BOUNDED CONVERGENCE THEOREM. *If $\{u_n\}_\infty^1$ is a nondecreasing sequence of integrable functions, convergent to a function u almost everwyhere then*

$$\lim_{n\to\infty} \int\limits_X u_n(x)\,d\mu(x) = \int\limits_X u(x)\,d\mu(x). \;\blacktriangledown$$

Let X_1 and X_2 be two locally compact spaces with measures μ_1 and μ_2, respectively. Then there exists on the space $X = X_1 \times X_2$ a measure ν such that

$$\nu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$$

for every measurable sets $A_i \subset X_i$. The measure ν is called the *tensor product* of measures μ_1 and μ_2 and is denoted by $\mu_1 \otimes \mu_2$.

7. FUBINI THEOREM

Version I. *If f: $X_1 \times X_2 \to R$ (or C) is $\mu_1 \otimes \mu_2$ integrable then the integrals*

$$I_{12} = \int_{X_1} d\mu_1(x_1) \int_{X_2} f(x_1, x_2) d\mu_2(x_2)$$

and

$$I_{21} = \int_{X_2} d\mu_2(x_2) \int_{X_1} f(x_1, x_2) d\mu_1(x_1)$$

exist and

$$\int_{X_1 \times X_2} f(x_1, x_2) d(\mu_1 \otimes \mu_2) = I_{12} = I_{21}.$$

Version II. *If f is $\mu_1 \otimes \mu_2$ measurable and $f \geqslant 0$ then all three above integral exists (finite or infinite) and are equal.* ▼

The Fubini Theorem implies the following

THEOREM ON INTEGRATION OF SERIES. *Let X be a Borel space with a Borel measure μ. Let $\{f_n(x)\}_1^\infty$ be a sequence of μ-measurable functions. If*

$$\int_X \sum_{n=1}^{\infty} |f_n| d\mu(x) \quad or \quad \sum_{n=1}^{\infty} \int |f_n(x)| d\mu(x)$$

exists, then

$$\int_X \sum_{n=1}^{\infty} f_n(x) d\mu(x) = \sum_{n=1}^{\infty} \int f_n(x) d\mu(x). \ ▼$$

8. *Various Results*

FRECHET AND RIESZ THEOREM. *To every linear continuous functional L in a Hilbert space H there correspond a vector $l \in H$ such that*

$$L(u) = (u, l) \quad for\ all\ u \in H. \ ▼$$

(For the proof see Maurin 1969, ch. I.)

Appendix B

Functional Analysis

§ 1. Closed, Symmetric and Self-Adjoint Operators in Hilbert Space

We begin with some basic properties of operators in the Hilbert space.

An operator A with the domain $D(A) \subset H$ is said to be *continuous* at a point $u_0 \big(u_0 \in D(A)\big)$, if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$||u-u_0|| \leqslant \delta, \; u \in D(A) \Rightarrow ||Au-Au_0|| < \varepsilon.$$

An operator A is said to be *bounded* if there exists a constant C such that

$$||Au|| \leqslant C||u|| \quad \text{for all } u \in D(A).$$

A bounded, linear operator is (uniformly) continuous. Indeed,

$$||Au-Au_0|| = ||A(u-u_0)|| \leqslant C||u-u_0||. \tag{1a}$$

Hence, if $||u-u_0|| \to 0$, then $||Au-Au_0|| \to 0$, i.e., $Au \to Au_0$. Conversely, if a linear operator A is continuous at a point u_0 (e.g., $u_0 = 0$), then A is bounded. In fact, let $||Au-Au_0|| < \varepsilon$ for $||u-u_0|| < \delta(\varepsilon)$. Then $||Av|| < \varepsilon$ for $||v|| \leqslant \delta(\varepsilon)$ by linearity of A. Because for every $w \in H$ we have $\left\|\frac{w}{||w||}\delta(\varepsilon)\right\| = \delta(\varepsilon)$, then

$$||Aw|| = \left\|A\frac{w}{||w||}\delta(\varepsilon)\right\|\frac{||w||}{\delta(\varepsilon)} < \varepsilon\frac{||w||}{\delta(\varepsilon)}. \tag{1b}$$

Setting $C = \varepsilon/\delta(\varepsilon)$, we obtain

$$||Aw|| < C||w||$$

for every $w \in D(A)$. Hence A is bounded. We see therefore that for linear operators in Hilbert space the continuity and boundedness are equivalent properties.

An operator A is said to be *positive definite*, if

$$(Au, u) \geqslant m||u||^2$$

for some $m > 0$ and all $u \in D(A)$, and is said to be *positive* if $(Au, u) \geqslant 0$ for all $u \in D(A)$. An operator A is said to be *closed* if from

$$u_m \in D(A), \quad \lim u_m = u, \quad \lim Au_m = v$$

it follows that

$$u \in D(A) \quad \text{and} \quad v = Au.$$

Notice the essential difference between continuous and closed operators: if A is continuous, then the existence of a limit

$$\lim_{n\to\infty} u_n \to u \in D(A)$$

implies the existence of a limit $\lim_{n\to\infty} Au_n$; on the other hand, if A is only closed, then from the convergence of a sequence

$$u_1, u_2, \ldots, u_m \in D(A), \tag{2}$$

it does not follow that the sequence

$$Au_1, Au_2, \ldots \tag{3}$$

is also convergent.

If A is not closed we define the *closure* $\bar{A}$ of A to be an operator with the following properties.

1° The domain $D(\bar{A})$ of $\bar{A}$ consists of all vectors $u \in H$ for which there exists at least one sequence (2) generating a convergent sequence (3).

2° The action of $\bar{A}$ is defined by the equality

$$\bar{A}u = \lim_{n\to\infty} Au_n, \quad u \in D(\bar{A}).$$

It follows from the definition of the closure that an operator A admits a closure $\bar{A}$, if and only if, from the relations

$$u_n \to 0, \qquad Au_n \to v$$

it follows that $v = 0$.

Let A be a linear operator in H (bounded or not). There exist pairs $v, v' \in H$ such that the equality

$$(Au, v) = (u, v') \tag{4}$$

is satisfied for every $u \in D(A)$. In fact, equality (4) is satisfied at least for $v = v' = 0$. It seems natural to set $v' = A^*v$ and call the operator A^* the *adjoint* of A. However, this definition of the adjoint operator A^* will be meaningful only if an element v' is uniquely determined by v. This condition will be satisfied if and only if $D(A)$ is dense in H. Indeed, if $D(A)$ is not dense in H and $w \neq 0$ is orthogonal to $D(A)$, then for every $u \in D(A)$ together with (4), we have

$$(Au, v) = (u, v'+w),$$

i.e., the symbol A^*v is meaningless. Conversely, if $D(A)$ is dense in H and if for any $u \in D(A)$

$$(Au, v) = (u, v_1'),$$

$$(Au, v) = (u, v_2'),$$

then for an arbitrary $u \in D(A)$ we have

$$(u, v_1'-v_2') = 0,$$

which is impossible if $v_1' \neq v_2'$. Therefore, if $D(A)$ is dense in H, then an operator A has the adjoint operator A^*: the domain $D(A^*)$ of A^* is the set of all $v \in H$, for which there exists $v' \in H$ satisfying eq. (4) for an arbitrary $u \in D(A)$. The action of A^* on $D(A^*)$ is given by the formula

$$A^*v = v'. \tag{5}$$

Notice that $v \in D(A^*)$ if and only if (Au, v) is a linear continuous functional on $D(A)$. The adjoint operator A^* has a series of interesting properties:

LEMMA 1. 1° *The operator A^* is linear.*

2° *The operator A^* is closed even if A is not closed.*

3° *If the operator A has a closure $\bar{A}$, then $(\bar{A})^* = A^*$.*

All of these properties directly follow from the definition of the adjoint operator A^*. Moreover, it can be proved that

$$A^{**} = \bar{A} \tag{6}$$

(cf. e.g. Stone 1932b, th. 2.9).

Let A, B be operators in H with $D(B) \supset D(A)$ and let

$$Bu = Au \quad \text{for } u \in D(A),$$

then the operator B is said to be an *extension* of A, and A is said to be a *reduction* of B. We shall write $B \supset A$. In particular the closure $\bar{A}$ is an extension of A. Note that

$$A \subset B \Rightarrow A^* \supset B^*. \tag{7}$$

Indeed, if $v \in D(B^*)$, then there exists $v' \in H$ such that $(Bu, v) = (u, v')$ for all $u \in D(B)$. If $A \subset B$, then also $(Au, v) = (u, v')$ for all $u \in D(A)$, because $Au = Bu$. Therefore $v' \in D(A^*)$ and $A^*v = v' = B^*v$. Hence $A^* \supset B^*$.

An operator A is said to be *symmetric* if $A^* \supset A$. In other words, a densely defined operator A is symmetric if

$$(Au, v) = (u, Av) \quad \text{for all } u, v \in D(A). \tag{8}$$

An operator A is said to be *self-adjoint* if $A^* = A$. Because A^* is closed (cf. lemma 1), then a self-adjoint operator is closed.

LEMMA 2. *Let A be a linear symmetric operator and $D(A) = H$. Then*

1° $A^* = A$.

2° *A is a bounded operator.*

PROOF: Ad 1°. Because $A^* \supset A$, then $D(A^*) \supset D(A) = H$. Hence $D(A^*) = H$ and consequently $A^* = A$.

Ad 2°. Because A is symmetric we have

$$(Au, v) = (u, Av), \quad u, v \in H.$$

Let $u_n \to u_0$ and $Au_n \to v_0$. For any $v \in H$, we have

$$(v_0, v) = \lim(Au_n, v) = \lim(u_n, Av)$$
$$= (u_0, Av) = (Au_0, v).$$

Hence $Au_0 = v_0$, i.e., A is continuous and therefore bounded. ▼

This lemma indicates that an unbounded self-adjoint operator cannot be defined on the whole Hilbert space H. Hence we might define them only on some dense domain in H. The selection of a proper dense domain for an unbounded operator is one of the most difficult problems of functional analysis and consequently also of quantum physics.

We shall now discuss the properties of symmetric operators and self-adjoint extensions of operators. We have

LEMMA 3. *An operator A having a symmetric extension $\tilde{A}$ is itself symmetric. Every symmetric extension $\tilde{A}$ of an operator A is a reduction of the operator A^*.*

PROOF: By assumption $\tilde{A} \supset A$ and $\tilde{A} \subset \tilde{A}^*$. Then $\tilde{A}^* \subset A^*$, by eq. (7), and consequently

$$A \subset \tilde{A} \subset \tilde{A}^* \subset A^*. \tag{9}$$

We also readily verify, using the definition of closed and symmetric operators and the properties of their domains that the closure $\bar{A}$ of a symmetric operator A is a symmetric operator, i.e.

$$(\bar{A})^* \supset \bar{A}. \quad \blacktriangledown \tag{10}$$

We know that only self-adjoint operators are proper candidates for physical observables. On the other hand, symmetric operators most often occur in quantum physics. Hence, the important question: when does a symmetric operator admit a self-adjoint extension? This problem can be solved with the help of the concept of *deficiency indices.*

Let A be a linear densely defined operator in a Hilbert space H, and let A^* be the adjoint of A. Set

$$D_+ = \{u \in D(A^*) : A^*u = iu\}, \qquad D_- = \{u \in D(A^*): A^*u = -iu\}. \tag{11}$$

The spaces D_+ and D_- are said to be the spaces of positive and negative deficiency of the operator A, respectively. Their dimensions (finite of infinite numbers) denoted by n_+ and n_-, respectively, are called *deficiency indices* of the operator A.

THEOREM 4. *Let A be a symmetric operator. Then*

1° *A has a self-adjoint extension if and only if $n_+ = n_-$.*

2° *If $n_+ = n_- = 0$, then the unique self-adjoint extension of A is its closure $\bar{A} = A^*$.*

(For the proof of the theorem cf. Dunford and Schwartz, vol. II, ch. 12, sec. 4.)

This theorem solves completely the problem of the existence of a self-adjoint extensions of a symmetric operator.

Let us note that with a given physical symmetric operator one can associate many self-adjoint extensions, and consequently many physical observables. It is evident that the selection of the proper representative for a physical observable must be a physical problem. In fact, the different self-adjoint extensions might have (in the same space H) different eigenvalues and different complete sets of orthogonal eigenfunctions; for example, consider the operator $d_\vartheta = i^{-1}\dfrac{d}{d\varphi}$ in $L^2(0, 2\pi)$ whose eigenfunctions u

$$d_\vartheta u = \lambda u$$

satisfy the boundary condition

$$u(2\pi) = \vartheta u(0), \qquad \vartheta = \exp(i\omega), \qquad \omega = \text{const} \in R.$$

The solutions have the form

$$u_n(\varphi) = \exp\left[i\left(n+\frac{\omega}{2\pi}\right)\varphi\right], \qquad n = 0, \pm 1, \pm 2, \dots \tag{12}$$

Hence, the eigenvalues of the operator d_ϑ are numbers $\lambda_n = n+\dfrac{\omega}{2\pi}$. It follows from the spectral theorem that every set of functions (12) for a fixed $\vartheta\{u_n\}$ forms a complete orthonormal set of functions in $L^2(0, 2\pi)$. Note that in quantum mechanics we select the extension d_ϑ with $\vartheta = 1$ by the requirement of the uniqueness of the wave function. This requirement is not however universal, because, for example, for half-integer spins we admit two-valued wave functions and some authors even consider infinite valued wave functions.

§ 2. Integration of Vector and Operator Functions

Let $u(t)$ be a function defined on the interval $[a, b] \subset R$ with values in a Hilbert space H. The Riemann integral of the function $u(t)$ is defined in the same manner as the Riemann integral of ordinary functions.

A subdivision Δ^i of the interval $[a, b]$ is a system of n_i points

$$a = t_0^{(i)} < t_{n_1}^{(i)} < \dots < t_{n_i}^{(i)} = b. \tag{1}$$

A sequence $\{\Delta^i\}$ of subdivisions is said to be normal if

$$\lim_{i\to\infty} \sup_{1\leqslant k\leqslant n_i} |t_k^{(i)} - t_{k+1}^{(i)}| = 0. \tag{2}$$

Set

$$S(u(t), \Delta^i, t_k^i) = \sum_{k=1}^{n} u(t_k)(t_k^{(i)} - t_{k-1}^{(i)}), \tag{3}$$

where t_k is an arbitrary point satisfying the inequalities $t_{k-1}^{(i)} \leqslant t_k \leqslant t_k^{(i)}$. If the limit

$$\lim_{i\to\infty} S(u(t), \Delta^i, t_k^i) \tag{4}$$

exists for an arbitrary normal sequence of subdivisions and for an arbitrary choice of points t_k, then this limit is called the *Riemann integral* of the function $u(t)$ and is denoted by

$$\int\limits_{[a,b]} u(t)\mathrm{d}t. \tag{5}$$

One can show, as in the case of ordinary functions, that this limit does not depend on the choice of the normal sequences of subdivisions or on the choice of the properties of t_k.

The Riemann integral (5) of a vector function $u(t)$ has all the properties of Riemann integrals of ordinary functions. In particular, we have

LEMMA 1. *A vector function $u(t)$ in H, continuous on the interval $[a, b] \subset R$, is integrable on $[a, b]$.*

PROOF: A bounded interval $[a, b]$ of R is compact. We know that a continuous map $t \to u(t)$ of compact set is uniformly continuous. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$||u(t_1| - u(t_2)|| < \varepsilon \quad \text{if } |t_1 - t_2| < \delta. \tag{6}$$

Take a subdivision Δ^i of the interval $[a, b]$ with a diameter smaller than δ. For every Δ^l, $l > i$, such that Δ^i is a subdivision of Δ^l, we have by eq. (6)

$$\begin{aligned} &||S(u(t), \Delta^l, \tau_k^l) - S(u(t), \Delta^i, t_k^i)|| \\ &= \Big\|\sum_{k=1}^{n_l} u(\tau_k)(\tau_k^{(l)} - \tau_{k-1}^{(l)}) - \sum_{k=1}^{n_l} u(t_k)(t_k^{(i)} - t_{k-1}^{(i)})\Big\| \\ &= \Big\|\sum_{k=1}^{n_l} (u(\tau_k) - u(t_k')(\tau_k^{(l)} - \tau_{k-1}^{(l)}))\Big\| \leqslant |b-a|\varepsilon, \end{aligned}$$

where $|\tau_k - t_k'| < \delta$. Consequently, for any subdivisions $\Delta^l, \Delta^s, l, s > i$, such that Δ^i is a subdivision of Δ^s and Δ^l, we have

$$\begin{aligned} &||S(u(t), \Delta^l, \tau_k^l) - S(u(t), \Delta^s, \tau_k^s)|| \\ &\quad \leqslant ||S(u(t), \Delta^l, \tau_k^l) - S(u(t), \Delta^i, t_k^i)|| + ||S(u(t), \Delta^s, \tau_k^s) - S(u(t), \Delta^i, t_k^i)|| \\ &\quad \leqslant 2|b-a|\varepsilon. \end{aligned}$$

Hence, a sequence $\Delta^i \to S(u(t), \Delta^i, \tau_k^i)$ is a Cauchy sequence. The completeness of H implies the existence of an element v in H such that

$$\lim_{i\to\infty} S(u(t), \Delta^i, \tau_k^i) = v. \;\blacktriangledown$$

LEMMA 2. *The Riemann integral of a vector function $u(t)$, $t \in [a, b] \subset R$, has the following properties:*

1° *Linearity: if $u_1(t)$ and $u_2(t)$ are integrable on $[a, b]$, then for $\alpha, \beta \in R$ the vector function $\alpha u_1(t) + \beta u_2(t)$ is integrable and*

$$\int_{[a,b]} (\alpha u_1(t)+\beta u_2(t))\mathrm{d}t = \alpha \int_{[a,b]} u_1(t)\mathrm{d}t+\beta \int_{[a,b]} u_2(t)\mathrm{d}t. \tag{7}$$

2° *If $u(t)$ is continuous on $[a, b]$, then $\|u(t)\|$ is integrable and*

$$\left\| \int_{[a,b]} u(t)\mathrm{d}t \right\| \leqslant \int_{[a,b]} \|u(t)\|\mathrm{d}t \leqslant C|a-b|, \tag{8}$$

where

$$C = \sup_{t\in[a,b]} \|u(t)\|.$$

3° *If A is a bounded linear operator in H and $u(t)$ is integrable on $[a, b] \subset R^1$, then $Au(t)$ is also integrable and*

$$\int_{[a,b]} Au(t)\mathrm{d}t = A \int_{[a,b]} u(t)\mathrm{d}t. \tag{9}$$

PROOF: The property 1° follows directly from the definition of the Riemann integral of a vector function. The first step in inequalities (8) follows also from the definition of the integral. The second step follows from the fact that the map $u \to \|u\|$ is continuous. Hence, the numerical continuous function $\|u(t)\|$ is bounded in the bounded interval $[a, b]$.

Ad 3°. A bounded linear operator is continuous by eq. 1 (1a). Hence, by eqs. (4) and (3) we have

$$\begin{aligned} A\cdot \int_{[a,b]} u(t)\mathrm{d}t &= A\lim_{i\to\infty} S(u(t), \Delta^{[i]}, t_k) \\ &= \lim_{i\to\infty} S(Au(t), \Delta^{(i)}, t_k) = \int_{[a,b]} Au(t)\mathrm{d}t. \blacktriangledown \end{aligned} \tag{10}$$

Remark: Let D be a closed bounded domain of R^n and $u(t), t \in D$, a vector function on D with values in H. Then, in a similar manner as for ordinary functions one readily shows that all previous results remain valid for $u(t), t \in D$. In particular,

$$\left\| \int_D u(t)\mathrm{d}t \right\| \leqslant \int_D \|u(t)\|\mathrm{d}t \leqslant CV_D, \tag{11}$$

where $C = \sup_{t\in D}\|u(t)\|$ and V_D is the volume of D.

Let A_t be a strongly continuous operator function in a closed bounded domain $D \subset R^n$. By definition, the vector function $u(t) = A_t u$ is continuous for any u in H. Set

$$\tilde{A}u = \int_D A_t u \mathrm{d}t. \tag{12}$$

Clearly, the operator $\tilde{A}$ is linear. It is also bounded. In fact, because the map

$\Phi: A \to ||A||$ is continuous, the numerical function $||A_t||$ is continuous in the closed, bounded domain D. Therefore, $\sup\limits_{t\in D}||A_t|| = C < \infty$. Hence,

$$||\tilde{A}u|| \leqslant \int ||A_t u|| \mathrm{d}t \leqslant ||u|| \int_D ||A_t|| \mathrm{d}t \leqslant CV_D ||u||. \tag{13}$$

A bounded, linear operator $\tilde{A}$ is said to be the *integral* of the operator function A_t over the domain D. We denote it as

$$\tilde{A} = \int_D A_t \mathrm{d}t. \tag{14}$$

We have by formula (12)

$$\left(\int_D A_t \mathrm{d}t\right) u = \int_D A_t u \mathrm{d}t \tag{15}$$

for any u in H. Moreover, by eq. (13), we have

$$\left\|\int_D A_t \mathrm{d}t\right\| \leqslant CV_D. \tag{16}$$

If B is a bounded operator, then by eq. (9) we have

$$B\int A_t \mathrm{d}t = \int BA_t \mathrm{d}t. \tag{17}$$

Similarly, one can define improper integrals over unbounded domains of strongly continuous operator function.

EXAMPLE. Let $G = \mathrm{SO}(2)$ and $H = L^2(G)$. Let T be the right regular representation of G in H, i.e., $T_x u(\varphi) = u(\varphi+x)$. The operator function

$$A_t = \frac{1}{2\pi}\exp(-\mathrm{i}mt)T_t, \quad t\in[0,2\pi], \quad m\in Z \tag{18}$$

is continuous because T is continuous. Hence, the integral

$$\tilde{A} = \frac{1}{2\pi}\int_G \exp(-\mathrm{i}mt)T_t \mathrm{d}t \tag{19}$$

is well-defined. The action of $\tilde{A}$ on any element $u(\varphi)$ in H gives the mth Fourier component of u. In fact,

$$\tilde{A}u = \frac{1}{2\pi}\int_G \exp(-\mathrm{i}mt)T_t u(\varphi)\mathrm{d}t = \frac{1}{2\pi}\int_G \exp(-\mathrm{i}mt)u(\varphi+t)\mathrm{d}t$$

$$= \frac{1}{2\pi}\exp(\mathrm{i}m\varphi)\int_G \exp(-\mathrm{i}m\psi)u(\psi)\mathrm{d}\psi = \exp(\mathrm{i}m\varphi)\hat{u}(m).$$

Similarly, all projection operators

$$P^\lambda_{ij} = \frac{\dim T^\lambda}{\mathrm{vol}\, G}\int_G \overline{D}^\lambda_{ij}(x)T_x \mathrm{d}x$$

considered in ch. 7, § 3, are integrals over G of continuous operator functions of the form

$$A_x = \bar{D}^{\lambda}_{ij}(x) T_x,$$

where $x \in G$ and $D^{\lambda}_{ij}(x)$ are the matrix elements of the irreducible representation T^{λ} of G which are continuous on G. If $G = R^1$, then the operator function

$$A_t = \exp(-\mathrm{i}\,\lambda t) T_t \tag{20}$$

is still continuous in G. However, the integral

$$\tilde{A} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-\mathrm{i}\,\lambda t) T_t \, \mathrm{d}t \tag{21}$$

does not provide an operator in H because the numerical function $\alpha(t) = \|A_t\| = 1$ is not integrable on the real line. In fact the quantity (21) represents an operator-valued distribution (cf. ch. 15.4).

Notice, however, that the unboundedness of the path of integration is not an obstacle in the construction of the integrable operator functions. Indeed, the operator function

$$A_t = \exp(-t^2) T_t$$

is continuous and $\alpha(t) = \|A_t\| = \exp(-t^2)$. Hence, the integral

$$\tilde{A} = \int_{-\infty}^{\infty} \exp(-t^2) T_t \, \mathrm{d}t \tag{22}$$

is well defined. The analytic vectors for a group representation are constructed by means of operators of the form (22) (cf. ch. 11, § 4).

§ 3. Spectral Theory of Operators

A. *Spectral Theorem*

The theory of the spectral decomposition of self-adjoint operators was developed mainly by Hilbert and v. Neumann. It provides extremely useful tools for the elaboration of representations of Lie groups and Lie algebras.

Let $[a, b]$ be a finite or infinite interval of the real line R. An operator function $E(\lambda)$, $\lambda \in [a, b]$, is said to be a *resolution of the identity* (or a spectral function), if it satisfies the following conditions:

$$\begin{aligned} &1^\circ\ E(\lambda)^* = E(\lambda). \\ &2^\circ\ E(\lambda)E(\mu) = E(\min(\lambda, \mu)). \\ &3^\circ\ E(\lambda+0) = E(\lambda). \\ &4^\circ\ E(-\infty) = 0,\ E(\infty) = I. \end{aligned} \tag{1}$$

Conditions 1° and 2° mean that $E(\lambda)$, $\lambda \in [a, b]$, are bounded hermitian operators of orthogonal projections (cf. ch. 7, § 3). For an interval $\Delta = [\lambda_1, \lambda_2] \subset [a, b]$

we shall denote the difference $E(\lambda_2)-E(\lambda_1)$ by $E(\Delta)$. If Δ_1 and Δ_2 are two such intervals, then, by condition 2°, we have

$$E(\Delta_1)E(\Delta_2) = E(\Delta_1 \cap \Delta_2). \tag{2}$$

In particular, if Δ_1 and Δ_2 have no common points, then

$$E(\Delta_1)E(\Delta_2) = 0, \tag{3}$$

i.e., the subspaces $H_1 = E(\Delta_1)H$ and $H_2 = E(\Delta_2)H$ are orthogonal.

Condition 3° means that the operator function $E(\lambda)$ is strongly right continuous. The convergence in 4° is meant in the strong sense, i.e.,

$$\lim_{\lambda\to-\infty} E(\lambda)u = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} E(\lambda)u = u \tag{4}$$

for every u in H.

EXAMPLE 1. Let G be the translation group of the real line R and $H = L^2(R^1)$. Let T be the right regular representation of G, i.e., $T_a u(x) = u(x+a)$. The operator function

$$E(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\lambda} \mathrm{d}\lambda \int_{-\infty}^{\infty} \exp(-\mathrm{i}\,\lambda a)T_a \mathrm{d}a \tag{5}$$

represents the resolution of the identity. We now directly verify the properties 1°–4° of eq. (1). For any $u(x)$ in H we have

$$E(\lambda)u(x) = \frac{1}{2\pi}\int_{-\infty}^{\lambda} \mathrm{d}\lambda \int_{-\infty}^{\infty} \exp(-\mathrm{i}\lambda a)u(x+a)\mathrm{d}a$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\lambda} \mathrm{d}\lambda \int_{-\infty}^{\infty} \exp[-\mathrm{i}\lambda(y-x)]u(y)\mathrm{d}y$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} \exp(\mathrm{i}\lambda x)\hat{u}(\lambda)\mathrm{d}\lambda. \tag{6}$$

Here

$$\hat{u}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-\mathrm{i}\,\lambda y)u(y)\mathrm{d}y \tag{7}$$

is the Fourier transform of the element u in H. Thus,

$$E(-\infty)u = 0 \quad \text{and} \quad E(\infty)u = u$$

for any u in H. Consequently, $E_{-\infty} = 0$ and $E_\infty = I$.

Ad 3°. By eq. (6) for any u in H we have

$$\lim_{\Delta\lambda\to 0} E(\lambda+\Delta\lambda)u = E(\lambda)u+\lim_{\Delta\lambda\to 0}\frac{1}{\sqrt{2\pi}}\int_{\lambda}^{\lambda+\Delta\lambda}\exp(\mathrm{i}\lambda x)\hat{u}(\lambda)\,\mathrm{d}\lambda = E(\lambda)u.$$

Hence, $E(\lambda+0) = E(\lambda)$.

Ad 2°. For any u in $L^2(R)$ we have

$$E(\lambda)E(\mu)u = \frac{1}{(2\pi)^2}\int_{-\infty}^{\lambda}\mathrm{d}\lambda\int_{-\infty}^{\infty}\exp(-\mathrm{i}\lambda a)\,T_a\mathrm{d}a\int_{-\infty}^{\mu}\mathrm{d}\mu\int_{-\infty}^{\infty}\exp(-\mathrm{i}\mu b)\,T_b u(x)\,\mathrm{d}b.$$

We can take the bounded operator T_a under the last integral sign. Then, $T_aT_bu(x) = T_{a+b}u(x) = u(x+a+b)$. Setting $y = x+a+b$, we obtain

$$\begin{aligned}
&E(\lambda)E(\mu)u\\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\lambda}\mathrm{d}\lambda\int_{-\infty}^{\infty}\exp(-\mathrm{i}\lambda a)\mathrm{d}a\int_{-\infty}^{\mu}\exp[\mathrm{i}\mu(x+a)]\mathrm{d}\mu\int_{-\infty}^{\infty}\exp(-\mathrm{i}\mu y)u(y)\,\mathrm{d}y\\
&= \frac{1}{2\pi}\int_{-\infty}^{\lambda}\mathrm{d}\lambda\int_{-\infty}^{\mu}\exp(\mathrm{i}\mu x)\,\delta(\lambda-\mu)\hat{u}(\mu)\,\mathrm{d}\mu\\
&= \int_{-\infty}^{\min(\lambda,\mu)}\exp(\mathrm{i}\mu x)\hat{u}(\mu)\,\mathrm{d}\mu = E(\min(\lambda,\mu))u.
\end{aligned}\tag{8}$$

The interchange of integrals relative to 'a' and 'μ' is justified by Fubini theorem (cf. app. A). Utilizing eq. (5) we have

$$E^*(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\lambda}\mathrm{d}\lambda\int_{-\infty}^{\infty}\exp(\mathrm{i}\lambda a)\,T_a^*\mathrm{d}a = \frac{1}{2\pi}\int_{-\infty}^{\lambda}\mathrm{d}\lambda\int_{-\infty}^{\infty}\exp(-\mathrm{i}\lambda b)\,T_b\mathrm{d}b = E(\lambda).$$

Here we have used the fact that T is unitary, i.e., $T_a^* = T_{a^{-1}} = T_{-a}$, and put $b = -a$. ▼

It follows from the formula (1) that, for any u in H, the function

$$\sigma_u(\lambda) \equiv (E(\lambda)u, u)\tag{9}$$

is a right continuous, non-decreasing function of bounded variation for which

$$\sigma_u(-\infty) = 0, \qquad \sigma_u(\infty) = \|u\|^2.$$

In fact, for $\mu < \lambda$,

$$(E(\mu)u, u) = \|E(\mu)u\|^2 = \|E(\mu)E(\lambda)u\|^2 \leqslant \|E(\lambda)u\|^2 = (E(\lambda)u, u).\tag{10}$$

The function $\sigma_u(\lambda)$ defines another function $\sigma_u(\Delta) = (E(\Delta)u, u)$ which is determined for any interval $\Delta \subset [a, b]$ and can be extended to all Borel subsets of $[a, b]$. The function $\sigma_u(\Delta)$ is positive by eq. (10). Moreover, it is denumerable additive. In fact, if

$$\Delta = \bigcup_{n=1}^{\infty} \Delta_n \quad \text{and} \quad \Delta_n \cap \Delta_m = 0 \quad \text{for } n \neq m,$$

then

$$E(\Delta) = \sum_{n=1}^{\infty} E(\Delta_n) \quad \text{and} \quad \sigma_u(\Delta) = \sum_{n=1}^{\infty} \sigma_u(\Delta_n). \tag{11}$$

The function $\sigma_u(\Delta)$ is called the *spectral measure.*

Let A be a self-adjoint operator in a Hilbert space H. We now state the fundamental theorem in spectral decomposition theory:

THEOREM 1. *Every self-adjoint operator in H has the representation*

$$A = \int_{-\infty}^{\infty} \lambda \mathrm{d}E(\lambda), \tag{12}$$

where $E(\lambda)$ is a spectral family, which is uniquely determined by the operator A. ▼

Because this is a classical result of functional analysis, we shall not give the proof here (cf., e.g., Maurin 1967, ch. 6). In (12) neither the domain of integration nor the operator function $\lambda \mathrm{d}E(\lambda)$ are bounded; it is therefore necessary to make precise the meaning of an integral of this type. The domain $D(A)$ of the operator A consists of all vectors u for which

$$(Au, Au) = \int_{-\infty}^{\infty} \lambda^2 \mathrm{d}\sigma_u(\lambda) < \infty. \tag{13}$$

For u in $D(A)$ the operator (12) is defined by the formula

$$Au = \int_{-\infty}^{\infty} \lambda \mathrm{d}\big(E(\lambda)u\big), \tag{14}$$

where this equality is understood in the weak sense, i.e.,

$$(Au, v) = \int_{-\infty}^{\infty} \lambda \mathrm{d}\big(E(\lambda)u, v\big)$$

for any v in H.

COROLLARY 1. *If Δ is any interval in $[a, b]$, then*

$$E(\Delta)A = AE(\Delta) = \int_{\Delta} \lambda \mathrm{d}E(\lambda), \tag{15}$$

i.e., the spectral function $E(\Delta)$ is permutable with A.

PROOF: By virtue of eq. (12)

$$E(\Delta)A = \int_{-\infty}^{\infty} \lambda E(\Delta) \mathrm{d}E(\lambda).$$

By eq. (2), $E(\Delta)\mathrm{d}E(\lambda) = 0$, if $\lambda \notin \Delta$ and $E(\Delta)\mathrm{d}E(\lambda) = \mathrm{d}E(\lambda)$ if $\lambda \in \Delta$. Therefore

$$E(\Delta)A = \int_{\Delta} \lambda \mathrm{d}E_{\lambda}$$

Similarly,

$$AE(\Delta) = \int_{\Delta} \lambda \mathrm{d}E(\lambda)$$

and consequently eq. (15) follows. ▼

Notice, that if $u \in H_{\Delta} \equiv E(\Delta)H$, where $\Delta = [\lambda, \mu]$ then, by eq. (15),

$$||Au - \lambda u|| \leqslant (\mu - \lambda)||u||. \tag{16}$$

Hence, if $\mu - \lambda$ is small, then u is 'almost an eigenvector' of the operator A. If

$$\bigcup_{i=1}^{\infty} \Delta_i = [a, b]$$

and $\Delta_i \cap \Delta_k = 0$ for $i \neq k$, then the space H can be represented as the orthogonal direct sum of subspaces $H_{\Delta_n} = E(\Delta_n)H$ in which the operator A 'almost' reduces to the multiplication operator.

Clearly, if A has only continuous spectrum, then all eigenvectors are outside the Hilbert space.

COROLLARY 2. *The set* $D = \{E(\Delta_i)u\}$, *where* Δ_i *runs over all finite intervals of* $[a, b]$ *and* u *runs over* H, *is dense in* H. *All powers* A^n, $n = 1, 2, \ldots,$ *are defined on* D.

PROOF: Let $\{\tilde{\Delta}_i\}_{i=1}^{\infty}$ be a collection of finite intervals such that

$$\bigcup_{i=1}^{\infty} \tilde{\Delta}_i = [a, b] \quad \text{and} \quad \tilde{\Delta}_i \cap \tilde{\Delta}_k = 0 \quad \text{for } i \neq k. \tag{17}$$

Then, for any v in H we have:

$$v = \lim_{N\to\infty} \sum_{i=1}^{N} E(\tilde{\Delta}_i)v.$$

Hence, the set $\{E(\tilde{\Delta}_i u)\}$, $u \in H$, is dense in H. Now, for any u in H, by eq. (15) we have

$$v = A^n E(\Delta)u = A^{n-1} \int_{\Delta} \lambda \mathrm{d}E(\lambda)u = A^{n-2} \int_{\Delta} \lambda^2 \mathrm{d}E(\lambda)u = \int_{\Delta} \lambda^n \mathrm{d}E(\lambda)u. \tag{18}$$

Since

$$||v||^2 \leqslant \int_{\Delta} |\lambda|^{2n} \, ||\mathrm{d}E(\lambda)u||^2 \leqslant \max_{\lambda\in\Delta} |\lambda|^{2n} |\Delta| \, ||u||^2, \tag{19}$$

we obtain that any vector $E(\Delta_i)u$ lies in the domain of A^n. ▼

In a given Hilbert space we have many resolutions of the identity associated with various self-adjoint operators. In applications, it is useful to know when a resolution of the identity $E(\lambda)$ can be associated with a self-adjoint operator A in H. This problem is solved by the following theorem.

THEOREM 2. *A resolution of the identity $E(\lambda)$ represents the spectral function of an operator A if and only if*

1° *$E(\Delta)$ reduces A for any interval $\Delta \subset [-\infty, \infty]$.*

2° *The condition $u \in (E(\lambda)-E(\mu))H$, $-\infty \leqslant \mu < \lambda \leqslant \infty$ implies the inequality*

$$\mu\|u\|^2 \leqslant (Au, u) \leqslant \lambda\|u\|^2. \tag{20}$$

(For the proof cf., e.g., Akhiezer and Glatzman 1966, § 75.)

EXAMPLE 2. Let $H = L^2(R)$; let $A = \dfrac{1}{\mathrm{i}}\dfrac{\mathrm{d}}{\mathrm{d}x}$ and let $D(A)$ consist of all functions $u(x)$ in $L^2(R)$ such that

1° $u(x)$ is absolutely continuous in every finite interval

$$\Delta \subset [-\infty, \infty],$$

2° $u'(x) = \dfrac{\mathrm{d}u}{\mathrm{d}x} \in L^2(R^1)$.

The functions u in $D(A)$ automatically satisfy the boundary condition i.e.,

$$\lim_{x\to-\infty} u(x) = \lim_{x\to\infty} u(x) = 0.$$

We readily verify that A is self-adjoint on $D(A)$.

We now show that the resolution of the identity

$$E(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\lambda} \mathrm{d}\lambda \int_{-\infty}^{\infty} \exp(-\mathrm{i}\lambda a)T_a \mathrm{d}a$$

given by eq. (5) is the spectral function for A. In fact, since $T_a u(x) = u(x+a) = \exp[a(\mathrm{d}/\mathrm{d}x)]u(x)$, the operators A and T_a commute. Hence, $E(\Delta)$ reduces A for any interval $\Delta \subset [-\infty, \infty]$. Moreover, by eq. (6)

$$(AE(\Delta)u, u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \lambda\exp(\mathrm{i}\lambda x)\hat{u}(\lambda)u(x)\mathrm{d}\lambda\mathrm{d}x$$

$$= \int_{\Delta} \lambda u(\lambda)\overline{u(\lambda)}\mathrm{d}\lambda \leqslant \max_{\lambda\in\Delta} \lambda\|u\|^2.$$

Hence, the condition 2° of th. 2 is also satisfied. Consequently $E(\lambda)$ is the spectral function of A. ▼

B. *Spectral Theory of Compact and Hilbert–Schmidt Operators*

A linear operator X: $H_1 \to H_2$ from a Hilbert space H_1 into a Hilbert space H_2 is said to be compact if it maps the unit ball in H_1 into a precompact set in H_2.

RELLICH–HILBERT–SCHMIDT THEOREM. *Let X be a linear bounded self-adjoint (or normal) operator in a Hilbert space H. Then X is compact if and only if the following conditions hold:*

(i) $X = \sum_{k=1}^{\infty} \lambda_k E_k$, *where E_k are mutually orthogonal projections on finite-dimensional subspaces $H_k = E_k H$ and $|\lambda_k| \to 0$ as $k \to \infty$,*

(ii) *for every $u_k \in H_k$ we have $Xu_k = \lambda_k u_k$,*

(iii) $H = \oplus H_k + H_0$ *where $H_0 = X^{-1}(0)$.* ▼

(For the proof cf. Maurin 1968, ch. I.)

We now introduce the important class of the so called Hilbert–Schmidt operators. Let H be a Hilbert space and let $\{e_i\}_1^\infty$ be an orthonormal basis in H. A linear bounded operator X in H is said to be a *Hilbert–Schmidt operator* if

$$|X| \equiv \sqrt{\sum_i \|Xe_i\|^2} < \infty .$$

HILBERT–SCHMIDT THEOREM. *A linear bounded self-adjoint operator X is Hilbert–Schmidt if and only if*

$$X = \sum_{k=1}^{\infty} \lambda_k E_k ,$$

where the projections E_k are of finite dimensions and

$$\Sigma_k (|\lambda_k|^2 \dim E_k) < \infty . ▼$$

(For the proof cf. Maurin 1968, ch. I.)

C. *Nuclear Variant of the Spectral Theorem*

In order to obtain a convenient interpretation of the eigenvectors in the case of continuous spectra we present in this section the so-called nuclear variant of the spectral theorem (cf. Maurin 1968, ch. II). We begin with the basic facts concerning nuclear spectral theory. We first recall the concept of direct integral of Hilbert spaces.

Let (Λ, μ) be a space with a measure. Consider a family $\{H(\lambda)\}_{\lambda \in \Lambda}$ of Hilbert spaces $H(\lambda)$, each equipped with a scalar product $(\cdot, \cdot)_\lambda$, and form the Cartesian product

$$\bigtimes_{\lambda \in \Lambda} H(\lambda).$$

We shall call elements $u, v \in \bigtimes_{\lambda \in \Lambda} H(\lambda)$ *vector fields*. We shall call $\Gamma = \{\overset{i}{e}\}_{i \in J}$ a *fundamental family of μ-measurable vector fields* if

1° for arbitrary $i, j \in J$ the function $\Lambda \to \lambda \to (\overset{i}{e}(\lambda), \overset{j}{e}(\lambda))_\lambda \in C$ is μ-measurable,

2° for every $\lambda \in \Lambda$, vectors $\{\overset{i}{e}(\lambda)\}_{i \in J}$ span the space $H(\lambda)$.

A field $u \in \bigtimes_{\lambda \in \Lambda} H(\lambda)$ is said to be *measurable* if all functions $\lambda \to (u(\lambda), \overset{i}{e}(\lambda))_\lambda$ are μ-measurable. It is evident that μ-measurable fields form a vector subspace of the space $\bigtimes_{\lambda \in \Lambda} H(\lambda)$.

A measurable field u is called *square integrable* if

$$\int \|u(\lambda)\|_\lambda^2 \,\mathrm{d}\mu(\lambda) < \infty.$$

Two measurable fields are said to be equivalent if they are equal μ-almost everywhere on Λ.

DEFINITION 1. The *direct integral* H of Hilbert spaces $H(\lambda)$ is the space of equivalence classes of measurable and integrable vector fields $\{u(\lambda)\}$ equipped with the scalar product

$$(u, v) = \int (u(\lambda), v(\lambda))_{H(\lambda)} \mathrm{d}\mu(\lambda). \blacktriangledown$$

Using the same arguments as in the proof of the classical Riesz–Fischer theorem one shows that H is complete. Consequently H is a Hilbert space: we shall denote it by the symbol

$$H = \int_\Lambda H(\lambda) \mathrm{d}\mu(\lambda).$$

EXAMPLES. 3′. Let $\Lambda = N$—the set of natural numbers and let $\mu(n) = 1$; then

$$H = \int_\Lambda H(\lambda) \mathrm{d}\mu(\lambda) = \bigoplus_{\lambda \in N} H(\lambda),$$

i.e. in this case the direct integral reduces to a direct sum.

3″. Let $H(\lambda) = \hat{H}$, for all $\lambda \in \Lambda$, where $\hat{H}$ is a Hilbert space; then

$$H = \int \hat{H}(\lambda) \mathrm{d}\mu(\lambda) = L^2(\Lambda, \mu, \hat{H}),$$

in particular if $\hat{H} = C$, then $H = L^2(\mu)$. $\blacktriangledown$

We give now a useful lemma implied by the measure desintegration theorem 4.3.2 and the concept of direct integral of Hilbert spaces. Let $X, Y, r, \mu, \tilde{\mu}$ and μ_y be such as in the formulation of th. 4.3.2. Then we have:

LEMMA 3. *Set* $H = L^2(X, \mu; \tilde{H})$, $H(y) = L^2(X, \mu_y, \tilde{H})$, $\tilde{H}$ *a Hilbert space. Then we can equip the field* $y \to H(y)$ *with the structure of* μ*-measurable fields of Hilbert spaces such that*

$$H = \int_Y H(y) \mathrm{d}\tilde{\mu}(y). \blacktriangledown$$

(For the proof cf. Mackey 1952, § 12.)

DIAGONAL AND DECOMPOSABLE OPERATORS

Let t be a measurable essentially bounded function, $t \in L^\infty(\Lambda, \mu)$ and let $I(\lambda)$ be the identity operator in $H(\lambda)$. The operator field

$$\lambda \to t(\lambda) I(\lambda) \in L(H(\lambda), H(\lambda)),$$

is called the *diagonal operator in the Hilbert space* $H = \int H(\lambda)\,\mathrm{d}\mu(\lambda)$. The diagonal operator field $\{t(\lambda) I(\lambda)\}$ defines an operator T in H by the formula

$$(Tu)(\lambda) = t(\lambda) u(\lambda), \quad u \in H.$$

One readily verifies that $||T|| = ||t||_\infty$.

An operator field $T(\cdot)\colon \Lambda \ni \lambda \to T(\lambda) \in L(H(\lambda), H(\lambda))$ is said to be *measurable* if all functions $\lambda \to (T(\lambda)\overset{i}{e}(\lambda), \overset{j}{e}(\lambda))_\lambda$, where $\{\overset{i}{e}\}$ is the fundamental family of vector fields, are measurable.

If $u(\cdot)$ and $T(\cdot)$ are measurable, then $\lambda \to T(\lambda)u(\lambda) \in H(\lambda)$ is a measurable vector field. Indeed since $(\overset{i}{e}(\lambda), T(\lambda)\overset{j}{e}(\lambda))_\lambda = (T^*(\lambda)\overset{i}{e}(\lambda), \overset{j}{e}(\lambda))_\lambda$, the vector field $\lambda \to T^*(\lambda)\overset{i}{e}(\lambda)$ is measurable. Hence $(T(\lambda)u(\lambda), \overset{i}{e}(\lambda))_\lambda = (u(\lambda), T^*\overset{i}{e}(\lambda))_\lambda$ is measurable. This implies that $\lambda \to T(\lambda)u(\lambda)$ is measurable.

We now introduce an important concept of a decomposable operator. Take $T(\lambda) \in L(H(\lambda), H(\lambda))$ such that the numerical function $||T(\cdot)|| = (\lambda \to ||T(\lambda)||_\lambda) \in L^\infty(\Lambda, \mu)$. Set $N = \operatorname*{ess\,sup}_{\lambda\in\Lambda} ||T(\lambda)||_\lambda$. The vector field $\lambda \to T(\lambda)u(\lambda)$, for every $u(\cdot) \in H$ is measurable and we have $||T(\lambda)u(\lambda)||_\lambda = N||u(\lambda)||_\lambda$, almost everywhere.

Consequently

$$\int ||T(\lambda)u(\lambda)||_\lambda^2 \,\mathrm{d}\mu(\lambda) \leqslant N^2 \int ||u(\lambda)||_\lambda^2 \,\mathrm{d}\mu(\lambda) = N^2 ||u||^2.$$

Denoting the vector field $\{(Tu)(\lambda)\}$ by Tu we have $Tu \in H$ and $||Tu|| \leqslant N||u||$. Thus $T = \{T(\lambda)\}$ represents a bounded operator in H. This operator is called a *decomposable operator* and is denoted by the symbol

$$T = \int_\Lambda T(\lambda)\,\mathrm{d}\mu(\lambda).$$

Clearly $||T|| = \operatorname*{ess\,sup}_{\lambda\in\Lambda} ||T(\lambda)||_\lambda = N$.

One readily verifies that the decomposable operators have the following properties:

$$\int (T(\lambda) + U(\lambda))\,\mathrm{d}\mu(\lambda) = \int T(\lambda)\,\mathrm{d}\mu(\lambda) + \int U(\lambda)\,\mathrm{d}\mu(\lambda),$$

$$\int T^*(\lambda)\,\mathrm{d}\mu(\lambda) = \left(\int T(\lambda)\,\mathrm{d}\mu(\lambda)\right)^*,$$

$$\int T(\lambda)\,\mathrm{d}\mu(\lambda) \int U(\lambda)\,\mathrm{d}\mu(\lambda) = \int T(\lambda)\,U(\lambda)\,\mathrm{d}\mu(\lambda).$$

If almost all $U(\lambda)$ are unitary then $\int U(\lambda)\mathrm{d}\mu(\lambda)$ is an unitary operator in H.

It is evident from the above properties that the set of decomposable operators forms a $*$-algebra in H.

A $*$-subalgebra $\mathscr{A}$ of $L(H, H)$ is said to be a *von Neumann algebra* if it satisfies one of the following (equivalent) conditions:

(i) $\mathscr{A}$ is closed in the weak operator topology of H.

(ii) $\mathscr{A}$ is closed in the strong operator topology of H.

(iii) $\mathscr{A}$ coincides with its bicommutant $\mathscr{A}''$.

THE VON NEUMANN THEOREM.

(i) *The algebra $\mathscr{D}$ of diagonal operators is a commutative von Neumann algebra.*

(ii) *The commutant $\mathscr{D}'$ of $\mathscr{D}$ is the von Neumann algebra $\mathscr{R}$ of decomposable operators in H, i.e.*

$$\mathscr{D}' = \mathscr{R}, \quad \mathscr{R}' = \mathscr{D}.$$

(For the proof cf. Maurin 1969, ch.I, § 6.)

Let $H(X)$ be a Hilbert space of functions with the domain X and let $A_1, A_2, \ldots, A_n$ be a set of self-adjoint, strongly commuting operators in $H(X)$ which contains an elliptic differential operator.

Let Φ be a dense, linear subset of H endowed with a nuclear topology, which is, stronger than the relative topology induced by H. This means, in particular that the natural embedding $i\colon \Phi \to H$ is continuous. Suppose that the space Φ is so chosen that the map $A_i\colon \Phi \to \Phi$ is a continuous one. Let Φ' be the conjugate space of linear continuous functionals over the space Φ. Then the triplet (Φ, H, Φ') is called the *Gel'fand triplet*. Let Λ denote a subset of E^n. Then we have

THE NUCLEAR SPECTRAL THEOREM.

1° *There exists a direct integral* $\hat{H} = \int_\Lambda \hat{H}(\lambda)\mathrm{d}\varrho(\lambda)$ *and a generalized Fourier transform F*

$$F\colon H \to FH \equiv \hat{H} = \int_\Lambda \hat{H}(\lambda)\mathrm{d}\varrho(\lambda), \tag{21}$$

which is given by the formula

$$\Phi \ni \varphi(x) \to \hat{\varphi}_k(\lambda) = (F\hat{\varphi})_k(\lambda) = \langle \varphi, e_k(\lambda)\rangle = \int_X \varphi(x)\overline{e_k(\lambda, x)}\mathrm{d}x, \tag{22}$$

where $k = 1, 2, \ldots, \dim \hat{H}(\lambda)$, *and* $e_k(\lambda) \in \Phi'$ *are generalized eigenvectors. If* $A_i\varphi \in \Phi$, $i = 1, 2, \ldots, n$, *then*

$$\big(A_j\varphi, e_k(\lambda)\big) = \hat{A}_j(\lambda)\langle \varphi, e_k(\lambda)\rangle \tag{23}$$

for ϱ-almost all $\lambda \in \Lambda$, where $\hat{A}_i(\lambda)$ is the spectrum of A_i. The eigenvectors $e_k(\lambda, x)$ are regular functions.*

2° *For every φ, ψ in Φ, the so-called Parseval equality is satisfied*:

$$(\varphi, \psi)_H = \int_\Lambda \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda)\overline{\hat{\psi}_k(\lambda)} = (\varphi, \psi)_{\hat{H}}. \tag{24}$$

3° *The generalized Fourier inversion formula* (*the spectral synthesis*) *of an element* $\varphi(x)$ *in* Φ *is*

$$\varphi(x) = \int_{\Lambda} \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda) e_k(\lambda, x) \mathrm{d}\varrho(\lambda). \tag{25}$$

(For the proof. cf. Maurin 1968, ch. II.)

Remark 1: If we drop the condition that the set $\{A_i\}_1^n$ contains an elliptic operator, then the nuclear spectral theorem still holds. However, the generalized eigenvectors, $e_k(\lambda, x)$, are in general no longer regular functions and we do not have a representation of $\hat{\varphi}_k(\lambda)$ as an integral over the x-space as given by eq. (22).

Remark 2: The so-called 'complete von Neumann spectral' theorem for a set $\{A_i\}_1^n$ of self-adjoint, strongly commuting operators states that the map $F: H \to \hat{H} = \int \hat{H}(\lambda)\mathrm{d}\varrho(\lambda)$ is unitary. Therefore, the Parseval equality (24) holds for any φ, ψ in H. However, we have a representation $\hat{\varphi}_k(\lambda) = \int \varphi(x)\overline{e_k(\lambda, x)}\mathrm{d}x$ only for elements φ in $\Phi(X)$.

The formula (23) requires some additional comments. Let A_i' be the natural extension of A given by the identity

$$\langle A_i\varphi, \psi'\rangle = \langle \varphi, A_i'\psi'\rangle, \tag{26}$$

where φ runs over Φ and ψ' is an element in Φ'. Eq. (26) means that the domain $D(A_i')$ is the extension of the domain $D(A_i)$ of A_i by those elements ψ' in Φ' for which identity (26) is satisfied, i.e. $A_i' \supset A_i^* = A_i$. Equation (23) can now be written concisely in the form

$$A_i' e_k(\lambda, x) = \hat{A}_i(\lambda) e_k(\lambda, x), \qquad k = 1, 2, \ldots, \dim \hat{H}(\lambda), \tag{27}$$

where $\hat{A}_i(\lambda)$ is the spectrum of A_i, $i = 1, 2, \ldots, n$. Formula (25) for the spectral synthesis allows us to write the following completeness relation, which is often used by physicists:

$$\int \mathrm{d}\varrho(\lambda) \sum_{k=1}^{\dim \hat{H}(\lambda)} e_k(\lambda, x)\overline{e_k(\lambda, y)} = \delta(x-y). \tag{28}$$

This integral is understood in the sense of the weak integral of the regular distributions $e_k(\lambda, x)\overline{e_k(\lambda, y)}$ on $X \times X$. The weak integral (28) applied to a function $\varphi(y) \in \Phi(X)$ gives the spectral synthesis (25) of φ, and applied to a function $\varphi(y)\overline{\psi(x)} \in \Phi(X) \times \Phi(X)$, it gives the Parseval equality (24).

Notice that the spectral theorem says nothing about the orthogonality of the generalized eigenvectors. However, in many cases the spectral function $\mathrm{d}\varrho(\lambda)$ is absolutely continuous with respect to the Lebesgue measure $\mathrm{d}\lambda$ on the set Λ.

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* We recall that a function $f(x)$ in $H(X)$ is said to be regular if $\varphi f \in \mathscr{D}(X)$ ($\mathscr{D}(X)$ is the Schwartz space) for every $\varphi \in \mathscr{D}(X)$.

This implies $d\varrho(\lambda) = \varrho(\lambda)d\lambda$ by the Radon–Nikodym theorem. The inverse formula (22) allows us to write the following orthogonality relation:

$$\int_X e_k(\lambda, x)\overline{e_{k'}(\lambda', x)}dx = \varrho^{-1}(\lambda)\,\delta(\lambda-\lambda')\,\delta_{k,k'}. \tag{29}$$

This is a generalization of the well-known orthogonality relation in ordinary Fourier analysis

$$\int_{-\infty}^{\infty} \exp(i\lambda x)\overline{\exp(i\lambda' x)}dx = 2\pi\delta(\lambda-\lambda').$$

In both cases these integrals are understood as weak integrals of distributions $e_k(\lambda, x), e_k(\lambda', x)$ defined in the subspace $\hat{\Phi}(\Lambda) = F[\Phi(X)]$.

The spectral synthesis (25) of an element $\varphi(X)$ in Φ suggests that it would be useful to introduce a space $H'(\lambda) \subset \Phi'$ isomorphic with $\hat{H}(\lambda)$, i.e.

$$\hat{H}(\lambda) \ni \{\hat{\varphi}_k(\lambda)\} \to \hat{\varphi}(\lambda) = \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda)\hat{e}_k(\lambda) \to$$

$$\to \varphi(\lambda, x) = \sum_{k=1}^{\dim \hat{H}(\lambda)} \hat{\varphi}_k(\lambda)e_k(\lambda, x) \in H'(\lambda) \subset \Phi'. \tag{30}$$

Here $\hat{e}_k(\lambda)$ are orthonormal basis vectors in $\hat{H}(\lambda)$, i.e., $(\hat{e}_k(\lambda), \hat{e}_{k'}(\lambda))_{\hat{H}(\lambda)} = \delta_{kk'}$, whereas $e_k(\lambda, x) \in \Phi'(X)$. Formula (30) shows that each $H'(\lambda)$ is a linear subset of Φ' and, by eqs. (27) and (30),

$$A_i'\varphi(\lambda, x) = \hat{A}_i(\lambda)\varphi(\lambda, x), \tag{31}$$

i.e. any element $\varphi(\lambda, x)$ in $H'(\lambda)$ is the generalized eigenvector of all A_i', $i = 1, 2, \ldots, n$. Formula (25) may now be concisely written in the form

$$\varphi(x) = \int_\Lambda \varphi(\lambda, x)d\varrho(\lambda). \tag{32}$$

It is interesting that the isomorphism given by (30) between the spaces $\hat{H}(\lambda)$ and $H'(\lambda)$ induces a finite scalar product in the spaces $H(\lambda) \subset \Phi'$. We shall illustrate this with two examples:

Example 4. Let $H = L^2(R^4)$, where R^4 is the four-dimensional Minkowski space. Let $A = \square = -\nabla^2 + \frac{\partial^2}{\partial t^2}$ be the wave operator in H. Let Φ be the Schwartz S-space and $\Phi' = S'(R^4)$. Then, all assumptions of the nuclear spectral theorem are satisfied. The generalized eigenvectors

$$\square e(\lambda, x) = \hat{\square}(\lambda)e(\lambda, x) \quad (k = 1) \tag{33}$$

are plane waves

$$e(\lambda, x) = (2\pi)^{-2}\exp(ipx), \quad p^2 = \lambda^2 = \hat{\square}(\lambda), \tag{34}$$

where the parameter λ plays the role of mass of a scalar particle.

The space $\hat{H}(\lambda)$, where

$$\lambda^2 = p_0^2 - \boldsymbol{p}^2, \tag{35}$$

is the momentum space in which the scalar product (for positive masses) is

$$(\hat{\varphi}, \hat{\psi})_{\hat{H}(\lambda)} = \int_{p_0 = [\boldsymbol{p}^2+\lambda^2]^{1/2}} p_0^{-1} \mathrm{d}^3\boldsymbol{p}\hat{\varphi}(p)\overline{\psi(p)}. \tag{36}$$

This scalar product induces the following formula in $H'(\lambda)$:

$$\begin{aligned}(\varphi, \psi)_{H'(\lambda)} &= \mathrm{i}\int_t \mathrm{d}^3x\,[\varphi(\lambda, x)\partial_t\overline{\psi(\lambda, x)} - (\partial_t\varphi(\lambda, x))\overline{\psi(\lambda, x)}] \\ &\equiv \mathrm{i}\int_t \mathrm{d}^3x\varphi(\lambda, x)\overleftrightarrow{\partial_t}\overline{\psi(\lambda, x)},\end{aligned} \tag{37}$$

where the symbol t affixed to the integral means that all functions under the integral sign are taken at the same time t. One readily verifies that the formula (37) possesses all the properties required from a scalar product and is conserved in time. The generalized Fourier expansion (25) in this case is just the ordinary Fourier expansion

$$\varphi(\lambda, x) = 2^{1/2}[2\pi]^{-3/2}\int \mathrm{d}^4p\exp(-\mathrm{i}px)\,\delta(p^2-\lambda^2)\theta(p_0)\varphi(p), \tag{38}$$

EXAMPLE 5. We have a similar result for the Dirac equation in the space $(L^2)^4(R^4)$. In fact, let

$$\gamma^\mu\partial_\mu\psi(\lambda, x) = \lambda\psi(\lambda, x), \tag{39}$$

where $\lambda = m$ and

$$\psi(\lambda, x) = \exp(-\mathrm{i}px)\begin{bmatrix}\psi_1(p)\\ \psi_2(p)\\ \psi_3(p)\\ \psi_4(p)\end{bmatrix}, \qquad p_\mu p^\mu = m^2 = \lambda^2.$$

The scalar product in $\hat{H}(\lambda)$*

$$(\varphi, \psi)_{\hat{H}(\lambda)} = \int_{p_0 = [\boldsymbol{p}^2+\lambda]^{1/2}} p_0^{-1}\mathrm{d}^3\boldsymbol{p}\sum_{\alpha=1}^{4}\hat{\varphi}_\alpha(p)\overline{\hat{\psi}_\alpha(p)}, \tag{40}$$

induces the following finite time invariant scalar product in $H'(\lambda)$

$$(\varphi, \psi)_{H'(\lambda)} = \int_t \mathrm{d}_3x\sum_{\alpha=1}^{4}\varphi_\alpha(\lambda, x)\overline{\psi_\alpha(\lambda, x)} = \int_{\sigma(t)} \mathrm{d}\sigma^\mu\varphi(x)\gamma^\mu\overline{\psi(x)}.\ \blacktriangledown \tag{41}$$

We would like to stress that in quantum physics we are more often interested n the spaces $H'(\lambda) \subset \Phi'$, than in the Hilbert spaces $\hat{H}(\lambda)$.

* In physical literature one uses the scalar product for the Dirac wave functions which is antilinear with respect to the first factor.

§ 4. Functions of Self-Adjoint Operators

Let A be a self-adjoint operator and $E(\lambda)$ the resolution of the identity associated with A. We want to define a function $f(A)$ of the operator A and elaborate an operational calculus for the collection of functions of A.

Let $f(\lambda)$ be a complex, continuous function defined on the real line. Then, for any u in H, the set of vectors $v(\lambda) = f(\lambda)E(\lambda)u$ represents a one-parameter (strongly) continuous curve in H. Thus, for any α, β satisfying the condition $-\infty < \alpha < \beta < \infty$, the integral

$$\int_{\alpha}^{\beta} f(\lambda)\mathrm{d}E(\lambda)u$$

can be defined as the Riemann integral considered in § 2, i.e.,

$$\int_{\alpha}^{\beta} f(\lambda)\mathrm{d}E(\lambda)u = \lim_{i\to\infty} \sum_{i} f(\lambda_i')[E(\lambda_{i+1})-E(\lambda_i)]u, \tag{1}$$

where

$$\alpha = \lambda_1 < \lambda_2 < \ldots < \lambda_n = \beta, \qquad \lambda_i' \in (\lambda_i, \lambda_{i+1}],$$

and

$$\max|\lambda_{i+1}-\lambda_i| \to 0 \quad \text{as } i \to \infty .$$

By the results of § 2, the integral (1) always exists.

We can also define an improper integral by the following formulae:

$$\int_{-\infty}^{\infty} f(\lambda)\mathrm{d}E(\lambda)u = \lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \int_{\alpha}^{\beta} f(\lambda)\mathrm{d}E(\lambda)u, \tag{2}$$

if such a limit of Riemann integrals exists. The problem of existence of the integral (2) is solved by the following theorem.

THEOREM 1. *Let $E(\lambda)$ be a resolution of the identity. Then, for a given u in H, the following conditions are equivalent*:

(i) $\displaystyle\int_{-\infty}^{\infty} f(\lambda)\mathrm{d}E(\lambda)u$ *exists*, (3)

(ii) $\displaystyle\int_{-\infty}^{\infty} |f(\lambda)|^2\mathrm{d}\|E(\lambda)u\|^2 < \infty$, (4)

(iii) $\displaystyle F(v) \equiv \int_{-\infty}^{\infty} f(\lambda)\mathrm{d}(E(\lambda)v, u)$ *is a bounded linear functional.* ▼ (5)

(For the proof cf. Yoshida 1965, ch. 11.)

We now show that a self-adjoint operator $f(A)$ can be associated with every real, continuous function $f(\lambda)$, $\lambda \in (-\infty, \infty)$. In fact, we have:

THEOREM 2. *Let $E(\lambda)$ be the resolution of the identity associated with a self-adjoint operator A, and let $f(\lambda)$ be a real, continuous function. The equality*

$$(f(A)u, v) = \int_{-\infty}^{\infty} f(\lambda) \mathrm{d}(E(\lambda)u, v), \tag{6}$$

where

$$u \in D = \left\{u \in H: \int_{-\infty}^{\infty} |f(\lambda)|^2 \mathrm{d}||E(\lambda)u||^2 < \infty\right\} \tag{7}$$

and v is any element in H, defines a self-adjoint operator $f(A)$ in H with $D(f(A)) = D$. Moreover, $f(A)E(\lambda) \supset E(\lambda)$, i.e., the operators $f(A)$ and $E(\lambda)$ commute. ▼

(For the proof cf. Yoshida 1965, ch. 11.)

§ 5. Essentially Self-Adjoint Operators

We know that the spectrum of a self-adjoint operator is real and that the corresponding eigenfunctions form a complete orthogonal set of functions. Hence, the self-adjoint operators defined in the Hilbert space of physical state vectors are proper candidates for physical observables. However, in the representation theory of Lie algebras self-adjoint operators are not the most convenient objects for the formulation of a series of interesting theorems. Hence, it is necessary to introduce a wider class of operators, which on the one hand still have good spectral properties and on the other hand allow us to express certain fundamental mathematical results of representation theory.

Consider first symmetric operators. The spectrum of a symmetric operator is also real and any two eigenvectors corresponding to different eigenvalues are orthogonal (cf., e.g., Akhiezer and Glazman 1966, p. 136). However, a symmetric operator A defined in the Hilbert space H of physical state vectors is not the proper candidate to be an observable. This is because the set of all eigenvectors of a symmetric operator A does not form a complete set, i.e., there exists state vectors in H which will be orthogonal to all eigenvectors of the operator A. On such vectors the observable A does not yield an eigenvalue.

If we want to replace a symmetric operator by its self-adjoint extension, we find that in many cases a given symmetric operator has many, or even infinitely many, self-adjoint extensions. Hence, in general, symmetric operators are not the proper candidates for observables. However, special symmetric operators, which have a unique self-adjoint extension, could be used in physics. Such operators are called essentially self-adjoint.

A symmetric operator A is said to be *essentially self-adjoint* (e.s.a.) if its closure $\bar{A}$ is self-adjoint, i.e., $(\bar{A})^* = \bar{A}$. These operators, on the one hand, admit through their closure a physical interpretation as observables and, on the other hand, they are convenient objects for stating a number of interesting theorems in the representation theory of Lie algebras

Let us now derive the simplest properties of essentially self-adjoint operators.

LEMMA 1. *An operator A is essentially self-adjoint if and only if $\bar{A} = A^*$.*

PROOF: Let A be e.s.a. Then, $\bar{A} = (\bar{A})^* = A^*$ by lemma 1 of § 1. Conversely, if $\bar{A} = A^*$, then $(\bar{A})^* = A^{**} = \bar{A}$ by eq. (6).▼

Note that by th. 4 in § 1 a symmetric operator is essentially self-adjoint if it has both deficiency indices equal zero.

LEMMA 2. *Let $A_k^* = A_k$, $k = 1, 2$, and let the spectral families of these operators mutually commute. Then, $A_1 \pm A_2$ is essentially self-adjoint.*

PROOF: The proof follows directly from the spectral theorem.▼

A useful criterion for essential self-adjointness is given in:

LEMMA 3. *Let A be symmetric operator and let $(A+I)^{-1}$ be bounded and densely defined. Then, A is essentially self-adjoint.*

PROOF: A bounded linear operator is continuous by eq. 1(1). Hence, the closed operator $(A+I)^{-1} = (\bar{A}+I)^{-1}$ is defined on the whole space. Consequently, $R(\bar{A}+I) = H$. The operator $\bar{A}+I$ is symmetric by eq. 1(16). Therefore, $\bar{A}+I$ is self-adjoint by lemma 1.2. Hence, $\bar{A} = (\bar{A}+I)-I$ is essentially self-adjoint by lemma 2.▼

We prove another useful result

LEMMA 4. *Let D be a dense linear manifold in a Hilbert space H. Let A and A be linear transformations whose domains are D and whose ranges are contained in D such that A' is contained in the adjoint of A. If $A'A$ is essentially self-adjoint, then the closure of A' is the adjoint of A.*

PROOF: We must show that the graph of A^* contains no non-zero element, orthogonal to the graph of A'. Suppose $\{a, b\}$ is an element of the graph of A^* that is orthogonal to the graph of A'. In other words, $b = A^*a$, but $(y, a)+$ $+(A'y, b) = 0$ for all y in D. If x is in D then Ax is in D and hence $(Ax, a)+$ $+(A'Ax, b) = 0$; that is, $(x, b)+(A'Ax, b) = 0$. But $1+A'A$ has a dense range. Consequently $b = 0$. So $(y, a) = 0$ for all y in D; therefore $a = 0$.▼

Let us note finally that properties of a given operator are, to some extent, under our control, as the following two striking examples show:

1° Th. 1.4 states that a symmetric operator with different deficiency indices has no self-adjoint extension. However, if we embed the original Hilbert space H in a larger space $\tilde{H}$, then we have

THEOREM 5. *Every symmetric operator A in H with arbitrary deficiency indices (n_+, n_-) can be extended to a self-adjoint operator A acting in a larger Hilbert space $\tilde{H} \supseteq H$.*▼

(For the proof of the theorem cf. B. Sz. Nagy 1955.)

Hence, if elements of a larger Hilbert space $\tilde{H}$ admit a physical interpretation as state vectors, we can, in principle, take any symmetric operator in H to be a physical observable.

2° The differential operator $d = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}\varphi}$ is unbounded in $H = L^2(0, 2\pi)$ and therefore discontinuous. This is, however, true only if we consider the continuity implied by the strong topology in the Hilbert space. However, in the space of distribution the differential operator d is continuous. To show this, we recall that a sequence $\{F_n\}$ of distributions is said to be *convergent* to a distribution F, if for an arbitrary test function ψ we have

$$\lim_{n\to\infty} (F_n, \psi) = (F, \psi).$$

Let $\psi \in C^\infty(0, 2\pi)$ and let $F_n \to F$. Then,

$$\left(\frac{\partial F_n}{\partial\varphi}, \psi\right) = \left(F_n, -\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right) \to \left(F, -\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right) = \left(\frac{\partial F}{\partial\varphi}, \psi\right).$$

Hence, if $F_n \to 0$, then $\frac{1}{\mathrm{i}}\frac{\mathrm{d}F_n}{\mathrm{d}\varphi} \to 0$, i.e., the operator d is continuous. ▼

These two examples indicate that we can considerably improve properties of operators, if we introduce a properly chosen structure in the carrier space.

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List of Important Symbols

$g_{\mu\nu}$ — space-time metric XIX
a^* — hermitian conjugation of a matrix or operator a XIX
$\bar{a}$ — complex conjugation of a matrix a XIX
$[\frac{1}{2}n]$ — XIX, 304
$\{\frac{1}{2}n\}$ — XIX, 304
$[X, Y]$ — Lie multiplication 1
$[M, N]$ — linear hull of vectors of the form $[X, Y]$, $X \in M$, $Y \in N$, $M \subset L$, $N \subset L$ 1
C^i_{jk} — structure constants of a Lie algebra or Lie group 2, 85
L^c — complex extension of a Lie algebra L 3
$\Phi(\xi, \eta)$ — bilinear form 4
$V_1 \dot{+} V_2 \dot{+} \ldots$ — direct sum of vector spaces V_i 5
$L_1 \oplus L_2 \oplus \ldots$ — direct sum of Lie algebras L_i 5
L/N — quotient Lie algebra of L with respect to N 6
$\mathrm{ad}X(Y) \equiv [X, Y]$ — 7
L_A — adjoint algebra of L 8
$L_1 \oplus\!\!\!\!\!\!\supset L_2$ — semidirect sum of two Lie algebras L_1 and L_2 9
(X, Y) — Killing form in a Lie algebra 12
L^α — subspace of the root α 21
$\Delta(L)$ — system of nonzero roots of a semisimple Lie algebra 21
$\Pi(L)$ — system of simple roots of a semisimple Lie algebra 24
$\mathrm{gl}(n, R)$ — 3
$\mathrm{gl}(n, C)$ — 4
$\mathrm{sl}(n, C)$ — 4
$O(2n+1, C)$ — 4
$O(2n, C)$ — 4
A_n, B_n, C_n, D_n — classical Lie algebras 4
$\mathrm{su}(n)$ — 34
$\mathrm{sl}(n, R)$ — 34
$\mathrm{su}(p, q)$ — 34
$\mathrm{su}^*(2n)$ — 34
$\mathrm{so}(2n)$ — 35
$\mathrm{so}(2n+1)$ — 35
$\mathrm{sp}(n)$ — 35
$\mathrm{sp}(n, R)$ — 35
$\mathrm{sp}(p, q)$ — 36
$A \cup B$ — union of two sets A and B 52
$A \cap B$ — intersection of two sets A and B 52
$\{X, \tau\}$ — topological space 52
$d(x, y)$ — distance in metric space 53
$S(x, r) = \{y \in X\colon d(x, y) < r\}$ — neighbourhood of a point x 54
$A' \equiv X \backslash A \equiv \{x\colon x \in X \text{ and } x \notin A\}$ — complement A' of a set $A \subset X$ 55

G^{HL} — carrier space for induced representation U_g^L of G 489
$E(Z)$ — transitive system of imprimitivity 494
D_G — Gårding space 496
$\hat{O}_{\hat{n}}$ — orbit of the character $\hat{n}$ 505
$\hat{O}$ — orbit of S in the product $N \circledS S$ 503, 507
K: W — space of double cosets 583
$T^G|_K$ — restriction of T representation of a group G to a subgroup K 602
$\mathrm{Im} f_n$ — image of a homomorphism f_n 619
$\mathrm{Ker} f_n$ — kernel of a homomorphism f_n 619
$x R y$ — x is in relation R with y 637
$x \prec y$ — relation of partial ordering 637
$\emptyset$ — empty set 638
$\mu_1 \otimes \mu_2$ — tensor product of measures μ_1 and μ_2 639
$\{f_n(x)\}_1^\infty$ — infinite sequence of functions 640
A^* — operator adjoint to A 642
A — closure of an operator A 642
n_+, n_- — deficiency indices of an operator 644
$E(\lambda)$ — resolution of the identity (spectral function) 649
(Λ, μ) — space Λ with measure μ 655
$\bigtimes_{\lambda \in \Lambda} H(\lambda)$ — Cartesian product of $H(\lambda)$ 655
$(\Phi H, \Phi')$ — Gel'fand triplet 658
$\Box = -\nabla^2 + \frac{\partial^2}{\partial t^2}$ — wave operator 660

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