Math 318 Paper Topics

Spring 2019

The following are possible topics for you to explore in your term paper. Many, if not all, of these topics are far too broad to fit into your paper; you will have to narrow them down (with David's and my help)! I am open to other topics of significant analytic content, be they pure or applied (to economics, to physics, to biology, etc.).

Dedekind Cuts: At the beginning of 317, we defined $\mathbb{R}$ by its axioms, i.e. as a complete ordered field. We never proved that such a field exists! One way to do this is to use Dedekind Cuts. You can also explore other constructions of the real numbers.

Starting points: [1, §8.6] and [10, Ch. 1]

Trig Functions and Exponentials: We never did define the sine and cosine functions, did we? Starting with their power series, you can explore some fundamental analytic properties, relate them to the "opposite over hypotenuse"-style definitions, look into Euler's formula $e^{it} = \cos t + i \sin t$, etc.

Starting points: [7, §5.4] and [10, Ch. 8]

Cesaro Summability: How do you sum the series $1 - 1 + 1 - 1 + 1 \cdots$? By our "limit of partial sums" definition, you can't, but if you take a different definition of the sum of a series, you get $\frac{1}{2}$. You should explore the definition of Cesaro summability and compare it to our old definition: in what cases are they the same? In what cases are they different? There are also generalizations of Cesaro summability, including Abel summability, that are interesting as well. More generally, you might want to explore so-called "Tauberian theorems."

Starting Point: [7, §5.10]

Baire's Theorem: We saw last semester that there is a function that is continuous on the irrationals but continuous on the rationals (Thomae's function) . . . but can you find a function that is continuous on the rationals but discontinuous on the irrationals? Baire's theorem will give you the answer, and has many applications beyond the one above! More generally, you can explore the Baire Category Theorem for normed vector spaces; this implies, for example, that "most" continuous functions are nowhere differentiable.

Starting points: [1, §3.5] and/or [1, §8.2], but I suggest ignoring the metric space material and just looking at the Baire Category Theorem at the end for normed vector spaces.

Weierstraß, revisited: We only scratched the surface of the Weierstraß Approximation Theorem, which says roughly that any continuous function can be approximated arbitrarily closely by polynomials. You can explore it more deeply in your project. For example, there are other proofs, including a proof due to Bernstein's that is related to probability theory. You can investigate *best* approximations of a function by polynomials — do does a best approximation (for a given degree) exist? Is it unique? You can also investigate approximation by (cubic) splines, which is used in computer graphics.

Starting points: [2, Ch. 10]

The Γ Function (or other "special" functions): You know how to compute factorials, like $3! = 3 \cdot 2 \cdot 1$, but what is $\frac{1}{2}!$? The Γ function extends the factorial, i.e. it is a smooth function with the property that $\Gamma(n) = (n-1)!$ for any natural number. There are also some interesting relationships with Euler's constant $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$, the Riemann ζ (zeta) function, and other interesting "special functions" that you can explore.

Starting points: [10, Ch. 8] and [1, §8.4]

The Implicit Function Theorem and Lagrange Multipliers: Why do Lagrange multipliers work? You need to use something called the "Implicit Function Theorem" (which belongs in the pantheon with IVT, EVT, MVT). You can explore this idea in more depth, or look into one of the many other applications of the Implicit Function Theorem. If you were intrigued by the our discussion about connections between linear algebra and calculus in $\mathbb{R}^n$ in Math 317, this is the project for you!

Starting points: [7, Ch. 7]

Discrete Dynamical Systems: Some systems, such as population models with a fixed reproductive cycle, are best modeled using a single function iterated time and again (e.g. if you start with a population y_0 and the population at time $n+1$ is given by $y_{n+1} = f(y_n)$, then $y_3 = f(f(f(y_0))))$. This model is a discrete dynamical system, or a difference equation. You could explore fixed points (x such that $f(x) = x$), periodic points (y such that the n^{th} iterate of f is y), "chaotic" behavior, and even Sharkovskii's Theorem. You could also look into invariant sets like the Sierpinski Gasket or into some more applied or computational directions. Frankly, this topic is enormous but quite exciting.

Starting points: [2, Ch. 11] or [3]

Topics in Probability: If you have taken Math 218, there are several ways to explore probability from a deeper analytic perspective. Some ideas are to look into the Central Limit Theorem, different types of convergence of random variables (in distribution, in norm, etc.), or this useful result: if two real-valued, nonnegative random variables X and Y satisfy $E[exp(-tX)] = E[exp(-tY)]$ for $t > 0$, then they have the same distribution.

Starting points: Notes from Math 328 or 396; [11] or [9].

Existence and Uniqueness for Solutions to ODEs: The basic theoretical question in the theory of ordinary differential equations (you should probably restrict yourself to examining first-order ODEs) is whether or not it has a solution, and if so, whether that solution is unique. You should start by looking into the contraction mapping theorem and its applications to existence and uniqueness theorem for ordinary differential equations.

Starting point: [2, §12.1–12.4 and §12.8].

Hausdorff Dimension: You can believe that the dimension of a line is 1 and the dimension of a plane is 2 — but what about the dimension of the Cantor set? Would you believe $\frac{\log 2}{\log 3}$? You can explore the Hausdorff measure (which can be used to measure the size of subsets of $\mathbb{R}^n$) and the Hausdorff dimension through a rigorous examination of the definitions and some fun examples (such as "fat" Cantor sets...).

Starting point: [1, §3.1], [8, Ch. 2]

The Brouwer Fixed Point Theorem: There are several analytic proofs of this awesome theorem, which says that any continuous map from the closed unit ball in $\mathbb{R}^n$ to itself has a fixed point, i.e. a point x such that $f(x) = x$.

Starting point: One analytic proof can be found in [4, §8.1], but this is not the only one! Please stay away from the (very cool, but not presently relevant) topological or combinatorial proofs (e.g. the Game of Hex or Sperner's Lemma).

Wavelets and Compression: Trig polynomials are not the only orthonormal basis for L^2, and there are some issues with them, such as slow convergence near a point of discontinuity. There are other bases called wavelets that are useful for problems for which Fourier series work badly, especially for image compression. You can to investigate what wavelets are, prove some of their basic properties, and perhaps discuss an application.

Starting Point: [2, Chapter 15]

Direct Methods for the Calculus of Variations: At the beginning of our discussion of the Calculus of Variations, I mentioned that there are more "direct" methods for finding minimizers: given a functional $\mathbb{F}$ that is bounded below, find a sequence of functions u_n so that $\mathbb{F}(u_n)$ converges to $\inf \mathbb{F}$. Hopefully, you can then prove that (a subsequence of) this sequence converges, and that the limit is, indeed, a minimizer. You should investigate one or more direct methods, such as the Ritz method or the method of finite differences.

Starting Point: [5, Chapter 8].

The Second Derivative Test for the Calculus of Variations: How do we know when a solution to the Euler-Lagrange equation is a local minimum? It would be nice to have something akin to the second derivative test in finite-dimensional optimization, and indeed, such a test exists. You should investigate this test (especially how it relates to the second derivative test in $\mathbb{R}^n$) and apply it to one of our classical variational problems.

Starting Points: [5, Ch. 5], [6]

References

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