TERM PAPER PROPOSAL

SIPENG ZHOU

1. Introduction

In this term paper, I am going to work on the interpolation and the best approximation of a given function. We know that polynomial functions can be easily processed in computers with finite iterations; meanwhile, they are nicely behaved so that we can easily compute their integrations on a certain interval or their nth derivatives at any given point, which are useful tools helping us to study the behavior and the properties of a function. However, under most circumstances, the function we confront is not as friendly as polynomial functions are; sometimes, it can be very harsh and difficult to process for computers (e.g. the computer itself actually doesn't know what $sin(x)$ means or how to process it). Therefore, we really want to transfer the object of study from the harsh function to a nicely behaved polynomial approximation to that given function. The two ways used very often are interpolation and polynomial approximation (which is a polynomial function or a composite of polynomial functions and thus, can be processed by computers).

2. Interpolation

We will start by working on the interpolants of a given function f (which is relatively easy). More precisely, we want to obtain spline approximations to data obtained by sampling the function f.

Example: $f(x) = sin^(x)$ on the closed interval $[-\pi, \pi]$.

Sample points: $(-\pi, 0), (-\pi/2, -1), (0, 0), (\pi/2, 1), (\pi, 0)$.

(a) We will use Lagrange interpolation formula to get the polynomial interpolant of the 5 given sample points on f. Let the polynomial interpolant be A(x).

(b) We will use cubic spline interpolation method to get the cubic spline interpolant of the 5 given sample points on f. Let the cubic spline interpolant be B(x).

We will compare A(x) and B(x) and determine which one is a better approximation in this case.

3. Best Approximation

In the previous section, we've worked on something specific. Now, we want to do something more general. Specifically, we are going to study the best polynomial approximation (for a given degree) to a function f. We want to argue that it exists and it is unique.

Here are the settings and theorems we are going to work through.

Suppose $f \in C^0[a, b]$.

Definition 3.1. Suppose $g \in \Phi$ for some $\phi \overset{\in}{\subset} C^0[a, b]$. Then g is the best approximation to f in Φ w.r.t. $\|\cdot\|$, if $\|f - g\| \leq \|f - h\|$ for all $h \in \Phi$.

In this paper, we will use $\|\cdot\|_\infty$.

Let P_n be the set of polynomials $p(x)$ of degree less than or equal to n.

Then

$$P_0 \subset P_1 \subset P_2 ...$$

Define, for any $p \in P_n$,

$$d(p, f) = \|p - f\|_\infty.$$

Let $d = d_n = d_n(f) = inf\{d(p, f) : p \in P_n\}$. Then $d \geq 0$.

Theorem 3.2. *(Chebyshev's Theorem of Best Approximation) There exists a polynomial $p(x)$ in P_n for which $d(p, f) = d$, i.e. p is the best approximation to f from P_n.*

Theorem 3.3. *The polynomial $p(x)$ of* **??** *is unique.*

References

[Gau] W. Gautschi. *Numerical Method* Birkhauser, 2012.
This is a source suggested by Prof. David. This source tells me what interpolation is and how to attain the interpolant for a given function through Lagrange formula, which I will use in my second section to work on my example $f = sin(x)$.

[Hol] M. Holmes *Introduction to Scientific Computing and Data Analysis* Springer, 2016.
This is a source I use in my applied math class. This source teaches me how to attain the interpolant for a given function through cubic spline interpolation, which I will use in my second section to work on my example $f = sin(x)$.

[Bur] J.G. Burkill *Lectures On Approximation By Polynomials* Tata Institute of Fundamental Research, Bombay, 1959.
This is a source I found online. This source provides me with information about the Chebyshev's theorem, and I will prove the existence theorem and the uniqueness theorem with the help of this source.