///, = TeX hints

///, = Citation hints

/// = Other comments.

RW = "Rewrite"... the math is fine, but style/grammar could use improvement

HAAR WAVELET AND ITS APPLICATION IN IMAGE COMPRESSION

JASON NGO

21st?

1. Introduction

Since the start of the 20th century, we have seen rapid development in the theory and applications of wavelets. As a mathematical tool, wavelets can be used to extract information from different kinds of data such as audio signals and images. As an attempt to explore wavelets at an introductory level, this paper will examine the first wavelet developed, called the Haar Wavelet, and discuss its applications in image compression.

odd parallel construction.

"To get a better idea of our goals"... (or the like)

First, recall that $L^2(\mathbb{R})$ is a vector space of square integrable functions, taken with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)dx.$$

Definition 1.1. The *Haar function* is the function $\Psi = \chi_{[0,0.5)} - \chi_{[0.5,1)}$. The *Haar system* is the family

citation?

?

$$\{\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k), \quad j,k \in \mathbb{Z}\}.$$

Note that each term in the Haar system is constructed by translating and/or dilating the original Ψ (see Figure 1). By this construction, since $\int_{\mathbb{R}} \Psi(x)dx = 0$, it follows that $\int_{\mathbb{R}} \Psi_{j,k}(x)dx = 0$ for $j,k \in \mathbb{Z}$. Furthermore, it is important to calculate the width and height for each $\Psi_{j,k}$. We have $\Psi(2^j - k) = 1$ when $0 \le 2^j - k < \frac{1}{2}$, or equivalently, $\frac{k}{2^j} \le x < \frac{k+\frac{1}{2}}{2^j}$. Therefore, in this interval, $\Psi_{j,k} = 2^j$. Similar calculations yield:

$$\Psi_{j,k} = \begin{cases} 2^{j/2} & \text{if } \frac{k}{2^j} \le x < \frac{k+\frac{1}{2}}{2^j}, \\ -2^{j/2} & \text{if } \frac{k+\frac{1}{2}}{2^j} \le x < \frac{k+1}{2^j}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Now, we will prove that the Haar function Ψ satisfies Definition 1.2 of an *orthonormal wavelet*.

One usually wants to be more informal + intuitive in the intro. Move this to a new §2 and replace w/ a high-level discussion (but keep the picture!)

Date: 2019-04-03.

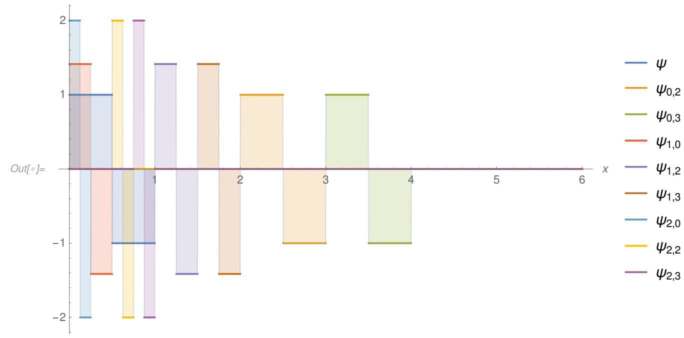

FIGURE 1. Elements of the Haar system

Definition 1.2 (Orthonormal Wavelet, [4, p. 303])**.** If $\Psi \in L^2(\mathbb{R})$, $\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k)$, and the set $\{\Psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, then Ψ is called an *orthonormal wavelet.*

where is the "then"

To prove the Haar system is an *orthonormal basis*, we have to show:

(1) The Haar system $\{\Psi_{j,k}\}$ is an orthonormal set, i.e. $\|\Psi_{j,k}\| = 1$ for all $j, k \in \mathbb{Z}$ and $\langle \Psi_{j,k}, \Psi_{j',k'} \rangle = 0$ for all $(j, k) \neq (j', k')$.
(2) The span of Haar system, denoted $\operatorname{span}\{\Psi_{j,k}\}$, is dense in $L^2(\mathbb{R})$.

In Section 2, we will first prove that the Haar system is orthonormal. Section 3 will show that it spans a dense subspace of $L^2(\mathbb{R})$. Finally, in Section 4, we will discuss the Haar wavelet's local property and its application in image compression.

can you briefly discuss the relationship btwn your source + your pf? (This goes for later thms, too)

2. ORTHONORMALITY

Each § should start w/ an overview that briefly states the goal of the section AND links back to your outline.

Lemma 2.1 ([2, p. 409])**.** *The Haar system is an orthonormal set in* $L^2(\mathbb{R})$.

Proof. First, it is easy to see that $\|\Psi\| = 1$. Let us compute:

$$\|\Psi_{j,k}\|^2 = \int_{\mathbb{R}} \left| 2^{j/2}\Psi(2^j x - k) \right|^2 dx = \int_{\mathbb{R}} 2^j \left| \Psi(2^j x - k) \right|^2 dx$$
$$= \int_{\mathbb{R}} |\Psi(y)|^2 \, dy = \|\Psi\|^2,$$

by change of variable $y = 2^j x - k$ in the integral. Therefore, all the elements in the Haar system has norm 1.

Now, we want to show the orthogonality. Consider $\Psi_{j,k}$ and $\Psi_{j,k'}$, for $k \neq k'$. Since these two elements have disjoint support[1], their inner product is 0. If $j < j'$, then either $\Psi_{j,k}$ and $\Psi_{j',k'}$ have disjoint supports (when $k \neq k'$),

[1]The *support* of a function, denoted supp, is the subset of the domain containing those elements which are not mapped to zero.

or supp $\Psi_{j',k'}$ is contained in an interval on which $\Psi_{j,k}$ is constant ~~to~~, (when $k = k'$). For the latter case, we have:

$$\langle \Psi_{j,k}, \Psi_{j',k'} \rangle = \int_{\mathbb{R}} 2^{j/2} \cdot \Psi_{j',k'} dx = 2^{j/2} \int_{\mathbb{R}} \Psi_{j',k'} dx = 0.$$

Hence, the set is orthonormal. □

Remark 2.2. The Haar function is a dyadic function, meaning that the dilations are taken to be powers of 2. Here, the factor $2^{j/2}$ is called the normalization factor, which is there so that the dilated and translated Haar function has norm 1. Although powers of 2 are a common choice, they are certainly not the most general one[2].

3. Haar Wavelet Basis for $L^2(\mathbb{R})$

Might be a good idea to state your Thm now, then use lemmas to build up to the proof, w/ some hints as to the final pf along the way

3.1. **Integral Transform.** Motivated by [1, p. 3] and [2, p. 516], we will set up the integral transform $P_n f$ below, which will act as the foundation to prove that any continuous function with compact support in $C_c(\mathbb{R})$, a dense subspace of $L^2(\mathbb{R})$[3], is in the closure of the span of $\{\Psi_{j,k}\}$.

Set $\phi(x) = \chi_{[0,1)}$. For $n \in \mathbb{Z}$, we define the integral kernel

$$K_n(x, y) = 2^n \sum_{k \in \mathbb{Z}} \phi(2^n x - k)\phi(2^n y - k).$$

Note that K_n is either equal to 2^n (when there is some $k \in \mathbb{Z}$ s.t. $2^n x - k$ and $2^n y - k$ lie in $[0, 1)$) or equal to 0 (otherwise). Equivalently, if $K_n = 2^n$, there is some k s.t.

$$x, y \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) = I_{n,k}.$$

Here, note that $K_n(x, y)$ is constant (equal to 2^n) on each dyadic interval of length 2^{-n}. Furthermore, for any given $n \in \mathbb{Z}$, the sets $\{I_{n,k}\}$ ~~is~~ are disjoint. Not sure I get what you're saying here.

We define the integral transform

$$P_n f(x) = \int_{\mathbb{R}} K_n(x, y) f(y) dy.$$

If $x \in \mathbb{R}$ then there is a unique $k_x \in \mathbb{Z}$ with $x \in I_{n,k_x}$ and

$$P_n f(x) = 2^n \int_{I_{n,k_x}} f(y) dy. \tag{3.1}$$

Since I_{n,k_x} has length 2^{-n}, we can say that $P_n f$ is the average value of a function f on a dyadic interval containing x.

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[2]See examples of non-dyadic wavelets and a discussion why they are more appropriate for statistical data analysis in [5].

[3]The proof that $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ can be found in [3, p. 326]. Good.

Notice that the integral kernel K_n is dependent on ~~the~~ ϕ. In order to write any function in terms of the Haar system $\{\Psi_{j,k}\}$, we must eliminate ϕ somehow from our calculations. The following Lemma 3.1 will write the integral kernels in terms of $\{\Psi_{j,k}\}$.

Lemma 3.1 ([4, p. 293])**.** *If $n \in \mathbb{Z}$, then*

For eqns. only.

$$K_{n+1} - K_n = \sum_{k\in\mathbb{Z}} \Psi_{n,k}(x)\Psi_{n,k}(y), \qquad x, y \in \mathbb{R}.$$

Proof. First, let us calculate the right-hand side. By Equation (1.1), $\Psi_{n,k}(x) = 2^{n/2}$ when $x \in I_{n+1,2k} = \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right)$ and $\Psi_{n,k}(x) = -2^{n/2}$ when $x \in I_{n+1,2k+1}$. Therefore, it follows that:

$$\Psi_{n,k}(x)\Psi_{n,k}(y) = \begin{cases} 2^n & \text{if } x, y \in I_{n+1,2k} \text{ or } x, y \in I_{n+1,2k+1}, \\ -2^n & \text{if one is in } I_{n+1,2k} \text{ and the other is in } I_{n+1,2k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, before calculating $K_{n+1} - K_n$, note that:

$$I_{n+1,2k} \cup I_{n+1,2k+1} = \left[\frac{2k}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right) = I_{n,k}.$$

Therefore, if x, y is in either $I_{n+1,2k}$ or $I_{n+1,2k+1}$, then $x, y \in I_{n,k}$ and $K_n(x, y) = 2^n$. Hence, for this case, we have:

$$K_{n+1}(x, y) - K_n(x, y) = 2^{n+1} - 2^n = 2^n.$$

By similar argument, if either x or y is in $I_{n+1,2k}$ and the other is in $I_{n+1,2k+1}$, then

$$K_{n+1}(x, y) - K_n(x, y) = 0 - 2^n = -2^n.$$

Otherwise, everything is 0.

Hence, we conclude $K_{n+1} - K_n = \sum_{k\in\mathbb{Z}} \Psi_{n,k}(x)\Psi_{n,k}(y)$. □

Remark 3.2. From this Lemma, the difference between two integral transforms can be written as the *inner product expansion* of the function f:

$$\begin{aligned} (P_{n+1} - P_n)f(x) &= \int_{\mathbb{R}} K_{n+1}(x, y)f(y)dy - \int_{\mathbb{R}} K_n(x, y)f(y)dy \\ &= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \Psi_{n,k}(x)\Psi_{n,k}(y)f(y)dy \\ &= \sum_{k\in\mathbb{Z}} \Psi_{n,k}(x) \int_{\mathbb{R}} \Psi_{n,k}(y)f(y)dy \\ &= \sum_{k\in\mathbb{Z}} \langle f, \Psi_{n,k}\rangle \Psi_{n,k}(x). \end{aligned} \tag{3.2}$$

The terms $\langle f, \Psi_{j,k}\rangle$ are called the the Haar coefficients of the expansion. One can easily notice that this inner product expansion is only one-sided, meaning we keep the dilation factor $j = n$ fixed and vary the translation factor k. In the following sections, we will abstract this expansion to a more general setting, varying both the translation and dilation factors.

3.2. **Two-sided Haar Representation.** From Equation 3.2, we can write

$$P_{n+1}f(x) - P_{-m}f(x) = \sum_{j=-m}^{n} \sum_{k\in\mathbb{Z}} \langle f, \Psi_{j,k}\rangle \Psi_{j,k}(x). \tag{3.3}$$

To prove the Haar inner product expansion on the right-hand side converges to a function f, we have to show that $P_{-m}f \to 0$ and $P_{n+1}f \to f$ as $m, n \to \infty$. We prove the two Lemmas in the same order below.

First, Lemma 3.3 tells us that the larger the intervals on which we average a continuous function, the smaller the supremum of the averaged function.

Lemma 3.3 (Average Function over Dyadic Intervals, [4, p. 295])**.** *If $f \in C_c(\mathbb{R})$, then $\|P_{-m}f\|_2 \to 0$ when $m \to \infty$.*

Proof. Suppose $f \in C_c(R)$ and $\mathrm{supp}(f) \subseteq [-K, K]$. There eixsts $M > 0$ s.t. $\exists f$ or g?
$\mathrm{supp}(g) \subset [-2^M, 2^M]$.

Now, if $m > M$, then $\mathrm{supp}(g) \subseteq I_{-m,-1} \cup I_{-m,0}$; Equation 3.1 holds only in these two intervals. We therefore compute:

$$\begin{aligned}
\|P_{-m}g\|_2^2 &= \int_{\mathbb{R}} |P_{-m}g(x)|^2 \, dx \\
&= \sum_{k\in\mathbb{Z}} \int_{I_{-m,k}} |P_{-m}g(x)|^2 dx \\
&= \sum_{k\in\mathbb{Z}} \int_{I_{-m,k}} \left| 2^{-m} \int_{I_{-m,k}} g(y)dy \right|^2 dx \\
&= 2^m \cdot 2^{-2m} \sum_{k\in\mathbb{Z}} \left| \int_{I_{-m,k}} g(y)dy \right|^2 dx \\
&= 2^{-m} \left(\left| \int_{I_{-m,-1}} g(y)dy \right|^2 dx + \left| \int_{I_{-m,-1}} g(y)dy \right|^2 dx \right) \\
&= 2^{-m} \left(\left| \int_{-K}^{0} g(y)dy \right|^2 dx + \left| \int_{0}^{K} g(y)dy \right|^2 dx \right).
\end{aligned}$$

Note that $\int_{-K}^{0} g(y)dy = \langle g(y), \chi_{[-K,0)}\rangle$; therefore, by Cauchy-Schwarz inequality, we have:

$$\begin{aligned}\|P_{-m}g\|_2^2 &\le 2^{-m}\left(K\|g\|_2^2 + K\|g\|_2^2\right)\\ &= 2^{-m}\cdot 2K\|g\|_2^2.\end{aligned}$$

As $m \to \infty$, the right-hand side converges to 0. Therefore, by Squeeze Theorem, it follows that $\|P_{-m}f\|_2 \to 0$. □

Lemma 3.4 (Convergence of Integral Transform, [2, p. 517]). *If $f \in C_c(\mathbb{R})$, then $P_n f \to f$ in the uniform nom as well as in the L^2 norm.*

Proof. Suppose $f \in C_c(\mathbb{R})$ and $\mathrm{supp}(f) \subset [-2^M, 2^M]$ for $M \ge 0$. Since f is continuous on a compact set, it is uniformly continuous.

Given $\epsilon > 0$. By uniform continuity, there exists $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Now, choose N such that $2^{-N} < \delta$; if $n > N$, then we have:

$$\begin{aligned}|P_n f(x) - f(x)| &= \left|2^n \int_{I_{n,k_x}} f(y)dy - f(x)\right|\\ &= \left|2^n \int_{I_{n,k_x}} f(y)dy - 2^n \int_{I_{n,k_x}} f(x)dy\right|\\ &\le 2^n \int_{I_{n,k_x}} |f(y) - f(x)|dy\\ &< 2^n \int_{I_{n,k_x}} \epsilon dy = \epsilon.\end{aligned}$$

Therefore, $P_n f$ converges to f uniformly. Furthermore,

$$\begin{aligned}\|P_n f - f\|_2^2 &= \int_{\mathbb{R}} |P_n f(x) - f(x)|^2 dx\\ &= \int_{-2^M}^{2^M} |P_n f(x) - f(x)|^2 dx\\ &\le \int_{-2^M}^{2^M} \|P_n f - f\|_\infty^2 dx = 2\cdot 2^M \|P_n - f\|_\infty^2.\end{aligned}$$

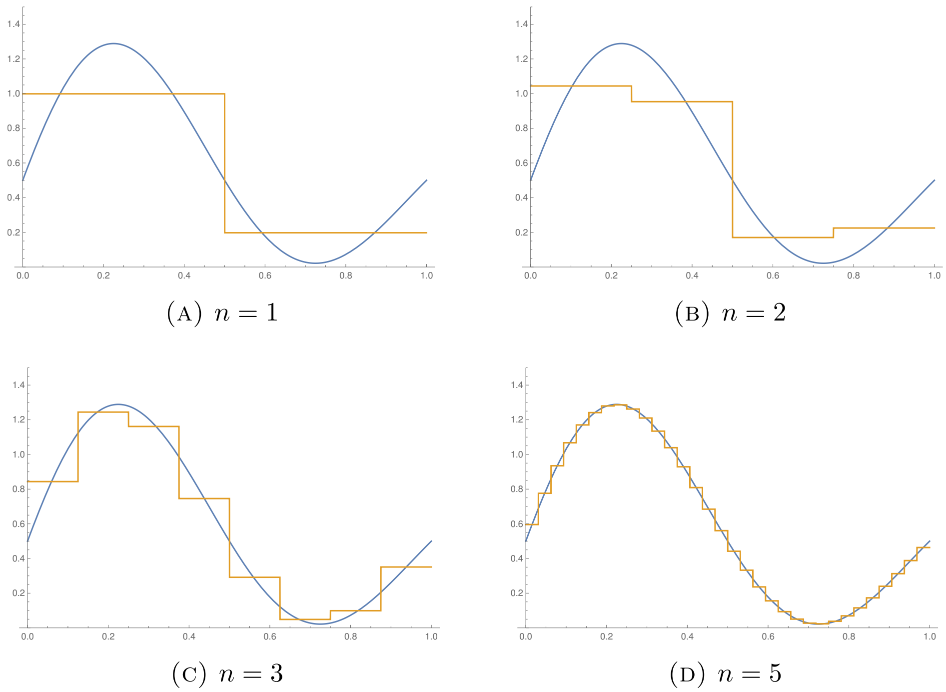
Note: $\|\cdot\|_2 \le C\|\cdot\|_\infty$ (for some C... this holds in general) (This is a HW assignment this week!)

Since the right hand side converges to 0, by Limit Comparison Test, it follows that $\|P_n f - f\|_2^2 \to 0$ as $n \to \infty$.

Hence, $P_n \to f$ both in the uniform norm and in the L^2 norm. □

To illustrate this convergence better, see Figure 2 for the sequence $\{P_n f\}$ approaching $f(x) = e^{-x}\sin 2\pi x$ on the unit interval.

Maybe do this example earlier to help the reader visualize what's going on in the proof.



(A) $n = 1$

(B) $n = 2$

(C) $n = 3$

(D) $n = 5$

FIGURE 2. Approximation of $f(x) = e^{-x} \sin 2\pi x$ by $P_n f(x)$

3.3. **Span of Haar System.**

Theorem 3.5 ([2, p. 411])**.** *The Haar system spans all of $L^2(\mathbb{R})$.*

Proof. By Equation 3.3, Lemma 3.3 and Lemma 3.4, it follows that for any $f \in C_c(\mathbb{R})$, f is in the span of the Haar system:

Closure of the

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, \Psi_{j,k} \rangle \Psi_{j,k}.$$

Since $C_c(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$, the Haar system spans $L^2(\mathbb{R})$. $\square$

Therefore, the Haar function Ψ is an orthonormal wavelet; we will now refer to it as the *Haar wavelet*.

4. Application to Image Compression

In this section, we discuss image compression as an application of the Haar wavelet basis. First, we define what we mean by compression. When compressing images, we want to discard the least significant details, keeping the original picture largely intact. Since an image is two-dimensional, it is helpful to think of image compression as discarding least significant details from the sets of row pixels and column pixels.

This feels a bit underexplained. How are you using wavelets in $\mathbb{R}^2$?

For instance, suppose we are given a function $f(x)$ expressed in terms of span of basis functions $\{u_i\}$:

$$f(x) = \sum_{i=0}^{M-1} c_i u_i(x).$$

The data set in this case consists of the coefficients c_i. We would like to find a function approximating $f(x)$ but requiring fewer coefficients, perhaps by using a different basis.

Can you be a bit more precise here?

Because Haar wavelet basis is orthonormal, we can easily calculate the coefficients and sort them in decreasing magnitude. Furthermore, due to its highly localization (the support for each function is located in a narrow dyadic interval), small coefficients can be discarded without losing much data.

Say a bit about "throwing away" coefs.

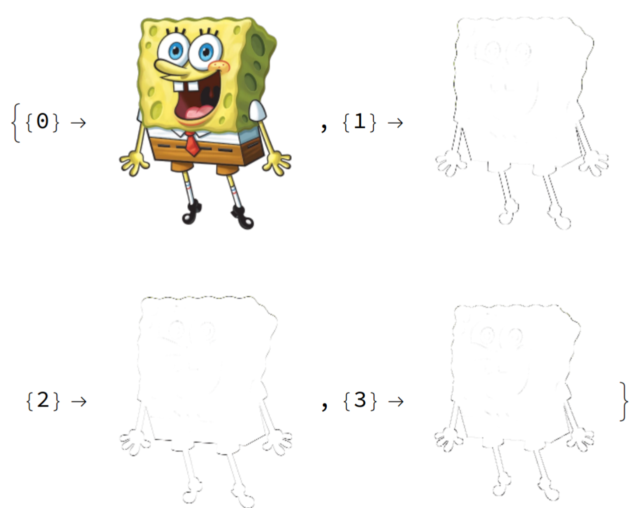


FIGURE 3. Compression with Haar Wavelet Basis

Figure 3 shows the four most significant coefficients when we apply image compression using Haar Wavelet Basis. The implementation of this compression algorithm can be found in [6].

REFERENCES

[1] Jordan Bell. "Haar wavelets and multiresolution analysis". 2014.
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[6] Eric J Stollnitz, AD DeRose, and David H Salesin. "Wavelets for computer graphics: a primer. 1". In: *IEEE Computer Graphics and Applications* 15.3 (1995), pp. 76–84.

Jason - Excellent start! I have a few structural suggestions above. Some of them will take space... but you're already @ the page limit. Anything else you can streamline? Better to do less, but do it better. You shouldn't have any barriers to moving to your final draft.

S +: The scope of this paper is spot on, with well-chosen topics, motivation, and details

C √+: Mathematics is presented clearly + correctly.

O √ : Ownership displayed in level of detail as well as examples. Maybe more of the latter?

W √+ : Careful statements, excellent notation. Could use some more overall structure (see pg 2)

C √+ : Citations are complete, but could use more explanation; see above.