INTRODUCTION TO WAVELETS IN IMAGE PROCESSING

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1. Introduction

Since the start of the 20th century, we have seen rapid development in the theory and applications of wavelets. As a mathematical tool, wavelets can be used to extract information from different kinds of data such as audio signals and images.

This paper will explore what wavelets are, introduce readers to the first orthonormal wavelet basis: the Haar system, approximate functions using Haar wavelets, and discuss the applications of wavelets in image compression.

2. Definitions

Definition 2.1. $L^2(\mathbb{R})$ *space* is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that the integral of f^2 over the whole real line is finite, taken with the norm

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Definition 2.2. A *wavelet* is a function $\Psi \in L^2(\mathbb{R})$ such that the set

$$\{\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

forms an orthonormal basis for $L^2(\mathbb{R})$. Sometimes Ψ is called the *mother wavelet*.

Observe that all $\Psi_{j,k}$ are generated by shifting and scaling[1] the wavelet Ψ and that each $\Psi_{j,k}$ is normalized so that $\|\Psi_{j,k}\|_2 = \|\Psi\|_2 = 1$ for all $j, k \in \mathbb{Z}$.

Now, let us explore one particular type of wavelet that has many applications in noise filtering and image processing: *The Haar Wavelet*. To get started, let $\chi_{[a,b)}$ denote the characteristic function[2] of the interval $[a, b)$.

Definition 2.3. The *Haar wavelet* is the function $\Psi = \chi_{[0,0.5)} - \chi_{[0.5,1)}$. The *Haar wavelet basis is the family* $\{\Psi_{j,k} : j, k \in \mathbb{Z}\}$.

See Figure 1 for some elements of the Haar wavelet basis, obtained by translating and/or stretching the Haar wavelet.

Now, our task is to prove that the Haar wavelet is in fact a wavelet, i.e. the Haar wavelet basis is an orthonormal basis for $L^2(\mathbb{R})$.

Date: 2019-01-21.

[1]The dilations here are taken to be powers of 2.

[2]The function defined to be identically one on $[a, b)$, and is zero elsewhere.

FIGURE 1. Elements of the Haar system: $\Psi, \Psi_{0,1}$ and $\Psi_{2,12}$

3. HAAR WAVELET BASIS

Lemma 3.1. *The Haar system is an orthonormal set in $L^2(\mathbb{R})$.*

Definition 3.2. Set $\varphi = \chi_{[0,1)}$. Define the sequence of partial sums to be:

$$H_n f(x) = \langle f, \varphi \rangle \varphi(x) + \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} \langle f, \Psi_{j,k} \rangle \Psi_{j,k}(x).$$

Lemma 3.3. *Let $f \in L^2(0,1)$. Then $H_n f$ converges to f in the L^2 norm. Consequently, the Haar system is an orthonormal basis for $L^2(0,1)$. Moreover, if f is continuous on $[0,1]$, then $H_n f$ converges uniformly to f.*

Theorem 3.4. *The Haar wavelet basis spans all of $L^($$\mathbb{R})$.*

4. APPROXIMATIONS USING HAAR WAVELETS

Theorem 4.1. *If $f \in C_0(\mathbb{R})$, then the series $\sum_{j,k \in \mathbb{Z}} \langle f, \Psi_{j,k} \rangle \Psi_{j,k}$ converges to f uniformly on $\mathbb{R}$.*

Example 4.2. Consider $f(x) = e^{-x} \sin 2\pi x$ for $x \in (0,1)$.
See Figure 2

FIGURE 2. Approximations of $f(x) = e^{-x} \sin 2\pi x$ using Haar wavelets

5. APPLICATION TO IMAGE PROCESSING

When compressing images, we want to discard the least significant details, keeping the original picture largely intact. Fortunately, wavelets can isolate and decompose a signal into low frequency part and high frequency part. Briefly discuss FBI Fingerprint Image Compression if there is space:

> Wavelet compression methods do not require dividing the image into smaller blocks because the desired localization properties are naturally built into the wavelet system.[2]

6. ANNOTATED BIBLIOGRAPHY

(1) Kenneth R. Davidson and Allan P. Donsig. *Real Analysis with Real Applications*. Prentice Hall, 2002. ISBN: 0-13-041647-9

This textbook is my primary source for proofs of orthonormal basis. Though terse, the book introduces the lemmas and proofs in a logical way, starting with proving the Haar system is orthonormal and then expand it to the basis for $L^2(\mathbb{R})$.

(2) Michael Frazier. *An introduction to wavelets through linear algebra.* Undergraduate texts in mathematics. Springer, 1999. ISBN: 978-0-387-98639-5

This textbook is very introductory, including a lot of examples and step-by-step proof for the properties of wavelets. Furthermore, it also has a nice description of the FBI Fingerprint Compression application using Haar Wavelets Analysis.

(3) Jonas Gomes and Luiz Velho. *From Fourier Analysis to Wavelets.* Springer International Publishing, 2015. ISBN: 978-3-319-22074-1. DOI: 10.1007/978-3-319-22075-8. URL: http://link.springer.com/10.1007/978-3-319-22075-8

Although this textbook is not very proof-heavy, it brings up a lot of cool wavelet examples that pertain to image compression, one of which being the Blur Derivative that I can talk about in my first draft.