1. Introduction

Since the start of the 20th century, we have seen rapid development in the theory and applications of wavelets. As a mathematical tool, wavelets can be used to extract information from different kinds of data such as audio signals and images.

This paper will explore what wavelets are; introduce readers to the first orthonormal wavelet basis, the Haar system; approximate functions using Haar wavelets; and discuss their applications in image compression.

2. Definitions

This section is based on paper [???].

Definition 2.1. $L^2(\mathbb{R})$ *space* is the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that the integral of f^2 over the whole real line is finite, taken with the norm

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Consider changing the notation to $\int_{\mathbb{R}}$ instead.

Definition 2.2. A set of functions $\{\ \varphi\}_{k=1}^{N}$ is said to be an *orthonormal basis* for the N-dimensional space V if:

(1) Every $u \in V$ can be written as $u = \sum_{k=1}^{N} c_k \varphi_k$ for some scalar c_k[1].
(2) $\langle \varphi_j, \varphi_k \rangle = 0$ if $j \neq k$.
(3) $\langle \varphi_j, \varphi_k \rangle = 1$ if $j = k$.

Definition 2.3. The *support* of a real-valued function f is the subset of the domain containing those elements which are are not mapped to zero.

Example: The *support* of the Haar wavelet is $[0, 1]$. Notation: $\operatorname{supp} \Psi = [0, 1]$.

Definition 2.4. A *wavelet* is a function $\Psi \in L^2(\mathbb{R})$ such that the set

$$\{\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

forms an orthonormal basis for $L^2(\mathbb{R})$. Sometimes Ψ is called the *mother wavelet*.

Observe that all $\Psi_{j,k}$ are generated by shifting and scaling[2] the wavelet Ψ and that each $\Psi_{j,k}$ is normalized so that $\|\Psi_{j,k}\|_2 = \|\Psi\|_2 = 1$ for all $j, k \in \mathbb{Z}$.

Now, let us explore one particular type of wavelet that has many applications in noise filtering and image processing: *The Haar Wavelet*.

> The child wavelets $\Psi_{j,k}$ are simply dilated and translated versions of the mother wavelet multiplied with the normalization factor 2j/2. The normalization factor is there so that the dilated and translated Haar function satisfies property (2) in the wavelet definition.

Definition 2.5. The *Haar wavelet* is the function $\Psi = \chi_{[0,0.5)} - \chi_{[0.5,1)}$, where χ is the characteristic function. The *Haar wavelet basis is the family* $\{\Psi_{j,k} : j, k \in \mathbb{Z}\}$.

See Figure 1 for examples of some elements of the Haar wavelet basis, obtained by translating and/or stretching the Haar wavelet.

Note that $\Psi_{j,k}$ has width of order 2^{-j}, and is centered about $k2^{-j}$. We can see that the Haar wavelet Ψ has the following properties:

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[1] We can easily prove that $c_k = \langle u, \varphi_k \rangle$, for $k = 1, 2, \cdots, N$.
[2] The dilations here are taken to be powers of 2.

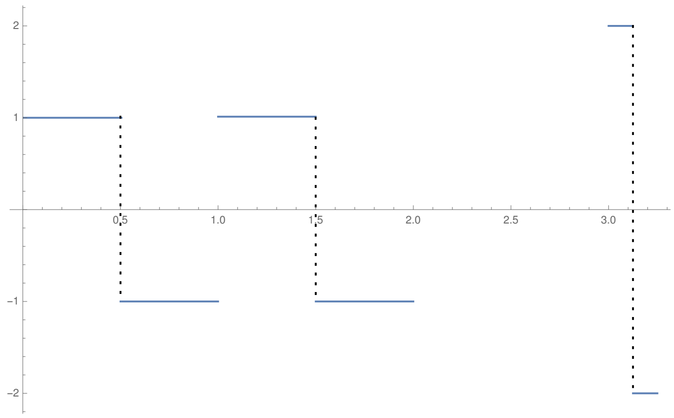

FIGURE 1. Elements of the Haar system: $\Psi, \Psi_{0,1}$ and $\Psi_{2,12}$

(1) $\int_{-\infty}^{\infty} \Psi(x)dx = 0$, which makes it a wave.
(2) $\int_{-\infty}^{\infty} \Psi(x)^2 dx = 1$, i.e. the norm is 1.

Now, our task is to prove that the Haar wavelet is in fact a wavelet, i.e. the Haar wavelet basis is an orthonormal basis for $L^2(\mathbb{R})$.

$$\begin{aligned}\mu(A+c) &= \mu\left(\bigcap_{n=1}^{\infty}(U_n + c)\right) \\ &= \mu\left(\left(\bigcap_{n=1}^{\infty} U_n\right) + c\right) \\ &= \mu\left(\bigcap_{n=1}^{\infty} U_n\right) \\ &= \mu(A).\end{aligned}$$