INTRODUCTION TO WAVELETS IN IMAGE PROCESSING

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1. Introduction

Since the start of the 20th century, we have seen rapid development in the theory and applications of wavelets. As a mathematical tool, wavelets can be used to extract information from different kinds of data such as audio signals and images. As an attempt to explore wavelets at an introductory level, this paper will examine the first wavelet developed, called the Haar Wavelet and discuss its applications in image processing.

First, recall that $L^2(\mathbb{R})$ is a vector space of square integrable functions, taken with the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)dx.$$

Definition 1.1. The *Haar function* is the function $\Psi = \chi_{[0,0.5)} - \chi_{[0.5,1)}$. The *Haar system* is the family

$$\{\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k), \qquad j, k \in \mathbb{Z}\}.$$

Note that each term in the Haar system is constructed by translating and/or dilating the original Ψ (see Figure 1). By this construction, since $\int_{\mathbb{R}} \Psi(x)dx = 0$, it follows that $\int_{\mathbb{R}} \Psi_{j,k}(x)dx = 0$ for $j, k \in \mathbb{Z}$. Furthermore, it is important to calculate the width and height for each $\Psi_{j,k}$. We have $\Psi(2^j - k) = 1$ when $0 \le 2^j - k < \frac{1}{2}$, or equivalently, $\frac{k}{2^j} \le x < \frac{k+\frac{1}{2}}{2^j}$. Therefore, in this interval, $\Psi_{j,k} = 2^j$. Similar calculations yield:

$$\Psi_{j,k} = \begin{cases} 2^{j/2} & \text{if } \frac{k}{2^j} \le x < \frac{k+\frac{1}{2}}{2^j}, \\ -2^{j/2} & \text{if } \frac{k+\frac{1}{2}}{2^j} \le x < \frac{k+1}{2^j}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Now, we will prove that the Haar function Ψ satisfies Definition 1.2 of an *orthonormal wavelet*.

Definition 1.2 (Orthonormal Wavelet, [4, p. 303])**.** If $\Psi \in L^2(\mathbb{R})$, $\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j x - k)$, and the set $\{\Psi_{j,k} : j, k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, then Ψ is called an *orthonormal wavelet*.

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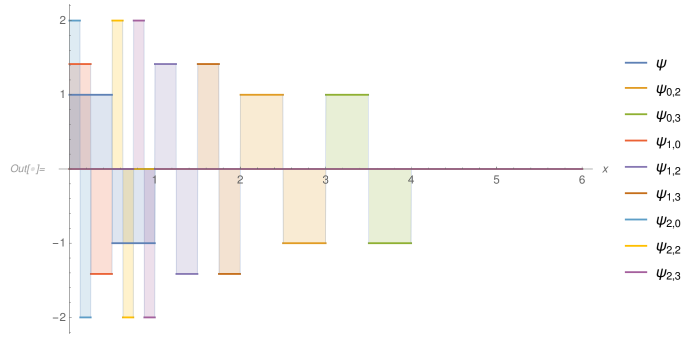


FIGURE 1. Elements of the Haar system

To prove the Haar system is an *orthonormal basis*, we have to show:

(1) The Haar system $\{\Psi_{j,k}\}$ is an orthonormal set, i.e. $\|\Psi_{j,k}\| = 1$ for all $j, k \in \mathbb{Z}$ and $\langle \Psi_{j,k}, \Psi_{j',k'} \rangle = 0$ for all $(j,k) \neq (j',k')$.
(2) The span of Haar system, denoted $\operatorname{span}\{\Psi_{j,k}\}$, is dense in $L^2(\mathbb{R})$.

In Section 2, we will first prove that the Haar system is orthonormal. Section 3 will show that it spans $L^2(\mathbb{R})$. Finally, in Section 4, we will discuss the local property of Haar wavelet and its application in image compression.

2. Orthonormality

Lemma 2.1 ([2, p. 409])**.** *The Haar system is an orthonormal set in $L^2(\mathbb{R})$.*

Proof. First, it is easy to see that $\|\Psi\| = 1$. Let us compute:

$$\begin{aligned}\|\Psi_{j,k}\|^2 = \int_{\mathbb{R}} \left|2^{j/2}\Psi(2^j x - k)\right|^2 dx &= \int_{\mathbb{R}} 2^j \left|\Psi(2^j x - k)\right|^2 dx \\ &= \int_{\mathbb{R}} |\Psi(y)|^2\, dy = \|\Psi\|^2,\end{aligned}$$

by change of variable $y = 2^j x - k$ in the integral. Therefore, all the elements in the Haar system has norm 1.

Now, we want to show the orthogonality. Consider $\Psi_{j,k}$ and $\Psi_{j,k'}$, for $k \neq k'$. Since these two elements have disjoint support[1], their inner product is 0. If $j < j'$, then either $\Psi_{j,k}$ and $\Psi_{j',k'}$ have disjoint supports (when $k \neq k'$), or supp $\Psi_{j',k'}$ is contained in an interval on which $\Psi_{j,k}$ is constant to (when $k = k'$). For the latter case, we have:

$$\langle \Psi_{j,k}, \Psi_{j',k'} \rangle = \int_{\mathbb{R}} 2^{j/2} \cdot \Psi_{j',k'} dx = 2^{j/2} \int_{\mathbb{R}} \Psi_{j',k'} dx = 0.$$

Hence, the set is orthonormal. □

[1]The *support* of a function, denoted supp, is the subset of the domain containing those elements which are not mapped to zero.

Remark 2.2. Haar function is a dyadic function, meaning that the dilations are taken to be powers of 2. Here, the factor $2^{j/2}$ is called the normalization factor, which is there so that the dilated and translated Haar function has norm 1. Although powers of 2 are a common choice, they are certainly not the most general one[2].

Now that we have proved the Haar system is orthonormal, in the following sections, we continue proving that the Haar function satisfies Definition 1.2 of *orthonormal wavelet*.

3. Haar Wavelet Basis for $L^2(\mathbb{R})$

3.1. **Integral Transform.** Motivated by [1, p. 3] and [2, p. 516], this section will set up the integral transform $P_n f$ below, which will act as the foundation to prove that any continuous function with compact support in $C_c(\mathbb{R})$, which is a dense subspace of $L^2(\mathbb{R})$[3], is in the span of $\{\Psi_{j,k}\}$.

Set $\phi(x) = \chi_{[0,1)}$. For $n \in \mathbb{Z}$, we define the integral kernel

$$K_n(x, y) = 2^n \sum_{k \in \mathbb{Z}} \phi(2^n x - k)\phi(2^n y - k).$$

Note that K_n is either equal to 2^n (when there is some $k \in \mathbb{Z}$ s.t. $2^n x - k$ and $2^n y - k$ lie in $[0, 1)$) or equal to 0 (otherwise). Equivalently, if $K_n = 2^n$, there is some k s.t.

$$x, y \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) = I_{n,k}.$$

Here, note that $K_n(x, y)$ is constant (equal to 2^n) on each dyadic interval of length 2^{-n}. Furthermore, for any given $n \in \mathbb{Z}$, the set $\{I_{n,k}\}$ is disjoint.

We define the integral transform

$$P_n f(x) = \int_{\mathbb{R}} K_n(x, y) f(y) dy.$$

If $x \in \mathbb{R}$ then there is a unique $k_x \in \mathbb{Z}$ with $x \in I_{n,k_x}$ and

$$P_n f(x) = 2^n \int_{I_{n,k_x}} f(y) dy. \tag{3.1}$$

Since I_{n,k_x} has length 2^{-n}, we can say that $P_n f$ is the average value of a function f on a dyadic interval.

Notice that the integral kernel K_n is dependent on the ϕ. In order to write any function in terms of the Haar system $\{\Psi_{j,k}\}$, we must eliminate ϕ somehow from our calculations. The following Lemma 3.1 will write the

[2]See examples of non-dyadic wavelets and a discussion why non-dyadic wavelets are more appropriate for statistical data analysis in [5].

[3]The proof for this claim can be found in [3, p. 326].

integral kernels solely in terms of the product of two functions from the Haar system, independent of the scaling function ϕ, a fact that proves to be useful in the subsequent sections.

Lemma 3.1 ([4, p. 293]). *If $n \in \mathbb{Z}$, then*

$$K_{n+1} - K_n = \sum_{k\in\mathbb{Z}} \Psi_{n,k}(x)\Psi_{n,k}(y), \qquad x, y \in \mathbb{R}.$$

Proof. First, let us calculate the right-hand side. By Equation 1.1, $\Psi_{n,k}(x) = 2^{n/2}$ when $x \in I_{n+1,2k} = \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right)$ and $\Psi_{n,k}(x) = -2^{n/2}$ when $x \in I_{n+1,2k+1}$. Therefore, it follows that:

$$\Psi_{n,k}(x)\Psi_{n,k}(y) = \begin{cases} 2^n & \text{if } x, y \in I_{n+1,2k} \text{ or } x, y \in I_{n+1,2k+1}, \\ -2^n & \text{if one is in } \ I_{n+1,2k} \text{ and the other is in } I_{n+1,2k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, before calculating $K_{n+1} - K_n$, note that:

$$I_{n+1,2k} \cup I_{n+1,2k+1} = \left[\frac{2k}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right) = I_{n,k}.$$

Therefore, if x, y is in either $I_{n+1,2k}$ or $I_{n+1,2k+1}$, then $x, y \in I_{n,k}$ and $K_n(x, y) = 2^n$. Hence, for this case, we have:

$$K_{n+1}(x, y) - K_n(x, y) = 2^{n+1} - 2^n = 2^n.$$

By similar argument, if either x or y is in $I_{n+1,2k}$ and the other is in $I_{n+1,2k+1}$, then

$$K_{n+1}(x, y) - K_n(x, y) = 0 - 2^n = -2^n.$$

Otherwise, everything is 0.

Hence, we conclude $K_{n+1} - K_n = \sum_{k\in\mathbb{Z}} \Psi_{n,k}(x)\Psi_{n,k}(y)$. □

Remark 3.2. From this Lemma, the difference between two integral transforms can be written as the *inner product expansion* of the function f:

$$\begin{aligned} (P_{n+1} - P_n)f(x) &= \int_{\mathbb{R}} K_{n+1}(x, y)f(y)dy - \int_{\mathbb{R}} K_n(x, y)f(y)dy \\ &= \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \Psi_{n,k}(x)\Psi_{n,k}(y)f(y)dy \\ &= \sum_{k\in\mathbb{Z}} \Psi_{n,k}(x) \int_{\mathbb{R}} \Psi_{n,k}(y)f(y)dy \\ &= \sum_{k\in\mathbb{Z}} \langle f, \Psi_{n,k}\rangle \Psi_{n,k}(x). \end{aligned} \tag{3.2}$$

The terms $\langle f, \Psi_{j,k}\rangle$ are called the the Haar coefficients of the expansion. One can easily notice that this inner product expansion is only one-sided, meaning we keep the dilation factor $j = n$ fixed and vary the translation factor k. In the following sections, we will abstract this expansion to a more general setting, varying both the translation and dilation factors.

3.2. **Two-sided Haar Representation.** From Equation 3.2, we can write

$$P_{n+1}f(x) - P_{-m}f(x) = \sum_{j=-m}^{n} \sum_{k\in\mathbb{Z}} \langle f, \Psi_{j,k}\rangle \Psi_{j,k}(x). \tag{3.3}$$

To prove the Haar inner product expansion on the right-hand side converges to a function f, we have to show that $P_{-m}f \to 0$ and $P_{n+1}f \to f$ as $m, n \to \infty$. We prove the two Lemmas in the same order below.

First, Lemma 3.3 tells us that the larger the intervals on which we average a continuous function, the smaller the supremum of the averaged function.

Lemma 3.3 (Average Function over Dyadic Intervals, [4, p. 295])**.** *If $f \in C_c(\mathbb{R})$, then $\|P_{-m}f\|_2 \to 0$ when $m \to \infty$.*

Proof. Suppose $f \in C_c(R)$ and $\text{supp}(f) \subseteq [-K, K]$. There eixsts $M > 0$ s.t. $\text{supp}(g) \subset [-2^M, 2^M]$.

Now, if $m > M$, then $\text{supp}(g) \subseteq I_{-m,-1} \cup I_{-m,0}$; Equation 3.1 holds only in these two intervals. We therefore compute:

$$\begin{aligned}
\|P_{-m}g\|_2^2 &= \int_{\mathbb{R}} |P_{-m}g(x)|^2\, dx \\
&= \sum_{k\in\mathbb{Z}} \int_{I_{-m,k}} |P_{-m}g(x)|^2 dx \\
&= \sum_{k\in\mathbb{Z}} \int_{I_{-m,k}} \left| 2^{-m} \int_{I_{-m,k}} g(y)dy \right|^2 dx \\
&= 2^m \cdot 2^{-2m} \sum_{k\in\mathbb{Z}} \left| \int_{I_{-m,k}} g(y)dy \right|^2 dx \\
&= 2^{-m} \left(\left| \int_{I_{-m,-1}} g(y)dy \right|^2 dx + \left| \int_{I_{-m,-1}} g(y)dy \right|^2 dx \right) \\
&= 2^{-m} \left(\left| \int_{-K}^{0} g(y)dy \right|^2 dx + \left| \int_{0}^{K} g(y)dy \right|^2 dx \right)
\end{aligned}$$

Note that $\int_{-K}^{0} g(y)dy = \langle g(y), \chi_{[-K,0)}\rangle$; therefore, by Cauchy-Schwarz inequality, we have:

$$\begin{aligned}\|P_{-m}g\|_2^2 &\leq 2^{-m}\left(K\|g\|_2^2 + K\|g\|_2^2\right) \qquad \text{(by)}\\ &= 2^{-m}\cdot 2K\|g\|_2^2\end{aligned}$$

As $m \to \infty$, the right-hand side converges to 0. Therefore, by Squeeze Theorem, it follows that $\|P_{-m}f\|_2 \to 0$. ☐

Lemma 3.4 (Convergence of Integral Transform, [2, p. 517])**.** *If $f \in C_c(\mathbb{R})$, then $P_n f \to f$ in the uniform nom as well as in the L^2 norm.*

Proof. Suppose $f \in C_c(\mathbb{R})$ and $\text{supp}(f) \subset [-2^M, 2^M]$ for $M \geq 0$. Since f is continuous on a compact set, it is uniformly continuous.

Given $\epsilon > 0$. By uniform continuity, there exists $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Now, choose N such that $2^{-N} < \delta$; if $n > N$, then we have:

$$\begin{aligned}|P_n f(x) - f(x)| &= \left|2^n \int_{I_{n,k_x}} f(y)dy - f(x)\right|\\ &= \left|2^n \int_{I_{n,k_x}} f(y)dy - 2^n \int_{I_{n,k_x}} f(x)dy\right|\\ &\leq 2^n \int_{I_{n,k_x}} |f(y) - f(x)|dy\\ &< 2^n \int_{I_{n,k_x}} \epsilon dy = \epsilon.\end{aligned}$$

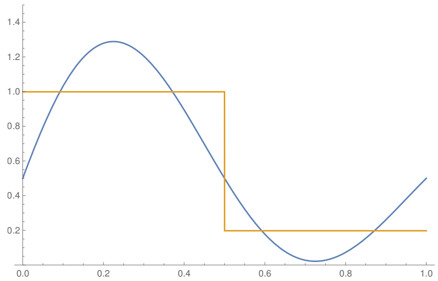
Therefore, $P_n f$ converges to f uniformly. Furthermore,

$$\begin{aligned}\|P_n f - f\|_2^2 &= \int_{\mathbb{R}} |P_n f(x) - f(x)|^2 dx\\ &= \int_{-2^M}^{2^M} |P_n f(x) - f(x)|^2 dx\\ &\leq \int_{-2^M}^{2^M} \|P_n f - f\|_\infty^2 dx = 2\cdot 2^M \|P_n - f\|_\infty^2.\end{aligned}$$

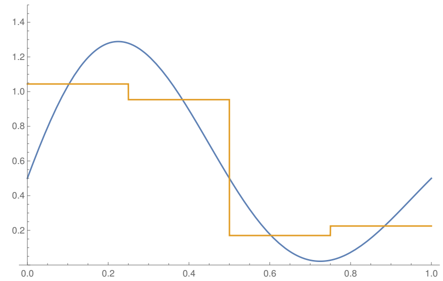
Since the right hand side converges to 0, by Limit Comparison Test, it follows that $\|P_n f - f\|_2^2 \to 0$ as $n \to \infty$.

Hence, $P_n \to f$ both in the uniform norm and in the L^2 norm. ☐

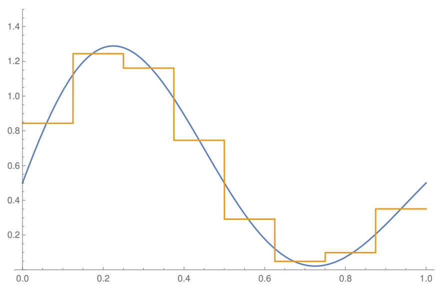
To illustrate this convergence better, see Figure 2 for the sequence $\{P_n f\}$ approaching $f(x) = e^{-x}\sin 2\pi x$ on the unit interval.



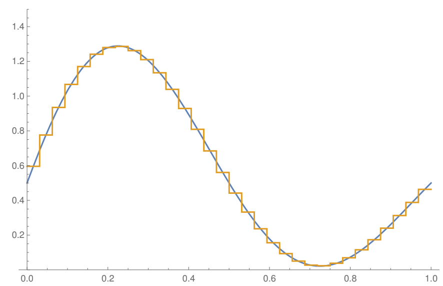
(A) $n = 1$



(B) $n = 2$



(C) $n = 3$



(D) $n = 5$

FIGURE 2. Approximation of $f(x) = e^{-x} \sin 2\pi x$ by $P_n f(x)$

3.3. **Span of Haar System.**

Theorem 3.5 ([2, p. 411])**.** *The Haar system spans all of $L^2(\mathbb{R})$.*

Proof. By Equation 3.3, Lemma 3.3 and Lemma 3.4, it follows that for any $f \in C_c(\mathbb{R})$, f is in the span of the Haar system:

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \langle f, \Psi_{j,k} \rangle \Psi_{j,k}.$$

Since $C_c(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$, it follows that the Haar system spans all of $L^2(\mathbb{R})$. $\square$

The Haar function Ψ is therefore an orthonormal wavelet, and we will now refer to it as the *Haar wavelet*.

4. Application to Image Compression

In this section, we discuss image compression as an application of the Haar wavelet basis. First, we define what we mean by compression. When compressing images, we want to discard the least significant details, keeping the original picture largely intact. Since an image is two-dimensional, it is helpful to think of image compression as discarding least significant details from the sets of data representing the pixels for the rows and and for the columns.

For instance, suppose we are given a function $f(x)$ expressed in terms of span of basis functions $\{u_i\}$:

$$f(x) = \sum_{i=0}^{M-1} c_i u_i(x).$$

The data set in this case consists of the coefficients c_i. We would like to find a function approximating $f(x)$ but requiring fewer coefficients, perhaps by using a different basis.

Because Haar wavelet basis is orthonormal, we can easily calculate the coefficients and sort them in decreasing magnitude. Furthermore, due to its highly localization (the support for each function is located in a narrow dyadic interval), we can discard small coefficients without losing much data.

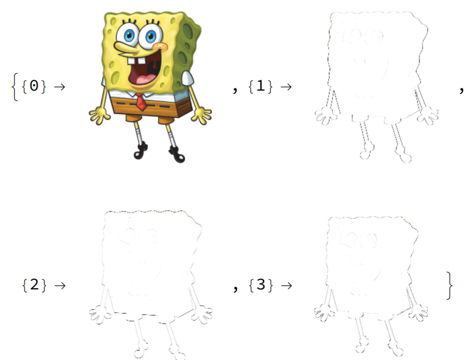


FIGURE 3. Compression with Haar Wavelet Basis

Figure 3 shows the most significant coefficients when we apply image compression using Haar Wavelet Basis. The implementation of this compression algorithms can be found in [6].

References

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