MODIFICATION OF THE LIPSCHITZ UNIQUENESS THEOREM WHEN $f(x_0, y_0) \neq 0$

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1. Introduction

This paper presents a modification of the traditional Lipschitz Uniqueness Theorem for a general initial value problem of an ordinary scalar differential equation. To state the general initial value problem, we consider an arbitrary neighborhood $\mathcal{N}$ of a point $(x_0, y_0) \in \mathbb{R}^2$, a mapping function $f : \mathcal{N} \to \mathbb{R}$, and the scalar initial value problem (1)

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{1}$$

An example of an initial value problem, which motivates this paper, is

$$y' = \cos(x) + x\sqrt[3]{y}, \quad y(0) = 0. \tag{2}$$

We define a solution to (1) to be a continuously differentiable function Y: $I \to \mathbb{R}$ for some interval I such that $x_0 \in I$, $Y(x_0) = y_0$, and for all $x \in I$, $(x, Y(x)) \in \mathcal{N}$ and $Y'(x) = f(x, Y(x))$. A unique solution to (1) exists if there exists an $\alpha > 0$ with the property that (1) has one solution defined on $[x_0 - \alpha, a_0 + \alpha]$ and any other solution to (1) defined on $I \to \mathbb{R}$ coincides with it on $I \cup [x_0 - \alpha, x_0 + \alpha]$ [Cid].

The traditional Lipschitz Uniqueness Theorem states that given the hypotheses of (1), if f satisfies the Lipschitz condition (to be defined below) with respect to the second variable on $\mathcal{N}$, then (1) has a unique solution. It is well-established that if f satisfies the Lipschitz condition with the respect to the first variable only, we are not guaranteed a unique solution to (1), as will be shown through an example in section 3. However, this paper will prove that if we know that for our initial condition $y(x_0) = y_0$, if $f(x_0, y_0) \neq 0$, then f being Lipschitz with respect to the first variable is enough to guarantee a unique solution to (1). In particular, we will show that despite (2) not being Lipschitz with respect to its second variable, and thus failing the hypotheses for the Traditional Lipschitz Uniqueness Theorem, (2) has a unique solution. The uniqueness of that solution can be proven to due our Modified Lipschitz Uniqueness Theorem and a corollary that follows from it.

2. Traditional Accounts of Uniqueness

As summarized in the introduction, we present the traditional Lipschitz Uniqueness Theorem as followed, as provided by and proved in [Chi].

For the purpose of this paper, f satisfying the Lipschitz condition with respect to the first variable will be defined as

$$\exists L > 0 \text{ such that if } (s, y), (x, y) \in \mathcal{N}, \text{ then } |f(s, y) - f(x, y)| \leq L|x - s|$$

for the given hypotheses of (1). The Lipschitz condition in the second variable is defined similarly.

Theorem 2.1 (Lipschitz's Uniqueness Theorem)**.** [Chi]
Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$. If f satisfies a Lipschitz condition with respect to the second variable on N, then (1) *has a unique solution.*

While this theorem proves that there exists a unique solution to many initial value problems, we claim that it does not show that there exists a unique solution to our motivating example (2) since (2) is not Lipschitz with respect to y, and thus fails the hypotheses for Lipschitz Uniqueness Theorem.

Claim 2.2. (2) *is not Lipschitz with respect to its second variable y.*

Proof. Proof by contradiction. Suppose that initial value problem (2) is Lipschitz with respect to its second variable y.

$$y' = \cos(x) + x\sqrt[3]{y}, \qquad y(0) = 0 \tag{2}$$

Then, there exists a Lipschitz constant L such that

$$\frac{|f(x, y) - f(x, z)|}{|y - z|} \leq L.$$

We can test this equality by using sequences y_n, z_n such that $y_n \to 0$ and $z_n \to 0$ as $n \to \infty$ where x is treated as a constant. Define $y_n = \frac{1}{n}$ and $z_n = -\frac{1}{n}$. Then, since we are assuming L is the Lipschitz constant, it should be the case that

$$\frac{|f(x, y_n) - f(x, z_n)|}{|y_n - z_n|} \leq L$$

However, we see that

$$\begin{aligned}
\frac{|f(x, y_n) - f(x, z_n)|}{|y_n - z_n|} &= \frac{\cos(x) + x\sqrt[3]{\frac{1}{n}} - \cos(x) - x\sqrt[3]{\frac{-1}{n}}}{\frac{1}{n} - \frac{-1}{n}} \\
&= \frac{x\sqrt[3]{\frac{1}{n}} - x\sqrt[3]{\frac{-1}{n}}}{\frac{2}{n}} \\
&= xn^{\frac{2}{3}} \leq L
\end{aligned}$$

Thus, this requires that our Lipschitz constant must be greater than or equal to a $xn^{\frac{2}{3}}$ for all $n \in \mathbb{N}$. As $n \to \infty$, $xn^{\frac{2}{3}} \to \infty$. Thus, $xn^{\frac{2}{3}}$ cannot be bound by a constant. Thus, we have reached a contradiction. Therefore, (2) is not Lipschitz with respect to its second variable y. □

Thus, (2) fails the hypotheses for the Traditional Lipschitz Uniqueness Theorem. However, we will show that (2) does have a unique solution by using a corollary of the Modified Lipschitz Uniqueness Theorem.

3. Modified Lipschitz Uniqueness Theorem

With the knowledge that (2) does not satisfy the criterion to show it has a unique solution via the traditional Lipschitz Uniqueness Theorem, we are now motivated to prove the Modified Lipschitz Uniqueness Theorem whose corollary will ultimately prove its solution's uniqueness.

Theorem 3.1 (Modified Lipschitz Uniqueness Theorem)**.** [Cid] *Let $\mathcal{N}$ be a neighborhood of a point (x_0, y_0) and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$. If $f(x_0, y_0) \neq 0$ and, moreover, f satisfies a Lipschitz condition with respect to the first variable on $\mathcal{N}$, then* (1) *has a unique solution.*

Notice that the hypothesis that $f(x_0, y_0) \neq 0$ is necessary since the problem

$$y' = 3y^{\frac{3}{2}}, \qquad y(0) = 0 \tag{3}$$

is Lipschitz with respect to the first variable x, but does not have a unique solution. In particular, we can see that $Y_1(x) = 0$ and $Y_2(x) = x^3$ are both solutions to (3) [Cid].

In order to prove the Modified Lipschitz Uniqueness Theorem, we will state and prove the following lemma.

Lemma 3.2. [Cid] *Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on N. If $f(x_0, y_0) \neq 0$, then* (1) *has a unique solution if and only if the problem*

$$x' = \frac{1}{f(x, y)}, \qquad x(y_0) = x_0 \tag{4}$$

has a unique solution.

This lemma states that if $f(x_0, y_0) \neq 0$, then (1) has a unique solution if and only if (4) does. Thus, instead of proving the Modified Lipschitz Uniqueness Theorem directly, this allows us to prove it by showing that if $f(x_0, y_0) \neq 0$ and (1) is Lipschitz with the first variable, then (4) has a unique solution. This is particularly convenient because (4) switches the order of the variables in (1) and thus the first variable of (1) is the second variable of (4). Thus, since the Traditional Lipschitz Uniqueness Theorem allows us to conclude that if (4) is Lipschitz with respect to its second variable, then (4) has a unique solution, this lemma will ultimately lead us to the conclusion that (1) has a unique solution.

Proof. The proof of this lemma will consist of first defining a neighborhood around the initial point (x_0, y_0) and the behavior of f and $\frac{1}{f}$ in it, then proving a claim that guarantees the existence of a solution to (4) if (1) has a solution and vice versa, and lastly proving a claim that uniqueness of a solution is preserved between (4) and (1).

Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$, let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$, and suppose $f(x_0, y_0) \neq 0$. Then, there exists a neighborhood around (x_0, y_0) where f has a constant sign, where either f is strictly positive or negative, and is bounded. Thus, $\frac{1}{f}$ has a constant sign as well. We can always take a compact neighborhood within any neighborhood around (x_0, y_0) where $\frac{1}{f}$ is bounded. Thus, we will assume for simplicity that f and $\frac{1}{f}$ are bounded on our neighborhood $\mathcal{N}$.

Claim 3.3. *If Y is a solution to* (1)*, then Y^{-1} is a solution to (*4*). Conversely, if X is a solution to* (4)*, then X^{-1} is a solution to* (1)*.*

To prove the first part of Claim 3.3, let $Y : I \to \mathbb{R}$ be a solution to (1). Thus, for all $x \in I$, $(x, Y(x)) \in \mathcal{N}$ and $Y'(x) = f(x, Y(x))$. Since f has a constant sign on $\mathcal{N}$, $Y'(x)$ has a constant sign on I. Now, we want to show that $X = Y^{-1}$ satisfies $X'(y) = \frac{1}{f(X(y),y)}$ and $X(y_0) = x_0$. Since Y' has a constant sign on $\mathcal{N}$, Y is strictly monotone. As proved in [Kir], that means that Y on $\mathcal{N}$ is a 1-1 function, which implies that Y^{-1} exists. By design of Y^{-1}, $Y(I)$ contains y_0 and $X(y_0) = Y^{-1}(y_0) = x_0$.

Now we can use the theorem of differentiation of inverse functions, which states that for an invertible function f, if f is differentiable at $f^{-1}(x)$ and $f'(f^{-1}(x))$ is not equal to zero, then f^{-1} is differentiable and $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$, which is proved in [Kir]. We can apply this theorem for all $y \in Y(I)$ to obtain the result we want:

$$X'(y) = (Y^{-1})'(y) = \frac{1}{Y'(Y^{-1}(y))}$$

Thus, by the definition of a solution to (1), $Y'(x) = f(x, Y(x))$, $Y'(Y^{-1}(x)) = f(Y^{-1}(y), YY^{-1}(y)) = f(Y^{-1}(y), y) = f(X(y), y)$. Therefore, $X'(y) = \frac{1}{f(X(y),y)}$, which satisfies the other part of the definition of a solution to (4). Thus, when Y solves (1), Y^{-1} solves (4).

The other direction of this claim is similar. We suppose that $X : I \to \mathbb{R}$ be a solution to (4), which implies that for all $y \in I$, for all $(X(y), y) \in \mathcal{N}$ and $X'(y) = \frac{1}{f(X(y),y)}$. Using the same process as above, we are able to conclude that this assumption implies that $Y(x_0) = y_0$ and $Y'(x) = f(x, Y(x))$. Thus, when X solves (4), X^{-1} solves (1). Therefore, we have proven the claim that if Y is a solution to (1), then Y^{-1} is a solution to (4) and conversely, if X is a solution to (4), then X^{-1} is a solution to (1).

Now we are ready to prove the uniqueness part of our lemma.

Claim 3.4. *If Y is the unique solution to* (1)*, then Y^{-1} is the unique solution to (4). Conversely, if X is the unique solution to* (4)*, then X^{-1} is the unique solution to* (1)*.*

To prove this claim and thus prove the lemma, suppose that Y is a unique solution to (1) on the interval $I = [x_0 - \alpha, x_0 + \alpha]$ for some $\alpha > 0$. We want to prove that Y^{-1} is the unique solution to (4) on the interval $J = [y_0 - \beta, y_0 + \beta]$ for a $\beta > 0$ small enough that $J \subset Y(I)$. Additionally, we want to prove that that if X is a solution to (4) defined on some interval $\tilde{J} \subset J$, then $X(\tilde{J}) \subset I$. We know there exists a small enough β such that $J \subset Y(I)$ since $\frac{1}{f}$ is bounded in $\mathcal{N}$. Let X be a solution to (4) on an interval $\tilde{J} \subset J$. Thus, by the claim above, X^{-1} is a solution to (1) on $X(\tilde{J})$ and $X(\tilde{J}) \subset I$. But since we supposed that Y is a unique solution to (1) on I and X^{-1} is a solution to (1) on a subset of I, it must be the case that $X^{-1} = Y$ on $X(\tilde{J})$. Therefore, $X = Y^{-1}$ on $\tilde{J}$. Thus, since Y is a unique solution to (1), by the claim above, Y^{-1} is a unique solution to (4).

As before, the other direction is similar. Thus, we can conclude that for the given hypotheses, if $f(x_0, y_0) \neq 0$, then (1) has a unique solution if and only if the problem (4) has a unique solution.

□

With that lemma proved, we can easily prove the Modified Lipschitz Uniqueness Theorem as done below.

Proof. In this proof, we show that $f(x_0, y_0)$ not equaling 0 and f being Lipschitz with respect to its first variable x implies (4) has a unique solution, which implies that (1) has a unique solution by Lemma 3.2. They key idea in this proof is to recognize that since the second variable of (4) is x, we can apply the Traditional Lipschitz Uniqueness Theorem to (4) to conclude that (4), and thus (1), has a unique solution.

Let $\mathcal{N}$ be a neighborhood of a point (x_0, y_0) and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$. Suppose $f(x_0, y_0) \neq 0$ and f satisfies a Lipschitz condition with respect to the first variable on $\mathcal{N}$, i.e., there exists an $L > 0$ such that $(s, y), (x, y) \in N$ implies $|f(s, y) - f(x, y)| \leq L|s - x|$. Since $f(x_0, y_0) \neq 0$, we can apply Lemma 3.2. Thus, it suffices to prove that (4) has a unique solution. Since f is continuous on $\mathcal{N}$ around (x_0, y_0), we know that $|f| \geq \frac{|f(x_0,y_0)|}{2}$. Define this as $r = \frac{|f(x_0,y_0)|}{2}$ where thus $r > 0$ since $f(x_0, y_0) \neq 0$. Assume that this holds on our neighborhood $\mathcal{N}$ around (x_0, y_0). Now, we will apply the Traditional Lipschitz Uniqueness Theorem. Thus, we want to show that f^{-1} is Lipschitz with respect to its second variable, which is x. Notice that for $(x, y), (s, y) \in \mathcal{N}$, we have

$$\left| \frac{1}{f(x, y)} - \frac{1}{f(s, y)} \right| = \left| \frac{f(s, y) - f(x, y)}{f(x, y) f(s, y)} \right|$$

Since f is Lipschitz with respect to the first variable x, $|f(x,y) - f(s,y)| \leq L|s-x|$ for some $L > 0$. Additionally, given $(x,y),(s,y) \in \mathcal{N}$, we know that $f(x,y)f(s,y) \geq r^2$. Thus

$$\left|\frac{1}{f(x,y)} - \frac{1}{f(s,y)}\right| = \left|\frac{f(s,y) - f(x,y)}{f(x,y)f(s,y)}\right| \leq \frac{L}{r^2}|s-x|$$

The Traditional Lipschitz Uniqueness Theorem applied to (4) would require that $\left|\frac{1}{f(x,y)} - \frac{1}{f(s,y)}\right| \leq T|s-x|$ for some constant $T > 0$ since in (4), the dependent or second variable is x. Thus, since $\frac{L}{r^2}$ is just another positive constant, we know that (4) has a unique solution. Therefore, by Lemma 3.2, (1) has a unique solution as well. □

4. Showing (2) Has a Unique Solution

Now that we have proven that the Modified Lipschitz Uniqueness Theorem holds, we can work to apply it to the initial value problem (2). In order to do that, however, we must prove the following corollary of the Modified Lipschitz Uniqueness Theorem.

Corollary 4.1. [Cid] *Let $\mathcal{N}$ be a neighborhood of a point $(x_0, y_0) \in \mathbb{R}^2$ and let $f : \mathcal{N} \to \mathbb{R}$ be continuous on $\mathcal{N}$. If $f(x_0, y_0) \neq 0$ and $\frac{\partial f}{\partial x}$ is continuous on $\mathcal{N}$, then* (1) *has a unique solution.*

Proof. Let $\mathcal{N}$' be a compact neighborhood around (x_0, y_0) such that $\mathcal{N}' \subset \mathcal{N}$. Let $L > 0$ be an upper bound of $\frac{\partial f}{\partial x}$ on $\mathcal{N}'$. For $(x,y),(s,y) \in \mathcal{N}'$ such that $x \neq s$, the mean value theorem guarantees that there exists a $r \in (x,s)$ such that

$$\left|\frac{\partial f}{\partial x}(r,y)\right| = \frac{|f(s,y) - f(x,y)|}{|x-s|}$$

Therefore,

$$|f(s,y) - f(x,y)| = \left|\frac{\partial f}{\partial x}(r,y)\right| |x-s| \leq L|x-s|$$

Thus, by the Modified Lipschitz Uniqueness Theorem, (1) has a unique solution. □

Finally, Corollary 4.1 allows us to prove (4) below has a unique solution despite it not being Lipschitz with respect to y, thus failing the necessary criteria for the Traditional Uniqueness Theorem. Recall $(2) y' = \cos(x) + x\sqrt[3]{y},\ y(0) = 0$. By Corollary 4.1, in order order to show that (2) is continuous, we need to show that $\frac{\partial f}{\partial x} = -\sin(x) + \sqrt[3]{y}$ is continuous on $\mathcal{N}$, some small neighborhood around (0,0). This can be done by applying laws of continuous functions as found in [Hof]. $g(x,y) = -\sin(x)$ and $h(x,y) = \sqrt[3]{y}$ are continuous functions. Since the sum of continuous functions in any Euclidean space, which of course includes $\mathbb{R}^2$, is continuous, $\frac{\partial f}{\partial x} = -\sin(x) + \sqrt[3]{y}$ is continuous. Therefore, by Corollary 4.1, we have shown that (2) has a unique solution.

References

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