COSSERAT MODEL FOR A CIRCULAR HELIX

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1. Helix formula

Given the upright helix $r(t) = (R\cos t, R\sin t, ct)$, where R is the radius of the helix and $2\pi c$ is a constant giving the vertical separation of the helix's loops. If s is the arc-length parameter of the helix, then we have:

$$s(t) = \int_0^t |r'(t)| d\tau = \int_0^t \sqrt{R^2 + c^2} d\tau = \sqrt{R^2 + c^2} t.$$

Therefore, we can reparametrize the center-line of the helix with respect to the arc-length parameter s as follows:

$$r(s) = \left(R\cos \frac{s}{\sqrt{R^2 + c^2}}, R\sin \frac{s}{\sqrt{R^2 + c^2}}, \frac{cs}{\sqrt{R^2 + c^2}} \right). \tag{1}$$

2. Directors

Here, we assume inextensibility, which requires d_3 to be tangent to the center-line; therefore, we have:

$$d_3(s) = \left(-\frac{R\sin\left(\frac{s}{\sqrt{c^2+R^2}}\right)}{\sqrt{c^2 + R^2}}, \frac{R\cos\left(\frac{s}{\sqrt{c^2+R^2}}\right)}{\sqrt{c^2 + R^2}}, \frac{c}{\sqrt{c^2 + R^2}} \right). \tag{2}$$

Now, since $\{d_1, d_2\}$ spans the plane orthogonal to the center-line, we can first choose d_1 to be a vector on the plane, then compute $d_2 = d_3 \times d_1$:

$$d_1(s) = \left(-\cos \frac{s}{\sqrt{R^2 + c^2}}, -\sin \frac{s}{\sqrt{R^2 + c^2}}, 0 \right) \tag{3}$$

$$d_2(s) = \left(\frac{c\sin\left(\frac{s}{\sqrt{c^2+R^2}}\right)}{\sqrt{c^2 + R^2}}, -\frac{c\cos\left(\frac{s}{\sqrt{c^2+R^2}}\right)}{\sqrt{c^2 + R^2}}, \frac{R}{\sqrt{c^2 + R^2}} \right). \tag{4}$$

3. Euler parameters q_i

From the master thesis, we know that the directors and Euler parameters at a point s are related through the equations

$$d_1 = \frac{1}{\| \boldsymbol{q} \|^2} \begin{pmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 \\ 2(q_1 q_2 + q_3 q_4) \\ 2(q_1 q_3 - q_2 q_4) \end{pmatrix} \tag{5}$$

$$d_2 = \frac{1}{\| \boldsymbol{q} \|^2} \begin{pmatrix} 2(q_1 q_2 - q_3 q_4) \\ -q_1^2 + q_2^2 - q_3^2 + q_4^2 \\ 2(q_1 q_4 + q_2 q_3) \end{pmatrix} \tag{6}$$

$$d_3 = \frac{1}{\| \boldsymbol{q} \|^2} \begin{pmatrix} 2(q_1 q_3 + q_2 q_4) \\ 2(-q_1 q_4 + q_2 q_3) \\ -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{pmatrix}. \tag{7}$$

With the additional requirement of $\| \boldsymbol{q} \|^2 = 1$, we can compute $\boldsymbol{q}(s)$:

$$q_1 = -\frac{R \sin\left(\frac{s}{\sqrt{c^2+R^2}}\right)}{2\sqrt{c^2+R^2}\sqrt{\frac{\left(\sqrt{c^2+R^2}+c\right)\left(\cos\left(\frac{s}{\sqrt{c^2+R^2}}\right)+1\right)}{\sqrt{c^2+R^2}}}},$$

$$q_2 = \frac{1}{2}\sqrt{\frac{\left(\sqrt{c^2+R^2}-c\right)\left(\cos\left(\frac{s}{\sqrt{c^2+R^2}}\right)+1\right)}{\sqrt{c^2+R^2}}},$$

$$q_3 = \frac{1}{2}\sqrt{\frac{\left(\sqrt{c^2+R^2}+c\right)\left(\cos\left(\frac{s}{\sqrt{c^2+R^2}}\right)+1\right)}{\sqrt{c^2+R^2}}},$$

$$q_4 = -\frac{R \sin\left(\frac{s}{\sqrt{c^2+R^2}}\right)}{2\sqrt{c^2+R^2}\sqrt{\frac{\left(\sqrt{c^2+R^2}-c\right)\left(\cos\left(\frac{s}{\sqrt{c^2+R^2}}\right)+1\right)}{\sqrt{c^2+R^2}}}}.$$

4. Strain parameters u_i

Given the directors, we can compute the strain parameters as follows:

$$u_1(s) = -d'3(s) \cdot d_2(s) = 0$$

$$u_2(s) = d'3(s) \cdot d_1(s) = \frac{R}{c^2+R^2}$$

$$u_3(s) = d'1(s) \cdot d_2(s) = \frac{c}{c^2+R^2}.$$