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Approximation of SDEs
A stochastic sewing approach

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Contents

1 Introduction

In this report, we are going to present three main results:

- Stochastic sewing lemma by Khoa Lê in [3], which we will present in Section 2.
- Weak rate of convergence for the Euler - Maruyama scheme with Hölder drifts by Teodor Holland in [2].
- Strong rate of convergence for the Euler - Maruyama scheme with Hölder drifts by Oleg Butkovsky, Konstantinos Dareiotis and Máté Gerenscér in [1].

Stochastic sewing lemma is a stochastic version of Gubinelli 's sewing lemma. Moreover, the essential elements of it have deep connections with the foundations of stochastic analysis and thus, it can offer a lot of potential applications in studying stochastic differential equations. In this report, we will only present the statement of this sewing lemma without proofs and show how the authors utilize it to prove some important bounds in their papers in Section 3 and Section 4.

For Section 3 and Section 4, we are about to present the weak and strong convergence rates of the Euler - Maruyama scheme for the SDEs with bounded and α-Hölder continuous drifts driven by multiplicative Brownian motion noise. In earlier results, it is well known that in our settings, the weak rate and strong rate are both $\alpha/2$. But we will see in our report that Section 3 and 4 offer more optimal rates which are $(1+\alpha)/2$ for the weak rate and $1/2$ for the strong rate.

2 Stochastic sewing lemma

In this section, we are going to introduce the Stochastic sewing lemma which was developed by Lê in [3]. This lemma will be useful in the proofs of some important bounds in Section 3 and Section 4.

For $0 \le S \le T \le 1$, we define the set $[S,T]_\le := \{(s,t) : S \le s \le t \le T\}$. Moreover, for a function $A_{\cdot,\cdot}$ defined on $[S,T]_\le$ and for times $s \le u \le t$, we set $\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}$.

Lê 's Stochastic sewing lemma is stated as follows

Lemma 2.1 (Stochastic sewing lemma, Theorem 2.3 in [3]). *Let $p \ge 2$, $0 \le S \le T \le 1$ and let $A_{\cdot,\cdot}$ be a function $[S,T]_\le \to L_p(\Omega, \mathbb{R}^d)$ such that for any $(s,t) \in [S,T]_\le$, the random vector $A_{s,t}$ is $\mathcal{F}_t$-measurable. Suppose that for some $\varepsilon_1, \varepsilon_2 > 0$ and C_1, C_2, the bounds*

(S1) $\|A_{s,t}\|_{L_p(\Omega)} \le C_1 |t-s|^{1/2+\varepsilon_1}$

(S2) $\|\mathbb{E}^s \delta A_{s,u,t}\|_{L_p(\Omega)} \le C_2 |t-s|^{1+\varepsilon_2}$

hold for all $S \le s \le u \le t \le T$. Then there exists a unique map $\mathcal{A} : [S,T] \to L_p(\Omega, \mathbb{R}^d)$ such that $\mathcal{A}_S = 0$, $\mathcal{A}_t$ is $\mathcal{F}_t$-measurable for all $t \in [S,T]$ and the following bounds hold for some constants $K_1, K_2 > 0$:

(S3) $\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L_p(\Omega)} \le K_1 |t-s|^{1/2+\varepsilon_1} \quad \forall (s,t) \in [S,T]_\le$

(S4) $\|\mathbb{E}^s(\mathcal{A}_t - \mathcal{A}_s - A_{s,t})\|_{L_p(\Omega)} \le K_2 |t-s|^{1+\varepsilon_2} \quad \forall (s,t) \in [S,T]_\le$

Moreover, there exists a constant K depending only of $\varepsilon_1, \varepsilon_2, d, p$ such that $\mathcal{A}$ satisfies the bound

$$\|\mathcal{A}_t - \mathcal{A}_s\|_{L_p(\Omega)} \le K C_1 |t-s|^{1/2+\varepsilon_1} + K C_2 |t-s|^{1+\varepsilon_2} \quad \forall (s,t) \in [S,T]_\le$$

3 Weak rate of convergence for the Euler - Maruyama scheme with Hölder drifts

3.1 Some preliminaries and notations

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a d-dimensional Brownian motion $(W_t)_{t\in[0,1]}$ for arbitrary $d \in \mathbb{Z}^+$. For times $t \in [0,1]$, we will consider the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0 \tag{3.1}$$

where the drift and diffusion coefficients satisfy the following conditions:

- $b : \mathbb{R}^d \to \mathbb{R}^d$ is bounded and α-Hölder continuous for some $\alpha \in (0,1)$.
- $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is bounded with bounded and Lipschitz-continuous first partial derivatives. Furthermore, σ satisfies the uniform ellipticity condition that $\sigma\sigma^T \geq \lambda I$ in the sense of positive definite matrices for some constant $\lambda > 0$.

We will moreover consider the Euler-Maruyama scheme with identical initial condition

$$dX^n_t = b(X^n_{\kappa_n(t)})dt + \sigma(X^n_{\kappa_n(t)})dW_t, \quad X^n_0 = x_0 \tag{3.2}$$

where $\kappa_n(t) := \frac{\lfloor nt \rfloor}{n}$ and $\lfloor \cdot \rfloor$ denotes integer part.

In the remainder of this subsection, we will introduce some standard notations that will be used throughout the report.

3.1.1 Hölder spaces

Let $A \subseteq \mathbb{R}^d$ and $(B, |\cdot|)$ be a normed space. For $\alpha \in (0,1]$ and $f : A \to B$, we define the α-Hölder seminorm of f by

$$[f]_{C^\alpha(A,B)} := \sup_{\substack{x,y\in A \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

For $\alpha \in (0, \infty)$, we then denote by $C^\alpha(A,B)$ the space of all functions such that for all $l \in (\mathbb{Z}^+)^d$ multiindices with $|l| < \alpha$, the derivative $\partial^l f$ exists and

$$\|f\|_{C^\alpha(A,B)} := \sum_{|l|<\alpha} \sup_{x\in A} |\partial^l f(x)| + \sum_{\alpha-1\leq|l|<\alpha} [\partial^l f]_{C^{\alpha-|l|}(A,B)} < \infty$$

We furthermore define $C^0(A,B)$ to be the space of all measurable functions $f : A \to B$ such that

$$\|f\|_{C^0(A,B)} := \sup_{x\in A} |f(x)| < \infty$$

The definition of Hölder spaces extends to negative α as follows: for $\alpha \in (-\infty, 0)$, we say that the distribution f is of class $C^\alpha(\mathbb{R}^d, \mathbb{R})$ if

$$\|f\|_{C^\alpha(\mathbb{R}^d,\mathbb{R})} := \sup_{\varepsilon \in (0,1]} \varepsilon^{-\alpha/2} \|\mathcal{P}_\varepsilon f\|_{C^0(\mathbb{R}^d,\mathbb{R})} < \infty$$

where $\mathcal{P}_\varepsilon f := p_\varepsilon * f$ denotes convolution with the standard heat kernel (3.6).
When no ambiguity can arise, we will simply write $C^\alpha(A)$ or C^α to mean $C^\alpha(A, B)$.

Remark. Using Hölder spaces, the regularity assumptions on our coefficients can be stated simply as follows: $b \in C^\alpha$ for some $\alpha \in (0, 1)$ and $\sigma \in C^2$.

3.1.2 Shorthands

Consider the filtration generated by W, that is $\mathcal{F}_t^W := \sigma(W_s : s \leq t)$ and its augmentation $\mathcal{F}_t := \sigma(\mathcal{F}_t^W \cup \mathcal{N})$ by the collection of null sets $\mathcal{N} := \{N \subseteq \Omega : N \subseteq F \text{ for some } F \subseteq \mathcal{F}_\infty^W \text{ such that } \mathbb{P}(F) = 0\}$. For $t \in [0, 1]$, we will write $\mathbb{E}^t(\cdot)$ as a shorthand for the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t)$.
Moreover, in proofs, we will write $f \lesssim g$ to mean that $f \leq Ng$ for some N constant where the dependence of N is specified in the statement we are proving.

3.2 Main result

The main result of this section is as follows

Theorem 3.1 (Theorem 2.1 in [2]). *Let $\alpha \in (0, 1)$, $g \in C^\alpha$ and fix $\varepsilon > 0$. Let X be the solution of (3.1) and X^n be given by (3.2). Then for all $n \in \mathbb{N}$, we have*

$$|\mathbb{E}g(X_1) - \mathbb{E}g(X_1^n)| \leq N n^{-\frac{1+\alpha}{2}+\varepsilon}$$

where N is a constant depending only on $d, \varepsilon, \alpha, \lambda, \|b\|_{C^\alpha}, \|\sigma\|_{C^2}, \|g\|_{C^\alpha}$.

3.3 On the method of the proof

The proof of the main result uses the idea of using the Feynmann-Kac framework to express the weak error as a comparison between the initial- and terminal-time values of the solution of a parabolic PDE evaluated at the approximation process in space. Using this method, the problem is reduced to having to prove quadrature estimates. Showing that the desired quadrature estimates hold requires the use of Stochastic sewing lemma, which we have presented in Section 2.

3.4 Some useful estimates

In this subsection, we will introduce some estimates that will be used throughout the proof of the main result. We will begin by recalling a well-known estimate on parabolic PDEs stated below.
Consider the parabolic PDE

$$\begin{cases} \partial_t u(t,x) + Lu(t,x) = 0 & \forall (t,x) \in [0,1) \times \mathbb{R}^d \\ u(1,x) = g(x) & \forall x \in \mathbb{R}^d \end{cases} \tag{3.3}$$

where the differential operator L is the infinitesimal generator of the solution X of (3.1), that is, for smooth $\phi : \mathbb{R}^d \to \mathbb{R}$, it is defined as

$$L\phi(x) := \frac{1}{2}\sum_{i,j=1}^{d} (\sigma(x)\sigma^T(x))_{ij}\partial_{x_i x_j}\phi(x) + \sum_{i=1}^{d} b_i(x)\partial_{x_i}\phi(x)$$

For the proof of the main result we will need estimates on the solution of (3.3). These can be obtained by using a Schauder-type estimate on the parabolic PDE corresponding to the differential operator $\mathscr{L}$ that is defined on smooth functions $\phi : \mathbb{R}^d \to \mathbb{R}$ by

$$\mathscr{L}\phi(x) := \sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i x_j}\phi(x) + \sum_{i=1}^{d} b_i(x)\partial_{x_i}\phi(x) + c(x)\phi(x) \tag{3.4}$$

where the coefficients satisfy the following assumption:

Assumption 3.2.

(i) There exist constants $K > 0$ and $\alpha \in (0,1)$ such that for all $i, j \in \{1, \dots, d\}$, we have $\|a_{ij}\|_{C^\alpha}, \|b_j\|_{C^\alpha}, \|c\|_{C^\alpha} < K$.

(ii) For $A(x) = (a_{ij}(x))_{1\le i,j\le d}$, there exists a positive constant μ such that $\langle A(x)\xi, \xi\rangle \ge \mu|\xi|^2$ for all $x, \xi \in \mathbb{R}^d$.

Let $\mathcal{C}^{1,2}$ denote the class of bounded continuous functions $u : [0,1] \times \mathbb{R}^d \to \mathbb{R}$ such that $\partial_t u, \nabla u, \nabla^2 u$ exist and are continuous on $(0,1) \times \mathbb{R}^d$. The following lemma can be found as Proposition 6.4.1 and Theorem 6.4.3 in [5].

Lemma 3.3 (A Schauder-type estimate for parabolic PDEs)**.** *Suppose that the operator (3.4) satisfies Assumption 3.2. For each bounded and continuous initial condition g, the Cauchy problem*

$$\begin{cases} \partial_t v(t,x) = \mathscr{L}v(t,x) & \forall (t,x) \in (0,1] \times \mathbb{R}^d \\ v(0,x) = g(x) & \forall x \in \mathbb{R}^d \end{cases}$$

admits a unique solution v in the class $\mathcal{C}^{1,2}$.
Moreover, for each $0 \leq \beta \leq \gamma \leq 2+\alpha$, there exists two positive constants N_1 and N_2 depending only on $\beta, \gamma, \alpha, d, K, \lambda$ such that

$$\|v(t,\cdot)\|_{C^\gamma(\mathbb{R}^d)} \leq N_1 t^{-\frac{\gamma-\beta}{2}} e^{N_2 t} \|g\|_{C^\beta(\mathbb{R}^d)}$$

The above lemma can be directly applied to the time reversal of the solution of (3.3) to obtain the following estimate.

Corollary 3.4 (A Schauder-type estimate for u)**.** *The parabolic PDE (3.3) admits a unique solution u in the class $\mathcal{C}^{1,2}$. Moreover, for each $0 \leq \beta \leq \gamma \leq 2+\alpha$, there exists two positive constants N_1 and N_2 depending only on $\|b\|_{C^\alpha}, \|\sigma\|_{C^\alpha}, \alpha, d, \lambda$ such that for all $t \in (0,1)$, we have*

$$\|u(t,\cdot)\|_{C^\gamma(\mathbb{R}^d)} \leq N_1 (1-t)^{-\frac{\gamma-\beta}{2}} e^{N_2(1-t)} \|g\|_{C^\beta(\mathbb{R}^d)}$$

Proof. Let $u(1-t,x) = v(t,x)$, then parabolic PDE (3.3) becomes

$$\begin{cases} \partial_t v(t,x) = Lv(t,x) & \forall (t,x) \in (0,1] \times \mathbb{R}^d \\ v(0,x) = g(x) & \forall x \in \mathbb{R}^d \end{cases}$$

It suffices to check that the operator L satisfies Assumption 3.2.
The fact that b satisfies (i) and $\sigma\sigma^T$ satisfies (ii) of Assumption 3.2 is obvious because of our conditions on the drift and diffusion coefficients of (3.1).
Now we are about to prove that for all $i,j \in \{1,\dots,d\}$, $\|(\sigma\sigma^T)_{ij}\|_{C^\alpha}$ is bounded. Indeed, denote $\sigma(x) := (a_{ij}(x))_{1\leq i,j\leq d}$ and we have

$$\begin{aligned}
\|(\sigma\sigma^T)_{ij}\|_{C^\alpha} &= \sup_{x\in\mathbb{R}^d} |(\sigma\sigma^T)_{ij}(x)| + \sup_{\substack{x,y\in\mathbb{R}^d \\ x\neq y}} \frac{|(\sigma\sigma^T)_{ij}(x) - (\sigma\sigma^T)_{ij}(y)|}{|x-y|^\alpha} \\
&= \sup_{x\in\mathbb{R}^d} \left| \sum_{k=1}^d (a_{ik}a_{jk})(x) \right| + \sup_{\substack{x,y\in\mathbb{R}^d \\ x\neq y}} \frac{\left| \sum_{k=1}^d (a_{ik}a_{jk})(x) - \sum_{k=1}^d (a_{ik}a_{jk})(y) \right|}{|x-y|^\alpha} \\
&\leq \sum_{k=1}^d \sup_{x\in\mathbb{R}^d} |(a_{ik}a_{jk})(x)| + \sum_{k=1}^d \sup_{\substack{x,y\in\mathbb{R}^d \\ x\neq y}} \frac{|(a_{ik}a_{jk})(x) - (a_{ik}a_{jk})(y)|}{|x-y|^\alpha}
\end{aligned}$$

For all $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
\frac{|(a_{ik}a_{jk})(x) - (a_{ik}a_{jk})(y)|}{|x-y|^\alpha} &\leq \frac{|a_{ik}(x)a_{jk}(x) - a_{ik}(x)a_{jk}(y)| + |a_{ik}(x)a_{jk}(y) - a_{ik}(y)a_{jk}(y)|}{|x-y|^\alpha} \\
&\leq \|\sigma\|_{C^0} \left(\frac{|a_{jk}(x) - a_{jk}(y)|}{|x-y|^\alpha} + \frac{|a_{ik}(x) - a_{ik}(y)|}{|x-y|^\alpha} \right) \\
&\leq \|\sigma\|_{C^0} (\|\sigma\|_{C^\alpha} + \|\sigma\|_{C^\alpha})
\end{aligned}$$

Therefore, $\sup_{\substack{x,y\in\mathbb{R}^d\\ x\neq y}} \frac{|(a_{ik}a_{jk})(x)-(a_{ik}a_{jk})(y)|}{|x-y|^\alpha}$ is bounded for all $k \in \{1,\ldots,d\}$. Combining this with the fact that $\sup_{x\in\mathbb{R}^d} |(a_{ik}a_{jk})(x)|$ is bounded by $\|\sigma\|_{C^0}$ for all $k \in \{1,\ldots,d\}$, we can obtain that for all $i,j \in \{1,\ldots,d\}$, $\|(\sigma\sigma^T)_{ij}\|_{C^\alpha}$ is bounded. $\square$

The following lemma (which has been proven via Malliavin-calculus in [1]) will be used in order to exploit the regularising property of convolution with the density of the Euler-Maruyama scheme. Consider the driftless scheme

$$\bar{X}_t^n = \sigma(\bar{X}_{\kappa_n(t)}^n)dW_t, \quad \bar{X}_0^n = y \in \mathbb{R}^d \tag{3.5}$$

Lemma 3.5. *Let $\sigma \in C^2$ with $\sigma\sigma^T \geq \lambda I$ for some constant $\lambda > 0$, let $\bar{X}^n$ be given by (3.5) and let $G \in C^1$. Then for all $t = 1/n, 2/n, \ldots, 1$ and $k = 1,\ldots,d$, ones have the bound*

$$|\mathbb{E}\partial_{x_k} G(\bar{X}_t^n)| \leq N\|G\|_{C^0}t^{-1/2} + N\|G\|_{C^1}e^{-cn}$$

with some constants $N = N(d,\lambda,\|\sigma\|_{C^2})$ and $c = c(d,\|\sigma\|_{C^2}) > 0$.

The last lemma of this section is Lemma 2.4 from [4].

Lemma 3.6. *Let $\beta \in \mathbb{R}\setminus\mathbb{Z}$. There exists a constant N depending on β such that for all $f \in C^\beta(\mathbb{R}^d)$,*

$$\|(1-\Delta)^{-1}f\|_{C^{\beta+2}(\mathbb{R}^d)} \leq N\|f\|_{C^\beta(\mathbb{R}^d)}$$

3.5 Quadrature estimates

We will begin by introducing some notations. Let u be the solution of parabolic PDE (3.3). Fix some $i \in \{1,\ldots,d\}$ and for convenience, let us denote

$$f_t := \partial_{x_i} u(t,\cdot)$$

Let furthermore $p_t : \mathbb{R}^d \to \mathbb{R}$ denote the standard heat kernel given by

$$p_t(x) := \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right) \tag{3.6}$$

for $x \in \mathbb{R}^d$ and $t > 0$ and by $p_0(x)$ we denote the Dirac-delta. More generally, for a positive semi-definite matrix $M \in \mathbb{R}^{d\times d}$, we denote by $p_M : \mathbb{R}^d \to \mathbb{R}$ the Gaussian density given by

$$p_M(x) := \frac{1}{(\det(2\pi M))^{1/2}} \exp\left(-\frac{1}{2}x^T M^{-1}x\right)$$

which coincides with p_t whenever $M = tI$ where I denotes the identity matrix.
Now we are going to present the main results and their proofs of this subsection.

Lemma 3.7 (Lemma 6.1 in [2]). *Let $p \geq 1$, $\varepsilon > 0$ and fix $T < 1$. Let $\bar{X}^n$ be given by (3.5). For all $n \in \mathbb{N}$ and $(s,t) \in [0,T]_{\leq}$, we have*

$$\left\| \int_s^t (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n)) f_r(\bar{X}_r^n)\, dr \right\|_{L_p} \leq N n^{-\frac{1+\alpha}{2}+\varepsilon} \Big(|1-T|^{-1/2} |t-s|^{1/2+\varepsilon} + |1-T|^{-1} |t-s|^{1+\varepsilon} \Big)$$

for some constant N depending only on $d, \alpha, p, \varepsilon, \lambda, \|b\|_{C^\alpha}, \|\sigma\|_{C^2}, \|g\|_{C^0}$.

Proof. This result is proved by using Lemma 2.1. Notice that by Hölder 's inequality, it suffices to show the bound for $p \geq 2$.
For $0 \leq s \leq t \leq T < 1$, let

$$A_{s,t} := \mathbb{E}^s \int_s^t (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n)) f_r(\bar{X}_s^n) dr$$

It can be checked that condition (S1) holds. Note that

$$\begin{aligned} \|A_{s,t}\|_{L_p} &= \left\| \int_s^t f_r(\bar{X}_s^n) \mathbb{E}^s (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n))\, dr \right\|_{L_p} \\ &\lesssim \int_s^t \|f_r\|_{C^0} \|\mathbb{E}^s (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n))\|_{L_p}\, dr \\ &\lesssim \int_s^t \|u(r,\cdot)\|_{C^1} \|\mathbb{E}^s (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n))\|_{L_p}\, dr \\ &\lesssim |1-T|^{-1/2} \|g\|_{C^0} \int_s^t \|\mathbb{E}^s (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n))\|_{L_p}\, dr \end{aligned} \tag{3.7}$$

Here Corollary 3.4 and the fact that $\|\partial_{x_i} u(t,\cdot)\|_{C^0} \leq \|u(t,\cdot)\|_{C^1}$ are used. We will prove the latter in Lemma 5.1 of Appendix 5. Next, the integrand will be bounded as follows.
Suppose that $\kappa_n(s) = \frac{k}{n}$ and consider the case when $r \geq \frac{k+4}{n}$. Then we have $\kappa_n(r) \geq s$ and thus,

$$\begin{aligned} \mathbb{E}^s (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n)) &= \mathbb{E}^s \mathbb{E}^{\kappa_n(r)} (b_i(\bar{X}_{\kappa_n(r)}^n + \sigma(\bar{X}_{\kappa_n(r)}^n)(W_r - W_{\kappa_n(r)})) - b_i(\bar{X}_{\kappa_n(r)}^n)) \\ &= \mathbb{E}^s \left(\mathbb{E} \left(b_i(x + \sigma(x)(W_r - W_{\kappa_n(r)})) - b_i(x) \right) \Big|_{x = \bar{X}_{\kappa_n(r)}^n} \right) \end{aligned}$$

Hence, defining $\tilde{g} : \mathbb{R}^d \to \mathbb{R}$ by

$$\tilde{g}(x) = \tilde{g}_r^n(x) := \int_{\mathbb{R}^d} (b_i(x+y) - b_i(x)) p_{\sigma\sigma^T(x)\delta}(y)\, dy$$

with $\delta := r - \kappa_n(r) \leq n^{-1}$ and using that the restriction of $\bar{X}^n$ to the gridpoints $\frac{1}{n}, \frac{2}{n}, \ldots, 1$ is a Markov process, it can be shown that

$$\begin{aligned}
\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)})) &= \mathbb{E}^s\tilde{g}(\bar{X}^n_{\kappa_n(r)}(y)) \\
&= \mathbb{E}^s\mathbb{E}^{\frac{k+1}{n}}\tilde{g}\left(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}+\frac{k+1}{n}}(y)\right) \\
&= \mathbb{E}^s\left(\mathbb{E}\tilde{g}\left(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x)\right)\Big|_{x=\bar{X}^n_{\frac{k+1}{n}}(y)}\right)
\end{aligned}$$

and thus,

$$|\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))| \le \sup_{x\in\mathbb{R}} |\mathbb{E}\tilde{g}(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x))| \tag{3.8}$$

We have

$$\|\tilde{g}\|_{C^{\alpha/2}} \lesssim \|\tilde{g}\|_{C^\alpha} \lesssim \|b\|_{C^\alpha} \tag{3.9}$$

where the last inequality is followed by Lemma 5.4 in Appendix 5.
By Lemma 3.6, for the solution $\tilde{u}$ of $(1-\Delta)\tilde{u} = \tilde{g}$, we have

$$\|\tilde{u}\|_{C^{2+\alpha/2}} \lesssim \|\tilde{g}\|_{C^{\alpha/2}} \qquad \text{and} \qquad \|\tilde{u}\|_{C^{1+2\varepsilon}} \lesssim \|\tilde{g}\|_{C^{-1+2\varepsilon}} \tag{3.10}$$

and thus from (3.8), we have

$$\begin{aligned}
\|\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))\|_{L_p} &\le \sup_{x\in\mathbb{R}^d} |\mathbb{E}(\tilde{u} - \Delta\tilde{u})(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x))| \\
&\le \sup_{x\in\mathbb{R}^d} |\mathbb{E}\tilde{u}(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x))| + \sup_{x\in\mathbb{R}^d} |\mathbb{E}\Delta\tilde{u}(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x))| \qquad (3.11)
\end{aligned}$$

We can see that

$$\sup_{x\in\mathbb{R}^d} |\mathbb{E}\tilde{u}(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x))| \le \|\tilde{u}\|_{C^0} \lesssim \|\tilde{u}\|_{C^1} \,|\kappa_n(r) - (k+1)/n|^{-1/2} \tag{3.12}$$

and using Lemma 3.5, we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} |\mathbb{E}\Delta\tilde{u}(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x))| &= \sup_{x\in\mathbb{R}^d} \left|\mathbb{E}\sum_{i=1}^d \partial_{x_ix_i}\tilde{u}(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x))\right| \\
&\le \sum_{i=1}^d \sup_{x\in\mathbb{R}^d} \left|\mathbb{E}\partial_{x_ix_i}\tilde{u}(\bar{X}_{\kappa_n(r)-\frac{k+1}{n}}(x))\right| \\
&\lesssim \sum_{i=1}^d \left(\|\partial_{x_i}\tilde{u}\|_{C^0}|\kappa_n(r) - (k+1)/n|^{-1/2} + \|\partial_{x_i}\tilde{u}\|_{C^1}e^{-cn}\right) \\
&\lesssim \|\tilde{u}\|_{C^1}|\kappa_n(r) - (k+1)/n|^{-1/2} + \|\tilde{u}\|_{C^2}e^{-cn} \qquad (3.13)
\end{aligned}$$

Here in the last inequality we use Lemma 5.1 of Appendix 5.

Using (3.10), (3.11), (3.12) and (3.13) yields

$$\begin{aligned}\|\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))\|_{L_p} &\lesssim \|\tilde{u}\|_{C^1}|\kappa_n(r) - (k+1)/n|^{-1/2} + \|\tilde{u}\|_{C^2}e^{-cn} \\ &\lesssim \|\tilde{u}\|_{C^{1+2\varepsilon}}|\kappa_n(r) - (k+1)/n|^{-1/2} + \|\tilde{u}\|_{C^{2+\alpha/2}}e^{-cn} \\ &\lesssim \|\tilde{g}\|_{C^{-1+2\varepsilon}}|\kappa_n(r) - (k+1)/n|^{-1/2} + \|\tilde{g}\|_{C^{\alpha/2}}e^{-cn} \qquad (3.14)\end{aligned}$$

We now need to bound $\|\tilde{g}\|_{C^{-1+2\varepsilon}}$. For convenience, we will use the notation $M(x,z) := \sigma\sigma^T(x-z)$. It can be seen that for $\gamma \in (0,1]$,

$$\begin{aligned}\mathcal{P}_\gamma\tilde{g}(x) &= \int_{\mathbb{R}^d} p_\gamma(z)\tilde{g}(x-z)\,dz \\ &= \int_{\mathbb{R}^d} p_\gamma(z)\int_{\mathbb{R}^d}(b_i(x-z+y) - b_i(x-z))\,p_{M(x,z)\delta}(y)\,dy\,dz \\ &= -\sum_{j=1}^d\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\gamma(z)p_{M(x,z)\delta}(y)\int_0^1 y_j\partial_{z_j}b_i(x-z+\theta y)\,d\theta\,dy\,dz \\ &= \sum_{j=1}^d\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{z_j}\left(p_\gamma(z)p_{M(x,z)\delta}(y)\right)y_j\int_0^1 b_i(x-z+\theta y)\,d\theta\,dy\,dz \\ &= \sum_{j=1}^d\left(I_1^j + I_2^j\right)\end{aligned}$$

Here in order to obtain the fourth line from the third line, integration by parts and the fact that the heat kernel $p_\gamma(z)$ vanishes exponentially as $|z|$ goes to infinity are used. Moreover, I_1^j and I_2^j are denoted as follows:

$$I_1^j := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{z_j}\left(p_\gamma(z)\right)p_{M(x,z)\delta}(y)y_j\int_0^1 b_i(x-z+\theta y)\,d\theta\,dy\,dz$$

and

$$I_2^j := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_\gamma(z)\partial_{z_j}\left(p_{M(x,z)\delta}(y)\right)y_j\int_0^1 b_i(x-z+\theta y)\,d\theta\,dy\,dz$$

As when bounding these, the j-dependence will not play a role, we will drop the superscript j and write I_1 and I_2 respectively. First, we need to bound I_1. Note that if we remove the y-dependency of b_i in I_1, then the integral would be zero, that is

$$\begin{aligned}\tilde{I}_1 &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{z_j}\left(p_\gamma(z)\right)p_{M(x,z)\delta}(y)y_j\int_0^1 b_i(x-z)\,d\theta\,dy\,dz \\ &= \int_{\mathbb{R}^d}\partial_{z_j}(p_\gamma(z))b_i(x-z)\left(\int_{\mathbb{R}^d}y_jp_{M(x,z)\delta}(y)\,dy\right)dz \\ &= 0\end{aligned}$$

Hence, I_1 can be bounded as follows

$$
\begin{aligned}
|I_1| &= |I_1 - \tilde{I}_1| \\
&= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{z_j} \left(p_\gamma(z)\right) p_{M(x,z)\delta}(y) y_j \int_0^1 \left(b_i(x - z + \theta y) - b_i(x-z)\right) d\theta \, dy \, dz \right| \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\partial_{z_j}(p_\gamma(z))| p_{M(x,z)\delta}(y) |y_j| \|b\|_{C^\alpha} |y|^\alpha \, dy \, dz \\
&\lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|z|}{\gamma} p_\gamma(z) p_{M(x,z)\delta}(y) |y|^{1+\alpha} \, dy \, dz \\
&\lesssim \frac{1}{\gamma} n^{-\frac{1+\alpha}{2}} \gamma^{1/2} \\
&\lesssim \gamma^{-1/2} n^{-\frac{1+\alpha}{2}}
\end{aligned}
$$

Here the fact that $|\partial_{z_j} p_\gamma(z)| \lesssim |z| \gamma^{-1} p_\gamma(z)$ and Lemma 5.2 of Appendix 5 are used.
In order to bound I_2, note that

$$
\begin{aligned}
&\partial_{z_j}(p_{M(x,z)\delta}(y)) \\
&= \partial_{z_j} \left(\det(2\pi M(x,z)\delta)^{-1/2} \exp\left(-\frac{1}{2} y^T (M(x,z)\delta)^{-1} y \right) \right) \\
&= -\frac{1}{2} \det(2\pi M(x,z)\delta)^{-3/2} \partial_{z_j} (\det(2\pi M(x,z)\delta)) \exp\left(-\frac{1}{2} y^T (M(x,z)\delta)^{-1} y \right) \\
&\quad + \det(2\pi M(x,z)\delta)^{-1/2} \exp\left(-\frac{1}{2} y^T (M(x,z)\delta)^{-1} y \right) \partial_{z_j} \left(-\frac{1}{2} y^T (M(x,z)\delta)^{-1} y \right) \\
&= -\frac{1}{2} \left(\frac{\partial_{z_j}(\det(M(x,z)))}{\det(M(x,z))} + \frac{\partial_{z_j}(y^T M(x,z)^{-1} y)}{\delta} \right) p_{M(x,z)\delta}(y)
\end{aligned}
\tag{3.15}
$$

We will prove in Lemma 5.3 of Appendix 5 that the following bounds hold

$$
\left| \frac{\partial_{z_j}(\det(M(x,z)))}{\det(M(x,z))} \right| \lesssim 1, \qquad \left| \frac{\partial_{z_j}(y^T M(x,z)^{-1} y)}{\delta} \right| \lesssim \delta^{-1} |y|^2 \tag{3.16}
$$

Using (3.15), I_2 can be decomposed as follows

$$
I_2 = I_{2,1} + I_{2,2}
$$

for

$$
I_{2,1} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_\gamma(z) \left(-\frac{1}{2} \frac{\partial_{z_j}(\det(M(x,z)))}{\det(M(x,z))} p_{M(x,z)\delta}(y) \right) y_j \int_0^1 b_i(x - z + \theta y) \, d\theta \, dy \, dz
$$

and

$$
I_{2,2} := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_\gamma(z) \left(-\frac{1}{2} \frac{\partial_{z_j}(y^T M(x,z)^{-1} y)}{\delta} p_{M(x,z)\delta}(y) \right) y_j \int_0^1 b_i(x - z + \theta y) \, d\theta \, dy \, dz
$$

The expression $I_{2,1}$ is easier to bound. By the same reasoning as the authors use for I_1, ones can see that if we removed the y-dependence of b_i in $I_{2,1}$, then the integral would be zero. Hence, using (3.16) gives

$$\begin{aligned}|I_{2,1}| &= \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\gamma(z)\left(-\frac{1}{2}\frac{\partial_{z_j}(\det(M(x,z)))}{\det(M(x,z))}p_{M(x,z)\delta}(y)\right) y_j \\ &\qquad\qquad \int_0^1 (b_i(x-z+\theta y) - b_i(x-z))\,d\theta\,dy\,dz\Big| \\ &\lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\gamma(z)p_{M(x,z)\delta}(y)\|b\|_{C^\alpha}|y|^\alpha|y_j|\,dy\,dz \\ &\lesssim n^{-\frac{1+\alpha}{2}}\end{aligned}$$

Finally, we need to bound $I_{2,2}$. It is less straightforward to see that if we removed the y-dependency of b_i in $I_{2,2}$, the integral would be zero. To show this, define

$$\begin{aligned}\tilde{I}_{2,2} &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\gamma(z)\left(-\frac{1}{2}\frac{\partial_{z_j}(y^T M(x,z)^{-1}y)}{\delta}p_{M(x,z)\delta}(y)\right) y_j \int_0^1 b_i(x-z)\,d\theta\,dy\,dz \\ &= \int_{\mathbb{R}^d} -\frac{1}{2}p_\gamma(z)b_i(x-z)\frac{1}{\delta}\left(\int_{\mathbb{R}^d} p_{M(x,z)\delta}(y)y_j\partial_{z_j}(y^T M(x,z)^{-1}y)\,dy\right)\,dz\end{aligned}$$

It can be shown that in this expression the integral with respect to y is always zero. To this end, note that

$$\begin{aligned}y_j\partial_{z_j}(y^T M(x,z)^{-1}y) &= y_j\partial_{z_j}\sum_{k=1}^d y_k(M(x,z)^{-1}y)_k \\ &= y_j\partial_{z_j}\sum_{k=1}^d y_k\sum_{l=1}^d (M(x,z)^{-1})_{kl}y_l \\ &= \sum_{k,l=1}^d y_jy_ky_l\partial_{z_j}(M(x,z)^{-1})_{kl}\end{aligned}$$

and therefore,

$$\int_{\mathbb{R}}^d y_j\partial_{z_j}(y^T M(x,z)^{-1}y)p_{M(x,z)\delta}(y)\,dy = \sum_{l,k=1}^d \partial_{z_j}(M(x,z)^{-1})_{kl}\int_{\mathbb{R}^d} y_jy_ky_lp_{M(x,z)\delta}(y)\,dy$$

To see that the integral in the right hand side expression is zero, note that

$$\int_{\mathbb{R}^d} y_jy_ky_lp_{M(x,z)\delta}dy = \mathbb{E}\left(Y_jY_kY_l\right)$$

with $Y \sim \mathcal{N}(0, M(x,z)\delta)$. Since Y is a Gaussian random vector with mean zero, so is

$$\tilde{Y} := (Y_j, Y_k, Y_l)$$

hence, $-\tilde{Y}$ has the same distribution as $\tilde{Y}$. Thus with $q : \mathbb{R}^3 \to \mathbb{R}$, $q(x_1, x_2, x_3) = x_1x_2x_3$, we have

$$\mathbb{E}\left(Y_jY_kY_l\right) = \mathbb{E}q(\tilde{Y}) = \mathbb{E}q(-\tilde{Y}) = \mathbb{E}\left((-Y_j)(-Y_k)(-Y_l)\right) = -\mathbb{E}q(\tilde{Y})$$

So since $\mathbb{E}q(\tilde{Y}) = -\mathbb{E}q(\tilde{Y})$, we have

$$\mathbb{E}\left(Y_jY_kY_l\right) = \mathbb{E}q(\tilde{Y}) = 0$$

We have shown that $\tilde{I}_{2,2} = 0$. Thus,

$$\begin{aligned}
|I_{2,2}| &= |I_{2,2} - \tilde{I}_{2,2}| \\
&= \left| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\gamma(z)\left(-\frac{1}{2}\frac{\partial_{z_j}(y^TM(x,z)^{-1}y)}{\delta}p_{M(x,z)\delta}(y)\right) y_j \int_0^1 (b_i(x-z+\theta y) - b_i(x-z))\,d\theta\,dy\,dz\right| \\
&\lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\gamma(z)\left|\frac{\partial_{z_j}(y^TM(x,z)^{-1}y)}{\delta}\right| p_{M(x,z)\delta}(y)|y_j|\|b\|_{C^\alpha}|y|^\alpha\,dy\,dz \\
&\lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\gamma(z)\delta^{-1}|y|^2 p_{M(x,z)\delta}(y)|y|^{1+\alpha}\,dy\,dz \\
&\lesssim \delta^{-1}\int_{\mathbb{R}^d} p_\gamma(z)\left(\int_{\mathbb{R}^d}|y|^{3+\alpha}p_{M(x,z)\delta}(y)\,dy\right)dz \\
&\lesssim \delta^{-1}\delta^{\frac{3+\alpha}{2}} \\
&\lesssim \delta^{\frac{1+\alpha}{2}} \\
&\lesssim n^{-\frac{1+\alpha}{2}}
\end{aligned}$$

where (3.16) is again used.
From the bounds on $|I_1|, |I_{2,1}|, |I_{2,2}|$, it can be concluded that

$$\|\mathcal{P}_\gamma\tilde{g}\|_{C^0} \lesssim \gamma^{-1/2}n^{-\frac{1+\alpha}{2}}$$

Note that we also have the trivial estimate

$$\|\mathcal{P}_\gamma\tilde{g}\|_{C^0} \lesssim \|\tilde{g}\|_{C^0} \lesssim \|b\|_{C^0}$$

Thanks to these two estimates, for $\beta = -1 + 2\varepsilon$, the following holds

$$\begin{aligned}
\|\tilde{g}\|_{C^\beta} &= \sup_{\gamma\in(0,1]}\gamma^{-\beta/2}\|\mathcal{P}_\gamma\tilde{g}\|_{C^0} \\
&\lesssim \sup_{\gamma\in(0,1]}\gamma^{-\beta/2}((\gamma^{-1/2}n^{-\frac{1+\alpha}{2}}) \wedge \|b\|_{C^0}) \\
&\lesssim \sup_{\gamma\in(0,1]}\gamma^{-\beta/2}((\gamma^{-1/2}n^{-\frac{1+\alpha}{2}}) \wedge 1) \\
&\lesssim \sup_{\gamma\in(0,n^{-(1+\alpha)}]}\gamma^{-\beta/2}\cdot 1 + \sup_{\gamma\in(n^{-(1+\alpha)},1]}\gamma^{-\beta/2-1/2}n^{-\frac{1+\alpha}{2}} \\
&\lesssim \left(n^{-(1+\alpha)}\right)^{1/2-\varepsilon} + \left(n^{-(1+\alpha)}\right)^{-\varepsilon}n^{-\frac{1+\alpha}{2}}
\end{aligned}$$

$$\lesssim n^{-\frac{1+\alpha}{2}+(1+\alpha)\varepsilon}$$

Using this to bound the first term of (3.14) and using (3.9) to bound the second term of (3.14) implies

$$\begin{aligned}
\|\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))\|_{L_p} &\lesssim n^{-\frac{1+\alpha}{2}+(1+\alpha)\varepsilon}|\kappa_n(r) - (k+1)/n|^{-1/2} + e^{-cn} \\
&\lesssim n^{-\frac{1+\alpha}{2}+(1+\alpha)\varepsilon}|\kappa_n(r) - (k+1)/n|^{-1/2} + n^{-1} \\
&\lesssim n^{-\frac{1+\alpha}{2}+(1+\alpha)\varepsilon}|\kappa_n(r) - (k+1)/n|^{-1/2} \qquad (3.17)
\end{aligned}$$

where the last line is followed by the fact that $\frac{\alpha+1}{2} < 1$ implying $n^{-\frac{1+\alpha}{2}}n^{(1+\alpha)\varepsilon} > n^{-1}$.
Recall that to obtain this bound, we assumed that $r \geq \frac{k+4}{n}$. Also, it can be proved that for any $r \geq s$,

$$\begin{aligned}
\|\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))\|_{L_p} &\leq \|b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)})\|_{L_p} \\
&\leq \|b\|_{C^\alpha}\|(\bar{X}^n_r - \bar{X}^n_{\kappa_n(r)})^\alpha\|_{L_p} \\
&\lesssim \|(\sigma(\bar{X}^n_{\kappa_n(r)})(W_r - W_{\kappa_n(r)}))^\alpha\|_{L_p} \\
&\lesssim \|\sigma\|^\alpha_{C^0}\|(W_r - W_{\kappa_n(r)})^\alpha\|_{L_p} \\
&\lesssim |r - \kappa_n(r)|^{\alpha/2} \lesssim n^{-\alpha/2} \qquad (3.18)
\end{aligned}$$

where the fact that if $Z \sim \mathcal{N}(0, \delta I)$, then $|Z|^\alpha \lesssim \delta^{\alpha/2}$, which we will prove in Lemma 5.2 of Appendix 5, are used.
$\|A_{s,t}\|_{L_p}$ is now ready to be bounded. First suppose that $t \geq \frac{k+4}{n}$. Then by (3.7), it can be shown that

$$\begin{aligned}
\|A_{s,t}\|_{L_p} &\lesssim |1-T|^{-1/2}\int_s^t \|\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))\|_{L_p}\,dr \\
&= |1-T|^{-1/2}\left(\int_s^{\frac{k+4}{n}} \|\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))\|_{L_p}\,dr + \int_{\frac{k+4}{n}}^t \|\mathbb{E}^s(b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))\|_{L_p}\,dr\right) \\
&=: |1-T|^{-1/2}(J_1 + J_2) \qquad (3.19)
\end{aligned}$$

Note that J_1 can be bounded using the bound (3.18)

$$\begin{aligned}
J_1 &\lesssim \int_s^{\frac{k+4}{n}} n^{-\alpha/2}\,dr \\
&= n^{-\alpha/2}\,|(k+4)/n - s| \\
&= n^{-\alpha/2}\,|(k+4)/n - s|^{1/2-\varepsilon}\,|(k+4)/n - s|^{1/2+\varepsilon} \\
&\leq n^{-\alpha/2}\,|(k+4)/n - k/n|^{1/2-\varepsilon}\,|t-s|^{1/2+\varepsilon} \\
&\lesssim n^{-\frac{1+\alpha}{2}+\varepsilon}|t-s|^{1/2+\varepsilon} \qquad (3.20)
\end{aligned}$$

Also, using the bound (3.17), J_2 can be bounded as follows

$$\begin{aligned}
J_2 &\lesssim \int_{\frac{k+4}{n}}^t n^{-\frac{1+\alpha}{2}+(1+\alpha)\varepsilon} |\kappa_n(r) - (k+1)/n|^{-1/2}\, dr \\
&\leq n^{-\frac{1+\alpha}{2}+(1+\alpha)\varepsilon} \int_{\frac{k+4}{n}}^t |r - (k+2)/n|^{-1/2}\, dr \\
&= n^{-\frac{1+\alpha}{2}+(1+\alpha)\varepsilon} \left[2|r-(k+2)/n|^{1/2}\right]_{r=\frac{k+4}{n}}^{r=t} \\
&\lesssim n^{-\frac{1+\alpha}{2}+(1+\alpha)\varepsilon} |t-s|^{1/2} \\
&\lesssim n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon} |t-s|^{1/2+\varepsilon} \qquad (3.21)
\end{aligned}$$

where the last inequality holds, because $n|t-s| \geq n|(k+4)/n - (k+1)/n| = 3$.
By (3.19),(3.20) and (3.21), the following bound holds for $t \geq \frac{k+4}{n}$

$$\|A_{s,t}\|_{L_p} \lesssim |1-T|^{-1/2} n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon} |t-s|^{1/2+\varepsilon}$$

Now we need to deal with the case $s \leq t \leq \frac{k+4}{n}$. The same estimate can be obtained simply by using (3.7) and the bound (3.18

$$\begin{aligned}
\|A_{s,t}\|_{L_p} &\lesssim |1-T|^{-1/2} \int_s^t \|\mathbb{E}^s(b(\bar{X}^n_r) - b(\bar{X}^n_{\kappa_n(r)}))\|_{L_p}\, dr. \\
&\lesssim |1-T|^{-1/2} \int_s^t n^{-\alpha/2}\, dr \\
&= |1-T|^{-1/2} n^{-\alpha/2} |t-s| \\
&= |1-T|^{-1/2} n^{-\alpha/2} |t-s|^{1/2-\varepsilon} |t-s|^{1/2+\varepsilon} \\
&\leq |1-T|^{-1/2} n^{-\alpha/2} |(k+4)/n - k/n|^{1/2-\varepsilon} |t-s|^{1/2+\varepsilon} \\
&\lesssim |1-T|^{-1/2} n^{-\frac{1+\alpha}{2}+\varepsilon} |t-s|^{1/2+\varepsilon} \\
&\lesssim |1-T|^{-1/2} n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon} |t-s|^{1/2+\varepsilon}
\end{aligned}$$

Thus condition (S1) holds with $\varepsilon_1 = \varepsilon$ and $C_1 = N|1-T|^{-1/2} n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon}$.
Now condition (S2) will be checked. Note that

$$\begin{aligned}
\mathbb{E}^s \delta A_{s,u,t} &= \mathbb{E}^s(A_{s,t} - A_{s,u} - A_{u,t}) \\
&= \mathbb{E}^s \Bigg(\mathbb{E}^s \int_s^t (b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)})) f_r(\bar{X}^n_s) dr - \mathbb{E}^s \int_s^u (b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)})) f_r(\bar{X}^n_s)\, dr \\
&\qquad - \mathbb{E}^u \int_u^t (b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)})) f_r(\bar{X}^n_u)\, dr \Bigg) \\
&= \int_u^t \mathbb{E}^s \mathbb{E}^u \left((b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))(f_r(\bar{X}^n_s) - f_r(\bar{X}^n_u)) \right)\, dr \\
&= \int_u^t \mathbb{E}^s \left((f_r(\bar{X}^n_s) - f_r(\bar{X}^n_u)) \mathbb{E}^u (b_i(\bar{X}^n_r) - b(\bar{X}^n_{\kappa_n(r)})) \right)\, dr
\end{aligned}$$

Therefore, by using Hölder 's inequality,

$$\begin{aligned}\|\mathbb{E}^s\delta A_{s,u,t}\|_{L_p} &\leq \int_u^t \|(f_r(\bar{X}_s^n)-f_r(\bar{X}_u^n))\mathbb{E}^u(b_i(\bar{X}_r^n)-b_i(\bar{X}_{\kappa_n(r)}^n))\|_{L_p}\,dr \\ &\leq \int_u^t \|f_r(\bar{X}_s^n)-f_r(\bar{X}_u^n)\|_{L_{2p}}\|\mathbb{E}^u(b_i(\bar{X}_r^n)-b_i(\bar{X}_{\kappa_n(r)}^n))\|_{L_{2p}}\,dr \\ &\leq \sup_{r\in[u,t]}\|f_r\|_{C^1}\|\bar{X}_s^n-\bar{X}_u^n\|_{L_{2p}}\int_u^t \|\mathbb{E}^u(b_i(\bar{X}_r^n)-b_i(\bar{X}_{\kappa_n(r)}^n))\|_{L_{2p}}\,dr \qquad (3.22)\end{aligned}$$

We need to bound each part of this expression. Using the previous result that is obtained while bounding $\|A_{s,t}\|$, it can be proved that

$$\begin{aligned}\int_u^t \|\mathbb{E}^u(b_i(\bar{X}_r^n)-b_i(\bar{X}_{\kappa_n(r)}^n))\|_{L_{2p}}\,dr &\lesssim |t-u|^{1/2+\varepsilon}n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon} \\ &\lesssim |t-s|^{1/2+\varepsilon}n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon} \qquad (3.23)\end{aligned}$$

Moreover, using the Schauder-estimate Corollary 3.4,

$$\sup_{r\in[u,t]}\|f_r\|_{C^1} \lesssim \sup_{r\in[u,t]}\|u(r,.)\|_{C^2} \lesssim \sup_{r\in[u,T]}|1-r|^{-1}\|g\|_{C^0} \lesssim |1-T|^{-1} \qquad (3.24)$$

It is easy to see that

$$\|\bar{X}_s^n-\bar{X}_u^n\|_{L_{2p}} \lesssim |t-s|^{1/2} \qquad (3.25)$$

Now using the bounds (3.23),(3.24) and (3.25), (3.22) can be bounded as follows

$$\begin{aligned}\|\mathbb{E}^s\delta A_{s,u,t}\|_{L_p} &\lesssim |1-T|^{-1}|t-s|^{1/2}|t-s|^{1/2+\varepsilon}n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon} \\ &\lesssim |1-T|^{-1}n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon}|t-s|^{1+\varepsilon}\end{aligned}$$

Hence, (S2) holds with $\varepsilon_2=\varepsilon$ and $C_2=N|1-T|^{-1}n^{-\frac{1+\alpha}{2}+(2+\alpha)\varepsilon}$.
Thus, there exists a map $\mathcal{A}:[0,T]\to L_p(\Omega,\mathbb{R}^d)$ satisfying the bounds (S3) and (S4). Let

$$\bar{\mathcal{A}}_t := \int_0^t (b_i(\bar{X}_r^n)-b_i(\bar{X}_{\kappa_n(r)}^n))f_r(\bar{X}_r^n)\,dr$$

It will be shown that conditions (S3) and (S4) hold with $\bar{\mathcal{A}}$ in place of $\mathcal{A}$. Condition (S3) holds because when using Corollary 3.4, ones can have for any $(s,t)\in[0,T]_{\leq}$,

$$\begin{aligned}\|\bar{\mathcal{A}}_t-\bar{\mathcal{A}}_s-A_{s,t}\|_{L_p} &= \Big\|\int_s^t (b_i(\bar{X}_r^n)-b_i(\bar{X}_{\kappa_n(r)}^n))f_r(\bar{X}_r^n)\,dr \\ &\qquad -\int_s^t \mathbb{E}^s((b_i(\bar{X}_r^n)-b_i(\bar{X}_{\kappa_n(r)}^n))f_r(\bar{X}_s^n))\,dr\Big\|_{L_p} \\ &\lesssim \|b\|_{C^0}\sup_{r\in[s,t]}\|f_r\|_{C^0}|t-s|\end{aligned}$$

$$\lesssim \sup_{r\in[s,t]} \|u(r,.)\|_{C^1}|t-s|$$
$$\lesssim |1-T|^{-1/2}|t-s|$$
$$\lesssim |1-T|^{-1/2}|t-s|^{1/2+\varepsilon}$$

Furthermore, condition (S4) holds because for any $(s,t) \in [0,T]_{\leq}$,

$$\begin{aligned}
\|\mathbb{E}^s(\bar{\mathcal{A}}_t - \bar{\mathcal{A}}_s - A_{s,t})\|_{L_p} &= \Bigg\|\mathbb{E}^s\Bigg(\int_s^t (b_i(\bar{X}_r^n) - b_i(\bar{X}^n_{\kappa_n(r)})f_r(\bar{X}_r^n)\,dr \\
&\qquad - \mathbb{E}^s\int_s^t (b_i(\bar{X}_r^n) - b_i(\bar{X}^n_{\kappa_n(r)})f_r(\bar{X}_s^n)\,dr\Bigg)\Bigg\|_{L_p} \\
&\leq \int_s^t \|(b_i(\bar{X}_r^n) - b_i(\bar{X}^n_{\kappa_n(r)})(f_r(\bar{X}_r^n) - f_r(\bar{X}_s^n))\|_{L_p}\,dr \\
&\lesssim \|b\|_{C^0}\int_s^t \|f_r\|_{C^1}\|\bar{X}_r^n - \bar{X}_s^n\|_{L_p}\,dr \\
&\lesssim \int_s^t \|u(r,.)\|_{C^2}\|\bar{X}_r^n - \bar{X}_s^n\|_{L_p}\,dr \\
&\lesssim |1-T|^{-1}\int_s^t |r-s|^{1/2}\,dr \\
&\lesssim |1-T|^{-1}|t-s|^{3/2} \\
&\lesssim |1-T|^{-1}|t-s|^{1+\varepsilon}
\end{aligned}$$

using that $\|\bar{X}_r^n - \bar{X}_s^b\| \lesssim |r-s|^{1/2}$ and Corollary 3.4.
Since conditions (S1) - (S4) of Lemma 2.1 are satisfied, by uniqueness of $\mathcal{A}$, we have that

$$\mathcal{A} = \bar{\mathcal{A}}$$

We complete the proof here. □

The result of Lemma 3.7 still holds if we consider X^n in place of the driftless scheme $\bar{X}^n$.

Lemma 3.8 (Lemma 6.2 in [2])**.** *Let $p \geq 1$, $\varepsilon > 0$, $T < 1$ and let X^n be given by (3.2). Then for all $n \in \mathbb{N}$ and $(s,t) \in [0,T]_{\leq}$, we have*

$$\left\|\int_s^t (b_i(X_r^n) - b_i(X^n_{\kappa_n(r)}))f_r(X_r^n)\,dr\right\|_{L_p} \leq Nn^{-\frac{1+\alpha}{2}+\varepsilon}\Big(|1-T|^{-1/2}|t-s|^{1/2+\varepsilon} + |1-T|^{-1}|t-s|^{1+\varepsilon}\Big)$$

for some constant N depending only on $d, \alpha, p, \varepsilon, \lambda, \|b\|_{C^\alpha}, \|\sigma\|_{C^2}, \|g\|_{C^0}$.

Proof. The proof is a standard argument via Girsanov 's theorem. By Hölder 's inequality, it suffices to show that the bound holds sufficiently large p. Let

$$\rho := \exp\left(-\int_0^1 (\sigma^{-1}b)(X^n_{\kappa_n(t)})\, dW_t - \frac{1}{2}\int_0^1 |(\sigma^{-1}b)(X^n_{\kappa_n(t)})|^2\, dt\right)$$

Construct a new probability measure $\tilde{\mathbb{P}}$ by

$$d\tilde{\mathbb{P}} = \rho\, d\mathbb{P}$$

By the assumptions on b and σ, we have that $\sigma^{-1}b$ is bounded, thus Girsanov 's theorem applies and then under $\tilde{\mathbb{P}}$, the process

$$\tilde{W}_t := W_t + \int_0^t (\sigma^{-1}b)(X^n_{\kappa_n(r)})\, dr, \quad t \in [0,1]$$

is an $\mathbb{F}$-Wiener process. Notice that

$$d\tilde{W}_t = dW_t + (\sigma^{-1}b)(X^n_{\kappa_n(t)})dt$$

This implies

$$\begin{aligned}\sigma(X^n_{\kappa_n(t)})d\tilde{W}_t &= \sigma(X^n_{\kappa_n(t)})dW_t + b(X^n_{\kappa_n(t)})dt\\ &= dX^n_t\end{aligned}$$

hence, the distribution of X^n under $\tilde{\mathbb{P}}$ coincides with the distribution of $\bar{X}^n$ under $\mathbb{P}$. For any continuous process Z, let us denote $\xi(Z) := \int_s^t (b(Z_r) - b(Z_{\kappa_n(r)}))f_r(Z_r)dr$. Then using Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}|\xi(X^n)|^p &= \tilde{\mathbb{E}}\left(|\xi(X^n)|^p\rho^{-1}\right)\\ &\leq \left(\tilde{\mathbb{E}}|\xi(X^n)|^{2p}\right)^{1/2}(\tilde{\mathbb{E}}\rho^{-2})^{1/2}\\ &= \left(\mathbb{E}|\xi(\bar{X}^n)|^{2p}\right)^{1/2}(\mathbb{E}\rho^{-1})^{1/2}\end{aligned}$$

From the assumptions on b and σ, it follows that ρ has finite moments of any order. Hence, $\mathbb{E}|\xi(X^n)|^p \lesssim \left(\mathbb{E}|\xi(\bar{X}^n)|^{2p}\right)^{1/2}$ and thus,

$$\|\xi(X^n)\|_{L_p} \lesssim \|\xi(\bar{X}^n)\|_{L_{2p}}$$

Now the bound we aimed to show follows from Lemma 3.7. □

Finally, in Corollary 3.9, Lemma 3.8 will be modified by extending the domain of integration from $[s,t]$ (where $0 \leq s \leq t \leq T < 1$) to $[0,1]$. The difficulty of this comes from the blowup of the bound in Lemma 3.8 as $T \to 1$. We will thus need to "trade" the $|t-s|$-dependence of the bound to extend the domain of integration as desired. This can be done using dyadic points.

Corollary 3.9 (Corollary 6.3 in [2]). *Let X^n be given by (3.2) and let $\varepsilon > 0$, $p \geq 1$. Then we have*

$$\left\| \int_0^1 (b_i(X_r^n) - b_i(X_{\kappa_n(r)}^n)) f(X_r^n)\, dr \right\|_{L_p} \leq N n^{-\frac{1+\alpha}{2}+\varepsilon}.$$

for some constant N depending only on $d, \alpha, p, \varepsilon, \lambda, \|b\|_{C^\alpha}, \|\sigma\|_{C^2}, \|g\|_{C^0}$.

Proof. For simplicity, let us use the notation $Y_r^n := (b_i(X_r^n) - b_i(X_{\kappa_n(r)}^n)) f(X_r^n)$
Note that

$$\begin{aligned} \left\| \int_0^{1-2^{-m}} Y_r^n\, dr \right\|_{L_p} &\leq \left\| \int_0^{1/2} Y_r^n\, dr \right\|_{L_p} + \left\| \int_{1/2}^{1-2^{-m}} Y_r^n dr \right\|_{L_p} \\ &=: A_1 + A_2(m) \end{aligned} \tag{3.26}$$

By Lemma 3.8, we have

$$A_1 = \left\| \int_0^{1/2} Y_r^n\, dr \right\|_{L_p} \lesssim n^{-\frac{1+\alpha}{2}}$$

To bound $A_2(m)$, note the elementary identity

$$\sum_{k=1}^m 2^{-k} = 1 - 2^{-m}$$

and therefore the collection of intervals $\{(\sum_{k=1}^j 2^{-k}, \sum_{k=1}^{j+1} 2^{-k}] : j \in \{1, \ldots, m-1\}\}$ forms a partition of the interval $(1/2, 1-2^{-m}]$. Using this and Lemma 3.8, the following bound holds

$$\begin{aligned} A_2(m) &= \left\| \int_{1/2}^{1-2^{-m}} Y_r^n\, dr \right\|_{L_p} \\ &= \left\| \sum_{j=1}^{m-1} \int_{\sum_{k=1}^j 2^{-k}}^{\sum_{k=1}^{j+1} 2^{-k}} Y_r^n\, dr \right\|_{L_p} \\ &\leq \sum_{j=1}^{m-1} \left\| \int_{\sum_{k=1}^j 2^{-k}}^{\sum_{k=1}^{j+1} 2^{-k}} Y_r^n\, dr \right\|_{L_p} \\ &\lesssim \sum_{j=1}^{m-1} n^{-\frac{1+\alpha}{2}+\varepsilon} \left(\left(1 - \sum_{k=1}^{j+1} \left(\frac{1}{2}\right)^k \right)^{-1/2} \left| \left(\frac{1}{2}\right)^{j+1} \right|^{1/2+\varepsilon} + \left(1 - \sum_{k=1}^{j+1} \left(\frac{1}{2}\right)^k \right)^{-1} \left| \left(\frac{1}{2}\right)^{j+1} \right|^{1+\varepsilon} \right) \\ &\lesssim n^{-\frac{1+\alpha}{2}+\varepsilon} \sum_{j=1}^{m-1} \left(\left(2^{-(j+1)}\right)^{-1/2} 2^{-(j+1)(1/2+\varepsilon)} + \left(2^{-(j+1)}\right)^{-1} 2^{-(j+1)(1+\varepsilon)} \right) \\ &\lesssim n^{-\frac{1+\alpha}{2}+\varepsilon} \sum_{j=1}^{m-1} \left(2^{\frac{j+1}{2} - \frac{j+1}{2} - \varepsilon(j+1)} + 2^{(j+1)-(j+1)-\varepsilon(j+1)} \right) \end{aligned}$$

$$\lesssim n^{-\frac{1+\alpha}{2}+\varepsilon} 2 \sum_{j=1}^{m-1} 2^{-\varepsilon(j+1)}$$
$$\lesssim n^{-\frac{1+\alpha}{2}+\varepsilon} 2^{1-\varepsilon} \sum_{j=1}^{m-1} (2^{-\varepsilon})^j$$

Therefore,

$$\begin{aligned}
\lim_{m\to\infty} A(m) &\lesssim n^{-\frac{1+\alpha}{2}+\varepsilon} 2^{1-\varepsilon} \sum_{j=0}^{\infty} (2^{-\varepsilon})^j \\
&\lesssim n^{-\frac{1+\alpha}{2}+\varepsilon} 2^{1-\varepsilon} \frac{1}{1-2^{-\varepsilon}} \\
&\lesssim \frac{2}{2^{\varepsilon}-1} n^{-\frac{1+\alpha}{2}+\varepsilon}
\end{aligned}$$

Now passing to the limit $m \to \infty$ in (3.26) and using the bounds on A_1, A_2 gives the result. $\square$

We are about to present the last lemma of this subsection. For notational convenience, we fix $i, j \in \{1, \dots, d\}$ and denote

$$f'_t := \partial_{x_i x_j} u(t, \cdot)$$

Lemma 3.10 (Lemma 6.4 in [2]). *Let X^n given by (3.2), $h \in C^2$ and $\varepsilon > 0$. Then for all $n \in \mathbb{N}$, we have*

$$\left| \mathbb{E} \int_0^1 (h(X_r^n) - h(X_{\kappa_n(r)}^n)) f'_r(X_r^n)\, dr \right| \leq N n^{-\frac{1+\alpha}{2}+\varepsilon}$$

for some constant N depending only on $d, \alpha, \lambda, \|b\|_{C^\alpha}, \|\sigma\|_{C^2} \|h\|_{C^2}, \|g\|_{C^\alpha}$.

Proof. Note that we have

$$\left| \mathbb{E} \int_0^1 (h(X_r^n) - h(X_{\kappa_n(r)}^n)) f'_r(X_r^n)\, dr \right| \leq \int_0^1 |\mathbb{E}(h(X_r^n) - h(X_{\kappa_n(r)}^n)) f'_r(X_r^n)|\, dr \tag{3.27}$$

We will now need to establish bounds on the integrand. Note that denoting $\delta := r - \kappa_n(r) \leq n^{-1}$, the following can be shown

$$\begin{aligned}
&|\mathbb{E}((h(X_r^n) - h(X_{\kappa_n(r)}^n)) f'_r(X_r^n))| \\
&= \Bigg| \mathbb{E}\mathbb{E}^{\kappa_n(r)} \Bigg[\Bigg(h\left(X_{\kappa_n(r)}^n + b(X_{\kappa_n(r)}^n)\delta + \sigma(X_{\kappa_n(r)}^n)(W_r - W_{\kappa_n(r)}) \right) - h\left(X_{\kappa_n(r)}^n \right) \Bigg) \\
&\qquad f'_r\left(X_{\kappa_n(r)}^n + b(X_{\kappa_n(r)}^n)\delta + \sigma(X_{\kappa_n(r)}^n)(W_r - W_{\kappa_n(r)}) \right) \Bigg] \Bigg| \\
&= \Bigg| \mathbb{E} \Bigg(\mathbb{E} \Bigg[\left(h\left(x + b(x)\delta + \sigma(x)(W_r - W_{\kappa_n(r)}) \right) - h(x) \right)
\end{aligned}$$

$$
\begin{aligned}
&\qquad f_r'\left(x+b(x)\delta+\sigma(x)(W_r-W_{\kappa_n(r)})\right)\Big]\Big]\Big|_{x=X^n_{\kappa_n(r)}}\Big)\Big|\\
&\le \sup_{x\in\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(h(x+b(x)\delta+y)-h(x))f_r'(x+b(x)\delta+y)p_{\sigma\sigma^T(x)\delta}(y)\,dy\right| \qquad (3.28)
\end{aligned}
$$

Decomposing the integral in the above expression in the following way

$$
\begin{aligned}
J &:= \int_{\mathbb{R}}(h(x+b(x)\delta+y)-h(x))f_r'(x+b(x)\delta+y)p_{\sigma\sigma^T(x)\delta}(y)\,dy\\
&= \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)f_r'(x+b(x)\delta+y)(h(x+b(x)\delta+y)-h(x+y))\,dy\\
&\quad + \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)f_r'(x+b(x)\delta+y)(h(x+y)-h(x))\,dy\\
&=: J_1+J_2 \qquad (3.29)
\end{aligned}
$$

Using the regularity of h, the integral J_1 is easier to bound as follows

$$
\begin{aligned}
|J_1| &= \left|\int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)f_r'(x+b(x)\delta+y)(h(x+b(x)\delta+y)-h(x+y))\,dy\right|\\
&\le \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)\|f_r'\|_{C^0}\|h\|_{C^1}\|b\|_{C^0}\delta\,dy\\
&\lesssim \|u(r,\cdot)\|_{C^2}n^{-1}\\
&\lesssim \|g\|_{C^\alpha}|1-r|^{-1+\alpha/2}n^{-1}\\
&\lesssim |1-r|^{-1+\alpha/2}n^{-1} \qquad (3.30)
\end{aligned}
$$

where the last inequality holds by Corollary 3.4. We will now need to bound J_2. To this end, note that

$$
\begin{aligned}
J_2 &= \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)f_r'(x+b(x)\delta+y)(h(x+y)-h(x))\,dy\\
&= \sum_{k=1}^d \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)f_r'(x+b(x)\delta+y)\int_0^1 \partial_{x_k}h(x+\theta y)y_k\,d\theta\,dy
\end{aligned}
$$

If we removed the y-dependence of f_r' and of $\partial_{x_k}h$, then the integral would be zero since $\int_{\mathbb{R}^d} y_k p_{\sigma\sigma^T(x)\delta}(y)\,dy=0$. Hence,

$$
\begin{aligned}
|J_2| &= \left|\sum_{k=1}^d \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)y_k\int_0^1\Big(f_r'(x+b(x)\delta+y)\partial_{x_k}h(x+\theta y)-f_r'(x+b(x)\delta)\partial_{x_k}h(x)\Big)\,d\theta\,dy\right|\\
&\lesssim \sup_k\left|J_{2,1}^k+J_{2,2}^k\right| \qquad (3.31)
\end{aligned}
$$

for

$$
J_{2,1}^k := \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)y_k\int_0^1 f_r'(x+b(x)\delta+y)\Big(\partial_{x_k}h(x+\theta y)-\partial_{x_k}h(x)\Big)\,d\theta\,dy
$$

and

$$J^k_{2,2} := \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y) y_k \int_0^1 \partial_{x_k} h(x)\Big(f'_r(x+b(x)\delta+y) - f'_r(x+b(x)\delta)\Big)\, d\theta\, dy$$

Using the regularity of h to bound $J^k_{2,1}$

$$\begin{aligned}
|J^k_{2,1}| &\leq \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)|y| \|f'_r\|_{C^0} \|h\|_{C^2} |y|\, dy \\
&\lesssim \|u(r,\cdot)\|_{C^2} \int_{\mathbb{R}} |y|^2 p_{\sigma\sigma^T(x)\delta}(y)\, dy \\
&\lesssim |1-r|^{-1+\alpha/2}\delta \\
&\lesssim |1-r|^{-1+\alpha/2} n^{-1} \qquad (3.32)
\end{aligned}$$

$J^k_{2,2}$ can be bounded using the α-Hölder-regularity of f'_r and Corollary 3.4. This will increase the blowup in time:

$$\begin{aligned}
|J^k_{2,2}| &\lesssim \int_{\mathbb{R}^d} p_{\sigma\sigma^T(x)\delta}(y)|y| \|h\|_{C^1} \|f'_r\|_{C^\alpha} |y|^\alpha\, dy \\
&\lesssim \|u(r,\cdot)\|_{C^{2+\alpha}} \delta^{\frac{1+\alpha}{2}} \\
&\lesssim |1-r|^{-1} n^{-\frac{1+\alpha}{2}} \qquad (3.33)
\end{aligned}$$

By (3.29),(3.30),(3.31),(3.32) and (3.33), we conclude that $J \lesssim |1-r|^{-1} n^{-\frac{1+\alpha}{2}}$. Hence, by (3.28), we have

$$|\mathbb{E}(h(X^n_r) - h(X^n_{\kappa_n(r)})) f'_r(X^n_r))| \lesssim |1-r|^{-1} n^{-\frac{1+\alpha}{2}} \qquad (3.34)$$

The right hand side converges at the desired rate, however it cannot be integrated on $[0,1]$ with respect to r. Therefore, we will also need to find an other bound with slower blowup at the terminal time. To this end, note that $J^k_{2,2}$ can be also bounded as follows

$$\begin{aligned}
|J^k_{2,2}| &\leq \int_{\mathbb{R}} p_{\sigma\sigma^T(x)\delta}(y)|y| \|h\|_{C^1} \|f'_r\|_{C^0}\, dy \\
&\lesssim \|u(r,\cdot)\|_{C^2} \int_{\mathbb{R}} |y| p_{\sigma\sigma^T(x)\delta}(y)\, dy \\
&\lesssim |1-r|^{-1+\alpha/2} n^{-1/2} \qquad (3.35)
\end{aligned}$$

So by (3.29),(3.30),(3.31),(3.32) and (3.35), we have

$$|\mathbb{E}(h(X^n_r) - h(X^n_{\kappa_n(r)})) f'_r(X^n_r))| \lesssim |1-r|^{-1+\alpha/2} n^{-1/2} \qquad (3.36)$$

The desired bound is finally ready to be shown. By (3.27),(3.34) and (3.36), it can be proved that for $a \in (0,1)$,

$$\left|\mathbb{E}\int_0^1 (h(X^n_r) - h(X^n_{\kappa_n(r)})) f'_r(X^n_r)\, dr\right| \lesssim \int_0^a (1-r)^{-1} n^{-\frac{1+\alpha}{2}}\, dr + \int_a^1 (1-r)^{-1+\alpha/2} n^{-1/2}\, dr$$

$$\lesssim n^{-\frac{1+\alpha}{2}} \left[-\log(1-r)\right]_{r=0}^{r=a} + n^{-1/2} \left[-(1-r)^{\alpha/2}\right]_{r=a}^{1}$$

thus choosing $a = 1 - \frac{1}{n}$, the following bound holds

$$\begin{aligned}\left|\mathbb{E}\int_0^1 (h(X_r^n) - h(X_{\kappa_n(r)}^n)) f_r'(X_r^n)\, dr\right| &\lesssim n^{-\frac{1+\alpha}{2}} \left(-\log\left(\frac{1}{n}\right) + 0\right) + n^{-1/2} \left(0 + \left(\frac{1}{n}\right)^{\alpha/2}\right) \\ &\lesssim \log(n) n^{-\frac{1+\alpha}{2}} \\ &\lesssim n^{-\frac{1+\alpha}{2}+\varepsilon}\end{aligned}$$

as required. □

3.6 Proof of the main result

Since the quadrature estimates are already established in subsection 3.5, the main result is now ready to be shown.

Proof of Theorem 3.1. Let u be the unique bounded solution of (3.3). By the Feynmann-Kac formula, we have

$$u(0, x_0) = \mathbb{E}g(X_1)$$

furthermore, by the terminal condition, we have

$$u(1, X_1^n) = g(X_1^n)$$

Hence, the weak error can be expressed as

$$\begin{aligned}d_g(X, X^n) &:= |\mathbb{E}g(X_1^n) - \mathbb{E}g(X_1)| \\ &= |\mathbb{E}g(X_1^n) - \mathbb{E}\mathbb{E}g(X_1)| \\ &= |\mathbb{E}\left(u(1, X_1^n) - u(0, x_0)\right)| \\ &= |\mathbb{E}\left(u(1, X_1^n) - u(0, X_0^n)\right)| \qquad (3.37)\end{aligned}$$

Applying Itô 's lemma gives

$$\begin{aligned}d_g(X, X^n) = \Bigg|\mathbb{E}\Bigg(\int_0^1 \Big(\partial_t u(r, X_r^n) + \sum_{i=1}^d \partial_{x_i} u(r, X_r^n) b_i(X_{\kappa_n(r)}^n) + \\ \frac{1}{2}\sum_{i,j=1}^d \partial_{x_i x_j} u(r, X_r^n) \sum_{k=1}^d \sigma_{ik}(X_{\kappa_n(r)}^n)\sigma_{jk}(X_{\kappa_n(r)}^n)\Big)\, dr\Bigg) \\ + \mathbb{E}\Bigg(\int_0^1 \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} u(r, X_r^n)\sigma_{ij}(X_{\kappa_n(r)}^n)\, dW_r^j\Bigg)\Bigg| \qquad (3.38)\end{aligned}$$

Note that since g has positive Hölder-regularity, by Corollary 3.4, the integrand $\partial_{x_i} u(t, X_t^n)\sigma_{ij}(X_{\kappa_n(t)}^n)$

is square-integrable on $\Omega \times [0,1]$ and thus in the above expression the stochastic integral term has zero expectation.

Hence, we are left with

$$d_g(X,X^n) = \left|\mathbb{E}\int_0^1 \left(\partial_t u(r,X_r^n) + \bar{L}u\left(r,X_r^n,X_{\kappa_n(r)}^n\right)\right)\,dr\right|$$

where the operator $\bar{L}$ is given by

$$\bar{L}\phi(t,x,\bar{x}) := \sum_{i=1}^d b(\bar{x})\partial_{x_i}\phi(x) + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^T)_{ij}(\bar{x})\partial_{x_ix_j}\phi(x)$$

Recall that by parabolic PDE (3.3), we have $\partial_t u = -Lu$. Hence, the weak error can be written as

$$d_g(X,X^n) = \left|\mathbb{E}\int_0^1 \left(\bar{L}u\left(r,X_r^n,X_{\kappa_n(r)}\right) - Lu\left(r,X_r^n\right)\right)\,dr\right|$$

Writing out the operators explicitly gives

$$\begin{aligned} d_g(X,X^n) = \Bigg|\mathbb{E}\Bigg(\int_0^1 \Big(\sum_{i=1}^d (b_i(X_{\kappa_n(r)}^n) - b_i(X_r^n)\partial_{x_i}u(r,X_r^n) \\ + \frac{1}{2}\sum_{i,j=1}^d ((\sigma\sigma^T)_{ij}(X_{\kappa_n(r)}^n) - (\sigma\sigma^T)(X_r^n))\partial_{x_ix_j}u(r,X_r^n)\Big)\,dr\Bigg)\Bigg| \end{aligned} \tag{3.39}$$

So by the triangle inequality, we can see that it suffices to show that the bound holds for

$$\left|\mathbb{E}\int_0^1 \left(\left(b_i(X_{\kappa_n(r)}^n) - b_i(X_r^n)\right)\partial_{x_i}u(r,X_r^n)\right)\,dr\right| \tag{3.40}$$

and for

$$\left|\mathbb{E}\int_0^1 \left(\left((\sigma\sigma^T)_{ij}(X_{\kappa_n(r)}^n) - (\sigma\sigma^T)_{ij}(X_r^n)\right)\partial_{x_ix_j}u(r,X_r^n)\right)\,dr\right| \tag{3.41}$$

for all i,j indices.

The desired bound now follows from Corollary 3.9 and Lemma 3.10. $\square$

4 Strong rate of convergence for the Euler - Maruyama scheme with Hölder drifts

4.1 Some preliminaries and notations

In this section, we consider the SDE and the Euler - Maruyama scheme with the same setting as the one in Section 3. We also use the same preliminaries and notations as we use in Section 3. Furthermore, we want to introduce one more notation: for $\alpha > 0$, $p \geq 2$ and $Q \subseteq \mathbb{R}^k$ for some $k \in \mathbb{Z}^+$, denote the space $C^\alpha(Q, L_p(\Omega))$ by $\mathscr{C}_p^\alpha, Q$ and for all $f \in C^\alpha(Q, L_p(\Omega))$, we denote

$$[\![f]\!]_{\mathscr{C}_p^\alpha,Q} := \|f\|_{C^\alpha(Q,L_p(\Omega))}$$

Now we would like to present the main result of this section.

4.2 Main result

Theorem 4.1 (Theorem 2.7 in [1]). *Let $\alpha \in (0,1)$, $\varepsilon > 0$, $\tau \in [0,1/2)$ and $p \geq 2$. Let X be the solution of (3.1) and X^n be given by (3.2). Then for all $n \in \mathbb{N}$, the following bound holds*

$$\|X - X^n\|_{C^\tau([0,1],L_p(\Omega))} \leq Nn^{-1/2+\varepsilon}$$

with some $N = N(p, d, \alpha, \varepsilon, \tau, \lambda, \|b\|_{C^\alpha}, \|\sigma\|_{C^2})$.

4.3 On the method of the proof

In order to prove the desired bound for $[\![X - X^n]\!]_{\mathscr{C}_p^\tau,[0,1]}$, it is reduced to prove the bounds for $[\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[0,1]}$, $[\![\mathscr{Q}^n]\!]_{\mathscr{C}_p^\tau,[0,1]}$ and $[\![\mathscr{R}^n]\!]_{\mathscr{C}_p^\tau,[0,1]}$ where

$$\begin{aligned}
\varphi_t^n &:= \int_0^t (b(X_{\kappa_n(r)}^n) - b(X_r))\, dr \\
\mathscr{Q}_t^n &:= \int_0^t (\sigma(X_r^n) - \sigma(X_r))\, dW_r \\
\mathscr{R}_t^n &:= \int_0^t (\sigma(X_{\kappa_n(t)}^n) - \sigma(X_t^n))\, dW_r
\end{aligned}$$

The authors utilize Burkholder-Davis-Gundy and Hölder 's inequalities to obtain the bounds for $[\![\mathscr{Q}^n]\!]_{\mathscr{C}_p^\tau,[0,1]}$ and $[\![\mathscr{R}^n]\!]_{\mathscr{C}_p^\tau,[0,1]}$. Meanwhile, proving the bound for $[\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[0,1]}$ requires a quadrature estimate and a regularization lemma which will be presented in subsection 4.4 and 4.5 respectively.

4.4 Quadrature estimates

Lemma 4.2 (Lemma 6.1 in [1]). *Let $\varepsilon_1 \in (0, 1/2)$, $\alpha \in (0, 1)$ and $p > 0$. Let $\bar{X}^n$ be given by (3.5). Then for all $f \in C^\alpha$, $0 \le s \le t \le 1$ and $n \in \mathbb{N}$, ones have the bound*

$$\Big\| \int_s^t (f(\bar{X}^n_r) - f(\bar{X}^n_{\kappa_n(r)}))\, dr \Big\|_{L_p(\Omega)} \le N n^{-1/2+2\varepsilon_1} |t-s|^{1/2+\varepsilon_1}$$

with some $N = N(\alpha, p, d, \varepsilon_1, \lambda, \|f\|_{C^\alpha}, \|\sigma\|_{C^2})$.

Proof. By Hölder 's inequality, it clearly suffices to prove the bound for $p \ge 2$.
We put for $0 \le s \le t \le 1$,

$$A_{s,t} := \mathbb{E}^s \int_s^t (f(\bar{X}^n_r) - f(\bar{X}^n_{\kappa_n(r)}))\, dr$$

Then for any $0 \le s \le u \le t \le 1$,

$$\begin{aligned}
\delta A_{s,u,t} &= A_{s,t} - A_{s,u} - A_{u,t} \\
&= \mathbb{E}^s \int_u^t (f(\bar{X}^n_r) - f(\bar{X}^n_{\kappa_n(r)}))\, dr - \mathbb{E}^u \int_u^t (f(\bar{X}^n_r) - f(\bar{X}^n_{\kappa_n(r)}))\, dr
\end{aligned}$$

Let us check that the conditions (S1) and (S2) of Stochastic sewing lemma 2.1 are satisfied. Note that

$$\mathbb{E}^s \delta A_{s,u,t} = 0$$

and so the condition (S2) trivially holds with $C_2 = 0$. As for (S1), a similar argument as the one in Lemma 3.7 will be used. Let $s \in [k/n, (k+1)/n)$ for some $k \in \mathbb{N}_0$. Suppose first that $t \in [(k+4)/n, 1]$. We write

$$|A_{s,t}| = |I_1 + I_2| := \Big| \Big(\int_s^{(k+4)/n} + \int_{(k+4)/n}^t \Big) \mathbb{E}^s \big(f(\bar{X}^n_r) - f(\bar{X}^n_{k_n(r)}) \big)\, dr \Big|$$

The bound for I_2 is shown as follows. Note that

$$\begin{aligned}
\mathbb{E}^s (f(\bar{X}^n_r) - f(\bar{X}^n_{\kappa_n(r)})) &= \mathbb{E}^s \mathbb{E}^{\kappa_n(r)} (f(\bar{X}^n_{\kappa_n(r)} + \sigma(\bar{X}^n_{\kappa_n(r)})(W_r - W_{\kappa_n(r)})) - f(\bar{X}^n_{\kappa_n(r)})) \\
&= \mathbb{E}^s \left(\mathbb{E} \left(f(x + \sigma(x)(W_r - W_{\kappa_n(r)})) - f(x) \right) \Big|_{x = \bar{X}^n_{\kappa_n(r)}} \right) \\
&= \mathbb{E}^s g(\bar{X}^n_{\kappa_n(r)}) \\
&= \mathbb{E}^s \mathbb{E}^{(k+1)/n} g(\bar{X}^n_{\kappa_n(r)})
\end{aligned} \tag{4.1}$$

where we define $g : \mathbb{R}^d \to \mathbb{R}$ by

$$g(x) = g^n_r(x) := \int_{\mathbb{R}^d} (f(x+y) - f(x)) p_{\sigma\sigma^T(x)\delta}(y)\, dy$$

with $\delta := r - \kappa_n(r) \le n^{-1}$. Hence, for $\varepsilon \in (0,1]$,

$$\begin{aligned}
\mathcal{P}_\varepsilon g(x) &= \int_{\mathbb{R}^d} p_\varepsilon(z) g(x-z)\,dz \\
&= \int_{\mathbb{R}^d} p_\varepsilon(z) \int_{\mathbb{R}^d} \left(f(x-z+y) - f(x-z)\right) p_{M(x,z)\delta}(y)\,dy\,dz \\
&= -\sum_{j=1}^d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\varepsilon(z) p_{M(x,z)\delta}(y) \int_0^1 y_j \partial_{z_j} f(x-z+\theta y)\,d\theta\,dy\,dz \\
&= \sum_{j=1}^d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_{z_j}\left(p_\varepsilon(z) p_{M(x,z)\delta}(y)\right) y_j \int_0^1 f(x-z+\theta y)\,d\theta\,dy\,dz \\
&= \sum_{j=1}^d \left(I_1^j + I_2^j\right)
\end{aligned} \tag{4.2}$$

where we denote $\sigma\sigma^T(x-z)$ by $M(x,z)$ for convenience. We moreover denote

$$I_1^j := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \partial_{z_j}\left(p_\varepsilon(z)\right) p_{M(x,z)\delta}(y) y_j \int_0^1 f(x-z+\theta y)\,d\theta\,dy\,dz \tag{4.3}$$

and

$$I_2^j := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_\varepsilon(z) \partial_{z_j}\left(p_{M(x,z)\delta}(y)\right) y_j \int_0^1 f(x-z+\theta y)\,d\theta\,dy\,dz \tag{4.4}$$

It is well known that

$$|\partial_{z_i} p_\varepsilon(z)| \lesssim |z|\varepsilon^{-1} p_\varepsilon(z) \tag{4.5}$$

Furthermore, as (3.15) in the proof of Lemma 3.7 we have

$$\begin{aligned}
|\partial_{z_i} p_{M(x,z)\delta}(y)| &= \Big|\frac{\partial_{z_i}(y^\intercal M^{-1}(x,z)y)}{2\delta} + \frac{\partial_{z_i}\det M(x,z)}{2\det M(x,z)}\Big| p_{M(x,z)\delta}(y) \\
&\lesssim (\delta^{-1}|y|^2 + 1) p_{M(x,z)\delta}(y)
\end{aligned} \tag{4.6}$$

where the last inequality holds because of (3.16). Therefore, by (4.2), (4.3), (4.4), (4.5), (4.6) and Lemma 5.2 in Appendix 5, it can be seen that

$$\begin{aligned}
\|\mathcal{P}_\varepsilon g\|_{C^0} &\lesssim \|f\|_{C^0} \int_{R^d}\int_{R^d} \left(\varepsilon^{-1}|z| + \delta^{-1}|y|^2 + 1\right)\left(|y| p_\varepsilon(z) p_{\sigma\sigma^\intercal(x-z)\delta}(y)\right) dy\,dz \\
&\lesssim \|f\|_{C^0}(\varepsilon^{-1/2}\delta^{1/2} + \delta^{1/2}) \\
&\lesssim \|f\|_{C^0}\varepsilon^{-1/2} n^{-1/2}
\end{aligned}$$

Ones also have the trivial estimate $\|\mathcal{P}_\varepsilon g\|_{C^0} \le 2\|f\|_{C^0}$ and combining these two bounds yields for all $\beta \in [-1,0)$,

$$\|g\|_{C^\beta} = \sup_{\varepsilon\in(0,1]} \varepsilon^{-\beta/2} \|\mathcal{P}_\varepsilon \tilde{g}\|_{C^0}$$

$$
\begin{aligned}
&\lesssim \sup_{\varepsilon\in(0,1]} \varepsilon^{-\beta/2}((\|f\|_{C^0}\varepsilon^{-1/2}n^{-1/2}) \wedge \|f\|_{C^0}) \\
&\lesssim \sup_{\varepsilon\in(0,1]} \varepsilon^{-\beta/2}((\varepsilon^{-1/2}n^{-1/2}) \wedge 1) \\
&\lesssim \sup_{\varepsilon\in(0,n^{-1}]} \varepsilon^{-\beta/2}\cdot 1 + \sup_{\varepsilon\in(n^{-1},1]} \varepsilon^{-\beta/2-1/2}n^{-1/2} \\
&\lesssim n^{\beta/2} + n^{\frac{\beta+1}{2}}n^{-1/2} \\
&\lesssim n^{\beta/2} \qquad (4.7)
\end{aligned}
$$

Note that the restriction of $\bar{X}^n_t(\cdot)$ to the gridpoints $t = 0, 1/n, \ldots, 1$ is a Markov process with state space $\mathbb{R}^d$. Therefore,

$$
\begin{aligned}
|\mathbb{E}^{(k+1)/n} g\big(\bar{X}^n_{\kappa_n(r)}(y)\big)| &= \left| \mathbb{E} g\big(\bar{X}^n_{\kappa_n(r)-(k+1)/n}(x)\big)\Big|_{x=\bar{X}^n_{(k+1)/n}(y)} \right| \\
&\le \sup_{x\in\mathbb{R}^d} |\mathbb{E} g\big(\bar{X}^n_{\kappa_n(r)-(k+1)/n}(x)\big)| \qquad (4.8)
\end{aligned}
$$

By Lemma 3.6, for the solution u of $(I-\Delta)u = g$, we have

$$
\|u\|_{C^{2+(\alpha/2)}} \le N\|g\|_{C^{\alpha/2}}, \qquad \|u\|_{C^{1+2\varepsilon_1}} \le N\|g\|_{C^{-1+2\varepsilon_1}} \qquad (4.9)
$$

Moreover, using a similar argument as (3.9), we can obtain that

$$
\|g\|_{C^{\alpha}/2} \lesssim \|f\|_{C^\alpha} \qquad (4.10)
$$

Hence, combining (4.8), Lemma 3.5, (4.9), (4.7) and (4.10) implies

$$
\begin{aligned}
|\mathbb{E}^{(k+1)/n} g\big(\bar{X}^n_{\kappa_n(r)}(y)\big)| &\le \sup_{x\in\mathbb{R}^d} |\mathbb{E}(u-\Delta u)\big(\bar{X}^n_{\kappa_n(r)-(k+1)/n}(x)\big)| \\
&\le \sup_{x\in\mathbb{R}^d} |\mathbb{E} u\big(\bar{X}^n_{\kappa_n(r)-(k+1)/n}(x)\big)| + \sup_{x\in\mathbb{R}^d} |\mathbb{E}\Delta u\big(\bar{X}^n_{\kappa_n(r)-(k+1)/n}(x)\big)| \\
&\lesssim \|u\|_{C^0} + \sup_{x\in\mathbb{R}^d} \left| \mathbb{E} \sum_{i=1}^d \partial_{x_i x_i} u\big(\bar{X}^n_{\kappa_n(r)-(k+1)/n}(x)\big) \right| \\
&\lesssim \|u\|_{C^0} + \sum_{i=1}^d \sup_{x\in\mathbb{R}^d} \big|\mathbb{E}\partial_{x_i x_i} u\big(\bar{X}^n_{\kappa_n(r)-(k+1)/n}(x)\big)\big| \\
&\lesssim \|u\|_{C^0} + \sum_{i=1}^d \big(\|\partial_{x_i} u\|_{C^0} |\kappa_n(r) - (k+1)/n|^{-1/2} + \|\partial_{x_i} u\|_{C^1} e^{-cn}\big) \\
&\lesssim \|u\|_{C^1} |\kappa_n(r) - (k+1)/n|^{-1/2} + \|u\|_{C^2} e^{-cn} \\
&\lesssim \|u\|_{C^{1+2\varepsilon_1}} |\kappa_n(r) - (k+1)/n|^{-1/2} + \|u\|_{C^{2+\alpha/2}} e^{-cn} \\
&\lesssim \|g\|_{C^{-1+2\varepsilon_1}} |\kappa_n(r) - (k+1)/n|^{-1/2} + \|g\|_{C^{\alpha/2}} e^{-cn} \\
&\lesssim n^{-1/2+\varepsilon_1} |\kappa_n(r) - (k+1)/n|^{-1/2}
\end{aligned}
$$

Putting this back into (4.1), ones obtain

$$\begin{aligned}
\|I_2\|_{L_p(\Omega)} &= \left\|\int_{(k+4)/n}^{t} \mathbb{E}^s\big(f(\bar{X}_r^n) - f(\bar{X}_{k_n(r)}^n)\big)\,dr\right\|_{L_p(\Omega)} \\
&= \left\|\int_{(k+4)/n}^{t} \mathbb{E}^s\mathbb{E}^{(k+1)/n} g(\bar{X}_{\kappa_n(r)}^n)\right\|_{L_p(\Omega)} \\
&\lesssim n^{-1/2+\varepsilon_1}\int_{(k+4)/n}^{t} |\kappa_n(r) - (k+1)/n|^{-1/2}\,dr \\
&\lesssim n^{-1/2+\varepsilon_1}\int_{(k+4)/n}^{t} |r - (k+4)/n|^{-1/2}\,dr \\
&\lesssim |t-s|^{1/2} n^{-1/2+\varepsilon_1} \\
&\lesssim |t-s|^{1/2+\varepsilon_1} n^{-1/2+2\varepsilon_1}
\end{aligned}$$

where the fact that $n^{-1} \leq |t-s|$ is used. The bound for I_1 is straightforward:

$$\begin{aligned}
\|I_1\|_{L_p(\Omega)} &\leq \int_s^{(k+4)/n} \|f(\bar{X}_r) - f(\bar{X}_{k_n(r)})\|_{L_p(\Omega)}\,dr \\
&\leq N\|f\|_{C^0} n^{-1} \\
&\lesssim n^{-1/2+\varepsilon_1}|t-s|^{1/2+\varepsilon_1}
\end{aligned}$$

Therefore,

$$\|A_{s,t}\|_{L_p(\Omega)} \lesssim n^{-1/2+2\varepsilon_1}|t-s|^{1/2+\varepsilon_1}$$

It remains to show the same bound for $t \in (s, (k+4)/n]$. Similarly to the above, we write

$$\begin{aligned}
\|A_{s,t}\|_{L_p(\Omega)} &\leq \int_s^t \|f(\bar{X}_r) - f(\bar{X}_{k_n(r)})\|_{L_p(\Omega)}\,dr \\
&\leq N\|f\|_{C^0}|t-s| \\
&\lesssim n^{-1/2+\varepsilon_1}|t-s|^{1/2+\varepsilon_1}
\end{aligned}$$

using that $|t-s| \leq 4n^{-1}$ and $\varepsilon_1 < 1/2$. Thus, (S1) holds with $C_1 = Nn^{-1/2+2\varepsilon_1}$.
Therefore, there exists a map $\mathcal{A} : [0,1] \to L_p(\Omega, \mathbb{R}^d)$ satisfying the bounds (S3) and (S4). Let

$$\bar{\mathcal{A}}_t := \int_0^t (f(\bar{X}_r^n) - f(\bar{X}_{\kappa_n(r)}^n))\,dr$$

It can be checked that conditions (S3) and (S4) hold with $\bar{\mathcal{A}}$ in place of $\mathcal{A}$.
We can see that

$$\bar{\mathcal{A}}_t - \bar{\mathcal{A}}_s - A_{s,t} = \int_s^t (f(\bar{X}_r^n) - f(\bar{X}_{\kappa_n(r)}^n))\,dr - \mathbb{E}^s\int_s^t (f(\bar{X}_r^n) - f(\bar{X}_{\kappa_n(r)}^n))\,dr$$

Therefore, $\mathbb{E}^s(\bar{\mathcal{A}}_t - \bar{\mathcal{A}}_s - A_{s,t}) = 0$ and then $\bar{\mathcal{A}}$ satisfies (S4) with $K_2 = 0$.

Moreover,

$$\begin{aligned}\|\bar{\mathcal{A}}_t - \bar{\mathcal{A}}_s - A_{s,t}\|_{L_p(\Omega)} &= \|\int_s^t (f(\bar{X}^n_r) - f(\bar{X}^n_{\kappa_n(r)}))\,dr - \mathbb{E}^s \int_s^t (f(\bar{X}^n_r) - f(\bar{X}^n_{\kappa_n(r)}))\,dr\|_{L_p(\Omega)} \\ &\lesssim \|f\|_{C^0}|t-s| \\ &\lesssim \|f\|_{C^0}|t-s|^{1/2+\varepsilon_1}\end{aligned}$$

This implies that $\bar{\mathcal{A}}$ satisfies (S3) with $K_2 = N\|f\|_{C^0}$. The proof is complete. □

The following lemma states that Lemma 4.2 still holds if we consider X^n in the place of the driftless scheme $\bar{X}^n$. In order to obtain this, the authors use a similar argument as Lemma 3.8 in Section 3, exploiting Gisarnov 's theorem.

Lemma 4.3 (Lemma 6.2 in [1]). *Let $\alpha \in (0,1)$ and take $\varepsilon_1 \in (0,1/2)$. Let X^n be given by (3.2). Then for all $f \in C^\alpha$, $0 \le s \le t \le 1$, $n \in \mathbb{N}$ and $p > 0$, ones have the bound*

$$\Big\| \int_s^t (f(X^n_r) - f(X^n_{\kappa_n(r)}))\,dr \Big\|_{L_p(\Omega)} \le N n^{-1/2+2\varepsilon_1}|t-s|^{1/2+\varepsilon_1} \tag{4.11}$$

with some $N = N(p, d, \alpha, \varepsilon_1, \lambda, \|b\|_{C^0}, \|f\|_{C^\alpha}, \|\sigma\|_{C^2})$.

4.5 A regularization lemma

In this subsection, we are going to present the regularization bound mentioned in subsection 4.3. The proof of this bound will exploit an estimate on the transition kernel $\bar{\mathcal{P}}$ of (3.1), that is, $\bar{P}_t f(x) = \mathbb{E} f(X_t(x))$ where $X_t(x)$ is the solution of (3.1) with initial condition $X_0(x) = x$. The following bound then follows from [6, Theorem 9/4/2].

Proposition 4.4. *Assume $b \in C^\alpha$, $\alpha > 0$ and $f \in C^{\alpha'}$, $\alpha' \in [0,1]$. Then for all $0 < t \le 1$, $x, y \in \mathbb{R}^d$, ones have the bounds*

$$|\bar{\mathcal{P}}_t f(x) - \bar{\mathcal{P}}_t f(y)| \le N\|f\|_{C^{\alpha'}}|x-y|t^{-(1-\alpha')/2}$$

with some $N = N(d, \alpha, \lambda, \|b\|_{C^\alpha}, \|\sigma\|_{C^1})$.

Lemma 4.5 (Lemma 6.4 in [1]). *Let $\alpha \in (0,1)$ and $\tau \in (0,1]$ satisfy*

$$\tau + \alpha/2 - 1/2 > 0 \tag{4.12}$$

Let X be the solution of (3.1) and φ be an adapted process. Then for all sufficiently small $\varepsilon_3, \varepsilon_4 > 0$, for all $f \in C^\alpha$, $0 \le s \le t \le 1$ and $p > 0$, ones have the bound

$$\Big\| \int_s^t (f(X_r) - f(X_r + \varphi_r))\,dr \Big\|_{L_p(\Omega)} \le N|t-s|^{1+\varepsilon_3} [\![\varphi]\!]_{\mathscr{C}^\tau_p,[s,t]} + N|t-s|^{1/2+\varepsilon_4} [\![\varphi]\!]_{\mathscr{C}^0_p,[s,t]}$$

with some $N = N(p, d, \alpha, \tau, \lambda, \|\sigma\|_{C^1})$.

Proof. Set for $s \le s' \le t' \le t$,

$$A_{s',t'} := \mathbb{E}^{s'} \int_{s'}^{t'} (f(X_r) - f(X_r + \varphi_{s'}))\, dr$$

Let us check the conditions of the Stochastic sewing lemma. We have

$$\delta A_{s',u,t'} = \mathbb{E}^{s'} \int_u^{t'} (f(X_r) - f(X_r + \varphi_{s'}))\, dr - \mathbb{E}^u \int_u^{t'} (f(X_r) - f(X_r + \varphi_u))\, dr$$

so $\mathbb{E}^{s'} \delta A_{s',u,t'} = \mathbb{E}^{s'} \hat{\delta} A_{s',u,t'}$ with

$$\begin{aligned}\hat{\delta} A_{s',u,t'} &:= \mathbb{E}^u \int_u^{t'} \Big(\big(f(X_r) - f(X_r + \varphi_{s'})\big) - \big(f(X_r) - f(X_r + \varphi_u)\big) \Big)\, dr \\ &= \int_u^{t'} (\bar{\mathcal{P}}_{r-u} f(X_u + \varphi_{s'}) - \bar{\mathcal{P}}_{r-u} f(X_u + \varphi_u))\, dr\end{aligned}$$

Invoking (4.4), we can write

$$|\hat{\delta} A_{s',u,t'}| \le N \int_u^{t'} |\varphi_{s'} - \varphi_u| |r - u|^{-(1-\alpha)/2}\, dr$$

Hence, using also Jensen 's inequality,

$$\begin{aligned}\|\mathbb{E}^{s'} \delta A_{s',u,t'}\|_{L_p(\Omega)} &= \|\mathbb{E}^{s'} \hat{\delta} A_{s',u,t'}\|_{L_p(\Omega)} \\ &\le \|\hat{\delta} A_{s',u,t'}\|_{L_p(\Omega)} \\ &\lesssim \left\| \int_u^{t'} |\varphi_{s'} - \varphi_u| |r - u|^{-(1-\alpha)/2}\, dr \right\|_{L_p(\Omega)} \\ &\lesssim \int_u^{t'} \|\varphi_{s'} - \varphi_u\|_{L_p(\Omega)} |r - u|^{-(1-\alpha)/2}\, dr \\ &\lesssim [\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]} |s' - u|^\tau \int_u^{t'} |r - u|^{-(1-\alpha)/2}\, dr \\ &\lesssim [\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]} |t' - s'|^{1+\tau-(1-\alpha)/2}\end{aligned}$$

The condition (4.12) implies that there exists some $\varepsilon_3 > 0$ sufficiently small such that

$$\tau - (1 - \alpha)/2 > \varepsilon_3$$

Thus, there exists some $\varepsilon_3 > 0$ such that

$$\|\mathbb{E}^{s'} \delta A_{s',u,t'}\|_{L_p(\Omega)} \lesssim [\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]} |t' - s'|^{1+\varepsilon_3}$$

Therefore, $A_{s',t'}$ satisfies condition (S2) with $C_2 = N[\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]}$.
Next, to bound $\|A_{s',t'}\|_{L_p(\Omega)}$, note that

$$
\begin{aligned}
|\mathbb{E}^{s'} f(X_r) - \mathbb{E}^{s'} f(X_r + \varphi_{s'})| &= |\bar{\mathcal{P}}_{r-s'} f(X_{s'}) - \bar{\mathcal{P}}_{r-s'} f(X_{s'} + \varphi_{s'})| \\
&\leq N|\varphi_{s'}||r-s'|^{-(1-\alpha)/2}
\end{aligned}
$$

Taking integration with respect to r and using Jensen 's inequality yields for any sufficiently small $\varepsilon_4 > 0$ ($\varepsilon_4 < \alpha/2$),

$$
\|A_{s',t'}\|_{L_p(\Omega)} \leq N|t'-s'|^{1/2+\varepsilon_4} [\![\varphi]\!]_{\mathscr{C}_p^0,[s,t]}
$$

Therefore, $A_{s',t'}$ satisfies condition (S1) with $C_1 = N[\![\varphi]\!]_{\mathscr{C}_p^0,[s,t]}$.
Hence, there exists a map $\mathcal{A} : [s,t] \to L_p(\Omega, \mathbb{R}^d)$ satisfying the bounds (S3) and (S4). Let

$$
\bar{\mathcal{A}}_{t'} := \int_0^{t'} (f(X_r) - f(X_r + \varphi_r))\, dr
$$

It can be shown that conditions (S3) and (S4) hold with $\bar{\mathcal{A}}$ in place of $\mathcal{A}$.
For condition (S4), ones have

$$
\begin{aligned}
\|\bar{\mathcal{A}}_{t'} - \bar{\mathcal{A}}_{s'} - A_{s',t'}\|_{L_p(\Omega)} &= \left\| \int_{s'}^{t'} (f(X_r) - f(X_r + \varphi_r))\, dr - \mathbb{E}^{s'} \int_{s'}^{t'} (f(X_r) - f(X_r + \varphi_{s'}))\, dr \right\|_{L_p(\Omega)} \\
&\lesssim \|f\|_{C^0} |t'-s'| \lesssim \|f\|_{C^0} |t'-s'|^{1/2+\varepsilon_4}
\end{aligned}
$$

using the fact that $\varepsilon_4 < \alpha/2$. This implies that condition (S4) is satisfied with $K_2 = N\|f\|_{C^0}$.
Furthermore,

$$
\begin{aligned}
\|\mathbb{E}^{s'}(\bar{\mathcal{A}}_{t'} - \bar{\mathcal{A}}_{s'} - A_{s',t'})\|_{L_p(\Omega)} &= \left\| \int_{s'}^{t'} (f(X_r + \varphi_{s'}) - f(X_r + \varphi_r))\, dr \right\|_{L_p(\Omega)} \\
&\lesssim \int_{s'}^{t'} \|f(X_r + \varphi_{s'}) - f(X_r + \varphi_r)\|_{L_p(\Omega)}\, dr \\
&\lesssim \int_{s'}^{t'} \|f\|_{C^\alpha} \||\varphi_{s'} - \varphi_r|^\alpha\|_{L_p(\Omega)}\, dr \\
&\lesssim \int_{s'}^{t'} \|f\|_{C^\alpha} \|\varphi_{s'} - \varphi_r\|^\alpha_{L_{\alpha p}(\Omega)}\, dr \\
&\lesssim \int_{s'}^{t'} \|f\|_{C^\alpha} \|\varphi_{s'} - \varphi_r\|^\alpha_{L_p(\Omega)}\, dr \\
&\lesssim \int_{s'}^{t'} \|f\|_{C^\alpha} [\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]} |r-s'|^{\tau\alpha}\, dr \\
&\lesssim \|f\|_{C^\alpha} [\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]} |t'-s'|^{\tau\alpha+1} \\
&\lesssim \|f\|_{C^\alpha} [\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]} |t'-s'|^{\varepsilon_3+1}
\end{aligned}
$$

where the last inequality is followed by the fact that $\tau - (1-\alpha)/2 > \varepsilon_3$ implying $\tau\alpha > \varepsilon_3$. Thus, condition (S3) is satisfied with $K_1 = N\|f\|_{C^\alpha}[\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]}$ and the proof is complete. □

4.6 Proof of the main result

Let us first recall the following simple fact:

If g is a predictable process, then by Burkholder-Davis-Gundy and Hölder 's inequalities, for $p \geq 2$, we have

$$\mathbb{E}\left|\int_s^t g_r\, dW_r\right|^p \leq N\mathbb{E}\int_s^t |g_r|^p\, dr|t-s|^{(p-2)/2}$$

with $N = N(p)$.

Fix $0 \leq s \leq t \leq 1$. Denote $G_u := \int_s^u g_r\, dW_r$ for $u \in [s,t]$. The above inequality implies

$$[\![G]\!]_{\mathscr{C}_p^{1/2-\varepsilon},[s,t]} \leq N\|g\|_{L_p(\Omega\times[s,t])} \tag{4.13}$$

whenever $p \geq 1/\varepsilon$.

Indeed, we have

$$[\![G]\!]_{\mathscr{C}_p^{1/2-\varepsilon},[s,t]} = \sup_{u\in[s,t]}\left\|\int_s^u g_r\, dW_r\right\|_{L_p(\Omega)} + \sup_{\substack{u,u'\in[s,t]\\ u\neq u'}}\frac{\left\|\int_u^{u'} g_r\, dW_r\right\|_{L_p(\Omega)}}{|u-u'|^{1/2-\varepsilon}} \tag{4.14}$$

We can see that for all $u, u' \in [s,t]$ and $p \geq 1/\varepsilon$,

$$\begin{aligned}
\left(\frac{\left\|\int_u^{u'} g_r\, dW_r\right\|_{L_p(\Omega)}}{|u-u'|^{1/2-\varepsilon}}\right)^p &= \frac{\mathbb{E}\left|\int_u^{u'} g_r\, dW_r\right|^p}{|u-u'|^{p/2-p\varepsilon}}\\
&\leq \frac{\mathbb{E}\left|\int_u^{u'} g_r\, dW_r\right|^p}{|u-u'|^{(p-2)/2}}\\
&\leq N\mathbb{E}\int_u^{u'} |g_r|^p\, dr\\
&\leq N\mathbb{E}\int_s^t |g_r|^p\, dr\\
&\leq N\|g\|^p_{L_p(\Omega\times[s,t])}
\end{aligned}$$

Therefore,

$$\sup_{\substack{u,u'\in[s,t]\\ u\neq u'}}\frac{\left\|\int_u^{u'} g_r\, dW_r\right\|_{L_p(\Omega)}}{|u-u'|^{1/2-\varepsilon}} \leq N\|g\|_{L_p(\Omega\times[s,t])} \tag{4.15}$$

Moreover, for all $u \in [s,t]$,

$$\Big\| \int_s^u g_r \, dW_r \Big\|^p_{L_p(\Omega)} = \mathbb{E} \Big| \int_s^u g_r \, dW_r \Big|^p \le N \mathbb{E} \int_s^u |g_r|^p \, dr |u-s|^{(p-2)/2}$$
$$\le N \mathbb{E} \int_s^t |g_r|^p \, dr$$
$$\le N \|g\|^p_{L_p(\Omega \times [s,t])}$$

Thus,

$$\sup_{u \in [s,t]} \Big\| \int_s^u g_r \, dW_r \Big\|_{L_p(\Omega)} \le N \|g\|_{L_p(\Omega \times [s,t])} \tag{4.16}$$

From (4.14), (4.15), (4.16), we obtain (4.13).

Now we are going to present the proof of the main result as follows.

Proof. Without loss of generality, we will assume that p is sufficiently large and τ is sufficiently close to $1/2$.

Let us rewrite the equation for X^n as

$$dX^n_t = b(X^n_{\kappa_n(t)}) \, dt + \big[\sigma(X_t) + (\sigma(X^n_t) - \sigma(X_t)) + R^n_r\big] \, dW_t$$

where $R^n_t := \sigma(X^n_{\kappa_n(t)}) - \sigma(X^n_t)$ is an adapted process such that for all $t \in [0,1]$, ones have

$$\begin{aligned} \|R^n_t\|_{L_p(\Omega)} = \|\sigma(X^n_{\kappa_n(t)}) - \sigma(X^n_t)\|_{L_p(\Omega)} &\le \|\sigma\|_{C^1} \|X^n_{\kappa_n(t)} - X^n_t\|_{L_p(\Omega)} \\ &\le N \|b(X^n_{\kappa_n(t)})(t - \kappa_n(t)) + \sigma(X^n_{\kappa_n(t)})(W_t - W_{\kappa_n(t)})\|_{L_p(\Omega)} \\ &\le N\|b\|_{C^0} |t - \kappa_n(t)| + N\|\sigma\|_{C^0} \|W_t - W_{\kappa_n(t)}\|_{L_p(\Omega)} \\ &\le N|t - \kappa_n(t)|^{1/2} \\ &\le N n^{-1/2} \end{aligned} \tag{4.17}$$

Let us denote

$$\begin{aligned} \varphi^n_t &:= \int_0^t (b(X^n_{\kappa_n(r)}) - b(X_r)) \, dr \\ \mathscr{Q}^n_t &:= \int_0^t (\sigma(X^n_r) - \sigma(X_r)) \, dW_r \\ \mathscr{R}^n_t &:= \int_0^t R^n_r \, dW_r \end{aligned}$$

Take some $0 \le S \le T \le 1$ and choose $\varepsilon_1 \in (0, \varepsilon/2)$. Then, taking into account (4.11), for any $S \le s < t \le T$,

$$\|\varphi^n_t - \varphi^n_s\|_{L_p(\Omega)} = \Big\| \int_s^t (b(X_r) - b(X^n_{\kappa_n(r)})) \, dr \Big\|_{L_p(\Omega)}$$

$$\leq \Big\| \int_s^t (b(X_r) - b(X_r^n))\, dr \Big\|_{L_p(\Omega)} + N|t-s|^{1/2+\varepsilon_1} n^{-1/2+\varepsilon} \tag{4.18}$$

We wish to apply Lemma 4.5, with $\varphi = \varphi^n + \mathscr{Q}^n + \mathscr{R}^n$.
Choose $\varepsilon_2 > 0$ sufficiently small such that $\tau = 1/2 - \varepsilon_2$ satisfies (4.12). Therefore, for sufficiently small $\varepsilon_3, \varepsilon_4 > 0$,

$$\begin{aligned}
\Big\| \int_s^t (b(X_r) - b(X_r^n))\, dr \Big\|_{L_p(\Omega)} &= \Big\| \int_s^t (b(X_r) - b(X_r + \varphi_r))\, dr \Big\|_{L_p(\Omega)} \\
&\leq N|t-s|^{1+\varepsilon_3} [\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]} + N|t-s|^{1/2+\varepsilon_4} [\![\varphi]\!]_{\mathscr{C}_p^0,[s,t]} \\
&\leq N|t-s|^{1/2+\varepsilon_4 \wedge (1/2+\varepsilon_3)} \big([\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[s,t]} + [\![\mathscr{Q}^n]\!]_{\mathscr{C}_p^\tau,[s,t]} + [\![\mathscr{R}^n]\!]_{\mathscr{C}_p^\tau,[s,t]} \big)
\end{aligned} \tag{4.19}$$

By (4.13), for sufficiently large p, it can be seen that

$$\begin{aligned}
[\![\mathscr{Q}^n]\!]_{\mathscr{C}_p^\tau,[s,t]} &\leq N \|\sigma(X^n) - \sigma(X)\|_{L_p(\Omega \times [0,T])} \\
&\leq N \left(\int_0^T \mathbb{E}|\sigma(X_r^n) - \sigma(X_r)|^p\, dr \right)^{1/p} \\
&\leq N \|\sigma\|_{C^1} \left(\int_0^T \mathbb{E}|X_r^n - X_r|^p\, dr \right)^{1/p} \\
&\leq N \|X^n - X\|_{L_p(\Omega \times [0,T])}
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
[\![\mathscr{R}^n]\!]_{\mathscr{C}_p^\tau,[s,t]} &\leq N \|R^n\|_{L_p(\Omega \times [0,T])} \\
&\leq N \left(\int_0^T \mathbb{E}|R_r^n|^p\, dr \right)^{1/p} \\
&\leq N \left(\int_0^T \|R_r^n\|_{L_p(\Omega)}^p\, dr \right)^{1/p} \\
&\leq N n^{-1/2}
\end{aligned} \tag{4.21}$$

where in the last inequality (4.17) is used.
From (4.19), (4.20), (4.21) and using $\tau < 1/2$, ones can obtain

$$\begin{aligned}
&\Big\| \int_s^t (b(X_r) - b(X_r^n))\, dr \Big\|_{L_p(\Omega)} \\
&\quad \leq N|t-s|^\tau |T-S|^{\varepsilon_5} \big([\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[S,T]} + \|X - X^n\|_{L_p(\Omega \times [0,T])} + n^{-1/2} \big)
\end{aligned}$$

with some $\varepsilon_5 > 0$. Combining with (4.18) gives

$$\begin{aligned}
\|\varphi_t^n - \varphi_s^n\|_{L_p(\Omega)} &\leq N|t-s|^{1/2+\varepsilon_1} n^{-1/2+\varepsilon} \\
&\quad + N|t-s|^\tau |T-S|^{\varepsilon_5} \big([\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[S,T]} + \|X - X^n\|_{L_p(\Omega \times [0,T])} + n^{-1/2} \big)
\end{aligned}$$

$$\leq N|t-s|^{\tau} n^{-1/2+\varepsilon}$$
$$+ N|t-s|^{\tau}|T-S|^{\varepsilon_5}\left([\![\varphi^n]\!]_{\mathscr{C}_p^{\tau},[S,T]} + \|X-X^n\|_{L_p(\Omega\times[0,T])} + n^{-1/2}\right)$$

where the last inequality is followed by the fact that $\tau < 1/2$.
Hence,

$$\sup_{\substack{s,t\in[S,T]\\ s\neq t}} \frac{\|\varphi_t^n-\varphi_s^n\|_{L_p(\Omega)}}{|t-s|^{\tau}} \leq Nn^{-1/2+\varepsilon} + N|T-S|^{\varepsilon_5}\left([\![\varphi^n]\!]_{\mathscr{C}_p^{\tau},[S,T]} + \|X-X^n\|_{L_p(\Omega\times[0,T])} + n^{-1/2}\right) \tag{4.22}$$

Moreover, for all $t \in [S,T]$,

$$\begin{aligned}
\|\varphi_t^n\|_{L_p(\Omega)} &\leq \|\varphi_S^n\|_{L_p(\Omega)} + \|\varphi_t^n - \varphi_S^n\|_{L_p(\Omega)} \\
&\leq \|\varphi_S^n\|_{L_p(\Omega)} + N|t-S|^{\tau} n^{-1/2+\varepsilon} \\
&\qquad + N|t-S|^{\tau}|T-S|^{\varepsilon_5}\left([\![\varphi^n]\!]_{\mathscr{C}_p^{\tau},[S,T]} + \|X-X^n\|_{L_p(\Omega\times[0,T])} + n^{-1/2}\right) \\
&\leq \|\varphi_S^n\|_{L_p(\Omega)} + Nn^{-1/2+\varepsilon} \\
&\qquad + N|T-S|^{\varepsilon_5}\left([\![\varphi^n]\!]_{\mathscr{C}_p^{\tau},[S,T]} + \|X-X^n\|_{L_p(\Omega\times[0,T])} + n^{-1/2}\right)
\end{aligned}$$

This implies that

$$\begin{aligned}
\sup_{t\in[S,T]} \|\varphi_t^n\|_{L_p(\Omega)} &\leq \|\varphi_S^n\|_{L_p(\Omega)} + Nn^{-1/2+\varepsilon} \\
&\qquad + N|T-S|^{\varepsilon_5}\left([\![\varphi^n]\!]_{\mathscr{C}_p^{\tau},[S,T]} + \|X-X^n\|_{L_p(\Omega\times[0,T])} + n^{-1/2}\right)
\end{aligned} \tag{4.23}$$

From (4.22) and (4.23), it can be proved that

$$\begin{aligned}
[\![\varphi^n]\!]_{\mathscr{C}_p^{\tau},[S,T]} &\leq N\|\varphi_S^n\|_{L_p(\Omega)} + |T-S|^{\varepsilon_5}[\![\varphi^n]\!]_{\mathscr{C}_p^{\tau},[S,T]} \\
&\qquad + N\|X-X^n\|_{L_p(\Omega\times[0,T])} + Nn^{-1/2+\varepsilon}
\end{aligned} \tag{4.24}$$

Fix an $m \in \mathbb{N}$ (not depending on n) such that $Nm^{-\varepsilon_5} \leq 1/2$. Whenever $|S-T| \leq m^{-1}$, the second term on the right-hand side of (4.24) can be therefore discarded and so ones in particular get

$$[\![\varphi^n]\!]_{\mathscr{C}_p^{\tau},[S,T]} \leq N\|\varphi_S^n\|_{L_p(\Omega)} + N\|X-X^n\|_{L_p(\Omega\times[0,T])} + Nn^{-1/2+\varepsilon} \tag{4.25}$$

and thus also

$$\|\varphi_T^n\|_{L_p(\Omega)} \leq N\|\varphi_S^n\|_{L_p(\Omega)} + N\|X-X^n\|_{L_p(\Omega\times[0,T])} + Nn^{-1/2+\varepsilon}$$

Iterating this inequality at most m times, ones therefore get

$$\begin{aligned}\|\varphi_t^n\|_{L_p(\Omega)} &\le N\|\varphi_0^n\|_{L_p(\Omega)} + N\|X - X^n\|_{L_p(\Omega\times[0,T])} + Nn^{-1/2+\varepsilon} \\ &\le N\|X - X^n\|_{L_p(\Omega\times[0,T])} + Nn^{-1/2+\varepsilon}\end{aligned} \tag{4.26}$$

for all $t \in [0, T]$.

From (4.20), (4.21) and (4.26), the following holds

$$\begin{aligned}\sup_{t\in[0,T]} \left\|X_t - X_t^n\right\|_{L_p(\Omega)}^p &= \sup_{t\in[0,T]} \left\|\varphi_t^n + \mathscr{Q}_t^n + \mathscr{R}_t^n\right\|_{L_p(\Omega)}^p \\ &\le \sup_{t\in[0,T]} \left(\left\|\varphi_t^n\right\|_{L_p(\Omega)} + \left\|\mathscr{Q}_t^n\right\|_{L_p(\Omega)} + \left\|\mathscr{R}_t^n\right\|_{L_p(\Omega)}\right)^p \\ &\le N \sup_{t\in[0,T]} \left\|\varphi_t^n\right\|_{L_p(\Omega)}^p \\ &\quad + N \sup_{t\in[0,T]} \left\|\mathscr{Q}_t^n\right\|_{L_p(\Omega)}^p + N \sup_{t\in[0,T]} \left\|\mathscr{R}_t^n\right\|_{L_p(\Omega)}^p \\ &\le N\|X_t - X_t^n\|_{L_p(\Omega\times[0,T])}^p + Nn^{-p(1/2-\varepsilon)} \\ &\le N \int_0^T \sup_{t\in[0,T]} \|X_t - X_t^n\|_{L_p(\Omega)}^p \, dt + Nn^{-p(1/2-\varepsilon)}\end{aligned}$$

Gronwall 's lemma then yields

$$\sup_{t\in[0,T]} \left\|X_t - X_t^n\right\|_{L_p(\Omega)} \le Nn^{-1/2+\varepsilon} \tag{4.27}$$

Putting (4.25)-(4.26)-(4.27) together, it can be obtained that

$$\begin{aligned}[\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[0,1]} &\le N \sum_{i=1}^{m-1} [\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[\frac{i}{m},\frac{i+1}{m}]} \\ &\le N \sum_{i=1}^{m-1} \left(\|\varphi_{\frac{i}{m}}^n\|_{L_p(\Omega)} + \|X - X^n\|_{L_p(\Omega\times[0,\frac{i+1}{m}])} + n^{-1/2+\varepsilon}\right) \\ &\le N \sum_{i=1}^{m-1} \left(\|X - X^n\|_{L_p(\Omega\times[0,\frac{i+1}{m}])} + n^{-1/2+\varepsilon}\right) \\ &\le Nn^{-1/2+\varepsilon}\end{aligned}$$

Therefore, recalling (4.20) and (4.21) again,

$$\begin{aligned}[\![X - X^n]\!]_{\mathscr{C}_p^\tau,[0,1]} &\le [\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[0,1]} + [\![\mathscr{Q}^n]\!]_{\mathscr{C}_p^\tau,[0,1]} + [\![\mathscr{R}^n]\!]_{\mathscr{C}_p^\tau,[0,1]} \\ &\le Nn^{-1/2+\varepsilon} + \sup_{t\in[0,1]} \left\|X_t - X_t^n\right\|_{L_p(\Omega)} \\ &\le Nn^{-1/2+\varepsilon}\end{aligned}$$

as desired. □

5 Appendix

Lemma 5.1. *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that $\nabla u, \nabla^2 u$ exist and are continuous on $\mathbb{R}^d$. Then for all $i \in \{1, \dots, d\}$, we have*

$$\|\partial_{x_i} u\|_{C^0} \lesssim \|u\|_{C^1}, \qquad \|\partial_{x_i} u\|_{C^1} \lesssim \|u\|_{C^2}$$

Proof. The second bound is obvious. Now we will prove the first bound as follows.
We have

$$\begin{aligned}\|\partial_{x_i} u\|_{C^0} &= \sup_{x\in\mathbb{R}^d} |\partial_{x_i} u(x)| \\ &= \sup_{x\in\mathbb{R}^d} \left|\lim_{h\to 0} \frac{u(x+he_i)-u(x)}{h}\right|\end{aligned}$$

For all $x \in \mathbb{R}^d$,

$$\begin{aligned}\left|\frac{u(x+he_i)-u(x)}{h}\right| = \frac{|u(x+he_i)-u(x)|}{|he_i|} &= \frac{|u(x+he_i)-u(x)|}{|x+he_i-x|} \\ &\le [u]_{C^1} \lesssim \|u\|_{C^1}\end{aligned}$$

Hence, $\|\partial_{x_i} u\|_{C^0} \lesssim \|u\|_{C^1}$. □

Lemma 5.2. *Let $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy $\sigma \in C^2$ and $\sigma\sigma^T \ge \lambda I$ for some positive constant λ. Suppose that $Z \sim \mathcal{N}(0, \sigma\sigma^T(x)\delta)$. Fix $\alpha > 0$, then $\mathbb{E}|Z|^\alpha \lesssim \delta^{\alpha/2}$ for all $x \in \mathbb{R}^d$.*

Proof. Take an arbitrary $x \in \mathbb{R}^d$.
Since $\sigma\sigma^T(x)$ is a positive definite matrix and $Z \sim \mathcal{N}(0, \sigma\sigma^T(x)\delta)$, $Z = (\sigma\sigma^T(x))^{-1/2}Y$ for $Y \sim \mathcal{N}(0, \delta I)$.
Moreover, we have

- $\sigma\sigma^T(x)$ is bounded for all $x \in \mathbb{R}^d$, then $(\sigma\sigma^T(x))^{1/2}$ is bounded for all $x \in \mathbb{R}^d$
- $\det((\sigma\sigma^T(x))^{1/2}) = \det(\sigma\sigma^T(x))^{1/2} \ge \lambda^{1/2}$ for all $x \in \mathbb{R}^d$

then $(\sigma\sigma^T(x))^{-1/2}$ is bounded for all $x \in \mathbb{R}^d$. Therefore, there exists a constant N not depending on x such that

$$\begin{aligned}\mathbb{E}|Z|^\alpha &= \mathbb{E}|(\sigma\sigma^T(x))^{-1/2}Y|^\alpha \\ &\le N\mathbb{E}|Y|^\alpha\end{aligned}$$

We can see that $Y \sim \mathcal{N}(0, \delta I)$, then $|Y|^2 \sim \delta\chi^2_d$. Hence, we obtain

$$\begin{aligned}\mathbb{E}|Z|^\alpha &\lesssim \mathbb{E}|Y|^\alpha \\ &\lesssim \delta^{\alpha/2}\mathbb{E}(\chi^2_d)^{\alpha/2}\end{aligned}$$

$$\lesssim \delta^{\alpha/2}$$

due to the fact that moments of χ_d^2 are finite for any positive orders. We complete our proof. $\square$

Lemma 5.3. *Let $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy $\sigma \in C^2$ and $\sigma\sigma^T \geq \lambda I$ for some positive constant λ. For convenience, let us denote $\sigma\sigma^T(x) := M(x)$. Then for all $x, y \in \mathbb{R}^d$, we have*

$$\left|\frac{\partial_{x_j}(\det(M(x)))}{\det(M(x))}\right| \lesssim 1, \qquad \left|\frac{\partial_{x_j}(y^T M(x)^{-1} y)}{\delta}\right| \lesssim \delta^{-1}|y|^2$$

Proof. For the first bound, we have for all $x \in \mathbb{R}^d$,

$$\left|\frac{\partial_{x_j}(\det(M(x)))}{\det(M(x))}\right| = \left|Tr\left(M(x)^{-1}\partial_{x_i} M(x)\right)\right| \lesssim 1$$

due to the fact that σ is bounded with bounded first partial derivatives and $\sigma\sigma^T \geq \lambda I$.
For the second bound, we have for all $x, y \in \mathbb{R}^d$,

$$\begin{aligned}\left|\partial_{z_j}(y^T M(x,z)^{-1} y)\right| &= \left|y^T \partial_{z_j}(M(x)^{-1})y\right| \\ &= \left|y^T(-M(x)^{-1})\partial_{x_i}(M(x))M(x)^{-1}y\right| \\ &\lesssim |y|^2\end{aligned}$$

since σ is bounded with bounded first partial derivatives and $\sigma\sigma^T \geq \lambda I$. Therefore, we obtain the second bound. $\square$

Lemma 5.4. *Let $b : \mathbb{R}^d \to \mathbb{R}$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfy $b \in C^\alpha$ for $\alpha \in (0,1)$, $\sigma \in C^2$ and $\sigma\sigma^T \geq \lambda I$ for some positive constant λ. For $x \in \mathbb{R}^d$, define*

$$g(x) := \int_{\mathbb{R}^d} (b_i(x+y) - b_i(x)) p_{\sigma\sigma^T(x)\delta}(y)\, dy$$

Then $\|g\|_{C^\alpha} \lesssim \|b\|_{C^\alpha}$.

Proof. We can see that

$$\begin{aligned}\|g\|_{C^\alpha} &= \| \int_{\mathbb{R}^d} (b_i(.+y) - b_i(.)) p_{\sigma\sigma^T(.)\delta}(y)\, dy\|_{C^\alpha} \\ &\leq \| \int_{\mathbb{R}^d} b_i(.+y) p_{\sigma\sigma^T(.)\delta}(y)\, dy\|_{C^\alpha} + \| \int_{\mathbb{R}^d} b_i(.) p_{\sigma\sigma^T(.)\delta}(y)\, dy\|_{C^\alpha} \\ &\lesssim \| \int_{\mathbb{R}^d} b_i(.+y) p_{\sigma\sigma^T(.)\delta}(y)\, dy\|_{C^\alpha} + \|b\|_{C^\alpha}\end{aligned}$$

Now we need to prove that

$$\| \int_{\mathbb{R}^d} b_i(.+y) p_{\sigma\sigma^T(.)\delta}(y)\, dy\|_{C^\alpha} \lesssim \|b\|_{C^\alpha}$$

Indeed, we have

$$\begin{aligned}
\| \int_{\mathbb{R}^d} b_i(. + z) p_{\sigma\sigma^T(.)\delta}(z)\, dz \|_{C^\alpha} &= \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} b_i(x+z) p_{\sigma\sigma^T(x)\delta}(z)\, dz \right| \\
&\quad + \sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{\left| \int_{\mathbb{R}^d} (b_i(x+z) p_{\sigma\sigma^T(x)\delta}(z) - b_i(y+z) p_{\sigma\sigma^T(y)\delta}(z))\, dz \right|}{|x-y|^\alpha} \\
&\lesssim \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |b_i(x+z)| p_{\sigma\sigma^T(x)\delta}(z)\, dz \\
&\quad + \sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{\int_{\mathbb{R}^d} |(b_i(x+z) - b_i(y+z)) p_{\sigma\sigma^T(x)\delta}(z)|\, dz}{|x-y|^\alpha} \\
&\quad + \sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{\int_{\mathbb{R}^d} |b_i(y+z)(p_{\sigma\sigma^T(x)\delta}(z) - p_{\sigma\sigma^T(y)\delta}(z))|\, dz}{|x-y|^\alpha} \\
&\lesssim \|b\|_{C^\alpha} + \|b\|_{C^0} \sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{\int_{\mathbb{R}^d} |p_{\sigma\sigma^T(x)\delta}(z) - p_{\sigma\sigma^T(y)\delta}(z)|\, dz}{|x-y|^\alpha}
\end{aligned}$$

It suffices to prove that $\sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{\int_{\mathbb{R}^d} |p_{\sigma\sigma^T(x)\delta}(z) - p_{\sigma\sigma^T(y)\delta}(z)|\, dz}{|x-y|^\alpha}$ is bounded. Using a similar argument as (3.15) and Lemma 5.3, we can see that for all $i \in \{1, \ldots, d\}$ and all $x, z \in \mathbb{R}^d$,

$$\begin{aligned}
|\partial_{x_j}(p_{\sigma\sigma^T(x)\delta}(z))| &= \left| -\frac{1}{2} \left(\frac{\partial_{x_j}(\det(\sigma\sigma^T(x)))}{\det(\sigma\sigma^T(x))} + \frac{\partial_{x_j}(z^T(\sigma\sigma^T(x))^{-1}z)}{\delta} \right) p_{\sigma\sigma^T(x)\delta}(z) \right| \\
&\lesssim (1 + \delta^{-1}|z|^2) p_{\sigma\sigma^T(x)\delta}(z)
\end{aligned}$$

Therefore, using the fact that if $Z \sim \mathcal{N}(0, \sigma\sigma^T(x)\delta)$, then $\mathbb{E}|Z|^\alpha \lesssim \delta^{\alpha/2}$ from Lemma 5.2, we have for all $x, y \in \mathbb{R}^d$ satisfying $|x-y| \leq 1$,

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|p_{\sigma\sigma^T(x)\delta}(z) - p_{\sigma\sigma^T(y)\delta}(z)|}{|x-y|^\alpha}\, dz &\leq \int_{\mathbb{R}^d} \frac{|p_{\sigma\sigma^T(x)\delta}(z) - p_{\sigma\sigma^T(y)\delta}(z)|}{|x-y|}\, dz \\
&\lesssim \int_{\mathbb{R}^d} \left(\sum_{i=1}^d |\partial_{x_i}(p_{\sigma\sigma^T(\xi)\delta}(z))|^2 \right)^{1/2} dz \\
&\lesssim \int_{\mathbb{R}^d} (1 + \delta^{-1}|z|^2) p_{\sigma\sigma^T(\xi)\delta}(z)\, dz \\
&\lesssim 2
\end{aligned}$$

for some $\xi \in \mathbb{R}^d$. Moreover, for all $x, y \in \mathbb{R}^d$ satisfying $|x-y| > 1$, we have

$$\int_{\mathbb{R}^d} \frac{|p_{\sigma\sigma^T(x)\delta}(z) - p_{\sigma\sigma^T(y)\delta}(z)|}{|x-y|^\alpha}\, dz \leq \int_{\mathbb{R}^d} |p_{\sigma\sigma^T(x)\delta}(z) - p_{\sigma\sigma^T(y)\delta}(z)|\, dz \leq 2$$

Thus, $\sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{\int_{\mathbb{R}^d} |p_{\sigma\sigma^T(x)\delta}(z) - p_{\sigma\sigma^T(y)\delta}(z)|\, dz}{|x-y|^\alpha}$ is bounded by 2 and we complete our proof. $\square$

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