The Girsanov theorem

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Preliminaries

Fatou 's lemma

Let X_n be random variables satisfying $X_n \geq 0$ a.s for all n. Then we have

$$\mathbb{E}(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} \mathbb{E}(X_n)$$

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Cauchy - Schwarz inequality

If $X, Y \in L^2$, then $XY \in L^1$ and we have

$$|\mathbb{E}(XY)| \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}$$

Preliminaries

Ito 's formula for semimartingale

Let X_t be a semimartingale with continuous paths and $f \in C^2$. Then for almost every ω

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\left\langle X\right\rangle_s$$

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Ito 's product formula

If X and Y are semimartingales with continuous paths, then

$$X_tY_t = X_0Y_0 + \int_0^t X_sdY_s + \int_0^t Y_sdX_s + \langle X, Y\rangle_t$$

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From now on, will we suppose that the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions.

Lemma 13.1

Suppose Y is a continuous local martingale with $Y_0 = 0$ and $Z_t = e^{Y_t - \frac{\langle Y \rangle_t}{2}}$. If $\langle Y \rangle_t$ is a bounded random variable for each t, then $\mathbb{E}(|Z_t|^p) < \infty$ for each $p > 1$ and each t.

Lemma 13.2

Suppose A_t is a continuous increasing process adapted to the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Let X be a bounded random variable, H be a bounded adapted process, $s < t$ and $B \in \mathcal{F}_s$. Then

$$\mathbb{E}\left(\int_s^t XH_r dA_r; B\right) = \mathbb{E}\left(\int_s^t \mathbb{E}[X|\mathcal{F}_r]H_r dA_r; B\right)$$

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Proposition 3.14

Suppose $\mathcal{F}_t$ is a filtration satisfying the usual conditions, A_t is an adapted process with paths that are increasing, right continuous with left limits and X is a non-negative integrable random variable. Then

$$\mathbb{E}\left(\int_0^t X dA_s\right) = \mathbb{E}\left(\int_0^t \mathbb{E}[X|\mathcal{F}_s] dA_s\right)$$

The Girsanov theorem

Let M_t is a non-negative continuous martingale with $M_0 = 1$ a.s. Define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by $\mathbb{Q}(A) = \mathbb{E}[M_t; A]$ if $A \in \mathcal{F}_t$.

- $\mathbb{Q}$ is well-defined since if $A \in \mathcal{F}_s \subset \mathcal{F}_t$, then

$$M_s = \mathbb{E}[M_t|\mathcal{F}_s] \to 1_A.M_s = \mathbb{E}[1_A.M_t|\mathcal{F}_s] \to \mathbb{E}[1_A.M_s] = \mathbb{E}[1_A.M_t]$$

- $\mathbb{Q}$ is a probability measure since
  - $\mathbb{Q}(A) \geq 0 \ \forall A$
  - $\mathbb{Q}(\Omega) = \mathbb{E}[M_t] = \mathbb{E}[M_0] = 1$ (as M_t is a martingale).
  - For a countable sequence of disjoint sets $A_i \in \mathcal{F}_t$, we have

$$\mathbb{Q}(\cup A_i) = \mathbb{E}[M_t.1_{\cup A_i}] = \sum_i \mathbb{E}[M_t.1_{A_i}] = \sum_i \mathbb{Q}(A_i)$$

The Girsanov theorem

Theorem (The Girsanov theorem)

Suppose W_t is a Brownian motion with respect to $\mathbb{P}$, H is bounded and predictable,

$$M_t = exp\left(\int_0^t H_r dW_r - \frac{1}{2}\int_0^t H_r^2 dr\right),$$

and

$$\mathbb{Q}(B) = \mathbb{E}_{\mathbb{P}}[M_t; B] \text{ if } B \in \mathcal{F}_t$$

Then $W_t - \int_0^t H_r dr$ is a Brownian motion with respect to $\mathbb{Q}$.

Theorem (Levy 's theorem)

Let M_t be a continuous local martingale with respect to a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions such that $M_0 = 0$ and $\langle M \rangle_t = t$. then M_t is a Brownian motion with respect to $\{\mathcal{F}_t\}$.

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Theorem

Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures such that $\mathbb{P}(A) = 0$ implies $\mathbb{Q}(A) = 0$ for all $A \in \mathcal{F}$. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that if $A \in \mathcal{F}$ and $\mathbb{P}(A) < \delta$, then $\mathbb{Q}(A) < \epsilon$.

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Applications

- The Girsanov theorem has many applications, including to financial mathematics.
- Here, we give an example of the use of the Girsanov theorem to compute the probability that Brownian motion crosses a line $a + bt$ by time t_0 where $a > 0$, i.e compute $\mathbb{P}(\exists t \leq t_0 : W_t = a + bt)$, where W is a Brownian motion.

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References

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