Approximation of SDEs - a stochastic sewing approach

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Table of contents

Table of Contents

Before stating the Stochastic sewing lemma, let us introduce some preliminaries and notations:

- Here, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$.
- For $S < T$, we denote by $[S, T]_{\leq}$ the simplex $\{(s, t) \in [S, T]^2 : s \leq t\}$.
- We will write $\mathbb{E}^t(\cdot)$ as a shorthand for the conditional expectation $\mathbb{E}(\cdot|\mathcal{F}_t)$.

Theorem 1.1 (Stochastic sewing lemma - Khoa Lê (2020))

Let $p \geq 2$, $0 \leq S \leq T \leq 1$ and let $A_{\cdot,\cdot}$ be a function $[S, T]_{\leq} \to L_p(\Omega, \mathbb{R}^d)$ such that for any $(s, t) \in [S, T]_{\leq}$, the random vector $A_{s,t}$ is $\mathcal{F}_t$-measurable. Suppose that for some $\varepsilon_1, \varepsilon_2 > 0$ and C_1, C_2, the bounds

(S1) $\|A_{s,t}\|_{L_p(\Omega)} \leq C_1|t-s|^{1/2+\varepsilon_1}$

(S2) $\|\mathbb{E}^s \delta A_{s,u,t}\|_{L_p(\Omega)} \leq C_2|t-s|^{1+\varepsilon_2}$ *where* $\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}$

hold for all $S \leq s \leq u \leq t \leq T$. Then there exists a unique map $\mathcal{A} : [S, T] \to L_p(\Omega, \mathbb{R}^d)$ such that $\mathcal{A}_S = 0$, $\mathcal{A}_t$ is $\mathcal{F}_t$-measurable for all $t \in [S, T]$ and the following bounds hold for some constants $K_1, K_2 > 0$:

(S3) $\|\mathcal{A}_t - \mathcal{A}_s - A_{s,t}\|_{L_p(\Omega)} \leq K_1|t-s|^{1/2+\varepsilon_1} \quad \forall (s,t) \in [S, T]_{\leq}$

(S4) $\|\mathbb{E}^s(\mathcal{A}_t - \mathcal{A}_s - A_{s,t})\|_{L_p(\Omega)} \leq K_2|t-s|^{1+\varepsilon_2} \quad \forall (s,t) \in [S, T]_{\leq}$

Moreover, there exists a constant K depending only on $\varepsilon_1, \varepsilon_2, d, p$ such that $\mathcal{A}$ satisfies the bound

$$(S5) \quad \|\mathcal{A}_t - \mathcal{A}_s\|_{L_p(\Omega)} \leq KC_1|t-s|^{1/2+\varepsilon_1} + KC_2|t-s|^{1+\varepsilon_2} \quad \forall (s,t) \in [S, T]_{\leq}$$

Table of Contents

1. Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a d-dimensional Brownian motion $(W_t)_{t\in[0,1]}$ for arbitrary $d \in \mathbb{Z}^+$. For times $t \in [0,1]$, we will consider the SDE
$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x_0 \tag{2.1}$$
where $b : \mathbb{R}^d \to \mathbb{R}$ is bounded and α-Hölder continuous for some $\alpha \in (0,1)$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is bounded with bounded and Lipschitz-continuous first partial derivatives.
Furthermore, we will assume that σ satisfies the uniform ellipticity condition that $\sigma\sigma^T \geq \lambda I$ in the sense of positive definite matrices for some constant $\lambda > 0$.
2. We will moreover consider the Euler - Maruyama scheme with identical initial condition
$$dX_t^n = b(X_{\kappa_n(t)}^n)dt + \sigma(X_{\kappa_n(t)}^n)dW_t, \quad X_0^n = x_0 \tag{2.2}$$
where $\kappa_n(t) := \frac{\lfloor nt \rfloor}{n}$ and $\lfloor \cdot \rfloor$ denotes integer part.

3. Notation
   - Hölder spaces
     Let $A \subset \mathbb{R}^d$ and $(B, |\cdot|)$ be a normed space. For $\alpha \in (0,1]$ and $f : A \to B$, we define the α-Hölder seminorm of f by

$$[f]_{\mathcal{C}^\alpha(A,B)} := \sup_{\substack{x,y \in A \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

For $\alpha \in (0,\infty)$, we then denote by $\mathcal{C}^\alpha(A,B)$ the space of all functions such that for all $l \in (\mathbb{Z}^+)^d$ multiindices with $|l| < \alpha$, the derivative $\partial^l f$ exists and

$$\|f\|_{\mathcal{C}^\alpha(A,B)} := \sum_{|l|<\alpha} \sup_{x \in A} |\partial^l f(x)| + \sum_{\alpha - 1 \leq |l| < \alpha} [\partial^l f]_{\mathcal{C}^{\alpha - |l|}(A,B)} < \infty \qquad (2.3)$$

We furthermore define $\mathcal{C}^0(A,B)$ to be the space of all measurable functions $f : A \to B$ such that

$$\|f\|_{\mathcal{C}^0(A,B)} := \sup_{x \in A} |f(x)| < \infty$$

When no ambiguity can arise, we will simply write $\mathcal{C}^\alpha(A)$ or $\mathcal{C}^\alpha$ to mean $\mathcal{C}^\alpha(A,B)$.
Remark: Using Hölder-spaces, the regularity assumptions on our coefficients can be stated simply as follows: $b \in \mathcal{C}^\alpha$ for some $\alpha \in (0,1)$ and $\sigma \in \mathcal{C}^2$.

3. Notation
   - Shorthands
     Consider the filtration generated by W, that is $\mathcal{F}_t^W := \sigma(W_s : s \leq t)$ and its augmentation $\mathcal{F}_t := \sigma(\mathcal{F}_t^W \cup \mathcal{N})$ by the collection of null sets $\mathcal{N} := \{N \subseteq \Omega : N \subseteq F \text{ for some } F \subseteq \mathcal{F}_\infty^W \text{ such that } \mathbb{P}(F) = 0\}$.

Theorem 2.1 (T. Holland (2022))

Let $\alpha \in (0,1)$, $g \in \mathcal{C}^\alpha$ *and fix* $\varepsilon > 0$. *Let* X *be the solution of (2.1) and* X^n *be given by (2.2). Then for all* $n \in \mathbb{N}$, *we have*

$$|\mathbb{E}g(X_1) - \mathbb{E}g(X_1^n)| \leq Nn^{-\frac{1+\alpha}{2}+\varepsilon}$$

where N *is a constant depending only on* $d, \varepsilon, \alpha, \lambda, \|b\|_{\mathcal{C}^\alpha}, \|\sigma\|_{\mathcal{C}^2}, \|g\|_{\mathcal{C}^\alpha}$.

Using the Feynmann-Kac formula, the weak error can be written as

$$d_g(X, X^n) = |\mathbb{E}\left(u(1, X_1^n) - u(0, X_0^n)\right)|$$

where u is the (unique bounded) solution of the parabolic PDE

$$\begin{cases} \partial_t u(t,x) + Lu(t,x) = 0 & \forall (t,x) \in [0,1) \times \mathbb{R}^d \\ u(1,x) = g(x) & \forall x \in \mathbb{R}^d \end{cases} \tag{2.4}$$

where L is the infinitesimal generator of the solution of (2.1), that is

$$L\phi(t,x) := \sum_{i=1}^{d} b_i(x)\partial_{x_i}\phi(x) + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^T)_{ij}(x)\partial_{x_i x_j}\phi(x)$$

for smooth $\phi : \mathbb{R}^d \to \mathbb{R}$. It can be shown by Ito 's formula that

$$d_g(X, X^n) = \left|\mathbb{E}\int_0^1 \left(\bar{L}u\left(r, X_r^n, X_{\kappa_n(r)}^n\right) - Lu(r, X_r^n)\right) dr\right|$$

where the operator $\bar{L}$ is the "frozen" generator associated with our SDE (2.1), that is

$$\bar{L}\phi(t,x,\bar{x}) := \sum_{i=1}^{d} b_i(\bar{x})\partial_{x_i}\phi(x) + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^T)_{ij}(\bar{x})\partial_{x_i x_j}\phi(x) \tag{2.5}$$

Writing out the operators L and $\bar{L}$ explicitly and using the triangle inequality gives that

$$d_g(X, X^n) \lesssim \sup_{i \in \{1,\dots,d\}} \left| \mathbb{E} \int_0^1 \left(\left(b_i(X^n_{\kappa_n(r)}) - b_i(X^n_r) \right) \partial_{x_i} u(r, X^n_r) \right) \, dr \right| + \sup_{i,j \in \{1,\dots,d\}} \left| \mathbb{E} \int_0^1 \left(\left((\sigma\sigma^T)_{ij}(X^n_{\kappa_n(r)}) - (\sigma\sigma^T)_{ij}(X^n_r) \right) \partial_{x_i x_j} u(r, X^n_r) \right) \, dr \right|$$

Therefore, in order to show that $d_g(X, X^n)$ converges with the desired rate, it suffices to show that the same rate of convergence holds for

$$\left| \mathbb{E} \int_0^1 \left(\left(b_i(X^n_{\kappa_n(r)}) - b_i(X^n_r) \right) \partial_{x_i} u(r, X^n_r) \right) \, dr \right| \tag{2.6}$$

and

$$\left| \mathbb{E} \int_0^1 \left(\left((\sigma\sigma^T)_{ij}(X^n_{\kappa_n(r)}) - (\sigma\sigma^T)_{ij}(X^n_r) \right) \partial_{x_i x_j} u(r, X^n_r) \right) \, dr \right| \tag{2.7}$$

for all i, j indices.

In order to prove the desired rate of convergence for (2.6) and (2.7), the authors have proved **Lemma 2.2**, **Lemma 2.3**, **Corollary 2.4** and **Lemma 2.5** which are stated and proven as follows
Let $\bar{X}^n$ be the driftless scheme given by

$$d\bar{X}^n_t = \sigma(\bar{X}^n_{\kappa_n(t)})dW_t, \quad \bar{X}^n_0 = y \in \mathbb{R}^d$$

Moreover, for convenience, let us denote

$$f_t := \partial_{x_i} u(t, \cdot)$$

Lemma 2.2 (T. Holland (2022))

Let $p \geq 1$, $\varepsilon > 0$ and $T < 1$. For all $n \in \mathbb{N}$ and $(s,t) \in [0,T]_{\leq}$, we have

$$\left\| \int_s^t (b_i(\bar{X}^n_r) - b_i(\bar{X}^n_{\kappa_n(r)}))f_r(\bar{X}^n_r)\, dr \right\|_{L_p} \leq Nn^{-\frac{1+\alpha}{2}+\varepsilon}\Big(|1-T|^{-1/2}|t-s|^{1/2+\varepsilon} + |1-T|^{-1}|t-s|^{1+\varepsilon}\Big) \quad (2.8)$$

for some constant N depending only on $d, \alpha, p, \varepsilon, \lambda, \|b\|_{\mathcal{C}^\alpha}, \|\sigma\|_{\mathcal{C}^2}, \|g\|_{\mathcal{C}^0}$.

Proof of Lemma 2.2
The proof of this lemma will exploit Stochastic sewing lemma that we have introduced in **Section 1**.
By Hölder 's inequality, it suffices to show the bound for $p \geq 2$.
For $0 \leq s \leq t \leq T < 1$, let

$$A_{s,t} := \mathbb{E}^s \int_s^t (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n)) f_r(\bar{X}_s^n) dr$$

The authors have shown that the process $A_{s,t}$ will satisfy condition (S1) with $\varepsilon_1 = \varepsilon$ and $C_1 = N|1 - T|^{-1/2} n^{-\frac{1+\alpha}{2} + (2+\alpha)\varepsilon}$ and condition (S2) with $\varepsilon_2 = \varepsilon$ and $C_2 = N|1 - T|^{-1} n^{-\frac{1+\alpha}{2} + (2+\alpha)\varepsilon}$.
Therefore, there exists a map $\mathcal{A} : [0, T] \to L_p(\Omega, \mathbb{R}^d)$ such that $\mathcal{A}_S = 0$, $\mathcal{A}_t$ is $\mathcal{F}_t$-measurable for all $t \in [0, T]$. Moreover, $\mathcal{A}$ satisfies the bounds (S3) and (S4) for some constants $K_1, K_2 > 0$.

Proof of Lemma 2.2 (continued)
Let

$$\bar{\mathcal{A}}_t := \int_0^t (b_i(\bar{X}_r^n) - b_i(\bar{X}_{\kappa_n(r)}^n)) f_r(\bar{X}_r^n)\, dr$$

It can be shown that the process $\bar{\mathcal{A}}$ also satisfies the bound (S3) with $K_1 = N\,|1-T|^{-1/2}$ and the bound (S4) with $K_2 = N\,|1-T|^{-1/2}$.
By the uniqueness of $\mathcal{A}$, we can conclude that $\bar{\mathcal{A}} = \mathcal{A}$.
This implies that $\bar{\mathcal{A}}$ also satisfies the bounded (S5) for some constant $K > 0$ depending only on $\varepsilon_1, \varepsilon_2, d, p$ and the proof is complete.

Lemma 2.3 states that **Lemma 2.2** still holds if we consider X^n in place of the driftless scheme $\bar{X}^n$. **Lemma 2.3** can be obtanied from **Lemma 2.2** via a standard argument using Girsanov 's theorem.

Lemma 2.3 (T. Holland (2022))

Let $p \geq 1$, $\varepsilon > 0$, $T < 1$ and let X^n be given by (2.2). Then for all $n \in \mathbb{N}$ and $(s,t) \in [0,T]_{\leq}$, we have

$$\left\| \int_s^t (b_i(X_r^n) - b_i(X_{\kappa_n(r)}^n)) f_r(X_r^n)\, dr \right\|_{L_p} \leq N n^{-\frac{1+\alpha}{2}+\varepsilon} \Big(|1-T|^{-1/2} |t-s|^{1/2+\varepsilon} + |1-T|^{-1} |t-s|^{1+\varepsilon} \Big)$$

for some constant N depending only on $d, \alpha, p, \varepsilon, \lambda, \|b\|_{\mathcal{C}^\alpha}, \|\sigma\|_{\mathcal{C}^2}, \|g\|_{\mathcal{C}^0}$.

In order to bound (2.6), we need to modify **Lemma 2.3** by extending the domain of integration from $[s,t]$ (where $0 \leq s \leq t \leq T < 1$) to $[0,1]$. This can be done using dyadic points.

Corollary 2.4 (T. Holland (2022))

Let X^n be given by (2.2) and let $\varepsilon > 0$, $p \geq 1$. Then, we have

$$\left\| \int_0^1 (b_i(X_r^n) - b_i(X_{\kappa_n(r)}^n)) f(X_r^n)\, dr \right\|_{L_p} \leq N n^{-\frac{1+\alpha}{2}+\varepsilon}$$

for some constant N depending only on $d, \alpha, p, \varepsilon, \lambda, \|b\|_{\mathcal{C}^\alpha}, \|\sigma\|_{\mathcal{C}^2}, \|g\|_{\mathcal{C}^0}$.

Finally, in order to bound (2.7), we have **Lemma 2.5** and this can be obtained by using convolution and a Schauder-type estimate for u.

Lemma 2.5 (T. Holland (2022))

Let X^n given by (2.2), *let $h \in \mathcal{C}^2$ and $\varepsilon > 0$. Then, for all $n \in \mathbb{N}$ we have*

$$\left| \mathbb{E} \int_0^1 (h(X_r^n) - h(X_{\kappa_n(r)}^n)) f_r'(X_r^n)\, dr \right| \leq N n^{-\frac{1+\alpha}{2}+\varepsilon}$$

for some constant N depending only on $d, \alpha, \lambda, \|b\|_{\mathcal{C}^\alpha}, \|\sigma\|_{\mathcal{C}^2} \|h\|_{\mathcal{C}^2}, \|g\|_{\mathcal{C}^\alpha}$.

- This result utilizes a new developed tool which is the Stochastic sewing lemma to obtain the desired weak rate of convergence.
- This result offers an improvement to earlier results: In earlier results, it is known that the weak rate in our setting is $\alpha/2$, this result have shown a more optimal rate which is $(1+\alpha)/2$.

Table of Contents

Theorem 3.1 (O. Butkovsky, K. Dareiotis, and M. Gerencsér (2021))

Let $\alpha \in (0,1)$, $\varepsilon > 0$, $\tau \in [0,1/2)$ *and* $p \geq 2$. *Let* X *be the solution of (2.1) and* X^n *be given by (2.2). Then for all* $n \in \mathbb{N}$, *the following bound holds*

$$\|X - X^n\|_{\mathcal{C}^\tau([0,1],L_p(\Omega))} \leq Nn^{-1/2+\varepsilon}$$

with some $N = N(p, d, \alpha, \varepsilon, \tau, \lambda, \|b\|_{\mathcal{C}^\alpha}, \|\sigma\|_{\mathcal{C}^2})$.

Given $Q \subset \mathbb{R}^k$ for some $k \in \mathbb{Z}^+$ and a function $f : Q \to L_p(\Omega)$ for $p \geq 2$. We denote

$$[\![f]\!]_{\mathscr{C}_p^\alpha, Q} := \|f\|_{\mathcal{C}^\alpha(Q, L_p(\Omega))}$$

Without loss of generality, we will assume that p is sufficiently large and τ is sufficiently close to $1/2$. Let us denote

$$\varphi_t^n := \int_0^t (b(X_{\kappa_n(r)}^n) - b(X_r))\, dr \tag{3.1}$$

$$\mathscr{Q}_t^n := \int_0^t (\sigma(X_r^n) - \sigma(X_r))\, dW_r \tag{3.2}$$

$$\mathscr{R}_t^n := \int_0^t (\sigma(X_{\kappa_n(t)}^n) - \sigma(X_t^n))\, dW_r \tag{3.3}$$

It is easy to see that $X_t^n - X_t = \varphi_t^n + \mathscr{Q}_t^n + \mathscr{R}_t^n$. Therefore, by triangle inequality,

$$[\![X - X^n]\!]_{\mathscr{C}_p^\tau, [0,1]} \leq [\![\varphi^n]\!]_{\mathscr{C}_p^\tau, [0,1]} + [\![\mathscr{Q}^n]\!]_{\mathscr{C}_p^\tau, [0,1]} + [\![\mathscr{R}^n]\!]_{\mathscr{C}_p^\tau, [0,1]}$$

Now we need to bound $[\![\varphi^n]\!]_{\mathscr{C}_p^\tau, [0,1]}$, $[\![\mathscr{Q}^n]\!]_{\mathscr{C}_p^\tau, [0,1]}$, $[\![\mathscr{R}^n]\!]_{\mathscr{C}_p^\tau, [0,1]}$. In order to obtain these bounds, the authors have proved the following **Lemma 3.2**, **Lemma 3.3** and **Lemma 3.4**.

Lemma 3.2 (O. Butkovsky, K. Dareiotis, and M. Gerencsér (2021))

Let $\varepsilon_1 \in (0, 1/2)$, $\alpha \in (0,1)$ and $p > 0$. Then for all $f \in \mathcal{C}^\alpha$, $0 \le s \le t \le 1$ and $n \in \mathbb{N}$, one has the bound

$$\Big\| \int_s^t (f(\bar{X}_r^n) - f(\bar{X}_{\kappa_n(r)}^n))\,dr \Big\|_{L_p(\Omega)} \le N \|f\|_{\mathcal{C}^\alpha} n^{-1/2+2\varepsilon_1} |t-s|^{1/2+\varepsilon_1}$$

with some $N = N(\alpha, p, d, \varepsilon_1, \lambda, \|\sigma\|_{\mathcal{C}^2})$.

Lemma 3.3 (O. Butkovsky, K. Dareiotis, and M. Gerencsér (2021))

Let $\alpha \in (0,1)$ and take $\varepsilon_1 \in (0, 1/2)$. Then for all $f \in \mathcal{C}^\alpha$, $0 \le s \le t \le 1$, $n \in \mathbb{N}$ and $p > 0$, one has the bound

$$\Big\| \int_s^t (f(X_r^n) - f(X_{\kappa_n(r)}^n))\,dr \Big\|_{L_p(\Omega)} \le N \|f\|_{\mathcal{C}^\alpha} n^{-1/2+2\varepsilon_1} |t-s|^{1/2+\varepsilon_1}$$

with some $N = N(\|b\|_{\mathcal{C}^0}, p, d, \alpha, \varepsilon_1, \lambda, \|\sigma\|_{\mathcal{C}^2})$.

Lemma 3.2 and **Lemma 3.3** are obtained by a similar argument as the one the authors have used to obtain **Lemma 2.2** and **Lemma 2.3** in **Section 2**.

Lemma 3.4 (O. Butkovsky, K. Dareiotis, and M. Gerencsér (2021))

Let $\alpha \in (0,1)$ *and* $\tau \in (0,1]$ *satisfy*

$$\tau + \alpha/2 - 1/2 > 0$$

Let φ *be an adapted process. Then for all sufficiently small* $\varepsilon_3, \varepsilon_4 > 0$, *for all* $f \in \mathcal{C}^\alpha$, $0 \leq s \leq t \leq 1$ *and* $p > 0$, *one has the bound*

$$\Big\| \int_s^t (f(X_r) - f(X_r + \varphi_r))\, dr \Big\|_{L_p(\Omega)} \leq N|t-s|^{1+\varepsilon_3} [\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]} + N|t-s|^{1/2+\varepsilon_4} [\![\varphi]\!]_{\mathscr{C}_p^0,[s,t]} \tag{3.4}$$

with some $N = N(p, d, \alpha, \tau, \lambda, \|\sigma\|_{\mathcal{C}^1})$.

Proof of Lemma 3.4
Set for $s \leq s' \leq t' \leq t$,

$$A_{s',t'} := \mathbb{E}^{s'} \int_{s'}^{t'} (f(X_r) - f(X_r + \varphi_{s'}))\, dr$$

The authors have shown that $A_{s',t'}$ satisfies condition (S1) with $C_1 = N[\![\varphi]\!]_{\mathscr{C}_p^0,[s,t]}$ and $\varepsilon_1 = \varepsilon_3$ satisfying $\varepsilon_3 < \alpha/2$ and condition (S2) with $C_2 = N[\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]}$ and $\varepsilon_2 = \varepsilon_4$ sufficiently small such that $\varepsilon_4 < \tau + \alpha/2 - 1/2$ of Stochastic sewing lemma.
Therefore, there exists a map $\mathcal{A} : [s,t] \to L_p(\Omega, \mathbb{R}^d)$ such that $\mathcal{A}_S = 0$, $\mathcal{A}_t$ is $\mathcal{F}_t$-measurable for all $t \in [s,t]$. Moreover, $\mathcal{A}$ satisfies the bounds (S3) and (S4) for some constants $K_1, K_2 > 0$.
Let

$$\bar{\mathcal{A}}_{t'} := \int_0^{t'} (f(X_r) - f(X_r + \varphi_r))\, dr$$

It can be shown that the process $\bar{\mathcal{A}}$ also satisfies the bound (S3) with $K_1 = N\|f\|_{\mathcal{C}^0}$ and the bound (S4) with $K_2 = N\|f\|_{\mathcal{C}^\alpha}[\![\varphi]\!]_{\mathscr{C}_p^\tau,[s,t]}$.
By the uniqueness of $\mathcal{A}$, it can be concluded that $\bar{\mathcal{A}} = \mathcal{A}$.
This implies that $\bar{\mathcal{A}}$ also satisfies the bounded (S5) for some constant $K > 0$ depending only on $\varepsilon_1, \varepsilon_2, d, p$ and the proof of **Lemma 3.4** is complete.

Back to the proof of the main result, we need to bound $[\varphi^n]_{\mathscr{C}^\tau_p,[0,1]}$, $[\mathscr{Q}^n]_{\mathscr{C}^\tau_p,[0,1]}$, $[\mathscr{R}^n]_{\mathscr{C}^\tau_p,[0,1]}$.
Take some $0 \leq S \leq T \leq 1$ and let $S \leq s < t \leq T$. Using BDG and Hölder 's inequalities yields for sufficiently large p,

$$[\mathscr{Q}^n]_{\mathscr{C}^\tau_p,[s,t]} \leq N\|X - X^n\|_{L_p(\Omega\times[0,T])} \tag{3.5}$$

$$[\mathscr{R}^n]_{\mathscr{C}^\tau_p,[s,t]} \leq N n^{-1/2} \tag{3.6}$$

Choose $\varepsilon_1 \in (0, \varepsilon/2)$. Applying **Lemma 3.3**,

$$\begin{aligned}\|\varphi^n_t - \varphi^n_s\|_{L_p(\Omega)} &= \Big\|\int_s^t (b(X_r) - b(X^n_{\kappa_n(r)}))\,dr\Big\|_{L_p(\Omega)}\\ &\leq \Big\|\int_s^t (b(X_r) - b(X^n_r))\,dr\Big\|_{L_p(\Omega)} + N|t-s|^{1/2+\varepsilon} n^{-1/2+\varepsilon}\end{aligned}$$

Choose $\varepsilon_2 > 0$ sufficiently small such that $\tau = 1/2 - \varepsilon_2$ satisfies $\tau + \alpha/2 - 1/2 > 0$.
Applying **Lemma 3.4** with $\varphi = \varphi^n + \mathscr{Q}^n + \mathscr{R}^n$,

$$\begin{aligned}\Big\|\int_s^t (b(X_r) - b(X^n_r))\,dr\Big\|_{L_p(\Omega)} &= \Big\|\int_s^t (b(X_r) - b(X_r + \varphi_r))\,dr\Big\|_{L_p(\Omega)}\\ &\leq N|t-s|^{1/2+\varepsilon_4\wedge(1/2+\varepsilon_3)}\big([\varphi^n]_{\mathscr{C}^\tau_p,[s,t]} + [\mathscr{Q}^n]_{\mathscr{C}^\tau_p,[s,t]} + [\mathscr{R}^n]_{\mathscr{C}^\tau p,[s,t]}\big)\end{aligned}$$

for sufficiently small $\varepsilon_3, \varepsilon_4 > 0$.

Therefore, for $\varepsilon_5 = \varepsilon_3 \wedge \varepsilon_4$,

$$\|\varphi_t^n - \varphi_s^n\|_{L_p(\Omega)} \leq N|t-s|^{1/2+\varepsilon_1} n^{-1/2+\varepsilon} + N|t-s|^{\tau}|T-S|^{\varepsilon_5}\left([\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[S,T]} + \|X - X^n\|_{L_p(\Omega\times[0,T])} + n^{-1/2}\right) \quad (3.7)$$

By the definition of $[\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[S,T]}$, it can be seen that whenever $|S-T| \leq m^{-1}$ for an $m \in \mathbb{N}$ (not depending on n) such that $Nm^{-\varepsilon_5} \leq 1/2$, ones have

$$[\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[S,T]} \leq N\|X - X^n\|_{L_p(\Omega\times[0,T])} + Nn^{-1/2+\varepsilon}$$

By Gronwall 's lemma,

$$\sup_{t\in[0,T]} \left\|X_t - X_t^n\right\|_{L_p(\Omega)} \leq Nn^{-1/2+\varepsilon}$$

Thus,

$$[\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[0,1]} \leq \sum_{i=0}^{m-1} [\![\varphi^n]\!]_{\mathscr{C}_p^\tau,[\frac{i}{m},\frac{i+1}{m}]} \leq Nn^{-1/2+\varepsilon}$$

Recall that we have shown that

$$[\![\mathscr{Q}^n]\!]_{\mathscr{C}_p^\tau,[s,t]} \leq N\|X - X^n\|_{L_p(\Omega\times[0,T])}$$

$$[\![\mathscr{R}^n]\!]_{\mathscr{C}_p^\tau,[s,t]} \leq Nn^{-1/2}$$

for all $0 \leq S \leq s < t \leq T \leq 1$. Therefore, the proof of the main theorem is complete.

This result of O. Butkovsky, K. Dareiotis, and M. Gerencsér offers several improvements to earlier results:

- In earlier results, the best known rate for our setting is only proven to be $\alpha/2$ while this result has shown that the optimal strong rate is $1/2$.
- All moments of the error can be treated in the same way and the error bound is uniform in time, showing that X and X_n are close as paths.

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