# The continuity of path
# Path properties of Brownian motion

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# Preliminaries

## Borel - Cantelli lemma

Let $A_n$ be a sequence of events. Then

- If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup A_n) = 0$
- If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and $A_n$ are independent, then $\mathbb{P}(\limsup A_n) = 1$

# Preliminaries

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## Fatou 's lemma

Let $X_n$ be random variables satisfying $X_n \geq 0$ a.s for all $n$. Then we have

$$\mathbb{E}(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} \mathbb{E}(X_n)$$

# Preliminaries

## Markov 's inequality

Let X be an random variable and $a > 0$. Then we have

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}$$

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## Application of martingales

If $W_t$ is a Brownian motion, then for $\lambda > 0$

$$\mathbb{P}(\sup_{s \leq t} |W_s| \geq \lambda) \leq 2e^{\frac{-\lambda^2}{2t}}$$

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## Holder continuous

A function $f : [0,1] \to \mathbb{R}$ is called Holder continuous of order $\alpha$ if there exists $M > 0$ s.t $|f(t) - f(s)| \leq M|t-s|^{\alpha}$ $\forall s, t \in [0,1]$.

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# The continuity of path

Let $\mathcal{D}_n = \{\frac{k}{2^n} : 0 \leq k \leq 2^n\}$ and $\mathcal{D} = \cup \mathcal{D}_n$.

**Theorem**

*Let $\{X_t : t \in \mathcal{D}\}$ be real-valued process. If there exists $c_1, \varepsilon, p > 0$ s.t*

$$\mathbb{E}(|X_t - X_s|^p) \leq c_1 |t - s|^{1+\varepsilon} \ \forall s, t \in \mathcal{D}$$

*Then*

1. *There exists $c$ depending on $c_1, \varepsilon, p$ s.t for $M > 0$*
$$\mathbb{P}(\sup_{s,t\in\mathcal{D}, s\neq t} \frac{|X_t - X_s|}{|t-s|^{\frac{\varepsilon}{4p}}} \geq M) \leq \frac{c}{M^p}$$
2. *$X_t$ is uniformly continuous on $\mathcal{D}$ a.s.*

# The continuity of path

**Theorem (Kolmogorov 's continuity theorem)**

*Let* $\{X_t : t \in [0,1]\}$ *be a real-valued process. If there exists* $c_1, \varepsilon, p > 0$ *s.t*

$$\mathbb{E}(|X_t - X_s|^p) \leq c_1 |t-s|^{1+\varepsilon} \ \forall s,t \in [0,1]$$

*Then there exists a version of the process* $X$ *which has continuous paths.*

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# Path properties of Brownian motion

**Theorem**

*With probability 1, the paths of Brownian motion are Holder continuous of order $\alpha < \frac{1}{2}$ on $[0, 1]$.*

# Path properties of Brownian motion

**Theorem (Law of iterated logarithm)**

*Let $W$ be a Brownian motion. We have*

$$\limsup_{t\to\infty} \frac{|W_t|}{\sqrt{2t\log\log t}} = 1 \ a.s$$

*and*

$$\limsup_{t\to 0} \frac{|W_t|}{\sqrt{2t\log\log \frac{1}{t}}} = 1 \ a.s$$

# Path properties of Brownian motion

**Theorem**

*With probability 1, the paths of Brownian motion are nowhere differentiable on* $[0, 1]$.

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