Stochastic processes

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Motivation

Stochastic processes play an important role in Mathematics and other sciences.

- Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner, i.e growth of a bacterial population, change of the price of a stock.

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Stochastic processes play an important role in Mathematics and other sciences.

- Stochastic processes are widely used as mathematical models of systems and phenomena that appear to vary in a random manner, i.e growth of a bacterial population, change of the price of a stock.
- Stochastic processes have applications in many disciplines such as biology, physics, computer science, telecommunications, finance.

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Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition

Let (E, τ) be a topological space and ε be the Borel σ-algebra on E (i.e. the σ-algebra generated by open sets of E).
X and Y are two random variables with values in E.
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Definition

Similarly, X and Y are two d-dimensional random vectors with values in E^d.
Then X and Y have the same law if $\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$ for all Borel subsets A of E^d.

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
A stochastic process $(X_t)_{t\in[0,\infty)}$ is a collection of random variables on $(\Omega, \mathcal{F})$ with values in a measurable space (E, ε), i.e.

$$\text{for all } t \in [0,\infty)\text{: } X_t : \Omega \to E \text{ is } \mathcal{F} \text{ - measurable}$$

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Remark:
Sometimes, a stochastic process $(X_t)_{t\in[0,\infty)}$ is considered as a map:

$$X : [0,\infty) \times \Omega \to E$$

s.t the random variables X_t is $\mathcal{F}$ - measurable for all $t \in [0,\infty)$.
We write $X_t = X_t(\omega) = X(t,\omega)$

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A filtration is a collection of σ-algebras $\mathcal{F}_t$ such that:

$$\mathcal{F}_t \subseteq \mathcal{F} \ \forall t \geq 0$$
$$\text{and } \forall s, t \geq 0\text{: } \mathcal{F}_s \subseteq \mathcal{F}_t \text{ if } s \leq t$$

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Definition

a) A stochastic process $(X_t)_{t\in[0,\infty)}$ is adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if X_t is $\mathcal{F}_t$ - measurable for all $t \geq 0$.
b) The natural filtration for the process $(X_t)_{t\geq 0}$: $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$.

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Remark: The process $(X_t)_{t\geq 0}$ is always adapted to its natural filtration.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition

Given a filtration $(\mathcal{F}_t)_{t\geq 0}$. We define $\mathcal{F}_\infty$ to be the σ-algebra generated by $\cup_{t\geq 0}\mathcal{F}_t$.

$$\text{We write } \mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$$

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A filtration $(\mathcal{F}_t)_{t\geq 0}$ is *right continuous* if $\mathcal{F}_{t+} = \mathcal{F}_t \ \forall t \geq 0$

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Remark: Most of the filtration we will come across will be right continuous.

Example of a non-right continuous filtration

Example

Let $\Omega = \{a, b\}$, $\mathcal{F} = 2^{\Omega}$, $\mathbb{P}(\{a\}) = \mathbb{P}(\{b\}) = \frac{1}{2}$. Define

$$X_t(\omega) = \begin{cases} 0 \textit{ if } \mathrm{t} \leq 1 \\ 0 \textit{ if } \mathrm{t} > 1 \textit{ and } \omega = a \\ \mathrm{t - 1} \textit{ if } \mathrm{t} > 1 \textit{ and } \omega = b \end{cases}$$

The natural filtration for the process X is not right continuous.

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition

a) A filtration $(\mathcal{F}_t)_{t\geq 0}$ is *complete* if $\mathcal{F}_t$ is *complete* for all t, i.e.:
$\forall A \in \mathcal{F}_t, \mathbb{P}(A) = 0$ then $B \in \mathcal{F}_t$ for all $B \subset A$.
b) A filtration is right continuous and complete is said to satisfy the *usual conditions*.

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Definition

The minimal augmented filtration generated by a process X is the smallest filtration that is right continuous and complete and with respect to which the process X is adapted.

Construct the minimal augmented filtration

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stochastic process $(X_t)_{t \geq 0}$. We construct the minimal augmented filtration of X as follows:

- Step 1: Take the natural filtration of X:

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where $\mathcal{N}$ is the set contains all subsets of the sets with probability 0
- Step 3: Let $\mathcal{F}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon}^0$

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a) Two stochastic processes X and Y are said to be indistinguishable if

$$\mathbb{P}(X_t = Y_t \ \forall t \geq 0) = 1$$

b) Two stochastic processes X and Y are versions of each other if for each $t \geq 0$:

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Remark:

- Two indistinguishable processes are versions of each other.
- However, two processes which are versions of each other are not necessarily indistinguishable.

Example of two process which are versions of each other but not indistinguishable

Example

$\Omega = [0,1]$, $\mathcal{F}$ is the Borel σ - algebra on $[0,1]$
$\mathbb{P}$ is the Lebesgue measure on $[0,1]$
i.e., $\mathbb{P}([a,b]) = b - a \ \forall 0 \leq a \leq b \leq 1$
Define

$$X_t = 0 \ \forall t \geq 0$$
$$Y_t = \begin{cases} 1 \ if \ \omega = t \\ 0 \ if \ \omega \neq t \end{cases}$$

Then X_t and Y_t are versions of each other, but not indistinguishable!

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Remark:

- Two indistinguishable processes are versions of each other.
- However, two processes which are versions of each other are not necessarily indistinguishable.
- If X and Y are discrete-time processes, then these two definitions are equivalent.

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Remark: Two stochastic processes X and Y are indistinguishable, then their paths are equal almost surely.

Definition

A process X is called *cadlag* if almost all its paths are *cadlag* (i.e. its paths are right continuous and has left limits except for a set of ω with probability 0)

Definition

Definition

Suppose X and Y are stochastic processes with continuous paths. Let $E = \mathcal{C}([0, \infty))$ together with the metric defined by

$$d(f, g) = sup_{t \geq 0} \mid f(t) - g(t) \mid$$

Define

$$\begin{aligned} \overline{X} : \Omega &\to E \\ \omega &\to X(t, \omega) \end{aligned}$$

$$\begin{aligned} \overline{Y} : \Omega &\to E \\ \omega &\to Y(t, \omega) \end{aligned}$$

Then $\overline{X}$ and $\overline{Y}$ are random variables with values in the metric space E. If $\overline{X}$ and $\overline{Y}$ have the same law, then we also say that stochastic processes X and Y have the same law.

Definition

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Two stochastic processes X and Y have the same *finite-dimensional distribution* if for every $n \geq 1$ and every $t_1 < \ldots < t_n$, the laws of $(X_{t_1}, \ldots, X_{t_n})$ and $(Y_{t_1}, \ldots, Y_{t_n})$ are equal.

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Remark: If two stochastic processes X and Y are versions of each other, then they have the same *finite-dimensional distribution*.

Kolmogorov continuity theorem

Definition

A function $f : [0, \infty) \to \mathbb{R}$ is said to be *Holder continuous of order* α if there exists a constant M such that for all $s, t \in [0, \infty)$:

$$| f(t) - f(s) | \leq M \mid t - s \mid^{\alpha}$$

Theorem

Let α, ε, $c > 0$. If a process $(X_t)_{t \in [0,1]}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies for $s, t \in [0, 1]$:

$$\mathbb{E}[| X_t - X_s |^{\alpha}] \leq c \mid t - s \mid^{1+\varepsilon}$$

then there exists a version of the process $(X_t)_{t \in [0,1]}$ that is a continuous process and whose paths are γ-Holder continuous for every $\gamma \in [0, \frac{\varepsilon}{\alpha})$.

Kolmogorov continuity theorem

Theorem

Let α, ε, $c > 0$. If a process $(X_t)_{t\in[0,+\infty)}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies for $s, t \in [0, +\infty)$:

$$\mathbb{E}[\mid X_t - X_s \mid^\alpha] \leq c \mid t - s \mid^{1+\varepsilon}$$

then there exists a version of the process $(X_t)_{t\in[0,+\infty)}$ that is a continuous process.

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Examples

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  - It is known as Mathematical model for the movement of a pollen immersed in water.
  - It is basic to the study of stochastic differential equations, financial mathematics, etc...

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  - It is known as Mathematical model for the movement of a pollen immersed in water.
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- Poisson process
  - It is first used to study the number of phone calls occurring in a certain period of time.
  - It is the building block for an important class of stochastic processes known as Levy processes.

References

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