Homework 1

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1 Exercise 1

	x_1	x_2	x_3	x_4	x_5	$\sum$
y_1	0.01	0.02	0.03	0.1	0.1	0.26
y_2	0.05	0.1	0.05	0.07	0.2	0.47
y_3	0.1	0.05	0.03	0.05	0.04	0.27
$\sum$	0.16	0.17	0.11	0.22	0.34	1

(a) The marginal distribution

- $p(x)$

$$\begin{aligned} p(X = x_1) &= \sum_{i=1}^{3} p(X = x_1, Y = y_i) \\ &= 0.01 + 0.05 + 0.1 \\ &= 0.16 \end{aligned}$$

$$\begin{aligned} p(X = x_2) &= \sum_{i=1}^{3} p(X = x_2, Y = y_i) \\ &= 0.02 + 0.1 + 0.05 \\ &= 0.17 \end{aligned}$$

$$
\begin{aligned}
p(X = x_3) &= \sum_{i=1}^{3} p(X = x_3, Y = y_i) \\
&= 0.03 + 0.05 + 0.03 \\
&= 0.11
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_4) &= \sum_{i=1}^{3} p(X = x_4, Y = y_i) \\
&= 0.1 + 0.07 + 0.05 \\
&= 0.22
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_5) &= \sum_{i=1}^{3} p(X = x_5, Y = y_i) \\
&= 0.1 + 0.2 + 0.04 \\
&= 0.34
\end{aligned}
$$

- $p(y)$

$$
\begin{aligned}
p(Y = y_1) &= \sum_{i=1}^{5} p(Y = y_1, X = x_i) \\
&= 0.01 + 0.02 + 0.03 + 0.1 + 0.1 \\
&= 0.26
\end{aligned}
$$

$$
\begin{aligned}
p(Y = y_2) &= \sum_{i=1}^{5} p(Y = y_2, X = x_i) \\
&= 0.05 + 0.1 + 0.05 + 0.07 + 0.2 \\
&= 0.47
\end{aligned}
$$

$$
\begin{aligned}
p(Y = y_3) &= \sum_{i=1}^{5} p(Y = y_3, X = x_i) \\
&= 0.1 + 0.05 + 0.03 + 0.05 + 0.04 \\
&= 0.27
\end{aligned}
$$

(b) The conditional distribution

- $p(x|Y = y_1)$

$$
\begin{aligned}
p(X = x_1|Y = y_1) &= \frac{p(X = x_1, Y = y_1)}{p(Y = y_1)} \\
&= \frac{0.01}{0.26} \\
&= \frac{1}{26} = 0.038
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_2|Y = y_1) &= \frac{p(X = x_2, Y = y_1)}{p(Y = y_1)} \\
&= \frac{0.02}{0.26} \\
&= \frac{1}{13} = 0.077
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_3|Y = y_1) &= \frac{p(X = x_3, Y = y_1)}{p(Y = y_1)} \\
&= \frac{0.03}{0.26} \\
&= \frac{3}{26} = 0.115
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_4|Y = y_1) &= \frac{p(X = x_4, Y = y_1)}{p(Y = y_1)} \\
&= \frac{0.1}{0.26} \\
&= \frac{5}{13} = 0.38
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_5|Y = y_1) &= \frac{p(X = x_5, Y = y_1)}{p(Y = y_1)} \\
&= \frac{0.1}{0.26} \\
&= \frac{5}{13} = 0.38
\end{aligned}
$$

- $p(x|Y = y_3)$

$$
\begin{aligned}
p(X = x_1|Y = y_3) &= \frac{p(X = x_1, Y = y_3)}{p(Y = y_3)} \\
&= \frac{0.1}{0.27} \\
&= \frac{10}{27} = 0.37
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_2|Y = y_3) &= \frac{p(X = x_1, Y = y_3)}{p(Y = y_3)} \\
&= \frac{0.05}{0.27} \\
&= \frac{5}{27} = 0.185
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_3|Y = y_3) &= \frac{p(X = x_1, Y = y_3)}{p(Y = y_3)} \\
&= \frac{0.03}{0.27} \\
&= \frac{1}{9} = 0.111
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_4 | Y = y_3) &= \frac{p(X = x_1, Y = y_3)}{p(Y = y_3)} \\
&= \frac{0.05}{0.27} \\
&= \frac{5}{27} = 0.185
\end{aligned}
$$

$$
\begin{aligned}
p(X = x_5 | Y = y_3) &= \frac{p(X = x_1, Y = y_3)}{p(Y = y_3)} \\
&= \frac{0.04}{0.27} \\
&= \frac{4}{27} = 0.148
\end{aligned}
$$

2 Exercise 2

$$
\begin{aligned}
E_Y[E_X[x|y]] &= E_Y\left[\sum_x xp(x|y)\right] \\
&= \sum_y \left[\sum_x xp(x|y)\right] p(y) \\
&= \sum_y \sum_x xp(x|y)p(y) \\
&= \sum_y \sum_x xp(x, y) \\
&= \sum_x \sum_y xp(x, y) \\
&= \sum_x x \sum_y p(x, y) \\
&= \sum_x xp(x) \\
&= E_X[X]
\end{aligned}
$$

3 Exercise 3

Let $P(X) = 0.207, P(Y) = 0.5$ and $P(X|Y) = 0.365$.

(a) Probability of using both X and Y

$$P(XY) = P(X|Y) \cdot P(Y) = 0.365 \times 0.5 = 0.1825$$

$\Rightarrow$ The probability that person uses both X and Y is 18.25%.

(b) Probability of using Y, given not using X

$$
\begin{aligned}
P(Y|\bar{X}) &= \frac{P(\bar{X}|Y) \cdot P(Y)}{P(\bar{X})} \\
&= \frac{(1 - P(X|Y) \cdot P(Y)}{1 - P(X)} \\
&= \frac{(1 - 0.365) \times 0.5}{1 - 0.207} \\
&= 0.4004
\end{aligned}
$$

⇒ The probability that person uses y in condition he does not use X is 40.04%.

4 Exercise 4

(a) In discrete case

$$
\begin{aligned}
V_X &= \sum_x (x - \mu)^2 P(x) \\
&= \sum_x (x^2 - 2x\mu + \mu^2) P(x) \\
&= \sum_x x^2 P(x) - 2\mu \sum_x x P(x) + \mu^2 \sum_x P(x) \\
&= E_X[x^2] - 2\mu^2 + \mu^2 \\
&= E_X[x^2] - \mu^2 \\
&= E_X[x^2] - (E_X[x])^2
\end{aligned}
$$

(b) In continuous case

$$
\begin{aligned}
V_X &= \int (x - \mu)^2 f(x) dx \\
&= \int (x^2 - 2x\mu + \mu^2) f(x) dx \\
&= \int x^2 f(x) dx - 2\mu \int x f(x) dx + \mu^2 \int f(x) dx \\
&= E_X[x^2] - 2\mu^2 + \mu^2 \\
&= E_X[x^2] - \mu^2 \\
&= E_X[x^2] - (E_X[x])^2
\end{aligned}
$$

5 Exercise 5

With 3 doors, there are 3 possible positions for the car. In this case, if you choose door 1, Monty will have the following ways to open the door having the goat:

- If the car is in door 1, Monty can choose either door 2 or door 3 with the probability of $\frac{1}{2}$.

- If the car is in door 2, Monty can not choose this door, he must pick the last one, door 3.
- If the car is in door 3, similar to the second case, Monty must choose door 2.

Let F be the event that the car is in door 1. and S be the event Monty open door 2 that have a goat. Then we have:

$$P(F) = \frac{1}{3}$$

After the first choice, there are 2 door left, so the probability that Monty choose a door having goat is $\frac{1}{2}$.

$$P(S|F) = \frac{1}{2}$$

As 3 cases listed above, the probability for the event S is:

$$P(S) = \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times 0 + \frac{1}{3} \times 1 = \frac{1}{2}$$

Apply Bayes's theorem, the probability that the car is in door 1 after Monty choose door 2 is:

$$\begin{aligned} P(F|S) &= \frac{P(S|F)P(F)}{P(S)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} \\ &= \frac{1}{3} \end{aligned}$$

Then the probability of the car is behind the door 3 is: $1 - \frac{1}{3} = \frac{2}{3}$. Therefore, if you switch your choice to door 3, you will have more chance (2/3) to pick the car.