MIT 18.06 - Problem set 1

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1 Section 1.2, prob. 23

- $\cos\beta = \frac{w_1}{\|w\|}$, $\sin\beta = \frac{w_2}{\|w\|}$
- $\cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = \frac{v_1 \cdot w_1 + v_2 \cdot w_2}{\|v\| \cdot \|w\|} = \frac{v \cdot w}{\|v\| \cdot \|w\|}$

2 Section 1.2, prob. 28

We call angles between $u - v$, $v - w$ and $w - u$ α, β and θ respectively. We have:

$$u \cdot v < 0 \Rightarrow \cos\alpha < 0 \Rightarrow \alpha > 90^\circ \tag{1}$$

$$v \cdot w < 0 \Rightarrow \cos\beta < 0 \Rightarrow \beta > 90^\circ \tag{2}$$

$$w \cdot u < 0 \Rightarrow \cos\theta < 0 \Rightarrow \theta > 90^\circ \tag{3}$$

Also, the three angles have to fulfill $\alpha + \beta + \theta = 360^\circ$, we can find a combination $\alpha = \beta = \theta = 120^\circ$ that fulfills all the conditions above. Therefore the three dot products can have negative values.

3 Section 1.3, prob. 4

$$w_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, w_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, w_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, x_1 w_1 + x_2 w_2 + x_3 w_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + 4x_2 + 7x_3 = 0 \\ 2x_1 + 5x_2 + 8x_3 = 0 \\ 3x_1 + 6x_2 + 9x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -4x_2 - 7x_3 \\ 2(-4x_2 - 7x_3) + 8x_3 = 0 \\ -4x_2 - 7x_3 + 2x_2 + 9x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -4x_2 - 7x_3 \\ -4x_2 - 3x_3 = 0 \\ -2x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -4x_2 - 7x_3 \\ -4x_3 - 3x_3 = 0 \\ x_2 = x_3 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_3 = 0 \\ x_2 = 0 \end{cases}$$

4 Section 1.3, prob. 13

5 × 5 centered difference matrix $C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

$Cx = b \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 - x_1 \\ x_4 - x_2 \\ x_5 - x_3 \\ -x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$

This equation can be only solved when $b_1+b_2+b_3+b_4+b_5 = 0$, each columns of C lies in that plane. Therefore the columns of C are dependent, which means the matrix is singular and not invertible.

5 Section 2.1, prob. 29

$u_2 = Au_1 = 1 \cdot \begin{bmatrix} .8 \\ .2 \end{bmatrix} + 0 \cdot \begin{bmatrix} .3 \\ .7 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$

$u_3 = Au_2 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$

One property of all four vectors is that all elements of these vectors have the sum of 1.

6 Section 2.2, prob. 20

combination

Third equation: $4x + y + 2z = 2$

7 Section 2.2, prob. 32

1. on the same plane
2. on the same line

3. A matrix that row 100 is the combination of row 1 and row 2

4. Row picture: all the planes are intersected at a line, not one specific point, so there is infinite intersection points of these planes that lie on that line. Column picture: the combination of all vectors does not fill up 100-axis plane.

8 Section 2.3, prob. 22

1. $(Ax)_3 = \sum a_{3j}x_j$

2. $(E_{21}A)_{21} = -A_{21}/A_{11}$

3. $(E_{21}(E_{21}A))_{21} = -2(E_{21}A)_{21}/(E_{21}A)_{11}$

4. $(E_{21}Ax)_1 = \sum a_{1j}x_j$

9 Section 2.3, prob. 29

There are some steps as below:

- Subtract row 1 from 2, row 2 from 3, row 3 from 4.
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

- Subtract row 2 from 3, row 3 from 4.
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- Subtract row 3 from 4
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

10 Section 2.4, prob. 32

$Ax_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, Ax_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, Ax_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

In the column picture, each column of AX is a combination of columns of A, or $AX = A\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix} = \begin{bmatrix}Ax_1 & Ax_2 & Ax_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

11 Section 2.4, prob. 36

1. $(AB)C$ has $2 \cdot 4 \cdot 7 + 2 \cdot 7 \cdot 10 = 196$ multiplications.

   $A(BC)$ has $4 \cdot 7 \cdot 10 + 2 \cdot 4 \cdot 10 = 360$ multiplications.

   Therefore, choose $(AB)C$.

2. $u, v, w \in \mathbb{R}^{n\times 1}$

   $(u^T v)w^T$ has $1 \cdot n \cdot 1 + 1 \cdot 1 \cdot n = 2n$ multiplications.

   $u^T(vw^T)$ has $n \cdot 1 \cdot n + 1 \cdot n \cdot n = 2n^2$ multiplications.

   Therefore, choose $(u^T v)w^T$.

3. $(AB)C$: $\frac{mnp+mpq}{mnpq} = q^{-1} + n^{-1}$.

   $A(BC)$: $\frac{mnq+npq}{mnpq} = p^{-1} + m^{-1}$

12 Section 2.5, prob. 7

1. $Ax = \begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 1\\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0\\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0\end{cases}$

   Because $a_{3i} = a_{1i} + a_{2i}$, we cannot solve the above equation system.

2. $b_3 = b_1 + b_2$

3. Third row will be all zeros