COMP20003 Workshop Week 9

Priority Queue
Heaps & Binary Heaps
Heap Sort

LAB:
- Heap Sort
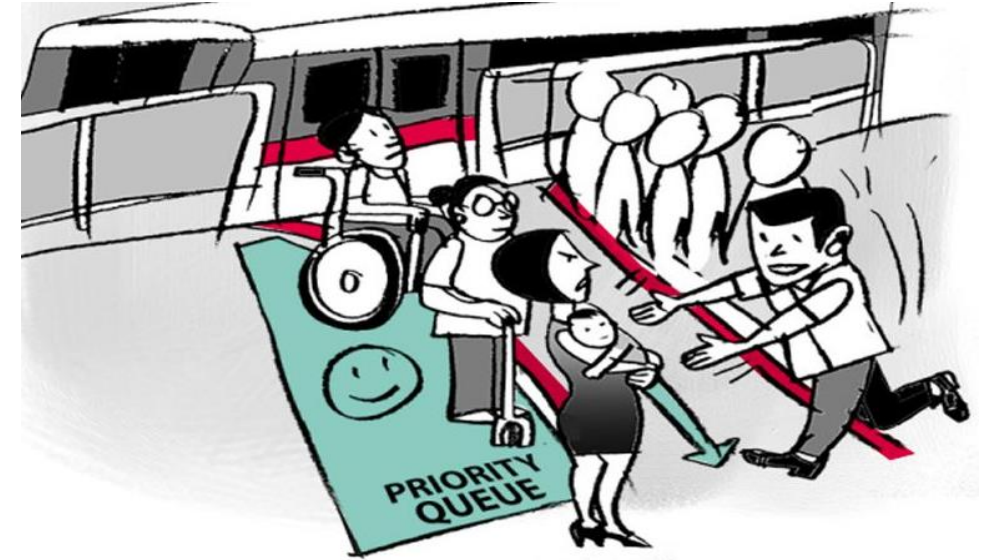
Important: Bring papers and pens to the last 3 workshops!

rear   front

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'Can I borrow your baby?...'

Queue: elements are dequeued in the FIFO order.

enqueue, dequeue can be easily implemented with O(1) time complexity using linked lists

Priority Queue : element with highest priority must be dequeued first!

How can we efficiently implement dequeue (and enqueue)?

Yet Another ADT: Priority Queue

PQ: queue, where each element is associated with a *priority* (or *weight*), and the elements will be *dequeued* following the order of priority.

'Can I borrow your baby?...'

Main operations:
- *enqueue*: `enPQ(PQ, item)` (supposing easy access to weight inside item), or

    `enPQ(PQ, weight, item)`
- *dequeue*: `dePQ(PQ)` removes & returns the highest-priority element of PQ. Normally named as

    `deleteMax(PQ)`, if *higher priority* means *bigger*, or

    `deleteMin(PQ)`, if *higher priority* means *smaller*
- *changeWeight*: change the weight of an item: `changeWeight(PQ, item)`

    or : `changeWeight(PQ, newWeight, item)`
- *create* : `makePQ()` – make an empty PQ or create a PQ from a set of items
- *check for being empty* : `isEmptyPQ(PQ)`

possible concrete data structures for PQ

Concrete Data Structure	Time complexity of			
	construction a PQ of n elements	enPQ	dePQ	peek
unsorted arrays or linked list				
sorted arrays or linked lists				
BST				
AVL				
hash table				

Example: priority= max

Unsorted array/list: 9   2   7   5   6   8   3

Sorted   array/list: 2   3   5   6   7   8   9

Related:
Ex 9.4

check your answers: possible concrete data structures for PQ

Concrete Data Structure	Time complexity of				Notes
	make PQ of n elements	enPQ	dePQ	peek	
unsorted arrays or linked list	O(n)	O(1)	O(n)	O(n)	
sorted arrays or linked lists	O(n logn)	O(n)	O(1)	O(1)	
BST	*the worst cases are the same as in sorted linked lists*				
AVL	O(nlogn)	O(logn)	O(logn)	O(logn)	space-inefficient (and poor cache locality) compared to Heap
hash table (using the priority as a key when hashing)	*Generally not efficient:* **O(n)** time for dequeue because a hash table is an *unordered data structure*. To find the highest-priority element (for peek or dequeue), the entire hash table must be scanned to locate the maximum value.				
Heap or **Binary Heap**	Θ(n)	O(logn)	O(logn)	O(1)	space efficient (great cache locality)

Peer Activity: Hashing Priorities?

Can an efficient priority queue that prioritises lower distances be implemented on a hash table with these constraints? Why?

    a. Yes, it can.

    b. No, it cannot.

Consider the following **constraints:**

- keys   supposing r is small
  - range is known ($0 \leq$ key $\leq r$)
  - hashed with the function

    hash(key) = key

- hash table

Index	0	1	...	r - 1	r
Key	...		...		

...

Peer Activity: Hashing Priorities?

Can an efficient priority queue that prioritises lower distances be implemented on a hash table with these constraints? Why?

a. Yes, it can.

b. No, it cannot.

Consider the following **constraints:**

- keys
  - unbounded ($\mathbb{N}_0$)
  - hashed with the function
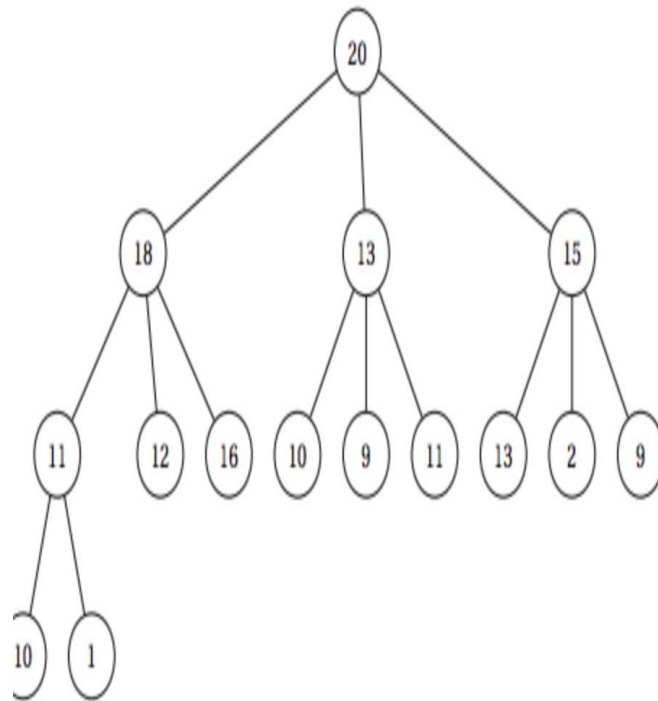
    `hash(key) = key % 101`
- hash table

Index	0	1	...	99	100
Key	...		...		

Binary Heap – An Efficient Data Structure for PQ

Heap?

Ternary Heap

Binary heap: an efficient implementation for priority queue

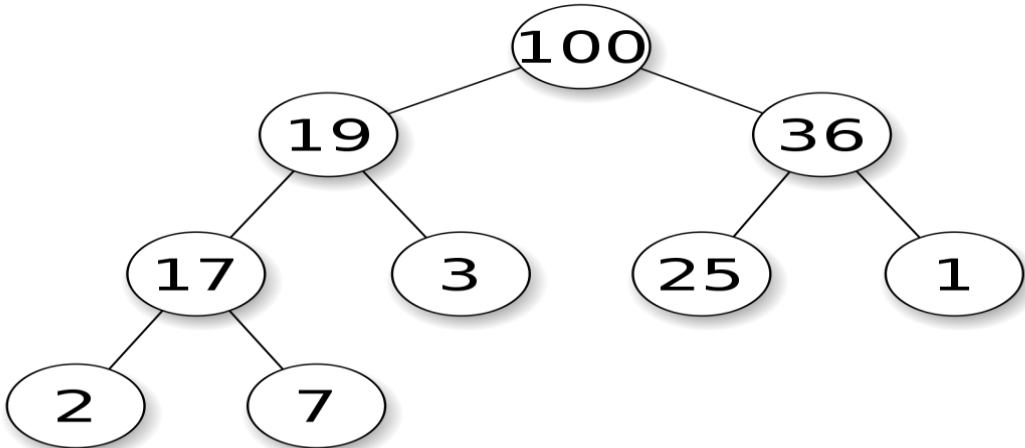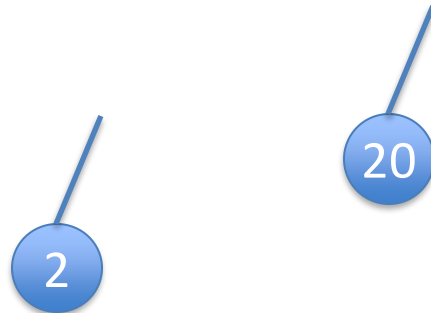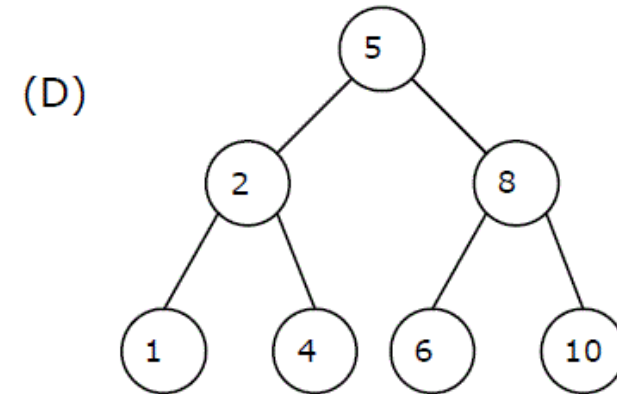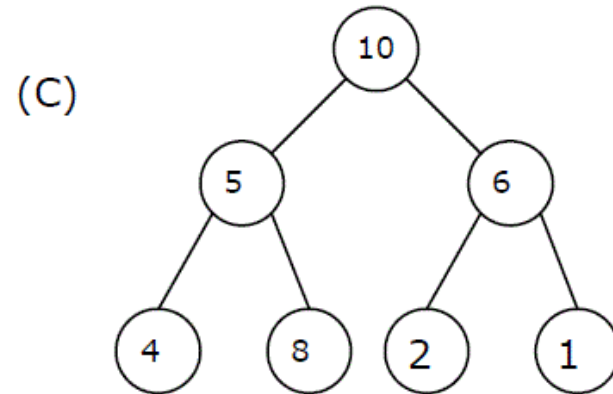Depending on the priority, we can have min-heap or max-heap
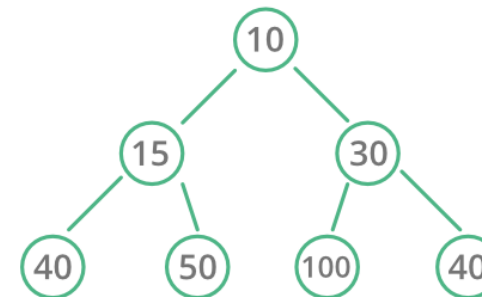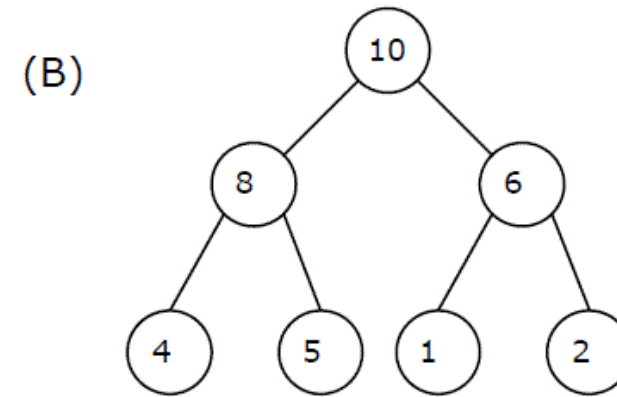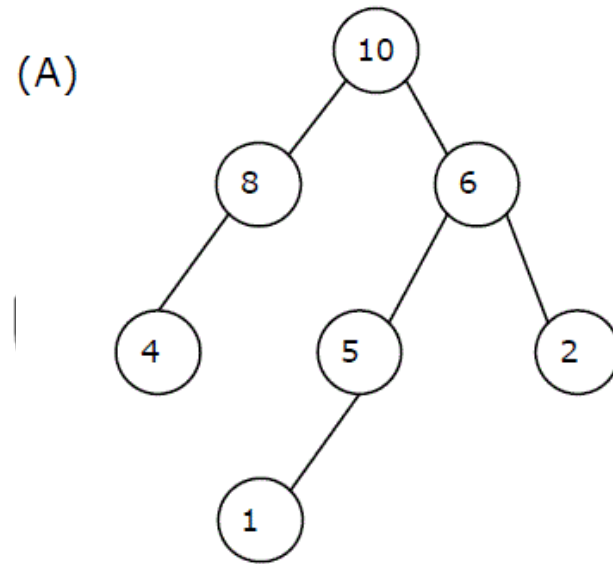

Min Heap
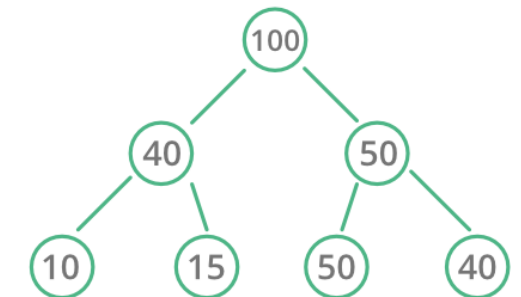
Max Heap

Heap: requirements

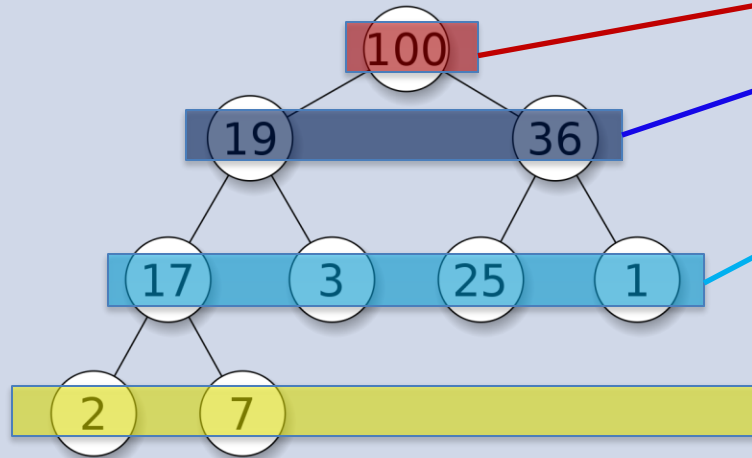| Example	Conditions
	1. The tree is complete. 2. *The heap property*: each node has a higher priority (here, is not smaller) than any of its descendants (or equivalently, just its children).

which one is a binary heap?

Binary Heap is implemented as an array!

Visualisation: as a complete binary tree	Implementation: using arrays

idx	0	1	2	3	4	5	6	7	8	9
val	✕	100	19	36	17	3	25	1	2	7

note: H[1] is for the root
H[0] not used

Heap is H[1..n]
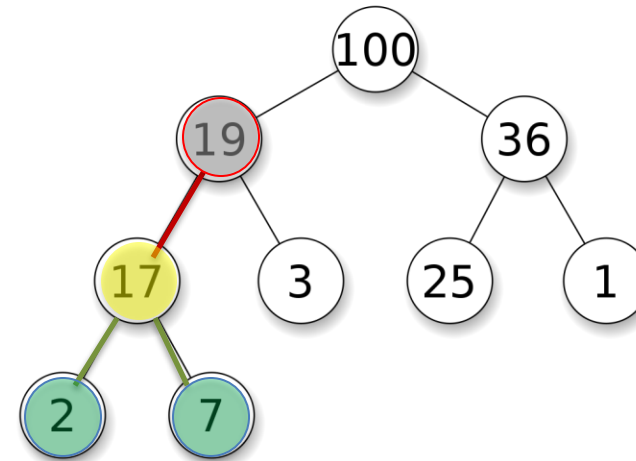- level i occupies 2^i cells in array H[1..n] from idex 2^i to $2^{i+1}-1$

Binary Heap is implemented as an array:
➔ efficient locating parent or children of the node at index i

- left child of H[i] is H[2*i]
- right child of H[i] is H[2*i+1]

parent of H[i] is H[i/2]

	4/2		i=4				4*2	4*2+1
idx	0	1	**2**	3	**4**	5	6	7	**8**	**9**
val	✕	**100**	**19**	**36**	**17**	**3**	**25**	**1**	**2**	**7**

Insert into a heap.

tree visualisation

insert

key to insert

add key to heap
violates heap order
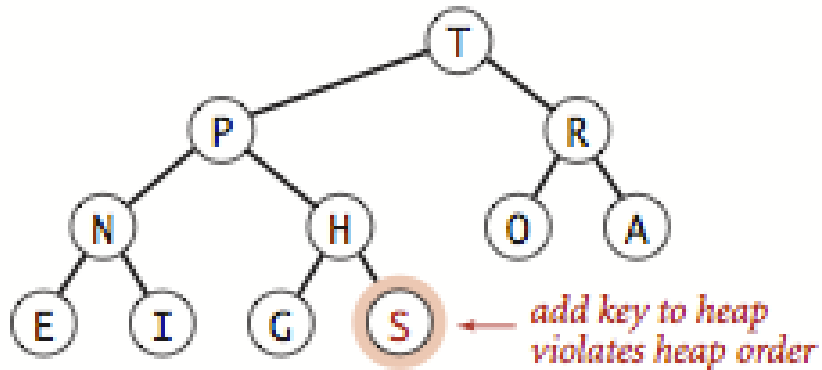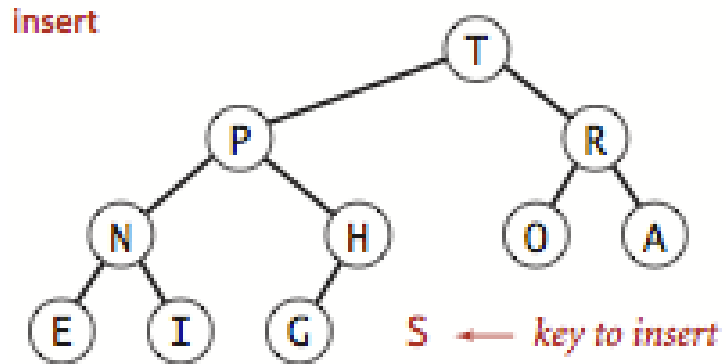
in the implemented array

index   1   2 3   4   5 6   7   8   9 10 11

H= [T,P,R,N,H,O,A,E,I,G,]

H has 10 elements

Insert S

Just added H[11]= S
parent of H[11] is H[11/2⌉ ie. H[5]

 index   5=11/2   11
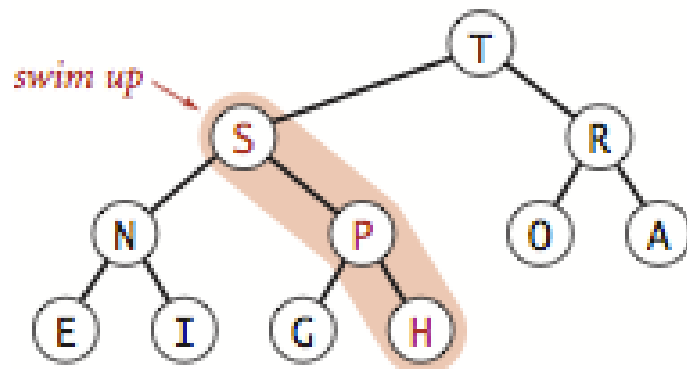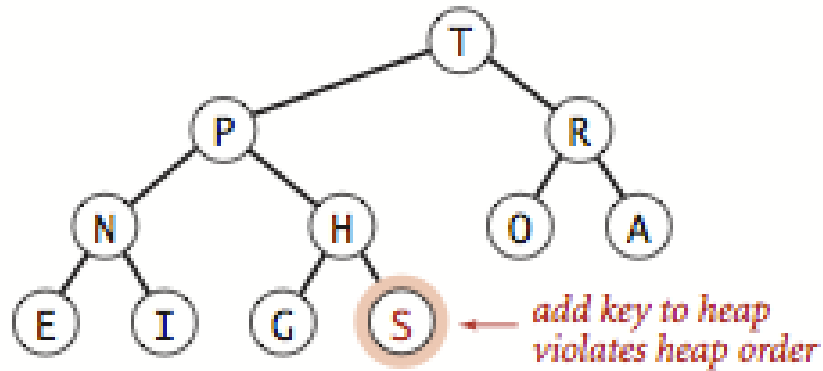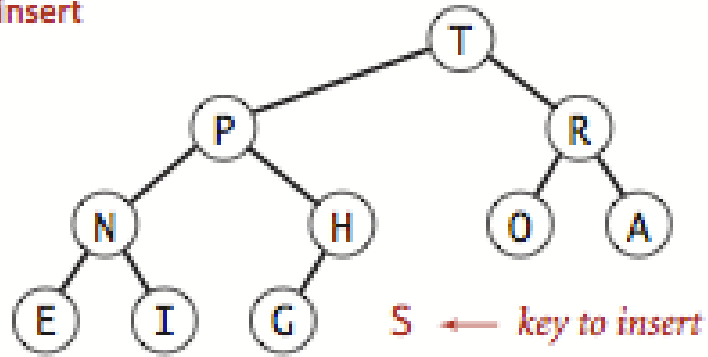
H= [T,P,R,N,H,O,A,E,I,C,S]

in this case H[11] and its parent H[5] violate
the heap order

 → need to repair using **upheap**

insert

T
P   R
N   H   O   A
E   I   G   S ← key to insert

T
P   R
N   H   O   A
E   I   G   S ← add key to heap violates heap order

swim up ↗
T
S   R
N   P   O   A
E   I   G   H

upheap
when a child node violates the heap order:
repeatedly swap the child with its parent (if exist) until having no violation

Complexity: $O(\ ?\ )$, $\theta(\ )$

Need to promote H[11] up using
upheap(h,i=11), which repeatedly swap node i with its parent.

2/2=1   2=5/2   5=11/2   11

[T,P,R,N,H,O,A,E,I,C,S]
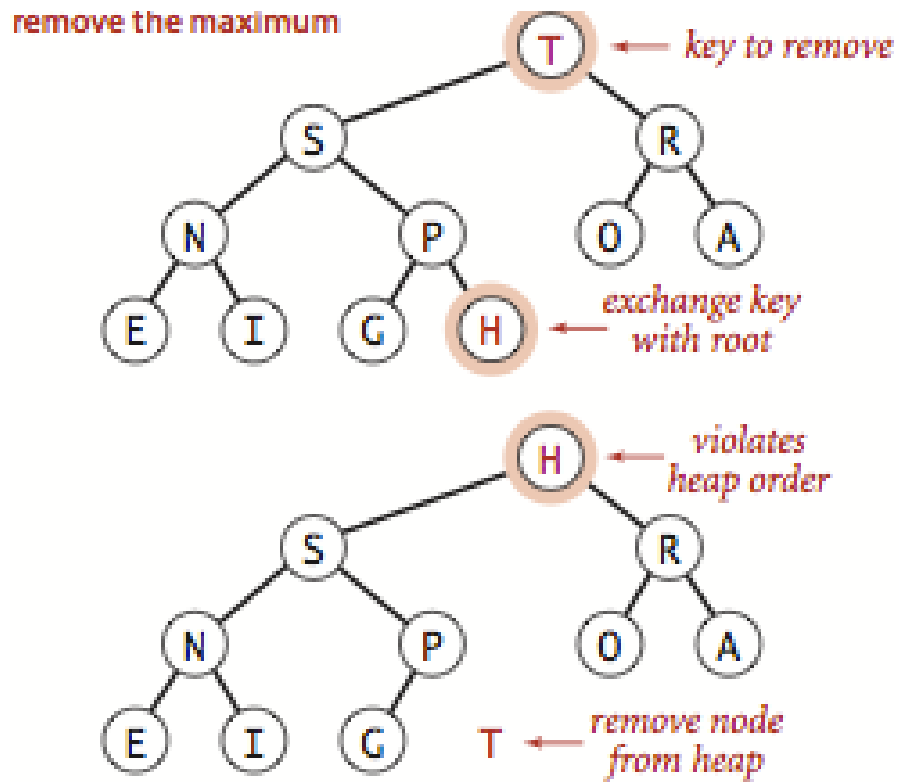   S   H

➔[T,P,R,N,S,O,A,E,I,C,H]
        S   P

➔[T,S,R,N,P,O,A,E,I,C,S]
            S

Complexity= ?

remove the maximum

T ← key to remove

H ← exchange key with root

H ← violates heap order

T ← remove node from heap
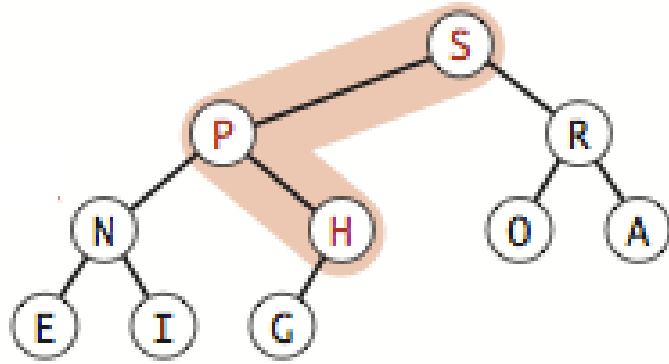
Heap= [T,S,R,N,P,O,A,E,I,G,H]

To remove (the heaviest, the root):
- swap root T with the last leaf H
- decrease number of elements in heap
- new root will likely violate the heap order: repair that by doing **downheap**

downheap= repeatedly swap node with its *heaviest child* until having no violation

Complexity: O(log n)

Notes: Here upheap(H, node) was used for insertion, and downheap(H, node) for deletion. The operations can be
performed for any node of the heap.
For example, when changing the priority of a node in a heap.

How to efficiently build a heap with n elements?

- Solution 1: insert each element into the (initially empty) heap, and do upheap after each insertion.

  Complexity: O(?)

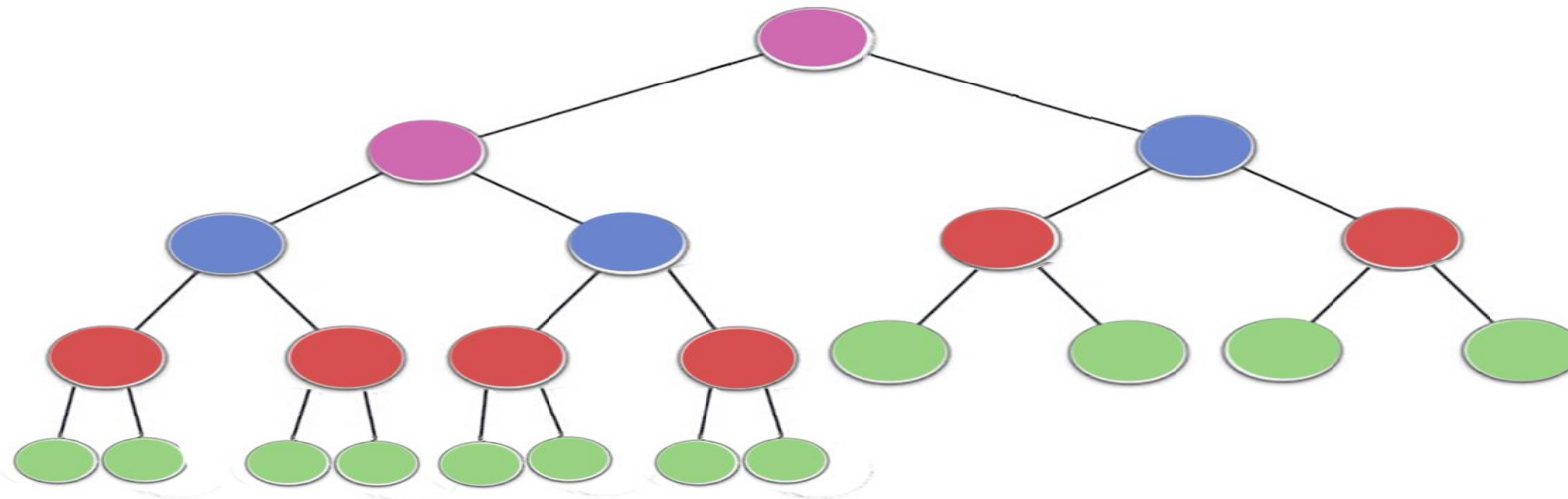**Solution 2:** populate the heap array with n elements in the input order, then turn the array to a heap (ie make it to satisfy the heap condition). Algorithm:

```
for (i= ??? ; i>0; i--) {
    // for i from last parent node to first node
        downheap(h, i);
```

Related:
Ex 9.3

= **Θ(n)** (see lectures and/or ask Google for a proof)

The operation is known as Heapify/Makeheap/ Bottom-Up Heap Construction

How? Complexity=?

Pre-questions:

- How to selection-sort by selecting the largest first?
- What is the complexity of selection sort?
- How can we have a faster select-the-largest?

Example: sort the keys: 20,3,60, 8,1,16

HeapSort summary

To sort an array A[1..n] in *increasing* order
1. Use heapify to turn A into a *maxheap*
2. while (heap A has more than 1 element):
   - delete root by :
     - first swap it with the last element of the heap, then
     - downheap the new root

Complexity of Heapsort =

$$= O(?)$$

Questions:
- What's the best case of heapsort?
- Is heapsort stable?

Example: using Heapsort, sort the keys: 20,3,60,8,1,16

Heapsort= sorting using a heap

How? Complexity=?

Example: sort the keys: 20,3,60, 8,1,16

Step 1: Turn the array to a maxheap using heapify

20 3 60 8 1 16

        60 > only child 16 : subtree 60 is a heap

20 3 60 8 1 16

      3 < larger child 8 : swap

20 8 60 3 1 16

       3 has no child, is a heap

20 8 60 3 1 16

swap 20 with larger child 60

60 8 20 3 1 16

      20 > only child 16 : 20 is a heap

   done

  complexity step 1: Θ(n)

Step 2: loop: a) swap root with the last
                     decrement heap length
              b) downheap(heap, position= 1)

60 8 20 3 1 16   n= 6

1a: swap(60,16)

16 8 20 3 1 60   n= 5

1b: downheap(16) : swap (16,20), done

20 8 16 3 1 60   n= 5

2a: swap(20,1)

1   8 16 3   20 60   n= 4

2b: downheap(1) : swap (1,16), done

16 8 1 3   20 60   n= 4

3ab: swap (16,3), downheap(3)= swap(3,8)

8 3 1   16 20 60   n= 3

4ab: swap (8,1), downheap(1)= swap(1,3)

3 1   8 16 20 60   n= 2

5ab: swap (3,1), downheap(3)= no swap needed

1 3 8 16 20 60   n= 1, done

DONE after 5= n-1 step

  complexity step 2= overall = O(n logn)

HeapSort summary

To sort an array A[1..n] in *increasing* order
1. Use heapify to turn A into a *maxheap*
2. while (heap A has more than 1 element):
   - delete root by :
     - first swap it with the last element of the heap, then
     - downheap(A,1): downheap the new root

Complexity=
= O(n logn)

Questions:
- What's the best case of heapsort? all elements equal, Θ(n)
- Is heapsort stable?   no (long-distance swap)

Heap & Heap Sort: Complexity Summary

Heap operations:

- upheap:

- downheap:

- insert/enPQ:

- deleteMax/deleteMin:

    note: changeWeight an be done in O(logn) time

- heapify:

- heapsort:

Heap & Heap Sort: Complexity Summary

Heap operations:

- upheap(H, pos):   O(log n)
- downheap(H, pos):   O(log n)
- insert/enPQ(H, key):   O(log n)
- deleteMax/deleteMin(H): O(log n)
- changeWeight(H, newkey, item) :   O(log n)

- heapify: θ(n)
- heapsort: O(n log n)   best case: θ(n)

Peer Activity: m^{th} Smallest Number

Does the upper-bound complexities

of these two algorithms differ?

    a.   Yes, they do.

    b.   No, they do not.

Assume that that $m \ll n$.

Consider an **unsorted algorithm** that:

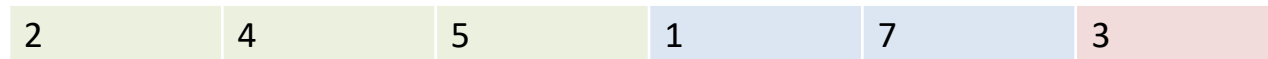- gets the mth smallest value
- from n unsorted values

Now consider a **sorted algorithm** that:

- sorts n values in ascending order
- indexes the mth value

Bottom-up merge sort improvement

- Monotonic increasing runs already sorted
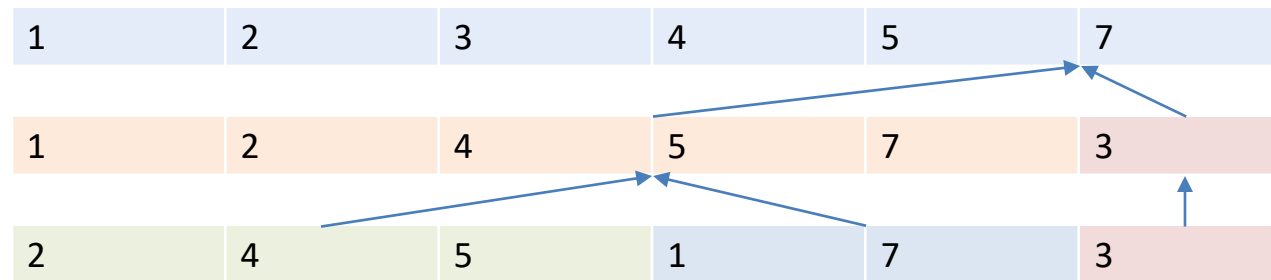- Insert monotonic runs into queue instead of singletons

| 2 | 4 | 5 | 1 | 7 | 3 |

Bottom-up merge sort improvement

- Best Case: Θ()
- Worst Case: Θ()
- If known k= number of monotonic runs: Θ()

1	2	3	4	5	7

1	2	4	5	7	3

2	4	5	1	7	3

W9.7: Implementing heapsort