Chapter 2

Sets, Functions, Sequences and Sums

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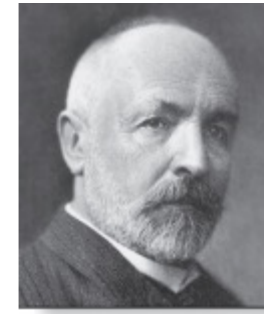
OUR GOAL:

- Use the language of set theory
- Some definitions: function, one-to-one function, onto function, bijection, inverse function
- Calculate sums of basic sequences

2.1. SETS



G. Cantor (1845-1918)

- An UNORDERED collection of objects/items called elements/members

{a, b, c} = {a, c, b} = {b, c, a}

Datatype or type in computer science. For example,

Boolean = {0, 1}

MEMBERS/ELEMENTS of a set

Symbol	Read	Symbol	Read
$a \in V$	a is in V	$a \notin V$	a is not in V
	a is a member of V		a is not a member of V
	a is an element of V		a is not an element of V
	a belongs to V		a doesn't belong to V

- ♥∈ {1, a, ♥} // ♥ is in {1, a, ♥}
- ♥∉ {♦, ≈, {♥}, f} // ♥ is not an element of this set
- $a \notin \{\{a\}, b\}$ $\{a\} \notin \{a, b\}$
- $\{a\} \in \{\{a\}, b\}$ // {a} is a member of {{a}, b}

WAYS TO DEFINE SETS

- **Method 1.** Roster method. List all members:

V = {a, i, o, e, u}

Z = {…, -2, -1, 0, 1, 2, …}

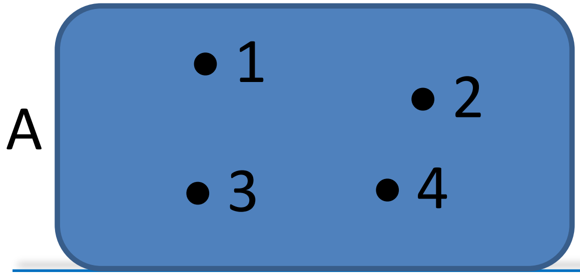
- **Method 2. Set builder notation.** Describe a set by **properties**

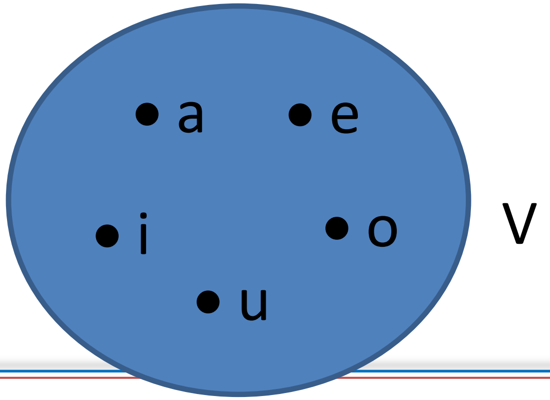
A = {x | P(x)}

A = {x | x is a positive integer and x < 5}={4,3,2,1}

Read: A is the set of positive integers such that less than 5.

- **Method 3. Venn diagram**





NULL SET. EMPTY SET

- **Null set/empty set:** set has no element.

A = {x | x is a month with 32 days}

A = { } // empty set

- Symbol: { } or ∅
- **NOTE:** ∅ belongs to {∅}.

So, {∅} is different from ∅

$$\varnothing \neq \{\varnothing\}$$

(Way to remember: empty folder ≠ folder containing empty folder)

SOME IMPORTANT SETS

- N: the set of natural numbers

  N = {0, 1, 2, 3, …}

- Z: the set of integers

  Z = {…, -2, -1, 0, 1, 2, …}

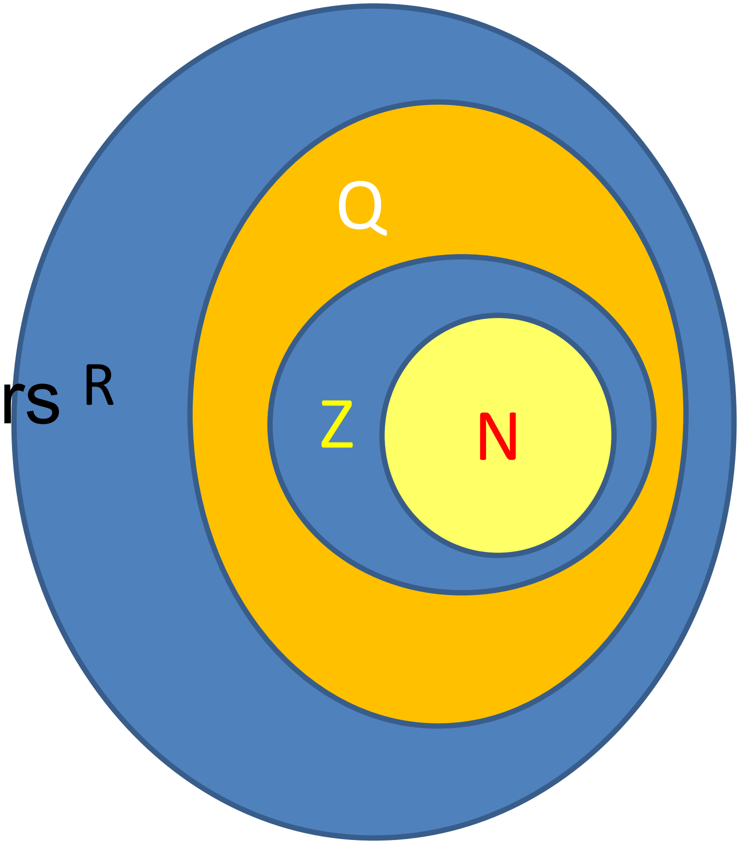
- Q = the set of rational numbers

  $Q = \{\frac{p}{q} \mid p, q \in Z, q \neq 0\}$

Q = {…, -1/3, 0, 2, -3, 5/7, …}

- R: the set of real numbers

  R = {…, $\sqrt{2}$, 1, -2/3, 0, π, …}



CARDINALITY OF A SET

- If a set has n elements, we say its cardinality is n.
- A = {a, b, c} ➔ cardinality of A, |A| = 3
- B = ∅ ➔ |B| = 0
- C = {∅} ➔ |C| = 1
- D = {a, {a}} ➔ |D| = 2
- E = {a, a, a, a} ➔ |E| = 1
- If V is the set of all vowels in English, then |V| = 5
- |N| = ∞

SUBSETS

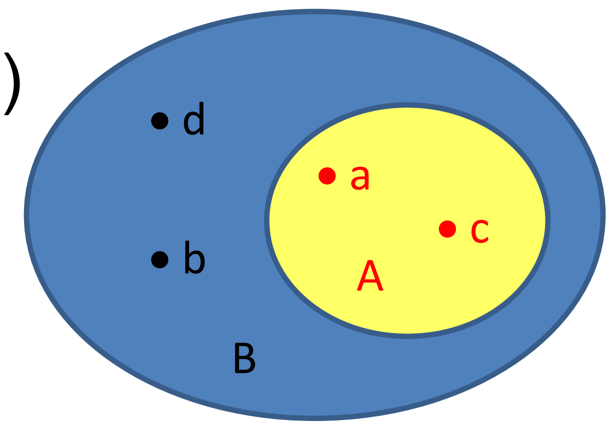
- If every element of A is in B, then we say A is a **SUBSET** of B.

$$\forall x(x \in A \rightarrow x \in B)$$

- Symbol: $A \subseteq B$

  $A \subset B$ (if $A \neq B$)

- Venn diagram:



Venn diagram for $A \subseteq B$

- Examples:

$\{a, c\} \subset \{a, b, c, d\}$

$N \subset Z$

$Z \subset Q$

$Q \subset R$

How to show $A \subseteq B$?

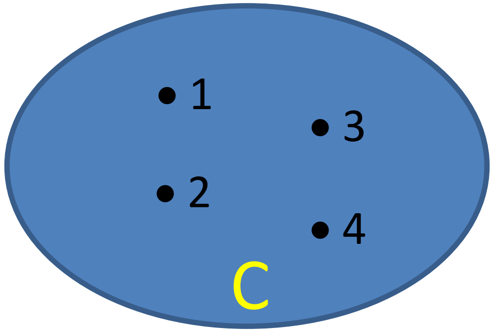
HOW TO SHOW THAT A $\subseteq$ B and A⊄B

- To show A $\subseteq$ B: show that if x∈A, then x∈B
- To show A⊂B: find only one x∈A such that x∉B

$$\neg(p \to q) \equiv \neg(\neg p \vee q) \equiv p \wedge (\neg q)$$

EQUALITY

- Two sets A and B are **equal** if they have the same elements no matter how they are defined.
- Symbol: A = B
- A = {x | x is a positive integer less than 5}
- B = {1, 2, 3, 4, 3, 3}
- D = {3, 2, 4, 1}
- A = B = C = D



EQUALITY

- To show that A = B:
- **Method 1.** Show that $A \subseteq B$ and $B \subseteq A$
- **Method 2.** Use set builder notation.
- **Method 3.** Use membership table.
- For examples, (see NEXT SEGMENT)

THEOREM

- $\varnothing \subseteq S$ for any set S

WHY?

- $S \subseteq S$ for any set S

POWER SET

- Given a set A
- The **POWER SET** of A, denoted by **P(A), is the set of all subsets of A.**

Sets	All Subsets	POWER SETS
A = {a}	∅, {a}	P(A) = { ??? }
B = {∗, b}	∅, {∗}, {b}, {∗, b}	P(B) = { ??? }

C = {1, 2, 3}	∅,
	{1}, {2}, {3},
	{1, 2}, {2, 3}, {1, 3},
	{1, 2, 3}
		P(C) = {
		???
		}

If A has n elements, then P(A) has 2^n elements.
$(|A| = n \rightarrow |P(A)| = 2^n)$

CARTESIAN PRODUCT (named after René Descartes)

- ORDERED PAIR: (a, b) $\neq$ (b, a) ~~{a, b}~~
- **Cartesian product** of two sets A and B:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

A	B	A×B
{a, b}	{1, 2, 3}	{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)}
{1, 2, 3}	{a, b}	{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}
{Nguyen, Le}	{Phuong, Lam}	{(Nguyen, Phuong), (Le, Phuong), (Nguyen, Lam), (Le, Lam)}
{ }	{a, b}	{ }

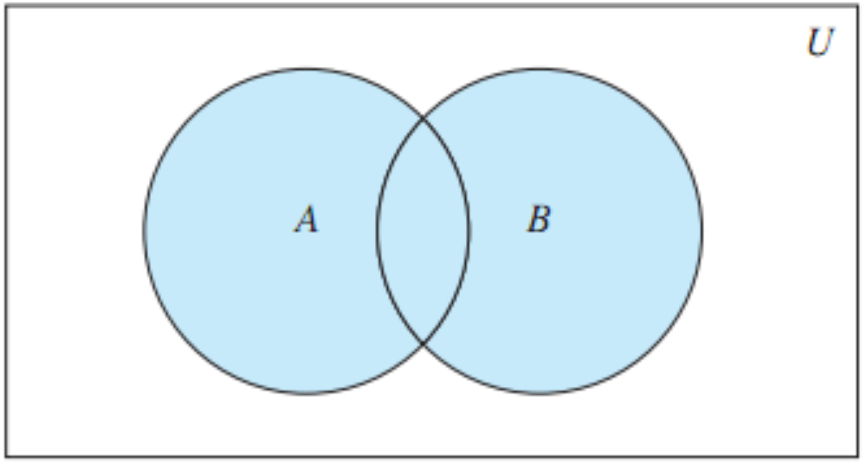
SET OPERATIONS.

Set operations	Symbols
Union	$A \cup B$
Intersection	$A \cap B$
Difference	$A - B$
Symmetric difference	$A \oplus B$
Complement	$\bar{A}$

UNION

- Symbol: $A \cup B$
- Set builder:

$A \cup B = \{x \mid x \in A \vee x \in B\}$



$A \cup B$ is shaded.

- Examples.

A	B	$A \cup B$
{a, b, c}	{c, d}	{a, b, c, d}
{a, b, c}	{a, b, c, d, e}	{a, b, c, d, e}
$\varnothing$	{1, 2, 3}	{1, 2, 3}
The set of all students who are from Long An	The set of all students who are from Phu Yen	The set of all students who are from Long An OR Phu Yen

MEMBERSHIP TABLE FOR A ∪ B

0: NOT A MEMBER OF THE SET
1: MEMBER OF THE SET

// Use a 1 bit to represent true and a 0 bit to represent false.

A	B	A ∪ B
0	0	0
1	0	1
0	1	1
1	1	1

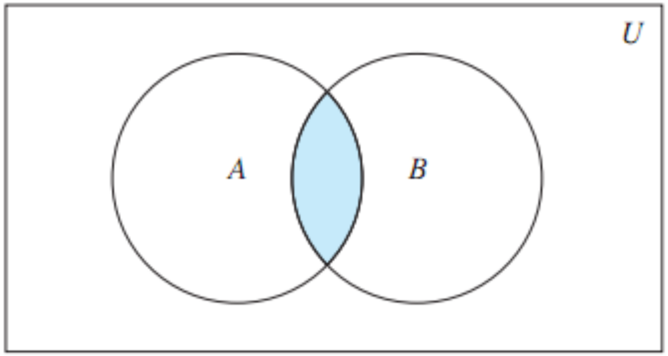
Consider x

Thinking about: ∪ is the same as ∨

INTERSECTION

- Symbol: $A \cap B$
- Set builder:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



$A \cap B$ is shaded.

A	B	$A \cap B$
{a, b, c}	{c, d, e}	{c}
{a, b, c}	{a, b, c, d, e}	{a, b, c}
{1, 2, 3}	{4, 5}	$\varnothing$ // **disjoint** sets
Set of all bit strings of length 4 start with 1	Set of all bit strings end with 00	Set of all bit strings of length 4 start with 1**AND** ends 00

MEMBERSHIP TABLE FOR A ∩ B

0: NOT A MEMBER OF THE SET
1: MEMBER OF THE SET

A	B	A ∩ B
0	0	?
1	0	?
0	1	?
1	1	?

CARDINALITY OF A∪B

$|A \cup B| = |A| + |B|$??? NOT TRUE

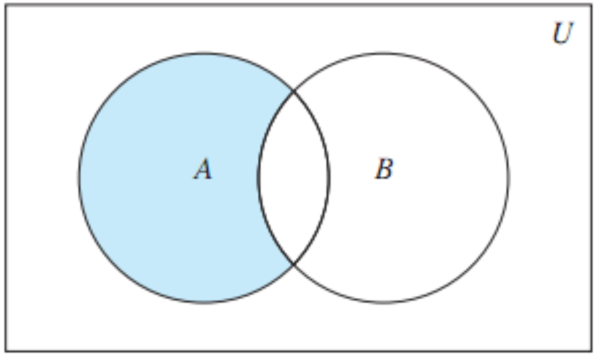
$|A \cup B| = |A| + |B| - |A \cap B|$

- For example
- $A = \{1, 2, 3, 4, 5\} \rightarrow |A| = 5$
- $B = \{4, 5, 6, 7\} \rightarrow |B| = 4$
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7\} \rightarrow |A \cup B| = 7$
- $A \cap B = \{4, 5\} \rightarrow |A \cap B| = 2$

DIFFERENCE

- Symbol: A – B   // difference of A and B
- Set builder:

$$A - B = \{x \mid x \in A \land x \notin B\}$$



$A - B$ is shaded.

A	B	A – B
{a, b, c}	{c, d, e}	{a, b}
{a, b, c}	{a, b, c, d, e}	{ }
{1, 2, 3}	{4, 5}	{1, 2, 3}
The set of all bit strings of length 4 begin with 1	The set of all bit strings of length 4 end with 00	The set of all bit string of length 4 begin with 1 AND NOT end with 00

MEMBERSHIP TABLE FOR A – B

0: NOT A MEMBER OF THE SET
1: MEMBER OF THE SET

A	B	A – B
0	0	0
1	0	1
0	1	0
1	1	0

COMPLEMENT OF A SET

- UNIVERSAL SET **U**: set contains all objects under consideration.
- Complement of a set A, denoted by $\bar{A}$ is the defined by:

$\bar{A}$ = U – A

$$x \in A \leftrightarrow x \notin \bar{A}$$

Example 1: U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

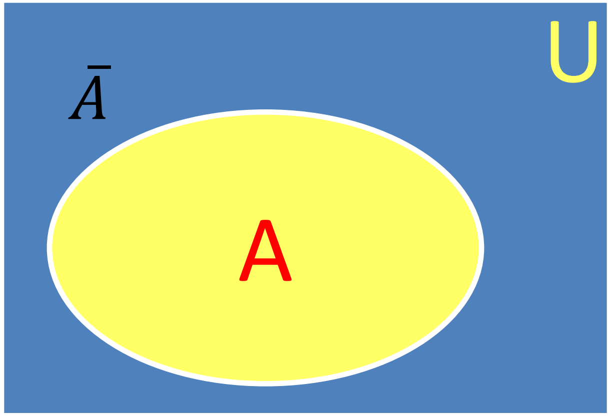
A = {1, 3, 5} ➔ $\bar{A}$ = {2, 4, 6, 7, 8, 9, 10}

Example 2: U is the set of all letters of the English alphabet.

A = {e, a, I, o, u}

➔ $\bar{A}$ = {b, c, d, f, g, h, j, k, l, m, n ,p, q, r, s, t, v, w, x, y, z}.

VENN DIAGRAM. MEMBERSHIP TABLE OF $\bar{A}$



A	$\bar{A}$
0	1
1	0

EXAMPLE

Show that A – B = A $\cap \overline{B}$

- **Method 1.** Use set builder.
- A – B = $\{x \mid x \in A \wedge x \notin \mathbf{B}\}$

  $= \{x \mid x \in A \wedge x \in \overline{B}\}$

  $= A \cap \overline{B}$
- **Method 2.** Membership table.

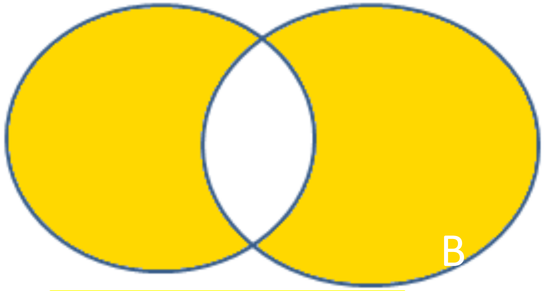
A	B	$\overline{B}$	A – B	A $\cap \overline{B}$
0	0	1	0	0
0	1	0	0	0
1	0	1	1	1
1	1	0	0	0

SYMMETRIC DIFFERENCE

- The **symmetric difference** of A and B is the set:

$$(A - B) \cup (B - A)$$

- Symbol: **A⊕B**



A ⊕ B is colored

MEMBERSHIP TABLE

A	B	A⊕B
0	0	0
0	1	1
1	0	1
1	1	0

SET IDENTITIES

Identity		Name (law)
$A \cap U = A$	$A \cup \varnothing = A$	Identit
$A \cup U = U$	$A \cap \varnothing = \varnothing$	Domination
$A \cup A = A$	$A \cap A = A$	Idempotent
$\overline{(\overline{A})} = A$		Complementation
$A \cup B = B \cup A$	$A \cap B = B \cap A$	Commutative
$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$	Associative
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive
$\overline{A \cap B} = \overline{A} \cup \overline{B}$	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan
$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$	Absorption
$A \cup \overline{A} = U$	$A \cap \overline{A} = \varnothing$	Complement

PROOFS OF $\overline{A \cap B} = \bar{A} \cup \bar{B}$

MEMBERSHIP TABLES

A	B	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$	$A \cap B$	$\overline{A \cap B}$
0	0	1	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	1	0	0	0	1	0

PROOFS OF $A \cap \bar{A} = \varnothing$, $A \cup \bar{A} = U$

A	$\bar{A}$	$A \cap \bar{A}$	$A \cup \bar{A}$
0	1	0	1
1	0	0	1

Other proof for $A \cup \bar{A} = U$.

To show that $A \cup \bar{A} = U$, we show that $A \cup \bar{A} \subseteq U$ and $U \subseteq A \cup \bar{A}$

- $A \cup \bar{A} \subseteq U$ is trivial because U is the universal set.
- To show $U \subseteq A \cup \bar{A}$, consider arbitrary x in U. Two possible cases:
  - If $x \notin A$, then $x \in \bar{A}$. So, $x \in A \cup \bar{A}$.
  - If $x \in A$, then $x \in A \cup \bar{A}$.

Therefore, $U \subseteq A \cup \bar{A}$.

Proved.

Bitwise AND/OR/XOR

A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.
Ex: 1 0101 1001 is a bit string of length nine.

Ex:

01 1011 0110

11 0001 1101

11 1011 1111 bitwise *OR*

01 0001 0100 bitwise *AND*

10 1010 1011 bitwise *XOR*

REPRESENTING SETS IN COMPUTER

Ex1. U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} // i = 1..10

A = {1, 3, 5}, B = {1, 2, 4, 5, 7, 9}

→ A = "1 0 1 0 1 0 0 0 0 0" // $[a_i]$: $a_i = 1$ iff $i \in A$

→ B = "1 1 0 1 1 0 1 0 1 0" // $[b_i]$: $b_i = 1$ iff $i \in B$

HOW TO FIND A∩B? ← bitwise AND

A∩B = "1 0 0 0 1 0 0 0 0 0" // $[c_i] := a_i \wedge b_i$

→ A∩B = {1, 5} // If $[c_i] = 1$ then print i in A∩B

Example 2.

U = {a, b, c, d, e}

A = {b, d}

→ A = "0 1 0 1 0"

REVIEW OF SECTION 2.1

- SETS
- WAYS TO DEFINE SETS: SET BUILDER, LISTING, VENN DIAGRAMS
- SET OPERATIONS: UNION, INTERSECTION, DIFFERENCE, SYMMETRIC DIFFERENCE, COMPLEMENT
- SET IDENTITIES
- COMPUTER REPRESENTATION OF SETS

OUR GOAL

- Why study functions, sequences and sums?
- What is a function?
- When does a function have an inverse?
- Some useful sums and sequences for estimating the complexity of algorithms.

Why study this section?

- Extremely important in mathematics and computer science
- Used in the definition of such discrete structures as sequences and strings
- Used to represent how long it takes a computer to solve problems of a given size
- Many computer programs are designed to calculate values of functions.
- Recursive functions, which are functions defined in terms of themselves, are used throughout computer science.
- To estimate the complexity of algorithms, we need to know some useful sums such as:

1 + 2 + 3 + … + n-1 // Sorting n integers by the Bubble sort algorithm

What is a function? Introduction.

- **Vending machine.**

If you press one button (input), how many tasks does the machine do (output)?

What is function?

- If you press the Pepsi button (input = Pepsi), you will receive a Pepsi can
- The next time, you press “Pepsi” again, you receive a Pepsi (NOT a Coca cola)

What is a function?

input	Machine	output
-2	How does it work?	4
3		9
-4		16
5		25

Output = $(\text{input})^2$

$f(x) = x^2$

Ways to describe how it works.

THE SAME

What is a function?

input	How does it work?	output
3		5
3		9

How to know the output when input is 3?

No one knows exactly.

➔ Not a function.

Function. Definitions.

$f: A \to B$ // Read: f is a function from A to B, f maps A to B

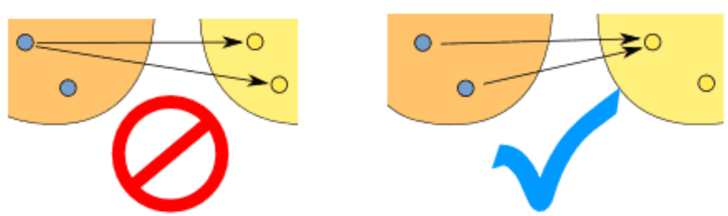
Each element a in A maps exactly one element b in B

Write b = f(a)

each input

Exactly one output

~~0 output~~
~~> 1 outputs~~



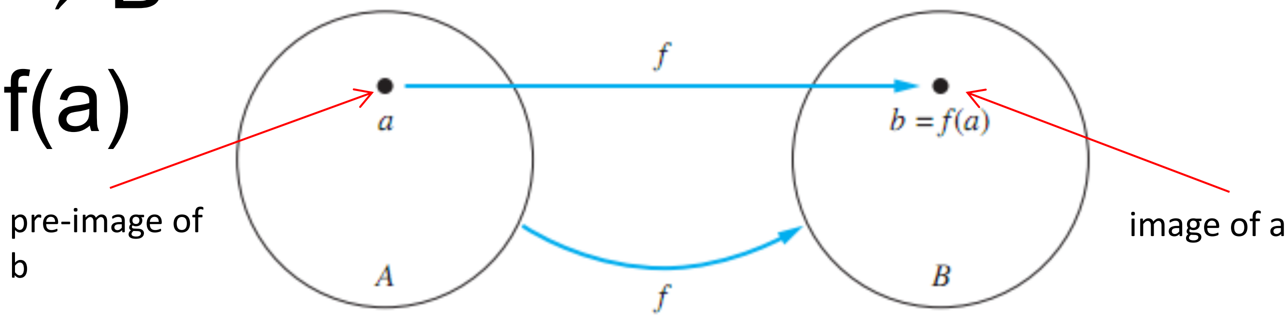
(one-to-many)
This is **NOT** OK in a function

(many-to-one)
But this **is** OK in a function

Functions/mappings/ transformation. Definitions.

$f: A \to B$

$b = f(a)$



Three parts of a function.

$f: A \to B$

❶ A: domain

❷ B: codomain

❸ Ways to determine the output.

For example, $f: \mathbf{Z} \to \mathbf{R}$, $f(m) = 1/(m^2 + 3)$

codomain vs **range**

- Codomain contains outputs
- Range: the set of all outputs
- Range is a subset Codomain (maybe equal)

Ways to define a function.

- By formulas:

$g(x) = 3$, $f(x) = x^2$

- By sets:

$f = \{(2, 3), (-3, 1), (4, 7)\}$

$g = \{(a, 1), (b, 4), (c, 3)\} \sim g(a)=1, g(b)=4, g(c)=3$

Domain: {a, b, c}
Range: {1, 4, 3}

- By Tables
- By words

A function in programming language (C)

codomain

```
float sm(float x)
```

domain

```
{
    float result;
    result = x+x;
    return result;
}
```

How the output is produced.

Not a function

- Two cases:
- No output for some input

f: R→ R; f(x) = 1/(x – 1)

- More than one outputs for one input

f: R → R

$$f(x) = \begin{cases} -x + 3 & if\ x > 2 \\ x + 1 & if\ x \leq 5 \end{cases}$$

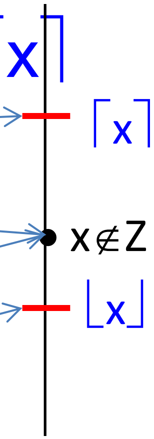
Some Important Functions

Ceiling function

$f: \mathbb{R} \to \mathbb{Z}$

$f(x) = \lceil x \rceil$ = smallest integer such that $x \le \lceil x \rceil$

$\lceil -4 \rceil = -4$, $\lceil 2.7 \rceil = 3$, $\lceil -3.7 \rceil = -3$



Floor function

$f: \mathbb{R} \to \mathbb{Z}$

$f(x) = \lfloor x \rfloor$ = largest integer such that $\lfloor x \rfloor \le x$

$\lfloor 3 \rfloor = 3$, $\lfloor 2.7 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$, $\lfloor 2/3 \rfloor = 0$

One-to-one/injective functions

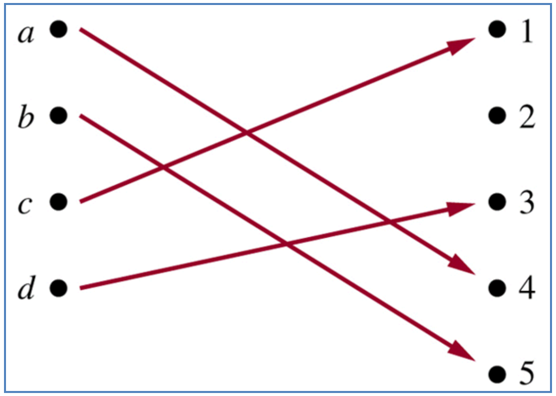
- f: A $\rightarrow$ B is one-to-one iff

different inputs $\rightarrow$ different outputs

Ex: $f(x) = x^3$

Different inputs: $x \neq y \rightarrow x^3 \neq y^3$: different outputs

$$\forall x \forall y (x \neq y \rightarrow f(x) \neq f(y))$$



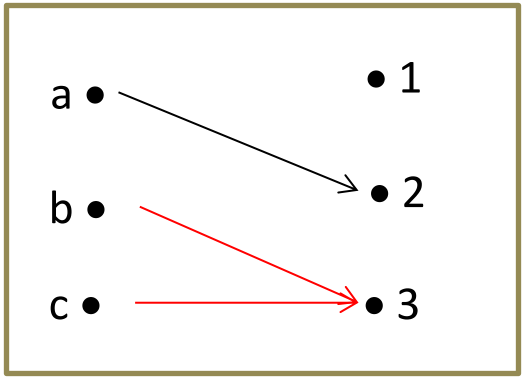
One-to-one

- Not one-to-one:

There are two different inputs that map to the same output

$$\exists x \, \exists y (x \neq y \wedge f(x) = f(y))$$

$$f(b) = f(c) = 3$$



Not one-to-one

Onto/surjective functions

- f: A $\rightarrow$ B is **onto** iff f(A) = B

  $\forall y \in B \exists x \in A (f(x) = y)$

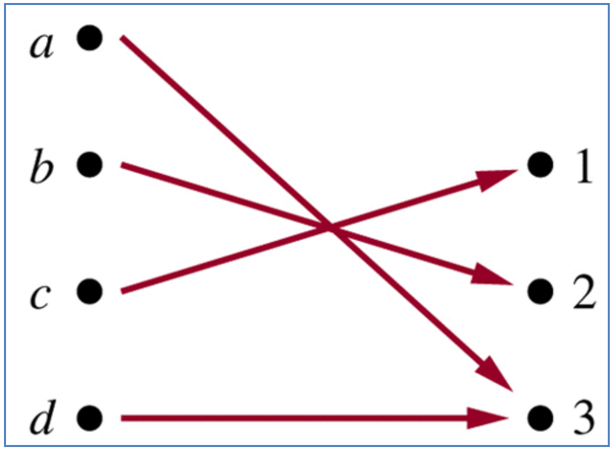
- For example,

  f(c) = 1, f(b) = 2, f(a) = 3

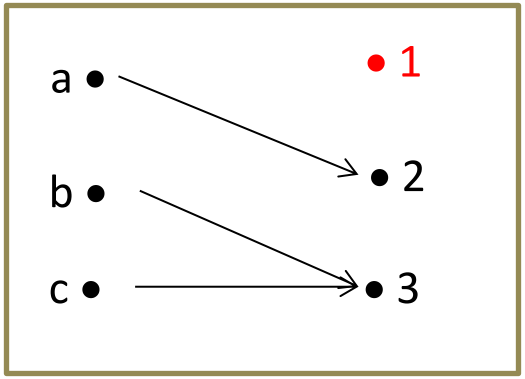
Not onto:

$\exists y \in B \forall x \in A (f(x) \neq y)$

f(a) $\neq$ 1, f(b) $\neq$ 1, f(c) $\neq$ 1



onto



Not onto

Bijection = one-to-one correspondence

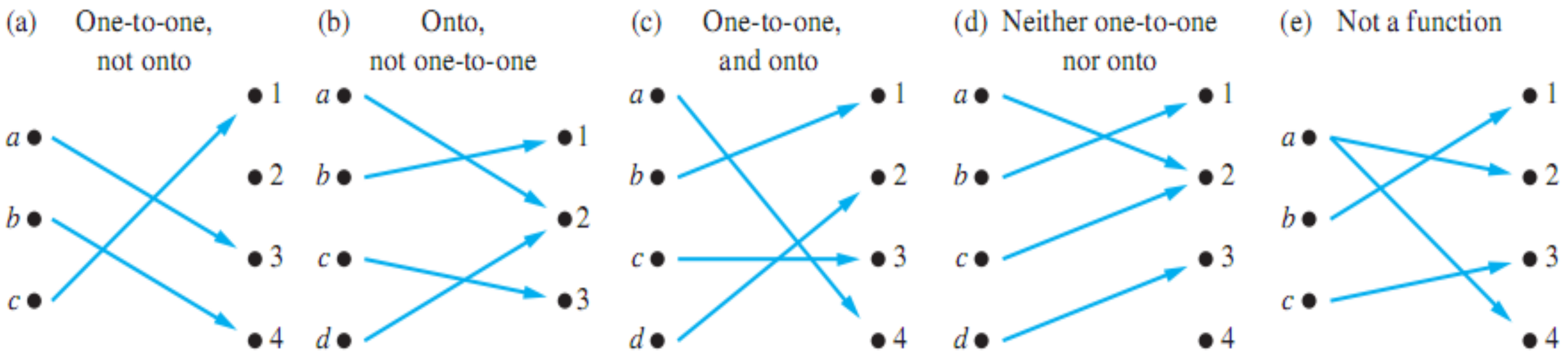
- A function is called a bijection if it is both onto and one-to-one.
- Bijection = onto + one-to-one
- Examples.

i) {(A, 10), (B, 11), (C, 12), (D, 13), (E, 14), (F, 15)} is a bijection. // hexadecimal

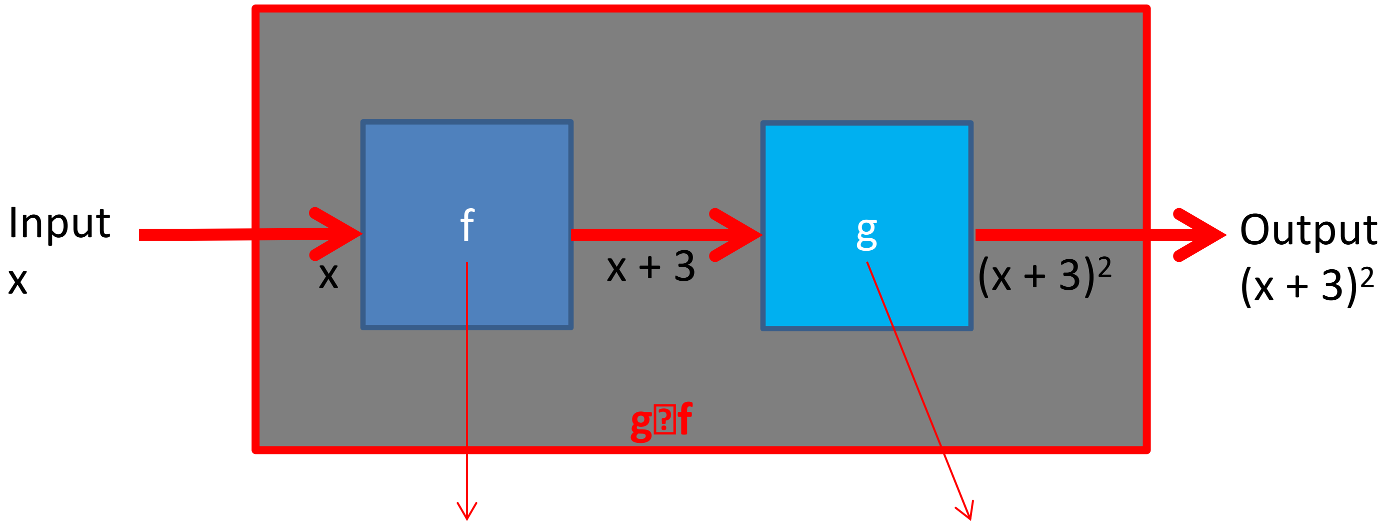
ii)

A	B	…	Z
65	66	…	90

is a bijection // ASCII code


(a) One-to-one, not onto
a
b
c
1
2
3
4
(b) Onto, not one-to-one
a
b
c
d
1
2
3
(c) One-to-one, and onto
a
b
c
d
1
2
3
4
(d) Neither one-to-one nor onto
a
b
c
d
1
2
3
4
(e) Not a function
a
b
c
1
2
3
4

Compositions of functions.



Ex. $f(x) = x + 3$ $g(x) = x^2$

Def. the composition of the functions f and g
Is defined by

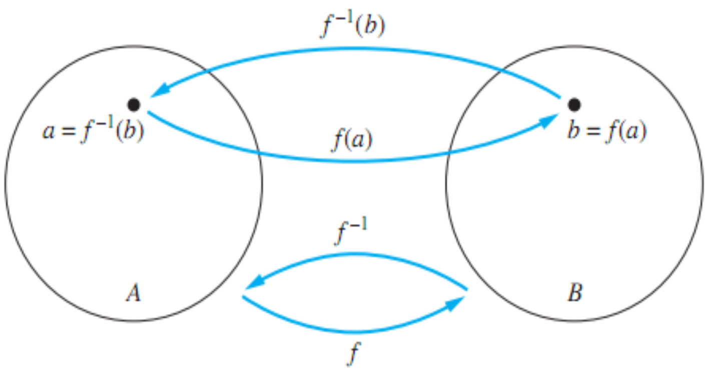
$(g \circ f)(x) = g(f(x))$

$(g \circ f)(x) = g(f(x)) = g(x + 3) = (x + 3)^2$

Inverse functions.

- f: A → B has an inverse iff f is a bijection
- The **inverse function** of f is denoted by f^{-1}.

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$



Examples.

1. f = {(A,10), (B,11), (C,12), (D,13), (E,14), (F,15)}

➔ f^{-1} = {(10,A), (11,B), (12,C), (13,D), (14,E), (15,F)}

2. $f(x) = 3x + 5 = y$ ➔ $x = (y-5)/3$

➔ $f^{-1}(y) = x = (y-5)/3$

Sequence & sums

- Arithmetic progression: (cấp số cộng)

  a, a + d, a + 2d, a + 3d, …, a + nd, …

  d: common difference

  3, 7, 11, 15, 19, 23, …, 3 + 4n, …

  ➔ $a_n = 3 + 4n$

- Geometric progression: (cấp số nhân)

  a, ar, ar^2, ar^3, …, ar^n, …

  r : common ratio

  2, 1, ½, ¼, 1/8, 1/16, …, $2.(1/2)^n$, …

  ➔ $b_n = 2(1/2)^n$

Some useful sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Sums

- Consider the following pseudocode segment

```
Procedure Hello(n: positive integer)
Begin
    For  i: = 1 to n
        For j:=1 to i
            Print "hello"
End
```

How many word hello will be printed?

Sums

i	j	Times of printing
1	1	1
2	1 2	2
3	1 2 3	3
…	…	…
n	1 .. n	n

$$1 + 2 + 3 + \ldots + n = n(n+1)/2$$

Summations

$$a_m + a_{m+1} + a_{m+2} + \ldots + a_n = \sum_{j=m}^{n} a_j = \sum\nolimits_{j=m}^{n} a_j = \sum\nolimits_{m \le j \le n} a_j$$

a : Sequence

j : Index of summation

m: Lower limit

n : Upper limit

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Cardinality

- Cardinality = number of elements in a set.
- The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B
- A set that is either finite or has the same cardinality as the set of positive integers is called ***countable***.
- A set that is not countable is called ***uncountable.***
- When a infinite set S is countable, we denote the cardinality of S is $|S| = \aleph_0$ (aleph null)
- For example, |⍰| = $\aleph_0$ because ⍰ is countable and infinite but ⍰ is uncountable and infinite, and we say |⍰| = $2^{\aleph_0}$

Examples

sets	countable	uncountable	cardinality
{a, b, ..., z}, $\{x \mid x^5 - 3x^2 - 11 = 0\}$, ...	✓	✗	< □
{0, 2, 4, ..., }	✓	✗	$\aleph_0$
N, Z^+, Z, Q, Z□Z, ...	✓	✗	$\aleph_0$
$\{x \mid 0 < x < 1\}$, R,...	✗	✓	$2^{\aleph_0}$

Summary

- Sets
- Set operations
- Functions
- Sequences
- Summations

THANKS