Chapter 1

The Foundations: Logic & Proofs

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Why study this chapter?

- Remove ambiguity of human languages
- Basis of all mathematical reasoning, and of all automated reasoning
- Important in design of computing machines
- Ubiquitous in computer science

Objectives – students can answer these questions after studying this topic:

1. What are **propositional logic and its applications**?
2. How to show two propositions are **equivalent**?
3. How to give a **negation** of a compound proposition?
4. What are predicates, quantifiers?
5. How to negate a the **universal (∀)** and **existential quantifications (∃)** of a **predicate** P(x)?
6. How to **translate** from English into logical expression?
7. How to use **rules of inference** to produce a valid argument?

Foundations of Logic: Overview

- Propositional logic:
  - Basic definitions.
  - Equivalence rules.
- Predicate logic:
  - Predicates.
  - Quantified predicate expressions.
  - Equivalences.

– Rules of Inference

What is Propositional Logic?

Propositional Logic is the logic of compound statements built from simpler statements using so-called *Boolean connectives.*

Some applications in computer science:

- Design of digital electronic circuits.
- Expressing conditions in programs.
- Queries to databases & search engines.

George Boole (1815-1864)

Propositions

- Propositions:
- 100% TRUE (T, 1) → proposition
- 100% FALSE (F, 0) → proposition
- Others → NOT a proposition

propositions	NOT propositions
• ***p*** = Ha Noi is not the capital of Vietnam. • ***q*** = 1 + 1 = 2. • ***r*** = You will pass this course. • h = Bà Tưng is one of Bà Trưng's descendants.	• What time is it? • Read this carefully. • x + 1 = 2. • Oh no!

Operators/connectives

- Old propositions ➔new propositions by using operators.
- On numbers: -3, 2 × 3, 7 + 2, 11/4
- On propositions: operators operate on **propositions** or truth values instead of on numbers.

Compound propositions

Statements	Operators/connectives
Michael's PC does not run Linux.	Negation (not)
He is young and strong.	And (conjunction)
I'm going to the cinema to see *Fast 8* on Wednesday **or** Thursday.	Xor (exclusive or)
Experience with C++ or Java is required.	Or (disjunction)
I will be shot if I know.	Implication
I go to school if and only if it doesn't rain.	If and only if (IFF)

Logical operators/connectives

Logical Operators	Notations	Nickname
Negation	$\neg$	not
Conjunction	$\wedge$	and
Disjunction	$\vee$	or
Exclusive or	$\oplus$	xor
Conditional statement	$\rightarrow$	implies
Biconditional statement	$\leftrightarrow$	If and only if

What is the negation of a proposition?

- p: proposition ➔ ¬p (or $\bar{p}$, read: "**NOT** p"): ***negation*** of p.

propositions	negations
• ***p***: Ha Noi **is not** the capital of Vietnam.	• ¬***p:*** Ha Noi **is** the capital of Vietnam.
• ***q:*** 1 + 1 = 2.	• ¬***q***: 1 + 1 ≠ 2.
• ***r***: You will pass this course.	• ¬***r***: You will **NOT** pass this course.

What is the truth table of a proposition?

- True = T = 1
- False = F = 0

The **Truth Table** for the Negation of a Proposition.

p	**¬*p***
T	F
F	T

***p*:** Ha Noi is not the capital of Vietnam.

¬*p*: Ha Noi is the capital of Vietnam.

What is the *conjunction* of propositions?

ΛND

- p, q are propositions
- The conjunction of p and q: $p \wedge q$, read "p Λnd q"

p	q	$p \wedge q$
T	**T**	**T**
T	F	F
F	T	F
F	F	F

The only **TRUE** case

- Ex.

p = "1+1 =3"

q = "3 > 2"

➔ $p \wedge q$ = "1+1=3 and 3 > 2"

Conjunction(AND). Examples

- It is below freezing (f) AND snowing (s).

  $f \wedge s$

- I know (k) but I say nothing (s). //but means AND

  $k \wedge s$

- Today is Monday (m) AND it is cold (c).

  $m \wedge c$

What is the *disjunction* of propositio

OR

- p, q: propositions
- The disjunction of p and q: $p \vee q$, read "p OR q"
- Ex.

p	q	$p \vee q$
F	F	F
T	F	T
F	T	T
T	T	T

The only **FALSE** case

p = "1+1 = 3"
q = "3 > 2"
➔ $p \vee q$ = "1+1= 3 or 3 > 2"

OR. Examples

- It is either snowing or below freezing (or both).

  $\underbrace{\text{It is either snowing}}_{s} \lor \underbrace{\text{below freezing}}_{f}$

- A password must have at least three digits (p) **or** be at least eight characters long (q).

  $p \lor q$

OR in daily usage may be different

- OR in logic is not necessarily exclusive, unlike daily usage.
- For example, Sack Mourinho or I will kill my cat.

→ Or in daily usage means "Exclusive or"

Exclusive Or

- Nickname: XOR
- Symbol: ⊕

p	q	p ⊕ q
F	F	F
F	T	T
T	F	T
T	**T**	**F**

DIFERENT FROM OR

Natural language is AMBIGUOUS // tối nghĩa

- He will join us **or** die ⊕
- Lunch includes soup **or** salad ∨

p	q	p **OR** q
F	F	F
F	T	T
T	F	T
T	**T**	**?**

∨ : … or …
⊕ : … or … **but not both**

What is the conditional statement $p \rightarrow r$?

- Symbol: $p \rightarrow r$ (IF p, THEN r)
- Nickname: implication
- Ex.

s	h	s → h
It is sunny.	It is hot.	IF it is sunny, THEN it is hot.

antecedent

consequent

$$p \rightarrow r$$

Implication. Truth table

- p → r can be true when p is false

implication

p = "Messi is Vietnamese"
r = "Messi is Asian"
p → r = "**If** Messi is Vietnamese, **then** Messi is Asian."

p	r	p → r
F	F	T
F	T	T
T	F	F
T	T	T

The only FALSE case

IMPLICATION - EXAMPLES

- If $\underbrace{1 + 1 = 2}_{T}$, then $\underbrace{2 + 2 = 5}_{F}$. **FALSE**
- If $\underbrace{1 + 1 = 3}_{F}$, then $\underbrace{2 + 2 = 4}_{T}$. **TRUE**
- If $\underbrace{1 + 1 = 3}_{F}$, then $\underbrace{2 + 2 = 5}_{F}$. **TRUE**
- If $\underbrace{\text{monkeys can fly}}_{F}$, then $\underbrace{1 + 1 = 3}_{F}$. **TRUE**

How to understand?

- More general than in the natural languages.
- In natural langue, we seldom say

"if 1+1 = 3, then monkeys can fly"

but in mathematics, this statement is true because p = "1+1 = 3" is false.

- A useful way to understand: think of an obligation or a contract.

How to understand?

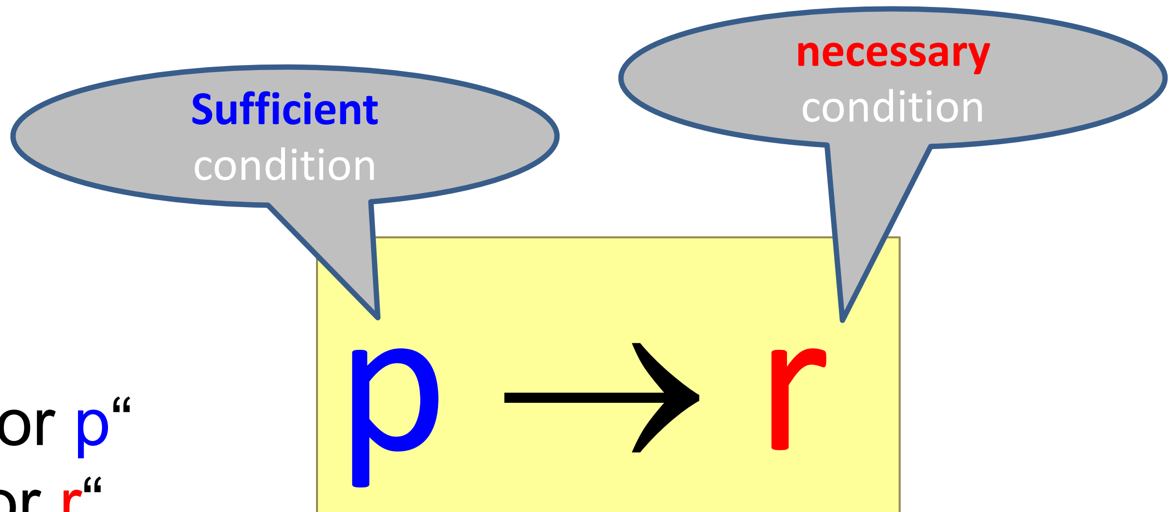
- **Ex.** Consider a statement that a friend might make:

"If you help me to pass the DM course, then I will invite you to go to cinema."

- Four possible cases:
  - (T → T = T): your friend passes the course, and he or she gives you an invitations ➔ TRUE.
  - (F → F = T): your friend does not pass the course, and you do not receive an invitation ➔ TRUE
  - (F → T = T): your friend does not pass the course, and you still receive an invitation ➔ TRUE
  - **(T → F = F):** However, if your friend passes the course but he or she does not give you an invitation, you will feel cheated. ➔ FALSE

“Fifty shades” to express this conditional statement

“**If** p, **then** r”
“If p, r”
“p **implies** r”
“r **if** p”
“p **only if** r”
“r **unless** $\neg$p”
“r is **necessary** for p“
“p is **sufficient** for r“
“r **whenever** p”
“**when** p, r”
“r **follows from** p”



AND MORE (will be considered in next segments)

Exercises. Rewrite the statements in form “If …, then …”

- It snows **whenever** the wind blows from the northeast.

➔ **If** the wind blows from the northeast, **then** it snows.

- The apple trees will bloom **if** it stays warm for a week.

➔ **If** it stays warm for a week, **then** the apple trees will bloom.

- That Vietnam win the 2018 AFF championship **implies that** they beat Malaysia.

➔ **If** Vietnam win the 2018 AFF championship, **then** they beat Malaysia.

Exercises. Rewrite the statements in form "If …, then …"

- To get tenure as a professor, it is **sufficient** to be world famous.

➔ **If** you are world famous, **then** you will get tenure as a professor.

- Your guarantee is good **only if** you bought your CD player less than 90 days ago.

➔ **If** your guarantee is good, **then** you must have bought your CD player less than 90 days ago.

- Jan will go swimming **unless** the water is too cold.

➔ **If** the water is not too cold, **then** Jan will go swimming.

Do yourself

- To be a citizen of this country, it is **sufficient** that you were born in the United States.
- Studying is sufficient for passing.
- It is **necessary** to have a valid password to log on to the server.
- I will remember to send you the address **only if** you send me an e-mail message.

CONVERSE, CONTRAPOSITIVE, AND INVERSE

- $q \rightarrow p$ is called the **converse** of $p \rightarrow q$
- $(\neg q) \rightarrow (\neg p)$ is called the **contrapositive** of $p \rightarrow q$
- $(\neg p) \rightarrow (\neg q)$ is called the **inverse** of $p \rightarrow q$.

Biconditional statement ↔

- Symbol: $p \leftrightarrow q$ (p IF AND ONLY IF q)
- common ways to express $p \leftrightarrow q$:
  - "p is **necessary and sufficient** for q"
  - "if p then q, and **conversely**"
  - "p iff q."

TRUTH TABLE

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

p = "You can take the flight"
q = "you buy a ticket "
$p \leftrightarrow q$ = "You can take the flight **if and only if** you buy a ticket."

Natural language is AMBIGUOUS

- Biconditionals are not always explicit in natural language
- "if and only if" is rarely used in common language; instead, "if, then" or an "only if" are used
  - In natural language,

"Catch me IF you can" // IF AND ONLY IF

→ Imprecise in natural language

→ In mathematics and logic, which precision is essential, we will always distinguish between $p \rightarrow q$ and $p \leftrightarrow q$

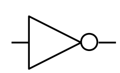
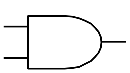
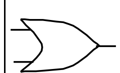
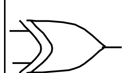
What is the precedence of logical operators?

(1) In parentheses from inner to outer
(2) $\neg$
(3) $\wedge$
(4) $\vee$
(5) $\rightarrow$
(6) $\leftrightarrow$

Examples.

- $\neg p \wedge q$ means $(\neg p) \wedge q$, not $\neg(p \wedge q)$
- $p \wedge q \vee r$ means $(p \wedge q) \vee r$, not $p \wedge (q \vee r)$
- $p \vee q \rightarrow r$ means $(p \vee q) \rightarrow r$, not $p \vee (q \rightarrow r)$

Some Alternative Notations

Name:	not	and	or	xor	implies	iff
Propositional logic:	$\neg$	$\wedge$	$\vee$	$\oplus$	$\rightarrow$	$\leftrightarrow$
Boolean algebra:	$\overline{p}$	pq	$+$	$\oplus$		
C/C++/Java (wordwise):	!	&&	||	!=		==
C/C++/Java (bitwise):	~	&	|	^		
Logic gates:						

Review 1.1

- Propositions: *p*, *q*, *r*, …
- Boolean operators/connectives: $\neg \wedge \vee \oplus \rightarrow \leftrightarrow$
- Truth table: 2^n rows for n propositions
- Precedence: (1) $\neg$ (2) $\wedge$ (3) $\vee$ (4) $\rightarrow$ (5) $\leftrightarrow$

- Exercises: 10, 12, 16, 31-39, 42, 43 P.14

Next:

- Your task: Readings: Equivalences
  - De Morgan's laws
  - Distributive Laws
  - $\neg(p \rightarrow q) \equiv p \wedge (\neg q)$

1.2 Logical equivalences. Introduction

- In numbers,

$a + b = b + a$ // commutative law

$(a + b)^2 = a^2 + 2ab + b^2$

$a(b + c) = ab + ac$ // distributive law

etc.

- In propositions:

$p \wedge q \equiv q \wedge p$ // commutative law

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ // distributive law

And more

What is a tautology? Contradiction?

- Tautology = always true
- Contradiction = always false
- How to know?

TRUTH TABLES

p	¬p	p ∨ (¬p)
F	T	T
T	F	T
Today is hot	Today is **not** hot	**Today is hot or not. // tautology**

Always true // tautology

Logical Equivalence

- If p $\leftrightarrow$ q is always true (= tautology), then p and q are **logically equivalent**.
- Symbol: p $\equiv$ q (or p $\Leftrightarrow$ q)

$$\overline{p \wedge q} \equiv \overline{p} \wedge \overline{q} \ ???$$

FALSE

$$\overline{p \wedge q} \equiv \overline{p} \vee \overline{q}$$

TRUE

Logical Equivalence

- **Example 1.**

De Morgan's Laws
$\neg(p \land q) \equiv \neg p \lor \neg q$
$\neg(p \lor q) \equiv \neg p \land \neg q$

Augustus De Morgan (1806-1871)

HOW TO KNOW?

Truth tables for $\neg (p \wedge q)$ and $\neg p \vee \neg q$

p	q	$\neg (p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
F	F	T	T	T	T
F	T	T	T	F	T
T	F	T	F	T	T
T	T	F	F	F	F

THESE TRUE VALUES AGREE

Example 2. Distributive laws

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Truth tables

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Example 3. Show that $p \to q$ and $\neg p \vee q$ are logically equivalent using truth tables.

- TRUTH TABLES

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
F	F	**T**	T	**T**
F	T	**T**	T	**T**
T	F	**F**	F	**F**
T	T	**T**	F	**T**

Equivalences

Equivalence		Name
$p \land T \equiv p$	$p \lor F \equiv p$	Identity laws
$p \lor T \equiv T$	$p \land F \equiv F$	Domination laws
$p \lor p \equiv p$	$p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$		Double negation law
$p \lor \neg p \equiv T$	$p \land \neg p \equiv F$	Negation laws

Equivalences

Equivalence		Name
$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$	Absorption laws

Note that De Morgan's laws extend to

$\neg(p_1 \vee p_2 \vee \cdots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$

and

$\neg(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \equiv \neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n$.

How to show two propositions are logically equivalent ($\equiv$)?

Two ways:

- **Truth tables:** use tables with 2^n rows ➔ the same truth values for all possible cases
- **Basic laws:**
  - De Morgan’s Laws
  - idempotent laws
  - associative laws
  - commutative laws
  - etc.

Constructing New Logical Equivalences

$p \rightarrow q \equiv \neg p \vee q$ ⇒ $\neg(p \rightarrow q) \equiv$???

Constructing New Logical Equivalences

Example 4:

Show that $\neg(p \rightarrow q) \equiv p \wedge \neg q$ WITHOUT using truth tables.

$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ by Example 3

$\equiv \neg(\neg p) \wedge \neg q$ by the second De Morgan law

$\equiv p \wedge \neg q$ by the dou

TRUTH TABLES

p	q	p → q	¬(p → q)	¬q	p ∧ ¬q
F	F	T	F	T	F
F	T	T	F	F	F
T	F	F	T	T	T
T	T	T	F	F	F

Constructing New Logical Equivalences

- Substitution Rule:

$p \to q \equiv \neg p \lor q$

$p \to (q \lor r) \equiv$???

$\neg p \lor (q \lor r)$

$(p \lor q) \to r \equiv$???

$\neg(p \lor q) \lor r$

DIY

Equivalences	Equivalences
$p \to q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \to q \equiv \neg q \to \neg p$	$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \vee q \equiv \neg p \to q$	$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$p \wedge q \equiv \neg (p \to \neg q)$	$\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$
$\neg(p \to q) \equiv p \wedge \neg q$	
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$	
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$	
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$	
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$	

How to find the **negation** of a compound proposition?

- De Morgan's Laws:

1. $\neg(p \wedge r) \equiv \neg p \vee \neg r$
2. $\neg(p \vee r) \equiv \neg p \wedge \neg r$

- Negation of implication:

$\neg(p \to r) \equiv p \wedge \neg r$

How to find the negation of a compound proposition?

- Example 1. Find the negation of the statement

"He is young and strong" → $p \wedge r$

Use De Morgan's Law

$\neg(p \wedge r) \equiv \neg p \vee \neg r$

→Negation: "He is not young or not strong"

How to find the negation of a compound proposition?

- Example 2. Find the negation of the statement

"Mia has a Sony smartphone or she has an iPhone" ➔ $p \lor r$

- Use De Morgan's Law $\neg(p \lor r) \equiv \neg p \land \neg r$

➔ Negation: "Mia does not have a Sony smartphone and she does not have an iPhone"

How to find the negation of a compound proposition?

- Example 3. Find the negation of the statement

"If he gets 4 on the final exam, then he will pass the course" ➔ $p \to r$

- Use the equivalence: $\neg(p \to r) \equiv p \land \neg r$

➔Negation: "He gets 4 on the final exam, but he will not pass the course"

Review 1.2

- Tautology = always true
- Contradiction = always false
- Equivalence: $p \equiv q$, q and q have the same truth values for all possible cases
- De Morgan's laws:
  - $\neg(p \land q) \equiv \neg p \lor \neg q$
  - $\neg(p \lor q) \equiv \neg p \land \neg q$
- Other Laws

- Exercises: 16-34 (even only)

1.4. Predicate Logic.

- No rules of propositional logic allow us to conclude the truth of the statement.

  Ex: "computer x is functioning properly"

Predicates – example 1: one-place predicate

Mickey / *is a mouse*.

$\underbrace{\text{Mickey}}_{\text{subject}}$ $\underbrace{\textit{is a mouse}}_{\text{predicate}}$

→ mouse(Mickey)

→ M(Mickey)

M denotes *"is a mouse"* – the *predicate* of the statement.

→ M(x) = "x is a mouse" // x: variable

→ M(Tom) = "Tom is a mouse"

Predicates – example 2: two-place predicate

"Mr. Bay *is the instructor of* the course DBI202"

→instructor(Mr. Bay, DBI202)

→I(Mr. Bay, DBI202)

I denotes *"is the instructor of"* – the *predicate* of the statement.

Consider the statement:

I(x, y) = "x is the instructor of the course y"

→ I(Mr. Khanh, DBI202) = **???**

"I = Mr. Khanh is the instructor of the course DBI202"

Predicates – example 3: three-place predicate

5 *is the sum of* 2 and 3.

→ *sum*(5, 2, 3)

→ *S*(5, 2, 3)

S denotes *"is the sum of"* – the *predicate* of the statement.

→ S(x, y, z) = "x = y + z"

→ S(5,2,3) = "5=2+3" is true.

→ S(3,2,5) = "3 = 2 + 5" is false.

What are **Quantifiers**?

- Used in quantifications
- In English, the words *all, some, every, each, any, many, none, few*, etc. are used.
- Two types:

- universal quantifier ∀
- existential quantifier ∃

Quantifiers – two types

- **Universal quantifier** $\forall$ ← for *All, for every, for each, for any,* for arbitrary, *etc.*

Symbol: $\forall x P(x)$

Read:

"for $\forall$ll x P(x)" or "for every x P(x)"

$\forall x P(x)$ = "P(x) for every element x in the domain."

Quantifiers – two types

$\exists$xistential quantifier $\exists$ ← *some, many, at least, few, etc.*

Notation: $\exists x P(x)$

Many ways to read:

"**There exists** an element x in the domain such that P(x)."

"**For some** x P(x)",

"For **at least one** x such that P(x)",

"**There is** an x such that P(x)"

$\exists x P(x)$ = "P(x) is true for at least one element in the domain."

Note that: no domain → not know $\exists x P(x)$ is true or false.

How to understand quantifiers?

Domain = domain of discourse = universe of discourse

Example.

- x: positive integers, less than 4 ➔ **domain** is {1, 2, 3}
- $P(x) =$ "$x^2 < 8$"
- $\forall x P(x)$
- $\equiv P(1) \wedge P(2) \wedge P(3)$

$\equiv$ "$1^2 < 8$"$\wedge$"$2^2 < 8$"$\wedge$"$3^2 < 8$" ➔ FALSE

A **counterexample**.
This is all we need to show that $\forall x P(x)$ is false.

- $\exists x P(x) = P(1) \vee P(2) \vee P(3)$

$\equiv$ "$1^2 < 8$"$\vee$"$2^2 < 8$"$\vee$"$3^2 < 8$" ➔ TRUE

$\exists$ versus $\forall$

Statement	When is true?	When is false?
$\exists x P(x)$ $\equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_i)$	P(x) is true for at least one x_i in domain	No value of x such that P(x) is true
$\forall x P(x)$ $\equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_i)$	P(x) is true for all x_i in domain	P(x) is false for at least one x_i in domain

Precedence of quantifiers

- $\forall$, $\exists$ have *higher precedence* than all logical operators.
- Examples:
  - $\forall x P(x) \vee Q(x)$ means $(\forall x P(x)) \vee Q(x)$,

not mean $\forall x(P(x) \vee Q(x))$.

  - $\neg\forall x P(x)$ means $\neg(\forall x P(x))$
  - $\neg\forall x(P(x)\rightarrow Q(x))$ means $\neg(\forall x(P(x) \rightarrow Q(x)))$

Scope of quantifiers

- **Free** variable: outside the scope of quantifiers
- **Bound** variables: inside the scope of quantifiers

$\forall x(x + y = 1)$

x is **bound** by $\forall x$
y is **free** // no value is assigned to y

$$\forall x(x + y = 1) \equiv \forall x(x + z = 1)$$

$$\exists x\, \boxed{(P(x) \land Q(x)) \lor} \forall x\, R(\boxed{x)}$$

Scope of $\exists x$

Scope of $\forall x$

Nested Quantifiers

$\forall x \exists y\ Q(x, y)$ vs $\exists y \forall x\ Q(x, y)$

THE SAME???

NOT OK

How to understand **nested quantifiers**?

Understand as loops

- Example 1.

Let x, y be in {1, 2, 3}

Q(x, y) = "x + y = 4"

➔ $\boxed{\forall x \exists y} Q(x, y)$

$\equiv \exists y Q(1,y) \wedge \exists y Q(2,y) \wedge \exists y Q(3,y)$

$\equiv \exists y$"1 + y = 4" $\wedge \exists y$"2 + y = 4" $\wedge \exists y$"3 + y = 4"

➔ T ∧ T ∧ T = T

➔ TRUE

How to understand **nested quantifiers**?

- Example 2.

Let x, y be in {1, 2, 3}

Q(x, y) = "x + y = 4"

→ $\exists x \forall y Q(x, y)$

$\equiv \forall y Q(1,y) \lor \forall y Q(2,y) \lor \forall y Q(3,y)$

→ $\forall y(1+y=4) \lor \forall y(2+y=4) \lor \forall y(3+y=4)$

→ $F \lor F \lor F = F$

→ FALSE

How to negate a statement?

- Use De Morgan's laws
- **Example 1:** For x in domain {1, 2, 3}

$\neg\forall x P(x)$

$\equiv \neg(\forall x P(x))$

$\equiv \neg((P(1)\wedge P(2)\wedge P(3))$

$\equiv \neg P(1)\vee \neg P(2)\vee \neg P(3)$

$\equiv \exists x \neg P(x)$

How to negate a statement?

Example 2

x: students in your class

P(x) = "x has taken a course in calculus"

"Every student in your class has taken a course in calculus"

➔ $\forall x\, P(x)$

Find the negation:

$\neg\forall x\, P(x) \equiv \exists x \neg P(x)$

➔ "There is a student in your class who has not taken a course in calculus."

How to negate a statement?

- Use De Morgan's laws
- **Example 3:** For x in domain {1, 2, 3}

$\neg\exists x P(x)$

$\equiv \neg(\exists x P(x))$

$\equiv \neg((P(1) \vee P(2) \vee P(3))$

$\equiv \neg P(1) \wedge \neg P(2) \wedge \neg P(3)$

$\equiv \forall x \neg P(x)$

How to negate a statement?

Example 4

- Find the negation of the statement $\exists x(x^2 = 2)$.

Solution.

$\neg\exists x(x^2 = 2) \equiv \forall x\neg(x^2 = 2) \equiv \forall x(x^2 \neq 2)$

➔ The truth value of this statement depends on the domain.

- If x is an integer: $\exists x(x^2 = 2)$is FALSE

and $\forall x(x^2 \neq 2)$ is TRUE

- If x is a real number: $\exists x(x^2 = 2)$is TRUE

and $\forall x(x^2 \neq 2)$ is FALSE

Expression	Equivalence	Expression	Negation
$\neg\exists x P(x)$	$\forall x\ \neg P(x)$	$\exists x P(x)$	$\forall x\ \neg P(x)$
$\neg\ \forall x P(x)$	$\exists x\ \neg P(x)$	$\forall x P(x)$	$\exists x\ \neg P(x)$

How to negate a statement?

- Use De Morgan's laws
- **Example 5:** For x, y in domain {1, 2, 3}

$\neg\forall x\exists y Q(x,y)$

$\equiv \neg(\exists y Q(1,y) \land \exists y Q(2,y) \land \exists y Q(3,y))$

$\equiv \neg\exists y Q(1,y) \lor \neg\exists y Q(2,y) \lor \neg\exists y Q(3,y)$

$\equiv \forall y\neg Q(1,y) \lor \forall y\neg Q(2,y) \lor \forall y\neg Q(3,y)$

$\equiv \exists x\forall y\neg Q(x,y)$

How to negate a statement? Example 6

- Find the negation of the statement

"Every student in this class plays some sport."

Solution.

x: student in this class

y: sport

$P(x, y)$ = "student x ***plays*** sport y"

"Every student in this class **plays** some sport."

➔ $\forall x \exists y P(x,y)$

➔ Negation: $\neg \forall x \exists y P(x,y) \equiv \exists x \forall y \neg P(x,y)$

➔ "There are some students in this class who don't play any sport.

Find the negations

- $\forall x\,(P(x) \to Q(x))$

Negation:

$\neg\forall x\,(P(x) \to Q(x))$

$\equiv\ \exists x \neg\,(P(x) \to Q(x))$

$\equiv\ \exists x\,(P(x) \land \neg Q(x))$

- $\forall x\,(P(x) \land Q(x))$

Negation:

$\neg\forall x\,(P(x) \land Q(x))$

$\equiv\ \exists x \neg\,(P(x) \land Q(x))$

$\equiv\ \exists x\,(\neg P(x) \lor \neg Q(x))$

How to **translate** a statement into logical expression?

- **Example 1.** Suppose the variables ***x*** represents **real numbers**, and

$E(x)$: x is even $\qquad\qquad$ $I(x)$: x is an integer.

Write the statement "Every integer is even" using these predicates and any needed quantifiers.

Ans. "Every integer is even " means

"For every number, if it is an integer, then it is even"

$$\rightarrow \underbrace{\text{"For every } \mathbf{x}}_{\forall x}\text{, } \mathbf{if}\ \underbrace{\mathbf{x} \text{ is an integer}}_{(\quad I(x)}\text{, } \mathbf{then}\ \underbrace{\mathbf{x} \text{ is even}}_{\rightarrow\quad E(x)\ )}\text{"}$$

$$\forall x\ (I(x) \rightarrow E(x))$$

How to **translate** a statement into logical expression?

Example 2. *x* : represents people

F(*x*): *x* is friendly; *T*(*x*): *x* is tall; *A*(*x*): *x* is angry.

- Bill is angry.

$$A(\text{Bill})$$

- Some people are not angry.

$$\exists x \neg A(x)$$

- All tall people are friendly.

  ➔ For every person x, if x is tall, then x is friendly.

$$\forall x\ (T(x) \to F(x))$$

How to **translate** a statement into logical expression?

Example 3. *x:* represents students; *y* represents courses, and

$T(x, y)$: *x* is taking *y* $P(x, y)$: *x* passed *y*.

Write the statement in good English. Do not use variables in your answers.

- $\forall x \exists y T(x, y)$

Every student is taking at least one course.

- $\exists y \forall x T(x, y)$

There is a course that all students are taking.

- $\forall x \exists y \neg P(x, y)$

Every student didn't pass at least one course.

DIY. Translating

Example 4. Assume that the universe for x is all people and the universe for y is the set of all movies.

$S(x, y)$: x saw y $L(x, y)$: x liked y $A(y)$: y won an award $C(y)$: y is a comedy.

- No comedy won an award.
- $S(\text{Lois, Fast 8}) \wedge \neg L(\text{Lois, Fast 8})$
- Some people have seen every comedy.
- $\neg\exists x\forall y(S(x, y) \rightarrow L(x, y))$
- Ben has never seen a movie that won an award.

Answers:

- $\forall y(C(y) \rightarrow \neg A(y))$
- Lois saw *Fast 8*, but didn't like it.
- $\exists x\forall y(C(y) \rightarrow S(x, y))$
- No one liked every movie he has seen.
- $\neg\exists y(S(\text{Ben}, y) \wedge A(y))$

Review 1.3, 1.4

- Predicates: $P(x)$, $Q(x, y)$, $R(x, y, z)$, …
- Quantifiers: $\forall$, $\exists$
- Translating from English into predicate logic.

- Exercises: 1, 4, 27, 32

Our Goal:

- Rules of inference:
  - How to produce valid arguments?
  - Fallacies

1.5.1- Definitions

An *argument* in propositional logic is a sequence of propositions.

- Proposition 1 // Hypothesis
- Proposition 2
- Proposition 3
- Proposition 4
- ....
- Final proposition // Conclusion

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables.

An argument (form) is *valid* if the truth of all its premises implies that the conclusion is true.

1.5.2- Rules Inferences

Rule	Tautology CHECK YOURSELF!	Name
p $\underline{p \rightarrow q}$ $\therefore q$	$[(p \rightarrow q) \wedge p] \rightarrow q$ **If Socrates is human, then Socrates is mortal.** Socrates is a human. **Therefore, Socrates is mortal.**	Modus ponens
$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$ She did not get a prize If she is good at learning she will get a prize $\therefore$ She is not good at learning	Modus tollens

1.5.2- Rules Inferences

Rule	Tautology	Name
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$ If the prime interest rate goes up then the stock prices go down. If the stock prices go down then most people are unhappy. Therefore, if the prime interest rate goes up then most people are unhappy.	Hypothetical syllogism

Rules Inferences…

Rule	Tautology	Name
$p \vee q$ $\underline{\neg p}$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$ Power puts off or the lamp is malfunctional. Power doesn’t put off. Therefore, the lamp is malfunctional	Disjunctive syllogism
$\underline{p}$ $\therefore p \vee q$	$p \rightarrow (p \vee q)$ It is below freezing now Therefore, It is below freezing now or raining now	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \rightarrow p$ It is below freezing now and raining now Therefore, It is below freezing now	Simplification

Rules Inferences…

Rule	Tautology	Name
p $\underline{q}$ $\therefore p \wedge q$	$[(p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$ It is not snowing OR Jasmine is skiing It is snowing OR Bart is playing hockey. Therefore, Jasmine is skiing OR Bart is playing hockey	Resolution

1.5.4- Rules of Inference for Quantified Statements

Rule	Name
$\underline{\quad\forall x P(x)}$ $\therefore P(c)$	Universal Instantiation
$\underline{\quad P(c) \text{ for arbitrary } c}$ $\therefore \forall x P(x)$	Universal generalization
$\underline{\exists x P(x)}$ $\therefore P(c)$ for some element c	Existential instantiation
$\underline{P(c) \text{ for some element } c}$ $\therefore \exists x P(x)$	Existential generalization

Fallacies… NOT OK

$$\begin{array}{l} \neg p \\ \underline{p \rightarrow q} \\ \therefore \neg q \ ??? \end{array}$$

If I can see God, then God exists

I cannot see God.

God doesn't exist because I can't see him.

$$\begin{array}{l} q \\ \underline{p \rightarrow q} \\ \therefore p \ ??? \end{array}$$

Penguins are black and white

Some old TV are black and white.

Therefore, some old TV are penguins.

$$\begin{array}{l} p \\ \underline{p \vee q} \\ \therefore \neg q \ ??? \end{array}$$

He knows C or Java. He showed his work in Java codes. Therefore, he doesn't know C.

$$\begin{array}{l} \underline{P(c) \text{ for some element } c} \\ \therefore \forall x P(x) \end{array}$$

Hitler was a vegetarian, therefore, I don't trust vegetarians

Summary – Review Questions

- What is a proposition? Give an example.
- What is the negation of a proposition? Give an example.
- What is the conjunction of propositions? Notation? Give an example.
- What is the disjunction of propositions? Notation? Give an example.
- What is the ex-clusive or? Notation. Give an example.
- What is the conditional statement? Notation? Give an example.
- What is the biconditional satement? Notation? Give an example.
- What is the precedence of logical operators? Give an example.

Summary – Review Questions

- What is a tautology? Give an example.
- When are two propositions called logically equivalent? Give an example.
- State the De Morgan's laws.
- What is the negation of $(p \rightarrow q)$?

Key Terms and Results

- **proposition**: a statement that is true or false
- **propositional variable**: a variable that represents a proposition
- **truth value**: true or false
- **truth table:** a table displaying all possible truth values of propositions
- ¬p (**negation** of p): truth value opposite

to the truth value of p

- **logical operators:** operators used to combine propositions

Logical operators

- p ∨ q (disjunction of p and q): the proposition “p or q,” which is true if and only if at least one of p and q is true
- p ∧ q (conjunction of p and q): the proposition “p and q,” which is true if and only if both p and q are true
- p ⊕ q (exclusive or of p and q): the proposition “p XOR q,” which is true when exactly one of p and q is true
- p → q (p implies q): the proposition “if p, then q,” which is false if and only if p is true and q is false