Chapter 7

Point Estimation of Parameters

Chapter 7: Point Estimation of Parameters

Learning objectives

1. Introduction
2. Sampling Distributions and the Central Limit Theorem

1.Introduction

Introduction

Let X is a random variable with probability distribution $f(x)$, which is characterized by the unknown $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$

For example, $X \sim N(\mu, \sigma^2)$ then $\theta = (\mu, \sigma^2)$.

How to "determine" the values of θ?

PROBLEMS IN ENGINEERING

In engineering, we often need to estimate

1. The mean μ of a single population.
2. The variance σ^2 (or standard deviation) of a single population.
3. The proportion *p* of items in a population that belong to a class of interest.
4. The difference in means of two populations, $\mu_1 - \mu_2$.
5. The difference in two population proportions, $p_1 - p_2$.

PROBLEMS IN ENGINEERING

The important results on point estimation

1. For μ, the estimate is the $\hat{\mu} = \bar{x}$, sample mean.
2. For σ^2, the estimate is $\hat{\sigma}^2 = s^2$, the sample variance.
3. For p, the estimate is $\hat{p} = x/n$, the sample proportion.
4. For $\mu_1 - \mu_2$, the estimate is $\hat{\mu}_1 - \hat{\mu}_2 = \bar{x}_1 - \bar{x}_2$
5. For $p_1 - p_2$, the estimate is $\hat{p}_1 - \hat{p}_2$

2. Sampling Distributions and the Central Limit Theorem

SAMPLING DISTRIBUTIONS

Definition — **Random Sample**

The random variables $X_1, \ldots, X_n$ are called a random sample of size n if

- The X_i's are independent
- Every X_i has the same probability distribution

Definition — **Statistic**

- A statistic $\hat{\Theta}$ is any function of the observations $X_1, \ldots, X_n$:

$$\hat{\Theta} = h(X_1, \ldots, X_n)$$

- The probability distribution of a statistic is called a sampling distribution.

SAMPLING DISTRIBUTIONS

Example — **Two important statistic**

- Sample mean $\overline{X}$

$$\overline{X} = \frac{X_1 + \ldots + X_n}{n}$$

- Sample variance S^2

$$S^2 = \frac{(X_1 - \overline{X})^2 + \ldots + (X_n - \overline{X})^2}{n-1}$$

Sampling Distributions

Theorem

In above example, if $(X_1, \ldots, X_n)$ is a random sample of size n take from a normal distribution $N(\mu, \sigma^2)$ then

- $\overline{X}$ has a normal distribution $N(\mu, \sigma^2/n)$
- $\dfrac{(n-1)S^2}{\sigma^2}$ has a chi-square distribution with n-1 degrees of freedom (see pages 273-274).

Sampling Distributions

Theorem — Central Limit Theorem

Let $(X_1, \ldots, X_n)$ is a random sample of size n take from a population with mean μ and finite variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

As $n \to \infty$, is the standard normal distribution.

Remark: The normal approximation for $\overline{X}$ depends on the sample size n.

Sampling Distributions

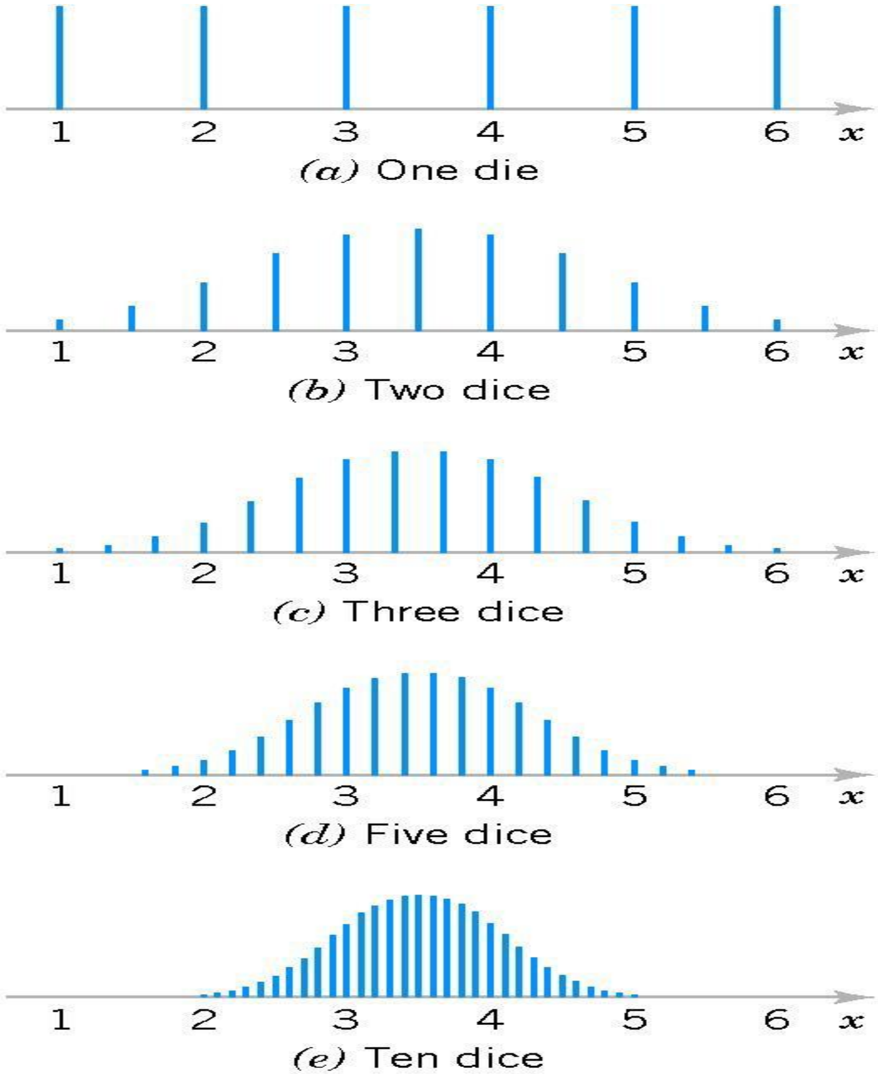

Figure 7-1 Distributions of average scores from throwing dice.

Sampling Distributions

Definition **Point Estimate**

- A **point estimate** of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$.
- The statistic $\hat{\Theta}$ is called the **point estimator**.

Two steps to find point estimation:

Step 1. Determine $\hat{\Theta}$ by using the theoretical results.

Steps 2. Calculate $\hat{\theta}$ from the experimental data.

Sampling Distributions

Example

Market researchers use the number of sentences per advertisement as a measure of readability for magazine advertisements. The following represents a random sample of the number of sentences found in 15 advertisements. Find a point estimate of the population mean, μ.

9 20 18 16 9 9 11 13 22 16 5 18 6 6 5

Step 1. $\hat{\Theta} = \dfrac{X_1 + \ldots + X_{15}}{15}$

Step 2. A point estimate for μ is

$$\hat{\mu} = \frac{9+20+\ldots+6+5}{15} = \frac{183}{15} = 12.2$$

Exercises 3

PVC pipe is manufactured with a mean diameter of 1.01 inch and a standard deviation of 0.003 inch. Find the probability that a random sample of $n = 9$ sections of pipe will have a sample mean diameter greater than 1.009 inch and less than 1.012 inch.

Exercises 4

Suppose that samples of size n = 25 are selected at random from a normal population with mean 100 and standard deviation 10. What is the probability that the sample mean falls in the interval from

$$\mu_{\overline{X}} - 1.8\sigma_{\overline{X}} \text{ to } \mu_{\overline{X}} + 1.0\sigma_{\overline{X}}$$

Exercises 5

A synthetic fiber used in manufacturing carpet has tensile strength that is normally distributed with mean 75.5 psi and standard deviation 3.5 psi. Find the probability that a random sample of $n = 6$ fiber specimens will have sample mean tensile strength that exceeds 75.75 psi.

Exercises 10

Suppose that the random variable X has the continuous uniform distribution

$$f(x)=\begin{cases}1, & 0\le x\le 1\\ 0, & x\notin[0,1]\end{cases}$$

Suppose that a random sample of n = 12 observations is selected from this distribution. What is the approximate probability distribution of $\overline{X}-6$ Find the mean and variance of this quantity.

7.2.5 **WP** A normal population has mean 100 and variance 25. How large must the random sample be if you want the standard error of the sample average to be 1.5?

7.2.6 WP VS The amount of time that a customer spends waiting at an airport check-in counter is a random variable with mean 8.2 minutes and standard deviation 1.5 minutes. Suppose that a random sample of $n = 49$ customers is observed. Find the probability that the average time waiting in line for these customers is

a. Less than 10 minutes
b. Between 5 and 10 minutes
c. Less than 6 minutes

7.2.7 WP SS VS A random sample of size $n_1 = 16$ is selected from a normal population with a mean of 75 and a standard deviation of 8. A second random sample of size $n_2 = 9$ is taken from another normal population with mean 70 and standard deviation 12. Let $\overline{X}_1$ and $\overline{X}_2$ be the two sample means. Find:

a. The probability that $\overline{X}_1 - \overline{X}_2$ exceeds 4

b. The probability that $3.5 \le \overline{X}_1 - \overline{X}_2 \le 5.5$

7.2.9 Data on the pH of rain in Ingham County, Michigan, are as follows:

5.47 5.37 5.38 4.63 5.37 3.74 3.71 4.96 4.64 5.11 5.65 5.39
4.16 5.62 4.57 4.64 5.48 4.57 4.57 4.51 4.86 4.56 4.61 4.32
3.98 5.70 4.15 3.98 5.65 3.10 5.04 4.62 4.51 4.34 4.16 4.64
5.12 3.71 4.64

What proportion of the samples has pH below 5.0?

Exercises 11

Suppose that X has the continuous uniform distribution

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x \notin [0,1] \end{cases}$$

Suppose that a random sample of n = 12 observations is selected from this distribution. What is the approximate probability distribution of $\overline{X} - 6$ Find the mean and variance of this quantity.