# Chapter 9
# GRAPHS

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# Objectives

9.1- Graphs and Graph Models

9.2- Graph Terminology and Special Types of Graphs

9.3- Representing Graphs and Graph Isomorphism – Đẳng cấu

9.4- Connectivity – Tính liên thông

9.5- Euler and Hamilton Paths

9.6- Shortest Path Problems

# 9.1- Graphs and Graph Models

**DEFINITION 1**

A *graph* $G = (V, E)$ consists of $V$, a nonempty set of *vertices* (or *nodes*) and $E$, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

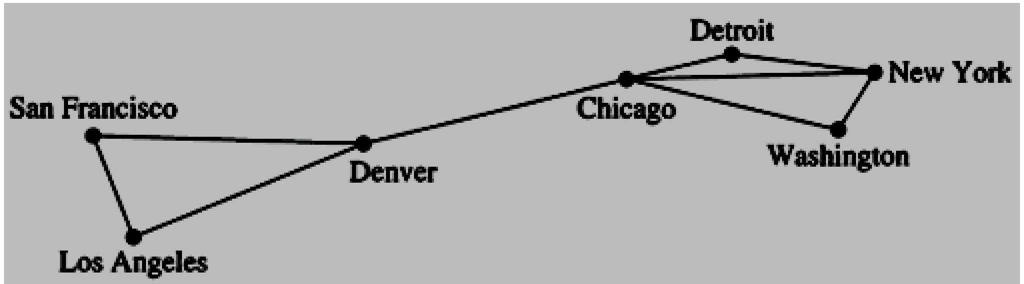



**FIGURE 1** A Computer Network.

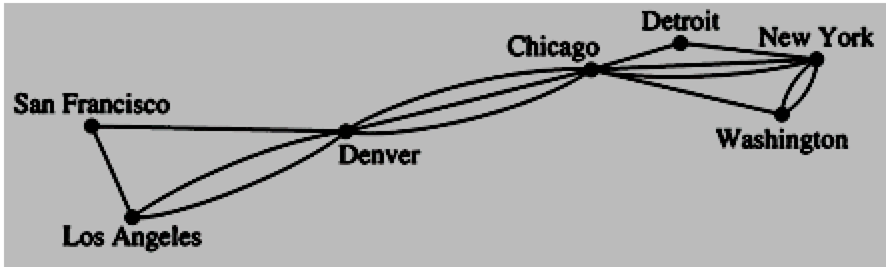



**FIGURE 2** A Computer Network with Multiple Links between Data Centers.

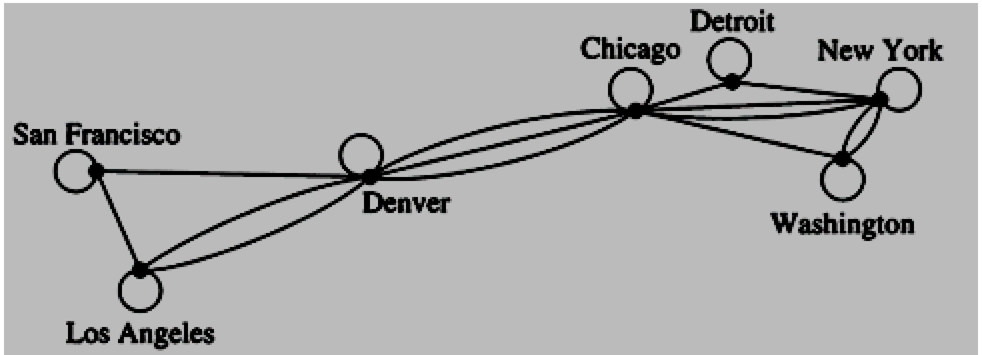



**FIGURE 3** A Computer Network with Diagnostic Links.

**Simple graph:** No two edges connect the same pair of vertices
**Multigraph**: Multiple edges connect the same pair of vertices
**Pseudograph**: Multigraph may have loops
**Undirected graph**: Each edge has no direction
**Directed graph**: Each edge has a determined direction

# Graphs and Graph Models….

**DEFINITION 2**

A *directed graph* (or *digraph*) $(V, E)$ consists of a nonempty set of vertices $V$ and a set of *directed edges* (or *arcs*) $E$. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair $(u, v)$ is said to *start* at $u$ and *end* at $v$.

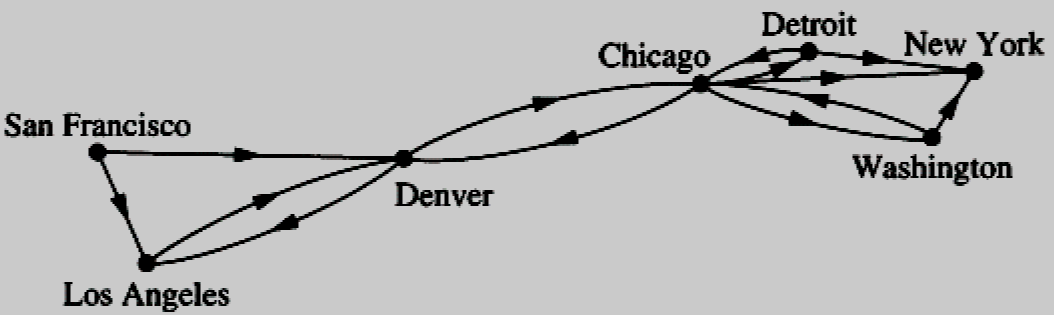



**FIGURE 4 A Communications Network with One-Way Communications Links.**

When there are m directed edges, each associated to an ordered pair of vertices (u,v), we say that (u,v) is an edge of multiplicity m

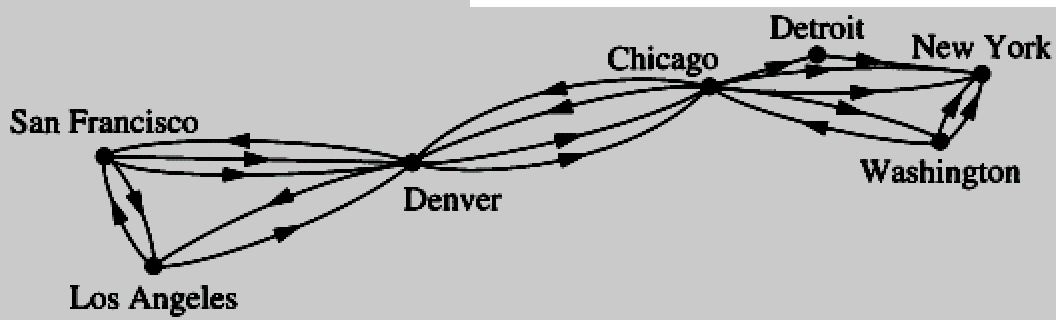



**FIGURE 5 A Computer Network with Multiple One-Way Links.**

# Graphs and Graph Models….

**TABLE 1 Graph Terminology.**

| *Type* | *Edges* | *Multiple Edges Allowed?* | *Loops Allowed?* |
|---|---|---|---|
| Simple graph | Undirected | No | No |
| Multigraph | Undirected | Yes | No |
| Pseudograph | Undirected | Yes | Yes |
| Simple directed graph | Directed | No | No |
| Directed multigraph | Directed | Yes | Yes |
| Mixed graph | Directed and undirected | Yes | Yes |

# Graphs and Graph Models….

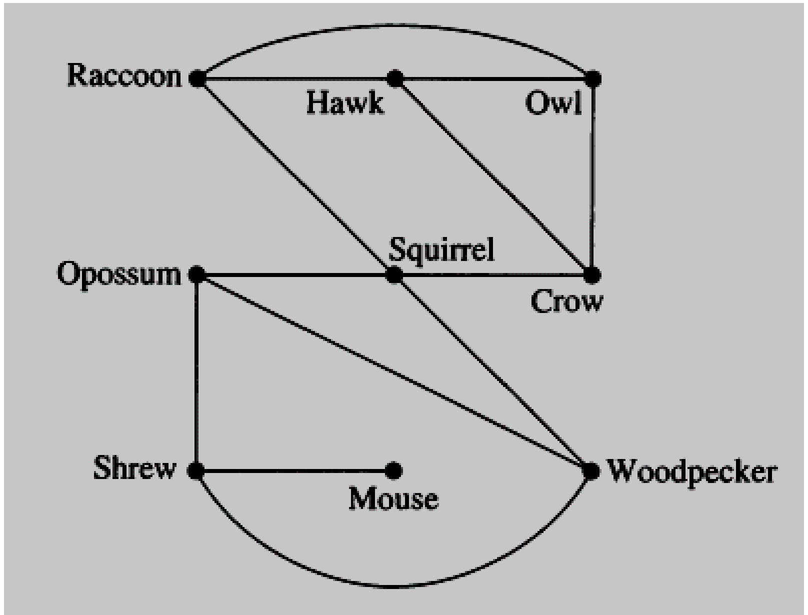



FIGURE 6 A Niche Overlap Graph.

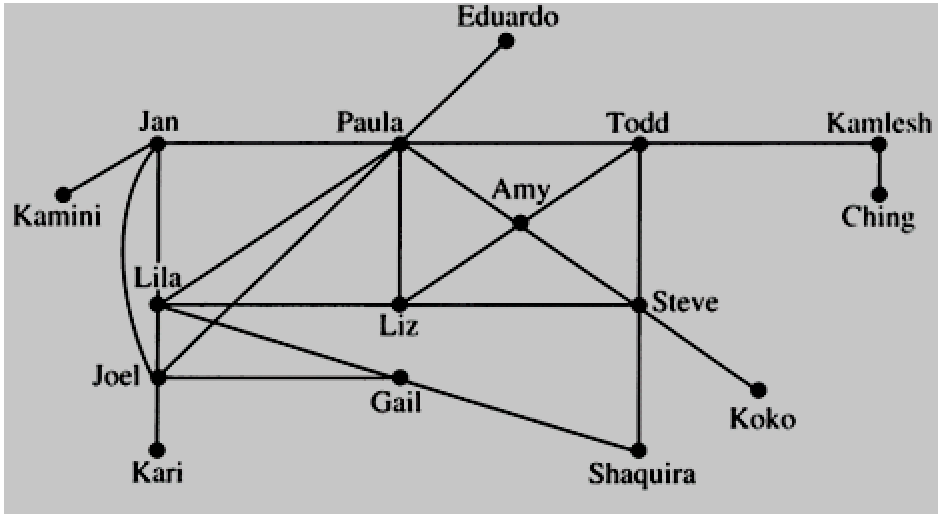
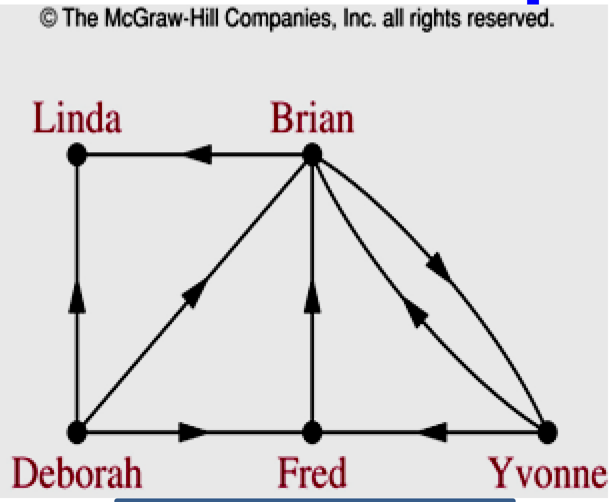



FIGURE 7 An Acquaintanceship Graph.

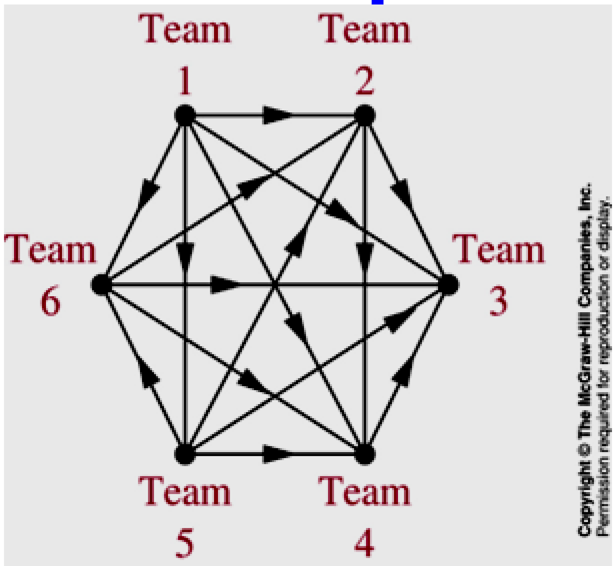
Niche Overlap Graph in Ecology
(sinh thái học) – Đồ thị lấn tổ

Acquaintanceship Graph
Đồ thị cho mô hình quan hệ giữa người

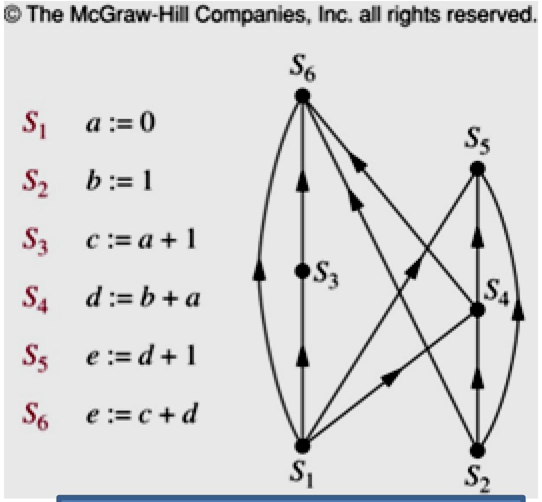
# Graphs and Graph Models….



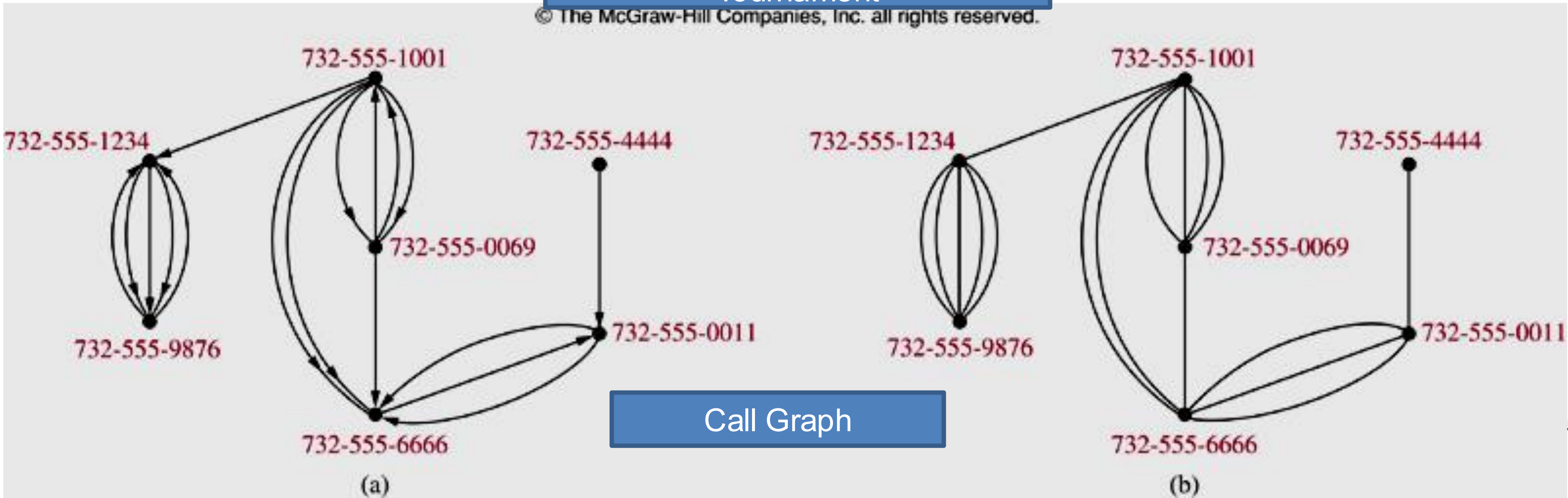

An Influence Graph




A Graph Model of a Round-Robin Tournament




An Precedence Graph

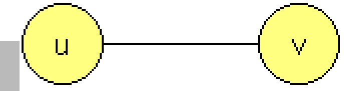



Call Graph

# 9.2- Graph Terminology and Special Types of Graphs

- Basic Terminology
- Some Special Simple Graphs
- Bipartite Graphs (Đồ thị lưỡng phân)
- Some Applications of Special Types of Graphs
- New Graphs From Old

# Basic Terminology




**DEFINITION 1**

Two vertices $u$ and $v$ in an undirected graph $G$ are called *adjacent* (or *neighbors*) in $G$ if $u$ and $v$ are endpoints of an edge of $G$. If $e$ is associated with $\{u, v\}$, the edge $e$ is called *incident with* the vertices $u$ and $v$. The edge $e$ is also said to *connect* $u$ and $v$. The vertices $u$ and $v$ are called *endpoints* of an edge associated with $\{u, v\}$.

**DEFINITION 2**

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex $v$ is denoted by $\deg(v)$.

Determine degrees of vertices in the following graphs:

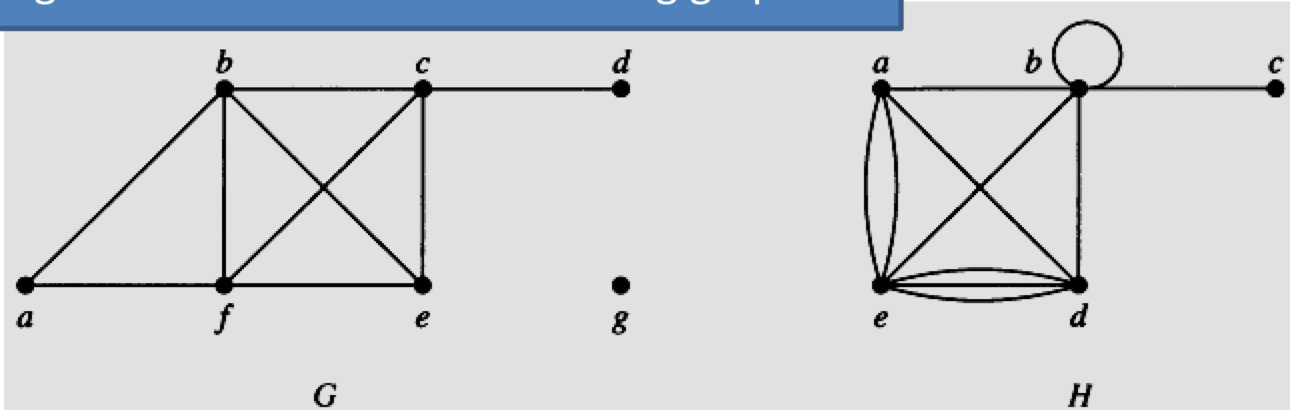



**FIGURE 1** The Undirected Graphs $G$ and $H$.

# Basic Terminology …

A vertex of degree zero is called **isolated.** It follows that an isolated vertex is not adjacent to any vertex. Vertex $g$ in graph $G$ in Example 1 is isolated. A vertex is **pendant** if and only if it has degree one. Consequently, a pendant vertex is adjacent to exactly one other vertex. Vertex $d$ in graph $G$ in Example 1 is pendant.

**THEOREM 1** **THE HANDSHAKING THEOREM** Let $G = (V, E)$ be an undirected graph with $e$ edges. Then

$$2e = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

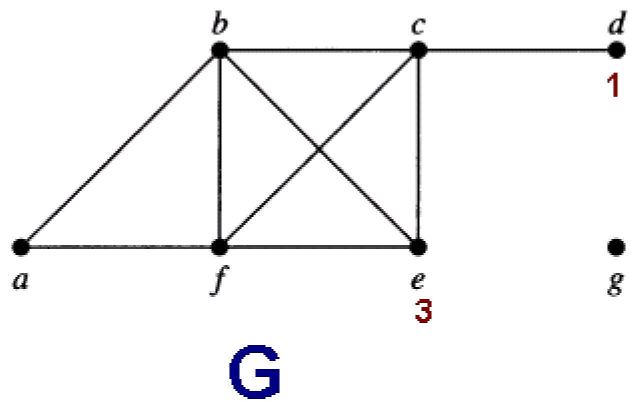
# Basic Terminology….

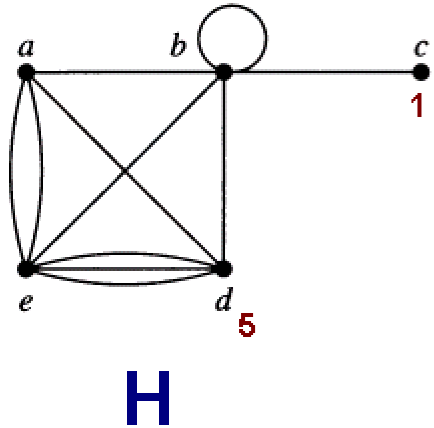
## THEOREM 2

An undirected graph has an even number of vertices of odd degree.

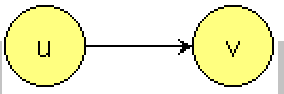
Proof: page 599

Check yourself:



# Basic Terminology….




**DEFINITION 3**

When $(u, v)$ is an edge of the graph $G$ with directed edges, $u$ is said to be *adjacent to* $v$ and $v$ is said to be *adjacent from* $u$. The vertex $u$ is called the *initial vertex* of $(u, v)$, and $v$ is called the *terminal* or *end vertex* of $(u, v)$. The initial vertex and terminal vertex of a loop are the same.

**DEFINITION 4**

In a graph with directed edges the *in-degree of a vertex* $v$, denoted by $\deg^-(v)$, is the number of edges with $v$ as their terminal vertex. The *out-degree of* $v$, denoted by $\deg^+(v)$, is the number of edges with $v$ as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

**In-degree $\deg^-(c)=3$**
**Out-degree $\deg^+(c)=2$**

# Basic Terminology….

**THEOREM 3**

Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v\in V} \deg^-(v) = \sum_{v\in V} \deg^+(v) = |E|.$$

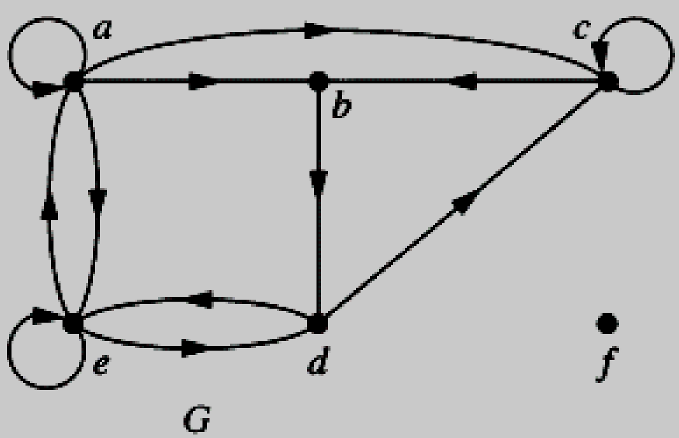



**FIGURE 2** The Directed Graph $G$.

| Vertex | In-degree | Out-degree |
|---|---|---|
| a | 2 | 4 |
| b | 2 | 1 |
| c | 3 | 2 |
| d | 2 | 2 |
| e | 3 | 3 |
| f | 0 | 0 |
| Sum | 12 | 12 |

# Some Special Simple Graphs

**Complete Graphs** The **complete graph on $n$ vertices,** denoted by $K_n$, is the simple graph that contains exactly one edge between each pair of distinct vertices. The graphs $K_n$, for $n = 1, 2, 3, 4, 5, 6$, are displayed in Figure 3.

Đồ thị đầy đủ

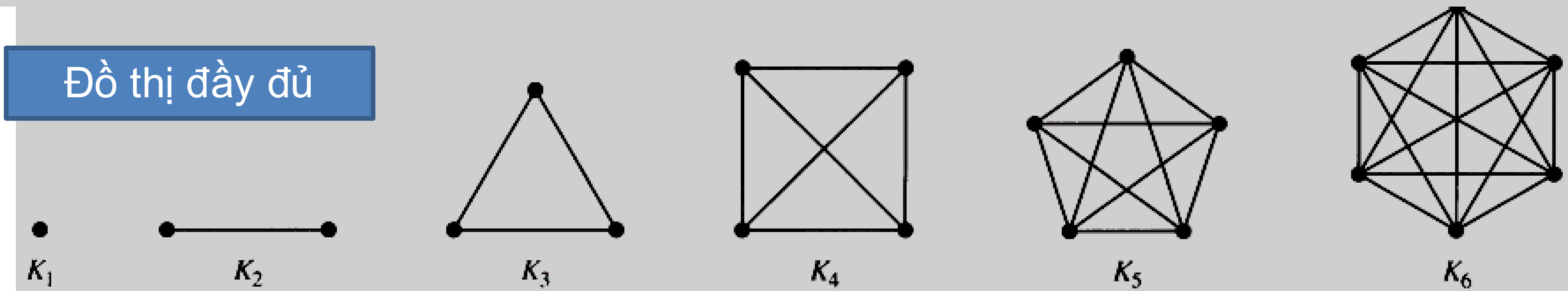
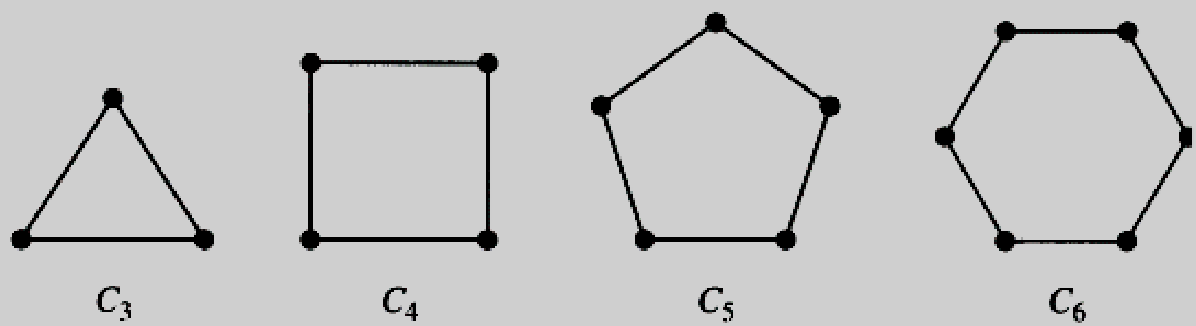



**FIGURE 3 The Graphs $K_n$ for $1 \leq n \leq 6$.**

**Cycles** The **cycle** $C_n$, $n \geq 3$, consists of $n$ vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles $C_3$, $C_4$, $C_5$, and $C_6$ are displayed in Figure 4.

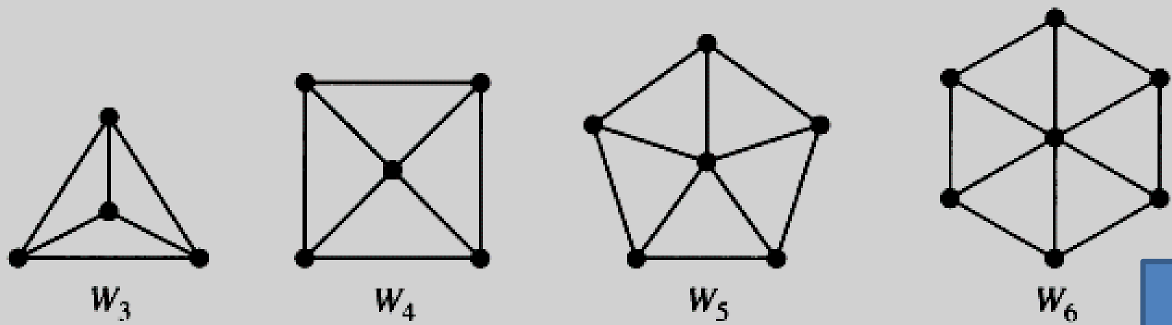



Đồ thị vòng

**FIGURE 4 The Cycles $C_3$, $C_4$, $C_5$, and $C_6$.**

# Some Special Simple Graphs

**Wheels** We obtain the **wheel** $W_n$ when we add an additional vertex to the cycle $C_n$, for $n \geq 3$, and connect this new vertex to each of the $n$ vertices in $C_n$, by new edges. The wheels $W_3$, $W_4$, $W_5$, and $W_6$ are displayed in Figure 5.




Đồ thị bánh xe

**FIGURE 5** **The Wheels $W_3$, $W_4$, $W_5$, and $W_6$.**

***n*-Cubes** The ***n*-dimensional hypercube,** or ***n*-cube,** denoted by $Q_n$, is the graph that has vertices representing the $2^n$ bit strings of length $n$. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position.

Đồ thị khối

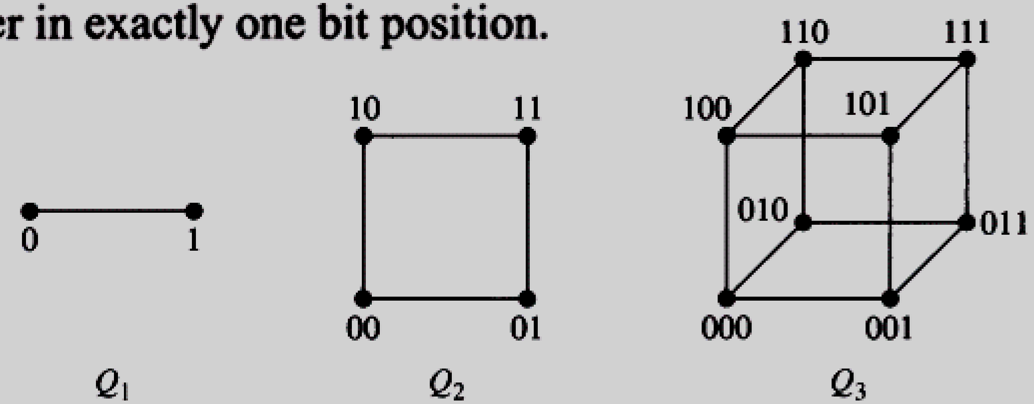


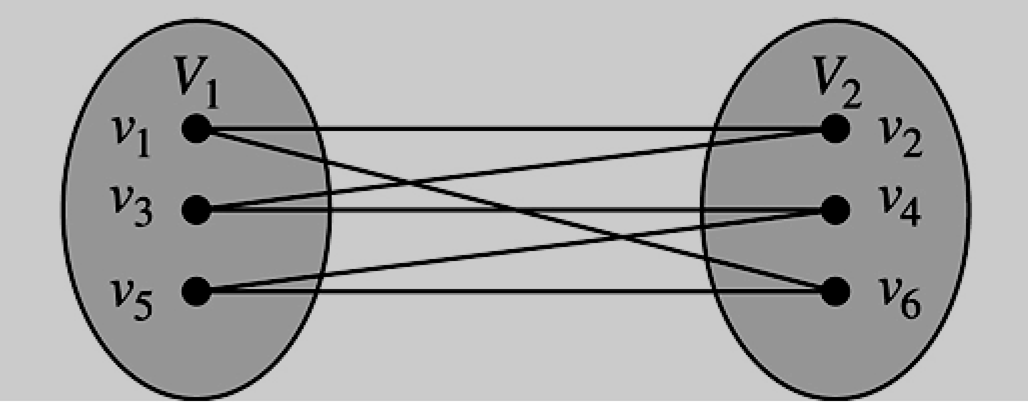

**FIGURE 6** **The *n*-cube $Q_n$ for $n = 1, 2$, and 3.**

# Bipartite Graphs – Đồ thị lưỡng phân

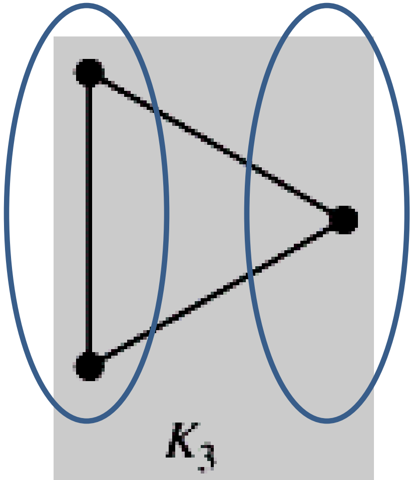
**DEFINITION 5**

A simple graph $G$ is called *bipartite* if its vertex set $V$ can be partitioned into two disjoint sets $V_1$ and $V_2$ such that every edge in the graph connects a vertex in $V_1$ and a vertex in $V_2$ (so that no edge in $G$ connects either two vertices in $V_1$ or two vertices in $V_2$). When this condition holds, we call the pair $(V_1, V_2)$ a *bipartition* of the vertex set $V$ of $G$.



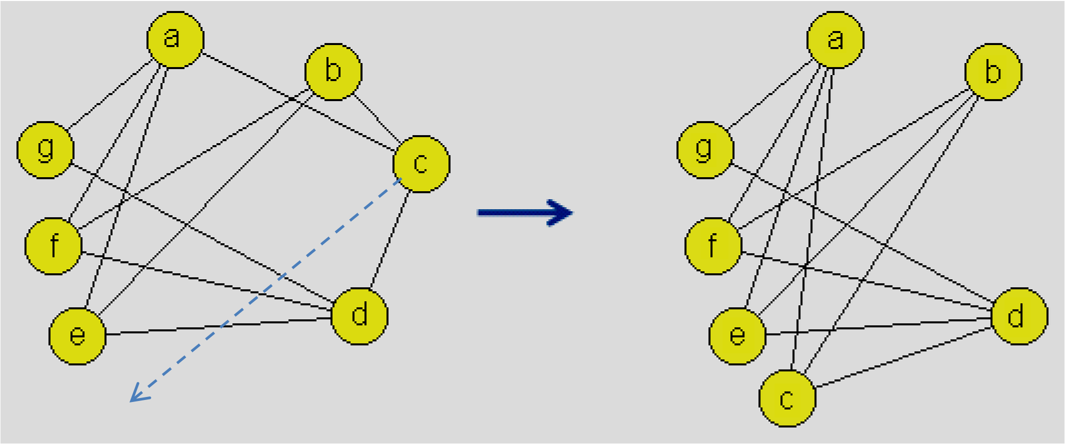



bipartite

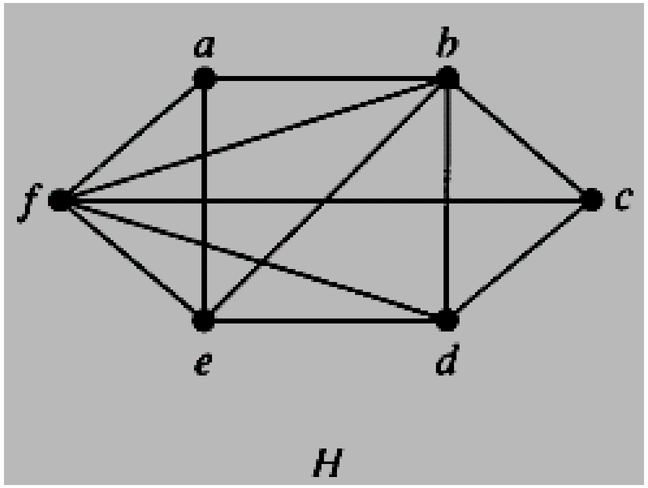



Non-bipartite

# Bipartite Graphs…




bipartite




| Edge considered | Set 1 | Set 2 |
|---|---|---|
| ab | a | b |
| ae | a | b, e |
| af | a | b, e, f |
| bc | ac | |
| be | a, c, e | |
| → **Non- bipartite** | | |
| | | |

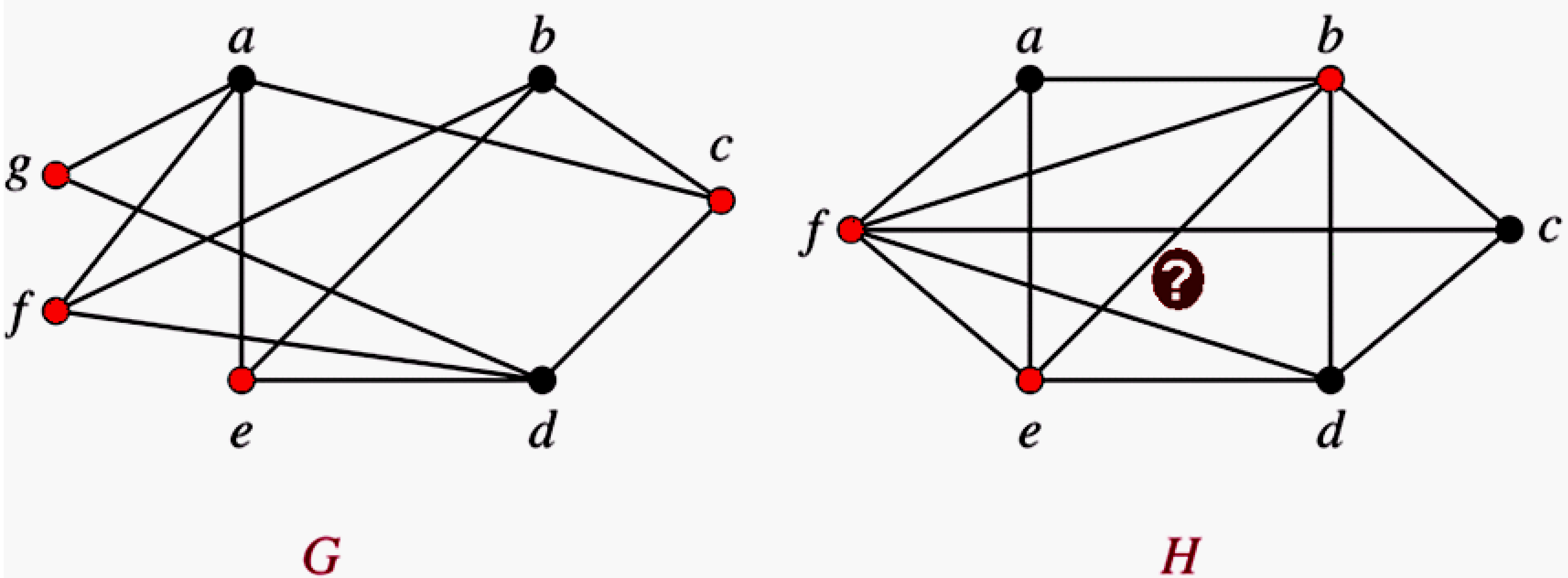
**Conflict**

# Bipartite Graphs…

**THEOREM 4** Proof: page 603

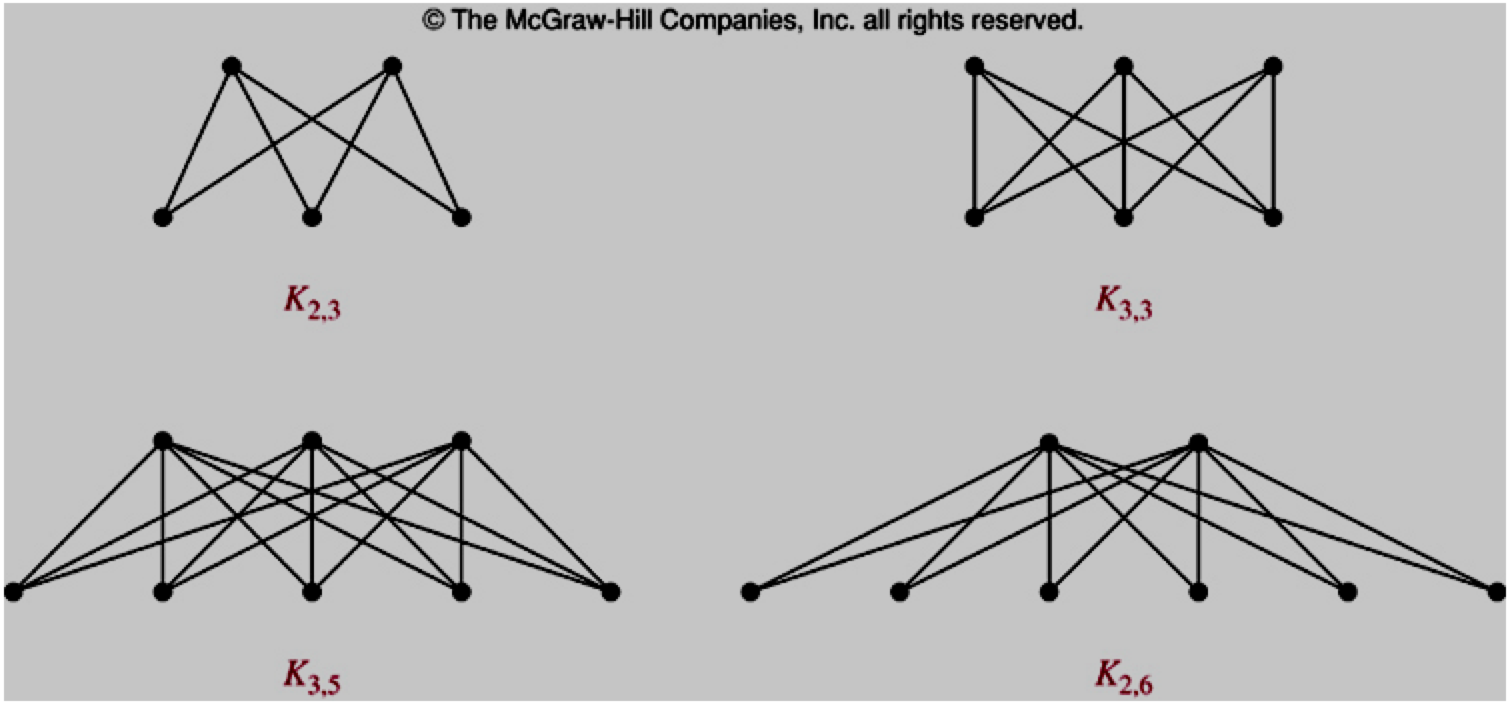
A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.





# Bipartite Graphs…

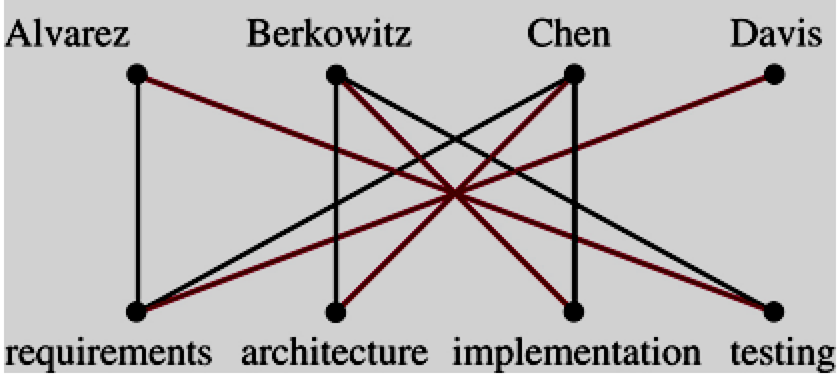
Some Complete Bipartite Graphs




Complete Bipartite Graphs $K_{m,n}$

# Some Applications of Special Types of Graphs





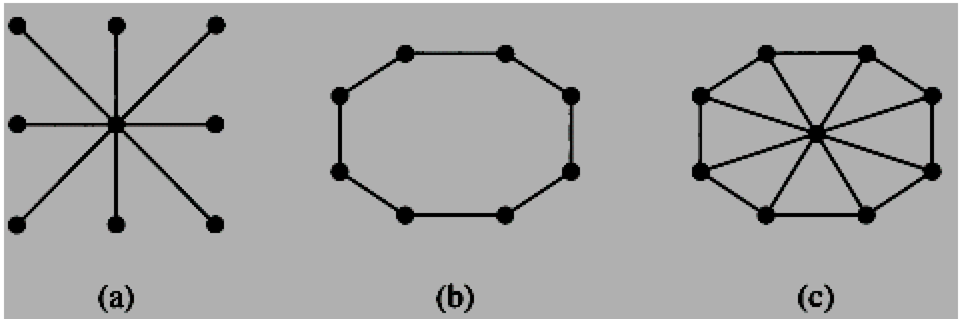




**FIGURE 11** Star, Ring, and Hybrid Topologies for Local Area Networks.

Figure 10: Modeling the jobs for which employees have been trained

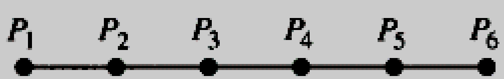



**FIGURE 12** A Linear Array for Six Processors.

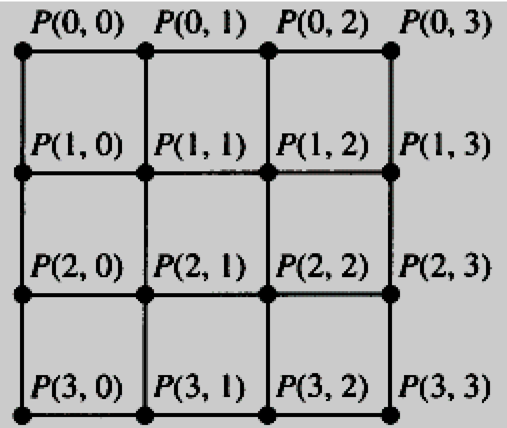



**FIGURE 13** A Mesh Network for 16 Processors.

# New Graphs From Old

**DEFINITION 6**

A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph $H$ of $G$ is a *proper subgraph* of $G$ if $H \neq G$.

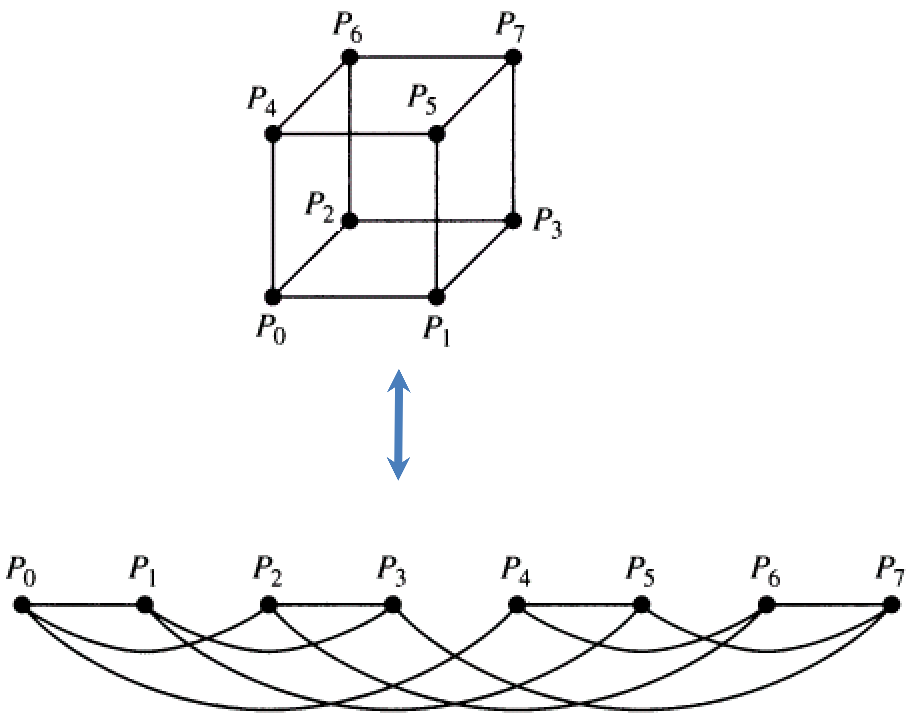



**FIGURE 14** **A Hypercube Network for Eight Processors.**

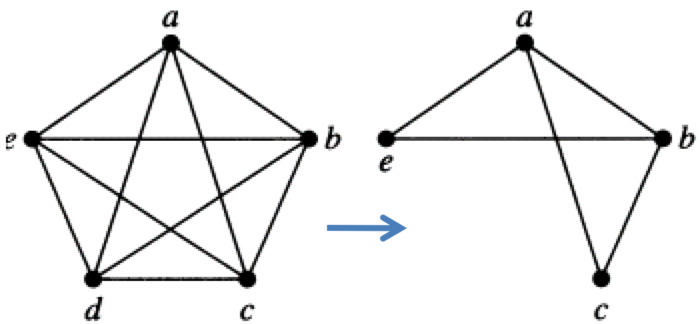



**FIGURE 15** **A Subgraph of $K_5$.**

# New Graphs From Old

**DEFINITION 7**

The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of $G_1$ and $G_2$ is denoted by $G_1 \cup G_2$.

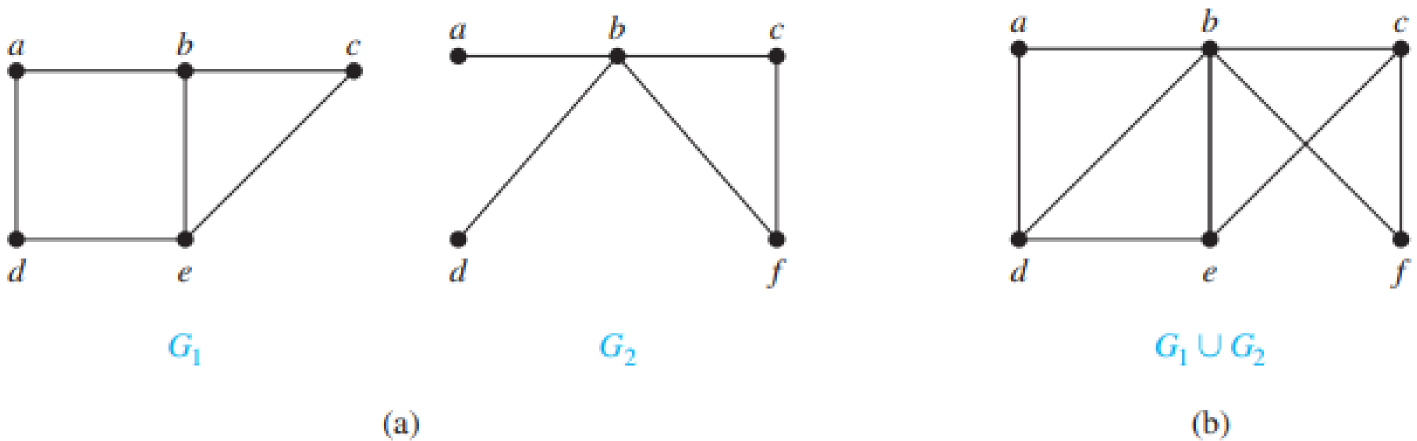
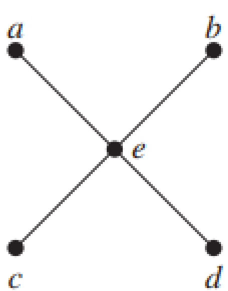


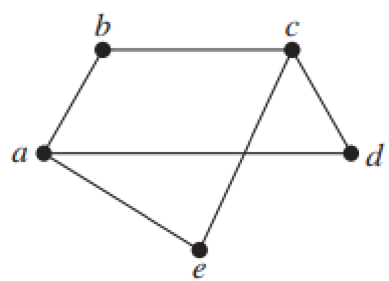

**FIGURE 16** (a) The Simple Graphs $G_1$ and $G_2$; (b) Their Union $G_1 \cup G_2$.

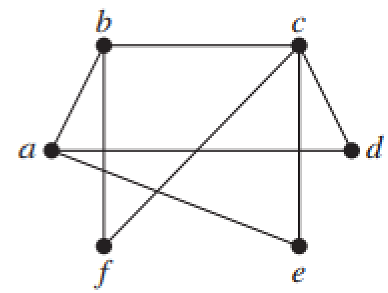
1. For which values of $n$ are these graphs bipartite?

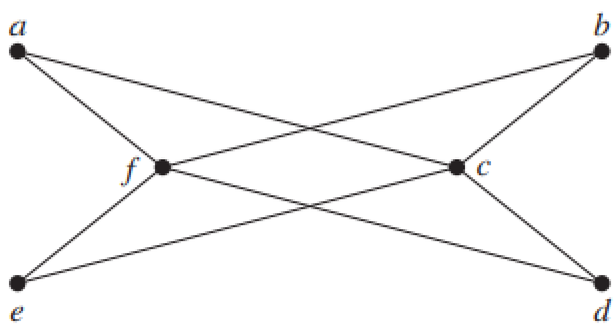
**a)** $K_n$ **b)** $C_n$ **c)** $W_n$ **d)** $Q_n$

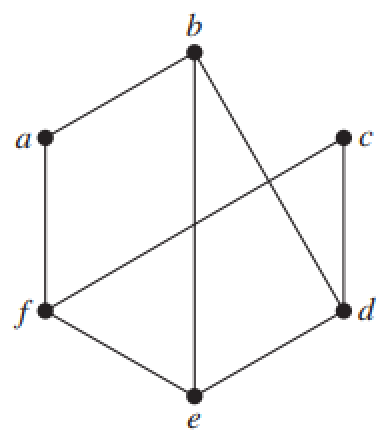
2. Determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.



The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in nonincreasing order.

3. Find the degree sequence of each of the following graphs.

**a)** $K_4$ **b)** $C_4$ **c)** $W_4$

**d)** $K_{2,3}$ **e)** $Q_3$

**4.** How many edges does a graph have if its degree sequence is 4, 3, 3, 2, 2? Draw such a graph.

**5.** How many edges does a graph have if its degree sequence is 5, 2, 2, 2, 2, 1? Draw such a graph.

**35.** How many vertices and how many edges do these graphs have?

**a)** $K_n$ **b)** $C_n$ **c)** $W_n$ **d)** $K_{m,n}$ **e)** $Q_n$

A sequence $d_1, d_2, \ldots, d_n$ is called **graphic** if it is the degree sequence of a simple graph.

**42.** Determine whether each of these sequences is graphic. For those that are, draw a graph having the given degree sequence.

**a)** 5, 4, 3, 2, 1, 0 **b)** 6, 5, 4, 3, 2, 1 **c)** 2, 2, 2, 2, 2, 2

**d)** 3, 3, 3, 2, 2, 2 **e)** 3, 3, 2, 2, 2, 2 **f )** 1, 1, 1, 1, 1, 1

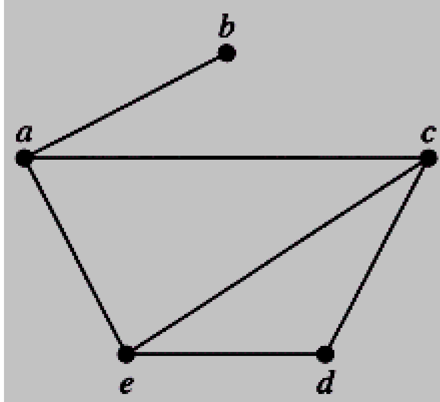
**g)** 5, 3, 3, 3, 3, 3 **h)** 5, 5, 4, 3, 2, 1

# 9.3- Representing Graphs and Graph Isomorphism ( sự đẳng cấu của đồ thị)

- Representing Graphs
- Adjacent Matrices – Ma trận kề
- Incidence Matrices – Ma trận liên thuộc
- Isomorphism of Graphs: Tính đồng dạng, đẳng cấu của đồ thị

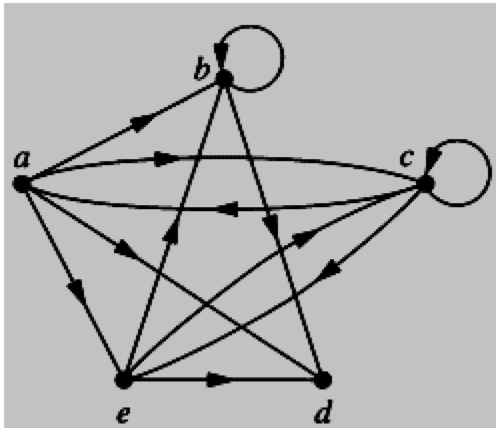
# Representing Graphs

- Using **Adjacency list**: For each vertex u in the graph, there is a list of vertex v which there is an edge between u and v.




| *Vertex* | *Adjacent Vertices* |
|---|---|
| *a* | *b, c, e* |
| *b* | *a* |
| *c* | *a, d, e* |
| *d* | *c, e* |
| *e* | *a, c, d* |

**FIGURE 1** A Simple Graph.

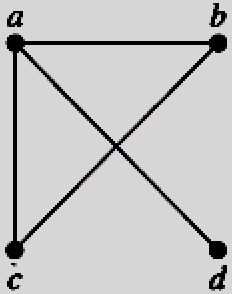



| *Initial Vertex* | *Terminal Vertices* |
|---|---|
| *a* | *b, c, d, e* |
| *b* | *b, d* |
| *c* | *a, c, e* |
| *d* | |
| *e* | *b, c, d* |

**FIGURE 2** A Directed Graph.

# Adjacent Matrices

$$A = [a_{ij}],\ a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$




$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

**FIGURE 3** Simple Graph.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

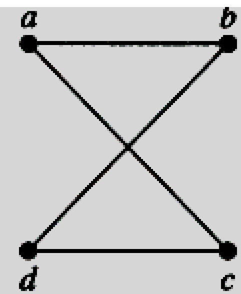
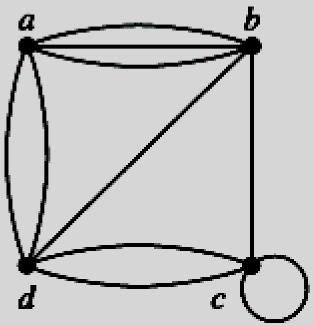



**FIGURE 4** A Graph with the Given Adjacency Matrix.



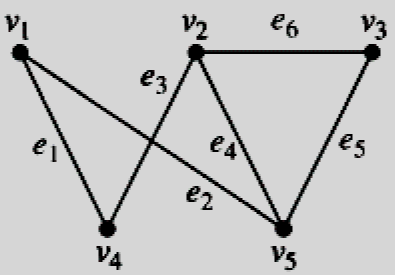

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

**FIGURE 5** A Pseudograph.

We need a trade-off between adjacency list and adjacency matrices because:
-Dense matrices : Memory is used efficient.
-sparse matrices: there is a waste in memory using.

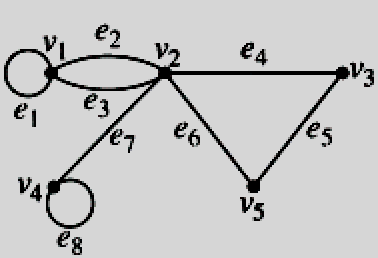
# Incidence Matrices (MT liên thuộc)

$$\mathbf{M} = [m_{ij}],\ m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$




$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

**FIGURE 6** An Undirected Graph.




$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

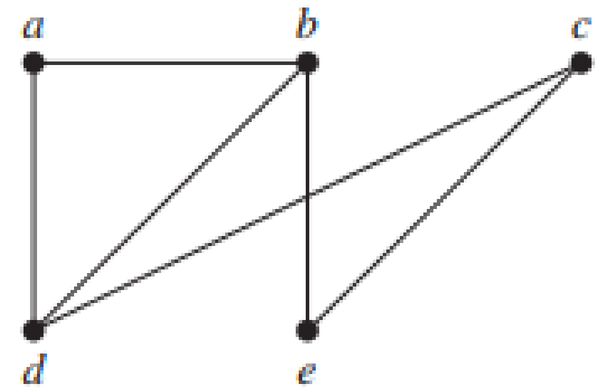
**FIGURE 7** A Pseudograph.

# Exercise

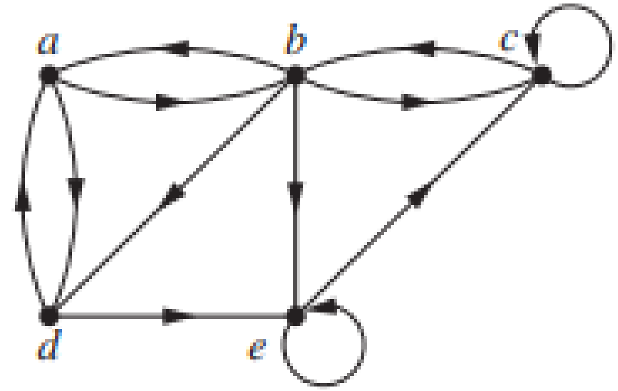
a) In Exercises 2,4 use an adjacency list to represent the given graph.

b) Represent the graph in Exercise 2,4 with an adjacency matrix.

**2.**




**4.**



# Exercise



In Exercises 16–18 draw an undirected graph represented by the given adjacency matrix.

**16.** $$\begin{bmatrix} 1 & 3 & 2 \\ 3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$$

**17.** $$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

**18.** $$\begin{bmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{bmatrix}$$

# Isomorphism of Graphs

- We may draw two graphs in the same way?
- In chemistry, different compounds can have the same molecular formula but can differ in structure. So, they can not be drawn the same way.

**DEFINITION 1**

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function $f$ from $V_1$ to $V_2$ with the property that $a$ and $b$ are adjacent in $G_1$ if and only if $f(a)$ and $f(b)$ are adjacent in $G_2$, for all $a$ and $b$ in $V_1$. Such a function $f$ is called an *isomorphism*.

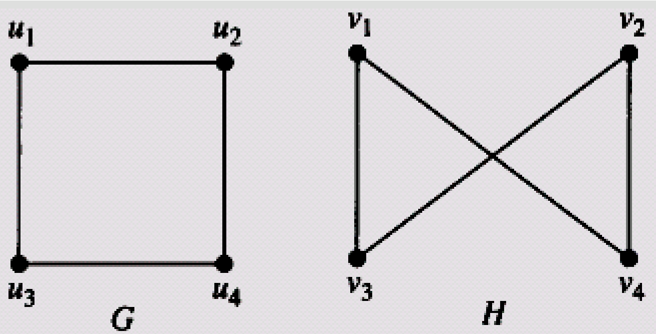



Figure 8: G and H are isomorphic

A permutation of vertices in H is similar with vertices in G → Complexity: O (n!) → It is often difficult to determine whether two graphs are isomorphic if number of vertices is large.

# Isomorphism of Graphs

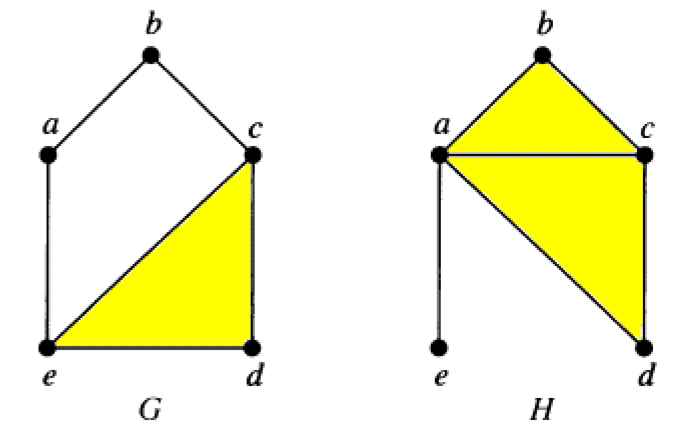



Figure 9: G and H are not isomorphic

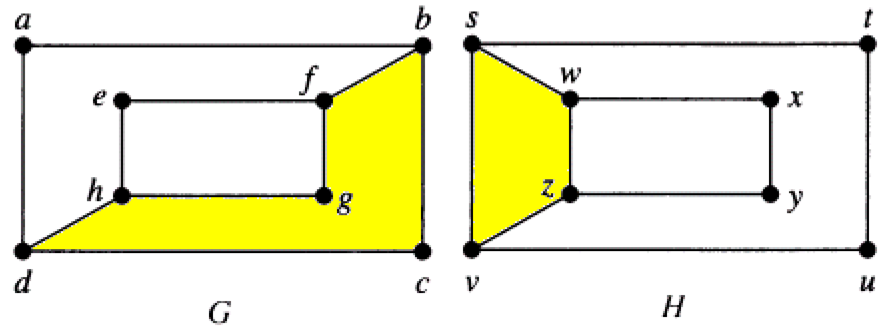



Figure 10: G and H are not isomorphic

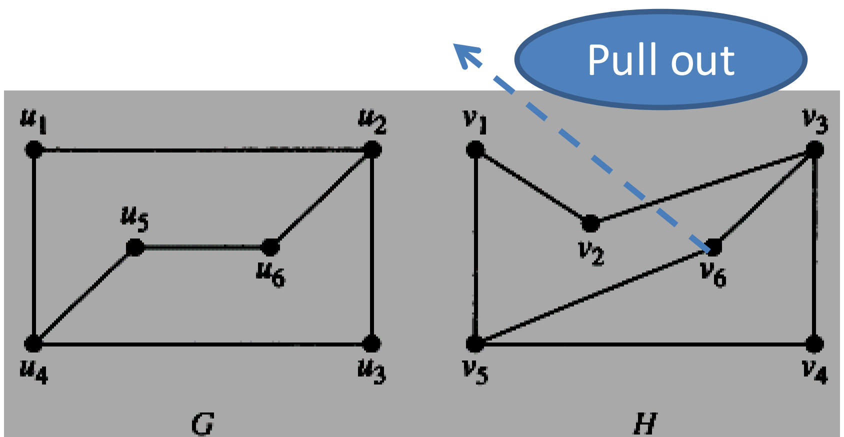



Figure 12: G and H are isomorphic

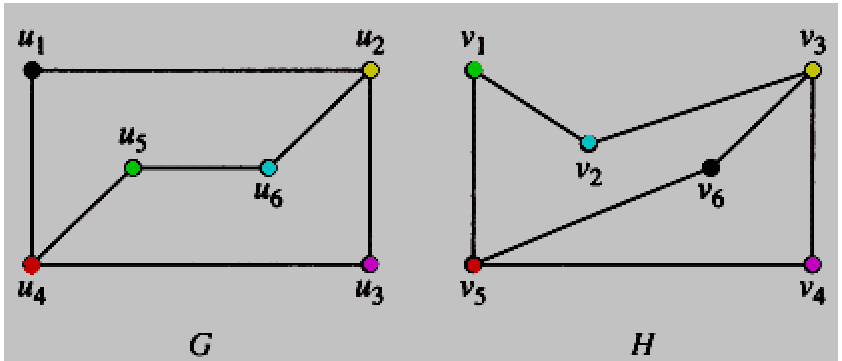


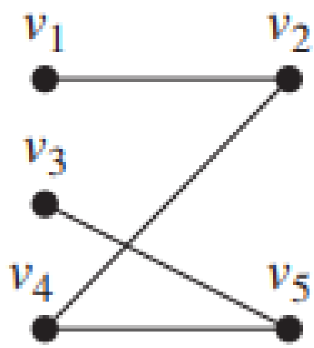

Figure 12: Function from G to H

# Exercise

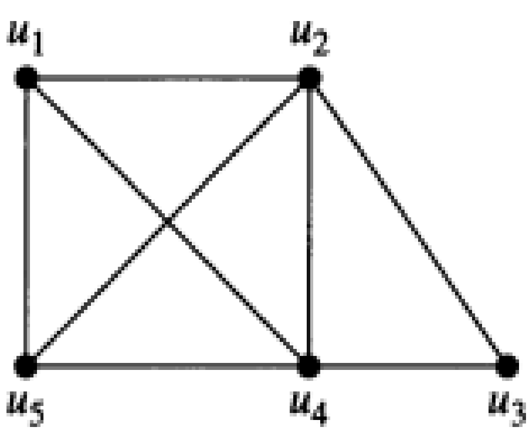


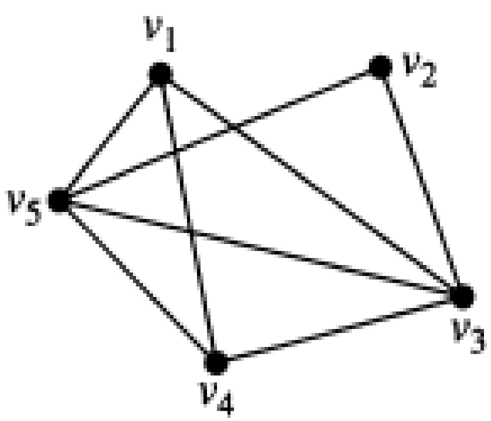
In Exercises 34–44 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

**34.**


**38.**

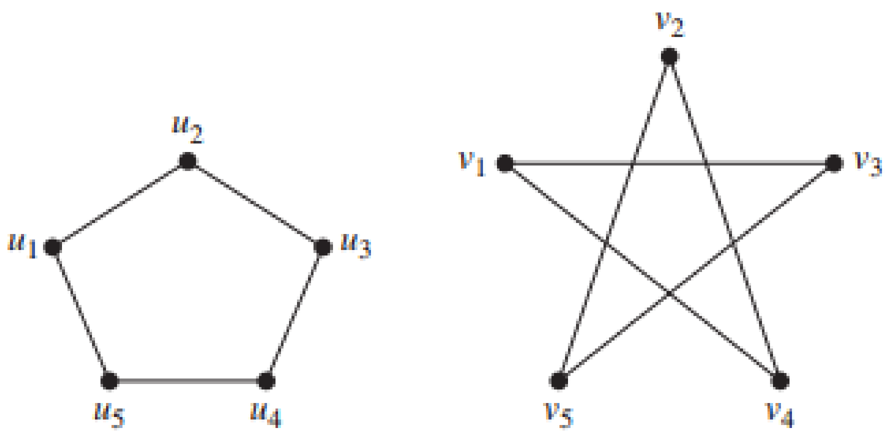


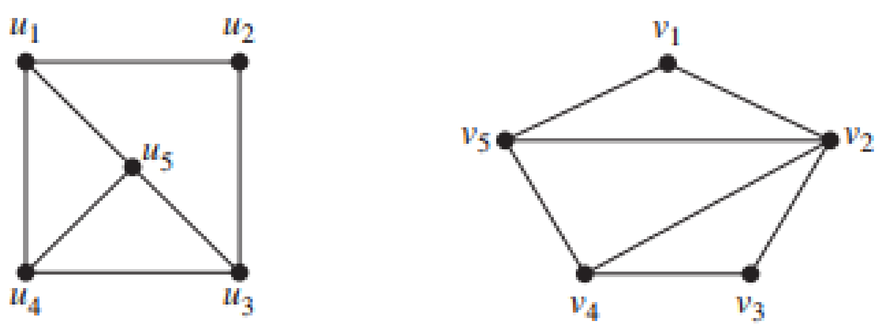
# Exercise

In Exercises 34–44 determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.

**35.**




**36.**



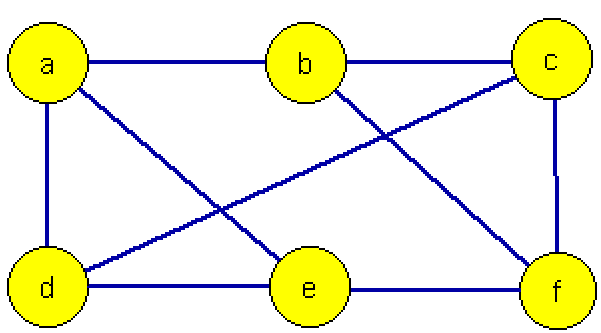
# 9.4- Connectivity (tính liên thông)

- Path
- Connectedness In Undirected Graphs
- Connectedness In Directed Graphs
- Paths and Isomorphism
- Counting Paths Between Vertices

# Path

**DEFINITION 1**

- Let $n$ be a nonnegative integer and $G$ an undirected graph.
- A *path of length $n$* from $u$ to $v$ in $G$ is a sequence of $n$ edges $e_1, \ldots, e_n$ of $G$ such that $e_1$ is associated with $\{x_0, x_1\}$, $e_2$ is associated with $\{x_1, x_2\}$, and so on, with $e_n$ associated with $\{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$.
- When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$ (because listing these vertices uniquely determines the path).
- The path is a *circuit* if it begins and ends at the same vertex, that is, if $u = v$, and has length greater than zero.
- The path or circuit is said to *pass through* the vertices $x_1, x_2, \ldots, x_{n-1}$ or *traverse* the edges $e_1, e_2, \ldots, e_n$.
- A path or circuit is *simple* if it does not contain the same edge more than once.

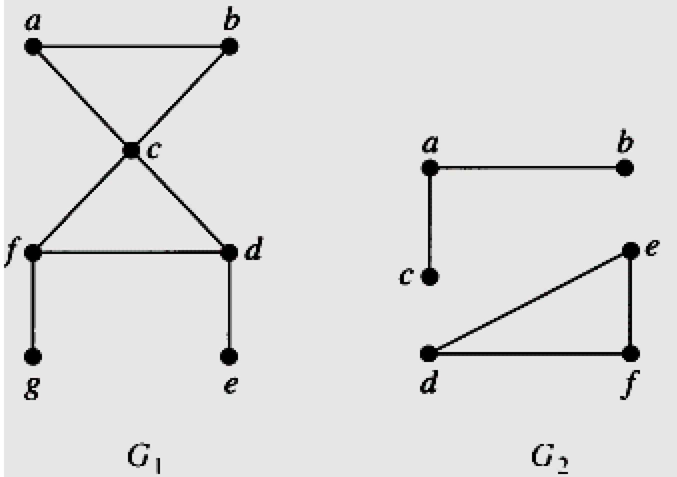



Determine the following ordered set of vertices whether it is a path and if it is a path, what is it's length?, is it a circuit?, is it simple?
{a,b,c,f,e}
{c,b,d,e,f}
{a,b,c,d,e,a,b,f}

# Connectedness In Undirected Graphs

**DEFINITION 3**

An undirected graph is called *connected* if there is a path between every pair of distinct vertices of the graph.




**THEOREM 1**

Proof: page 625

There is a simple path between every pair of distinct vertices of a connected undirected graph.

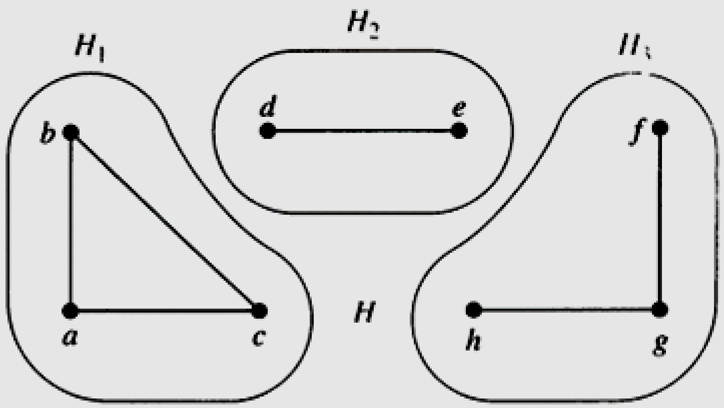
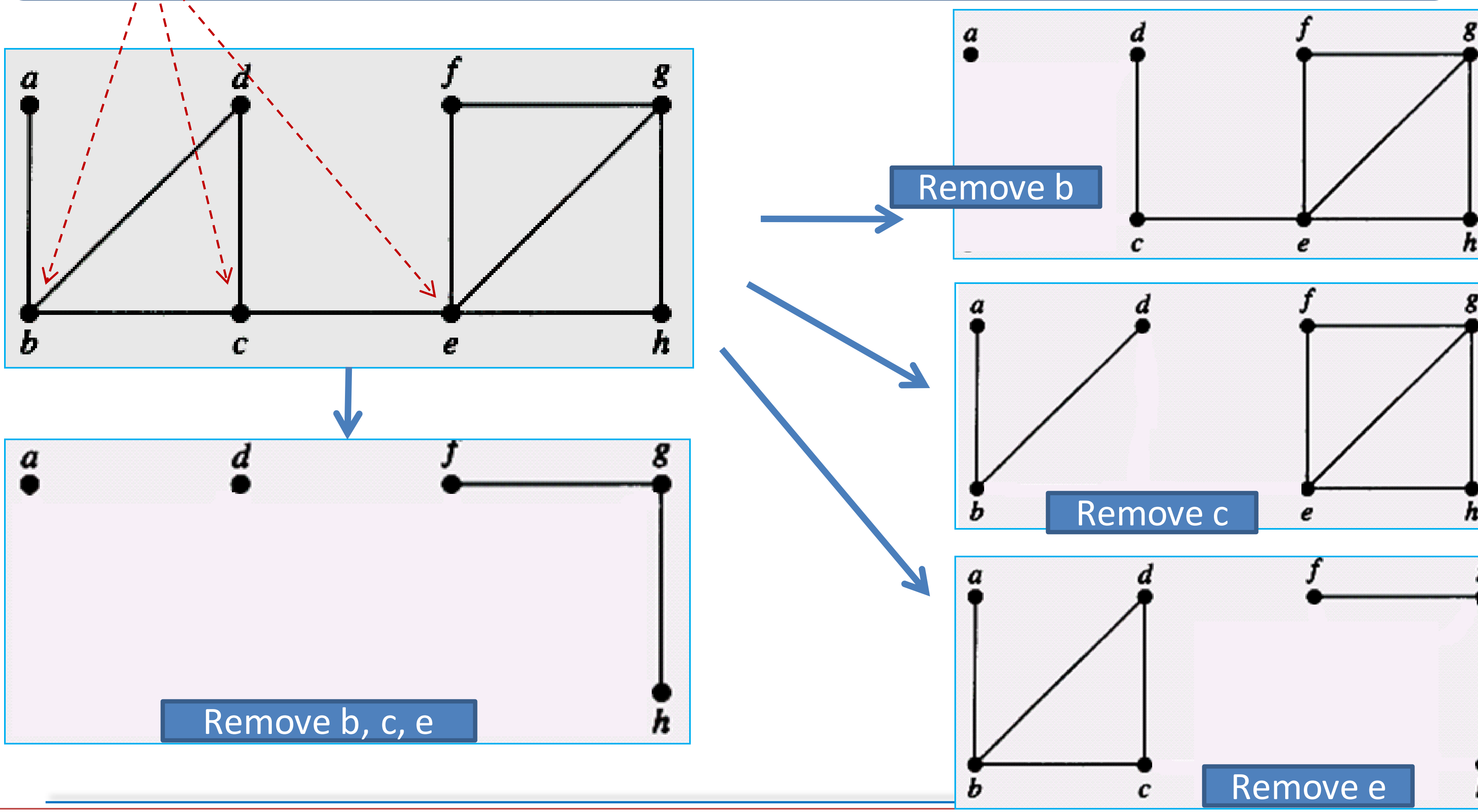



**FIGURE 3** The Graph $H$ and Its Connected Components $H_1$, $H_2$, and $H_3$.

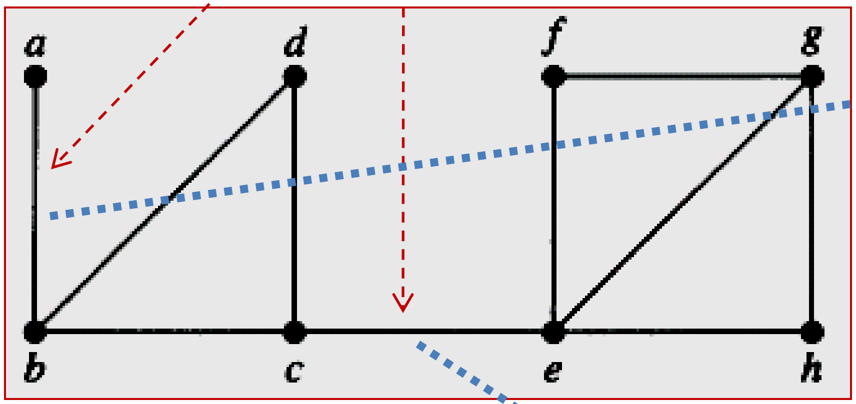
# Connectedness In Undirected Graphs

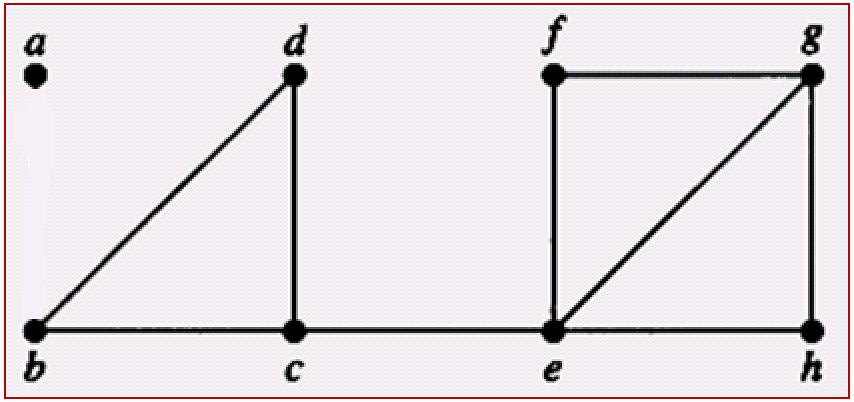
**Cut vertex** (articulation point – điểm khớp): It's removal will produce disconnected subgraph from original connected graph.

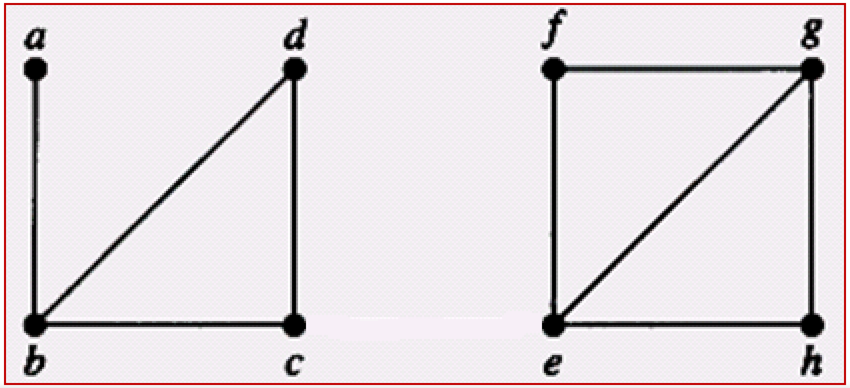


# Connectedness In Undirected Graphs

**Cut edge** (bridge): It's removal will produce subgraphs which are more connected components (thành phần liên thông) than in the original graph



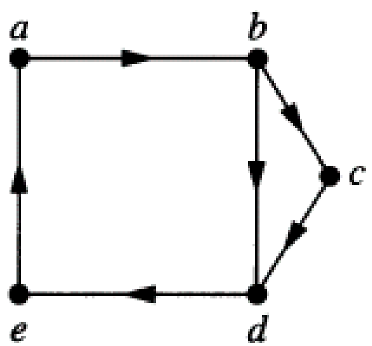
# Connectedness In Directed Graphs

**DEFINITION 4**

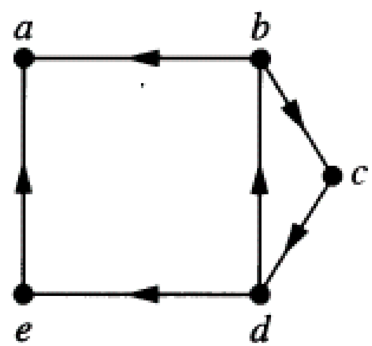
A directed graph is *strongly connected* if there is a path from $a$ to $b$ and from $b$ to $a$ whenever $a$ and $b$ are vertices in the graph.

**DEFINITION 5**

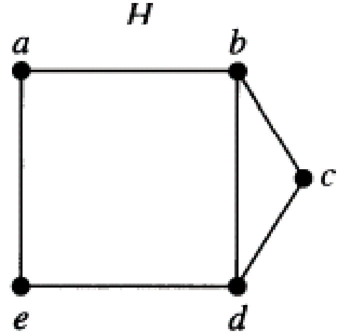
A directed graph is *weakly connected* if there is a path between every two vertices in the underlying undirected graph.




$G$




$H$




underlying undirected graph of $H$

-G is strongly connected → G is weakly also.
- H is not strongly connected and it is weakly connected.
-( you can verify these )

# Path and Isomorphism

- Using path to determine whether two graphs are isomorphic:

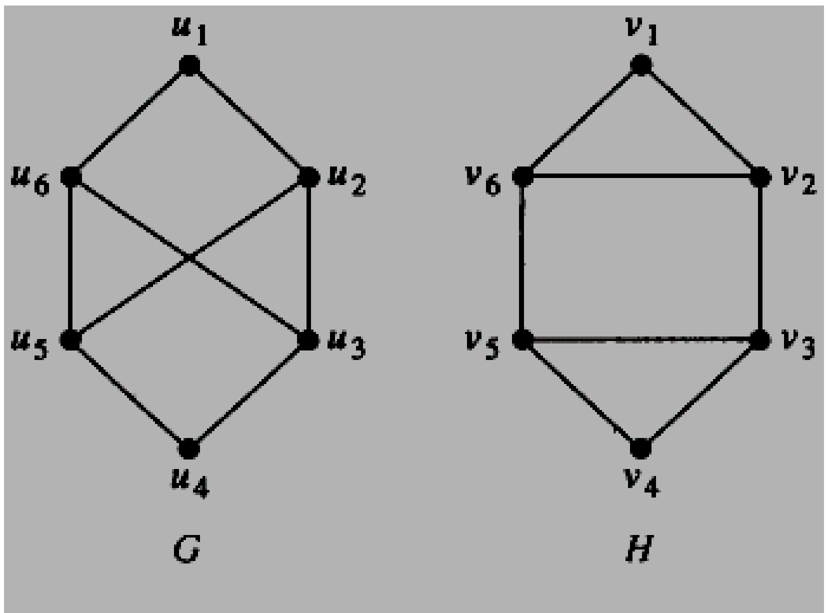



**FIGURE 6** The Graphs *G* and *H*.

G and H have the same:
-Number of vertices
-Number of edges
- 2 vertices degree 2
- 2 vertices degree 3
But:
H has a loop with minimum length 3
and
G has a loop with minimum length 4
➔ They are not isomorphic.

# Path and Isomorphism

- Using path to determine whether two graphs are isomorphic:

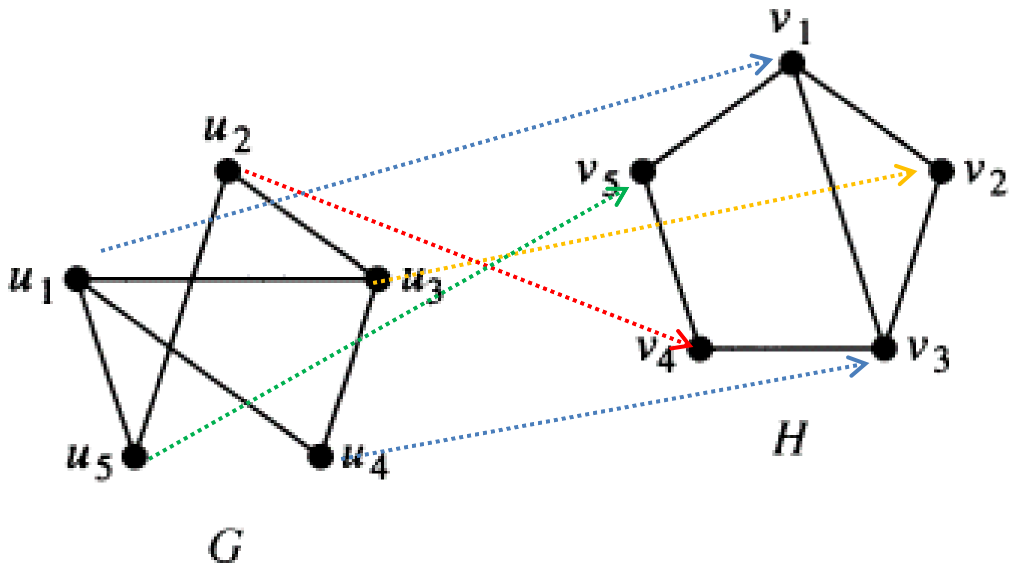



FIGURE 7 The Graphs *G* and *H*.

They are isomorphic.

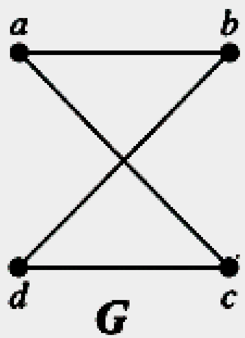
# Counting Paths Between Vertices

**THEOREM 2** Proof: page 628

Let $G$ be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length $r$ from $v_i$ to $v_j$, where $r$ is a positive integer, equals the $(i, j)$th entry of $\mathbf{A}^r$.

**EXAMPLE 14**

How many paths of length four are there from $a$ to $d$ in the simple graph $G$?




*Solution:* The adjacency matrix of $G$ (ordering the vertices as $a, b, c, d$) is

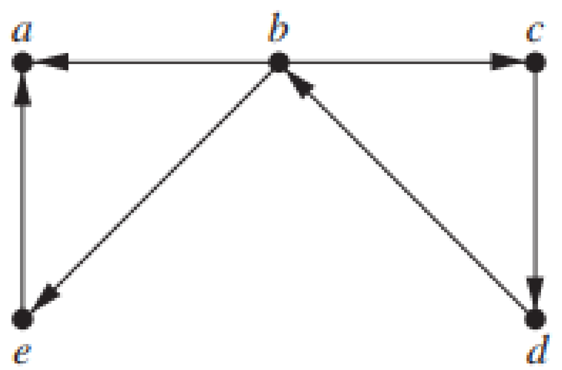
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \mathbf{A}^{\boxed{4}} = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Result= 8

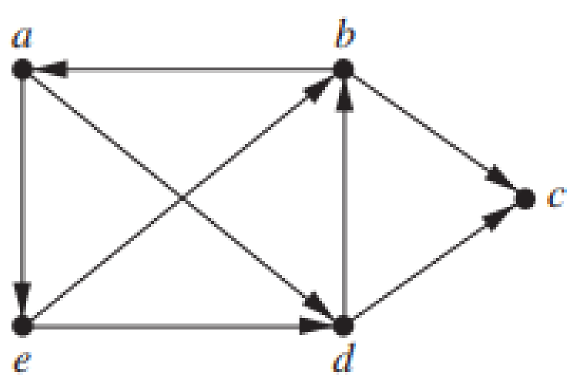
$a, b, a, b, d$
$a, b, a, c, d$
$a, b, d, b, d$
$a, b, d, c, d$
$a, c, a, b, d$
$a, c, a, c, d$
$a, c, d, b, d$
$a, c, d, c, d$

**11.** Determine whether each of these graphs is strongly connected and if not, whether it is weakly connected.

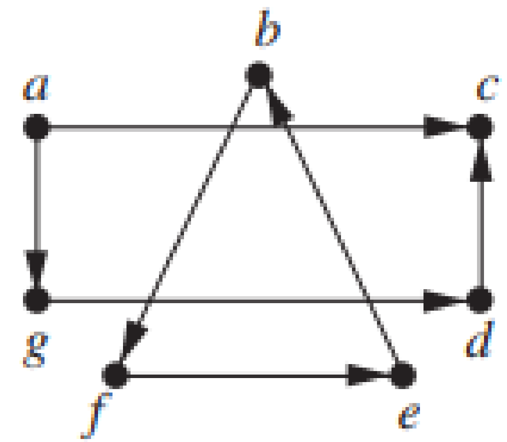
**a)**



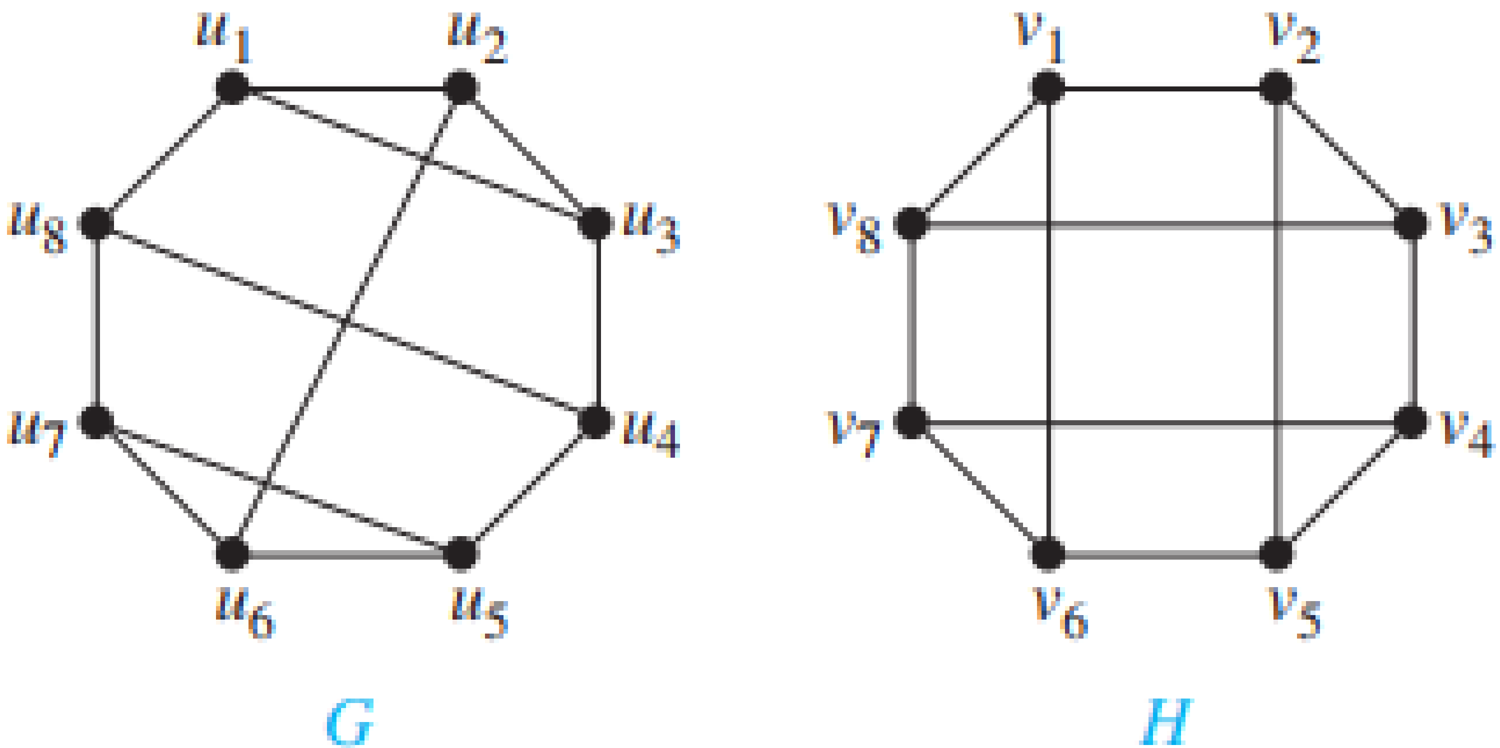

**b)**




**c)**

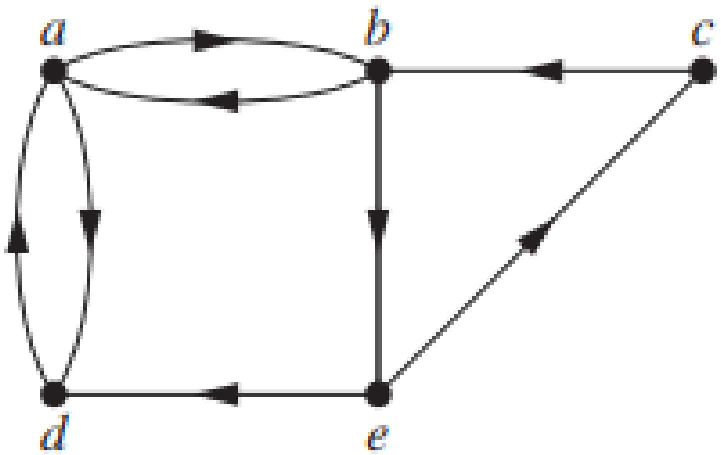


**21.** Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



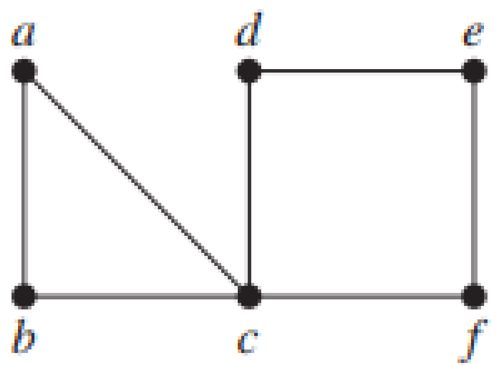
27. Find the number of paths from *a* to *e* in the directed graph in Exercise 2 of length

**a)** 2. **b)** 3. **c)** 4. **d)** 5. **e)** 6. **f)** 7

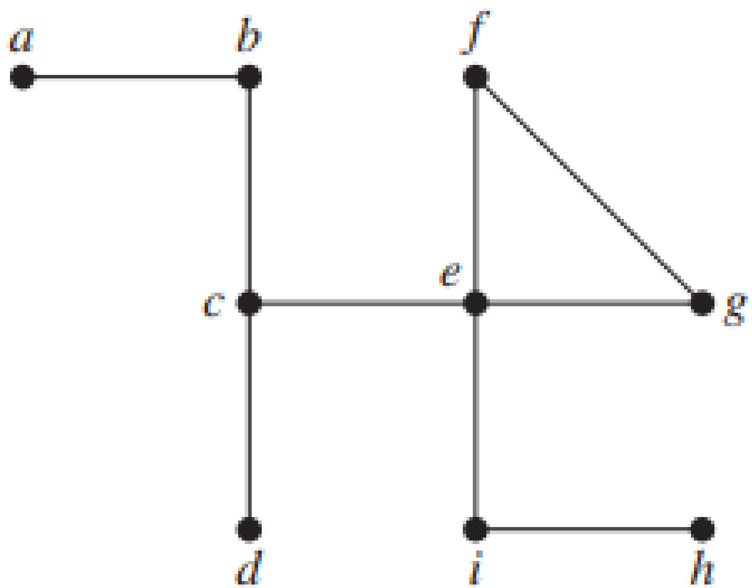


find all the cut vertices of the given graph.

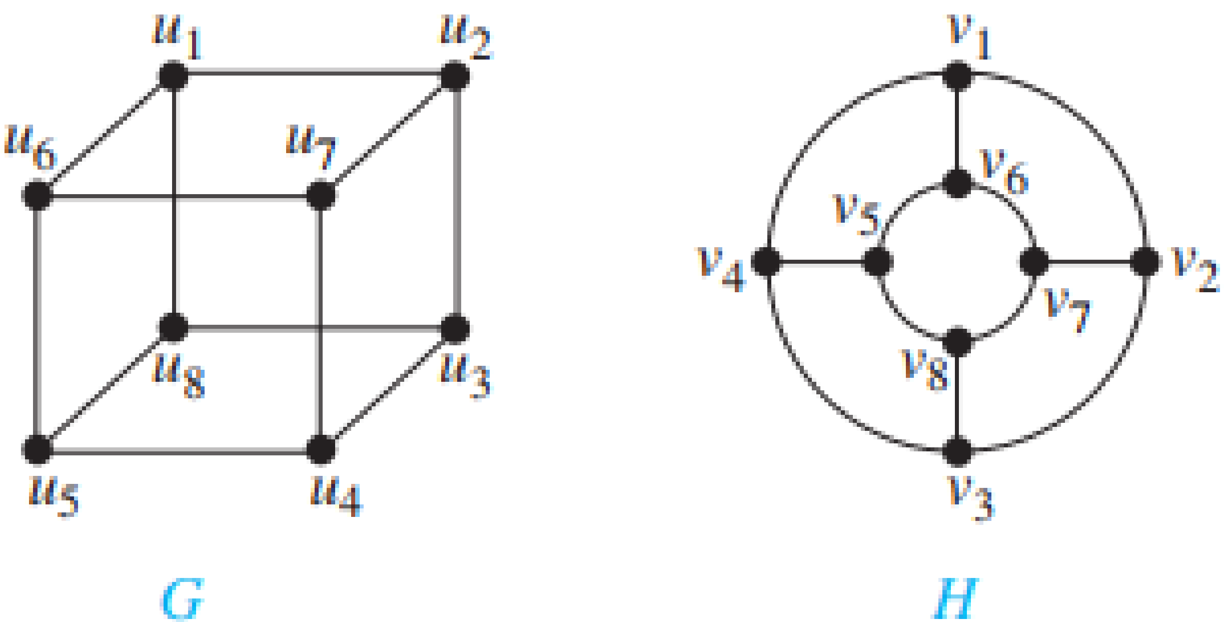
31.




33.



**23.** Use paths either to show that these graphs are not isomorphic or to find an isomorphism between them.



# 9.5- Euler and Hamilton Path

- Euler Paths and Circuit
  - Paths and circuits contains all Edges of a graph.
- Hamilton Paths and Circuit
  - Paths and circuits contains all vertices of a graph.

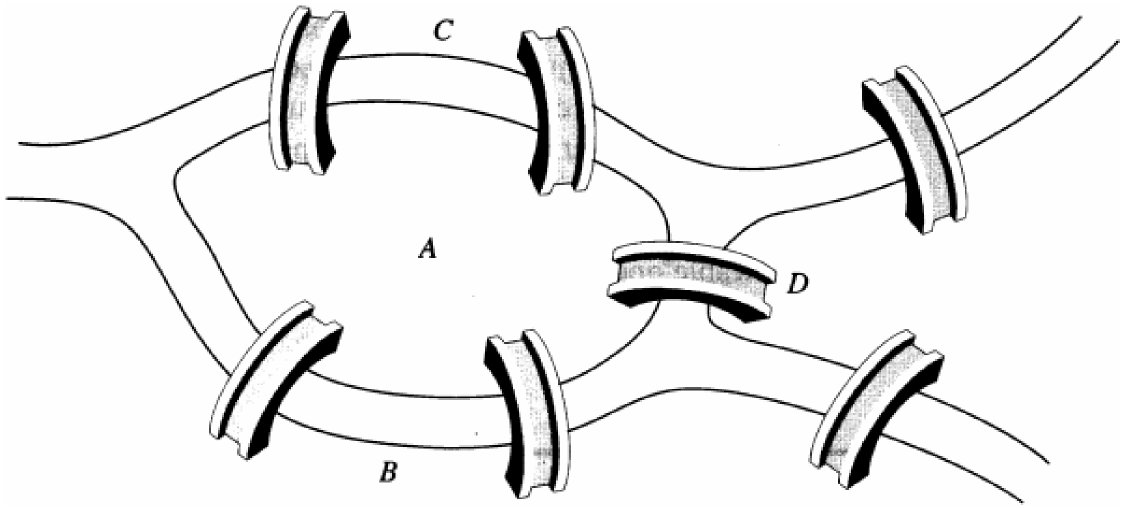



FIGURE 1 The Seven Bridges of Königsberg.

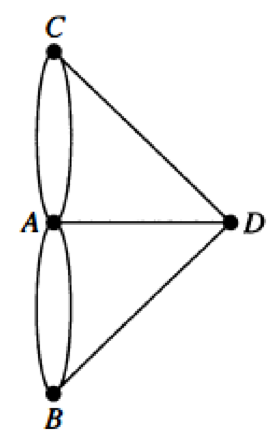



FIGURE 2 Multigraph Model of the Town of Königsberg.

# Euler Paths and Circuits

**DEFINITION 1**

An *Euler circuit* in a graph $G$ is a simple circuit containing every edge of $G$. An *Euler path* in $G$ is a simple path containing every edge of $G$.

**EXAMPLE 1** Which of the undirected graphs in Figure 3 have an Euler circuit? Of those that do not, which have an Euler path?

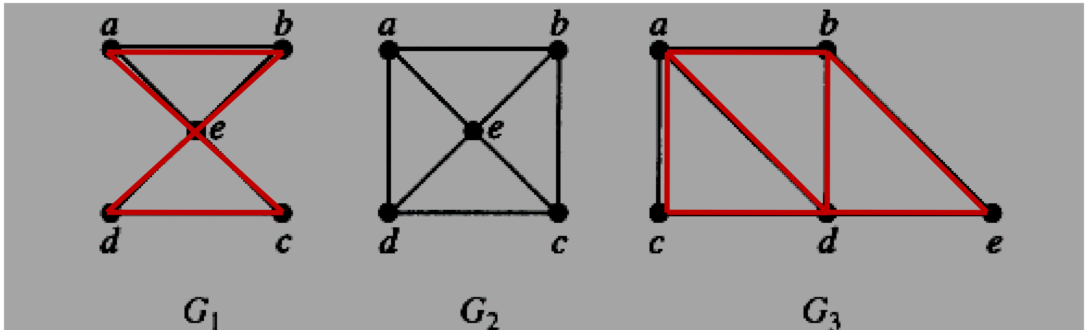



**FIGURE 3** The Undirected Graphs $G_1$, $G_2$, and $G_3$.

$G_1$: has an Euler circuit:
a,e,c,d,e,b,a
G3: has no Euler circuit but it has Euler path a,c,d,e,b,d,a,b.
G2 has no Euler circuit and also has no Euler path.

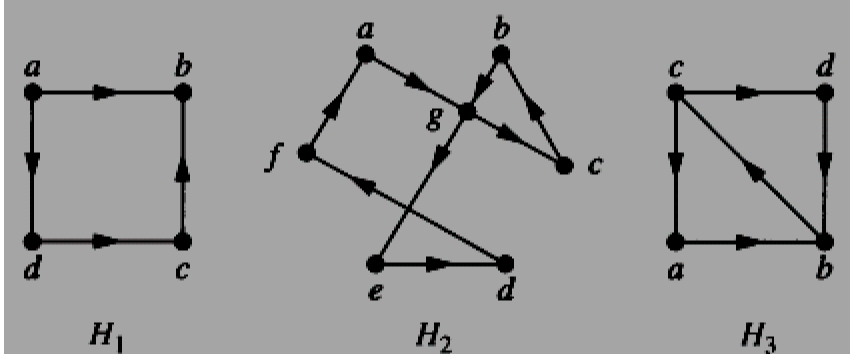



Figure 4: Directed Graphs

**EXAMPLE 2**
Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?

# Euler Paths and Circuits

**THEOREM 1**

A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

**ALGORITHM 1 Constructing Euler Circuits.**

**procedure** *Euler*($G$: connected multigraph with all vertices of even degree)
*circuit* := a circuit in $G$ beginning at an arbitrarily chosen
  vertex with edges successiv
  returns to this vertex
$H$ := $G$ with the edges of this ci
**while** $H$ has edges
**begin**
  *subcircuit* := a circuit in $H$
    also is an endpoint of a
  $H$ := $H$ with edges of *subc*
    removed
  *circuit* := *circuit* with *subci*
    vertex
**end** {*circuit* is an Euler circuit}

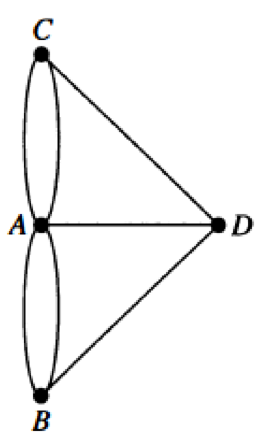
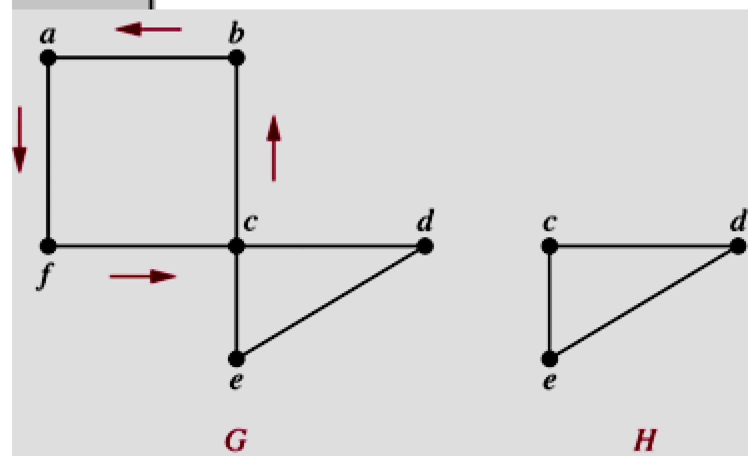



**FIGURE 2 Multigraph Model of the Town of Königsberg.**



# Euler Paths and Circuits

**THEOREM 2**

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

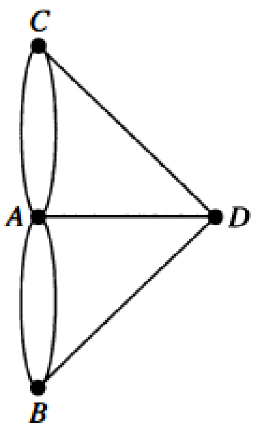



**FIGURE 2** **Multigraph Model of the Town of Königsberg.**

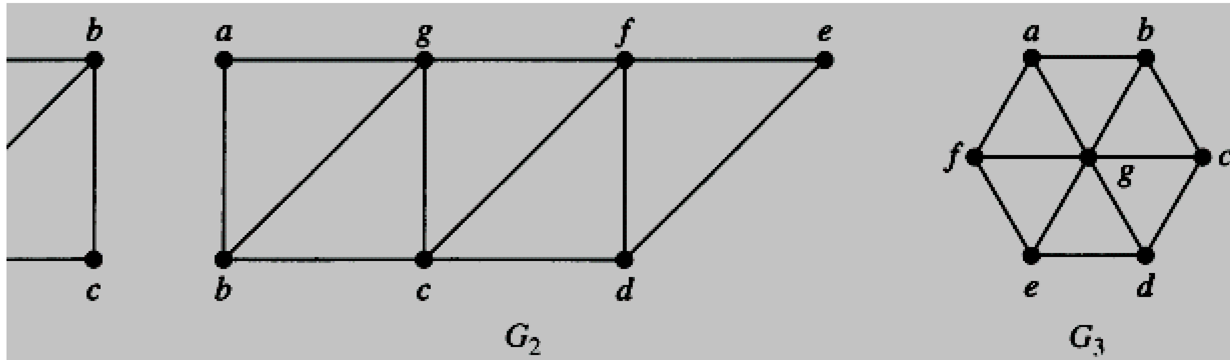



**FIGURE 7** **Three Undirected Graphs.**

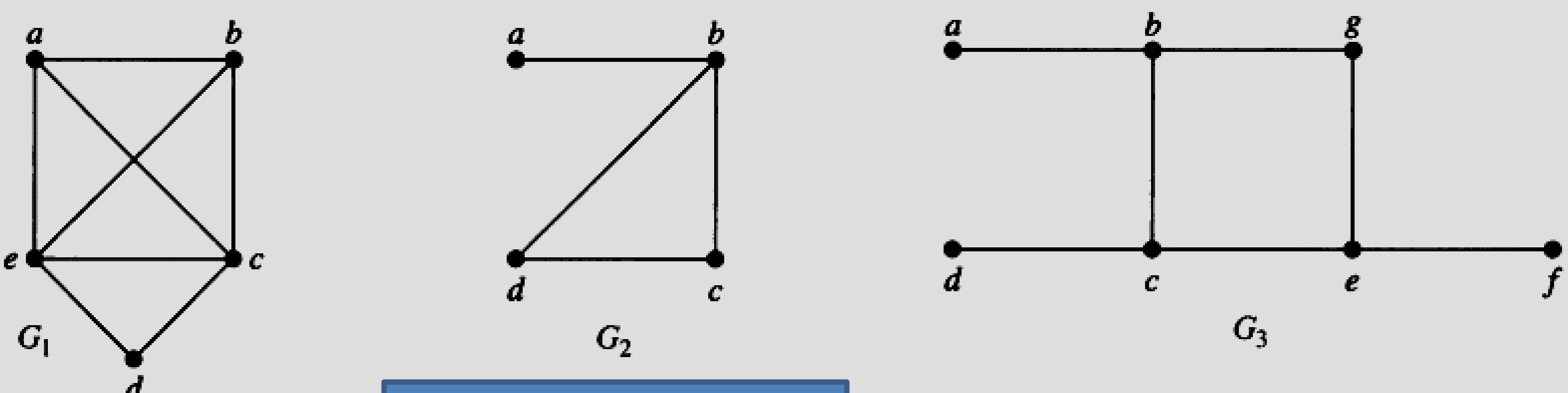
- $G_1$ contains exactly two vertices of odd degree, namely b and d. So, they are endpoints of Euler path: b,d,c,b,a,d or b,c,d,b,a,d, . . .
- Do similarly on $G_2$
- $G_3$ have six vertices of add degree. So, $G_3$ has no Euler path

# Hamilton Paths and Circuits

**DEFINITION 2**

A simple path in a graph $G$ that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph $G$ that passes through every vertex exactly once is called a *Hamilton circuit*. That is, the simple path $x_0, x_1, \ldots, x_{n-1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, \ldots, x_{n-1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, \ldots, x_{n-1}, x_n$ is a Hamilton path.

**FIGURE 10 Three Simple Graphs.**




Hamilton circuit:
(a,b,c,d,e,a)
(a,b,e,d,c,a)
. . .

Hamilton circuit: none
Hamilton Paths:
(a,b,c,d)
(a,b,d,c)
. . .

Hamilton circuit: none
Hamilton Path: none

# Hamilton Paths and Circuits

There are no known simple necessary and sufficient criteria for the existence of Hamilton circuits. However, many theorems are known that give sufficient conditions for the existence of Hamilton circuits.

Also, certain properties can be used to show that a graph has no Hamilton circuit. For instance, a graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in the circuit. Moreover, if a vertex in the graph has degree two, then both edges that are incident with this vertex must be part of any Hamilton circuit. Also, note that when a Hamilton circuit is being constructed and this circuit has passed through a vertex, then all remaining edges incident with this vertex, other than the two used in the circuit, can be removed from consideration. Furthermore, a Hamilton circuit cannot contain a smaller circuit within it.

**THEOREM 3**

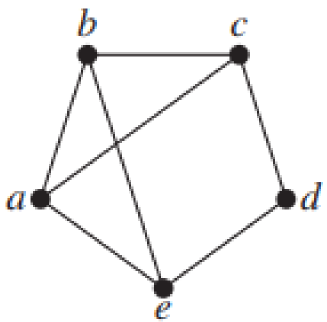
**DIRAC'S THEOREM** If $G$ is a simple graph with $n$ vertices with $n \geq 3$ such that the degree of every vertex in $G$ is at least $n/2$, then $G$ has a Hamilton circuit.

**THEOREM 4**

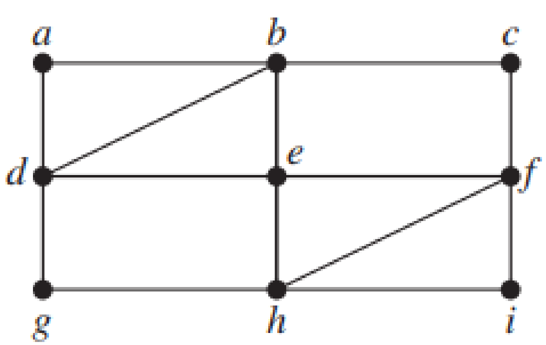
**ORE'S THEOREM** If $G$ is a simple graph with $n$ vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices $u$ and $v$ in $G$, then $G$ has a Hamilton circuit.

Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.

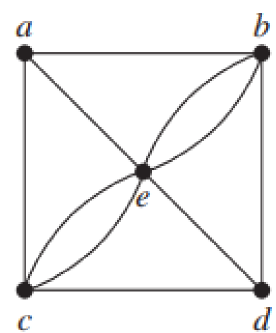
**1.**



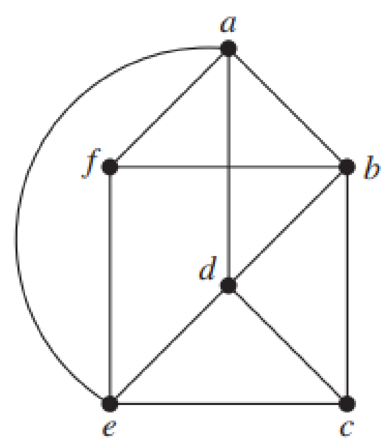

**2.**



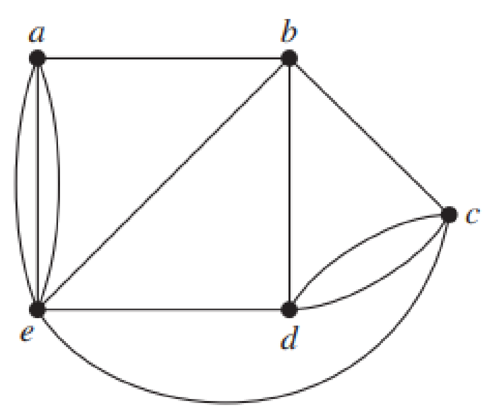

**3.**




**4.**

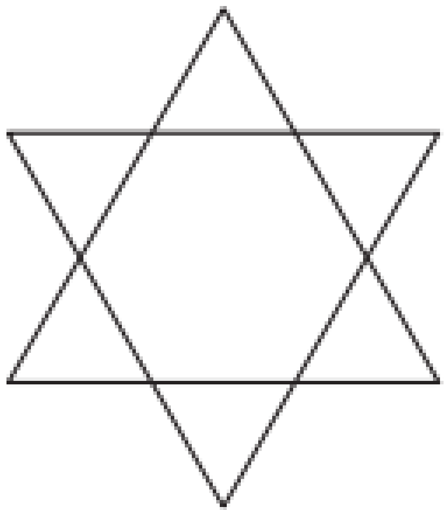



**5.**

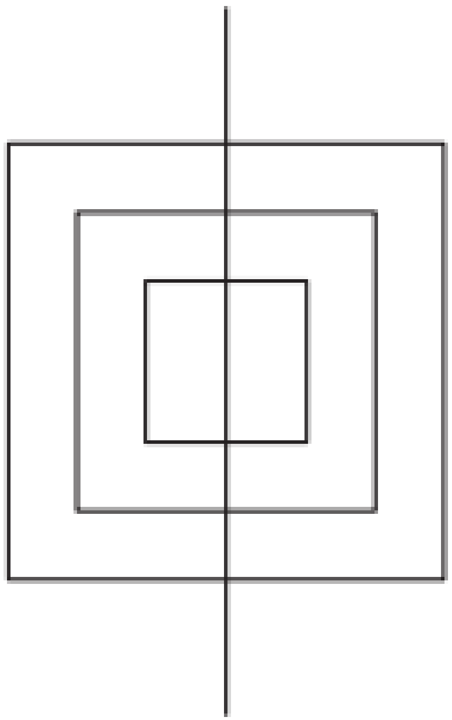


Determine whether the picture shown can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.

13.

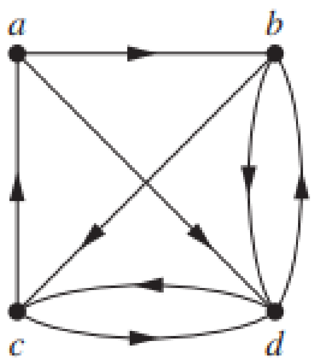


14.

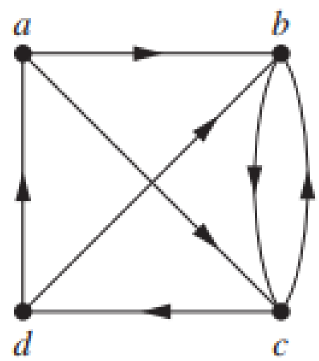


Exercises 18–20 determine whether the directed graph shown has an Euler circuit. Construct an Euler circuit if one exists. If no Euler circuit exists, determine whether the directed graph has an Euler path. Construct an Euler path if one exists.

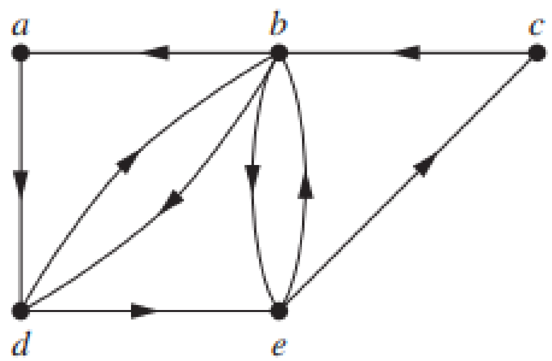
**18.**




**19.**

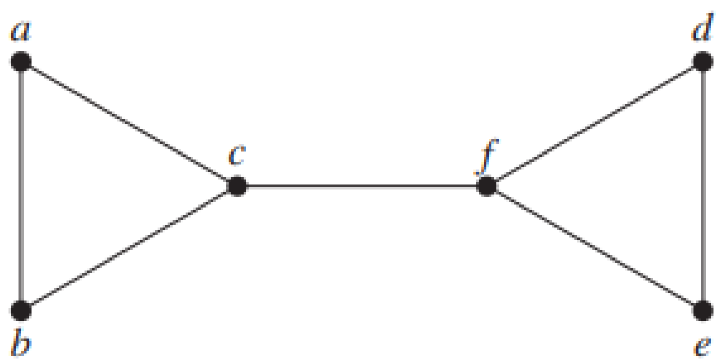



**20.**

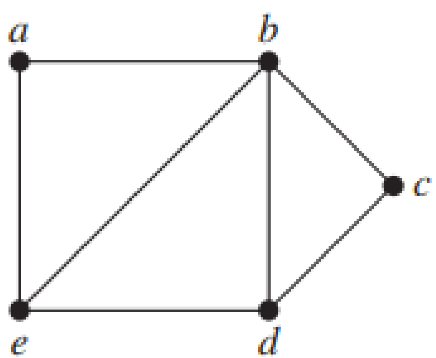


Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.

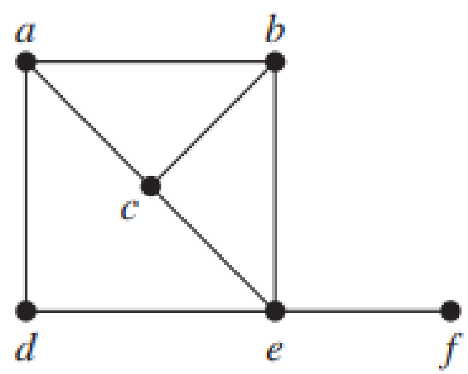
**30.**




**31.**



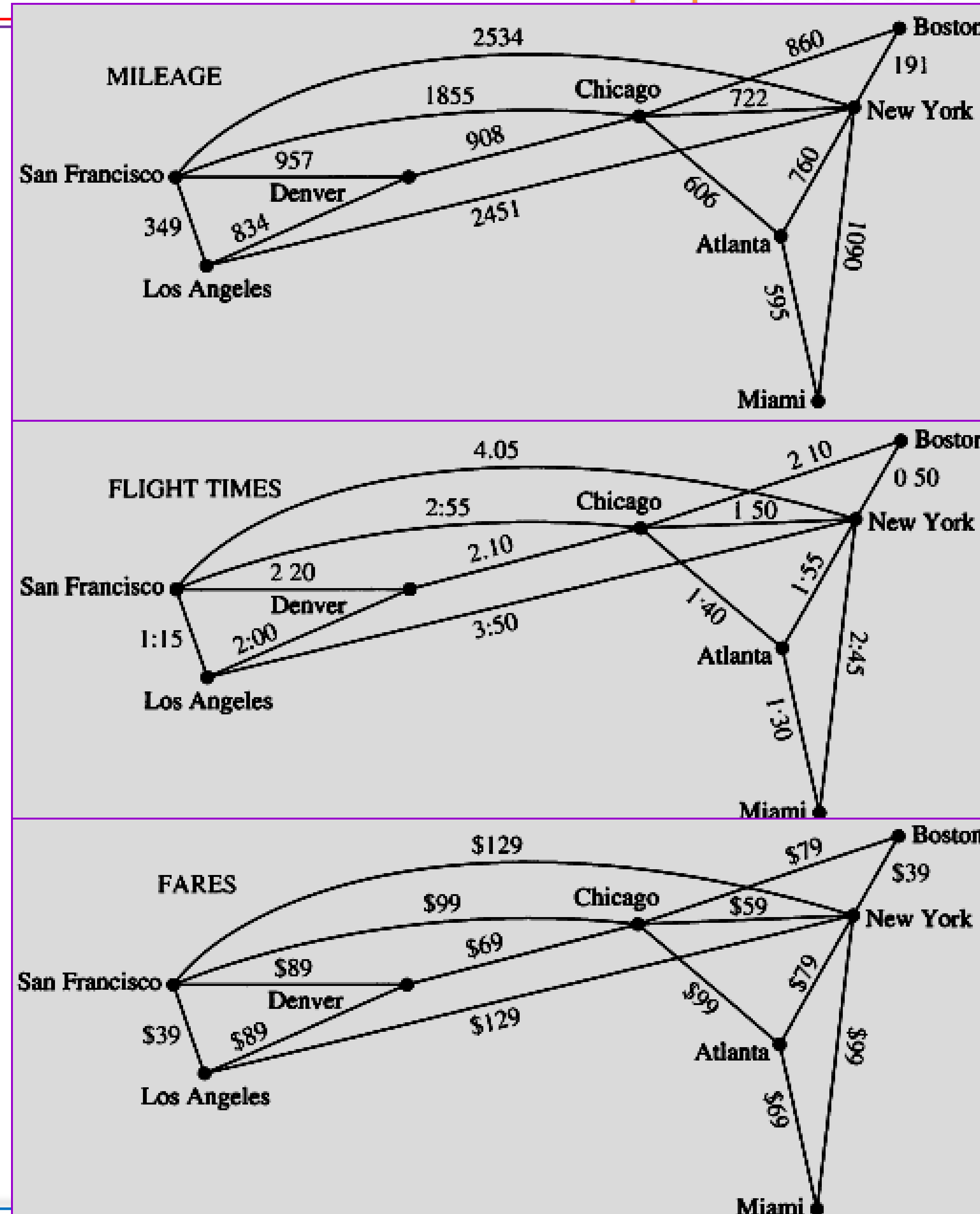

**32.**



# 9.6- Shortest Path Problems

- Introduction
- A Shortest Path Algorithm
- The Traveling Salesman Problem

# Introduction



# Introduction

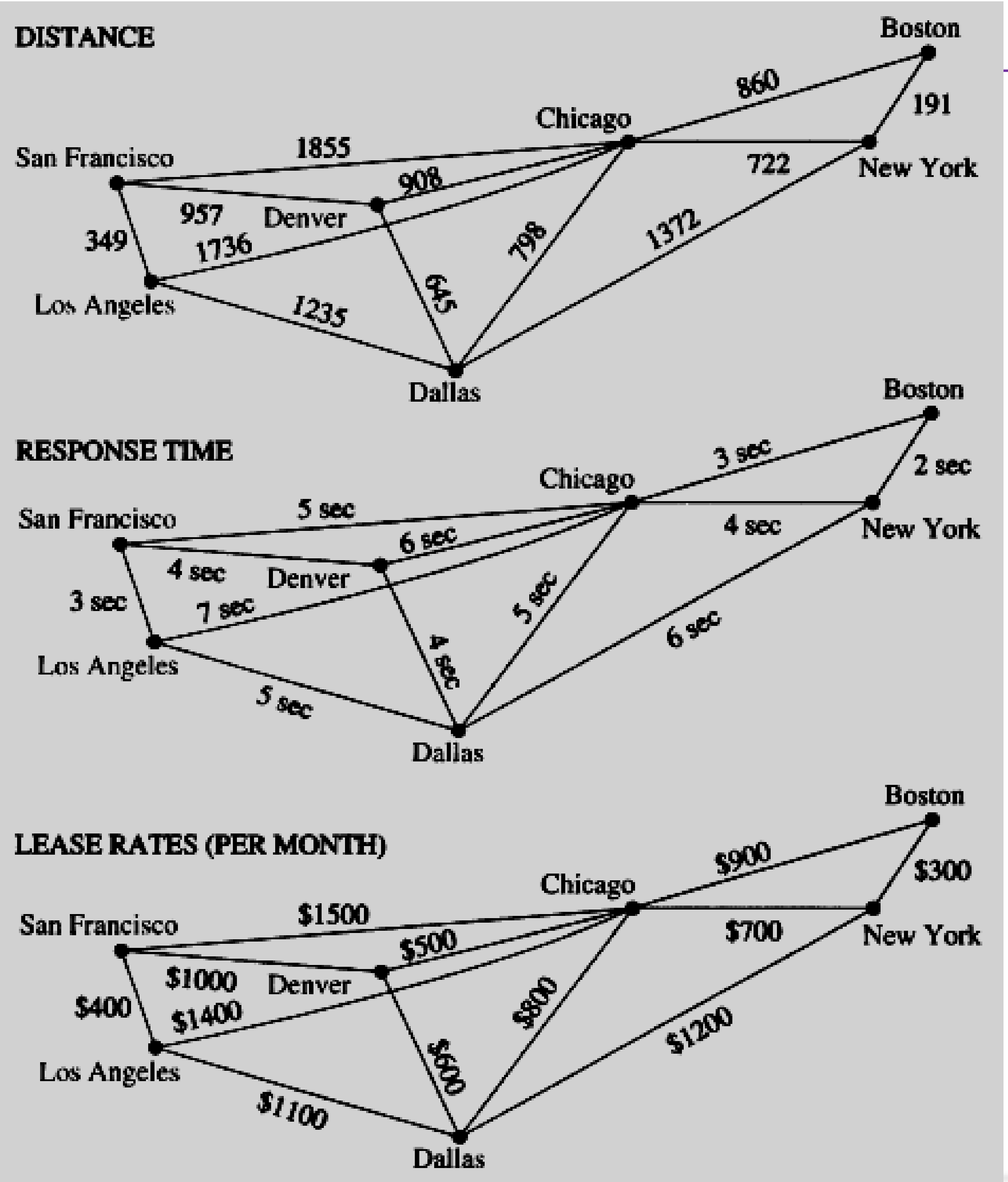



FIGURE 2 Weighted Graphs Modeling a Computer Network.

# Shortest Path Problems…
# Dijkstra Algorithm

**ALGORITHM 1** Dijkstra's Algorithm.

**procedure** *Dijkstra*($G$: weighted connected simple graph, with all weights positive)
{$G$ has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in $G$}
**for** $i := 1$ **to** $n$ $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of $a$ is 0 and all other labels are $\infty$, and $S$ is the empty set}
**while** $z \notin S$
**begin**
  $u :=$ a vertex not in $S$ with $L(u)$ minimal
  $S := S \cup \{u\}$
  **for** all vertices $v$ not in $S$
    **if** $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
  {this adds a vertex to $S$ with minimal label and updates the labels of vertices not in $S$}
**end** {$L(z) =$ length of a shortest path from $a$ to $z$}

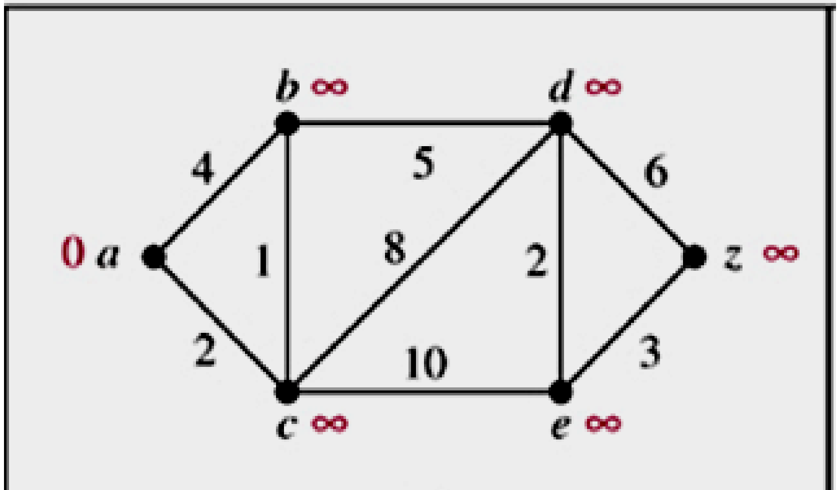
**THEOREM 1**

Dijkstra's algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.

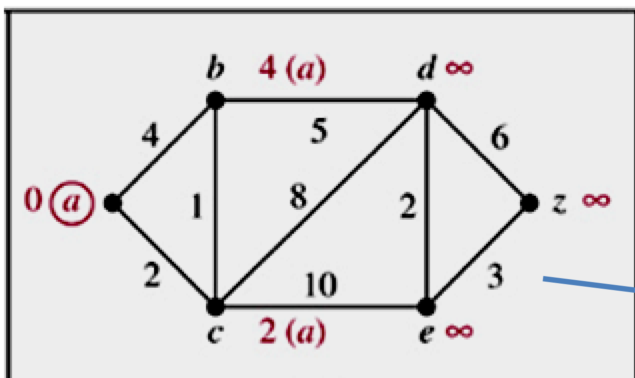
**THEOREM 2**

Dijkstra's algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with $n$ vertices.

# Shortest Path Problems…




(a)



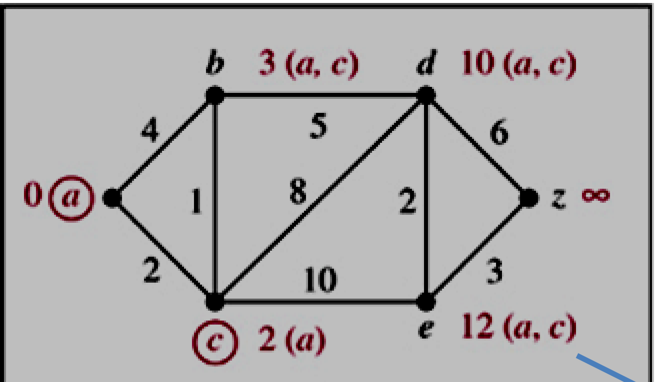

(b)

**while (z ∉ S)**

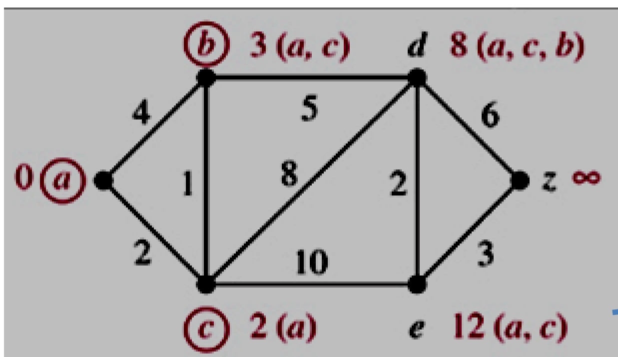
| L(a) | L(b) | L(c) | L(d) | L(e) | L(z) | S |
|---|---|---|---|---|---|---|
| 0 | ∞ | ∞ | ∞ | ∞ | ∞ | {a} |
| [redacted] | ∞ | ∞ | ∞ | ∞ | ∞ | {a} |
| [redacted] | 4 | 2 | ∞ | ∞ | ∞ | |

Examining all vertex connected to a but not in S

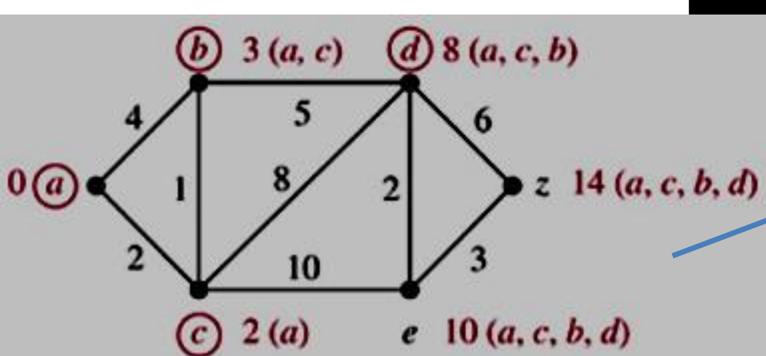
# Shortest Path Problems…



(c)



(d)



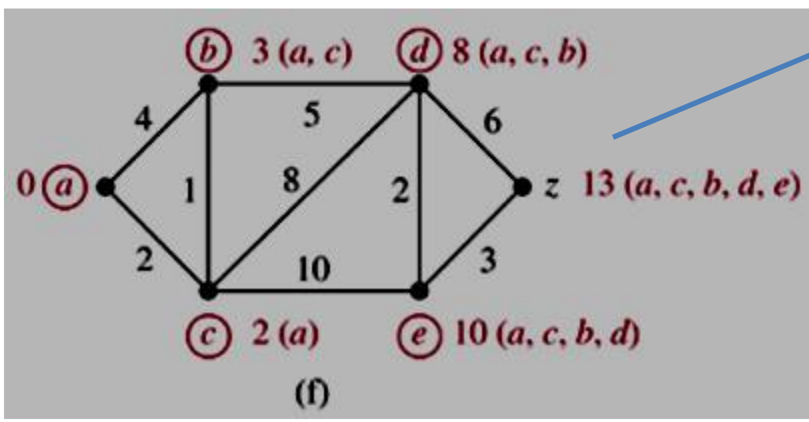
(e)

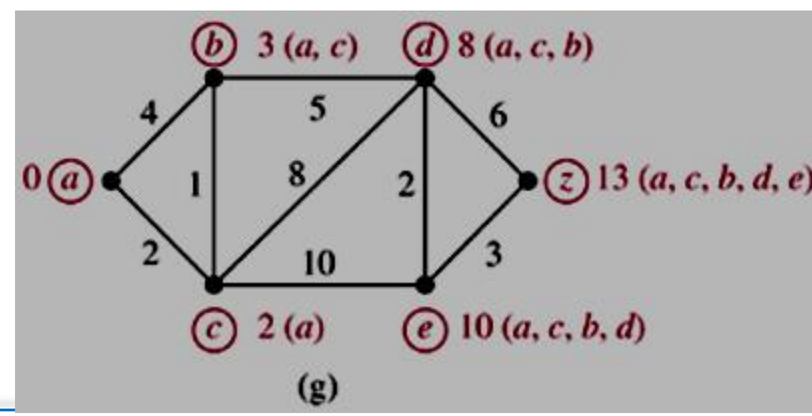
| L(a) | L(b) | L(c) | L(d) | L(e) | L(z) | S |
|---|---|---|---|---|---|---|
| **0** | ∞ | ∞ | ∞ | ∞ | ∞ | {**a**} |
| | ∞ | ∞ | ∞ | ∞ | ∞ | {**a**} |
| | 4 | **2** | ∞ | ∞ | ∞ | {**a,c**} |
| | (cb) 2+1 = **3** | | (cd) 2+8 =10 | (ce) 2+10=12 | ∞ | |
| | | | 10 | 12 | ∞ | {**a,c,b**} |
| | | | (bd) 3+5 =**8** | 12 | ∞ | {**a,c,b,d**} |
| | | | | (de) 8+2=**10** | (dz) 8+6= 14 | {**a,c,b,d,e**} |
| | | | | | | |

# Shortest Path Problems…

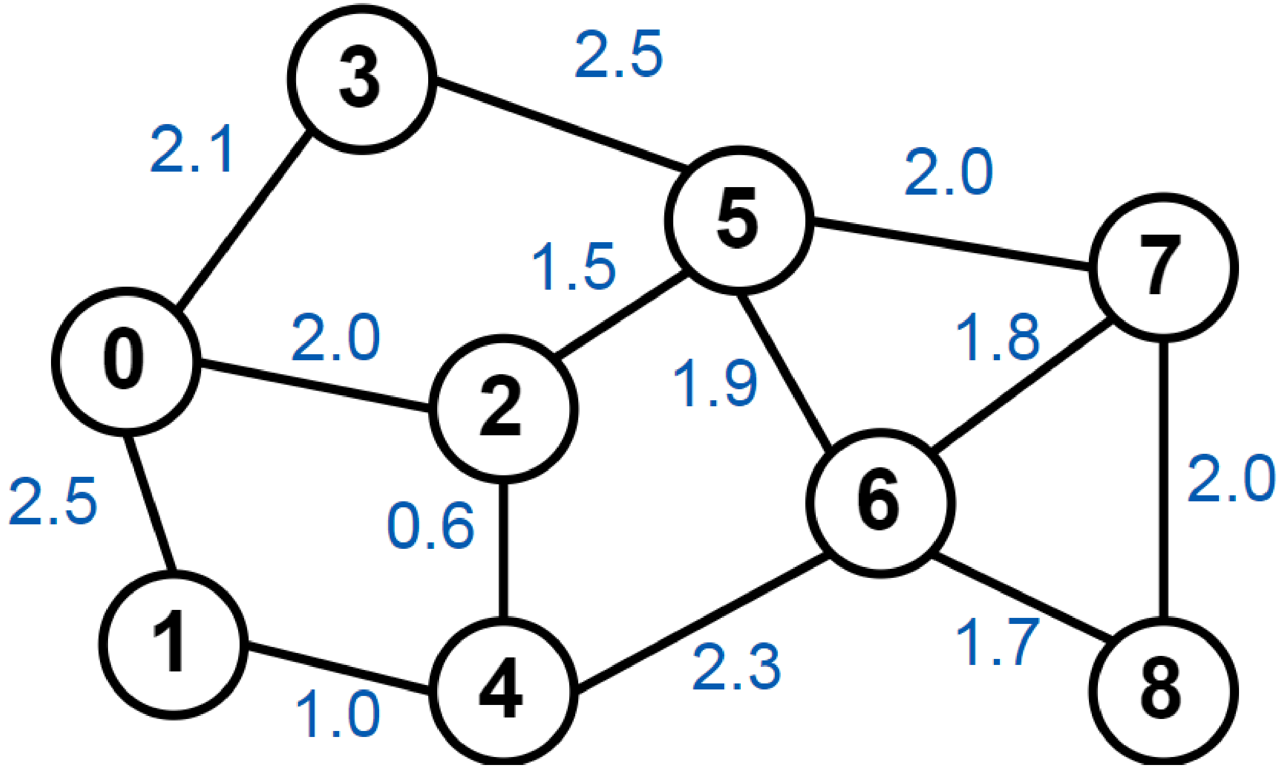
| L(a) | L(b) | L(c) | L(d) | L(e) | L(z) | S |
|---|---|---|---|---|---|---|
| [redacted] | [redacted] | [redacted] | [redacted] | (de) 8+2=**10** | (dz) 8+6= 14 | {**a,c,b,d,e**} |
| [redacted] | [redacted] | [redacted] | [redacted] | [redacted] | (ez) 10+3 =**13** | {**a,c,b,d,e,z**} |

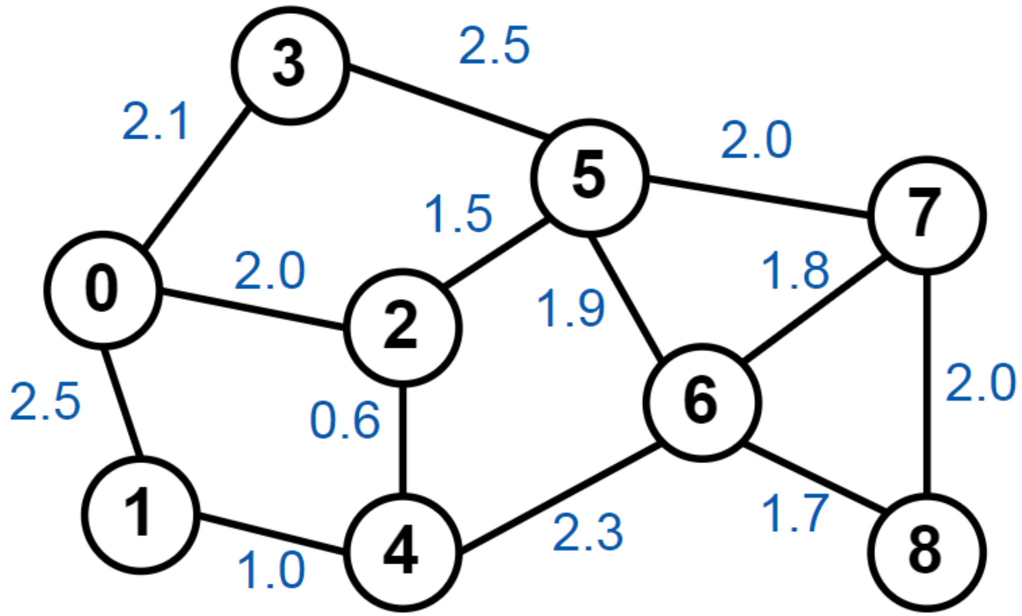
**Stop**



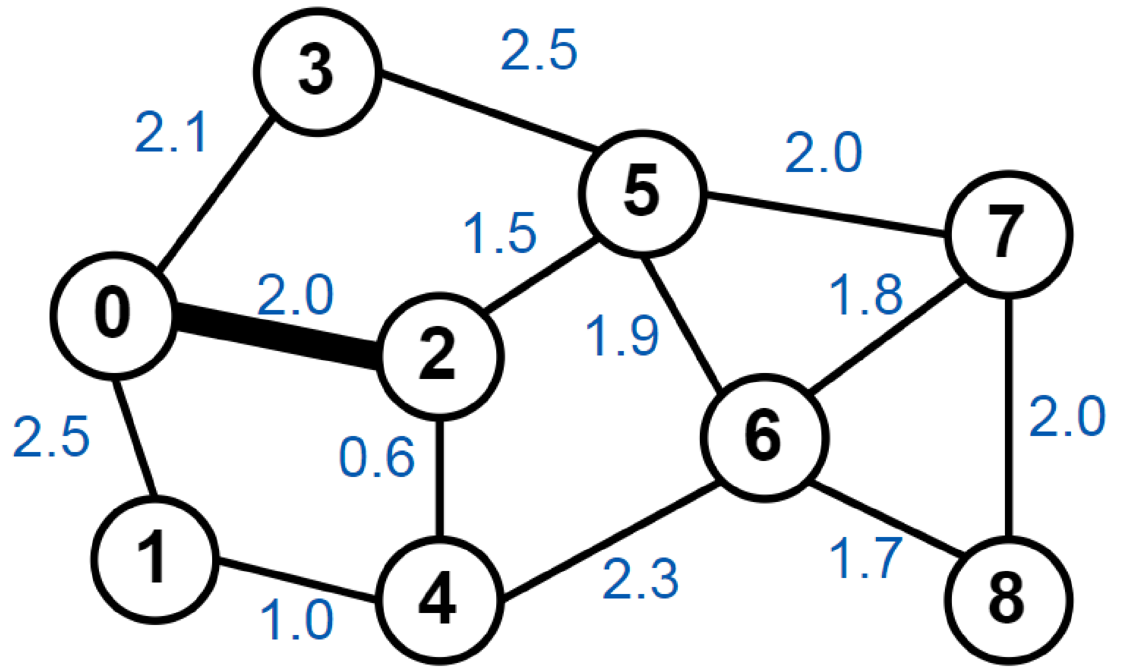



Find the length of a shortest path between *a* and *z* in the given weighted graph.

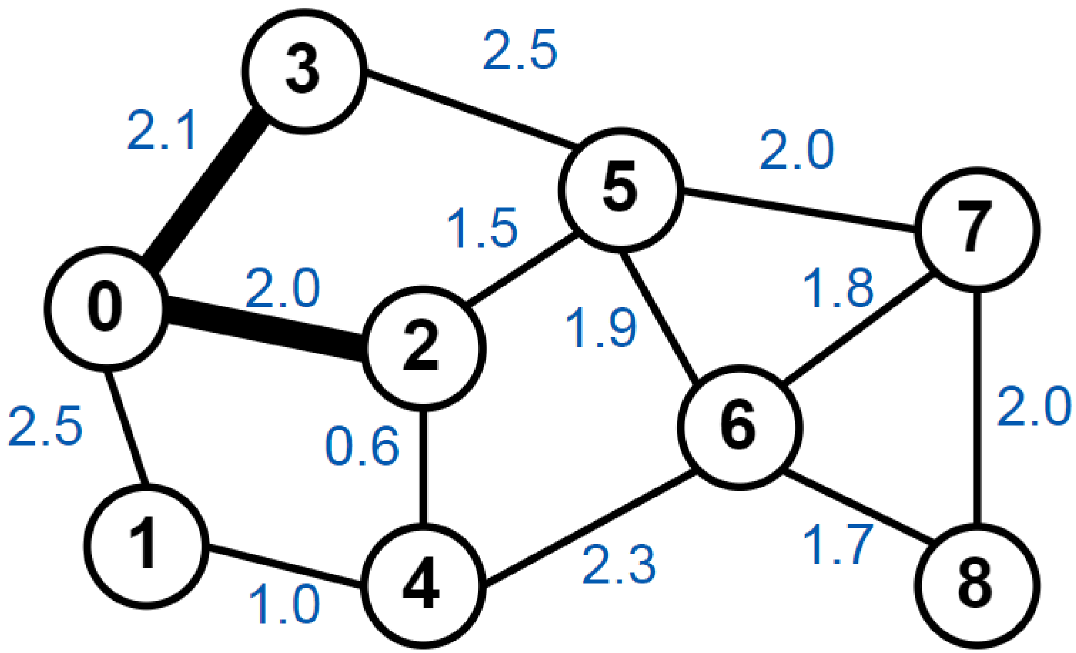




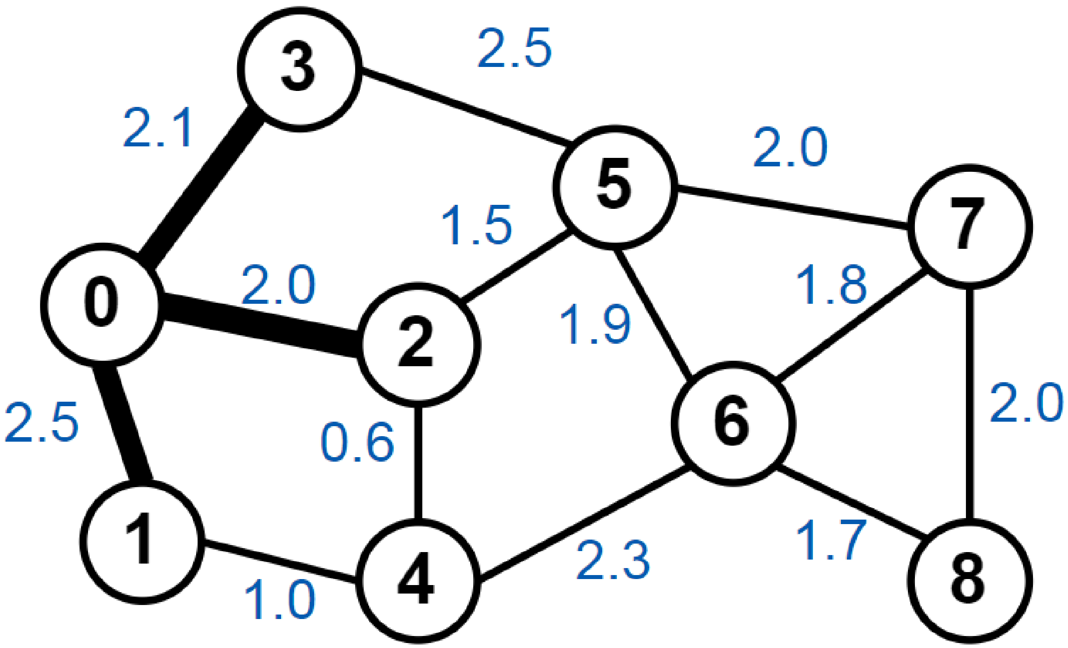

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **0** | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ |




| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **0** | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ |
| – | $(2.5, 0)$ | $\mathbf{(2.0, 0)}$ | $(2.1, 0)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ |




| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **0** | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) |
| – | (2.5, 0) | **(2.0, 0)** | (2.1, 0) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) |
| – | (2.5, 0) | – | **(2.1, 0)** | (2.6, 2) | (3.5, 2) | (∞,–) | (∞,–) | (∞,–) |




| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **0** | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) |
| – | (2.5, 0) | **(2.0, 0)** | (2.1, 0) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) |
| – | (2.5, 0) | – | **(2.1, 0)** | (2.6, 2) | (3.5, 2) | (∞,–) | (∞,–) | (∞,–) |
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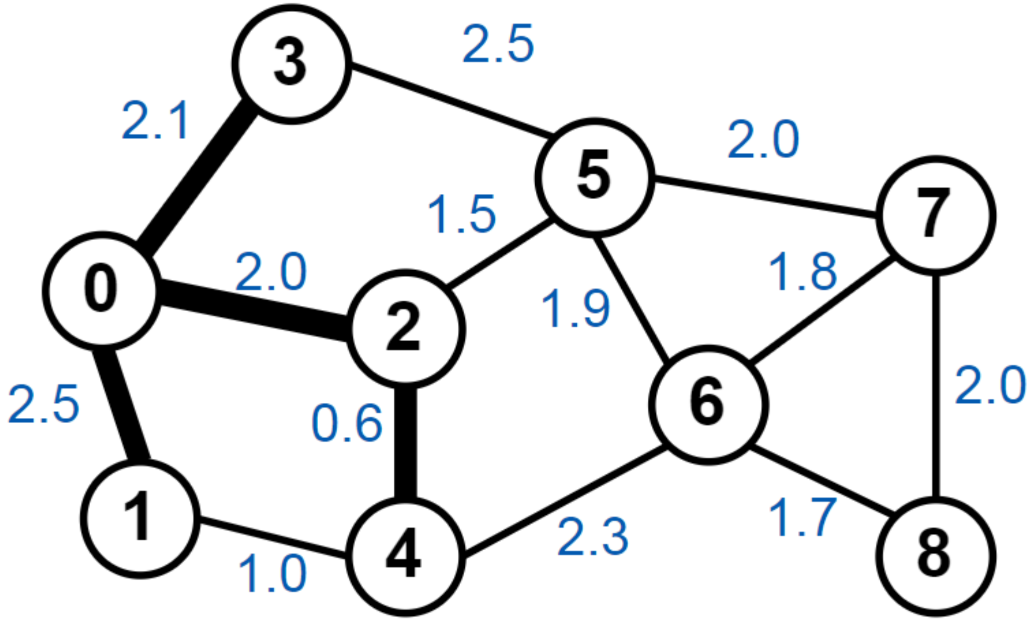


| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
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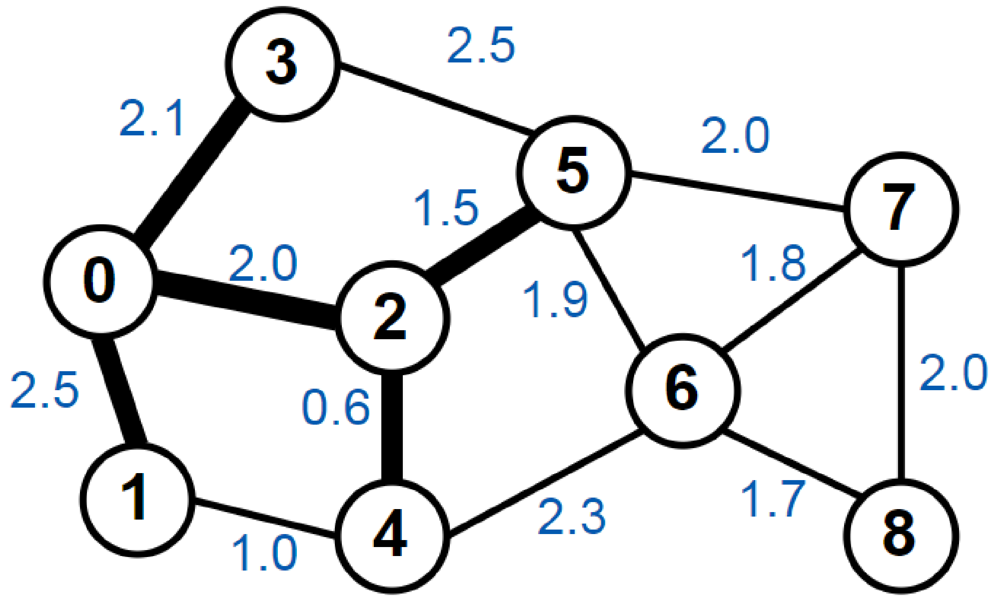
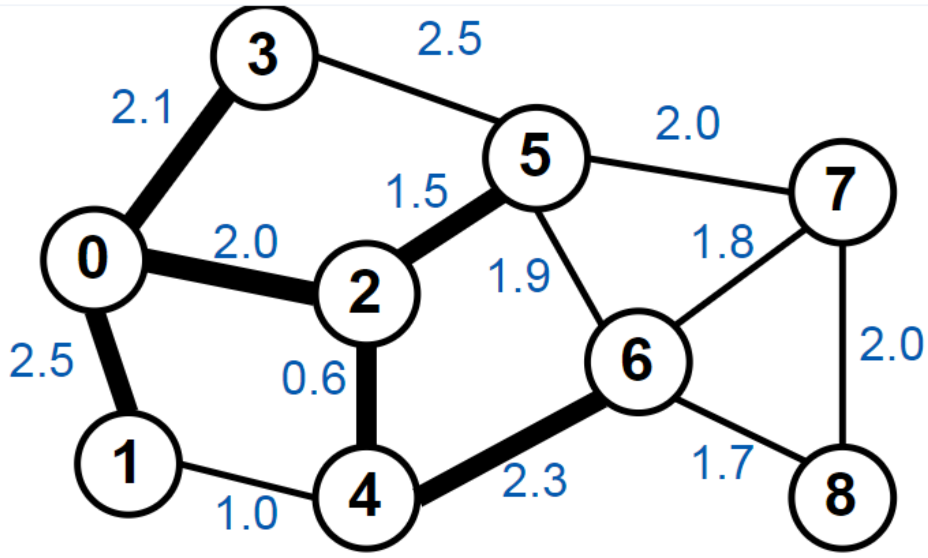
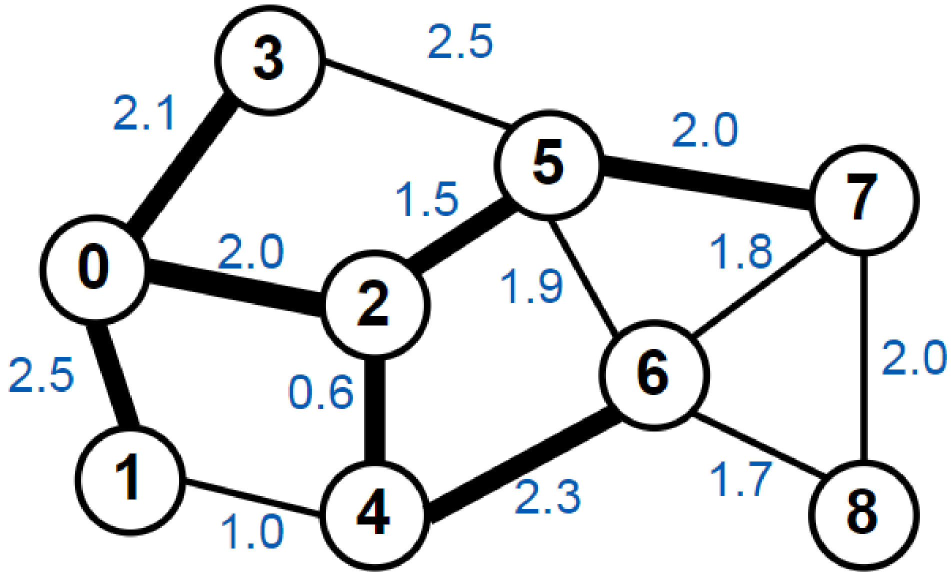
| – | (2.5, 0) | – | **(2.1, 0)** | (2.6, 2) | (3.5, 2) | $(\infty,-)$ | $(\infty,-)$ | $(\infty,-)$ |
| – | **(2.5, 0)** | – | – | (2.6, 2) | (3.5, 2) | $(\infty,-)$ | $(\infty,-)$ | $(\infty,-)$ |
| – | – | – | – | **(2.6, 2)** | (3.5, 2) | $(\infty,-)$ | $(\infty,-)$ | $(\infty,-)$ |



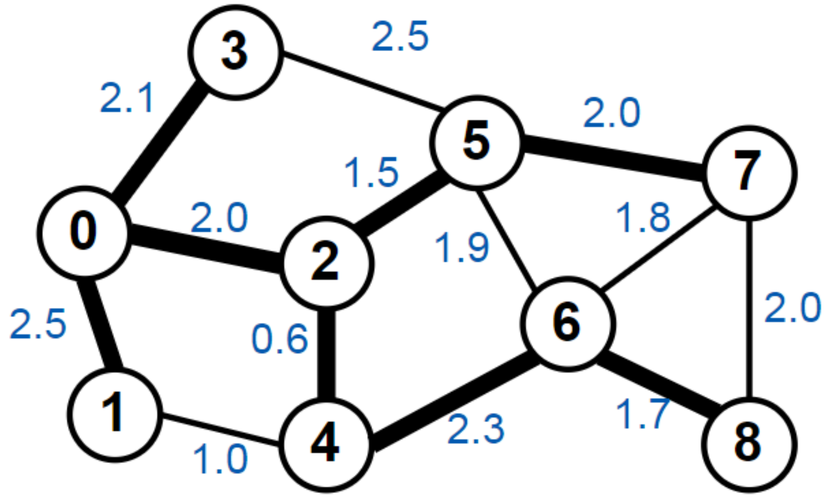

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **0** | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ |
| – | (2.5, 0) | **(2.0, 0)** | (2.1, 0) | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ |
| – | (2.5, 0) | – | **(2.1, 0)** | (2.6, 2) | (3.5, 2) | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ |
| – | **(2.5, 0)** | – | – | (2.6, 2) | (3.5, 2) | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ |
| – | – | – | – | **(2.6, 2)** | (3.5, 2) | $(\infty, -)$ | $(\infty, -)$ | $(\infty, -)$ |
| – | – | – | – | – | **(3.5, 2)** | (4.9, 4) | $(\infty, -)$ | $(\infty, -)$ |




| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **0** | (∞,−) | (∞,−) | (∞,−) | (∞,−) | (∞,−) | (∞,−) | (∞,−) | (∞,−) |
| − | (2.5, 0) | **(2.0, 0)** | (2.1, 0) | (∞,−) | (∞,−) | (∞,−) | (∞,−) | (∞,−) |
| − | (2.5, 0) | − | **(2.1, 0)** | (2.6, 2) | (3.5, 2) | (∞,−) | (∞,−) | (∞,−) |
| − | **(2.5, 0)** | − | − | (2.6, 2) | (3.5, 2) | (∞,−) | (∞,−) | (∞,−) |
| − | − | − | − | **(2.6, 2)** | (3.5, 2) | (∞,−) | (∞,−) | (∞,−) |
| − | − | − | − | − | **(3.5, 2)** | (4.9, 4) | (∞,−) | (∞,−) |
| − | − | − | − | − | − | **(4.9, 4)** | (5.5, 5) | (∞,−) |



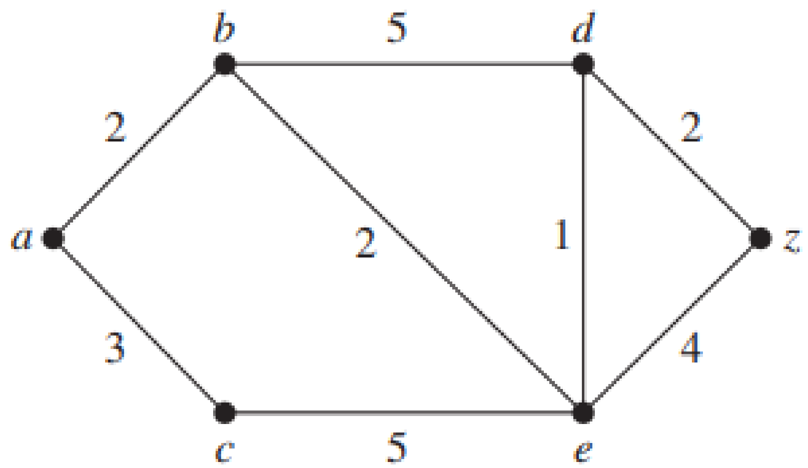

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **0** | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) |
| – | (2.5, 0) | **(2.0, 0)** | (2.1, 0) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) |
| – | (2.5, 0) | – | **(2.1, 0)** | (2.6, 2) | (3.5, 2) | (∞,–) | (∞,–) | (∞,–) |
| – | **(2.5, 0)** | – | – | (2.6, 2) | (3.5, 2) | (∞,–) | (∞,–) | (∞,–) |
| – | – | – | – | **(2.6, 2)** | (3.5, 2) | (∞,–) | (∞,–) | (∞,–) |
| – | – | – | – | – | **(3.5, 2)** | (4.9, 4) | (∞,–) | (∞,–) |
| – | – | – | – | – | – | **(4.9, 4)** | (5.5, 5) | (∞,–) |
| – | – | – | – | – | – | – | **(5.5, 5)** | (6.6, 6) |



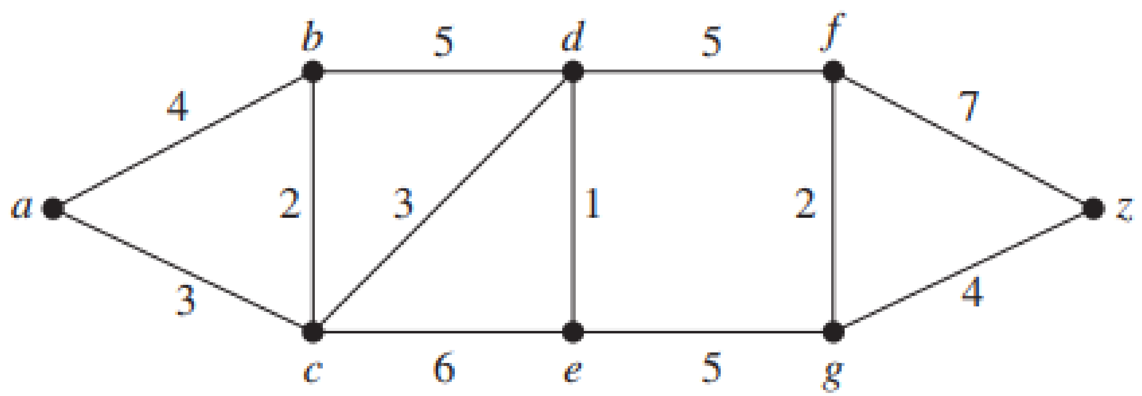

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **0** | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) |
| – | (2.5, 0) | **(2.0, 0)** | (2.1, 0) | (∞,–) | (∞,–) | (∞,–) | (∞,–) | (∞,–) |
| – | (2.5, 0) | – | **(2.1, 0)** | (2.6, 2) | (3.5, 2) | (∞,–) | (∞,–) | (∞,–) |
| – | **(2.5, 0)** | – | – | (2.6, 2) | (3.5, 2) | (∞,–) | (∞,–) | (∞,–) |
| – | – | – | – | **(2.6, 2)** | (3.5, 2) | (∞,–) | (∞,–) | (∞,–) |
| – | – | – | – | – | **(3.5, 2)** | (4.9, 4) | (∞,–) | (∞,–) |
| – | – | – | – | – | – | **(4.9, 4)** | (5.5, 5) | (∞,–) |
| – | – | – | – | – | – | – | **(5.5, 5)** | (6.6, 6) |
| – | – | – | – | – | – | – | – | **(6.6, 6)** |

Find the length of a shortest path between *a* and *z* in the given weighted graph.

**2.**




**3.**



# The Traveling Salesman Problem

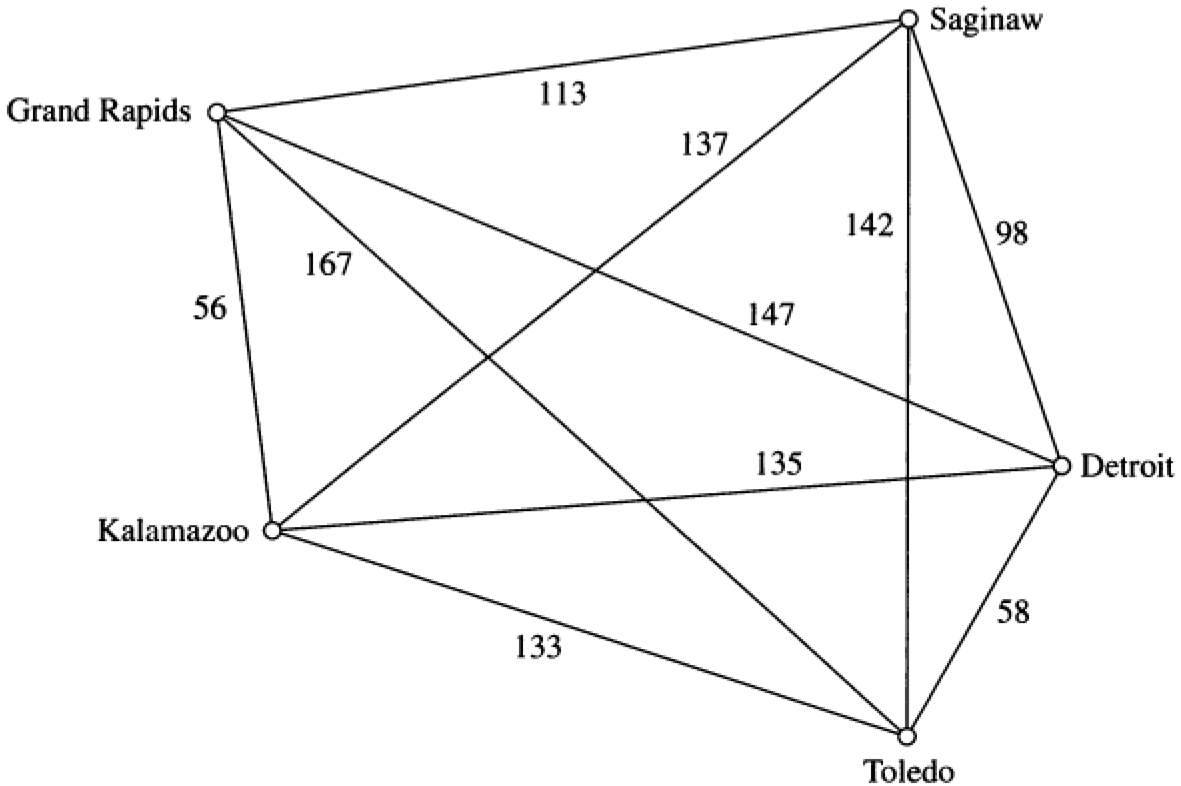



**FIGURE 5** **The Graph Showing the Distances between Five Cities.**

Salesman starts in one city (ex. Detroit). He wants to visit n cities exactly once and return to his starting point (Detroit). In which order should he visit theses cities to travel the minimum total distance ?

# The Traveling Salesman Problem

| *Route* | *Total Distance (miles)* |
|---|---|
| Detroit–Toledo–Grand Rapids–Saginaw–Kalamazoo–Detroit | 610 |
| Detroit–Toledo–Grand Rapids–Kalamazoo–Saginaw–Detroit | 516 |
| Detroit–Toledo–Kalamazoo–Saginaw–Grand Rapids–Detroit | 588 |
| Detroit–Toledo–Kalamazoo–Grand Rapids–Saginaw–Detroit | 458 |
| Detroit–Toledo–Saginaw–Kalamazoo–Grand Rapids–Detroit | 540 |
| Detroit–Toledo–Saginaw–Grand Rapids–Kalamazoo–Detroit | 504 |
| Detroit–Saginaw–Toledo–Grand Rapids–Kalamazoo–Detroit | 598 |
| Detroit–Saginaw–Toledo–Kalamazoo–Grand Rapids–Detroit | 576 |
| Detroit–Saginaw–Kalamazoo–Toledo–Grand Rapids–Detroit | 682 |
| Detroit–Saginaw–Grand Rapids–Toledo–Kalamazoo–Detroit | 646 |
| Detroit–Grand Rapids–Saginaw–Toledo–Kalamazoo–Detroit | 670 |
| Detroit–Grand Rapids–Toledo–Saginaw–Kalamazoo–Detroit | 728 |

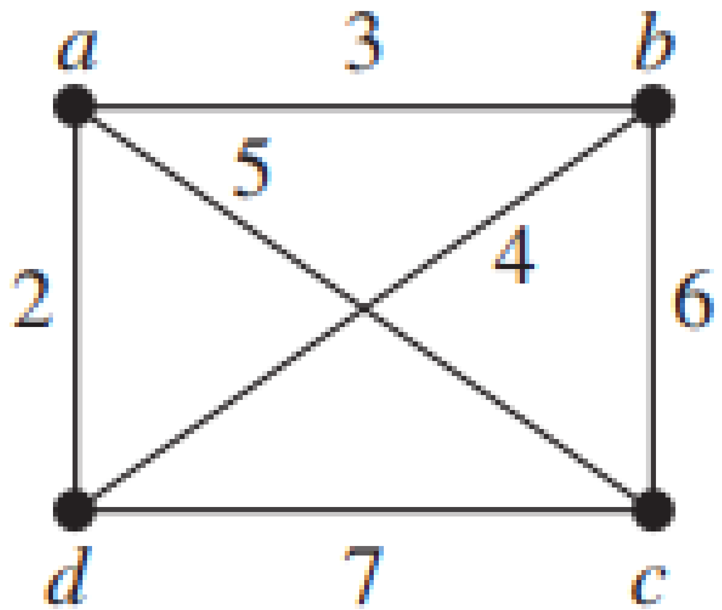
-The problem is equivalent to asking for a Hamilton circuit with minimum total weight .

- How many way do we have to examine to solve the problem if there are n vertex in the graph? ➔ **Exhaustive search technique**

- (n-1) (n-2) (n-2) … 3.2.1 = (n-1)! ➔ Complexity

-Approximation algorithm: W <= W' <= cW ( Part III—Graph Theory/Apps_Ch15.pdf)

Solve the traveling salesperson problem for this graph by finding the total weight of all Hamilton circuits and determining a circuit with minimum total weight.



# Summary

- 9.1- Graphs and Graph Models
- 9.2- Graph Terminology and Special Types of Graphs
- 9.3- Representing Graphs and Graph Isomorphism
- 9.4- Connectivity
- 9.5- Euler and Hamilton Paths
- 9.6- Shortest Path Problems