Chapter 2: Probability

Identify sample spaces and events, and use basic rules to calculate the probability of an event

Chapter 2: Probability

LEARNING OBJECTIVES

1. Sample Spaces and Events
2. Interpretations of Probability
3. Addition Rules
4. Conditional Probability
5. Multiplication and Total Probability Rules
6. Independence
7. Bayes’ Theorem

Sample Spaces

Definition Random experiment

- An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment.**
- The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S.
- An **event** is a subset of the sample space of a random experiment.

Sample spaces

Example

Random experiment: Roll a die

Sample space: $S = \{1, 2, 3, 4, 5, 6\}$

Event: $E_1 = \{\text{Die is even}\} = \{2, 4, 6\}$

$E_2 = \{\text{Die is odd}\} = \{1, 3, 5\}$

Sample spaces

Tree Diagrams

Sample spaces can also be described graphically with **tree diagrams.**

- When a sample space can be constructed in several steps or stages, we can represent each of the n_1 ways of completing the first step as a branch of a tree.
- Each of the ways of completing the second step can be represented as n_2 branches starting from the ends of the original branches, and so forth.

Sample spaces

Example

A probability experiment consists of tossing a coin and then rolling a six-sided die. Describe the sample space.

Example

Each message in a digital communication system is classified as to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.

Sample spaces

Basic Set Operations

The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.

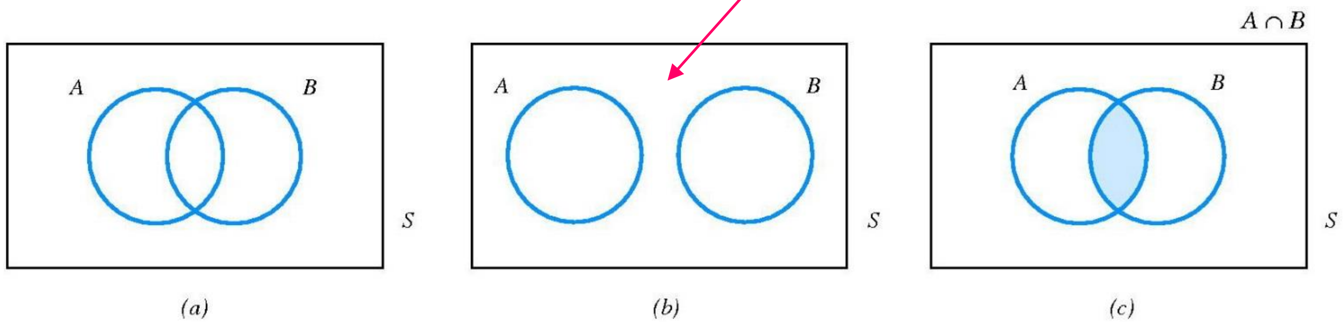
The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.

The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event E as E'.

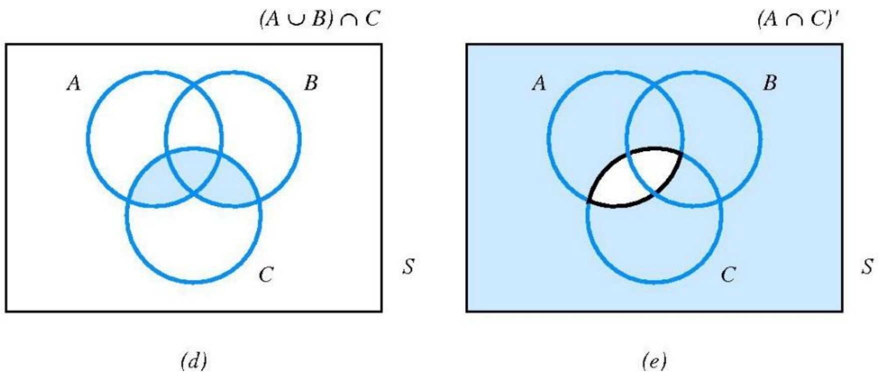
Sample spaces

Venn Diagrams

mutually exclusive



Sample space S with events A and B



Sample spaces

Important properties:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

$$A = (A \cap B) \cup (A \cap B')$$

Probability

Introduction

There are three approaches to assessing the probability of an uncertain event:

1. **a priori classical probability**: the probability of an event is based on prior knowledge of the process involved.

2. **empirical classical probability**: the probability of an event is based on observed data.

3. **subjective probability**: the probability of an event is determined by an individual, based on that person's past experience, personal opinion, and/or analysis of a particular situation.

Probability

Equally Likely Outcomes

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

1. **a priori classical probability**

$$\text{Probability of Occurrence} = \frac{X}{T} = \frac{\text{number of ways the event can occur}}{\text{total number of possible outcomes}}$$

2. **empirical classical probability**

$$\text{Probability of Occurrence} = \frac{\text{number of favorable outcomes observed}}{\text{total number of outcomes observed}}$$

Probability

Example	priori classical probability
Find the probability of selecting a face card (Jack, Queen, or King) from a standard deck of 52 cards.	

Probability

Example empirical classical probability

Find the probability of selecting a male taking statistics from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

Probability

Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \le P(E) \le 1$

(3) For two events E_1 and E_2 with $E_1 \cap E_2 = \varnothing$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Addition Rules

The special addition rule

1. If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

2. A collection of events, $E_1, E_2, \ldots, E_k$ is said to be **mutually exclusive** if for all pairs,

$$E_i \cap E_j = \varnothing$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k)$$

Addition Rules

The general addition rule

1. Two events: *A* and *B* are any events,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2. Three events:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

Addition Rules

Example

Find the probability of selecting a male or a statistics student from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

Example

John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company *A* is 0.8, and his probability of getting an offer from company *B* is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

Example

What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

Example

If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colors?

Example

If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

Conditional Pro.

To introduce conditional probability, consider an example involving manufactured parts.

Let D denote the event that a part is defective and let F denote the event that a part has a surface flaw.

Then, we denote the probability of D given, or assuming, that a part has a surface flaw as $P(D|F)$. This notation is read as the **conditional probability** of D given F, and it is interpreted as the probability that a part is defective, given that the part has a surface flaw.

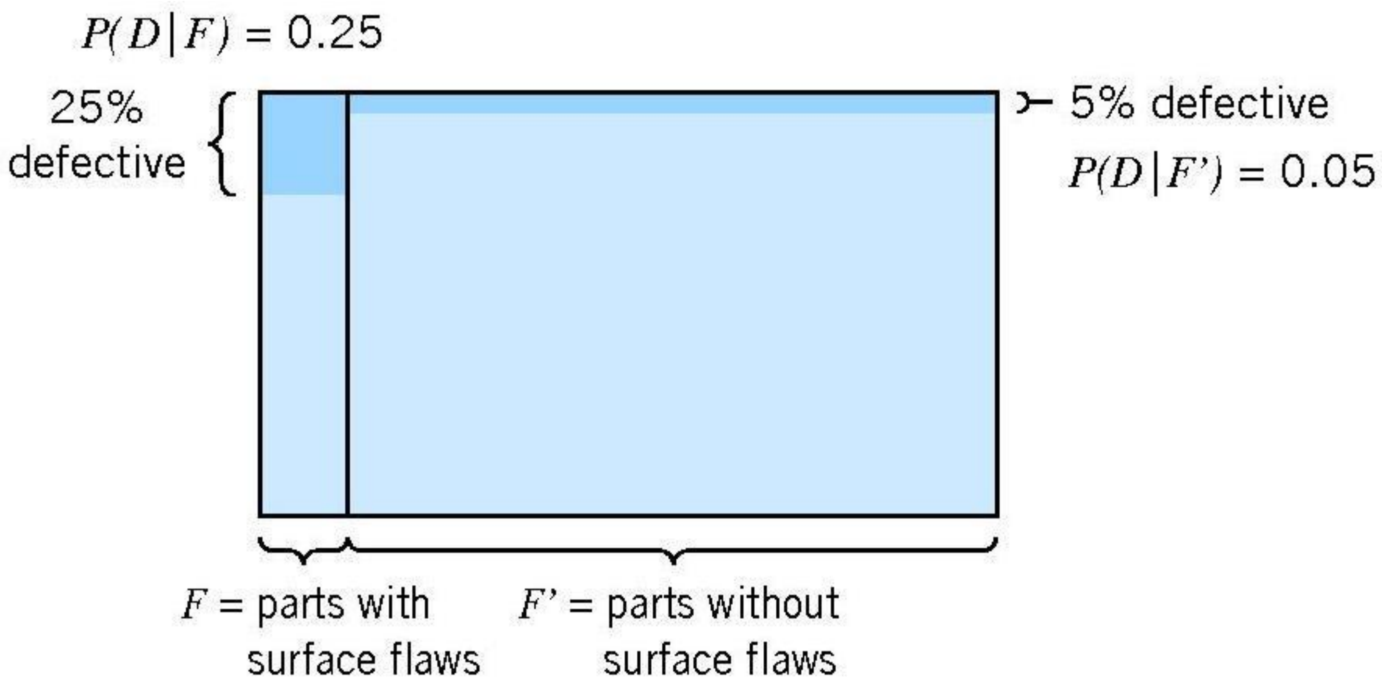
Conditional Pro.

Example 2-17(page 32)

TABLE 2.2 Parts Classified

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	28
	No	30	342	372
	Total	40	360	400

Conditional Pro.



Conditional Pro.

Definition

The conditional probability of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = P(B \cap A)/P(A)$$

for $P(A) > 0$.

Special case: all outcomes are equally likely

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

Conditional Pro.

Example

TABLE 2.2 **Parts Classified**

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	28
	No	30	342	372
	Total	40	360	400

Conditional Pro.

Example

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both.

What is the probability that a car has a CD player, given that it has AC?

Conditional Pro.

Example

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane

(a) arrives on time, given that it departed on time,

(b) departed on time, given that it has arrived on time.

Conditional Pro.

Example

Consider an industrial process in the textile industry in which strips of a particular type of cloth are being produced. These strips can be defective in two ways, length and nature of texture. For the case of the latter, the process of identification is very complicated. It is known from historical information on the process that 10% of strips fail the length test, 5% fail the texture test, and only 0.8% fail both tests. If a strip is selected randomly from the process and a quick measurement identifies it as failing the length test, what is the probability that it is texture defective?

Multiplication Rule

Theorem 2.10:

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

Theorem 2.12:

If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k)$$

$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

Multiplication Rule

Example

The probability that an automobile battery subject to high engine compartment temperature suffers low charging current is 0.7. The probability that a battery is subject to high engine compartment temperature is 0.05.

The probability that a battery is subject to low charging current and high engine compartment temperature is

$$P(C \cap T) = P(C|T)P(T) = 0.7 \times 0.05 = 0.035$$

C={a battery suffers low charging current}

T ={a battery is subject to high engine compartment temperature}

Multiplication Rule

Example

The probability that the first stage of a numerically controlled machining operation for high-rpm pistons meets specifications is 0.90. Failures are due to metal variations, fixture alignment, cutting blade condition, vibration, and ambient environmental conditions. Given that the first stage meets specifications, the probability that a second stage of machining meets specifications is 0.95. What is the probability that both stages meet specifications?

Example

Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Example

One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

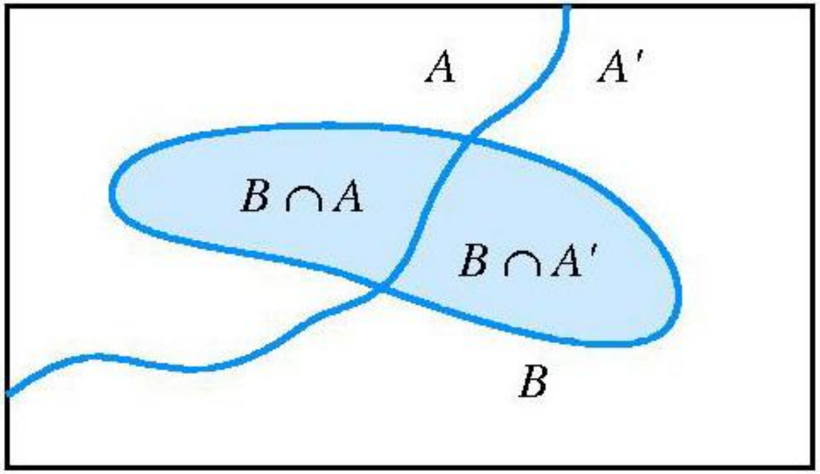
Example

Three cards are drawn in succession, without replacement, from an ordinary deck of playing cards. Find the probability that the event $A_1 \cap A_2 \cap A_3$ occurs, where A_1 is the event that the first card is a red ace, A_2 is the event that the second card is a 10 or a jack, and A_3 is the event that the third card is greater than 3 but less than 7.

Total Pro. Rule

Partition of an event

Figure 2-15
Partitioning an event into two mutually exclusive subsets.



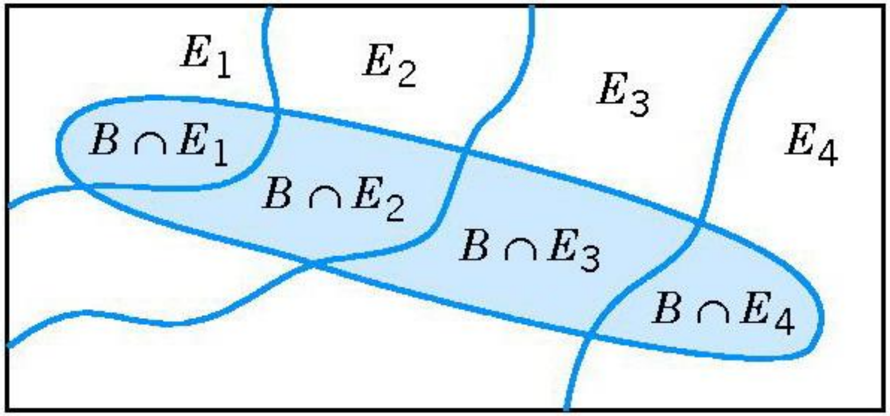


Figure 2-16
Partitioning an event into several mutually exclusive subsets.

$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Total Pro. Rule

Total Probability Rule: two events

$$P(B) = P(BA) + P(BA') = P(A)P(B|A) + P(A')P(B|A')$$

Total Probability Rule: multiple events

$$E_1 \cup E_2 \cup \ldots E_k = S$$

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k) \end{aligned}$$

Example. Semiconductor Contamination

Consider the contamination discussion at the start of this section. The information is summarized here

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not high	0.8

Let *F* denote the event that the product fails, and let *H* denote the event that the chip is exposed to high levels of contamination.

Example

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Independence

Definition

Two events is called independent if any one of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

(2) $P(A \cap B) = P(A)P(B)$

(3) $P(B|A) = P(B)$

Proposition: If A and B are independent events, then so are events A and B’, events A’ and B, and events A’ and B’.

Independence

Example 2-31 (page 52)

A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected at random, without replacement, from the batch. Let A = {the first part is defective}, and let B = {the second part is defective}.

Independence

Definition

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$

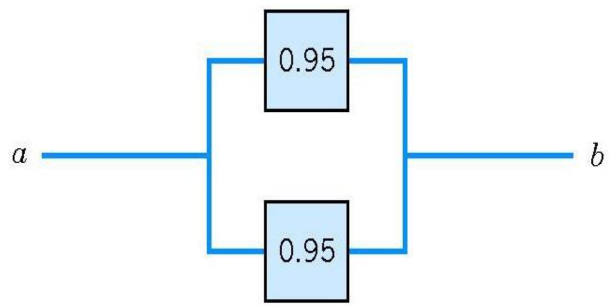
$$P(E_{i_1} \cap E_{i_2} \cap \ldots \cap E_{i_k}) = P(E_{i_1})P(E_{i_2})\ldots P(E_{i_k})$$

Exercise

Two coins are tossed. Let A denote the event "at most one head on the two tosses," and let B denote the event "one head and one tail in both tosses." Are A and B independent events?

Example 2-24 Independence

The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



Example

A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Bayes' Theorem

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$\boxed{P(E_1 \mid B) = \frac{P(B \mid E_1)P(E_1)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \ldots + P(B \mid E_k)P(E_k)} \quad \text{for } P(B) > 0}$$

In special case:

$$\boxed{P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \qquad \text{for } P(B) > 0}$$

Example

Suppose that $P(A \mid B) = 0.7$, $P(A) = 0.5$, and $P(B) = 0.2$.

Determine $P(B \mid A)$.

Example

In a state where cars have to be tested for the emission of pollutants, 25% of all cars emit excessive amount of pollutants. When tested, 99% of all cars that emit excessive amount of pollutants will fail, but 17% of all cars that do not emit excessive amount of pollutants will also fail. What is the probability that a car that fails the test actually emits excessive amounts of pollutants?

Example

Two events A and B are such that $P[A \cap B] = 0.15$, $P[A \cup B] = 0.65$, and $P[A|B] = 0.5$. Find $P[B|A]$.

Example

In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. If a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

Bayes' Theorem

Example

A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$P(D|P_1) = 0.01,\ P(D|P_2) = 0.03,\ P(D|P_3) = 0.02,$$

where $P(D|P_j)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

Bayes' Theorem

Example

Customers are used to evaluate preliminary product designs. In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products.

a. What is the probability that a product attains a good review?

b. If a new design attains a good review, what is the probability that it will be a highly successful product?

c. If a product does not attain a good review, what is the probability that it will be a highly successful product?