Chapter 7

Point Estimation of Parameters

Chapter 7: Point Estimation of Parameters

Learning objectives

1. Introduction
2. Sampling Distributions and the Central Limit Theorem

1.Introduction

Introduction

Let X is a random variable with probability distribution $f(x)$, which is characterized by the unknown $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$

For example, $X \sim N(\mu, \sigma^2)$ then $\theta = (\mu, \sigma^2)$.

How to "determine" the values of θ?

PROBLEMS IN ENGINEERING

In engineering, we often need to estimate

1. The mean μ of a single population.
2. The variance σ^2 (or standard deviation) of a single population.
3. The proportion p of items in a population that belong to a class of interest.
4. The difference in means of two populations, $\mu_1 - \mu_2$.
5. The difference in two population proportions, $p_1 - p_2$.

PROBLEMS IN ENGINEERING

The important results on point estimation

1. For μ , the estimate is the $\hat{\mu} = \bar{x}$, sample mean.
2. For σ^2 , the estimate is $\hat{\sigma}^2 = s^2$, the sample variance.
3. For p, the estimate is $\hat{p} = x/n$, the sample proportion.
4. For $\mu_1 - \mu_2$, the estimate is $\hat{\mu}_1 - \hat{\mu}_2 = \bar{x}_1 - \bar{x}_2$
5. For $p_1 - p_2$, the estimate is $\hat{p}_1 - \hat{p}_2$

2. Sampling Distributions and the Central Limit Theorem

SAMPLING DISTRIBUTIONS

Definition **Random Sample**

The random variables $X_1, \ldots, X_n$ are called a random sample of size n if

- The X_i's are independent
- Every X_i has the same probability distribution

Definition **Statistic**

- A statistic $\hat{\Theta}$ is any function of the observations $X_1, \ldots, X_n$:

$$\hat{\Theta} = h(X_1, \ldots, X_n)$$

- The probability distribution of a statistic is called a sampling distribution.

SAMPLING DISTRIBUTIONS

Example — Two important statistic

- Sample mean $\overline{X}$

$$\overline{X} = \frac{X_1 + \ldots + X_n}{n}$$

- Sample variance S^2

$$S^2 = \frac{(X_1 - \overline{X})^2 + \ldots + (X_n - \overline{X})^2}{n-1}$$

Sampling Distributions

Theorem

In above example, if $(X_1, \ldots, X_n)$ is a random sample of size n take from a normal distribution $N(\mu, \sigma^2)$ then

- $\bar{X}$ has a normal distribution $N(\mu, \sigma^2/n)$
- $\dfrac{(n-1)S^2}{\sigma^2}$ has a chi-square distribution with n-1 degrees of freedom (see pages 273-274).

Sampling Distributions

Theorem Central Limit Theorem

Let (X_1, ..., X_n) is a random sample of size n take from a population with mean μ and finite variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

As $n \to \infty$, is the standard normal distribution.

Remark: The normal approximation for $\overline{X}$ depends on the sample size n.

Sampling Distributions

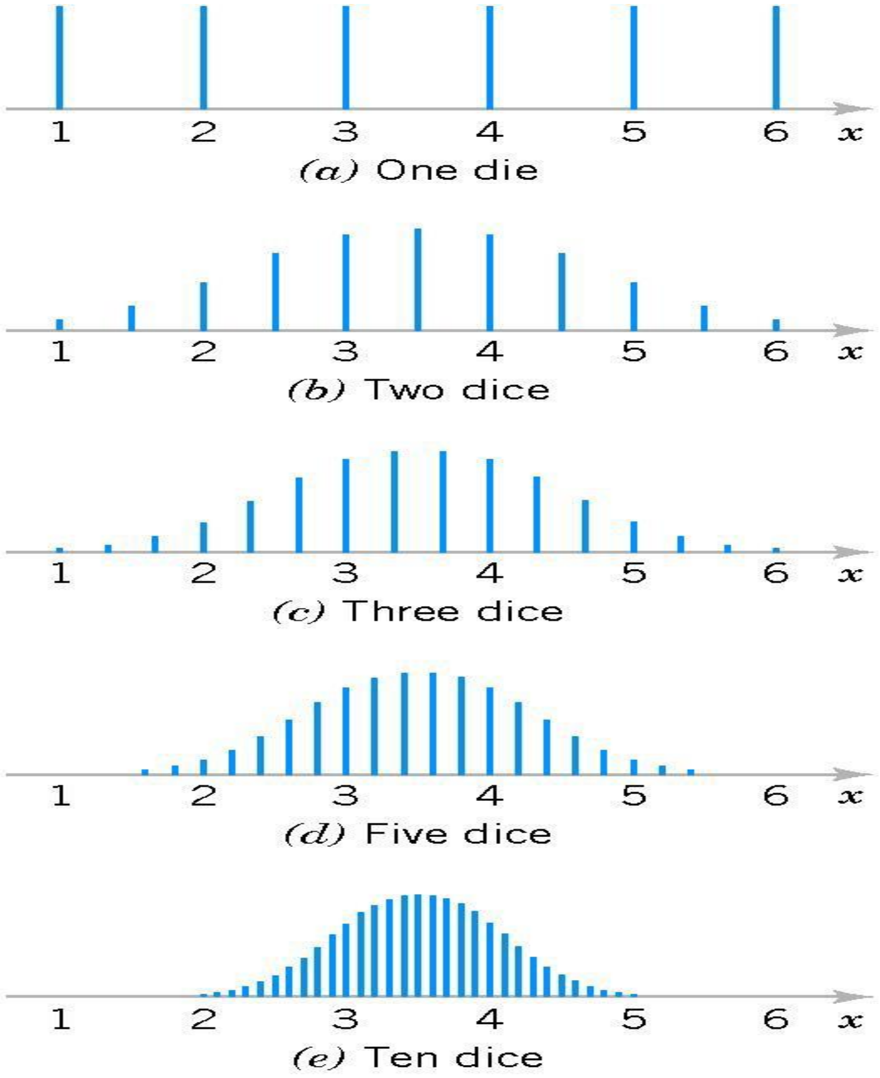

Figure 7-1 Distributions of average scores from throwing dice.

Sampling Distributions

Definition **Point Estimate**

- A **point estimate** of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$.
- The statistic $\hat{\Theta}$ is called the **point estimator**.

Two steps to find point estimation:

Step 1. Determine $\hat{\Theta}$ by using the theoretical results.

Steps 2. Calculate $\hat{\theta}$ from the experimental data.

Sampling Distributions

Example

Market researchers use the number of sentences per advertisement as a measure of readability for magazine advertisements. The following represents a random sample of the number of sentences found in 15 advertisements. Find a point estimate of the population mean, μ.

9 20 18 16 9 9 11 13 22 16 5 18 6 6 5

Step 1. $\hat{\Theta} = \dfrac{X_1 + \dots + X_{15}}{15}$

Step 2. A point estimate for μ is

$$\hat{\mu} = \frac{9+20+\dots+6+5}{15} = \frac{183}{15} = 12.2$$

Exercises 3

PVC pipe is manufactured with a mean diameter of 1.01 inch and a standard deviation of 0.003 inch. Find the probability that a random sample of *n* = 9 sections of pipe will have a sample mean diameter greater than 1.009 inch and less than 1.012 inch.

$\mu = 1{,}01 \quad \sigma = 0{,}003$

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0{,}003}{3} = 0{,}001$

$z_1 = \frac{x - \mu}{\sigma} = -1 \; ; \; z_2 = 2$

$P(z < 2) - P(z < -1) = 0{,}81859$

Exercises 4

Suppose that samples of size n = 25 are selected at random from a normal population with mean 100 and standard deviation 10. What is the probability that the sample mean falls in the interval from

$$\mu_{\overline{X}} - 1.8\sigma_{\overline{X}} \text{ to } \mu_{\overline{X}} + 1.0\sigma_{\overline{X}}$$

$Z = \frac{X - \mu}{\sigma_{\bar{x}}} = \frac{X - \mu}{\sigma / \sqrt{n}}$

$\mu = 100$

$n = 25$

$\sigma = 10$

$X \sim N(\mu, \sigma^2) \Rightarrow \overline{X} \sim N(100, 4)$

$\mu = 100$

$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{10}{5} = 2$

$\mu_{\bar{x}} - 1{,}8\sigma_{\bar{x}} = 96{,}4$

$\mu_{\bar{x}} + \sigma_{\bar{x}} = 101{,}8$

$P(X < 101{,}8) - P(X < 96{,}4)$

$= P\left(Z < \frac{101{,}8 - 100}{2}\right) - P\left(Z < \frac{96{,}4 - 100}{2}\right)$

$= 0{,}78$

Exercises 5

A synthetic fiber used in manufacturing carpet has tensile strength that is normally distributed with mean 75.5 psi and standard deviation 3.5 psi. Find the probability that a random sample of *n* = 6 fiber specimens will have sample mean tensile strength that exceeds $\geq$ 75.75 psi.

$\mu = 75,5$

$\sigma = 3,5$

$n = 6$

$$P(X \geq 75,75) = 1 - P(X < 75,75)$$

$$= 1 - P\left(Z < \frac{75,75 - 75,5}{3,5/\sqrt{6}}\right)$$

$$= 0,431$$

$\mu_{\bar{x}} = \mu_x$

$\sigma_{\bar{x}} = \dfrac{\sigma_x}{\sqrt{n}}$

Exercises 10

Suppose that the random variable X has the continuous uniform distribution

$$f(x)=\begin{cases}1, & 0\le x\le 1\\ 0, & x\notin[0,1]\end{cases}$$

Suppose that a random sample of n = 12 observations is selected from this distribution. What is the approximate probability distribution of $\overline{X}-6$ Find the mean and variance of this quantity.

$\mu_x = \frac{a+b}{2} = \frac{1}{2}$

$\sigma_x = \frac{(b-a)^2}{12} = \frac{1}{12}$

$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}} = \frac{1}{12} \cdot \frac{1}{12} = \frac{1}{144}$

7.2.5 **WP** A normal population has mean 100 and variance 25. How large must the random sample be if you want the standard error of the sample average to be 1.5?

$\mu = 100$

$\sigma_x^2 = 25$

$\sigma_x = 5$.

$\sigma_{\bar{x}} = 1,5 = \frac{\sigma_x}{\sqrt{n}}$

$1,5 = \frac{5}{\sqrt{n}}$

$n = \left(\frac{5}{1,5}\right)^2 = 12$?

7.2.6 WP VS The amount of time that a customer spends waiting at an airport check-in counter is a random variable with mean 8.2 minutes and standard deviation 1.5 minutes. Suppose that a random sample of $n = 49$ customers is observed. Find the probability that the average time waiting in line for these customers is

a. Less than 10 minutes **b.** Between 5 and 10 minutes

c. Less than 6 minutes

c) $\mu = 8{,}2$

$\sigma = 1{,}5$

$n = 49$

$P\left(Z < \dfrac{6 - 8{,}2}{1{,}5/7}\right)$

$= P(Z < A)$

$= 0$

7.2.7 WP SS VS A random sample of size $n_1 = 16$ is selected from a normal population with a mean of 75 and a standard deviation of 8. A second random sample of size $n_2 = 9$ is taken from another normal population with mean 70 and standard deviation 12. Let $\overline{X}_1$ and $\overline{X}_2$ be the two sample means. Find:

a. The probability that $\overline{X}_1 - \overline{X}_2$ exceeds 4

b. The probability that $3.5 \le \overline{X}_1 - \overline{X}_2 \le 5.5$

b) $\overline{X}_1 - \overline{X}_2 \sim N(5, 20)$

$P\left(Z \le \frac{5,5 - 5}{\sqrt{20}}\right) - P\left(Z \le \frac{3,5 - 5}{\sqrt{20}}\right)$

$= 0,17585$

7.2.9 Data on the pH of rain in Ingham County, Michigan, are as follows:

5.47	5.37	5.38	4.63	5.37	3.74	3.71	4.96	4.64	5.11	5.65	5.39
4.16	5.62	4.57	4.64	5.48	4.57	4.57	4.51	4.86	4.56	4.61	4.32
3.98	5.70	4.15	3.98	5.65	3.10	5.04	4.62	4.51	4.34	4.16	4.64
5.12	3.71	4.64									

What proportion of the samples has pH below 5.0?

Exercises 11

Suppose that X has the continuous uniform distribution

$$f(x)=\begin{cases}1, & 0\le x\le 1\\ 0, & x\notin[0,1]\end{cases}$$

Suppose that a random sample of n = 12 observations is selected from this distribution. What is the approximate probability distribution of $\overline{X}-6$ Find the mean and variance of this quantity.