Chapter 4

Continuous Random Variables and Probability Distributions

Chapter 4: Continuous Random Variables and Probability Distributions

LEARNING OBJECTIVES

4.1. Continuous random variable

4.2. Probability Distributions and Probability density function

4.3. Cumulative distribution function

4.4. Mean and Variance

4.5. Continuous Uniform Distribution

4.6. Continuous Normal Distribution

4.7. Normal approximation to the Binomial and Poisson.

4.8. Exponential Distribution

4.1. Continuous random variable

CONTINUOUS RANDOM VARIABLE

Example

height of students in class

weight of students in class

time it takes to get to school

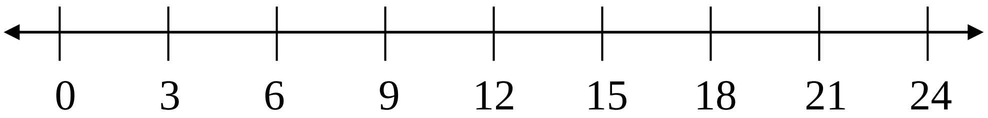
distance traveled between classes

CONTINUOUS RANDOM VARIABLE

Definition

A continuous random variable is a random variable whose possible values includes in an interval of real numbers.

Hours spent studying in a day



The time spent studying can be any number between 0 and 24.

4.2. Probability density function

PROBABILITY DENSITY FUNCTION

Definition

The **probability density function (pdf)** of a continuous random variable *X* is a function such that

(1) $f(x) \geq 0 \qquad \forall \ x$

(2) $\displaystyle\int_{-\infty}^{+\infty} f(x)dx = 1$

(3) $\displaystyle P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$ for any *a* and *b*.

PROBABILITY DENSITY FUNCTION (PDF)

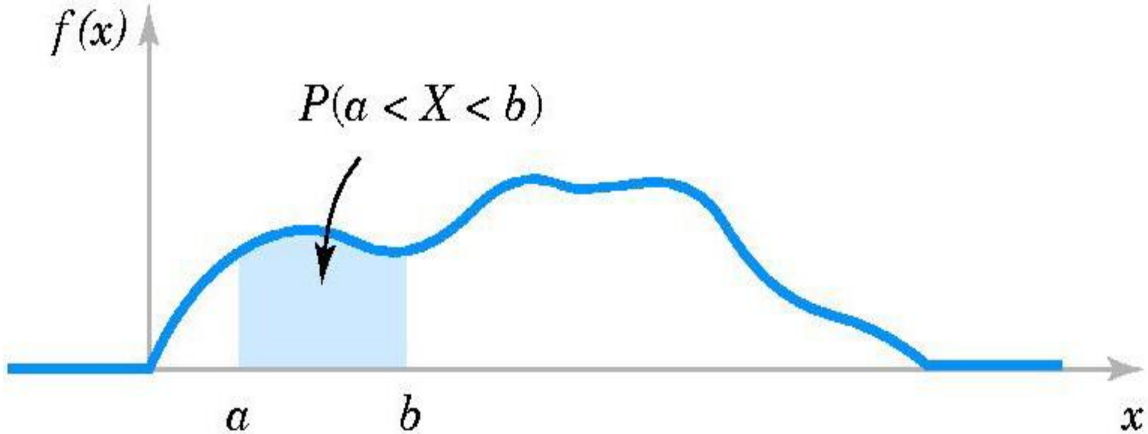


Figure 4-2 Probability determined from the area under $f(x)$.

Property

If X is a continuous random variable then for any x_1 and x_2 we have

$$\begin{aligned} P(x_1 \le X \le x_2) &= P(x_1 \le X < x_2) \\ &= P(x_1 < X \le x_2) \\ &= P(x_1 < X < x_2) \end{aligned}$$

PDF

Example

Suppose that the error in the reaction temperature, in ◦C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere \end{cases}$$

(a) Verify that $f(x)$ is a density function.

(b) Find $P(0 < X \leq 1)$.

PDF

Example

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function

$$f(x) = 20e^{-20(x-12.5)},\ x \geq 12.5$$

(a) If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped?

(b) What proportion of parts is between 12.5 and 12.6 millimeters?

Exercises 1 PDF

Suppose that $f(x) = e^{-x}$ for $0 < x$. Determine the following

a. $P(1 < X)$ **b.** $P(1 < X < 2.5)$ **c.** $P(X = 3)$

d. $P(X < 4)$ **e.** $P(3 \le X)$

f. x such that $P(x < X) = 0.10$

g. x such that $P(X \le x) = 0.10$

Exercises 2 PDF

Suppose that $f(x) = 0.5 \cos x$ for $-\pi/2 < x < \pi/2$.
Determine the following:

a. $P(X < 0)$ **b.** $P(X < -\pi/4)$

c. $P(-\pi/4 < X < \pi/4)$ **d.** $P(X > -\pi/4)$

e. x such that $P(X < x) = 0.95$

Exercises 3 PDF

The waiting time for service at a hospital emergency department (in hours) follows a distribution with probability density function

$$f(x) = 0.5\exp(-0.5x) \text{ for } 0 < x.$$

Determine the following:

a. $P(X < 0.5)$

b. $P(X > 2)$

c. Value x (in hours) exceeded with probability 0.05.

4.3. Cumulative distribution function

CUMULATIVE DISTRIBUTION FUNCTION (CDF)

Definition

The **cumulative distribution function (cdf)** of a continuous random variable *X* is

$$F(x) := \int_{-\infty}^{x} f(t)dt$$

for -∞ < x < +∞.

As an immediate consequence of Definition, one can write the two results

$$P(a < X < b) = F(b) - F(a) \text{ and } f(x) = \frac{dF(x)}{dx}$$

if the derivative exists.

Example

Let $f(x)=\begin{cases}0 & x<12.5\\ 20e^{-20(x-12.5)} & x\geq 12.5\end{cases}$. Find $F(x)$.

Solution: a) For $x\geq 12.5$

$$F(x)=\int_{-\infty}^{x} f(t)\,dt=\int_{12.5}^{x} 20e^{-20(t-12.5)}dt$$

$$=-e^{-20(t-12.5)}\Big|_{12.5}^{x}=1-e^{-20(x-12.5)}$$

Therefore,

$$F(x)=\begin{cases}0 & x<12.5\\ 1-e^{-20(x-12.5)} & x\geq 12.5\end{cases}$$

PDF

Example

Suppose that the error in the reaction temperature, in ◦C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere \end{cases}$$

a) Find $F(x)$,

b) And use it to evaluate $P(0 < X \leq 1)$.

Example

The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate *b*. The DOE has determined that the density function of the winning (low) bid is

$$f(y) = \begin{cases} \dfrac{5}{8b} & \dfrac{2}{5}b \le y \le 2b \\ 0 & \textit{elsewhere.} \end{cases}$$

Find $F(y)$ and use it to determine the probability that the winning bid is less than the DOE's preliminary estimate *b*.

Exercises 1

Suppose that the cumulative distribution function of the random variable X is

$$F(x)=\begin{cases}0 & x<0\\ 0.25x & 0\le x<5\\ 1 & 5\le x\end{cases}$$

Determine the following:

a. $P(X < 2.8)$ **b.** $P(X > 1.5)$

c. $P(X < -2)$ **d.** $P(X > 6)$

Exercises 2

Determine the cumulative distribution function for the distribution:

a) $f(x) = e^{-x}$ for $0 < x$.

b) $f(x) = 0.5 \cos x$ for $-\pi/2 < x < \pi/2$.

Exercises 3

Determine the probability density function for each of the following cumulative distribution functions.

a) $F(x) = 1 - e^{-2x}$ $x > 0$.

b)

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2x & 0 \le x < 4 \\ 0.04x + 0.64 & 4 \le x < 9 \\ 1 & 9 \le x \end{cases}$$

4.4. Mean and Variance

Mean & Variance

Definition

Suppose X is a continuous random variable with probability density function $f(x)$.

The **mean** or **expected value** of X is defined by

$$\mu = E(X) := \int_{-\infty}^{+\infty} xf(x)dx$$

The **variance** of X is defined by

$$\sigma^2 = V(X) := \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2$$

The **standard deviation** of X is σ

Mean & Variance

Example

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \dfrac{20000}{x^3}, & x > 100 \\ 0, & elsewhere. \end{cases}$$

Find the expected life of this type of device.

We have

$$E(X) = \int_{100}^{\infty} x \frac{20000}{x^3} dx = 200$$

Therefore, we can expect this type of device to last, *on average*, 200h

MEAN AND VARIANCE

Example: The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable *X* having the probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2 \\ 0, & elsewhere. \end{cases}$$

Find the mean and variance of *X*.

Example MEAN AND VARIANCE

Assume that X is a continuous random variable with the following probability density function

$$f(x) = \begin{cases} 20e^{-20(x-12,5)} & x \geq 12.5 \\ 0 & x < 12.5 \end{cases}$$

Find the mean and variance of X.

MEAN AND VARIANCE

Expected Value of a Function of a Continuous Random Variable

$$E\,h(X) = \int_{-\infty}^{+\infty} h(x)\,f(x)\,dx$$

Example

Let X be a random variable with density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere. \end{cases}$$

Find the expected value of g(X) = 4X+3.

Exercises

1. Suppose that the probability density function of the length of computer cables is f (x) = 0.1 from 1200 to 1210 millimeters.

a. Determine the mean and standard deviation of the cable length.

b. If the length specifications are $1195 < x < 1205$ millimeters, what proportion of cables is within specifications?

2. The thickness of a conductive coating in micrometers has a density function of $600x^{-2}$ for $100\ \mu m < x < 120\ \mu m$.

a. Determine the mean and variance of the coating thickness.

b. If the coating costs $0.50 per micrometer of thickness on each part, what is the average cost of the coating per part?

4.5. Continuous Uniform Distribution

Uniform

Continuous uniform random variable over interval [*a*, *b*]

pdf:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0 & otherwise \end{cases}$$

Mean and Variance:

$$\mu = EX = \frac{a+b}{2} \quad , \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12}$$

cdf:

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & b \ge b \end{cases}$$

Example

CONTINUOUS UNIFORM RANDOM VARIABLE

The random variable X has a continuous uniform distribution on [4.9, 5.1]. The probability density function of X is $f(x) = 5,\ 4.9 \le x \le 5.1$. What is the probability that a measurement of current is between 4.95 and 5.0 milliamperes?

Example CONTINUOUS UNIFORM RANDOM VARIABLE

Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval [0, 4].

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3h?

4.6. Continuous Normal Distribution

Normal

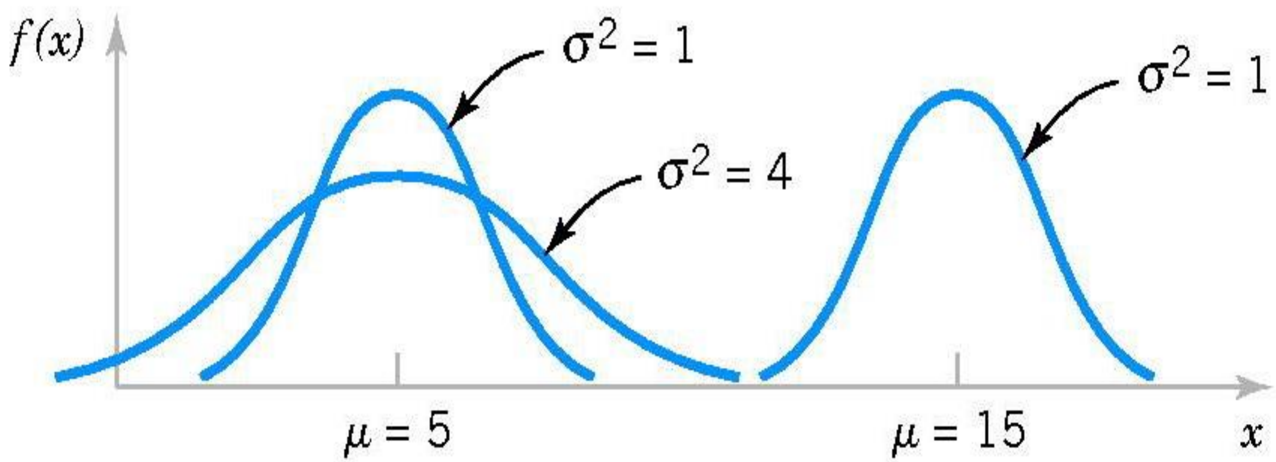
Normal random variable X ~ $N(\mu, \sigma^2)$

pdf:

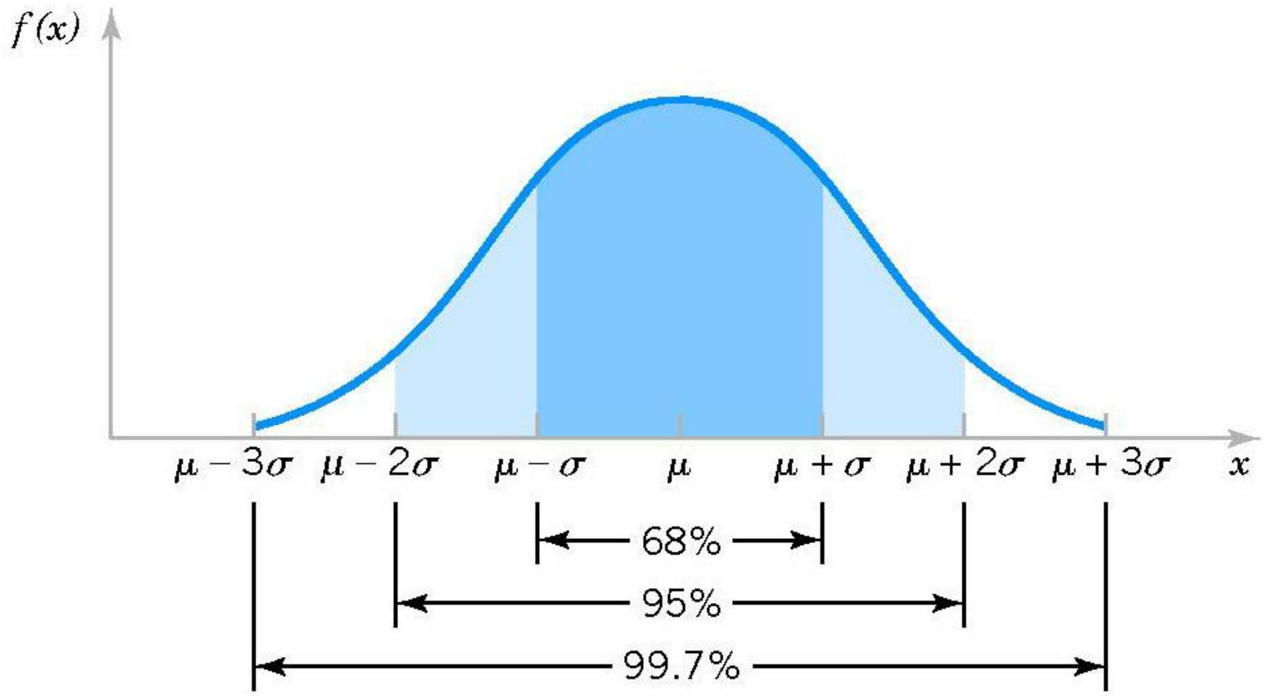
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < +\infty$$

Mean and Variance:

$E(X) = \mu$ $\qquad$ $V(X) = \sigma^2$



3σ-rule



Normal

Standard Normal Random Variable $Z \sim N(0,1)$

pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \quad , \quad -\infty < x < +\infty$$

cdf:

$$\Phi(z) = \int_{-\infty}^{z} \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}} dx.$$

Standardizing

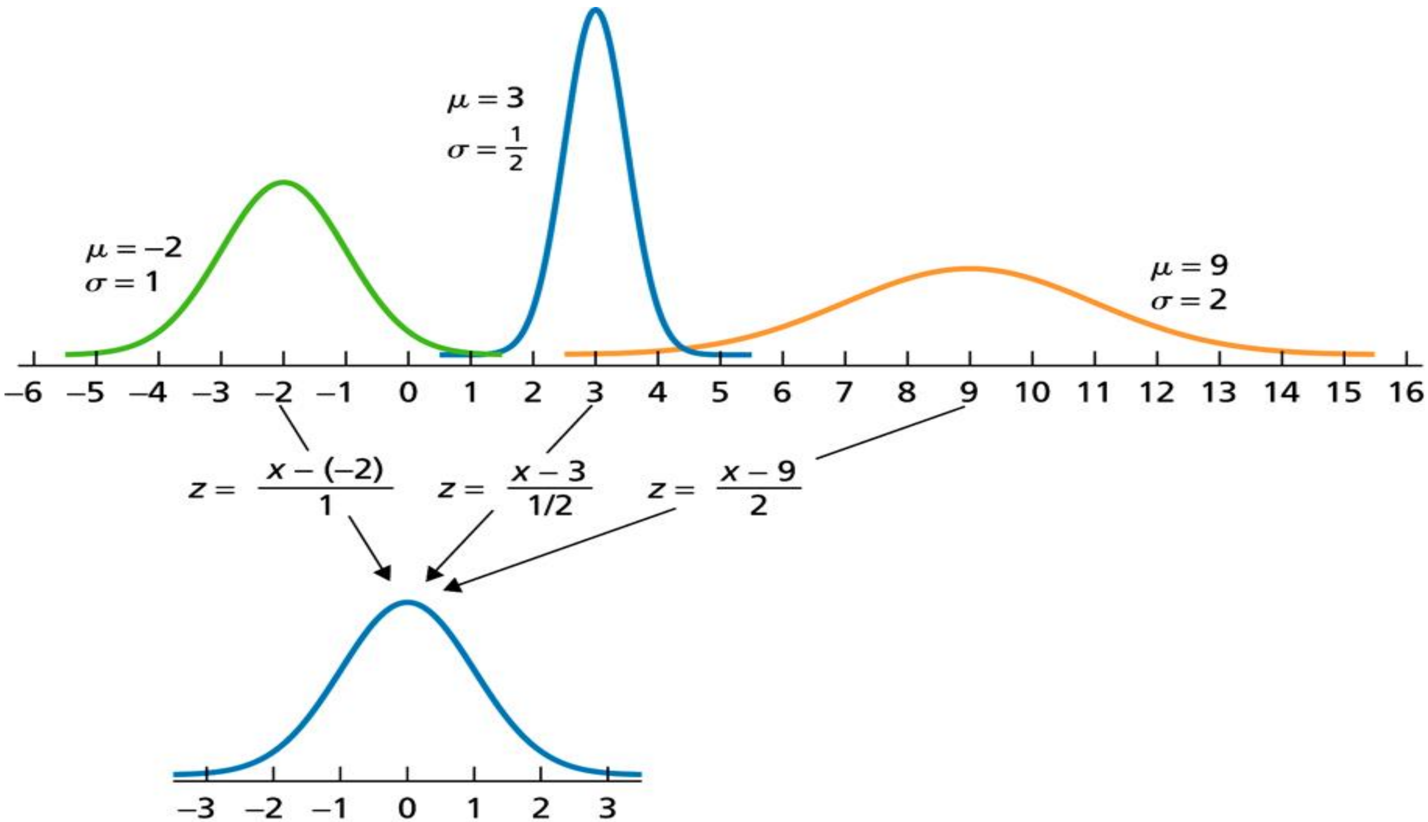
If X is a normal random variable $X \sim N(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma}$$

is a standard normal random variable $N(0, 1)$.

Normal

Standardizing



Normal

Finding Probabilities

Problem 1: x is given known, find $P(X < x)$.

Problem 2: $P(X < x) = p$ is given known, find x.

Using Excel

1. $Z \sim N(0,1)$

To find P(Z < z) when given z: =NORMSDIST(z)

To find z when P(Z < z) = *p*: =NORMSINV(p)

2. $X \sim N(\mu, \sigma^2)$

To find P(X < x) when given x: =NORMDIST(x, μ, σ,1)

To find x when P(X < x) = *p*: =NORMINV(p, μ, σ).

Example

(a) Let $X \sim N(34, 144)$. Find $P(X < 43)$ and $P(24 < X < 37)$.

(b) Let $Z \sim N(0,1)$. Find the value of z to $P(Z > z) = 0.95$

(a) $P(X < 43) = \mathrm{P}(Z<(43-34)/12) = \mathrm{P}(Z<0.75) = 0.7734$

$P(24 < X < 37) = P(X < 37) - P(X < 24)$

$= \mathrm{P}(Z<0.25) - \mathrm{P}(Z< -0.83)$

$= 0.5987 - 0.2023 =0.3964$

(b) $\mathrm{P}(Z > z) = 0.951 \Leftrightarrow 1- \mathrm{P}(Z < z) = 0.95 \Rightarrow \mathrm{P}(Z < z) = 0.05$

$\mathrm{P}(Z< -1.65) = 0.05 \Rightarrow z = -1.65$

Normal

Finding Probabilities: Using Table II

Table II Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350

Example

Assume that the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2. What is the probability that a measurement exceeds 13 milliamperes?

Example

An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours

Exercises 1

Assume that X is normally distributed with a mean of 10 and a standard deviation of 2. Determine the following:

a. $P(X < 13)$

b. $P(X > 9)$

c. $P(6 < X < 14)$

d. $P(2 < X < 4)$

e. $P(-2 < X < 8)$

Exercises 2

Assume that X is normally distributed with a mean of 10 and a standard deviation of 2. Determine the value for x that solves each of the following:

a. $P(X > x) = 0.5$

b. $P(X > x) = 0.95$

c. $P(x < X < 10) = 0$

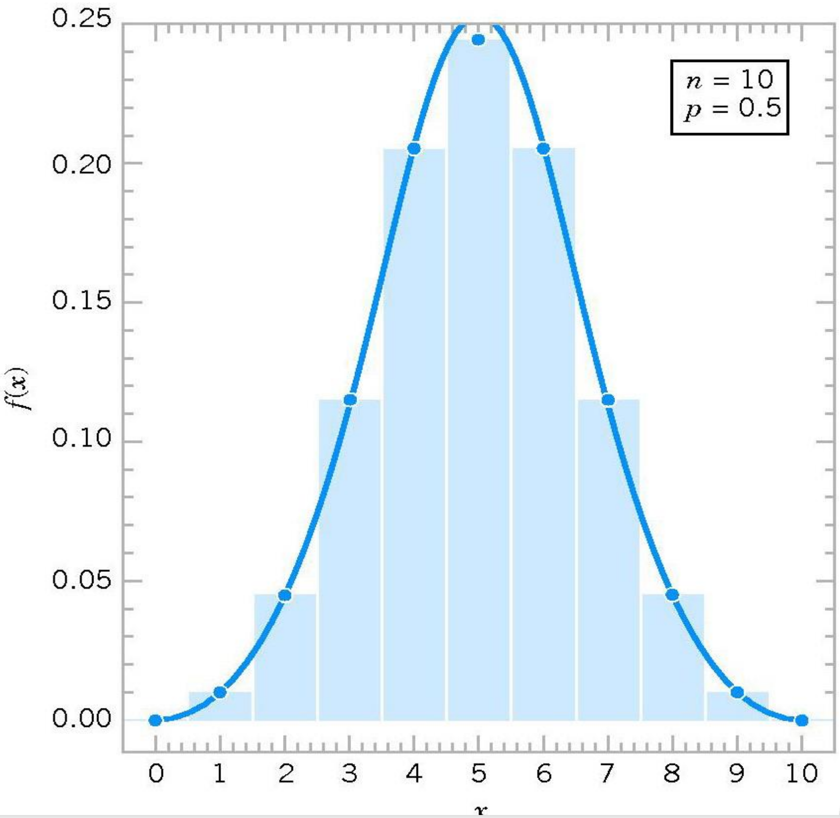
d. $P(-x < X - 10 < x) = 0.95$

e. $P(-x < X - 10 < x) = 0.99$

4.7. Normal approximation to the Binomial and Poisson.

NORMAL APPROXIMATION

Under certain conditions, the normal distribution can be used to approximate the binomial distribution and the Poisson distribution.



NORMAL APPROXIMATION

If $X \sim B(n, p)$ then random variable

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately a standard random variable $N(0,1)$.

$$P(X \le x) = \mathrm{P}\left(X \le x + 0.5\right) \approx P\left(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$$P(X \ge x) = \mathrm{P}\left(x - 0.5 \le X\right) \approx P\left(Z \ge \frac{x - 0.5 - np}{\sqrt{np(1-p)}}\right)$$

Remark: The approximation is good for $np > 5$ and $n(1-p) > 5$.

NORMAL APPROXIMATION

Example

Let $X \sim B(16\times10^6, 10^{-5})$. Find the probability $P(X > 150)$.

NORMAL APPROXIMATION

Example

The probability that a patient recovers from a rare blood disease is 0.4. If 100 people are known to have contracted this disease, what is the probability that fewer than 30 survive?

NORMAL APPROXIMATION

Example

A multiple-choice quiz has 200 questions, each with 4 possible answers of which only 1 is correct. What is the probability that sheer guesswork yields from 25 to 30 correct answers for the 80 of the 200 problems about which the student has no knowledge?

NORMAL APPROXIMATION

Normal Approximation to the Poisson Distribution

If X is a Poisson random variable $P(\lambda)$ then

$$Z = \frac{X - \lambda}{\sqrt{\lambda}}$$

is approximately a standard random variable $N(0,1)$.

$$P(X \le x) = P(Z \le \frac{x + 0.5 - \lambda}{\sqrt{\lambda}})$$

The approximation is good for **λ > 5.**

NORMAL APPROXIMATION

Example

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that less than 950 particles are found?

NORMAL APPROXIMATION

Example

Suppose that X is a Poisson random variable with $\lambda = 6$.

a. Compute the exact probability that X is less than four.

b. Approximate the probability that X is less than four and compare to the result in part (a).

c. Approximate the probability that $8 < X < 12$.

NORMAL APPROXIMATION

Example

Suppose that X has a Poisson distribution with a mean of 64. Approximate the following probabilities:

(a) $P\,(X > 72)$

(b) $P\,(X < 64)$

(c) $P\,(60 < X \leq 68\ )$

4.8. Exponential Distribution

Exponential Distribution

The random variable X that equals the distance between successive events from a Poisson process with mean number of events $\lambda > 0$ per unit interval is an **exponential random variable** with parameter λ. The probability density function of X is

$$f(x) = \lambda e^{-\lambda x}, \qquad 0 \le x < \infty$$

Exponential Distribution

Mean and Variance

If the random variable X has an exponential distribution with parameter λ,

$$\mu = E(X) = \frac{1}{\lambda}$$

$$\sigma^2 = V(X) = \frac{1}{\lambda^2}$$

Example

Exponential Distribution

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour.

a) What is the probability that there are no log-ons in an interval of 6 minutes?

b) What is the probability that the time until the next log-on is between 2 and 3 minutes?

Example

Exponential Distribution

Suppose that X has an exponential distribution with $\lambda = 2$. Determine the following:

(a) $P(X \leq 0)$

(b) $P(X \geq 2)$

(c) $P(X \leq 1)$

(d) $P(1 < X < 2)$

(e) Find the value of x such that $P(X < x) = 0.05$