Example optimization problem

Machine Learning — CS 4641/7641

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1 Optimization and KKT Example

Optimization is a fundamental concept in mathematics and engineering that seeks to find the best solution to a problem within a given set of constraints. In this tutorial, we will explore the use of Karush-Kuhn-Tucker (KKT) conditions and Lagrangian functions to solve a simple optimization problem. Let's consider an example with a single variable x.

We will minimize a function $f(x)$ subject to a single constraint $g(x)$. This will make it easier to illustrate the use of KKT conditions and the Lagrangian function. Here's the modified problem:

1.1 Minimize

$$f(x) = x^2 - 4x$$

Subject to Constraint:

$$g(x) = x - 2 \leq 0$$

Now, let's use KKT conditions and the Lagrangian function to find the optimal value of x that minimizes $f(x)$ while satisfying the constraint.

1.2 Lagrangian Function

The Lagrangian function for this problem is defined as follows:

$$L(x, \lambda) = x^2 - 4x + \lambda(x - 2)$$

Note: for maximization, expression will be $L(x, \lambda) = x^2 - 4x - \lambda(x - 2)$.

1.3 KKT Conditions

We have one constraint, so we only need to consider one set of KKT conditions:

- Stationarity Condition:

$$\frac{\partial L}{\partial x} = 2x - 4 + \lambda = 0$$

- Primal Feasibility:

$$x - 2 \leq 0$$

- Dual Feasibility:

$$\lambda \geq 0$$

- Complementary Slackness:

$$\lambda(x - 2) = 0$$

1.4 Solution Procedure

1. Formulate the Lagrangian Function:

   We start by forming the Lagrangian function for our minimization problem:

   $$L(x, \lambda) = x^2 - 4x + \lambda(x - 2)$$

2. Find Partial Derivatives:

   Calculate the partial derivatives of the Lagrangian function with respect to x and λ:

   $$\frac{\partial L}{\partial x} = 2x - 4 - \lambda$$

   $$\frac{\partial L}{\partial \lambda} = x - 2$$

3. Apply KKT Conditions and Determine the Candidate Points:

   Solve the system of equations formed by the stationary condition, primal feasibility, dual feasibility, and complementary slackness condition to find the value of x and λ. The constraint $g(x)$ may be active or inactive. Active constraints impose only equality $x - 2 = 0$ while inactive constraints can include $\leq$ or $\geq$ like $x - 2 \leq 0$. Think about how active vs. inactive affects the values of λ as dictated in the complementary slackness condition. Remember that λ and the constraint $g(x)$ can't both be 0.

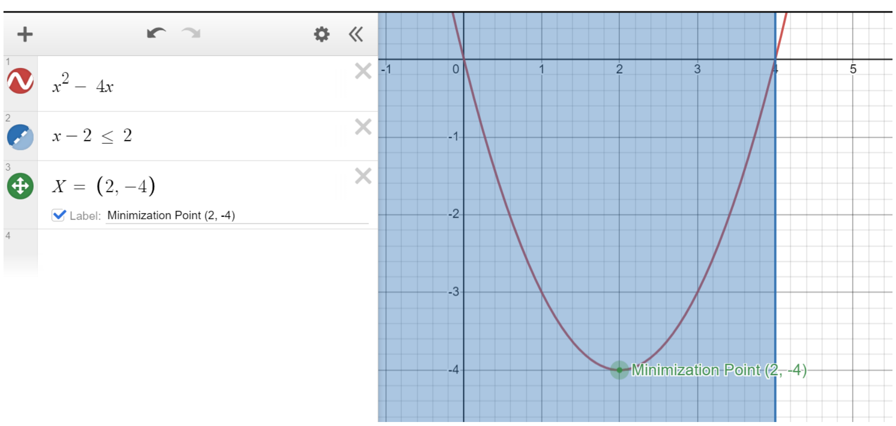
   For both $g(x)$ active and $g(x)$ inactive, try to calculate x and λ. There may not be a valid point (one of the 4 KKT conditions is violated) for both scenarios, but there must be at least one valid point over all scenarios.

4. Evaluate the Objective Function:

   Calculate the value of the objective function $f(x)$ using the value of x obtained in step 3.

2 Plotting

By solving the simple problem above, we have the conditions satisfied by one value of x which is when $x = 2$. We plot it with the constraints and the objective function $f(x)$.



Here, we have used Desmos because it is handy for plotting functions in 2D. You can use tools like Math3D to plot complex 3D functions. Make sure you provide a view of the maximization or minimization point with the constraints or if possible share the graph link in our solution along with a picture.