Measure Theory- OSE Bootcamp 2019

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1 Chapter 1

1.1 Exercise 1.3

1. No, $\mathcal{G}_1$ is not an algebra. $\mathcal{G}_1$ is not closed under complements. For a given $a \in \mathbb{R}$, let A be defined as $A = (-a, a)$ which is an open set in $\mathbb{R}$. Then, $A^c = (-\infty, -a] \cup [a, \infty)$ is not an open set in $\mathbb{R}$. In fact, it is closed since any sequence in this set, converges to a point in the set itself (contains all its limit points). Therefore, condition (ii) is violated and hence, this is not an algebra. Moreover, this is not a σ-algebra.

2. Yes, $\mathcal{G}_2$ is an algebra. To show that $\mathcal{G}_2$ is an algebra, we sequentially follow the steps. For the configuration of sets provided here, pick a=b and one gets $(a, b] = \emptyset \in \mathcal{G}_2$. Next, we show that $\mathcal{G}_2$ is also closed under complements and finite unions. For $a, b \in \mathbb{R}$, for any interval of the form $(a, b]^c = (-\infty, a] \cup (b, \infty), (-\infty, b]^c = (b, \infty), (a, \infty)^c = (-\infty, a] \in \mathcal{G}_2$. Furthermore, $(-\infty, b]^c = (b, \infty) \in \mathcal{G}_2$, and $(a, \infty)^c = (-\infty, a] \in \mathcal{G}_2$. Moreover, $\mathcal{G}_2$ is closed under finite unions. A finite union of intervals of the form $(a, b], (-\infty, b]$, and (a, ∞) will still yield us a finite union of intervals of the same form. Therefore, $\mathcal{G}_2$ is an algebra. However, by definition this $\mathcal{G}_2$ is a finite union of the intervals of the form expressed above. Therefore, a countable union does not belong to $\mathcal{G}_2$. Hence, it is not a σ-algebra.

3. Yes, it is both an algebra and a σ-algebra. It is easy to follow this from (2.) above. The only addition is that $\mathcal{G}_3$ is now a countable union of intervals rather than finite unions, and therefore, admits the requirement for $\mathcal{G}_3$ to be a σ-algebra.

1.2 Exercise 1.7

As Jan discussed in class, the empty set and the set itself is the smallest algebra since it follows all the properties that an algebra has to satisfy. Similarly, the power set is the largest sigma algebra and it satisfies all the requirements one needs an algebra and a σ-algebra to satisfy.

1.3 Exercise 1.10

We are given that $\{S_\alpha\}$ be a family of σ-algebras on X. We need to show that $\cap_\alpha S_\alpha$ is also a σ-algebra.

1. Trivial to see that $\emptyset \in \cap_\alpha S_\alpha$. The intersection of S_α contains $\emptyset$ because it is an intersection of σ-algebras. By definition, any sigma algebra contains $\emptyset$.

2. Now we show that $\cap_\alpha S_\alpha$ is closed under complements.

   Let $X \in \cap_\alpha S_\alpha$. By definition, $X \in S_\alpha$ for all α, i.e, X is in $\{S_\alpha\}$ for all α. Since, each S_α is a σ-algebra, the complement of $X^c \in S_\alpha$ for all α. Therefore, $X^c \in \cap_\alpha S_\alpha$. Hence, $\cap_\alpha S_\alpha$ is closed under complements.

3. Finally, we show that $\cap_\alpha S_\alpha$ is closed under countable unions.

   Let $X_1, X_2, ... \in \cap_\alpha S_\alpha$. For illustration, α could belong to the set of natural numbers. Therefore, $X_1, X_2, ... \in S_\alpha$ for every α. Since, each S_α is a σ-algebra, which implies that countable union of $\cup_{n=1}^{\infty} X_n \in S_\alpha$ for every α. Therefore, $\cup_{n=1}^{\infty} X_n \in \cap_\alpha S_\alpha$,i.e,belongs to the intersection of $\cap_\alpha S_\alpha$ which shows that $\cap_\alpha S_\alpha$ is closed under countable unions.

1.4 Exercise 1.22

Proposition 1. *μ is monotone, i.e, if $A, B \in \mathcal{S}$, $A \subset B$, then $\mu(A) \leq \mu(B)$.*

Proof. Following the proofs outlined in the notes, we can write $B = (A) \cup (B \cap A^c)$. Using the measure property, we have $\mu(B) = \mu(A) + \mu(B \cap A^c)$. Since a measure is always non-negative, we have, $\mu(A) \leq \mu(B)$. ∎

Proposition 2. *μ is countably subadditive: if $\{A_i\}_{i=1}^{\infty} \subset \mathcal{S}$, then $\mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.*

Proof. Define $A = \cup_{i=1}^{\infty} A_i$ and then sequentially we let $\{B_i\}_{i=1}^{\infty}$ to be defined as $B_1 = A_1$, $B_2 = A_2 \cap A_1^c \ldots B_i = A_i \cap (\cup_{n=1}^{i-1} A_i)^c$. First, by the way of definition, $B_i \cap B_j = \emptyset$ for all $i \neq j$. Moreover, it can be shown that $A \subseteq \cup_{i=1}^{\infty} B_i$ and $\cup_{i=1}^{\infty} B_i \subseteq A$, therefore, $A = \cup_{i=1}^{\infty} B_i$. Finally, $B_i \subset A_i$. By proposition 1, we have that $\mu(B_i) \leq \mu(A_i)$. Hence by appealing to disjointness and Proposition 1,

$$\mu(\cup_{i=1}^{\infty} B_i) = \mu(A) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i) \tag{1}$$

∎

1.5 Exercise 1.23

Let $(X, \mathcal{S}, \mu)$ be a measure space and $B \in \mathcal{S}$. Let λ be a mapping from a set $\mathcal{S}$ to R_+. In particular, $\lambda(A) = \mu(A \cap B)$.

Proposition 3. $\lambda(\emptyset) = 0$

It is easy to see that the intersection of an empty set with B is the empty set itself. In particular and by definition it follows, $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.

Proposition 4. $\lambda(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \lambda(A_i)$ *for any* $\{A_i\}_{i=1}^{\infty} \subset \mathcal{S}$ *such that* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$.

This is similar in spirit to the question above.
For any $\{A_i\}_{i=1}^{\infty} \subset \mathcal{S}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, we have

$$\begin{aligned}
\lambda(\cup_{i=1}^{\infty} A_i) &= \mu((\cup_{i=1}^{\infty} A_i) \cap B) && \text{(By def.)} \quad (2)\\
&= \mu(\cup_{i=1}^{\infty} (A_i \cap B)) && \text{(associativity)} \quad (3)\\
&= \sum_{i=1}^{\infty} \mu(A_i \cap B) = \sum_{i=1}^{\infty} \lambda(A_i \cap B) && \text{(By disjointness and additivity)} \quad (4)
\end{aligned}$$

1.6 Exercise 1.26 (Continuous from below)

Let $A = \cap_{n=1}^{\infty} A_n$. We proceed to give a proof of (ii).

Let $B_n = A_1 - A_n$ for $n \in \mathbb{N}$ and let $B = \cup_{n=1}^{\infty} B_n$. Now, with B being the infinite union of such sets, it is easy to see that they form a collection of an increasing sequence, i.e., $\{B_n\}_{n=1}^{\infty}$ is an increasing sequence.

Therefore,

$$\begin{aligned}
\mu(A_1) - \mu(A) &= \mu(A_1 - A)\\
&= \mu(B)\\
&= \mu(\cup_{n=1}^{\infty} B_n)\\
&= \lim_{n \to \infty} \mu(B_n)\\
&= \lim_{n \to \infty} (\mu(A_1) - \mu(A_n))\\
&= \mu(A_1) - \lim_{n \to \infty} \mu(A_n))
\end{aligned}$$

Hence, since $\mu(A_1)$ is a constant, eliminating it we have our desired statement.

2 Chapter 2

2.1 Exercise 2.10

We can easily show this by considering the statements in Theorem 2.8. B can be written as $B = (B \cap E) \cup (B \cap E^c)$. By the property of countable subadditivity, $\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$. For the conclusion to follow, therefore, we have $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

2.2 Exercise 2.14

Carathéodory property gives us that $\mathcal{M}$ is a σ-algebra. Not only that, $\mathcal{M}$ is also a collection of Lebesgue measurable sets which contain all open sets. By appealing to the definition, $\mathcal{B}(X)$ is defined as the intersection of all σ-algebra containing open sets, therefore, we must have that $\mathcal{B}(X) \subset \mathcal{M}$.

3 Chapter 3

3.1 Exercise 3.1

Proposition 5. *Every countable subset of the real line has Lebesgue measure* 0.

Proof. Let $a \in R$. Then, $a \subset [a-\epsilon, a+\epsilon]$ holds for every $\epsilon > 0$ and so $\lambda(a) \leq \lambda(a-\epsilon, a+\epsilon) = 2\epsilon$ for every $\epsilon > 0$. Therefore, $\lambda(a) = 0$ holds for every $a \in R$. If $A = (a_1, a_2, ...a_n)$ is a countable set, then it is easy to observe that $\lambda(A) = 0$. ∎

3.2 Exercise 3.7

To prove, $1 = 2 = 3 = 4$.

1. $\{x \in X : f(x) < a\} \in \mathcal{M}$
2. $\{x \in X : f(x) \geq a\} \in \mathcal{M}$
3. $\{x \in X : f(x) > a\} \in \mathcal{M}$
4. $\{x \in X : f(x) \leq a\} \in \mathcal{M}$

Proof. We are given that $\{x \in X : f(x) < a\} \in \mathcal{M}$. Using the fact that $\mathcal{M}$ is closed under complements, and the observation that $f^{-1}([a, \infty)) = (f^{-1}(-\infty, a))^c$. we can easily see $f^{-1}([a, \infty)) \in \mathcal{M}$. Therefore, $(1) \implies (2)$. Hence, (1.) proved.

We are given that $\{x \in X : f(x) \geq a\} \in \mathcal{M}$. Using the property of inverse and some of the steps shown above in the exercises, one can express $f^{-1}((a,\infty)) = \cap_{n=1}^{\infty} f^{-1}([a - \frac{1}{n}, \infty))$. Since $\mathcal{M}$ is closed under countable intersections and each of these belong to $\mathcal{M}$. Therefore, we have $f^{-1}(a,\infty) \in \mathcal{M}$. Hence, (2) $\implies$ (3)

Similar to 1 above. Hence, (3) $\implies$ (4).

Similar to 2 above. All one needs to do is to replace is to observe that $f^{-1}((-\infty,a)) = \cap_{n=1}^{\infty} f^{-1}((-\infty, a + \frac{1}{n}))$. Rest of the proof follows 2.. ∎

3.3 Exercise 3.10

Given 2 and 4, we proceed to prove 1.

1. Take 4 first. We have, $F(f(x) + g(x)) = f(x) + g(x)$. By continuity and application of Theorem 3.9 we have F to be continuous and measurable. Therefore, $f + g$ is measurable.
2. Similarly, we have $F(f(x) + g(x)) = f(x)g(x)$. Again, in the same spirit as above we have $f \cdot g$ is measurable.
3. Now, given that f and g are measurable functions on $(X, \mathcal{M})$ and by appealing to closedness of $\mathcal{M}$ under countable intersectons, we have that $\max(f(x), g(x))$ is measurable.
4. Follows from the proof above with the just an inequality twist.
5. Again $\{x \in X : |f(x)| > a\} = \{x \in X : f(x) < -a\} \cup \{x \in X : f(x) > a\}$. $\mathcal{M}$ is closed under countable unions, and since both are contained in $\mathcal{M}$, therefore, $\{x \in X : |f(x)| > a\} \in \mathcal{M}$, so that $|f(x)|$ is measurable.

3.4 Exercise 3.17 (Uniform convergence)

This follows exactly from notes. Given f is bounded and we take an $\epsilon > 0$. Owing to its boundedness, there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in X$. Hence, $x \in E_i^M$ for some i and all $x \in X$. Moreover, for an $N \in \mathbb{R}$ and $N \geq M$ we have $\frac{1}{2^N} < \epsilon$. Therefore, $n \geq N$, $||s_n(x) - f(x)|| < \epsilon$ for all $x \in X$. Hence, proved.

4 Chapter 4

4.1 Exercise 4.13

All we need to show here is that $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite. And by prop 4.5 it follows.

Firstly, we have $||f|| = f^+ + f^-$ and are non-negative. Further, $||f|| < M$ on E, then $0 \leq f^+ < M$ and $0 \leq f^- < M$ on E.

Finally, by Proposition 4.5, we derive,

$$\int_E f^+ d\mu < M\mu(E) < \infty$$
$$\int_E f^- d\mu < M\mu(E) < \infty$$

4.2 Exercise 4.14

One can easily prove this otherwise using contrapositive and arrive at a contradiction where $f \notin \mathscr{L}^1(\mu, E)$.

4.3 Exercise 4.15

Again appealing to Proposition 4.7 and by appealing to definition of Lebesgue integral, we will show

$$\int_E f d\mu \leq \int_E g d\mu \tag{5}$$

We have $f, g \in \mathscr{L}^1(\mu, E)$. We use simple functions to progress further. We define the set of simple functions $B(f) = \{z : 0 \leq z \leq f\}$. WLOG, $f \leq g$. Appealing to 4.7, we have $B(f^+) \subset B(g^+)$ and $B(g^-) \subset B(f^-)$. Therefore, $\int_E f^+ d\mu \leq \int_E g^+ d\mu$ and $\int_E f^- d\mu \geq \int_E g^- d\mu$. Hence,

$$\int_E f d\mu = \int_E (f^+ d\mu - f^- d\mu) \leq \int_E (g^+ d\mu - g^- d\mu) = \int_E g d\mu \tag{6}$$

Finally,

$$\int_E f d\mu \leq \int_E g d\mu \tag{7}$$

4.4 Exercise 4.16

We start with a simple function $s(x) = \sum_{i=1}^{N} c_i \chi_{E_i}$, where $E_i \in \mathcal{M}$. Also, we pickup a set A s.t. $A \subset E \in \mathcal{M}$. By the monotonicity of measures, we have that $\mu(A \cap E_i) \leq \mu(E \cap E_i)$ for all i. Hence, we have,

$$\int_A s d\mu \leq \int_E s d\mu \tag{8}$$

Appealing to Definition 4.2, we have by Equation (8) that,

$$\int_A f d\mu \leq \int_E f d\mu \tag{9}$$

We can easily establish by virtue of properties of f that $\int_E f d\mu < \infty$. Finally, it follows that $\int_A f d\mu < \infty$, which in turn implies $\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$, so that $f \in \mathscr{L}^1(\mu, A)$.

4.5 Exercise 4.21

Given Proposition 4.6, we have,

$$\int_{A-B} f d\mu = 0. \tag{10}$$

Recall that f^+ and f^- are non-negative $\mathcal{M}$-measurable functions because $f \in \mathscr{L}^1$. Further, by Theorem 4.19 we have that $\mu_1(A) = \int_A f^+ d\mu$ and $\mu_2(A) = \int_A f^- d\mu$ as measures on $\mathcal{M}$.

$$\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \mu_1(A) - \mu_2(A) \tag{11}$$

Applying a tool used in exercise 1, we have $A = (A - B) \cup B$ and on further refinement, we have that $\mu_i(A) = \mu_i(B)$ for $i = 1, 2$ because $\mu(A - B) = 0$. Therefore,

$$\int_A f d\mu = \mu_1(B) - \mu_2(B) = \int_B f d\mu \tag{12}$$

Henceforth,

$$\int_A f d\mu \leq \int_B f d\mu \tag{13}$$

5 Conclusion

This concludes the solutions for the exercises.