Course Assignment for Advanced Macroeconomics

I analyze a simple version of Clementi and Palazzo (2016)'s model of the firm lifecycle (which itself builds on Hopenhyan (1992) and Hopenhayn and Rogerson (1993)). The setup of it replicates my code for Khan-Thomas. I study the steady state of the model without aggregate shocks.

The model is as follows. There are two groups of firms. The first group of *incumbent firms* behaves similarly to the Khan and Thomas (2008) model, except they face convex capital adjustment costs rather than fixed costs. Specifically, each of these incumbent firms has access to a decreasing returns to scale production function $y_{jt} = e^{\varepsilon_{jt}} k_{jt}^{\theta} n_{jt}^{\nu}$, where y_{jt} is output, ε_{jt} is idiosyncratic productivity, k_{jt} is the firm's capital stock, n_{jt} is the firm's labor input, and $\theta + \nu < 1$. Idiosyncratic productivity ε_{jt} follows a Markov chain described below. Firms accumulate capital according to the accumulation equation $k_{jt+1} = (1-\delta)k_{jt} + i_{jt}$. Capital accumulation incurs the convex adjustment cost $-\frac{\phi}{2}\left(\frac{i_{jt}}{k_{jt}}\right)^2 k_{jt}$, in units of output. At the beginning of each period, incumbent firms must pay a fixed cost c_f units of output to remain in operation. A firm that does not pay this fixed cost does not produce, sells its entire capital stock with value $(1-\delta)k$, and permanently exits the economy.

There is a continuum of the second group of firms, the *potential entrants*. These firms are ex-ante identical. At the beginning of each period, each firms decides whether to pay a fixed entry cost c_e and enter the economy. If a potential entrant enters the economy, it draws a value for idiosyncratic productivity ε_{jt} from some distribution ν and begins as an incumbent firm with $k_{jt} = 0$. Assume that there are no adjustment costs at $k_{jt} = 0$. Further, assume there is free entry among potential entrants, which implies that the expected value from entering is less than or equal to the entry cost c_e, with equality if entry actually takes place. In equations, this condition is $c_e \leq \int v(\varepsilon, 0)\nu(d\varepsilon)$, with equality if $m^* > 0$ (where $v(\varepsilon, k)$ is the value function of an incumbent firm and m^* is the mass of entrants in equilibrium).

Finally, there is a representative household with preferences over consumption C_t and labor supply N_t represented by the expected utility function $\mathbb{E}_0 \beta^t (\log C_t - aN_t)$, where β and a are parameters. Output can be used for consumption, investment, capital adjustment costs, entry costs, or operating costs.

A steady state recursive competitive equilibrium of this economy is a set of incumbent value functions $v(\varepsilon, k)$, policy rules $k'(\varepsilon, k)$ and $n(\varepsilon, k)$, a mass of entrants per period m^*, a measure of active firms at the beginning of the period $g^*(\varepsilon, k)$, and real wage w^* such that (i) incumbent firms maximize their firm value; (ii) the free entry condition holds; (iii) the labor market clears; and (iv) the measure of active firms $g^*(\varepsilon, k)$ is stationary.

I further calibrate the fixed costs c_f and c_e . Throughout, the following parameter values are fixed: $\theta = 0.21$, $\nu = 0.64$, $\delta = 0.1$, $\beta = 0.96$, and $\varphi = 0.5$. Assume that the distribution of idiosyncratic shocks ε_{jt} follows the AR(1) process $\varepsilon_{jt+1} = \rho\varepsilon_{jt} + \omega_{jt+1}$ where $\omega_{jt+1} \sim N(0, \sigma^2)$ with parameters $\rho = 0.9$ and $\sigma = 0.02$. Numerically, I approximate this process with a Markov chain with $n_\varepsilon = 10$ points using Tauchen's method (a quantecon installation is in the code). We finally, assume the distribution of new entrants' productivity $\nu(\varepsilon)$ is the stationary distribution associated with this process.

1. Step 1.I first define the recursive competitive steady state equilibrium of this model a-la-Khan and Thomas.

2. Step 2. Before computing for the steady state of this model, it will be convenient to compute the steady state equilibrium of the model in which there is a representative firm with the same production function as the hetero-geneous firms and productivity equal to the mean value of ε according to its stationary distribution. I assume the representative firm rents capital and labor in competitive input markets as in the RBC model from lecture. I compute the steady state capital stock K^*_{rep} and steady state wage w^*_{rep} of the representative agent model. Further, I assume that steady state labor supply is $N^*_{\text{rep}} = 0.6$.

3. Step 3. We now begin solving the heteroge-neous firm model with the incumbent firms' decision rules. The Bellman equation of incumbent firms' optimization problem is

$$v(\varepsilon, k) = \max\{(1-\delta)k, v^1(\varepsilon, k) - c_f\} \tag{1}$$

$$v^1(\varepsilon, k) = \max_{k', n} \varepsilon k^\theta n^\nu - w^* n - (k' - (1-\delta)k) - \frac{\varphi}{2}\left(\frac{k'}{k} - (1-\delta)\right)^2 k + \beta\mathbb{E}[v(\varepsilon', k')] \tag{2}$$

   where w^* is the steady state real wage. The first Bellman equation corresponds to the exit decision and the second corresponds to the optimal decisions of continuing firms. For now, I set $w^* = w^*_{\text{rep}}$; in Part 4 i will solve for the true steady state wage. I will also need a parameter value for c_f. For now, set $c_f = 0.01$; in Step 6 I will calibrate this parameter to match the average exit rate in the economy.

   I Plot the value function, capital accumulation policy function, and an indicator for whether the firm will continue. The graphs are in the Jupyter notebook. Further, the labor decision n can be solved in closed form from the first order condition. Since I assume there is no capital adjustment costs at $k = 0$, I should solve that gridpoint separately from the rest. Once I have solved for the value function for all points $k > 0$, I can compute the object $v(\varepsilon, 0) \equiv \widehat{v}(\varepsilon) = \max_{k'} -k' + \beta\mathbb{E}[v(\varepsilon', k')]$.

4. Step 4. I study equilibria of this model in which there is entry and exit, i.e., $m^* > 0$. In this case, the free-entry condition must hold with equality, i.e. $c_e = \widehat{v}(\varepsilon)\nu(d\varepsilon)$. The real wage w^* is solved by casting this problem as a root-finding problem $F(w^*) = 0$, where the function F performs the following computations:

   - Solves for the incumbent's firm decision rules $v(\varepsilon, k)$ given the value w^* (as discussed in Part 3).
   - Compute the value function of new entrants, $\widehat{v}(\varepsilon)$ (again, as discussed in Part 3).
   - Return the value $c_e - \int \widehat{v}(\varepsilon)\nu(d\varepsilon)$.

   For some particular values , it is possible that there is no solution to the equation $F(w^*) = 0$. I guess this is an indication that the entry costs are so high that firms will never find it profitable to enter the economy.

5. Step 5. Now we will solve for the mass of entrants m^* and the stationary measure of incumbent firms $g^*(\varepsilon, k)$. As discussed by Hopenhayn and Rogerson (1993), the stationary measure is linearly homogenous in the amount of entry m^*. That is, if we double the amount of entrants m^*, then you will double the total mass of firms in the economy but the fraction of firms with any particular pair (ε, k) is constant. Therefore, we proceed in two steps: first, solve for the stationary measure with $m^* = 1$; and second, solve for the mass of entrants m^* itself.

   (a) *Step 1: solve for stationary measure with* $m^* = 1$ In principle, the stationary distribution must satisfy the law of motion

   $$g^*(\varepsilon', k') = \int X(\varepsilon, k)Pr(\varepsilon'|\varepsilon)\mathbb{1}\{k'(\varepsilon, k) = k'\}dg^*(\varepsilon, k) + m^* \Pr(\nu = \varepsilon') \qquad (3)$$

   where $X(\varepsilon, k)$ is an indicator variable for whether a firm (ε, k) will survive (i.e., choose not to exit the economy).

   I iterate on a numerical approximation of this equation with $m^* = 1$. In order to do so, we assume that the capital stock k takes values in a discrete grid $\mathbf{k} = (k_1, ..., k_N)$ where $k_i > k_{i-1}$ and n_k is the total number of grid points. Under this assumption, the measure $g(\varepsilon, k)$ can be represented by a $n_\varepsilon \times n_k$ matrix or, in more compact form, a $n_\varepsilon n_k \times 1$ vector.

I further approximate the stationary distribution with a probability mass function over the discretized state space $g(\varepsilon_i, k_j)$. This function must satisfy the discretized law of motion

$$g(\varepsilon_k, k_l) = \sum_{i=1}^{n_\varepsilon} \sum_{j=1}^{N} \Pr(\varepsilon' = \varepsilon_k | \varepsilon = \varepsilon_i) \mathbb{1}\{k'_{ij} \in k_l\} X(\varepsilon_i, k_j) g(\varepsilon_i, k_j) + m^* \Pr(\nu(\varepsilon) = \varepsilon_k) \tag{4}$$

where the notation $\{\mathbb{1}k'_j \in k_l\}$ means that k_l is the closest gridpoint to $k'(\varepsilon_i, k_j)$.

We compute the stationary measure $\mathbf{g}$ for m^* by iterating on (4) with $m* = 1$ (i.e., plug in the current iteration for $\mathbf{g}$ on the RHS and get the new iteration on th LHS).

Plot the stationary measure corresponding to $m^* = 1$, $g(\varepsilon, k)$. If there is a positive mass of firms at either end of the state space, this may very well indicate the grid bounds on the capital stock k are too narrow. If there is empty space at either end of the state space, this indicates that the grid bounds are too wide. One must adjust the grid bounds appropriately.

(b) *Step 2: solve for the mass of entrants* m^* As discussed above, the equilibrium measure of firms is $g^*(\varepsilon, k) = m^* g(\varepsilon, k)$, where m^* is the equilibrium mass of entrants. This mass of entrants m^* must be consistent with labor market clearing, i.e. aggregate labor demand $N^{\mathrm{d}}(m^*)$ equals aggregate labor supply $N^{\mathrm{s}}(m^*)$.

As discussed in Part 1, a common way to calibrate business cycle models is to choose the disutility of labor supply, a, so that steady state equilibrium labor supply is $N^{\mathrm{s}}(m^*) = 0.6$. This calibration strategy will simplify the computation of the equilibrium mass of entrants m^*. First, we will compute the mass of entrants m^* such that aggregate labor demand $N^{\mathrm{d}}(m^*) = 0.6$. Second, given this value for the mass of entrants m^*, we will use the household's first order condition to back out the parameter a which ensures $N^{\mathrm{s}}(m^*) = 0.6$. With indivisible labor, the labor supply of the household is simply $\frac{w^*}{C^*} = a$, so this task amounts to compute C^*.

We define a function F(m*) that has input a guess of mass of entrants m^* and outputs excess aggregate labor demand. This function $F(m^*)$ should perform the following steps.

- Compute the stationary mass of firms $g^*(\varepsilon, k) = m^* g(\varepsilon, k)$, where $g(\varepsilon, k)$ is the object computed in step 1.

- Compute aggregate labor demand $N^{\mathrm{d}}(m^*) = \int n(\varepsilon, k) dg^*(\varepsilon, k)$, where $n(\varepsilon, k)$ is the decision rule from Part 2.
- Return $N^{\mathrm{d}}(m^*) - 0.6$.

Solving for F(m*)=0 gives us a measure.

The household's first order condition for labor is $w^* C = a$, where C is aggregate consumption. That makes us get the aggregate consumption using the resource constraint and using the firm's output, investment decisions, adjustment costs, and entry/exit costs, integrated against the stationary distribution and the implied value of a.

6. Step 6. So far, I have used ad-hoc values of the operating cost c_f and entry cost c_e. Let's loosely calibrate these parameters to target
(i) the annual exit rate of 10% and (ii) the average size of new firms to be 40% the average size of all firms. To do so, consider parameter values on the coarse grids
$c_e = \{0.01, 0.05, 0.1, 0.2, 0.5\}$ and $c_f = \{0.01, 0.05, 0.1, 0.2, 0.5\}$. We find the combination that gets closest to the empirical targets (i) and (ii).

7. **Analyze calibrated model** Lastly and the final objective, I analyze the behavior of firms in this model where I compare my results with Hseih and Klenow (2009), Cooper and Haltiwanger (2006).