HW 4

6) a) $A_{m\times n}$ ; $P = A^TA$ $Q = AA^T$

$$y^TPy = y^TA^TAy = (Ay)^T(Ay) = \|Ay\|_2^2 \geq 0$$

$$z^TQy = z^TAA^Tz = \|A^Tz\|_2^2 \geq 0 \rightarrow \text{Hence proved}$$

$y^TPy \Rightarrow$ let $y$ be replaced by an eigen vector $\boxed{v}$.

$$\Rightarrow v^TPv = v^T\lambda v = \lambda \cdot v^Tv = \lambda\|v\|_2^2$$

(as $\lambda$ is scalar)

Now $\lambda\underbrace{\|v\|_2^2}_{\geq 0} \geq 0$ (as it is true for any $y$)

$\therefore \boxed{\lambda \geq 0}$ Similarly, for Q.

b) $Pu = \lambda u \Rightarrow A^TAu = \lambda u$

Now, $QAu = A\underbrace{A^TAu}_{\lambda u} = A\lambda u = \lambda Au$.

$\therefore Q(Au) = \lambda(Au)$ $\therefore$ $Au$ is eigen vector with eigen value $\lambda$ for Q.

$\boxed{u = n\times 1}$

$Qv = \mu v \Rightarrow AA^Tv = \mu v$.

$$PA^Tv = A^T\underbrace{AA^Tv}_{\mu v} = A^T\mu v = \boxed{\mu A^Tv}$$

$\therefore P(A^Tv) = \mu(A^Tv)$ $\therefore$ $A^Tv$ is an eigen vector of P with eigen value $\mu$.

$\boxed{v = m\times 1}$

c) $Qv_i = \lambda v_i$ • $u_i = \dfrac{A^Tv_i}{\|A^Tv_i\|_2}$

$$\Rightarrow Au_i = \frac{AA^Tv_i}{\|A^Tv_i\|_2} = \frac{\lambda v_i}{\|A^Tv_i\|_2} \qquad \frac{\overset{\geq 0}{\lambda}}{\underset{\geq 0}{\|A^Tv_i\|_2}} = \gamma_i \geq 0$$

$Qv_i = \lambda v_i$

non-negative

$\exists \; \boxed{\gamma_i \geq 0}$ s.t. $\boxed{Au_i = \gamma_i v_i}$

6) d) $m \leq n$. $A_{m\times n}$ ; $v_i \Rightarrow m\times 1$

$u_i \Rightarrow n\times 1$

$$U = [v_1 | v_2 | v_3 | \dots | v_m]_{m\times m}$$

$$V = [u_1 | u_2 | u_3 | \dots | u_m]_{n\times m}$$

$$A = U \overset{m\times m}{\Gamma} V^T$$

$$\underbrace{[\bar{v}_1 | \bar{v}_2 | \bar{v}_3 | \dots | \bar{v}_m]_{m\times m} \begin{bmatrix} \gamma_1 & 0 & \cdots \\ 0 & \gamma_2 & 0 \\ \vdots & \vdots & \ddots \\ & 0 & \quad \gamma_m \end{bmatrix}_{m\times m}}_{U\Gamma} \underbrace{\begin{bmatrix} — u_1 — \\ — u_2 — \\ \vdots \\ — u_m — \end{bmatrix}_{m\times n}}_{V^T}$$

$$U\Gamma \Rightarrow (\Gamma^T U^T)^T$$

$$\Gamma^T = \Gamma,$$

$$\therefore (\Gamma^T U^T)^T = (\Gamma U^T)^T$$

(From prev $q^n$)
$\gamma_i \bar{v}_i = A\bar{u}_i$

$$\Rightarrow \begin{bmatrix} \gamma_1 & 0 & 0 & \cdots \\ 0 & \gamma_2 & & \\ 0 & & \ddots & \\ \vdots & & & \gamma_m \end{bmatrix} \begin{bmatrix} — \bar{v}_1 — \\ — \bar{v}_2 — \\ \vdots \\ — \bar{v}_m — \end{bmatrix} = \begin{bmatrix} \gamma_1 \bar{v}_1 \\ \vdots \\ \gamma_1 \bar{v}_m \end{bmatrix} = \begin{bmatrix} A\bar{u}_1 \\ A\bar{u}_2 \\ \vdots \\ A\bar{u}_m \end{bmatrix}$$

$$\therefore (\Gamma U^T)^T = \begin{bmatrix} A\bar{u}_1 \\ A\bar{u}_2 \\ A\bar{u}_m \end{bmatrix}^T = [A\bar{u}_1 \ A\bar{u}_2 \ \dots \ A\bar{u}_m]$$

$$\therefore U\Gamma V^T = [A\bar{u}_1 \ A\bar{u}_2 \ \dots \ A\bar{u}_m]_{m\times m} \begin{bmatrix} — u_1 — \\ — u_2 — \\ \vdots \\ — u_m — \end{bmatrix}_{m\times n}$$

$$U\Gamma V^T = \underset{m\times n}{A} \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{bmatrix}_{n\times m} \begin{bmatrix} - & u_1 & - \\ - & u_2 & - \\ & \vdots & \\ - & u_m & - \end{bmatrix}_{m\times n}$$

$$= AVV^T \quad \text{(In slides)}$$

Now, $VV^T = I$, so, $\boxed{U\Gamma V^T = AI = A}$

Hence, Proved.

$\cancel{A = I}$