5) $\mathcal{X} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots \bar{x}_N\}$. $(\bar{x}_i \in \mathbb{R}^d)$

$$\bar{x} = \frac{\sum \bar{x}_i}{N} \quad \text{(Mean)}$$

So, our objective function (to be maximized) is :

$\boxed{\bar{f}^T C \bar{f}}$ with the following constraints:

① $\bar{e} \cdot \bar{f} = 0$ i.e. $\bar{e}^T \bar{f} = \boxed{\bar{f}^T \bar{e} = 0}$

② $\bar{f} \cdot \bar{f} = 1$ i.e. $\boxed{\bar{f}^T \bar{f} = 1}$

Now, we use the concept of Lagrange multipliers to get our final modified function to be optimized:

$$J(\bar{f}) = \bar{f}^T C \bar{f} - \lambda(\bar{f}^T \bar{f} - 1) - \mu(\bar{f}^T \bar{e})$$

Taking derivative of J wrt. $\bar{f}$, and setting it to 0 we get:

$$2(C\bar{f} - \lambda \bar{f}) - \mu \bar{e} = 0 \quad \ldots\ldots ①$$

Now, premultiplying by $\bar{e}^T$ gives us:

$$2(\bar{e}^T C \bar{f} - \lambda \bar{e}^T \bar{f}) - \mu \bar{e}^T \bar{e} = 0$$

$\bar{e}^T \bar{e} = 1$; Also: $\bar{e}^T \bar{f} = 0$; So:

$$2(\bar{e}^T C \bar{f}) = \mu.$$

$$\Rightarrow 2((C^T \bar{e})^T \bar{f}) = \mu$$

$C^T = C$ (as symmetric)

$$\Rightarrow 2(C\bar{e})^T \bar{f} = \mu$$

Since $\bar{e}$ is eigenvector of C with largest eigenvalue, say λ_1

$$\Rightarrow \mu = 2\lambda_1 \bar{e}^T \bar{f}$$

$\bar{e}^T \bar{f} = 0$; So:

$$\Rightarrow \mu = 2\lambda_1(0) \quad \boxed{\mu = 0}.$$

So, now, eq (1) becomes:

$$2C\bar{f} - 2\lambda \bar{f} = 0$$

$$\Rightarrow C\bar{f} = \lambda \bar{f}$$

$\bar{f}$ is an eigenvector of C.

Now, $\bar{f}^T C \bar{f}$ is to be maximized

and $\bar{f}^T C \bar{f} = \lambda \bar{f}^T \bar{f} = \underset{\text{eigen value}}{\lambda}$.

So, $\bar{f}$ has to be an eigenvector with largest possible eigen value and simultaneously perpendicular to $\bar{e}$ (where $\bar{e}$ is the eigenvector corresponding to the largest eigenvalue. Hence, we must have $\bar{f} \neq \bar{e}$ & $\bar{f}$ maximing $\bar{f}^T C \bar{f}$, i.e. as large eigenvalue as possible. Thus, $\bar{f}$ has to be the eigenvector corresponding to 2^{nd} largest eigen value.

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