Some problems on Brownian motions

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Abstract

We solve some problems in Brownian motion that will be useful in other math-finance topics.

Contents

1 Evolution of square of GBM

• We know empirically that a stock price $S(t)$ evolves as a Geometric Brownian Motion:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW(t) \tag{1}$$

where the drift μ represents interest rate, σ represents volatility in the market and $W(t)$ is the Canonical Brownian Motion or Standard Wiener Process with mean 0 and variance t.

• Question – How does S^2 evolve.

2 Independent increment

• What is the distribution of $B(t)$ given $B(0) = 0$ and $B(s) = 10$ with $0 \leq s \leq t$?

Soln. $B(t) - B(s) \sim \mathcal{N}(0, t - s)$ is independent of $B(s) = 10$ implying $B(t) \sim \mathcal{N}(10, t - s)$.

3 Brownian Bridge

• A standard Wiener process is "tied down" at the origin: $W(0) = 0$. In a Brownian bridge, the process is "tied down" at both ends, $t = 0$ and $t = T$: $B(0) = B(T) = 0$. In a slight generalization, we can also consider $B(T)$ taking the value of a known constant. B(t) = W(t) - t/T*W(T)

• What is the distribution of $B(t)$ given $B(0) = 0$ and $B(s) = 10$ with $0 \leq t \leq s$?

• This is a trickier question. Before answering this, let us try a different question coming from the binomial model. It will give us the intuition for this problem.

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