UNIVERSITY OF TEXAS AT AUSTIN

## Problem set #4
### Happy Thanksgiving!

4.1. **One binomial period.** Provide your <u>final answer</u> only for the following problems.

**Problem 4.1.** (5 points) A non-dividend-paying stock, currently priced at \$125 per share, can either go up by \$25 or down \$25 in a year. Consider a one-year European call option with a strike price of \$135. The continuously-compounded risk-free interest rate is 8%. Use a one-period binomial model to determine the current price $V_C(0)$ of the call option.

    (a) About 8.60
    (b) About 9.52
    (c) About 9.76
    (d) About 9.81
    (e) None of the above.

**Solution: (d)**
In this problem, $T = h = 1$. The risk-neutral probability of the stock price going up is
$$p^* = \frac{e^r - (100/125)}{(150/125) - (100/125)} \approx 0.71.$$
Using the risk-neutral pricing formula, we get
$$V_C(0) = e^{-rT} p^* (150 - 135) = e^{-0.08} \cdot 0.71 \cdot 15 \approx 9.81.$$

**Problem 4.2.** (5 points) Consider a non-dividend-paying stock with the initial price of $S(0) = 100$. Assume that the annual risk-free continuously compounded interest rate equals $r = 0.05$. Let the annualized standard deviation of the sontinuously compounded stock return, i.e., the **volatility** be $\sigma = 0.25$.
Using a one-period forward binomial tree, calculate the price of a one-year at-the-money European call on this underlying asset.

    (a) \$11.07
    (b) \$12.46
    (c) \$13.38
    (d) \$14.58
    (e) None of the above.

<span style="color:red">Because at the down node, the payoff is guaranteed to be 0 (since it's below the strike), so we don't need the exact value of d to compute the option's value</span>

**Solution: (d)**
By the definition of the forward binomial tree, with the given data,
$$u = e^{(r-\delta)h + \sigma\sqrt{h}} = e^{0.05 + 0.25} = e^{0.3} \approx 1.35, \quad d = e^{(r-\delta)h - \sigma\sqrt{h}} = e^{-0.2} < 1.$$
We do not care about the actual value of $d$ since the option is at-the-money and we get the payoff of 0 at the lower node regardless of the actual value. Also, we do not need $d$ explicitly to calculate the risk-neutral probability in this model, since
$$p^* = \frac{1}{1 + e^{\sigma\sqrt{h}}} = \frac{1}{1 + e^{0.25}} = 0.4378.$$
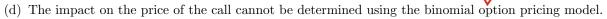Finally, by risk-neutral pricing
$$V_C(0) = e^{-0.05}(135 - 100) \cdot 0.4378 \approx 14.58.$$

**Problem 4.3.** (5 points) Consider the one-period binomial option pricing model. Let $V_C(0) > 0$ denote the price of a European call on a stock which pays continuous dividends. What is the impact on the value of European call option prices if the company decides to increase the dividend yield paid to the shareholders?

    (a) The call-option price will drop.

If the stock pays dividends, the stock price tends to drop when the dividend is paid.
So:
Higher dividend yield → lower expected stock price in the future → less valuable call option.

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(b) The call-option price will increase.

(c) The call-option price will always remain constant.

(d) The impact on the price of the call cannot be determined using the binomial option pricing model. ✓

(e) There is not enough information provided.

**Solution: (a)**

Let $\delta < \tilde{\delta}$ be the two dividend yields. Then, the risk-neutral price of the European call on the stock with the dividend yield $\delta$ equals

$$V_C(0) = e^{-rT}[p^*(S_u - K)_+ + (1 - p^*)(S_d - K)_+]$$

with $p^* = (e^{(r-\delta)h} - d)/(u - d)$. On the other hand, the risk-neutral price of the European call on the stock with the dividend yield $\tilde{\delta}$ equals

$$\tilde{V}_C(0) = e^{-rT}[\tilde{p}^*(S_u - K)_+ + (1 - \tilde{p}^*)(S_d - K)_+]$$

with $\tilde{p}^* = (e^{(r-\tilde{\delta})h} - d)/(u - d)$. We have

$$\delta < \tilde{\delta} \quad \Rightarrow \quad e^{(r-\delta)h} > e^{(r-\tilde{\delta})h} \quad \Rightarrow p^* > \tilde{p}^* \quad \Rightarrow \quad V_C(0) > \tilde{V}_C(0).$$

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✓✓ **4.2. Two binomial periods.** Please, provide your **complete solution** to the following problem(s):

**Problem 4.4.** (10 points)

Consider a two-period binomial model for the stock price with both periods of length one year. Let the initial stock price be $S(0) = 100$ and assume that the stock pays no dividends. Let the up and down factors be $u = 1.25$ and $d = 0.75$, respectively. Let the continuously compounded interest rate be $r = 0.05$ per annum.

Roger is interested in purchasing a **chooser** option with the provision that he can choose if the option is a put or a call after one year. The strike for this option is \$100 and the expiry date is two years.

Using the above binomial tree, find the price of the chooser option.

**Solution:** With the fiven $u$ and $d$, we get the following tree modeling the stock price

The risk-neutral probability of the stock price going up is

$$p^* = \frac{e^{0.05} - 0.75}{1.25 - 0.75} = 2(e^{0.05} - 0.75) \approx 0.6025.$$

We can price the chooser option in question in two ways.

*Method I.* One way is to consider all of the possible payoffs for both the put and the call and see which one the rational investor would choose at time one depending on whether he is in the "up" or the "down" node.

In the "up" node, the value of the call is

$$e^{-0.05}[0.6025 \cdot 56.25] = 32.24.$$

In the same node, the value of the put is

$$e^{-0.05}[(1 - 0.6025) \cdot 6.25] = 2.36.$$

So, a prudent investor would choose for his option to become a call if he/she is in the "up" node and the value of his chooser option at this node is 32.24.

Similarly, at the "down" node, the value of the call is zero, while the value of the put is

$$e^{-0.05}[0.6025 \cdot 6.25 + (1 - 0.6025) \cdot 43.75] = 20.12.$$

So, at this node, the rational investor chooses for the option to become a put and, thus, chooser option is worth 20.12.

Finally, the time-0 value of the chooser option is

$$e^{-0.05}[0.6025 \cdot 32.24 + (1 - 0.6025) \cdot 20.12] = 26.08.$$

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*Method II.* The **alternative method** involves the pricing formula for chooser options (see Sample MFE Problem #25). Here, the time-0 price of a chooser option can be written as

$$V_C(0, T, K) + V_P(0, t^*, Ke^{-r(T-t^*)})$$

where $V_C(0, T, K)$ stands for the time-0 price of a European call where $K$ denotes the strike and $T$ is the expiration date and where $V_P(0, t^*, Ke^{-r(T-t^*)})$ stands for the time-0 price of a European put where $Ke^{-r(T-t^*)}$ is the strike and $t^*$ is the expiration date.

In the current problem,

$$V_C(0, 2, 100) = e^{-0.05*2}56.26 \cdot 0.6025^2 = 18.48,$$

and

$$V_P(0, 1, 95.12) = e^{-0.05}20.12 \cdot (1 - 0.6025) = 7.61.$$

So, the price of the chooser option is $18.48 + 7.61 = 26.09$. The difference in cents between these two answers is due to rounding errors.

4.3. **American-option pricing.** Provide your **complete solution** to the following problems.

**Problem 4.5.** (10 points) For a two-period binomial model, you are given that:
  (1) each period is one year;
  (2) the current price of a non-dividend-paying stock $S$ is $S(0) = \$20$;
  (3) $u = 1.3$, with $u$ as in the standard notation for the binomial model;
  (4) $d = 0.9$, with $d$ as in the standard notation for the binomial model;
  (5) the continuously compounded risk-free interest rate is $r = 0.05$.
Find the price of an American call option on the stock $S$ with $T = 2$ and the strike price $K = \$23$.

**Solution:**   Since the stock does not pay dividends, we can price the option as if it were European, i.e., without taking into account the possibility of early exercise.

The risk-neutral probability is

$$p^* = \frac{e^{0.05} - 0.9}{1.3 - 0.9} = 0.3782.$$

When one constructs the two-period binomial tree, one gets

$$S_u = 26, \ S_d = 17,$$

$$S_{uu} = 33.80, \ S_{ud} = S_{dd} = 23.4, \ S_{dd} = 16.2.$$

Hence, the payoffs at the end of the second period are

$$V_{uu} = 10.80, \quad V_{ud} = 0.4, \quad V_{dd} = 0.$$

So, taking the expected value at time 0 of the payoff with respect to the risk-neutral probability, we get that the price of the call should be

$$e^{-0.05 \times 2}[V_{uu} \cdot (p^*)^2 + V_{ud} \cdot 2p^*(1 - p^*)] = e^{-0.1}[10.8 \times 0.3782^2 + 0.4 \times 2 \times 0.3782 \times (1 - 0.3782)] = 1.568.$$

**Problem 4.6.** (10 points) The current price of a non-dividend-paying stock is \$100 per share and its volatility is given to be 0.25.

The continuously-compounded, risk-free interest rate equals 0.06.

Consider a \$110-strike, one-year American put on the above stock. Use a two-period forward binomial stock-price tree to calculate the current price of the American put.

**Solution:** By the definition of the forward binomial tree, we obtain

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = e^{0.03+0.25\sqrt{\frac{1}{2}}} \approx 1.2297,$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = e^{0.03-0.25\sqrt{\frac{1}{2}}} \approx 0.8635.$$

in our usual notation. The binomial tree modeling the stock price is
    The risk-neutral probability of the stock price going up in a single period equals

$$p^* = \frac{e^{(r-\delta)h}-d}{u-d} = \frac{e^{0.03}-0.8635}{1.2297-0.8635} = 0.4559.$$

Should the American option not be exercised early the possible payoffs would be

$$V_{uu} = 0, V_{ud} = 110 - 106.18 = 3.82, V_{dd} = 110 - 74.56 = 35.44.$$

It is not sensible to exercise the American put at the up node, so the value of the American put equals the continuation value at the up node. We get

$$V_u^A = CV_u = e^{-0.03}(1 - 0.4559) \times 3.82 = 2.017.$$

At the down node, the value of immediate exercise is

$$IE_d = 110 - 86.35 = 23.65.$$

On the other hand, the continuation value at the down node equals

$$CV_d = e^{-0.03}[0.4559 \times 3.82 + (1 - 0.4559) \times 35.44] = 20.4031.$$

We conclude that the American put's value at the down node equals the value of immediate exervise, i.e., $V_d^A = 23.65$.
    Should the option be exercised at time−0, the payoff would be 10. The continuation value at the root node is

$$CV_0 = e^{-0.03}[0.4559 \times 2.017 + (1 - 0.4559) \times 23.65] = 13.38.$$

So, the price we were looking for is $13.38.

**Problem 4.7.** (10 points) *Source: Problem 10.10 from "Derivatives Markets" by McDonald.*
Let $S(0) = 100, K = 95, r = 0.08, \sigma = 0.3, \delta = 0$ and $T = 1$ (in the usual notation). You should use a 3−period forward binomial tree for the above stock to price an American call option on $S$. Let this price be denoted by $V_C^A(0)$. Then,

**Solution:** **The Pedestrian Method**
The risk-neutral probability is $p^* = 0.46$.
    We can calculate for the different nodes of the tree.
    Then, going backwards through the tree:

|                          | node uu   | node ud = du | node dd |
|--------------------------|-----------|--------------|---------|
| delta                    | 1         | 0.8966       | 0       |
| $B$                      | −92.5001  | −79.532      | 0       |
| call premium             | 56.6441   | 15.0403      | 0       |
| value of early exercise  | 54.1442   | 10.478       | 0       |

Using these values at the previous node and at the initial node yields:

|                          | $t = 0$   | node $d$  | node $u$  |
|--------------------------|-----------|-----------|-----------|
| delta                    | 0.7400    | 0.4870    | 0.9528    |
| $B$                      | −55.7190  | −35.3748  | −83.2073  |
| call premium             | 18.2826   | 6.6897    | 33.1493   |
| value of early exercise  | 5         | 0         | 27.1250   |

Please note that in all instances the value of immediate exercise is smaller than the continuation value, the (European) call premium. Therefore, the value of the European call and the American call are identical.

### The Efficient Method

Again, we calculate the values at different nodes in the tree. Then, we realize that the stock pays no dividends and that the price of the American call is the same as the price of the European call. We get

$$e^{-0.08}[(182 - 95) \cdot 0.46^3 + (128 - 95) \cdot 3 \cdot 0.46^2 \cdot 0.54] \approx 18.2596.$$

The answer is a bit different from the answer using the first method, since there are various rounding errors in both calculations. But, they are both good enough!

**Problem 4.8.** (15 points)
Consider a one-period forward binomial model for the stock-price movement over the following year. The current stock price is $S(0) = 100$, its dividend yield is 0.05 and its volatility is 0.3 The continuously compounded risk-free interest rate is given to be 0.05.

Consider American call options on this stock with the expiration date at the end of the period/year.

What is the maximal (rounded to the nearest dollar) strike price $K$ for which there is early exercise?

**Solution:** In our usual notation, $u = e^{0.3} = 1.35$ and $d = 0.74$. The risk-neutral probability is

$$p^* = \frac{1}{1 + e^{\sigma}} = 0.425.$$

Then, the continuation value at the root node is

$$V_C(0) = e^{-0.05}[0.425(135 - K)_+ + 0.575(74 - K)_+]$$

as a function of $K$. The early-exercise condition is

$$100 - K > V_C(0).$$

It is evident that in order for early exercise to occur it must be that $K < 100$. So, let us focus on the possible solutions to the above inequality in the interval $(74, 100)$ first. For such $K$, the above inequality becomes

$$100 - K > e^{-0.05} \times 0.425(135 - K).$$

Note the absence of the "positive part" in the last expression. The $K$ which satisfy this inequality are such that

$$100 - 54.57 > 0.596K \quad \Rightarrow \quad 76 > K.$$

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4.4. **Properties of American-option prices.** Provide your <u>final answer</u> only for the following problems.

**Problem 4.9.** (2 points) If the interest rate is 0, then it never makes sense to early exercise an American call option on a stock with a positive dividend yield.

**Solution: FALSE**

<span style="color:red">when a company pays divident, its stock value decreases by the amount it paid as divided on after exidivident date</span>

**Problem 4.10.** (5 points) Which of the following American-type options will **never** be exercised early to get strictly higher profit?

<span style="color:red">Call options lose value because they give the right to buy at a fixed price, and if the stock price drops, they become less valuable.</span>

(a) Put on a dividend-paying stock

(b) Call on a dividend-paying stock   <span style="color:red">Put options gain value because they give the right to sell at a fixed price, and a lower stock price makes them more valuable.</span>

(c) Call on a non-dividend-paying stock

(d) Put on a non-dividend-paying stock

(e) All of the above should be exercised early in a certain scenario.

**Solution: (c)**