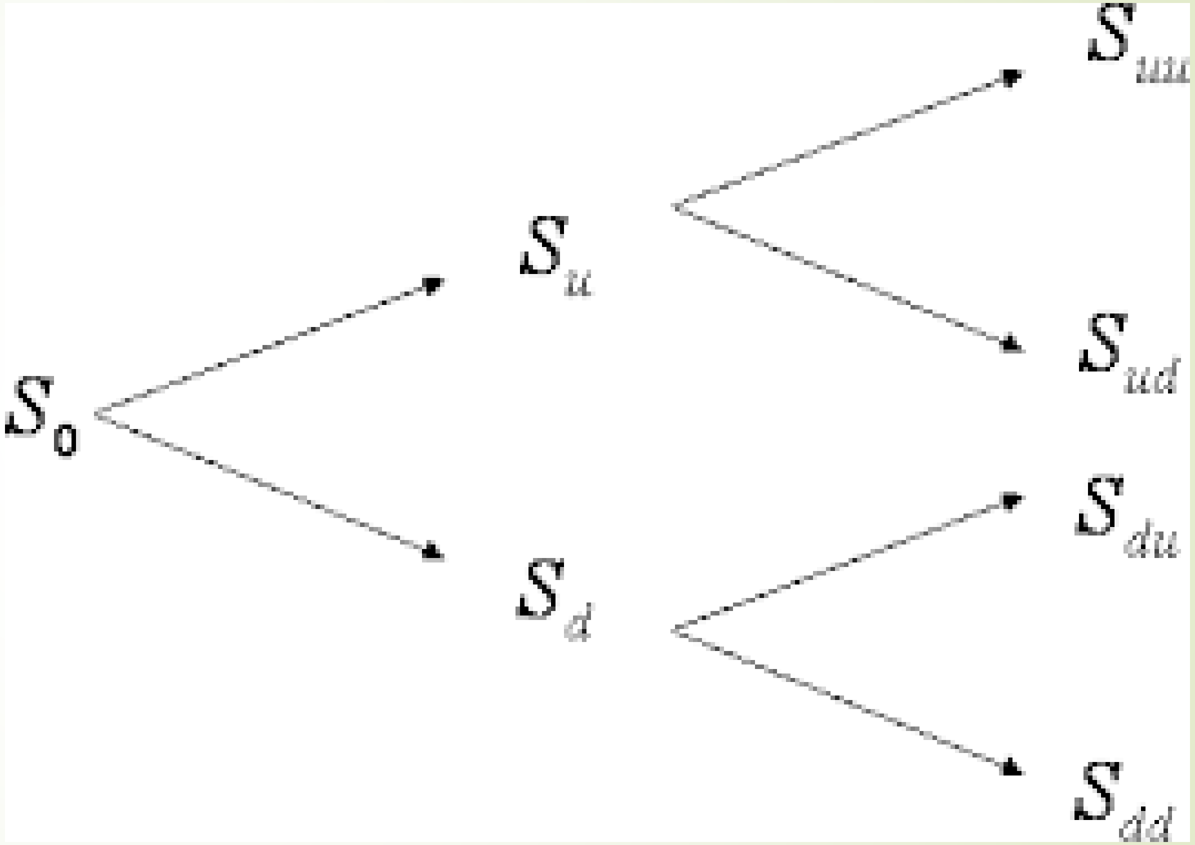
Option Pricing Models: Binomial Model



portfolio is of same stock

Philosophy of option pricing

It is possible to annul the volatility in the price of the derivative by a counter position in the underlying asset and vice versa.

delta = hedge ratio

portfolio

Derivative - Δ*Underlying Asset = Risk-free asset

Derivative = Δ*Underlying Asset + Risk-free asset

The philosophy of option pricing says that we can remove this risk by creating a special kind of portfolio. This portfolio combines:

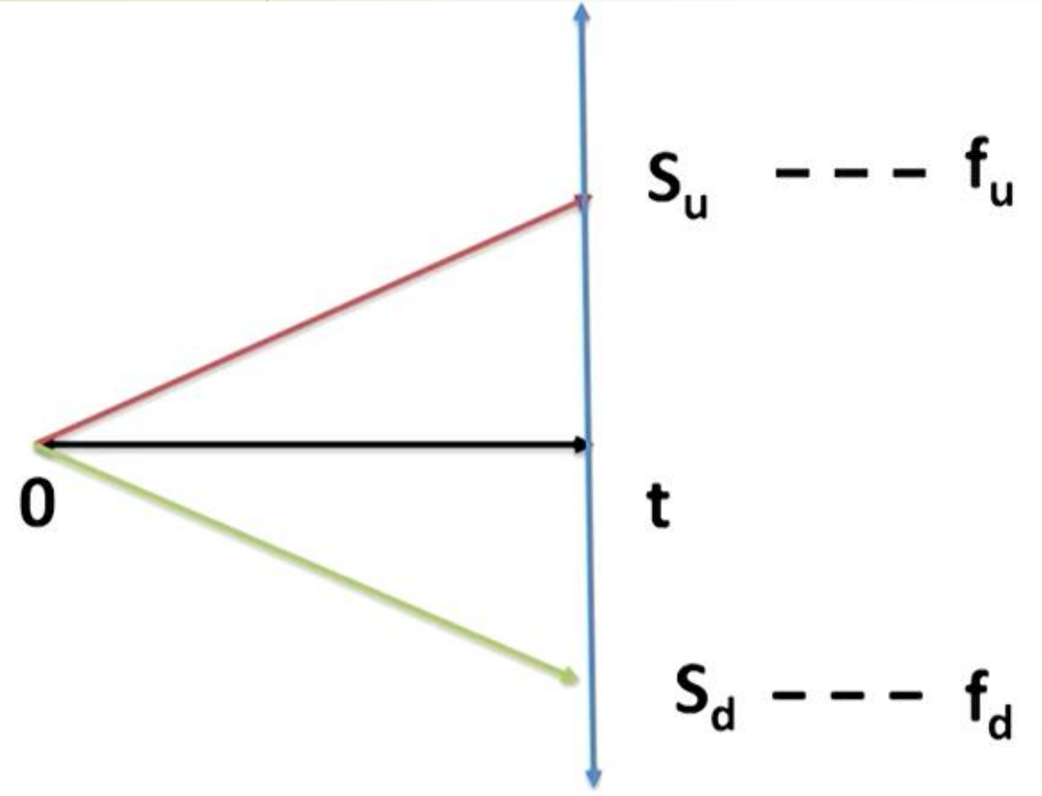
A position in the option (like owning or selling the option).
A position in the stock itself (buying or short-selling the stock

So, in simple words:

We cancel out the risk by combining the option and the stock in the right way.
This makes the portfolio grow at a safe, predictable rate.
That’s how we figure out the option’s fair value.

We adjust the number of shares of the stock and the option in a way that their gains and losses balance each other out, no matter whether the stock price goes up or down

The Binomial model



$S_u = S_0 * u$

$S_d = S_0 * d$

We construct a portfolio V at t=0 consisting of:

1. one unit of derivative, costing f_0 at t=0
2. Δ units of stock costing S_0 per unit

The value of this portfolio at t=0 is $V_0 = f_0 + \Delta S_0$

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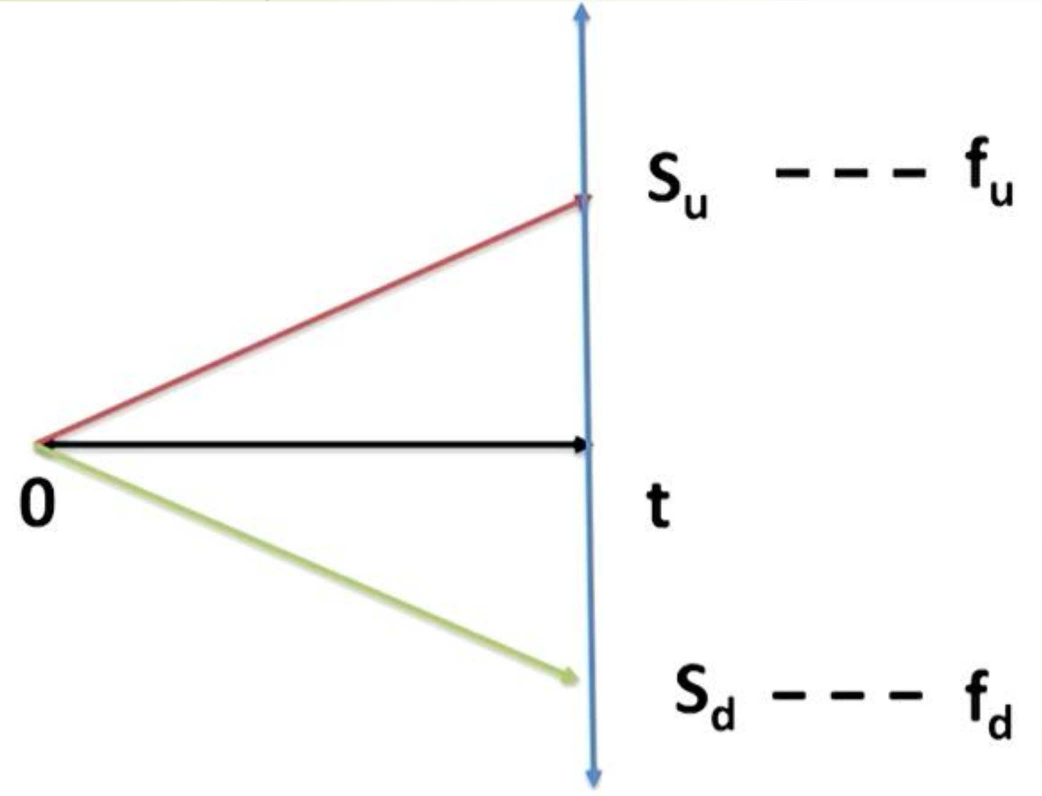
The Binomial model



The value of this portfolio V at t=T will be

$V_u = f_u + \Delta S_u$ if the stock price goes up

$V_d = f_d + \Delta S_d$ if the stock price goes down

If the portfolio V is to be riskless

$V_u = f_u + \Delta S_u = f_d + \Delta S_d = V_d$

$\Delta = -(f_u - f_d)/(S_u - S_d)$

The Binomial Model

$$f_0 = V_0 - \Delta S_0 = V_T e^{-rT} - \Delta S_0 = (f_u + \Delta S_u)e^{-rT} - \Delta S_0$$

$$= e^{-rT}\frac{f_d S_u - f_u S_d}{S_u - S_d} + \frac{f_u - f_d}{S_u - S_d} S_0$$

$$= e^{-rT}\left[f_d \boxed{\frac{S_u - S_0 e^{rT}}{S_u - S_d}} + f_u \boxed{\frac{S_0 e^{rT} - S_d}{S_u - S_d}}\right]$$

$= e^{-rT}[f_d q_d + f_u q_u]$ where $q_d = \frac{u - e^{rT}}{u-d}$, $\boxed{q_u = \frac{e^{rT} - d}{u-d}}$

p = probability

p of the stock moving up

$q_d + q_u = 1$

$0 \le q_d, q_u \le 1$

u = S(u) / S(0)

d = S(d) / S(0)

The Binomial Model

$$q_d = \frac{S_u - S_0 e^{rT}}{S_u - S_d} < 0?$$

$$\frac{S_u - S_0 e^{rT}}{S_u - S_d} < 0 \Rightarrow \boxed{S_u < S_0 e^{rT}}$$

The above is not possible since risky asset's maximum possible value is less than risk-free return at t = T

$$q_d + q_u = 1$$
$$0 \leq q_d, q_u \leq 1$$

q_d, q_u satisfiy axioms of probability. They are some probabilities.

We adjust the number of shares of the stock and the option in a way that their gains and losses balance each other out, no matter whether the stock price goes up or down

The Expectation of The Stock Price

q_d, q_u are some probabilities

$$E_q(S_T) = S_d q_d + S_u q_u = S_d \frac{S_u - S_0 e^{rT}}{S_u - S_d} + S_u \frac{S_0 e^{rT} - S_d}{S_u - S_d} = S_0 e^{rt}$$

These probabilities reflect the probabilities of stock price movements in a risk neutral world i.e. in a world where risk has no significance to each and every investor and return is the only investment criterion. Hence, these probabilities are also called risk neutral probabilities.

The current value of the option is the expected value of payoffs from the option taken according to the risk-neutral probability measure, discounted at the risk-free rate.

$$price\ of\ derivative = e^{-rt} E_q(f(S_T))$$ = e^(-rt) * pay_off * q(u)

q(u) = (e^(6/100*0.25))*100 - 70 / 150-70

Example

The current price of a stock is 100. After 3 months, it be either 150 or 70. A call option with expiry date is 3 months from today with strike price 110. What will be the price of the call option? 6% is the risk-free interest rate.

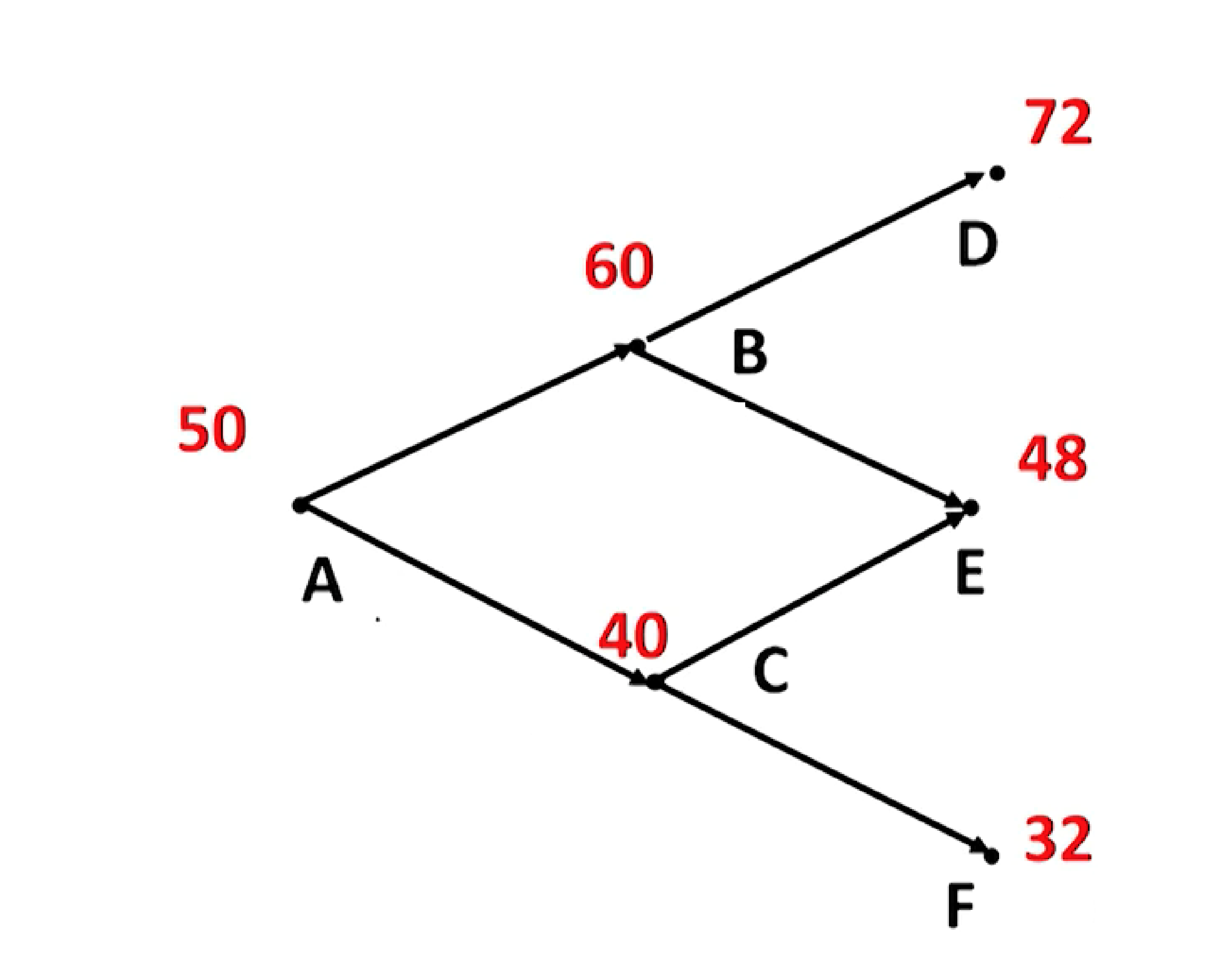
Stock Price		100
Upper		150
Lower		70
r		6%
T		0.25
Strike Price		110
Payoff	=Upper-Strike	40
		0
q(u)	=(EXP(interest*time)*Stock-Lower)/(Upper-Lower)	0.39
call price	=EXP(-interest*time)*PayoffU*qu	15.52

2-Steps Binomial

Consider a 2-year European put with a strike price of 52 on a stock whose current price is 50.

In each time step (of one year) the stock price either moves up by 20% or moves down by 20%.

Let the risk-free interest rate be 5%.



pay off call option = max(St - K,0)
pay off put option = max(K-St,0)

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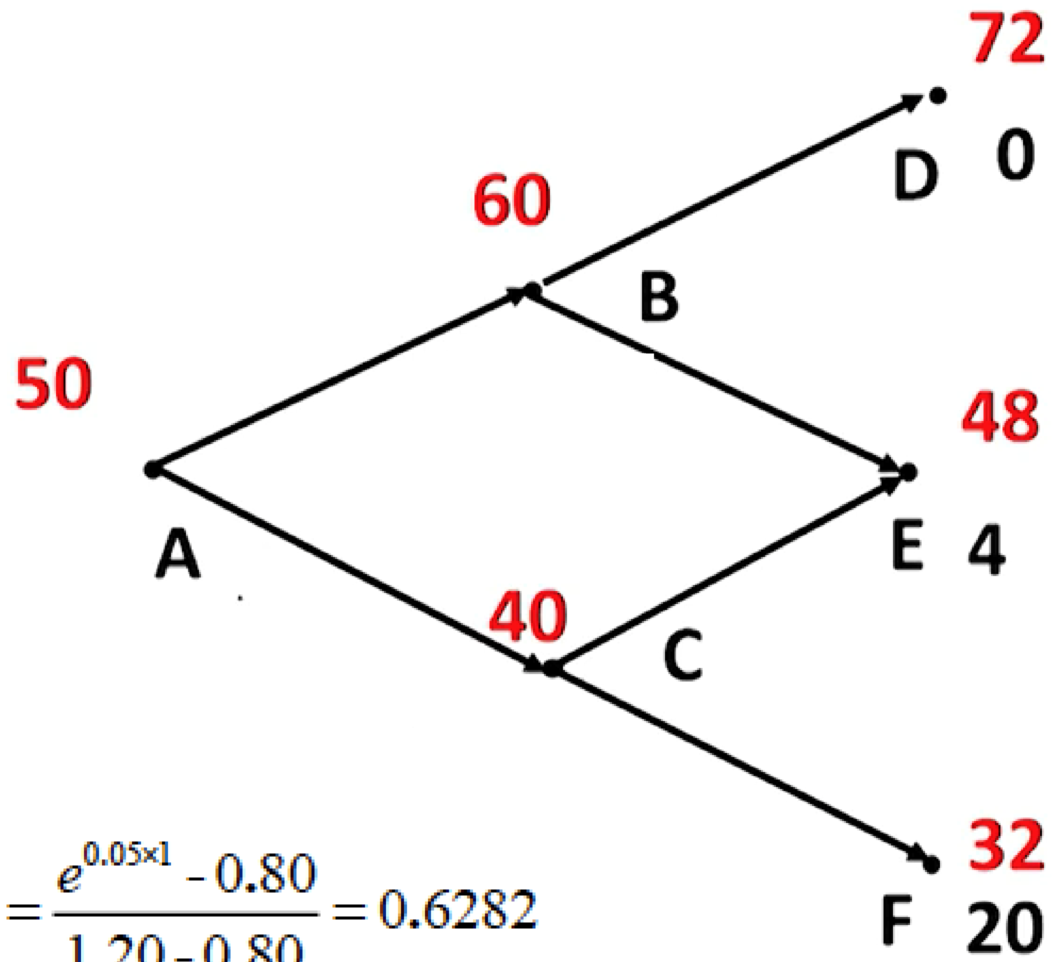

$$q_u = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05\times 1} - 0.80}{1.20 - 0.80} = 0.6282$$

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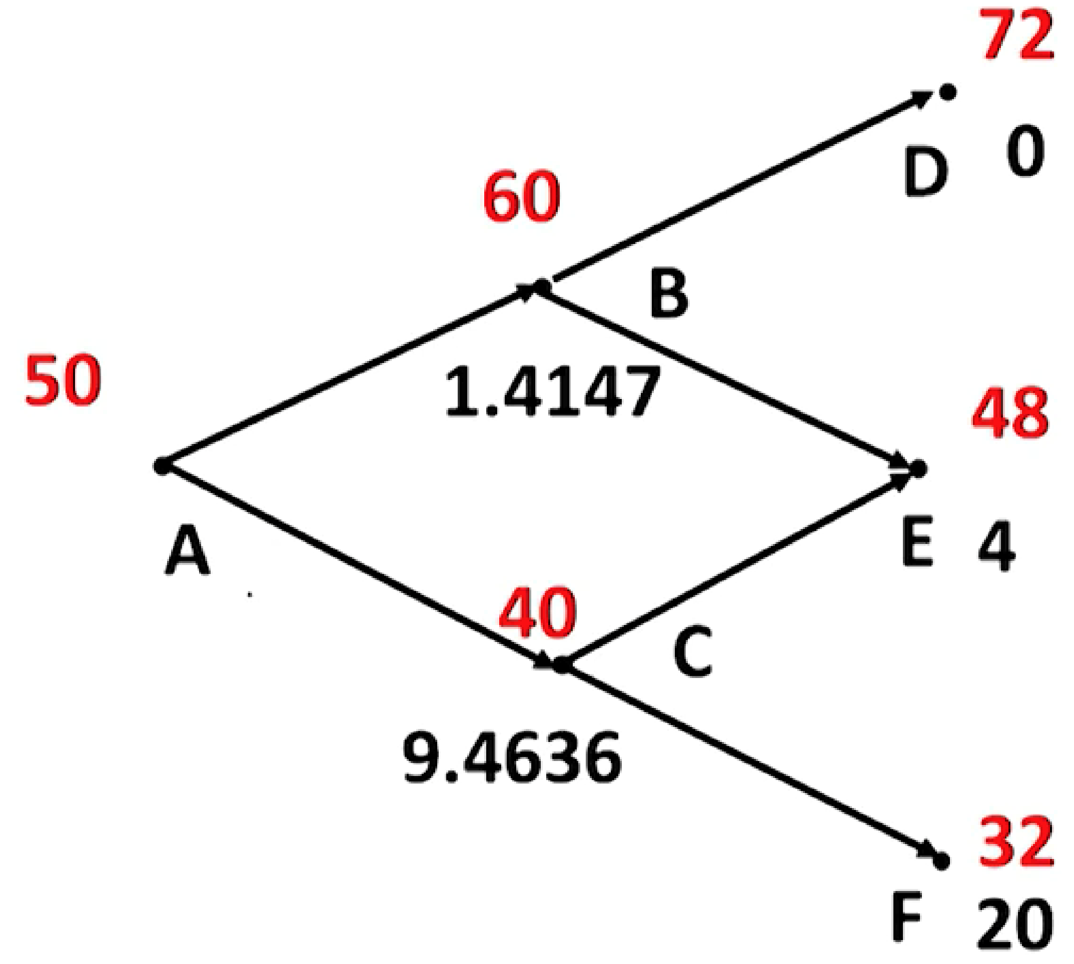

$$f_u = e^{-rT}\left[qf_{uu} + (1-q)f_{ud}\right] = 0.9512(0.6282\times 0 + 0.3718\times 4) = 1.4147$$

$$f_d = e^{-rT}\left[qf_{ud} + (1-q)f_{dd}\right] = 0.9512(0.6282\times 4 + 0.3718\times 20) = 9.4636$$

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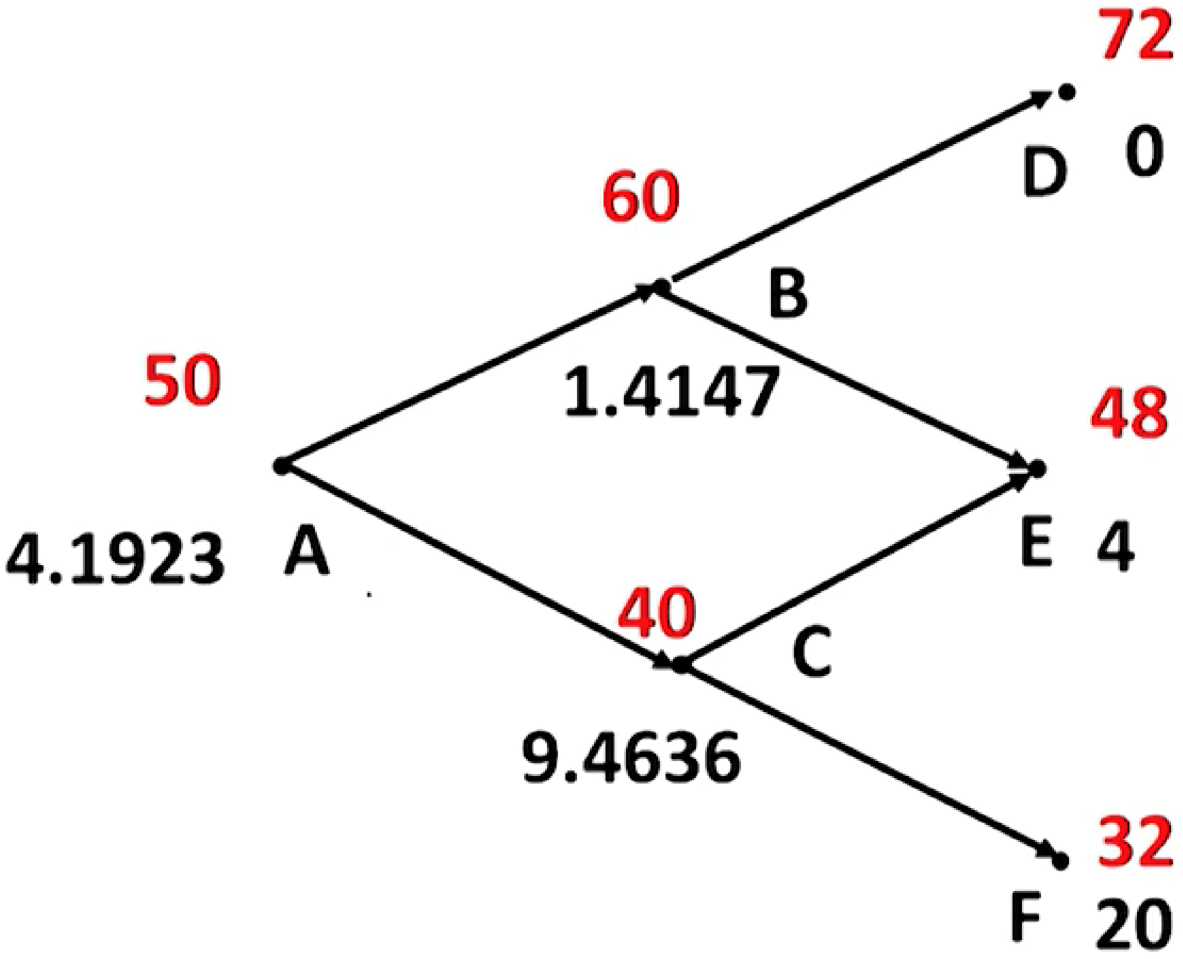

$$f_0 = e^{-rT}\left[qf_u + (1-q)f_d\right] = 0.9512(0.6282\times1.4147 + 0.3718\times9.4636) = 4.1923$$