Introduction to Option pricing

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Introduction

We will discuss via an example as to why we need a separate theory for pricing option. Why does the usual principle price equals expected returns not work in this case?

We will discuss the main ingredients of option pricing theory via an artificial example in discrete time and focus on key ideas such as No Arbitrage and Complete Markets and show its connections with Martingale theory.

A question

We begin with a question. Let S_t be the model for the price of stock of a company and consider a European call option with terminal time T and strike price K. If the rate of interest is r, then the expected discounted gain from the option is

$$E_P\left[\frac{(S_T - K)^+}{\exp\{rT\}}\right].$$

Should this not be the price of the option? Why should Equivalent martingale measure enter the picture ?

An Example

We will now consider a concrete example and use it to illustrate the ideas that play an important role in option pricing. This is an artificial example. Its role is only to explain the notions such as No Aribitrage, hedging strategy, complete markets, etc.

To avoid technicalities, we will consider a discrete model.

Standard Assumptions

We will consider a Frictionless market:

(1) there are no transaction costs (in buying or selling shares)

(2) : the rate of interest on investments is same as that on loans(or there are riskless bonds which carry a fixed rate of return available for trade).

Example

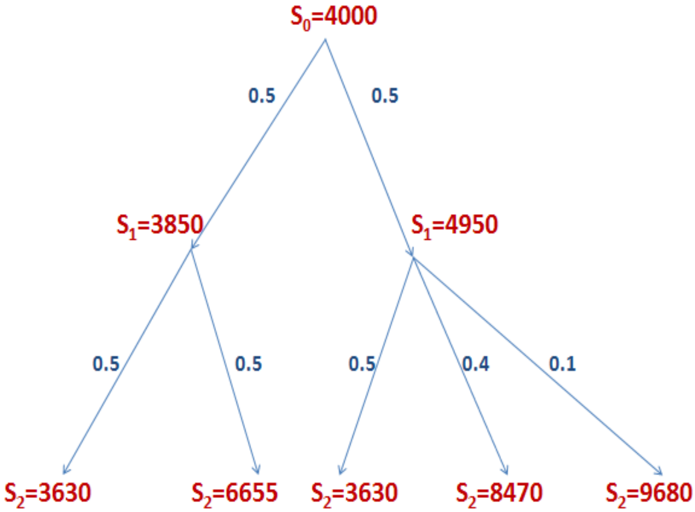
We will consider a company whose shares are trading at the initial time $(t = 0)$@ Rs S_0 per share. We assume that trading is allowed only at the end of first year or at the end of second year.

Let S_1 and S_2 denote the price of a share at the end of years 1 and 2 respectively.

Let us assume that the bonds carry interest at 10% per year.

Stochastic Model for S_0, S_1, S_2:

$$
\begin{aligned}
P(S_0 = 4000) &= 1\\
P(S_1 = 4950) &= 0.5\\
P(S_1 = 3850) &= 0.5\\
P(S_2 = 9680 \mid S_1 = 4950) &= 0.1\\
P(S_2 = 8470 \mid S_1 = 4950) &= 0.4\\
P(S_2 = 3630 \mid S_1 = 4950) &= 0.5\\
P(S_2 = 6655 \mid S_1 = 3850) &= 0.5\\
P(S_2 = 3630 \mid S_1 = 3850) &= 0.5
\end{aligned}
$$


S_0=4000
0.5
0.5
S_1=3850
S_1=4950
0.5
0.5
0.5
0.4
0.1
S_2=3630
S_2=6655
S_2=3630
S_2=8470
S_2=9680

European Call Option

Suppose that also selling in the market is European call option on these shares, with terminal time $T = 2$ years, stike price $K = 6050$.

At what price should this option be traded in a market in equilbrium (which means enough buyers and sellers will be there in the market at this price)?

At a first glance it would appear, at least to those familiar with probability theory, that the option price must be the discounted expected return.

Discounted Expected Return

Thus the gain from the option is

$$\max(S_2 - 6050,\ 0)$$

This gain is due at the end of 2 years. Its worth at time zero (with rate of interest 10%) is

$$\max(S_2 - 6050,\ 0)/1.21$$

Discounted Expected Return

Thus the expected gain is

$$g = E(\max(S_2 - 6050,\ 0)/1.21,)$$

Here, $P(S_2 = 9680) = .05$**,** $P(S_2 = 8470) = .2$**,** $P(S_2 = 6655) = .25$**,** $P(S_2 = 3630) = .5$**.**

This leads to $g = 675$**.**

Can the price of the option be Rs. 675?

Can Option price be 675?

As yet we have not talked of what do we mean by price of an option. However, we will show that price cannot be 675.

Suppose that options are trading @ 675 so there are buyers as well as sellers at this price.

An investor A decides to buy 100 options by investing Rs. 67500.

Can Option price be 675?.......

Mr. B decides to invest the same amount Rs. $67500 (= x_0)$ **at time** 0**; buy** $\pi_0 = 75$ **shares @ Rs. 4000 by borrowing the shortfall (short selling the bond).**

At the end of the year, if the price is $S_1 = 4950$**, B sells 10 shares to bring down his holding to** $\pi_{11} = 65$**, using the proceeds to settle part of this loan. If** $S_1 = 3850$**, he sells 50 shares: now** $\pi_{12} = 25$**, again paying off loan with the money received.**

Can Option price be 675?.....

Denoting the bonds held by the investor at time 0 **by** ξ_0 **and at time 1 by** ξ_{11} **if** $S_1 = 4950$**,** ξ_{12} **if** $S_1 = 3850$ **(negative** ξ **means loan),** ξ**'s are determined by**

$$
\begin{aligned}
\xi_0 &= x_0 - \pi_0 \times 4000 \\
\xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\
\xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12}.
\end{aligned}
$$

Can Option price be 675?.....

For the invester B, $x_0 = 67500$, $\pi_0 = 75$, $\pi_{11} = 65$, $\pi_{12} = 25$ and these equations give $\xi_0 = -232500$, $\xi_{11} = -206250$, $\xi_{12} = -63250$.

Before proceeding further, let us note that a trading strategy is determined by $x_0, \pi_0, \pi_{11}, \pi_{12}$ which in tern determine $\xi_0, \xi_{11}, \xi_{12}$.

Can Option price be 675?.....

The table given below shows the net worth of the holdings of A, B in each of the five possible outcomes of (S_1, S_2): (A's assets are 100 options and no liabilities; B's assets are $\pi_{11}(\pi_{12})$ shares and a deposit of $\xi_{11}(\xi_{12})$ made at time 1 if $S_1 = 4950(S_1 = 3850)$).

Net worth of A and B at the end of year 2:

Outcome: $(S_1, S_2) =$	**A**	**B**
(4950, 9680)	**3,63,000**	**4,02,325**
(4950, 8470)	**2,42,000**	**3,23,675**
(4950, 3630)	**0**	**9,075**
(3850, 6655)	**60,500**	**96,800**
(3850, 3630)	**0**	**21,175**

Can Option price be 675?.....

Note that while both A, B made the same initial investment, namely 67500, but B has done better than A in each possible outcome of the stock prices. So whatever A was buying is ourpriced.

Thus option price must be less than 675!

Can Option price be 675?.....

Not convinced? Assume that there are enough buyers and sellers for the options at Rs 675.
An investor C devises a strategy as follows: Sell 100 options @ 675 per option to collect Rs. 67500 and then follow the strategy of B : $x_0 = 67500$, $\pi_0 = 75$, $\pi_{11} = 65$, $\pi_{12} = 25$.
The net worth of C's is the difference between 3rd column and 2nd column in the previous table and is given by:

Net worth of C at the end of year 2:

Outcome (S_1, S_2)	C
(4950, 9680)	39,325
(4950, 8470)	81,675
(4950, 3630)	9,075
(3850, 6655)	26,300
(3850, 3630)	21,175

Option price < 675

Thus, C would make a profit in each of the five outcomes without making any initial investment.

Many investors would like to follow this strategy and make money without taking any risk, disturbing the equilibrium as soon there would be no buyers for the option. Thus the equilibrium price has to be less than Rs 675.

Arbitrage opportunity

An Arbitrage opportunity is a strategy that involves no initial investment and for which the net worth of holdings (at some time in future) is non-negative for each possible outcome and strictly positive for at least one possible outcome.

In the example above, if the price of the option were 675, the strategy of C outlined above is an example of an Arbitrage opportunity.

Arbitrage opportunity

As explained above, if an Arbitrage opportunity exists, it would disturb the equlibrium as all investors would like to replicate the same. Thus, in a market in equilibrium, Arbitrage opportunities do not exist.

This is known as the principle of No Arbitrage abbriviated as *NA*.

NA plays an extremely important role in pricing of derivative securities.

What is the price p of the option?

In the example discussed above we can conclude

$$NA \Rightarrow p < 675$$

Can we get a lower bound? Can we determine p completey on the basis of NA?

What is the price p of the option?...

Now, let us consider another invester D's strategy : $x_0 = 30000$, $\pi_0 = 50$, $\pi_{11} = 45$, $\pi_{12} = 15$. **The equations**

$$
\begin{aligned}
\xi_0 &= x_0 - \pi_0 \times 4000 \\
\xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\
\xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12}.
\end{aligned}
$$

yield : $\xi_0 = -1,70,000$ $\xi_{11} = -162,250$, $\xi_{12} = -52,250$. **In each of the outcomes, the net worth of D's holding is given by**

What is the price p of the option?...

Outcome $(S_1, S_0) =$	**Worth of D's holding**
(4950, 9680)	**9680** $\pi_{11} + 1.1\xi_{11} = 2,57,125$
(4950, 8470)	**8470** $\pi_{11} + 1.1\xi_{11} = 2,02,675$
(4950, 3630)	**3630**$\pi_{11} + 1.1\xi_{11} = -15,125$
(3850, 6655)	**6655** $\pi_{12} + 1.1\xi_{12} = 42,350$
(3850, 3630)	**3630** $\pi_{12} + 1.1\xi_{12} = -3,025$

What are 100 options worth?

Outcome $(S_1, S_0) =$	Worth of 100 options
(4950, 9680)	(9680 -6050)*100 = 3,63,000
(4950, 8470)	(8470 -6050)*100 = 2,42,000
(4950, 3630)	0
(3850, 6655)	(6655 -6050)*100 = 60,500
(3850, 3630)	0

D's Net worth vs 100 options

Outcome: $(S_1, S_2) =$	**D**	**100 Options**
(4950, 9680)	**2,57,125**	**3,63,000**
(4950, 8470)	**2,02,675**	**2,42,000**
(4950, 3630)	**-15,125**	**0**
(3850, 6655)	**42, 350**	**60,500**
(3850, 3630)	**-3,025**	**0**

What is the price p of the option?...

Note that for each outcome, D's holdings are worth less than the worth of100 options. Thus, 100 options are worth more than 30000 - the investment at time zero that is needed for D's strategy. Thus $p > 300$.

If $p \leq 300$, then the strategy consisting of buying 100 options and $\pi_0 = -50$, $\pi_{11} = -45$, $\pi_{12} = -15$ would be an arbitrage opportunity (note that the π's are (-1) times the corresponding π in D's strategy.

What is the price p of the option?...

So we have shown that

$$NA \Rightarrow 300 < p < 675.$$

Can we narrow the interval $(300, 675)$ for the option price any further? We need to list all possible trading strategies an investor might follow.

What is the price p of the option?...

It can be seen that a trading strategy consists of $(x_0, \pi_0, \pi_{11}, \pi_{12})$. Then $\xi_0, \xi_{11}, \xi_{12}$ are determined by

$$
\begin{aligned}
\xi_0 &= x_0 - \pi_0 \times 4000 \\
\xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\
\xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12}.
\end{aligned}
$$

What is the price p of the option?...

It follows that if for a trading strategy $(x_0, \pi_0, \pi_{11}, \pi_{12})$ **(with** $\xi_0, \xi_{11}, \xi_{12}$ **determined by equations on preceeding page)**

$$\begin{aligned}
9680\pi_{11} + 1.1\xi_{11} &\geq 363000 \\
8470\pi_{11} + 1.1\xi_{11} &\geq 242000 \\
3630\pi_{11} + 1.1\xi_{11} &\geq 0 \\
6655\pi_{12} + 1.1\xi_{12} &\geq 60500 \\
3630\pi_{12} + 1.1\xi_{12} &\geq 0.
\end{aligned}$$

then $100p \leq x_0$.

What is the price p of the option?...

Likewise if $(x_0, \pi_0, \pi_{11}, \pi_{12})$, **(with** $\xi_0, \xi_{11}, \xi_{12}$**) determined as before) satisfy**

$$\begin{aligned}
9680\pi_{11} + 1.1\xi_{11} &\leq 363000 \\
8470\pi_{11} + 1.1\xi_{11} &\leq 242000 \\
3630\pi_{11} + 1.1\xi_{11} &\leq 0 \\
6655\pi_{12} + 1.1\xi_{12} &\leq 60500 \\
3630\pi_{12} + 1.1\xi_{12} &\leq 0.
\end{aligned}$$

then $100p \geq x_0$

LP problem for upper bound

minimize x_0 **subject to**

$$
\begin{aligned}
\xi_0 &= x_0 - \pi_0 \times 4000 \\
\xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\
\xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12} \\
9680\pi_{11} + 1.1\xi_{11} &\geq 3630 \\
8470\pi_{11} + 1.1\xi_{11} &\geq 2420 \\
3630\pi_{11} + 1.1\xi_{11} &\geq 0 \\
6655\pi_{12} + 1.1\xi_{12} &\geq 605 \\
3630\pi_{12} + 1.1\xi_{12} &\geq 0.
\end{aligned}
$$

LP problem for upper bound

The optimum value x^+ of the LP problem given above is an upper bound for p. If for the optimum solution, even one of the inequalities is a strict inequality then $p < x^+$.

Here the variables $x_0, \pi_0, \pi_{11}, \pi_{12}, \ \xi_0, \xi_{11}, \xi_{12}$ are unrestrained, i.e. these are not required

Likewise, consider the LP problem:

maximize x_0 **subject to**

$$
\begin{aligned}
\xi_0 &= x_0 - \pi_0 \times 4000 \\
\xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\
\xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12} \\
9680\pi_{11} + 1.1\xi_{11} &\leq 3630 \\
8470\pi_{11} + 1.1\xi_{11} &\leq 2420 \\
3630\pi_{11} + 1.1\xi_{11} &\leq 0 \\
6655\pi_{12} + 1.1\xi_{12} &\leq 605 \\
3630\pi_{12} + 1.1\xi_{12} &\leq 0.
\end{aligned}
$$

LP problem for lower bound

The optimum value x^{-} of the LP problem given above is a lower bound for p. If for the optimum solution, even one of the inequalities is a strict inequality then $p > x^{+}$.

Here the variables $x_0, \pi_0, \pi_{11}, \pi_{12}, \xi_0, \xi_{11}, \xi_{12}$ are unrestrained, i.e. these are not required

Note that the two problems are identicle: minimize is replaced by maximize and inequalities are reversed: $\geq$ is replaced by $\leq$.

Back to the example:

The optimum solution to the first problem is $x_0 = 500, \pi_0 = 0.80, \pi_{11} = 0.60, \pi_{12} = 0.20, \xi_0 = -2700, \xi_{11} = -1980, \xi_{12} = -660$ **with a strict inequality. Thus** $p < 500$.

The optimum solution to the second problem is $x_0 = 425, \pi_0 = 0.65, \pi_{11} = 0.50, \pi_{12} = -660$ **with a strict inequality. Thus** $p > 425$ **we thus conclude that**

$$425 < p < 500.$$

What is the price p of the option?...

NA yields

$$425 < p < 500.$$

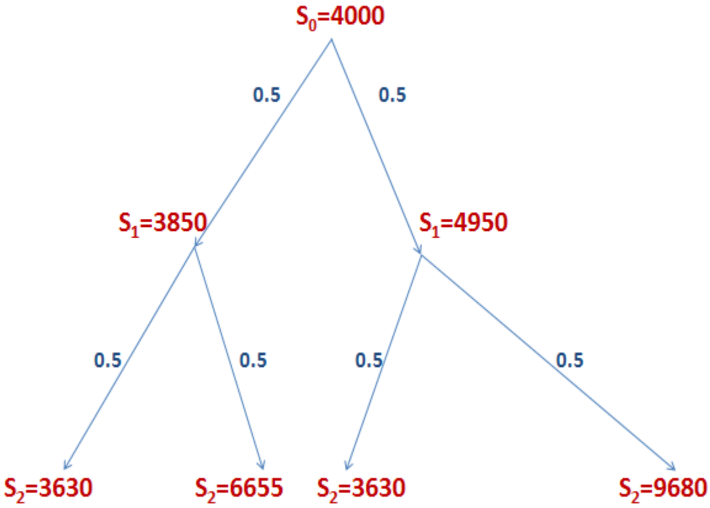
We can not reduce the interval further based on NA. In particular, we cannot deduce a unique option price based on the No Arbitrage principle.

Alternate Scenario

Let us explore an alternate scenerio. Suppose that instead of $P(S_2 = 9680 \mid S_1 = 4950) = 0.1$ **and** $P(S_2 = 8470 \mid S_1 = 4950) = 0.4$ **one has**

$$P(S_2 = 9680 \mid S_1 = 4950) = 0.5$$

Note (in the new picture given next) that each node has exactly two child nodes.


S_0=4000
0.5
0.5
S_1=3850
S_1=4950
0.5
0.5
0.5
0.5
S_2=3630
S_2=6655
S_2=3630
S_2=9680

Alternate Scenario.........

This time the LP problem for upper and lower bounds are modification of the earlier problems: one constrain (corresponding to the price 8470) are dropped.

Alternate Scenario... Upper bound

The LP problem for upper bound:

minimize x_0 **subject to**

$$
\begin{aligned}
\xi_0 &= x_0 - \pi_0 \times 4000 \\
\xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\
\xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12} \\
9680\pi_{11} + 1.1\xi_{11} &\geq 3630 \\
3630\pi_{11} + 1.1\xi_{11} &\geq 0 \\
6655\pi_{12} + 1.1\xi_{12} &\geq 605 \\
3630\pi_{12} + 1.1\xi_{12} &\geq 0.
\end{aligned}
$$

Alternate Scenario... Lower bound

The LP problem for lower bound:

maximize x_0 **subject to**

$$
\begin{aligned}
\xi_0 &= x_0 - \pi_0 \times 4000 \\
\xi_{11} &= (4950\pi_0 + 1.1\xi_0) - 4950\pi_{11} \\
\xi_{12} &= (3850\pi_0 + 1.1\xi_0) - 3850\pi_{12} \\
9680\pi_{11} + 1.1\xi_{11} &\leq 3630 \\
3630\pi_{11} + 1.1\xi_{11} &\leq 0 \\
6655\pi_{12} + 1.1\xi_{12} &\leq 605 \\
3630\pi_{12} + 1.1\xi_{12} &\leq 0.
\end{aligned}
$$

Alternate Scenario.................

In this case, both the problems have solution with all constrains being equalities and the optimum values are the same: The following is the common optimal solution:

$x_0 = 500$, $\pi_0 = 0.80$, $\pi_{11} = 0.60$, $\pi_{12} = 0.20$, $\xi_0 = -2700$, $\xi_{11} = -1980$ **and** $\xi_{12} = -660$.

Thus, $x^+ = x^- = 500$ **and so for the modified problem, the option price must be 500.**

Replicating strategy

In fact, here the strategy $x_0 = 500$, $\pi_0 = 0.80$, $\pi_{11} = 0.60$, $\pi_{12} = 0.20$, $\xi_0 = -2700$, $\xi_{11} = -1980$ and $\xi_{12} = -660$ does something interesting: for all the possible values of S_1, S_2, the worth of the portfolio from this strategy is exactly the same as the worth of the call option.

The strategy is called replicating strategy (or sometimes hedging strategy).

Replicating strategy......

When a replicating strategy exists, the price of the option is uniquely determined by NA principle: it equals the initial investment required for the replicating strategy.

Another Scenario.........

Likewise, if instead of $P(S_2 = 9680 \mid S_1 = 4950) = 0.1$ **and** $P(S_2 = 8470 \mid S_1 = 4950) = 0.4$ **one has**

$$P(S_2 = 8470 \mid S_1 = 4950) = 0.5$$

then again, the two LP problems have identicle solution: and a replicating strategy exists : $x_0 = 425$, $\pi_0 = 0.65$, $\pi_{11} = 0.50$, $\pi_{12} = 0/2$,$\xi_0 = -2175$, $\xi_{11} = -1650$ **and** $\xi_{12} = -660$. **Thus in this case, the option price is** $p = 425$.

Do probabilities of outcomes matter?

Note that in the fomulation of the Linear Programming problems, the probabilites of the outcomes did not appear at all and so the bounds for the option price did not depend upon the probabilities of the various outcomes.

This is so because we are matching the returns for each outcome and so it doesn't matter as to with what probability an outcome occurs.

We have also seen that the bounds did depend upon the set of possible outcomes.

Option price is not expected discounted gain

Since the bounds did not depend upon the probabilites of occurances, it follows that even when the bounds coincide, the price can be more or less than the expected discounted gain.

Option price is not expected discounted gain....

So that went wrong with the reasoning price = expected gain. The reason is that along with the option, another comodity, namely the shares of the same company, one also available in the market and of course, the shares are correlated with the option - and thus we need to valuate the option in terms of a basket consisting of money and shares.

If the shares of the company were not being traded but only the options were being sold, then perhaps the expected (discounted) gain can be taken as the price (if the utility is taken as linear).

More questions

In general, how does one know if the upper bound and lower bound for option price will be equal or not?

In what generality can we prove that the bounds must be consistant: namely, upper bound is not smaller than lower bound.

And if the bounds are equal, how does one compute it from the specified model?

Let us consider a general model for stock prices in discrete time. Without loss of generality, let us assume that unit of time is a day and that stock prices change every afternoon at 2:00 pm The investors are allowed to trade in the morning, at 11:00 am, at the prices prevailing then (same as the previous evening).
The price of the stock on k^{th} afternoon is denoted by S_k. S_0 is assumed to be deterministic, $S_0 = s_0$. Also the face value of the bond is 1 on morning of day zero. The interest rate is r per day, due at 2:00 pm, so that the value of the bond on k^{th} day at 2:00 pm is R^k where $R = (1 + r)$.

General Discrete Model

For $k \geq 0$, let π_k denote the number of shares a specified investor decides to hold on the morning of k^{th} day and ξ_k denote the number of bonds he decides to hold. Thus for $k \geq 1$ if

$$\pi_k \geq \pi_{k-1},$$

he buys $\pi_k - \pi_{k-1}$ shares on the morning of k^{th} day and if

$$\pi_k < \pi_{k-1}$$

, then he sells $\pi_{k-1} - \pi_k$ shares (on the morning of k^{th} day). We have similar interpretation for bonds.

Clearly, π_k, ξ_k should depend only on

$$\{S_0, S_1, S_2, \ldots, S_{k-1}\}$$

for $k \geq 1$. For when choosing π_k, ξ_k the only information the investor has is $\{S_i : i \leq k-1\}$. π_0, ξ_0 are required to be constants. We express this as

$$\pi_k = \phi_k(S_0, S_1, \ldots, S_{k-1}) \tag{1}$$

$$\xi_k = \psi_k(S_0, S_1, \ldots, S_{k-1}) \tag{2}$$

where ψ_k, ϕ_k are real valued functions on $\mathbb{R}^k$, $k \geq 1$.

Of course to implement such a trading strategy, the investor may need to put in extra money on certain days and may have surplus on others. We are going to consider a special class of trading strategies, called **self-financing strategies**. These are trading strategies where there is no money put in and there is no surplus on any day except for the initial investment x . Thus on any given day, the investor only moves his money from shares to bonds or vice-versa. The shares and bonds held by an investor together is, known as his **portfolio**.

On the k^{th} morning $(k \geq 1)$, the investors portfolio is worth

$$\pi_{k-1} S_{k-1} + \xi_{k-1} R^{k-1}$$

and he needs

$$\pi_k S_{k-1} + \xi_k R^{k-1}$$

to implement his trading strategy. Since the strategy is assumed to be self financing, it follows that these two quantities must be equal, i.e.,

$$\pi_{k-1} S_{k-1} + \xi_{k-1} R^{k-1} = \pi_k S_{k-1} + \xi_k R^{k-1}, \quad k \geq 1. \tag{3}$$

Thus for any $j \geq 1$ (writing $\beta = R^{-1}$)

$$\xi_j = \xi_{j-1} + (\pi_{j-1} - \pi_j) S_{j-1} \beta^{j-1}. \tag{4}$$

Also, $x = \xi_0 + \pi_0 S_0$ is the initial investment. We conclude using (4) that

$$\xi_j = x - \xi_0^1 S_0 + \sum_{i=1}^{j} (\pi_{i-1} - \pi_i) S_{i-1} \beta^{i-1}. \tag{5}$$

It is clear that in a self-financing strategy, the investor only chooses π_k for $k \geq 0$ and together with x , this determines ξ_k via (5).

Let V_k denote the worth of the portfolio on the evening of k^{th} day. Then $V_k = \pi_k S_k + \xi_k R^k$. Let us rewrite this as

$$\beta^k V_k = \pi_k S_k \beta^k + \xi_k$$

and using (5) we get

$$\begin{aligned} \beta^k V_k &= \pi_k \beta^k S_k + x - \pi_0 S_0 + \textstyle\sum_{j=1}^{k} (\pi_{j-1} - \pi_j) S_{j-1} \beta^{j-1} \\ &= x + \textstyle\sum_{j=1}^{k} \pi_j (S_j \beta^j - S_{j-1} \beta^{j-1}). \end{aligned} \tag{6}$$

Here $\tilde{V}_k = \beta^k V_k$ is the discounted value process and if we define $\tilde{G}_k = \sum_{j=1}^{k} \pi_j (S_j \beta^j - S_{j-1} \beta^{j-1})$ then $\tilde{G}_k$ represents the discounted gains process. The equation (6) can be recast then as

$$\tilde{V}_k = x + \tilde{G}_k$$

that is the (discounted) value of the portfolio from a self financing strategy is equal to the initial investment plus the (discounted) gain from the strategy.

We will further assume that each S_k takes only finitely many values and that we are considering a time horizon of N days. Let $\mathcal{S}$ denote the set

$$\left\{(s_0, s_1, \ldots, s_N) \in \mathbb{R}^{N+1} : P(S_0 = s_0, \ldots, S_N = s_N) > 0\right\}.$$

Then $\mathcal{S}$ is a finite set and

$$P((S_0, \ldots, S_N) \in \mathcal{S}) = 1.$$

We will now assume without loss of generality that the underlying probability space is $\mathcal{S}$, P is a probability measure on $\mathcal{S}$ and $S_0, S_1, \ldots, S_N$ are given by

$$S_i(s_0, \ldots, s_N) = s_i.$$

A self financing strategy is represented by

$$\theta = \{x, \phi_1, \ldots, \phi_N\},$$

where ϕ_k is a function on $\mathbb{R}^k$. θ determines π_j, ξ_j as seen earlier (via (1) and (5)).

For a self financing strategy θ , the worth of the portfolio $V_k(\theta)$ on the k^{th} evening is given by

$$V_k(\theta)(s_0,\ldots,s_N) = \Big[x + \sum_{j=1}^{k} \phi_j(s_0,\ldots,s_{j-1})(s_j\beta^j - s_{j-1}\beta^{j-1})\Big] R^k$$

and so discounted worth is

$$\tilde{V}_k(\theta)(s_0,\ldots,s_N) = x + \sum_{j=1}^{k} \phi_j(s_0,\ldots,s_{j-1})(s_j\beta^j - s_{j-1}\beta^{j-1})$$

In this context, an arbitrage opportunity is a self financing strategy $\theta = (0, \phi)$ such that

$$\tilde{V}_N(\theta)(s_0, \ldots, s_N) \geq 0 \quad \text{for all } (s_0, \ldots, s_N) \in \mathcal{S} \tag{7}$$

and

$$\tilde{V}_N(\theta)(s_0^1, \ldots, s_N^1) > 0 \quad \text{for some } (s_0^1, \ldots, s_N^1) \in \mathcal{S}. \tag{8}$$

The principle of no arbitrage here means that if a stategy θ satisfies (7), then it cannot satisfy (8).

How do we verify if "no arbitrage" is true for a given model? We will see that Martingales enter the picture out of nowhere. First note that if under a probability measure Q on $\mathcal{S}$, $S_k\beta^k$ is a martingale then $\tilde{V}_k$ is also a martingale since

$$\tilde{V}_k(\theta)(s_0, \ldots, s_N) = x + \sum_{j=1}^{k} \phi_j(s_0, \ldots, s_{j-1})(s_j\beta^j - s_{j-1}\beta^{j-1})$$

i.e.

$$\tilde{V}_k = \sum_{j=1}^{k} \phi_j(S_0, \ldots, S_{j-1})(S_j\beta^j - S_{j-1}\beta^{j-1})$$

Suppose for some $i, (\bar{s}_0, \ldots, \bar{s}_i)$

$$P(S_{i+1} \geq RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) = 1. \tag{9}$$

$$P(S_{i+1} > RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) > 0. \tag{10}$$

Take $\theta = \{0, \phi_1, \ldots, \phi_N\}$ defined as follows: ϕ_j identically equal to zero for $j \neq (i+1)$ and

$$\phi_{i+1}(s_0, \ldots, s_N) = \begin{cases} 1 & \textbf{if } (s_0, \ldots, s_i) = (\bar{s}_0, \ldots, \bar{s}_i) \\ 0 & \textbf{otherwise.} \end{cases}$$

Then one has

$$\tilde{V}_N(\theta)(s_0, \ldots, s_N) = \left(\frac{s_{i+1}}{R^{i+1}} - \frac{\bar{s}_i}{R^i}\right) 1_{\{s_0 = \bar{s}_0, \ldots, s_i = \bar{s}_i\}}.$$

Since we have assumed that

$$P(S_{i+1} \geq RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) = 1.$$
$$P(S_{i+1} > RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) > 0$$

this is an arbitrage opportunity. Thus these two equations cannot be simultaneously true.

Similarly, we can show that if for any i , $\bar{s}_0, \ldots, \bar{s}_i$

$$P(S_{i+1} \geq RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) = 1.$$
$$P(S_{i+1} > RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) > 0$$

cannot be simultaneously true.

In other words, we have that for any i , $(\bar{s}_0, \ldots, \bar{s}_i)$ such that $P(S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) > 0$,

$$P(S_{i+1} = RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) < 1 \tag{11}$$

implies

$$P(S_{i+1} > RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) > 0 \tag{12}$$

and

$$P(S_{i+1} < RS_i \mid S_0 = \bar{s}_0, \ldots, S_i = \bar{s}_i) > 0. \tag{13}$$

For $0 \leq M \leq N$, let

$$\mathcal{S}^{(M)} = \{(s_0, \ldots, s_M) : P(S_0 = s_0, \ldots, S_M = s_M) > 0\}$$

and

$$\mathcal{S}^* = \cup_{M=0}^{N-1} \mathcal{S}^{(M)}.$$

We will denote elements of $\mathcal{S}^*$ by α
(so that $\alpha = (s_0, \ldots, s_i)$ for some $i, 0 \leq i < N$) .
The set $\mathcal{S}^*$ denotes the set of all possible histories of the stock process.

For $\alpha = (s_0, \ldots, s_i) \in \mathcal{S}^*$, let $\ell_\alpha = s_i$ and $k_\alpha = Rs_i$

$$C_\alpha = \left\{ s : (s_0, \ldots, s_i, s) \in \mathcal{S}^{(i+1)} \right\}.$$

Then the observations made earlier can be rewritten as: If No Arbitrage holds, then for all $\alpha \in \mathcal{S}^*$, either C_α is singleton and the only element is k_α or

$$\exists a_\alpha, b_\alpha \in C_\alpha \; : \; a_\alpha > k_\alpha > b_\alpha$$

Let $\mathcal{F}_i$ denote the finite field $\sigma\{S_j : 0 \leq j \leq i\}$.

Theorem

The following are equivalent.

(i) *No arbitrage holds for the given model.*

(ii) *for all $\alpha \in \mathcal{S}^*$, either C_α is singleton and the only element is k_α or*

$$\exists a_\alpha, b_\alpha \in C_\alpha \,:\, a_\alpha > k_\alpha > b_\alpha$$

(iii) *There exists a probability measure Q on $\mathcal{S}$ such that $\{S_i\beta^i, \mathcal{F}_i\}$ is a Q -martingale and*

$$Q((s_0, \ldots, s_N) > 0 \quad \forall (s_0, \ldots, s_N) \in \mathcal{S}. \tag{14}$$

Proof: We have proven earlier that (i) implies (ii). Now suppose (ii) holds. We will construct Q. We will show that there exist

$$p_\alpha(s), \alpha \in \mathcal{S}^*, s \in C_\alpha$$

such that $0 < p_\alpha(s) \leq 1$ **,**

$$\sum_{\alpha \in C_\alpha} p_\alpha(s) = 1 \tag{15}$$

and

$$\sum_{\alpha \in C_\alpha} s p_\alpha(s) = \ell_\alpha R. \tag{16}$$

Then we can get a probability measure Q such that

$$Q(S_k = s \mid (S_0, \ldots, S_{k-1}) = \alpha) = p_\alpha(s).$$

Define

$$Q((s_0, \ldots, s_N)) = p_{(s_0,\ldots,s_{N-1})}(s_N) \cdot \\ p_{(s_0,\ldots,s_{N-2})}(s_{N-1}) \ldots p_{s_0}(s_1). \quad (17)$$

We can check that for $\alpha = (s_0, \ldots, s_i)$

$$E[S_{i+1} \mid S_0 = s_0, \ldots, S_i = s_i] = \sum_{s \in C_\alpha} s p_\alpha(s) = s_i R$$

and hence conclude that $\{S_j \beta^j, \mathcal{F}_j\}$ is a Q -martingale.

It remains to choose p_α satisfying

$$\sum_{\alpha \in C_\alpha} p_\alpha(s) = 1$$

$$\sum_{\alpha \in C_\alpha} s p_\alpha(s) = \ell_\alpha R.$$

If C_α is a singleton, then the only element must be $k_\alpha = \ell_\alpha R$ and hence taking $p_\alpha(\ell_\alpha R) = 1$ we can check that the required conditions hold.

If C_α has more than one element, then take

$$a_\alpha, b_\alpha \in C_\alpha,$$

$a_\alpha > k_\alpha = \ell_\alpha R$ and $b_\alpha < k_\alpha = \ell_\alpha R$.
Let $D_\alpha = C_\alpha - \{a_\alpha, b_\alpha\}$. For $0 \leq \epsilon < 1$, take $p^\epsilon_\alpha(s) = \epsilon$ for $s \in D_\alpha$. Let $p^\epsilon_\alpha(a_\alpha), p^\epsilon_\alpha(b_\alpha)$ be solutions to the equations

$$p^\epsilon_\alpha(a_\alpha) + p^\epsilon_\alpha(b_\alpha) = 1 - \epsilon(\# D_\alpha) \tag{18}$$

$$a_\alpha p^\epsilon_\alpha(a_\alpha) + b_\alpha p^\epsilon_\alpha(b_\alpha) = \ell_\alpha R - \epsilon \cdot \Big[\textstyle\sum_{s \in D_\alpha} s\Big]. \tag{19}$$

Since $a_\alpha > b_\alpha$, these equations admit a unique solution.

The unique solution is given by

$$p_\alpha^\epsilon(a_\alpha) = \frac{\ell_\alpha R - b_\alpha - \epsilon \sum_{s \in D_\alpha}(s - b_\alpha)}{a_\alpha - b_\alpha} \tag{20}$$

$$p_\alpha^\epsilon(b_\alpha) = \frac{a_\alpha - \ell_\alpha R - \epsilon \sum_{s \in D_\alpha}(a_\alpha - s)}{a_\alpha - b_\alpha}. \tag{21}$$

It is clear from these expressions that for suitably small $\epsilon > 0$, $p_\alpha^\epsilon(a_\alpha) > 0$ and $p_\alpha^\epsilon(b_\alpha) > 0$. Thus for sufficiently small $\epsilon > 0$, $\{p_\alpha^\epsilon(s) : \alpha \in C_\alpha\}$ satisfies the required conditions. This completes the proof of (ii) implies (iii).

Suppose (iii) holds. Let Q be as in (iii). Then as noted earlier, for every strategy θ ,

$$\tilde{V}_k(\theta)(S_0, \ldots, S_N)$$

is a Q -martingale.

Hence

$$E^Q[\tilde{V}_k(\theta)(S_0, \ldots, S_N)] = V_0(\theta). \tag{22}$$

Let θ be a strategy such that $V_0(\theta) = 0$ and

$$V_N(\theta)(s_0, \ldots, s_N) \geq 0 \quad \text{for all } (s_0, \ldots, s_N) \in \mathcal{S}. \tag{23}$$

We also have

$$E^Q[V_k(\theta)(S_0, \ldots, S_N)]\beta^k = V_0(\theta) = 0.$$

Since $Q((s_0, \ldots, s_N)) > 0 \ \forall (s_0, \ldots, s_N) \in \mathcal{S}$ we conclude

$$V_N(\theta)(s_0, \ldots, s_N) = 0 \ \forall (s_0, \ldots, s_N) \in \mathcal{S}$$

Thus arbitrage opportunities do not exist.

As noted earlier, this completes the proof.

Let us note here that when the number of elements in any C_α is more than two, we have two (in fact infinitely many) distinct choices of $\epsilon > 0$ for which $p^\epsilon_\alpha(a_\alpha) > 0$, $p^\epsilon_\alpha(b_\alpha) > 0$ and we get two distinct measures Q^1 and Q^2 satisfying requirements in part (iii).

Conversely, it is easy to see that if cardinality of each C_α is at most two, then for $\alpha = (s_0, \ldots, s_i)$

$$p_\alpha(s) = Q(S_{i+1} = s \mid S_0 = s_0, \ldots, S_i = s_i)$$

is uniquely determined by

$$\sum_{\alpha \in C_\alpha} p_\alpha(s) = 1$$

$$\sum_{\alpha \in C_\alpha} s p_\alpha(s) = \ell_\alpha R.$$

and hence the probability measure Q is uniquely determined. The measure Q is called an Equivalent Martingale Measure (EMM).

European call option

Let us consider the discrete model for stock prices. **We will assume that arbitrage opportunities do not exist.** Consider a European call option with striking price K and termianal time N. The holder of the option contract makes a profit of $(S_N - K)^+$ at the terminal time N. Suppose p is the price of such an option contract in the marketplace, so that there are buyers as well as sellers at this price.

Recall that an investment strategy is represented as

$$\theta = \{x, \phi_1, \ldots, \phi_N\}$$

where ϕ_k is a function on $\mathbb{R}^k$. We will shorten it by writing $\Phi = (\phi_1, \ldots, \phi_N)$ so that an investment strategy is represented as

$$\theta = \{x, \Phi\}.$$

Suppose x is such that there exists an investment strategy $\theta = (x, \Phi)$ such that

$$V_N(\theta)(s_0, \ldots, s_N) \geq (s_N - K)^+ \quad \forall (s_0, \ldots, s_N) \in \mathcal{S}. \tag{24}$$

Then x must be $\geq p$:
if $x < p$, consider the strategy of selling an option contract at price p, and following strategy $\bar{\theta} = (p, \Phi)$ for investing on shares and bond.

At time N, the assets are $V_N(\bar{\theta})$ and liabilities are $(S_N - K)^+$. Thus the net worth is

$$
\begin{aligned}
& V_N(\bar{\theta})(s_0, \ldots, s_N) - (s_N - K)^+ \\
& \quad = \; V_N(\theta)(s_0, \ldots, s_N) - (s_N - K)^+ + p - x \\
& \quad \geq \; p - x \\
& \quad > \; 0.
\end{aligned}
\tag{25}
$$

This is an arbitrage opportunity. Since arbitrage opportunities do not exist, x must be greater than or equal to p.

Let A^+ be the set of all x such that there exists $\theta = (x, \Phi)$ satisfying

$$V_N(\theta)(s_0, \ldots, s_N) \geq (s_N - K)^k \quad \forall (s_0, \ldots, s_N) \in \mathcal{S}$$

and let

$$x^+ = \inf A^+.$$

We thus have proven

$$x^+ \geq p. \tag{26}$$

Similarly, taking A^- to be the set of all x such that there exists $\theta = (x, \Phi)$ satisfying

$$V_N(\theta)(s_0, \ldots, s_N) \leq (s_N - K)^+ \;\; \forall (s_0, \ldots, s_N) \in \mathcal{S} \quad (27)$$

and $x^- = \sup A^-$, we can conclude

$$x^- \leq p \quad (28)$$

We have thus obtained upper and lower bounds for the price of the option using only the stipulation that arbitrage opportunities do not exist. When the upper and lower bounds coincide, they determine the price uniquely.

Another way to look at this is: Let the market consisting of the stock and bond satisfy "No arbitrage". Then in order that the augumented market consisting of stock, bond and the option be free of arbitrage oportunities, the option price p must satisfy $p \leq x^+$ and $p \geq x^-$.

We first observe that $x^+ \geq x^-$.

Indeed, if Q is a probability measure such that $S_i\beta^i$ is a Q- martingale, then for a self financing strategy $\theta = (x, \Phi)$

$$E_Q\big[V_N(\theta)(S_0, \ldots, S_N)\beta^N\big] = x \tag{29}$$

and hence it follows that if $x_1 \in A^+$, $x_2 \in A^-$, then

$$x_2 \le E^Q\big[(S_N - K)^+\beta^N\big] \le x_1.$$

Thus,

$$x^- \le E^Q\big[(S_N - K)^+\beta^N\big] \le x^+.$$

Hence, $x^+ \ge x^-$.

We will now analyse a special case where $x^- = x^+$ and as a consequence, the price is completely determined by the principle of no arbitrage.

Recall our observation that if NA holds and if

$$\#C_\alpha = 1 \ \textbf{ or } \ 2 \quad \forall \alpha \in \mathcal{S}^*$$

then the equivalent martingale measure is unique.

Theorem

Suppose that $\boldsymbol{NA}$ *holds and that*

$$\#\boldsymbol{C}_\alpha = 1 \;\; or \;\; 2 \quad \forall \alpha \in \boldsymbol{\mathcal{S}}^*. \tag{30}$$

Let $\boldsymbol{Q}$ *be the unique EMM. Let*

$$\hat{x} = \boldsymbol{E}^{\boldsymbol{Q}}[(\boldsymbol{S_N} - \boldsymbol{K})^+ \boldsymbol{R}^{-\boldsymbol{N}}]. \tag{31}$$

Then $\exists\, \theta = (x, \Phi)$ *(a self financing strategy) such that*

$$\boldsymbol{V_N}(\theta)(s_0, \ldots, s_{\boldsymbol{N}}) = (s_{\boldsymbol{N}} - \boldsymbol{K})^+ \quad \forall (s_0, \ldots, s_{\boldsymbol{N}}) \in \boldsymbol{\mathcal{S}}. \tag{32}$$

As a consequence $x^+ = x^- = \hat{x}$.

Proof: For $\alpha \in \mathcal{S} = \mathcal{S}^{(N)}$, let

$$g(\alpha) = (S_N(\alpha) - K)^+. \tag{33}$$

We define ϕ_N, ξ_N and g on $\mathcal{S}^{(N-1)}$ as follows. Fix $\alpha = (s_0, \ldots, s_{N-1}) \in \mathcal{S}^{(N-1)}$.
If $\#C_\alpha = 1$, then $C_\alpha = \{s_{N-1}R\}$ and define

$$\begin{aligned} \phi_N(\alpha) &= 0, \\ \xi_N(\alpha) &= R^{-N} g((s_0, \ldots, s_{N-1}, Rs_{N-1})). \end{aligned}$$

If $\# C_\alpha = 2$, then $C_\alpha = \{a_\alpha, b_\alpha\}$ with $a_\alpha > s_{N-1}R > b_\alpha$. Consider the equations

$$ya_\alpha + zR^N = g((s_0, \ldots, s_{N-1}, a_\alpha))$$

$$yb_\alpha + zR^N = g((s_0, \ldots, s_{N-1}, b_\alpha))$$

in variables y, z. These equations admit a unique solution as $a_\alpha > b_\alpha$. At time $N-1$, if the investor had an amount $yS_{N-1} + zR^{N-1}$, then buying y stocks and z bonds, an investor would exactly have the amount $g((s_0, \ldots, s_{N-1}, b_\alpha))$ at time N irrespective of the outcome of S_N.

So, define $\phi_N(\alpha) = y$, $\pi_N(\alpha) = z$ and

$$g(\alpha) = \phi_N(\alpha) S_{N-1} + \xi_N(\alpha) R^{N-1}.$$

If at time $N-1$ an investor had an amount $g((s_0, \ldots, S_{N-1}))$, he/she can follow a strategy of investing

$\phi_N((s_0, \ldots, S_{N-1}))$ on the stock and $\xi_N((S_0, \ldots, S_{N-1}))$ on the bonds and end up at time N with exactly the same reward as the one from the option irrespective of the outcome of S_N.

We now define $\phi_{i+1}(\alpha)$, $\xi_{i+1}(\alpha)$ and $g(\alpha)$ for $\alpha \in \mathcal{S}^{(i)}$, $0 \leq i \leq N-2$ by backward induction.
Having defined ϕ_{j+1}, ξ_{j+1} and g for $j > i, i \leq N-2$, we define these for $j = i$ as follows.
Fix $\alpha = (s_0, \ldots, s_i) \in \mathcal{S}^{(i)}$. If $C_\alpha = \{Rs_i\}$ then

$$\phi_{i+1}(\alpha) = 0 \tag{34}$$

$$\xi_{i+1}(\alpha) = R^{-(i+1)} g((s_0, \ldots, s_i, Rs_i)). \tag{35}$$

On the other hand, if $C_\alpha = \{a_\alpha, b_\alpha\}$ (with $a_\alpha > b_\alpha$) then $\phi_{i+1}(\alpha)$ and $\xi_{i+1}(\alpha)$ be the unique solutions to the equations

$$\phi_{i+1}(\alpha) a_\alpha + \xi_{i+1}(\alpha) R^{i+1} = g((s_0, \ldots, s_i, a_\alpha)) \tag{36}$$

$$\phi_{i+1}(\alpha) b_\alpha + \xi_{i+1}(\alpha) R^{i+1} = g((s_0, \ldots, s_i, b_\alpha)). \tag{37}$$

Then define

$$g(\alpha) = \phi_{i+1}(\alpha) s_i + \xi_{i+1}(\alpha) R^i. \tag{38}$$

Take $\theta = \{\phi_i : 1 \leq i \leq N\}$ and $\hat{x} = g((s_0))$.

By construction one has, for $(s_0, \ldots, s_i, s_{i+1}) \in \mathcal{S}^{(i+1)}$

$$\phi_{i+1}((s_0, \ldots, s_i))s_{i+1} + \xi_{i+1}((s_0, \ldots, s_i))R^{i+1} = g((s_0, \ldots, s_i, s_{i+1})), \quad (39)$$

$$\phi_{i+1}((s_0, \ldots, s_i))s_i + \xi_{i+1}((s_0, \ldots, s_i))R^{i} = g((s_0, \ldots, s_i)). \quad (40)$$

Thus (recall $\beta = R^{-1}$)

$$\phi_{i+1}((s_0, \ldots, s_i))(s_{i+1}\beta^{i+1} - s_i\beta^i) = g((s_0, \ldots, s_i, s_{i+1}))\beta^{i+1} - g((s_0, \ldots, s_i))\beta^i. \quad (41)$$

Hence summing over i we get

$$\sum_{i=0}^{N-1} \phi_{i+1}((s_0, \ldots, s_i))(s_{i+1}\beta^{i+1} - s_i\beta^i) \tag{42}$$

$$= \sum_{i=0}^{N-1} \left[g((s_0, \ldots, s_i, s_{i+1}))\beta^{i+1} - g((s_0, \ldots, s_i))\beta^i \right] \tag{43}$$

$$= g((s_0, \ldots, s_N))\beta^N - g(s_0) \tag{44}$$

$$= (s_N - K)^+ = \hat{x}. \tag{45}$$

Let $\theta = (\hat{x}, \Phi)$. Then we have

$$V_N(\theta)(s_0, \ldots, s_N) = (s_N - K)^+ \quad \forall (s_0, \ldots, s_N) \in \mathcal{S},$$

$$\hat{x} = E^Q[(S_N - K)^+ R^{-N}].$$

It follows that

$$\hat{x} \in A^+ \text{ so that } x^+ \leq \hat{x}$$

and

$$\hat{x} \in A^- \text{ so that } x^- \geq \hat{x}.$$

Since $x^- \leq x^+$, it follows that

$$x^- = x^+ = \hat{x}.$$

Complete markets

Consider a contract that entitles the holder of the contract to receive a payoff of $f(S_0, \ldots, S_N)$ at time N, where f is a specified inon-negative function. This contract is called a contingent claim. We will denote this claim by $CC(f; N)$. Thus European call option is the contingent claim with

$$f(s_0, \ldots, s_N) = (s_N - K)^+.$$

We can consider other examples of derivative securities such as: the payoff to the holder of the contract at time N is

(a) S_N^2

(b) $\max_{i \leq N} \mid S_i - K \mid^+$

(c) $S_1 + S_2 + \ldots + S_N$

(d) $(K - S_N)^+$.

All these are examples of contingent claims.

The contingent claim in (d) above is the European put option with terminal time N and striking price K and is traded in European markets: it entitles its buyer to sell one share at the striking price K at terminal time N if he so wishes. The contingent claims corresponding to the other examples are not traded on the markets.

In analogy with the European option case, let us define $A^+(f;N)$ to be the set of $x \in \mathbb{R}$ such that there exists a self financing strategy θ such that

$$V_N(x,\theta)(s_0,\ldots,s_N) \geq f(s_0,\ldots,s_N) \tag{46}$$

for all $(s_0,\ldots,s_N) \in \mathcal{S}$ and let

$$x^+(f;N) = \inf A^+(f;N).$$

$A^-(f;N)$ is defined simillarly with the $\geq$ in (46) replaced by $\leq$ and then define

$$x^-(f;N) = \sup A^-(f;N).$$

Here again, the principle of no arbitrage implies that $x^+(f; N)$ is an upper bound and $x^-(f; N)$ is a lower bound for the price of the contingent claim $CC(f; N)$. The claim $CC(f; N)$ is said to be attainable if there exists a strategy $\theta = (\hat{x}, \Phi)$ such that

$$V_N(\hat{x}, \Phi)(s_0, \ldots, s_N) = f(s_0, \ldots, s_N) \quad \forall (s_0, \ldots, s_N) \in \mathcal{S}. \tag{47}$$

This means owning $CC(f : N)$ at time 0 is the same as having an amount $\hat{x}$ at time 0 : Following strategy $\theta = (\hat{x}, \Phi)$, the portfolio $V_N(\hat{x}, \Phi)$ can be matched with the payoff of the contingent claim for every outcome of the stock process.

Note that if

$$V_N(\hat{x}, \Phi)(s_0, \ldots, s_N) = f(s_0, \ldots, s_N) \quad \forall (s_0, \ldots, s_N) \in \mathcal{S}$$

holds, then $\hat{x}$ belongs to $A^+(f; N)$ as well as to $A^-(f; N)$. Thus the price of the claim $CC(f; N)$ must be $\hat{x}$.

Further, using $E^Q[V_N(\hat{x}, \Phi)] = R^N \hat{x}$ for an EMM Q it follows that the price $\hat{x}$ of an attainable claim $CC(f; N)$ is given by

$$\hat{x} = R^{-N} E^Q[f(S_0, \ldots, S_N)] \tag{48}$$

where Q is any EMM (any probability measure on $\mathcal{S}^{(N)}$ such that $\beta^i S_i$ is a Q-martingale).

The market consisting of the bond and the stock $\{S_i\}$ is said to be complete if every contingent claim is attainable.

As observed above, in a complete market, prices of all contingent claims are completely determined by the principle of No Arbitrgae.

If the market if not complete (i.e. incomplete) then the the principle of No Arbitrage gives bounds $x^{+}(f; N)$ and $x^{-}(f; N)$ for the price of the contingent claim $CC(f; N)$

The following theorem characterizes completeness of market consisting of a bond and a stock.

Theorem

Consider a market consisting of a bond and a stock $\{S_k\}$. *Assume that arbitrage opportunities do not exist. Let* $\mathcal{E}(P)$ *denote the class of probability measures* Q *on* $\mathcal{S}^{(N)}$ $\beta^i S_i$ *is a* Q*- martingale and*

$$Q\{\alpha\} > 0 \quad \forall \alpha \in \mathcal{S}^{(N)}.$$

Then the following are equivalent

(a) *The market is complete.*

(b) $\mathcal{E}(P)$ *is a singleton.*

(c) $\forall \alpha \in \mathcal{S}^*, \# C_\alpha \in \{1, 2\}$.

Proof: We have seen earlier that $\mathcal{E}(P)$ is nonempty in view of our assumption of No Arbitrage.
First, we will prove (a) $\Rightarrow$ (b). Suppose market is complete. Let $Q^1, Q \in \mathcal{E}(P)$. Define $g : \mathcal{S} \to \mathbb{R}$ as follows:

$$g(s_0, \ldots, s_N) = \frac{Q^1((s_0, \ldots, s_N))}{Q((s_0, \ldots, s_N))} \quad (s_0, \ldots, s_N) \in \mathcal{S}. \tag{49}$$

By completeness of the market, there exists a strategy $\theta = (x, \Phi)$ such that the contingent claim g is attained at time N *i.e.*

$$g(s_0, \ldots, s_N) = x + \sum_{j=0}^{N-1} \phi_{j+1}(s_0, \ldots, s_j)(\beta^{j+1} s_{j+1} - \beta^j s_j). \quad (50)$$

Let $\mathcal{F}_i = \sigma(S_0, \ldots, S_i)$. For $i \geq 0$, define

$$h^i(s_0, \ldots, s_i) = \frac{Q^1(\{S_0 = s_0, \ldots, S_i = s_i\})}{Q(\{S_0 = s_0, \ldots, S_i = s_i\})}.$$

It is easy to verify that $Z_i = h^i(S_0, \ldots, S_i)$ is a Q-martingale w.r.t. $(\mathcal{F}_i)$. Since $g = h^N$ one has

$$Z_N = x + \sum_{j=0}^{N-1} \phi_{j+1}(S_0, \ldots, S_j)(\beta^{j+1} S_{j+1} - \beta^j S_j). \quad (51)$$

Using the fact that Z_j, $\beta^j S_j$ are Q-martingales (w.r.t. $(\mathcal{F}_i)$) it follows that

$$Z_i = [x + \sum_{j=0}^{i-1} \phi_{j+1}(S_0, \ldots, S_j)(\beta^{j+1} S_{j+1} - \beta^j S_j)]$$

and hence

$$Z_{i+1} - Z_i = \phi_{i+1}(s_0, \ldots, s_i)(\beta^{i+1} S_{i+1} - \beta^i S_i) \quad (52)$$

Next, we will show that $\beta^j S_j Z_j$ is a Q-martingale.

$$
\begin{aligned}
&E^Q[\beta^{i+1} S_{i+1} Z_{i+1} 1_{\{S_0=s_0,\ldots,S_i=s_i\}}] \\
&\quad = \sum_{s_{i+1},\ldots,s_N} \beta^{i+1} s_{i+1} h^{i+1}(s_0,\ldots,s_i) Q((s_0,\ldots,s_N)) \\
&\quad = \sum_{s_{i+1}} \beta^{i+1} s_{i+1} h^{i+1}(s_0,\ldots,s_i) Q(S_0=s_0,\ldots,S_i=s_i) \\
&\quad = h^i(s_0,\ldots,s_i) \cdot \sum_{s_{i+1}} \beta^{i+1} s_{i+1} Q^1(S_0=s_0,\ldots,S_i=s_i) \\
&\quad = h^i(s_0,\ldots,s_i) \beta^i s_i \cdot Q(S_0=s_0,\ldots,S_i=s_i).
\end{aligned}
$$

Here, we have used the definition of h^i, h^{i+1} and the fact that $\beta^j S_j$ is a Q^1-martingale.

We have thus proved

$$E^Q\left[\beta^{i+1}S_{i+1}Z_{i+1} \mid S_0,\ldots,S_i\right] = \beta^i S_i Z_i. \tag{53}$$

In the steps that follow, we use this relation along with

$$Z_{i+1} - Z_i = \phi_{i+1}(s_0,\ldots,s_i)(\beta^{i+1}S_{i+1} - \beta^i S_i)$$

and the fact that $\beta^j S_j$, Z_j are Q-martingales w.r.t. $(\mathcal{F}_i)$. Let us write $\delta_{i+1} = \phi_{i+1}(S_0,\ldots,S_i)$ for convenience. Note that δ_{i+1} is $(\mathcal{F}_i)$ measurable.

$$
\begin{aligned}
E^Q&((Z_{i+1} - Z_i)^2) \\
&= E^Q((Z_{i+1} - Z_i)\delta_{i+1}(\beta^{i+1}S_{i+1} - \beta^i S_i)) \\
&= E^Q[\delta_{i+1}E^Q((Z_{i+1} - Z_i)(\beta^{i+1}S_{i+1} - \beta^i S_i) \mid \mathcal{F}_i)] \\
&= E^Q[\delta_{i+1}(\beta^i S_i Z_i - \beta^i S_i Z_i - \beta^i S_i Z_i + \beta^i S_i Z_i)] \\
&= 0.
\end{aligned}
$$

We thus have $Z_{i+1} = Z_i$ Q-a.s. for all i, and as a consequence $Z_N = Z_0$ Q-a.s.

By definition of $\mathcal{E}(P)$, every singleton $\alpha \in \mathcal{S}$ has positive Q-probability. It thus follows that Z_N is a constant, which means g is a constant function. In turn, this yields $Q^1 = Q$.

Thus we have proved, (a) $\Rightarrow$ (b). We have already observed that (b) implies (c). It remains to prove (c) $\Rightarrow$ (a). We have seen attainability of European call option if (c) holds. The proof of attainability in the general case is essentially the same: Given a contingent claim $f(S_0, \ldots, S_N)$, let $g = f$

Proceeding as in the earlier case we obtain a strategy π for which

$$V_N(x, \pi)(s_0, \ldots, s_N) = f(s_0, \ldots, s_N).$$

Thus if (c) holds, every contingent claim is attainable.

We have thus seen that in the context of discrete time finite state space models for stock prices, the No Arbitrage condition and completeness of markets can be characterised by the sets C_α. However, this is not suitable for generalisation to continuous time real valued models. On the other hand, both these notions can be captured by Equivalenet Martingale measures (EMM): NA is same as existance of EMM and uniqueness is same as completeness of markets.

American option

The options traded on the American stock markets are different from the European options discussed earlier: the American option can be exersied by the holder at any time before the terminal time.

Thus in case of an American call option with terminal time N and striking price K, the holder of the contract can, if he so wishes, exercise his option at time $n \leq N$ and buy one share at price K, making a profit of $(S_n - K)^+$ at time n.

Of course, the decision to exercise the option at time n or not has to be based on actual information available at that time, namely $S_0, \ldots, S_n$.
If τ denotes the (random) time at which the option is exercised, the event $\{\tau = n\}$ should depend only on $S_1, \ldots, S_n$ and then τ must be a stopping time, i.e. $\{\tau = n\}$ should belong to the σ-field

$$\mathcal{F}_n = \sigma(S_0, S_1, \ldots, S_n).$$

Since we are considering the case of discrete random variables $S_0, S_1, \ldots, S_N$ each taking only finitely many values, $\mathcal{F}_n$ is actually a field generated by atoms

$$\{S_0 = s_0, S_1 = s_1, \ldots, S_n = s_n\}.$$

Thus the American call option can be described as follows: the holder can exercise his option at any stopping time $\tau \leq N$ and make a profit of $(S_\tau - K)^+$. In case of American put option, the holder can sell one share at price K at any stopping $\tau \leq N$ making a profit of $(K - S_\tau)^+$. In order to consider American call and put options at the same time, we will consider the following American type security:
The holder of the security can exercise it at any stopping time $\tau \leq N$ to make a profit of Z_τ, where $Z_n = h_n(S_0, S_1, \ldots, S_n)$. The security is denoted by $(\{h_n\}; N)$.

We will regard h_n as a function on $\mathcal{S}$ defined by

$$h_n(s_0, \ldots, s_N) := h_n(s_0, \ldots, s_n).$$

For a stopping time $\tau \leq N$ (on $\mathcal{S}$),

$$h_\tau(s_0, \ldots, s_N) := h_{\tau(s_0,\ldots,s_N)}(s_0, \ldots, s_N).$$

We assume that arbitrage opportunities do not exist, so that $\exists Q \in \mathcal{E}(P)$.

Let $B^+(\{h_n\}; N)$ consist of $y \in [0, \infty)$ such that there exists a strategy (y, Φ) with

$$V_k(y, \Phi)(s_0, \ldots, s_N) \geq h_k(s_0, \ldots, s_N) \; \forall (s_0, \ldots s_N) \in \mathcal{S} \forall k \tag{54}$$

and let $B^-(\{h_n\}; N)$ consist of $y \in [0, \infty)$ such that there exists a strategy (y, Φ) and a stopping time τ such that

$$V_\tau(y, \Phi)(s_0, \ldots, s_N) \leq h_\tau(s_0, \ldots, s_N) \quad \forall (s_0, \ldots, s_N) \in \mathcal{S}. \tag{55}$$

Notice the assymetry in the two definitions.

Now define

$$y^{+}(\{h_n\}; N) = \inf B^{+}(\{h_n\}; N)$$

$$y^{-}(\{h_n\}; N) = \sup B^{-}(\{h_n\}; N).$$

We will prove below that these are the upper and lower bounds for the American type contingent claim.

If the price p of the American type security $(\{h_n\}; N)$ is less than $y^-(\{h_n\};\ N)$, then there exists a strategy (y, Φ) with $p < y$ and a stopping time τ such that

$$V_\tau(y, \Phi)(s_0, \ldots, s_N) \leq h_\tau(s_0, \ldots, s_N)\ \forall (s_0, \ldots, s_N) \in \mathcal{S}$$

An investor can buy the security at price p and invest the amount $-p$ on the stock market consisting of the bond and the stock $\{S_k\}$ following the strategy $-\Phi$.

At time τ, he would exercise the American security, liquidate his investments on the stock and bond. Thus, starting from zero investment, his net assets at time τ would be

$$\begin{aligned} h_\tau + V_\tau(-p, -\pi) &= h_\tau - V_\tau(p, \pi) \\ &= h_\tau - \{V_\tau(y, \pi) + (p - y)R^\tau\} \\ &\geq (y - p)R^\tau. \end{aligned}$$

This would be an arbitrage opportunity.
Hence, $y^-(\{h_n\}; N)$ is a lower bound for the price.

On the other hand, if $p > y$ for $y \in B^+(\{h_n\}; N)$, let π be such that for all k

$$V_k(y, \Phi)(s_0, \dots, s_N) \geq h_k(s_0, \dots, s_N) \quad \forall (s_0, \dots s_N) \in \mathcal{S}.$$

An investor can sell the American security at price p and invest p on the stock following the strategy π. If the buyer exercises his option at time τ, the investor can liquidate his investments at time τ as well and his net assets then are

$$\begin{aligned} V_\tau(p, \pi) - h_\tau &= V_\tau(y, \pi) + (p - y) \cdot R^\tau - h_\tau \\ &\geq (p - y) R^\tau. \end{aligned}$$

making this an arbitrage opportunity.

Thus $y^{+}(\{h_n\}; N)$ is an upper bound for the price of the American security. The following result implies that these constraints on the price are consistent.

Theorem

Let $Q \in \mathcal{E}(P)$. Then for any stopping time $\tau \leq T$,

$$y^{-}(\{h_n\}; N) \leq \sup_{\tau \leq T} E^{Q}[Z_{\tau}\beta^{\tau}] \leq y^{+}(\{h_n\}; N)$$

Proof: The assertion follows from the observation that for any y, π

$$V_n(y,\pi)(S_0,\ldots,S_N)\cdot\beta^n$$

is a Q-martingale (with mean y) and hence for any stop time τ

$$y = E^Q(V_\tau(y,\pi)(S_0,\ldots,S_N)\beta^\tau).$$

Let us note that when an investor buys the American option, he can choose the stopping time τ at which he can exercise his option, whereas the seller of the American option has to be prepared for any choice of τ which may be made by the buyer. This asymmetry is reflected in the definition of $B^{+}(\{h_n\}; N)$ and $B^{-}(\{h_n\}; N$.

Let us briefly consider the corresponding European security, namely the contingent claim $h_N(S_0, \ldots, S_N)$. As we have seen earlier, the upper bound $x^+ = x^+(h_N, N)$ and lower bound $x^- = x^-(h_N, N)$ for the price of the contingent claim $h_N(S_0, \ldots, S_N)$ are given by

$$x^+ = \inf A^+(h_N, N) \quad x^- = \sup A^-(h_N, N)$$

where

$A^{+}(h_N, N)$ consists of $y \in [0, \infty)$ such that there exists π such that

$$V_N(y,\pi)(s_0,\ldots,s_N) \geq h_N(s_0,\ldots,s_N) \quad \forall (s_0,\ldots,s_N) \in \mathcal{S} \tag{56}$$

and $A^{-}(h_N, N)$ consists of $y \in [0, \infty)$ such that there exists π such that

$$V_N(y,\pi)(s_0,\ldots,s_N) \leq h_N(s_0,\ldots,s_N) \quad \forall (s_0,\ldots,s_N) \in \mathcal{S}. \tag{57}$$

If y, π are such that $\forall k$

$$V_k(y, \Phi)(s_0, \ldots, s_N) \geq h_k(s_0, \ldots, s_N) \quad \forall (s_0, \ldots s_N) \in \mathcal{S}$$

then it is true for $k = N$ and so $y \in A^+(h_N, N)$. Thus $B^+(\{h_n\}; N) \subseteq A^+(h_N, N)$ and as a consequence

$$y^+(\{h_n\}; N) \geq x^+(h_N, N) \tag{58}$$

On the other hand if $y \in A^{-}(h_N, N)$ and y, π are such that

$$V_N(y, \Phi)(s_0, \ldots, s_N) \leq h_N(s_0, \ldots, s_N) \quad \forall (s_0, \ldots, s_N) \in \mathcal{S}$$

then we can take $\tau = N$ and then

$$y \in B^{-}(\{h_n\}; N).$$

Thus $A^{-}(h_N, N) \subseteq B^{-}(\{h_n\}; N)$ implying

$$x^{-}(h_N, N) \leq y^{-}(\{h_n\}; N) \tag{59}$$

We will prove later that in a complete market,

$$y^+(\{h_n\}; N) = y^-(\{h_n\}; N).$$

First we will consider an American call option $(h_n(s_0, \ldots, s_n) = (s_n - K)^+)$ and show that if

$$x^+(h_N, N) = x^-(h_N, N)$$

for the corresponding European call option, then

$$y^+(\{h_n\}; N) = y^-(\{h_n\}; N) = x^+(h_N, N) = x^-(h_N, N).$$

Theorem

Suppose that

$$x^{+}((s_N - K)^{+}, N) = x^{-}((s_N - K)^{+}, N) = \tilde{x} \qquad (60)$$

Then

$$y^{+}(\{(s_n - K)^{+}\}; N) = y^{-}(\{(s_n - K)^{+}\}; N) = \tilde{x} \qquad (61)$$

Proof: Let us write $x^{+}, y^{+}, x^{-}, y^{-}$ for the upper and lower bounds of the European call option and American call option. Likewise we will drop $(\{h_n\}; N)$ and (h_N, N) from the notation $A^{+}, A^{-}, B^{+}, B^{-}$.

We will prove that $A^+ = B^+$ and as a consequence

$$x^+((s_N - K)^+, N) = y^+(\{(s_n - K)^+\}, N). \tag{62}$$

This along with the inequalities

$$x^- \leq y^-,\ y^- \leq y^+$$

would give the required result.

We have seen earlier that $B^+ \subseteq A^+$. Let $y \in A^+$ and π be such that

$$V_N(y, \pi)(S_0, \ldots, S_N) \geq (S_N - K)^+$$

Let $Q \in \mathcal{E}(P)$. Recall assuption of no arbitrage.
Since $V_n(y,\pi)(S_0,\ldots,S_n)\beta^n$ is a Q-martingale we get

$$V_n(y,\pi)(S_0,\ldots,S_n)\beta^n \geq E^Q[(S_N - K)^+ \mid \mathcal{F}_n]\beta^N.$$

Using Jensen's inequality and the fact that $S_n\beta^n$ is a Q martingale we get (recall $R > 1$ and $\beta = R^1$)

$$\begin{aligned} V_n(y,\pi)(S_0,\ldots,S_n)\beta^n &\geq (E^Q(S_N \mid \mathcal{F}_n) - K)^+\beta^N \\ &\geq (E^Q(S_N\beta^N \mid \mathcal{F}_n) - K\beta^N)^+ \\ &= (S_n\beta^n - K\beta^N)^+ \\ &= (S_n - K\beta^{N-n})^+\beta^n \\ &\geq (S_n - K)^+\beta^n. \end{aligned}$$

We thus have proved that

$$V_n(y,\pi)(S_0,\ldots,S_N) \geq (S_n - K)^+ \quad \text{a.s. } Q. \tag{63}$$

By the definition of $\mathcal{E}(P)$, $Q(S_0 = s_0, \ldots, S_N = s_N) > 0$ for all $(s_0, \ldots, s_N) \in \mathcal{S}$, and hence

$$V_n(y,\pi)(s_0,\ldots,s_N) \geq (s_n - K)^+ \quad \forall (s_0,\ldots,s_N).$$

Thus $y \in B^+$. As a consequence $A^+ = B^+$.
This gives $x^+ = y^+$. The result follows.

Let us return to an American type security $(\{h_n,\}, N)$. The following is the main result of this section. The following result shows that in a complete market, $y^+ = y^-$ and hence the price of the the American security $(\{h_n\}, N)$ is uniquely determined by the no arbitrage principle. Recall our notation:

$$Z_n = h_n(S_0, \ldots, S_n) = h_n(S_0, \ldots, S_N).$$

Theorem

Suppose that arbitrage opportunities do not exist and that the market is complete. Let Q be the unique EMM. Let

$$\tilde{y} = \sup_{\tau \leq T} E^{Q}[Z_{\tau}\beta^{\tau}] \tag{64}$$

where the supremum is taken over all stopping times $\tau \leq T$. Then

$$y^{+}(\{h_n\}, N) = y^{-}(\{h_n\}, N) = \tilde{y} \tag{65}$$

Proof: Let us define $\{Y_n : n \le N\}$ by backward induction as follows:

$$Y_N = Z_N \beta^N \tag{66}$$

and having defined $\{Y_i : n+1 \le i \le N\}$, define

$$Y_n = \max\{Z_n \beta^n, E(Y_{n+1} \mid \mathcal{F}_n)\}. \tag{67}$$

By construction $\{Y_n, \mathcal{F}_n\}$ is a supermartingale, i.e.

$$Y_n \ge E(Y_{n+1} \mid \mathcal{F}_n)$$

$$Y_n \ge Z_n \beta^n.$$

For $n \geq 1$, define

$$D_n = \sum_{i=0}^{n-1} \{ Y_i - E^Q(Y_{i+1} \mid \mathcal{F}_i) \}.$$

Then

$$Y_n = Y_0 + M_n - D_n \tag{68}$$

with $D_n \geq D_{n-1} \geq 0$, where

$$M_n = \sum_{i=1}^{n} (Y_i - E^Q(Y_i \mid \mathcal{F}_{i-1})). \tag{69}$$

Then M_n is a Q-martingale.

Consider the contingent claim

$$(Y_0 + M_N)R^N.$$

Using completeness of the market, get y, Φ such that

$$V_N(y, \Phi)(S_0, \ldots, S_N) = (Y_0 + M_N)R^N \tag{70}$$

Since S_0 is a constant and $\mathcal{F}_0 = \sigma(S_0)$, Y_0 is a constant. Multiplying both sides in (70) by β^N (recall $\beta = R^{-1}$ and taking expectation w.r.t. Q we get $y = Y_0$. Here we have used that M_n as well as $\tilde{V}_n = V_n\beta^n$ are Q-martingales.

Further, using the fact that $V_n(y,\Phi)(S_0,\ldots,S_n)\beta^n$ and M_n are Q-martingales, we conclude

$$V_n(y,\Phi)(S_0,\ldots,S_n)\beta^n = y + M_n. \tag{71}$$

Then

$$\begin{aligned} Y_n &= y + M_n - D_n \\ V_n(y,\pi)(S_0,\ldots,S_n)\beta^n &= y + M_n \\ D_n &\geq 0 \end{aligned}$$

imply that for all n

$$V_n(y,\Phi)(S_0,\ldots,S_n)\beta^n \geq Y_n \geq Z_n\beta^n. \tag{72}$$

We thus conclude that $y \in B^+$. Also for any stopping time τ

$$EV_\tau(y,\Phi)(S_0,\ldots,S_n)\beta^\tau = y,$$

and

$$V_n(y,\Phi)(S_0,\ldots,S_n)\beta^n \geq Z_n\beta^n$$

gives

$$E_Q(Z_\tau\beta^\tau) \leq y. \tag{73}$$

Define

$$\sigma = \min\Big\{\inf\{i < N : Y_i > E(Y_{i+1} \mid \mathcal{F}_i\}, N\Big\}.$$

Then σ is a stop time. By the definition of σ, $D_i 1_{\{\sigma=i\}} = 0$ and so

$$Y_i 1_{\{\sigma=i\}} = (y + M_i) 1_{\{\sigma=i\}}. \tag{74}$$

Also, if for $i < N$, $Y_i > E(Y_{i+1} \mid \mathcal{F}_i)$ then $Y_i = Z_i \beta^i$ and $Y_N = Z_N \beta^N$. It follows that

$$Y_i 1_{\{\sigma=i\}} = Z_i \beta^i 1_{\{\sigma=i\}}. \tag{75}$$

Together, (74) and (75) imply

$$Z_\sigma \beta^\sigma = (y + M_\sigma) \tag{76}$$

and as a consequence

$$E^{Q}(Z_{\sigma}\beta^{\sigma}) = y. \tag{77}$$

We have seen that

$$E_{Q}[V_{\tau}(y,\Phi)(S_0,\ldots,S_n)\beta^{\tau}] = y \quad \forall \tau$$

$$V_{n}(y,\Phi)(S_0,\ldots,S_n)\beta^{n} \geq Z_{n}\beta^{n} \quad \forall n$$

Thus,

$$V_{\sigma}(y,\pi(S_0,\ldots,S_N)) = Z_{\sigma}\beta^{\sigma} \tag{78}$$

and thus $y \in B^{-}$. So $y \in B^{+} \cap B^{-}$. Hence $y = y^{+} = y^{-}$.