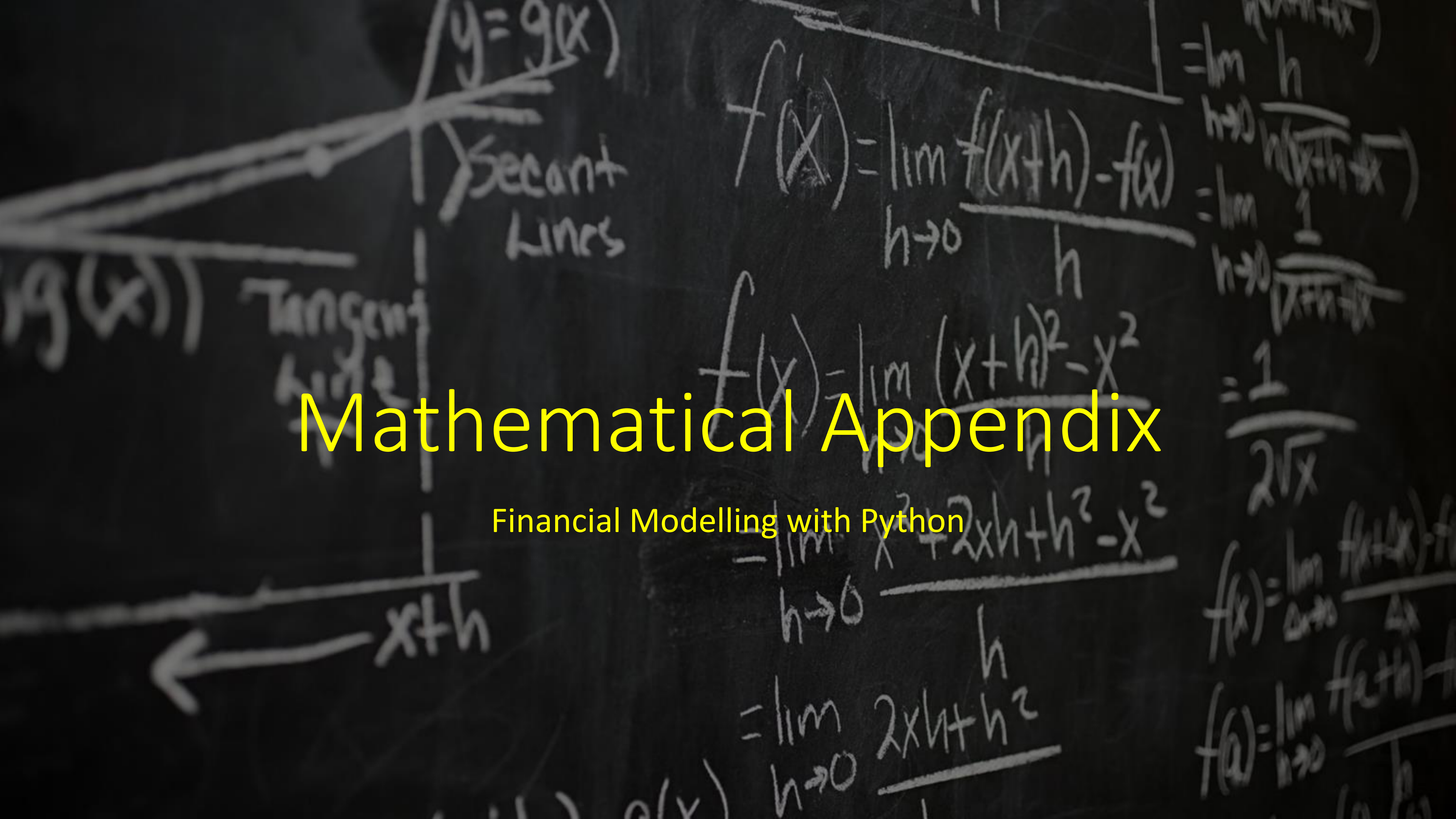


Mathematical Appendix

Financial Modelling with Python

Types of evolutionary processes

- Deterministic Process
- Chaotic Process
- Stochastic Process

Stochastic Process

A stochastic process is a process that evolves in time in a random manner. Thus, it can be represented by a sequence of random variables indexed with reference to the instants of time at which the process evolves.

We can, thus, represent a stochastic process as a sequence $\{X_t; t \geq 0\}$ of random variables defined on a suitable probability space.

Markov Process

A Markov process is a stochastic process, and with the property that the future is independent of the past, given the present.

The next step of the process, given its current position at a particular point in time, is independent of the history or the path of how the process reached the current position.

Markov Process

Under the Markov assumption, all the joint probabilities can be expressed as products of just two independent probabilities

- The single-time probability $P_1(j, t_0)$ - the probability of the system being in state j at time t_0

- The two-time conditional probability $P_2(k, t|j, t_0)$ – the probability of the system moving to state k at time t given that it was in state j at time t_0.

$$P_n(j_n, t_n; j_{n-1}, t_{n-1}; \ldots; j_1, t_1) = P_n(j_n, t_n | j_{n-1}, t_{n-1}; \ldots; j_1, t_1) \times$$
$$P_{n-1}(j_{n-1}, t_{n-1}; j_{n-2}, t_{n-2}; \ldots; j_1, t_1)$$
$$= P_n(j_n, t_n | j_{n-1}, t_{n-1}; \ldots; j_1, t_1) \times P_{n-1}(j_{n-1}, t_{n-1} | j_{n-2}, t_{n-2}; \ldots; j_1, t_1)$$
$$\times P_{n-2}(j_{n-2}, t_{n-2}; j_{n-3}, t_{n-3}; \ldots; j_1, t_1)$$
$$= P_n(j_n, t_n | j_{n-1}, t_{n-1}; \ldots; j_1, t_1) \times P_{n-1}(j_{n-1}, t_{n-1} | j_{n-2}, t_{n-2}; \ldots; j_1, t_1)$$
$$\times \ldots \times P_2(j_2, t_2 | j_1, t_1) \times P_1(j_1, t_1)$$

For Markov Process

$$P_n\left(j_n, t_n \middle| j_{n-1}, t_{n-1}; \ldots; j_1, t_1\right) = P_2\left(j_n, t_n \middle| j_{n-1}, t_{n-1}\right) \ \forall n \geq 2$$

Therefore, for a Markov process

$$P_n\left(j_n, t_n; j_{n-1}, t_{n-1}; \ldots; j_1, t_1\right) = P_2\left(j_n, t_n \middle| j_{n-1}, t_{n-1}\right) \times$$

$$P_2\left(j_{n-1}, t_{n-1} \middle| j_{n-2}, t_{n-2}\right) \times \ldots \times P_2\left(j_2, t_2 \middle| j_1, t_1\right) \times P_1\left(j_1, t_1\right)$$

Discrete time processes

- Stochastic processes usually evolve with time. They are, therefore, indexed with reference to points on the timeline.
- In discrete time processes, time is assumed to evolve in discontinuous jumps i.e. in discrete steps of a certain length.
- Discrete time can be represented by a lattice with lattice points labelled by integers.

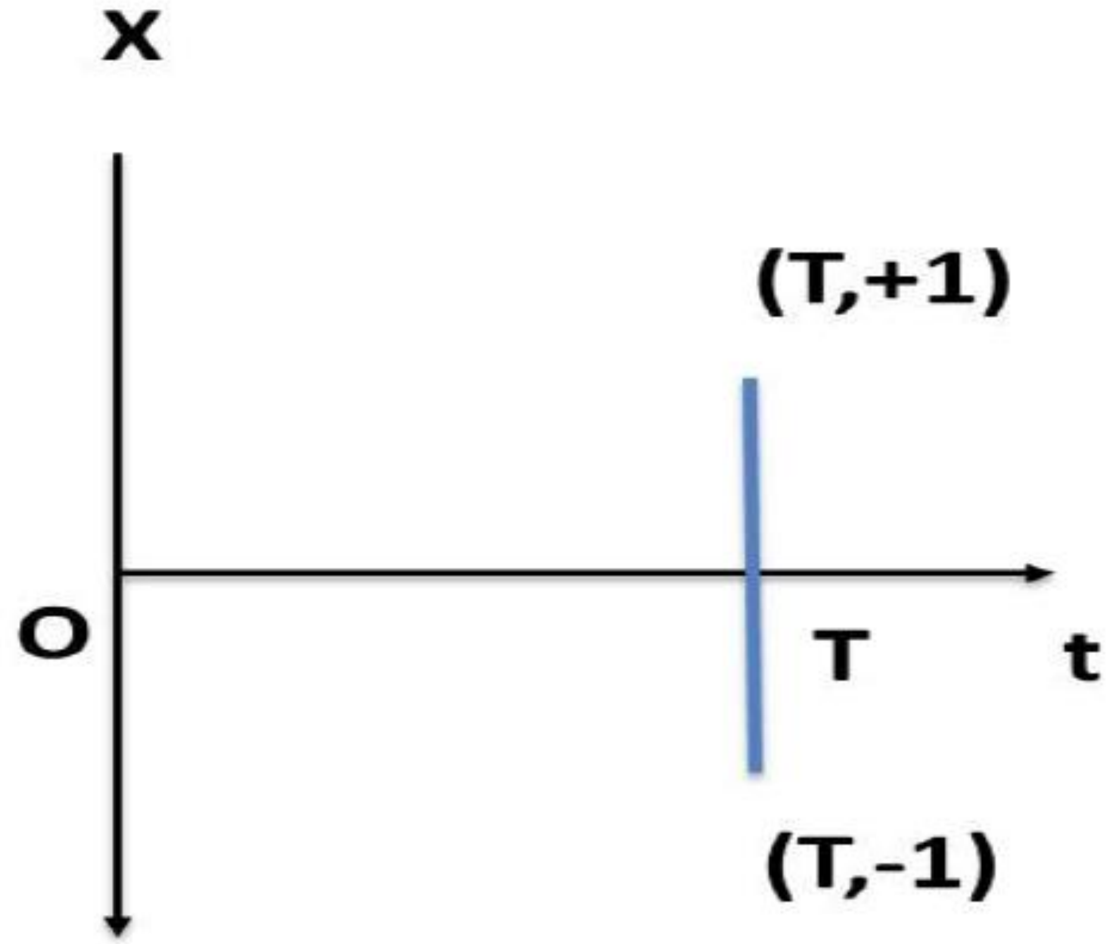
One Step Unbiased Unscaled Random Walk

Consider a stochastic variable X_1 whose initial state ($t=0$) is represented by the origin ($X=0$ at $t=0$).

Let time be modelled discretely with time step T so that the first time step is at $t=T$ at which point the variable X_1 will make its first and only move.

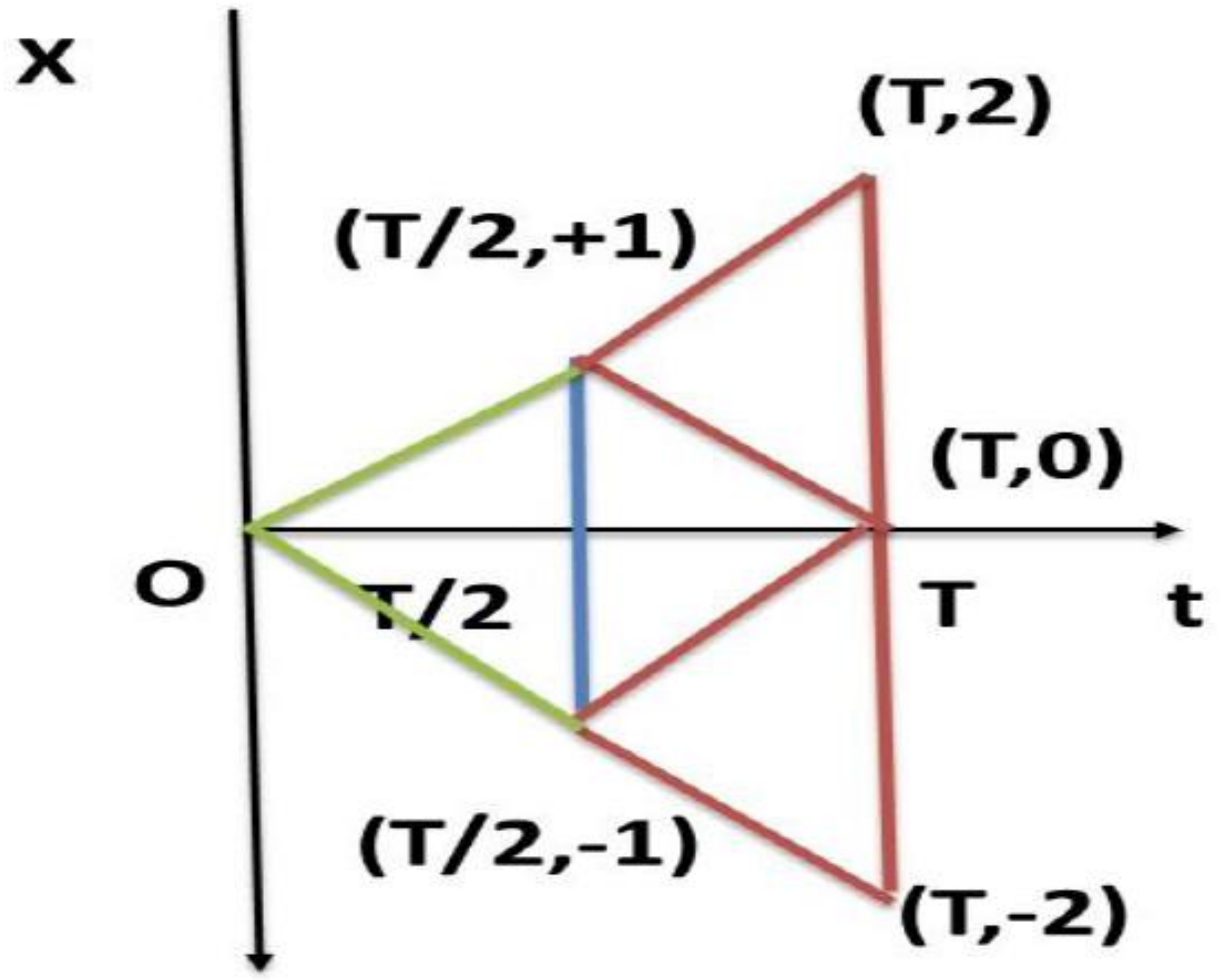
Let the move of the variable X_1 also be discrete of with step size 1 unit.

One Step Unbiased Unscaled Random Walk



$W_1(T)=X_1=\pm1$; $E[W_1(T)] = E(X_1) = ½ *-1 + ½ *+1= 0$,
$E(X_1^2) = ½ *(-1)^2 + ½ *(+1)^2= 1$ so that $\sigma_1^2=1$.

Two Step Random Walk



$W_2(T)=W_2(T/2)+X_2=X_1+X_2=0,\pm2$, $E[W_2(T)]=E(X_1+X_2)=0$,

$E[W_2(T)]^2=E(X_1+X_2)^2= E(X_1)^2+ E(X_2)^2+ 2E(X_1X_2) =2$, since $E(X_1X_2) =E(X_1)\, E(X_2) =0$

n-Step Random Walk

$$W_n(T) = W_n\left(n-1.\frac{T}{n}\right) + X_n = W_n\left(n-2.\frac{T}{n}\right) + X_{n-1} + X_n = \sum_{i=1}^{n} X_i$$

$$E\left[W_n(T)\right] = E\left[\sum_{i=1}^{n} X_i\right] = \left[\sum_{i=1}^{n} E(X_i)\right] = 0$$

$$E\left[W_n(T)\right]^2 = E\left[\sum_{i=1}^{n} X_i\right]^2 = E\left[\sum_{i=1}^{n} X_i^2\right] = \left[\sum_{i=1}^{n} E(X_i^2)\right] = n$$

$$\sigma_T^2 = n$$

Scaled Unbiased Random Walk

$$Y_i = \sqrt{\frac{T}{n}} X_i$$

$$Y_i = \sqrt{\frac{T}{n}} X_i \text{ are independent identically distributed random variables}$$

$$Y_i = \begin{cases} +\sqrt{\frac{T}{n}} \text{ with } p\left(Y_i = +\sqrt{\frac{T}{n}}\right) = 1/2 \\ -\sqrt{\frac{T}{n}} \text{ with } p\left(Y_i = -\sqrt{\frac{T}{n}}\right) = 1/2 \end{cases}$$

$$E\left[W_n(T)\right]^2 = E\left[\sum_{i=1}^{n} Y_i\right]^2 = E\left[\sum_{i=1}^{n} Y_i^2\right] = \left[\sum_{i=1}^{n} E\left(Y_i^2\right)\right] = \left[\sum_{i=1}^{n} E\left(X_i \sqrt{\frac{T}{n}}\right)^2\right] = \frac{T}{n}\left[\sum_{i=1}^{n} E\left(X_i^2\right)\right] = T$$

$$\sigma_T^2 = T$$

Scaled Unbiased Random Walk

arbitrary point t=t* in (0,T)

$$W_n\left(t^*\right)=W_n\left(\frac{nt^*}{T}\cdot\frac{T}{n}\right)=\sum_{i=1}^{nt^*/T}Y_i=\sqrt{\frac{T}{n}}\sum_{i=1}^{nt^*/T}X_i$$

$$E\left[W_n\left(t^*\right)\right]=0;E\left[W_n\left(t^*\right)\right]^2=\frac{T}{n}\left(\frac{nt^*}{T}\right)=t^*\ \textit{so that}\ Var\left[W_n\left(t^*\right)\right]=t^*$$

Central Limit Theory

Let X_i; i=1,2,…,*n* be independent identically distributed random variables **each** with finite mean and variance μ and σ^2 respectively. Then the following expression is distributed as a standard normal variate.

$$Z_n = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}}$$

Application of CLT to the Random Walk

$$\mu_i = E(X_i) = 0;\ E\left(X_i^2\right) = 1,\ \sigma_i^2 = 1\ \forall\ i$$

Hence, by Central Limit Theorem $\lim_{n\to\infty} \dfrac{\sum_{i=1}^{nt/T} X_i - \frac{nt}{T}\mu}{\sqrt{\frac{nt}{T}\sigma^2}}$

$$= \lim_{n\to\infty} \frac{\sum_{i=1}^{nt/T} X_i}{\sqrt{nt/T}} = \lim_{nt/T\to\infty} \frac{\sum_{i=1}^{nt/T} X_i}{\sqrt{nt/T}} \xrightarrow{\text{distribution}} N(0,1)$$

so that $W(t) = W_\infty(t) = \lim_{nt/T\to\infty} \sqrt{\dfrac{T}{n}} \sum_{i=1}^{nt/T} X_i \xrightarrow{\text{distribution}} N(0,t)$

Properties of Brownian Motion

The process W = (W(t), t >= 0) is said to have the following properties

- CONTINUITY: W(t) is continuous and W(0) = 0
- DISTRIBUTION: W(t) is distributed as a Normal(0, t)
- DISTRIBUTION OF INCREMENTS: The increment W(s+t) – W(s) is distributed as a normal N(0, t) and is independent of the history of what the process did up to time s
- REPRESENTATION: We can express an increment of BM as $dW(t) = z\sqrt{dt}$ where z is standard normal distribution
- DIFFERENTIABILITY: The process W t is not differentiable at any point t

RANDOM WALKS	TIME LENGTH /LAYER SPACING	JUMP SIZE	EXPECTATION	VARIANCE
SINGLE STEP RW	T	±1	0	1
TWO STEP RW	T/2	±1	0	2
n-STEP RW	T/n	±1	0	n
n-STEP SCALED RW	T/n	$\pm\sqrt{(T/n)}$	0	T
n-STEP SCALED RW	T/n	$\pm\sigma\sqrt{(T/n)}$	0	σ^2T
n-STEP SCALED RW WITH DRIFT	T/n	$(\mu T/n)\pm \sigma\sqrt{(T/n)}$	μT	σ^2T
Brownian Motion	->0	->0	0	T
Scaled BM	->0	->0	0	σ^2T

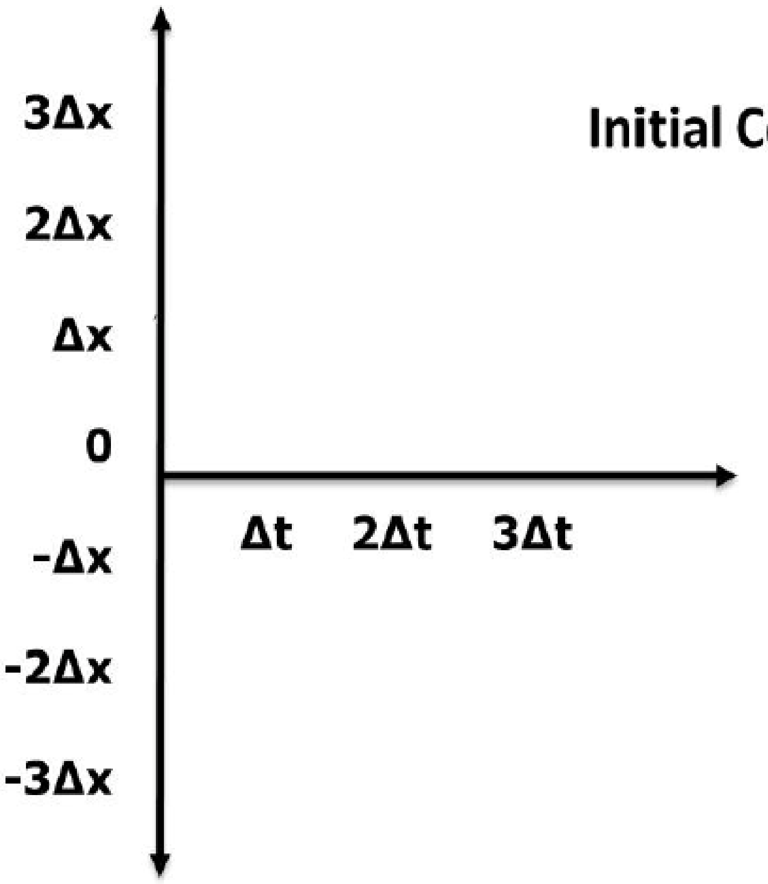
BM with drift

The infinitesimal increment of Brownian motion with drift is given by

$dx = \mu dt + \sigma dW_t = \mu dt + \sigma z\sqrt{(dt)}$ where z is the standard normal variate.

Clearly, $E(dx) = \mu dt$ and $Var\ (dx) = \sigma^2\ (dt)$

BM with Diffusion



Initial Condition: $P(x,0)=\delta(x)$

- Let p=probability that the randomly moving particle moves one step to the right in the next step i.e. the probability that the coordinate increases by Δx;
- q=probability that the randomly moving particle moves one step to the left in the next step i.e. the probability that the coordinate decreases by Δx;
- r=1-p-q=probability that the randomly moving particle stays where it is at time t.
- p,q,r are assumed constant over the length of the walk i.e. for all time and space steps.
- P(X,t)=probability of finding the particle at the coordinate x=X at time t=t.

$$P(X,t+\Delta t) = pP[X-\Delta x,t] + qP[X+\Delta x,t] + (1-p-q)P(X,t)$$

$$P(X,t+\Delta t) - P(X,t) = pP[X-\Delta x,t] + qP[X+\Delta x,t] - (p+q)P(X,t)$$

Now, in the limiting case, LHS can be written as:

$$P(X,t+\Delta t) - P(X,t) = \Delta t\left[\frac{\partial P(X,t)}{\partial t}\right]$$

For the RHS, $pP[X-\Delta x,t] + qP[X+\Delta x,t] - (p+q)P(X,t)$

$$= \frac{1}{2}(p+q)\{P[X-\Delta x,t] - 2P(X,t) + P[X+\Delta x,t]\}$$

$$+ \frac{1}{2}(p-q)\{P[X-\Delta x,t] - P[X+\Delta x,t]\}$$

$$\frac{\partial P(X,t)}{\partial t} = \frac{(\Delta x)^2}{2\Delta t}(p+q)\frac{\partial^2 P(X,t)}{\partial X^2} - \frac{\Delta x}{\Delta t}(p-q)\frac{\partial P(X,t)}{\partial X}$$

For an unbiased random walk p=q= ½

$$\frac{\partial P(X,t)}{\partial t} = \frac{(\Delta x)^2}{2\Delta t}\frac{\partial^2 P(X,t)}{\partial X^2}$$

$$\frac{\partial P(X,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 P(X,t)}{\partial X^2}$$

The diffusion equation for the unbiased random walk

The marks in a class are believed to follow a normal distribution with a mean of 60 and a variance of 144. If the total number of students in the class is 100, what is the number of students who have obtained greater than 78 marks?

- Let X represent the marks obtained by various students in the class. Then it is given that X is N(60,144). We need to find P(X>78)
- The standard normal variate Z corresponding to X=78 is (78-60)/12 = 1.5
- Hence, we need to find P(Z>1.5)=0.067
- Since there are 100 students in the class, number of students above 78 =6.7 i.e. 6 students

A particle starts executing standard Brownian motion at $t=0$. What is the probability that the particle will be less than 10 units away from the origin after 100 units of time?

- The spectrum of possible values of a BM $W(t)$ at time t after initiation is normally distributed with a mean of 0 and a variance of t i.e. a standard deviation of $\sqrt{(t)}$.
- Hence, the possible values after time $t=100$ are normally distributed with a mean of 0 and standard deviation of $\sqrt{(100)} = 10$.
- We need to find the probability that $-10<W(100)<+10$ i.e. $P(-10<X<+10)$ where $W(100)=X$ is $N(0,100)$.
- $P(-1<z<+1) = 0.6826$

A particle executes scaled Brownian motion with drift in one dimension. The drift rate per unit time is 0.0001 units. The variance rate is 0.01. Calculate the probability that the particle is more than 2 units on the positive side of its initial position (origin) after 2500 units of time.

- The spectrum of possible values of a BM with drift i.e. $dx=\mu dt+\sigma dW_t=\mu dt+\sigma z\sqrt{(dt)}$
- At time dt after initiation is normally distributed with a mean of μdt and a variance of $\sigma^2 dt$
- Here $\mu=0.0001$, $\sigma^2=0.01$ and $t=2500$
- Hence, the possible values after time $t=2500$ are normally distributed with a mean of 0.25 and standard deviation of $\sqrt{(0.01 * 2500)} = 5$
- Thus, we need to find out $P(X>2.00)$ where X is $N(0.25,25)$
- Converting to standard normal variate $P(X>2.00)=P[Z>(2.00-0.25)/5]=P(Z>0.35)=0.3632$

Let x be defined as the drift diffusion stochastic (Ito) process: dx = a(x,t)dt + b(x,t)dW

Ito's Lemma: Let G(x,t) be continuous & at least twice differentiable function of a stochastic variable x as defined above and time t, then we have:

$$dG = \left(a.\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 . \frac{\partial^2 G}{\partial x^2} \right) dt + b.\frac{\partial G}{\partial x} dW_t$$

Consider a continuous and differentiable function $G(x,t)$ *of a stochastic variable* x *and* t *where* x *satisfies* $dx = a(x,t)dt + b(x,t)dW$. *Taylor expansion of* $G(x,t)$ *is*

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}dx^2 + \frac{\partial^2 G}{\partial x \partial t}dxdt + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}dt^2 + \ldots$$

$$= \frac{\partial G}{\partial x}\left[a(x,t)dt + b(x,t)dW\right] + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\begin{bmatrix} a(x,t)dt + \\ b(x,t)dW \end{bmatrix}^2$$

$$+ \frac{\partial^2 G}{\partial x \partial t}\left[a(x,t)dt + b(x,t)dW\right]dt + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}dt^2 + \ldots$$

$We\ have\ dx^2 = \left[a(x,t)dt + b(x,t)dW\right]^2$

$= \left[a(x,t)dt + b(x,t)z\sqrt{dt}\right]^2 = b^2z^2dt + h.o\ terms\ in\ dt$

$Now,\ E\left(z^2\right) = 1\ whence\ E\left(b^2z^2dt\right) = b^2dt.$

$Also\ Var\ \left(z^2\right) = 2\ whence\ Var\left(b^2z^2dt\right) = 2b^4dt^2 = 0$

if higher order terms than dt are neglected.

Now the Variance of a stochastic variable in

time dt is proportional to dt not dt^2.

Since $Var\left(b^2z^2dt\right) = 2b^4dt^2$ *is proportional to* dt^2

it is too small to have a stochastic component.

Hence, in the limit that higher order terms in dt

are neglected, $\left(b^2z^2dt\right)$ *and hence,* dx^2 *may be*

considered non-stochastic with a value of b^2dt.

Thus, we get the Ito Lemma as:

$$dG = \left(a.\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2}b^2.\frac{\partial^2 G}{\partial x^2}\right)dt + b.\frac{\partial G}{\partial x}dW$$

Let $G(S,t)=\ln S$ where $dS=\mu S dt+\sigma S dW$. Using Ito's Lemma find the drift and diffusion terms and the distribution of G(S,t).

Ito's Lemma states that if G(x,t) is a twice differentiable function of x where x is given by dx=adt+bdW, then

$$dG=\left(a.\frac{\partial G}{\partial x}+\frac{\partial G}{\partial t}+\frac{1}{2}b^2.\frac{\partial^2 G}{\partial x^2}\right)dt+b.\frac{\partial G}{\partial x}dW_t$$

In our problem $G(S,t)=\ln S$, $\frac{\partial G}{\partial S}=\frac{\partial \ln S}{\partial S}=\frac{1}{S}$, $\frac{\partial^2 G}{\partial S^2}=\frac{\partial}{\partial S}\left(\frac{1}{S}\right)=-\frac{1}{S^2}$, $\frac{\partial G}{\partial t}=\frac{\partial \ln S}{\partial t}=0$, $a=\mu S$, $b=\sigma S$. Putting the values, we get:

$$dG=d(\ln S)=\left(\mu S\frac{1}{S}-\frac{1}{2}\sigma^2 S^2\frac{1}{S^2}\right)dt+\sigma S\frac{1}{S}dW_t=\left(\mu-\frac{1}{2}\sigma^2\right)dt+\sigma dW_t$$

This shows that the drift term of $G(S,t)=\ln S$ is $\left(\mu-\frac{1}{2}\sigma^2\right)$ and the diffusion term is σ.

Further, $d(\ln S)$ is normally distribution with a mean of $\left(\mu-\frac{1}{2}\sigma^2\right)dt$ and a variance of $\sigma^2 dt$

.

Equivalently $\ln S_T$ is $N\left[\ln S_0+\left(\mu-\frac{1}{2}\sigma^2\right)T,\sigma^2 T\right]$.

Stock Price Distributions

Stock prices are assumed to follow a Markov process. The Markov property of stock prices is consistent with the weak form of market efficiency i.e. that the current market price encapsulates its entire past history and, therefore, the current price is the only relevant information for future price prediction. Whatever is today's stock price encapsulates everything or incorporates all information about the prior behaviour of the price.

Stock Price Distributions

The expected instantaneous percentage return i.e. the percentage change in price (dS/S) over an infinitesimal time interval dt required by investors from a stock is independent of the stock's price.

$$E(r) = E\left[\frac{1}{dt}\left(\frac{dS}{S}\right)\right] = \mu \text{ or } E\left(\frac{dS}{S}\right) = \mu dt$$

$$\frac{dS}{S} = \mu dt + X \quad (\text{where } X \text{ is a random variable with zero mean})$$

Stock Price Distributions

The variance of instantaneous percentage return over infinitesimal time dt is constant and independent of the stock price.

$$Var(r) = Var\left(\frac{1}{dt}\frac{dS}{S}\right) = \sigma^2 \text{ so that } SD\left(\frac{dS}{S}\right) = \sigma\sqrt{dt}$$

$$\frac{dS}{S} = \mu dt + \sigma dW = \mu dt + \sigma z \sqrt{dt}$$

Distribution of Returns

$$\ln S_T \xrightarrow{\text{distribution}} N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$$

$$\text{it follows that } r_{\ln} = \frac{1}{T}\ln\frac{S_T}{S_0} \xrightarrow{\text{distribution}} N\left(\left(\mu - \frac{1}{2}\sigma^2\right), \frac{\sigma^2}{T}\right)$$