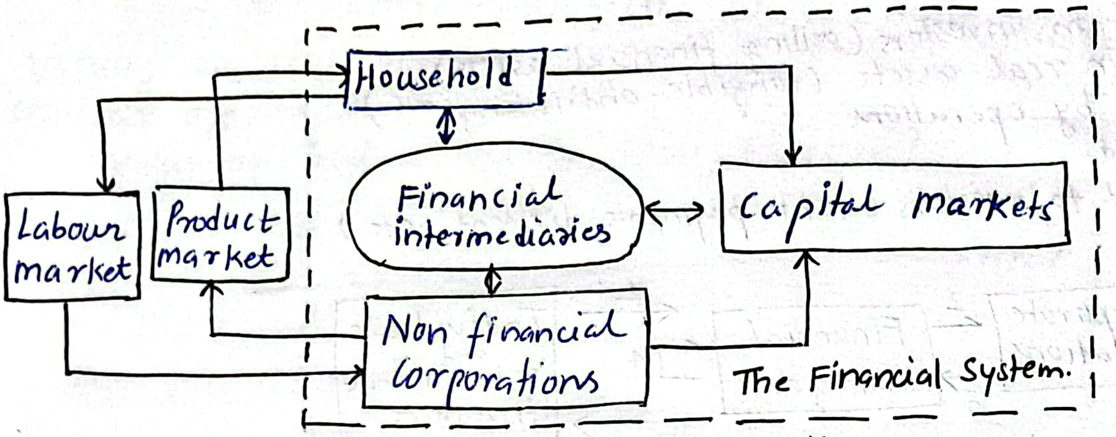
Introduction to Mathematical finance:

A flow model of the economy:



All business activities reduce to two functions:
→ Valuation of assets (Real/Financial/tangible/intangible)
→ management of assets (acquiring/selling).

You cannot manage what you can measure.
Valuation is starting point for management. Once value is established, management is easier. (Bidding example).

Objectives + Valuation ⇒ Decision.

Valuation is generally independent of objectives
Role of financial markets and the "price discovery" process.

Fundamental challenges of Finance:

Valuation
How are financial assets valued?
How should financial assets be valued?
How do financial markets determine asset value?
How well do financial market work?

Management.
How much should I save/spend?
What should I buy/sell?
When should I buy/sell?
How should I finance the transaction?

Balance sheet and Income Statement Perspectives

Balance sheet: snapshot of financial status quo (stock)
Income statement: rate of change of status quo (flow)
] Remember Bathtub example.

Financial status: Balance sheet
Financial decision: Income statement.

Balance sheet:

Assets	Liabilities
cash capital Intangibles	Equity Debt.
Value	Value

Income statement:

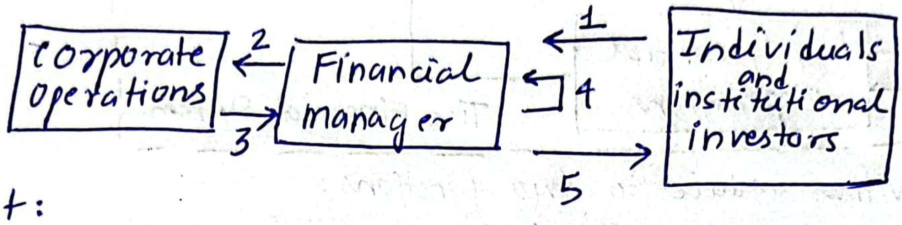
source of funds = Uses of funds

$$\Delta S + \Delta B + NI = I + D + T + C$$

The framework of financial Analysis.

Corporate Financial Decisions.

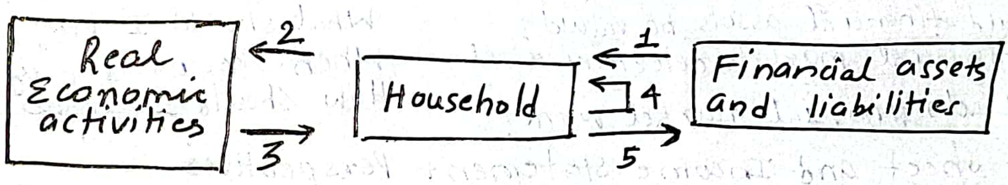
1) Cash raised from investors (selling financial assets)
2) Cash invested in real assets (tangible and intangible)
3) Cash generated by operations
4) Cash reinvested
5) Cash returned to investors (debt payments, divident etc).



management:
Real Investment: 2,3
Financing : 1,4
Payout : 5
Risk Management: 1,5
Objective: Create and maximise shareholder value.

Personal Financial decision.

1) Cash raised from financial institutions
2) Cash invested in real assets
3) Cash generated by labour supply
4) Cash consumed and reinvested in real assets
5) Cash invested in financial assets.



management:
1) Real investment 2,3
2) Consumption / Financing 1,4
3) Saving / Investment 5
4) Risk Management 1,5
5) Objective: Maximise lifetime "Happiness" or expected utility.

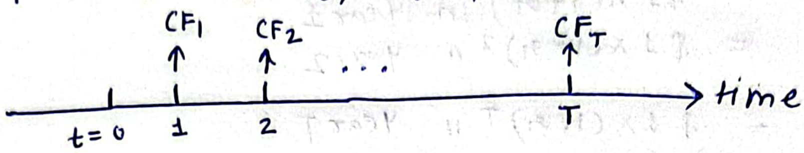
Two other factors that make finance challenging
1) Time: Cashflows now are different from cashflow later
2) Risk: Under perfect certainty, finance theory is complete.

Asset: An asset is a sequence of cashflows

$$Asset_t = \{CF_t, CF_{t+1}, CF_{t+2}, \cdots\}$$

Valuing an asset requires valuing a sequence of cashflows. Sequence of cashflows are the "building block" of finance.

$$\text{Value of } Asset_t = V_t(CF_t, CF_{t+1}, CF_{t+2}, \cdots)$$



The present Value operator:

Cashflows at different dates are different. Past and future cannot be combined without first converting them. Once "exchange rates" are given, combining cashflows is trivial. A numeraire date should be picked; typically t=0 or "today". Cashflows can then be converted to present value.

$$V_0(CF_1, CF_2, CF_3 \cdots) = \left(\frac{\$_1}{\$_0}\right) \times CF_1 + \left(\frac{\$_2}{\$_0}\right) \times CF_2 + \cdots$$

NPV (Net present Value): Net of initial cost or Investment

→ Can be captured by date 0 cashflow CF_0.

$$V_0(CF_0, CF_1, \cdots) = CF_0 + \left(\frac{\$_1}{\$_0}\right) \times CF_1 + \left(\frac{\$_2}{\$_0}\right) \times CF_2 + \cdots$$

- If there is any initial investment then $CF_0 < 0$
- Note that any CF_t can be negative (future cost).
- V_0 is completely general expression for net present value.

example: Exchange rate / discount factor → comes from market place.

$$\left(\frac{\$_1}{\$_0}\right) = 0.90, \left(\frac{\$_2}{\$_0}\right) = 0.80$$

What is the net present value of a project requiring a current investment of \$10MM with cashflows of \$5MM in year1 and \$7MM in Year 2?

$$NPV_0 = -\$10 + 0.90 \times 5 + 0.8 \times 7 = \$0.10$$

What is the NPV of a project requiring an investment of \$8MM in Year. with a cashflow of \$2MM immediately and a CF of \$5MM in Year 1?

$$NPV_0 = \$2 + 0.9 \times \$5 - 0.8 \times \$8 = \$0.10$$

Implicit Assumptions / Requirements for NPV Calculations

- Cashflows are known.
- Exchange rates are known.
- No friction in currency conversion.

The time value of money: What determines the growth of \$1 over T years? \$1 today should be worth more than \$1 in the future.
→ People want money today (impatience).
→ supply and Demand.
Opportunity cost of capital r.

$$\$1 \text{ in Year } 0 = \$1 * (1+r) \text{ in Year } 1$$
$$" = \$1 \times (1+r)^2 \text{ '' Year } 2$$
$$\vdots$$
$$" = \$1 \times (1+r)^T \text{ '' Year } T$$

\$1 in Year-T should be worthless than \$1 today.

$$\$1/(1+r) \text{ in Year } 0 = \$1 \text{ in Year } 1$$
$$\$1/(1+r)^2 \text{ '' } = \$1 \text{ in Year } 2$$
$$\vdots$$
$$\$1/(1+r)^T \text{ '' } = \$1 \text{ in Year } T$$

We now have explicit expression for V_0:

$$V_0 = CF_0 + \frac{1}{(1+r)} CF_1 + \frac{1}{(1+r)^2} CF_2 + \cdots$$

Using this expression, any cashflow can be valued.
Take positive NPV projects, reject negitive NPV projects.

The Perpetuity:

Perpetuity Pays constant cashflow C forever.

$$PV = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \cdots$$ infinite GP, sum = $a/1-r$

$$\boxed{PV = \frac{C}{r}}$$

- r is the interest rate at present
- assumption: r is constant through years.

Growing Perpetuity Pays growing cashflow $C(1+g)^t$ forever.

$$PV = \frac{C}{(1+r)} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots$$

$$\boxed{PV = \frac{C}{r-g}}, \quad r > g$$

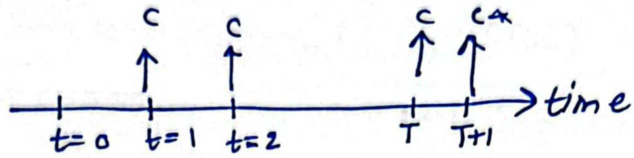
The Annuity: Pays constant cashflows C for T periods.

$$\boxed{PV = \frac{C}{r} - \frac{C}{r} \cdot \frac{1}{(1+r)^T}}$$

$$= C \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^T}\right] = C \times ADF(r, T)$$

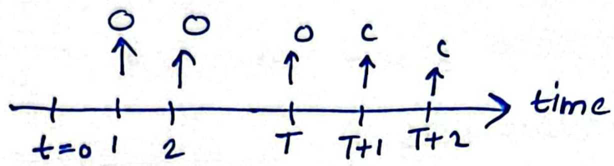
Diagram:

Perpetuity.



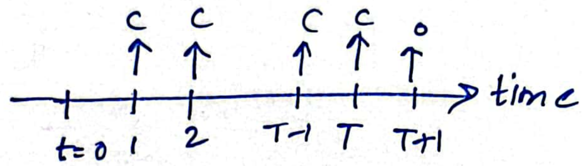
(minus)

Date T perpetuity.



(equals)

T-period Annuity



Compounding: Interest may be credited or charged more often than annually.

EAR: Effective Annual Rate.

APR: Annual Percentage rate.

Typical compounding conventions:

Let r denote APR, n periods of compounding r/n is per period rate for each period.

Effective annual rate (EAR) is $\boxed{r_{EAR} = (1 + r/n)^n - 1}$

* Compound Interest:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

n: no. of compounding periods.

t: time in years.

Imf Lecture 2:

→ Binomial model for pricing
→ Risk Neutral vs Real World problem.
→ Forward, Options.

$$S \to Su, \quad S \to Sd$$

option: $(p)\ Su-k$, $(1-p)\ 0$

price: $\dfrac{p(Su-k)}{1+r\Delta t}$, $\dfrac{pf(Su)+q\, f(Sd)}{1+r\Delta t}$.

Replication portfolio

$\boxed{\alpha \text{ Stock} + \beta \text{ cash}}$ → $\beta(1+r\Delta t) + \alpha Su$, $\beta(1+r\Delta t) + \alpha Sd$

$$\beta(1+r\Delta t) + \alpha Su = Su-k \qquad \beta(1+r\Delta t) + \alpha Sd = 0$$

calculate α, β.

$$\beta = \frac{-\alpha Sd}{1+r\Delta t}, \quad \alpha = \frac{Su-k}{Su-Sd}.$$

$$S \to Su - a, \quad S \to Sd - b$$

$(\alpha S + \beta \text{cash})$ $\quad \alpha Su + \beta(1+r\Delta t) = a$

$\alpha Sd + \beta(1+r\Delta t) = b$

$$\alpha S + \beta\text{cash} = \big((\ )\, a + (\ )\, b\big)(1+r\Delta t)^{-1}$$

↳ present value of $\alpha S + \beta$cash.

$$\alpha = \frac{a-b}{Su-Sd}, \quad \beta = \frac{aSd - bSu}{(1+r\Delta t)(Sd-Su)}$$

(wrong) → recalculated next page.

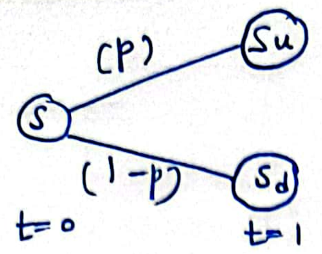
$$\text{coef of } a = \frac{S}{Su-Sd} + \frac{Sd}{(1+r\Delta t)(Sd-Su)} = \frac{1}{u-d}\left(1 - \frac{d}{r\Delta t}\right)$$

$$\text{coef of } b = \frac{-S}{Su-Sd} - \frac{Su}{r\Delta t\,(Sd-Su)} = \frac{1}{u-d}\left(-1 + \frac{u}{r\Delta t}\right)$$

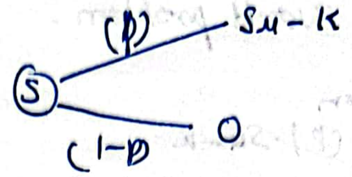
coef of a + coef of b = 1

prob.

Discrete Processes.



(i) stock



(ii) option.

Consider a general portfolio, / replication portfolio:

$\longrightarrow \alpha\,[\text{stocks}] + \beta\,[\text{cash}] \Rightarrow \alpha S + \beta \rightarrow$ value of portfolio at time $t=0$.

$$\boxed{\alpha S + \beta} \begin{cases} \xrightarrow{p} \alpha S_u + \beta(1+r\Delta t) & \text{— (1)} \\ \xrightarrow{(1-p)} \alpha S_d + \beta(1+r\Delta t) & \text{— (2)} \end{cases}$$

$t=0$ $\qquad$ $t=1$

In case of options: Two unknowns, α, β.

$\alpha S_u + \beta(1+r\Delta t) = S_u - K \quad , \quad \alpha S_d + \beta(1+r\Delta t) = 0$

$\alpha S_u - \alpha S_d = S_u - K \qquad\qquad \beta = \dfrac{-\alpha S_d}{(1+r\Delta t)}$

$$\boxed{\alpha = \frac{S_u - K}{S_u - S_d}} \qquad = \frac{-(S_u-K)S_d}{(S_u-S_d)(1+r\Delta t)}$$

$$\boxed{\beta = \frac{(K-S_u)S_d}{(S_u-S_d)(1+r\Delta t)}}$$

$S \xrightarrow{(p)} S_u = a$

$S \xrightarrow{(1-p)} S_d = b$

$t=0 \qquad t=1$

$$\alpha S + \beta \begin{cases} \longrightarrow \alpha S_u + \beta(1+r\Delta t) \\ \longrightarrow \alpha S_d + \beta(1+r\Delta t) \end{cases}$$

$t=0 \qquad t=1$

$\alpha S_u + \beta(1+r\Delta t) = a$ — (3)

$\alpha S_d + \beta(1+r\Delta t) = b$ — (4)

$\alpha S + \beta$ can also be written as:

$$\alpha S + \beta = \Big[\underbrace{(\quad)\, a + (\quad)}_{}\, b\Big](1+r\Delta t)^{-1}$$

coefficient of a, b are probabilities, whose sume $=1$.

from ③ and ④

Let us calculate α, β.

$$\alpha S_u + b - \alpha S_d = a$$

$$\alpha (S_u - S_d) = a - b$$

$$\boxed{\alpha = \frac{a-b}{S_u - S_d}}$$

$$\beta = \frac{b - \alpha S_d}{(1 + r\Delta t)}$$

$$= \frac{b - \frac{(a-b)S_d}{(S_u - S_d)}}{(1 + r\Delta t)}$$

$$= \frac{bS_u - bS_d - aS_d + bS_d}{(1+r\Delta t)(S_u - S_d)}$$

$$\boxed{\beta = \frac{bS_u - aS_d}{(1+r\Delta t)(S_u - S_d)}}$$

* let us calculate coeff of a, b.) $(\alpha S + \beta)(1 + r\Delta t)$

$$(\alpha S + \beta)(1+r\Delta t) = \left[\frac{(a-b)S}{S_u - S_d} + \frac{bS_u - aS_d}{(1+r\Delta t)(S_u - S_d)}\right](1 + r\Delta t)$$

$$\text{coef of } a = \frac{S}{S_u - S_d} - \frac{S_d}{(1+r\Delta t)(S_u - S_d)}$$

$$= \frac{1}{S_u - S_d}\left[S - \frac{S_d}{1 + r\Delta t}\right] \quad \text{—— ⑤}$$

$$\text{coef of } b = \frac{-S}{S_u - S_d} + \frac{S_u}{(1+r\Delta t)(S_u - S_d)}$$

$$= \frac{1}{S_u - S_d}\left[\frac{S_u}{(1+r\Delta t)} - S\right] \quad \text{—— ⑥}$$

Add ⑤ and ⑥.

$$\frac{1}{S_u - S_d}\left[S - \frac{S_d}{1+r\Delta t} + \frac{S_u}{(1+r\Delta t)} - S\right]$$

carry on.

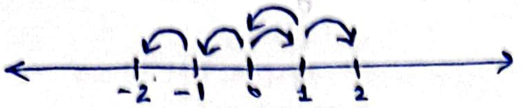
$$\frac{1}{S_u - S_d}\left[\frac{S_u - S_d}{1 + r\Delta t}\right] = \frac{1}{1+r\Delta t} \times (1 + r\Delta t) = 1.$$

coefficient of a, b are probabilities.

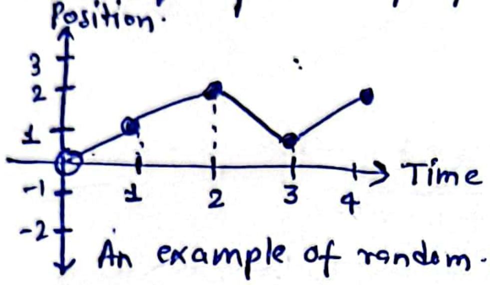
IMF Lecture 5

Random Walks, markov Processes.

1 D discrete random walk.



- can go either left or right
- can go to only neighbouring points.
- with equal probability (symmetric).



An example of random-walk.

Set of paths till time 3 = $8 = 2^3$ with equal probability.

Stocastic processes ← indexed random variables.

$x = 1$ ← deterministic variable

y ↗ 1 with prob 1/2 ↘ −1 " " 1/2 ← random variable.

$y_1, y_2, y_3 \dots$ ← Series of random variables indexed with time; stocastic process.

Question:

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$-2 \quad -1 \quad 0 \quad 1 \quad 2$$

x_i ← denote position at time i

$x_0 = 0$

You stop if you reach either 2 or −2

- What is the probability that you stop at 2? Ans. 1/2
- What is the prob that you stop at 3?
- What is the expected duration of the walk?

Answer: $P(0)$ = is the probability to hit 3 before −2 starting from 0.

$q(0) = 1 - P(0)$; is the probability to hit −2 before 3 starting from 0.

We know, $P(3) = 1$, ⇒ $P(-2) = 0$.

$$P(0) = \frac{1}{2}P(1) + \frac{1}{2}(P(-1))$$

$$\Rightarrow P(i) = \frac{1}{2}P(i+1) + \frac{1}{2}P(i-1) \quad -2 < i < 3$$

$$\Rightarrow \frac{1}{2}(P(i) - P(i-1)) = \frac{1}{2}(P(i+1) - P(i))$$

⇒ constant increments

$$-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$$
$$0 \qquad\qquad\qquad\qquad 1$$

↳ 5 equal parts.

$$\Rightarrow P(0) = \frac{2}{5} \leftarrow \text{prob you stop at } 3$$

$$q(0) = 3/5$$

first passage problems/hitting time.

$$E[Pos(t = i+1) \mid P(t = i)] = P(t = i)$$

↳ martingall.

Above problem is a martingall, we can use above formula to calculate $P(0)$.

$$P(0) \times 3 + (1 - P(0)) \times (-2) = 0$$
$$P(0) = 2/5$$

Now what is the expected duration? We know, $E\,d(i)$ = expected duration starting from i.

i = −2, −1, 0, 1, 2, 3.

$d(3) = 0$, $d(-2) = 0$.

$$d(0) = 1 + \frac{1}{2}d(1) + \frac{1}{2}d(-1)$$

$$\Rightarrow d(i) = 1 + \frac{1}{2}d(i+1) + \frac{1}{2}d(i-1)$$

$$\frac{1}{2}\underbrace{(d(i) - d(i-1))}_{f(i)} = 1 + \frac{1}{2}[d(i+1) - d(i)].$$

$$\frac{1}{2}f(i) = 1 + \frac{1}{2}f(i+1)$$

$$f(i) = 2 + f(i+1)$$

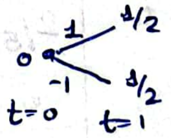
$S_t = x_1 + x_2 + \dots + x_t$. → x_i are independent

Variance: $E(x^2) - [E(x)]^2$ ← 0

$$= (-2)^2 \times \frac{3}{5} + (3)^2 \times \frac{2}{5}$$

$$= \frac{12}{5} + \frac{18}{5} = 6$$

variance at each step.



$$Var = \frac{1}{2} \times 1^2 + \frac{1}{2} \times (-1)^2 = 1$$

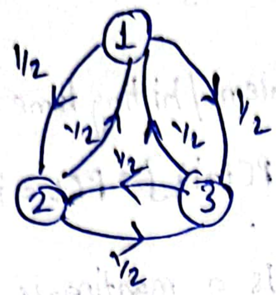
No. of steps/duration = 6

Applicable when actions are independent at each time step. $Var(S_t) = \sum_{i=1}^{t} Var(x_i)$

Markov processes.

→ memoryless

→ Future depends only on the present and not on history.

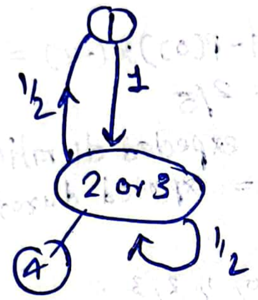


Start from ① and stop when you reach ① again.

$$t(1) = 1 + \frac{1}{2} t(2) + \frac{1}{2} t(3)$$

$$t(2) = 1 + \frac{1}{2} t(3)$$

$$t(3) = 1 + \frac{1}{2} t(2)$$



$$t(1) = 1 + t(4)$$

$$t(4) = 1 + \frac{1}{2} t(4)$$

$$\Rightarrow t(4) = 2$$

$$\boxed{t(1) = 3}$$

Coupon collector problem

① ② ③ ④ ⑤ ← 5 kind of coupon.

at each time step, you collect one coupon at random. You stop once your collection has all the 5 coupons.

Q: Expected stopping time =

State: # of different kinds of coupons at each time step.

at start we have zero coupon.

↗	0	1	2	3	4	5
0	0	1	0	0	0	0
1	0	1/5	4/5	0	0	0
2	0	0	2/5	3/5	0	0
3	0	0	0	3/5	2/5	0
4	0	0	0	0	4/5	1/5
5	0	0	0	0	0	1

Find $S(0) = ?$, $S(5) = 0$

$$S(4) = 1 + \frac{1}{5} S(5) + \frac{4}{5} S(4)$$

$$S(4) = 5$$

$$S(3) = 1 + \frac{2}{5} S(4) + \frac{3}{5} S(3)$$

$$S(3) = 1 + 2 + \frac{3}{5} S(3)$$

$$S(3) \frac{2}{5} = 3$$

$$S(3) = \frac{15}{2} = 7.5$$

$$S(2) = 1 + \frac{3}{5} S(3) + \frac{2}{5} S(2)$$

$$\frac{3}{5} S(2) = 1 + \frac{3}{5} \times 7.5$$

$$S(2) = \frac{5.5 \times 5}{3}$$

$$S(0) = 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} + \frac{5}{5}$$

harmonic sum

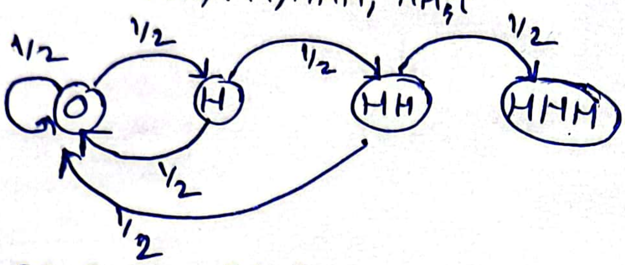
Keep tossing a coin, stop when you get HHH, what is the expected stopping time.

States : 0, H, HH, $S(0) = ?$

I : H, T

II : HT, TH, TT, HH

III : HTH, HTT, THH, THT, TTH, TTT, HHH, HHT



$S(3) = 0, \quad S(2) = 1 + \frac{1}{2} S(3) + \frac{1}{2} S(0)$

$$\boxed{S(2) = 1 + \frac{1}{2} S(0)}$$

$S(1) = 1 + \frac{1}{2} S(2) + \frac{1}{2} S(0)$

$= 1 + \frac{1}{2}\left(1 + \frac{1}{2} S(0)\right) + \frac{1}{2} S(0)$

$S(1) = 1 + \frac{1}{2} + \frac{1}{4} S(0) + \frac{1}{2} S(0)$

$S(0) = 1 + \frac{1}{2} S(1) + \frac{1}{2} S(0)$

$\frac{1}{2} S(0) = 1 + \frac{1}{2}\left[1 + \frac{1}{2} + \frac{1}{4} S(0) + \frac{1}{2} S(0)\right]$

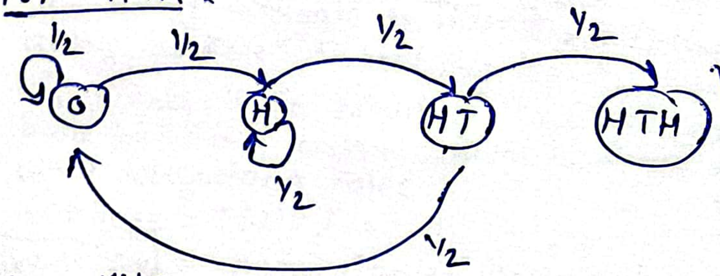
$\frac{1}{2} S(0) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} S(0) + \frac{1}{4} S(0)$

$= 1 + \frac{3}{4} + \frac{3}{8} S(0)$

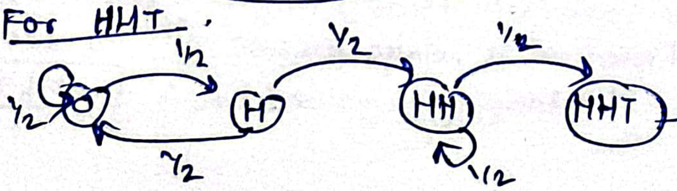
$\frac{1 S(0)}{16} = \frac{7}{4}$

$S(0) = \frac{16 \times 7}{4} = 28$ (check if it is correct).

For HTH.



For HHT.



Calculate stopping time for both.