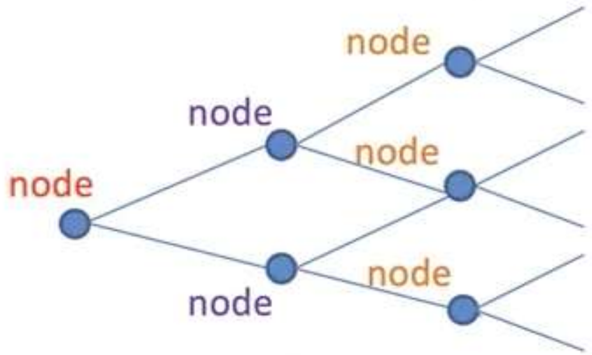
Interest Rate Modelling

Binomial Tree

- A binomial model is a model that assumes that interest rates can take **only one of two possible values** in the next period.
- A **node** is a point in time when interest rates can take **one of two** possible paths.
  - At **time 0**, there is one root node with the **current interest rate**.
  - From there it leads to **two other nodes** which show the alternative interest rates in the next period.
  - As time goes, more and more possibilities emerge resulting in some kind of a **complex network** that can extend several periods into the future.



Backward Induction

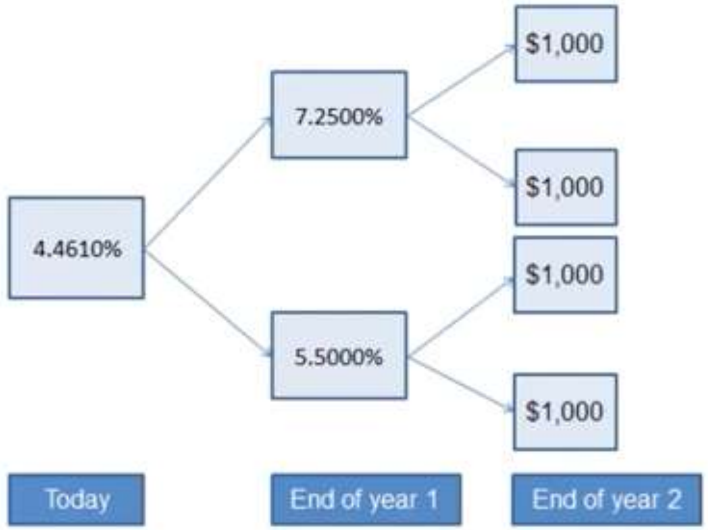
- Bonds are redeemed at par. Therefore, we start at maturity, fill in those values, and **work back from right to left** to find the bond's value at the desired node.
- For a zero-coupon bond, the only cash flow occurs at **maturity**.
    - To find the value of a bond at a given node in a binomial tree we find the **average** of the present values of the two possible values from the next period.
- The appropriate discount rate is the **forward rate** associated with the node under analysis.

One of the most important rules underlying the construction of binomial trees is that there should be **no arbitrage**.

Valuing an Option Free Bond

- Suppose we have a binomial tree of a **$1,000 face value zero-coupon bond** with **two years** remaining to maturity.
  - The blocks on the far right give the bond's par value.
  - Regular time value buttons: FV = 1,000; N = 2; I = 4.461; PMT = 0; Solve for PV = 916.41 (computed based on simple assumptions!).
- Assuming that the **bond's market price is $900**, and that up and down moves have **equal probabilities**, we can demonstrate that the following tree is arbitrage free using the concept of backward induction.
- **So how do we go about it?**

Valuing an Option Free Bond



- Consider the value of the bond at the upper node for period 1, $\mathbf{V_{1,U}}$:
  - $V_{1,U} = \frac{\$1{,}000 \times 0.5 + \$1{,}000 \times 0.5}{1.0725} = \$932.40$
- Similarly, the value of the bond at the lower node for period 1, $\mathbf{V_{1,L}}$:
  - $V_{1,L} = \frac{\$1{,}000 \times 0.5 + \$1{,}000 \times 0.5}{1.055} = \$947.87$
- From this point, we calculate the current value of the bond at node 0, V_0:
  - $V_0 = \frac{\$932.40 \times 0.5 + \$947.87 \times 0.5}{1.04461} = \$900$

Terminologies

- **Short term rate models** are the models used to describe the evolution of short rates.
- **Short rates are spot rates**
  - The short rate (r_t) is the **continuously compounded, annualized** interest rate at which an entity can **borrow money** for an infinitesimally short period of time.

- If we measure the **movement of the interest rates**, we will conclude that it has a **mean**/average.
  - **Drift** is the rate at which the average **changes.**

Model 1 – Normally Distributed without Drift

- Model 1 is used in cases where there is **no drift** and interest rates are **normally distributed**.
- Under this model, the continuously compounded, instantaneous rate r_t is assumed to evolve according to the following equation:

$$dr = \sigma dw$$

- where:
    - *dr* = **change in interest rates** over small time interval, *dt*
    - *dt* = **small time interval** (measured in years) (e.g., one month = 1/12, 2 months = 2/12, and so forth)
    - σ = **annual basis-point volatility** of rate changes
    - *dw* = **normally distributed random variable** with mean 0 and standard deviation $\sqrt{dt}$

Model 1 – Normally Distributed without Drift

Example: Estimating the Change in Short-Term Rate

- Suppose that:
  - The current value of the **short-term rate is 5.26%,**
  - **Volatility** equals **115 basis points** per year, and that
  - The **time interval** under consideration is **one month**.
- Mathematically, $r_0 = 5.26\%$; $\sigma = 1.15\%$; and $dt = 1/12$.
- A month passes and the **random variable *dw***, with its zero mean and its standard deviation of $\sqrt{\frac{1}{12}}$ (or 0.2887), happens to take on a **value of 0.25**.
- Determine the **short-term rate after one month**.

Solution

- The change in the short-term rate is given by:
  - $dr = \sigma dw$
  - $= 1.15\% \times 0.25 = 0.2875\%$
- New short-term rate = 5.26% + 0.2875% = 5.55%
  - Since the short-term rate **started at 5.26%,** the short-term rate **after a month is 5.55%.**

Model 1 – Normally Distributed without Drift

Standard Deviation

- Since the **expected value of *dw* is zero**, it follows that the **expected change** in the rate, or the drift, **is zero**.
- Since the standard deviation of the normally distributed variable, *dw,* is $\sqrt{dt}$, the **standard deviation of the change in the rate** is:

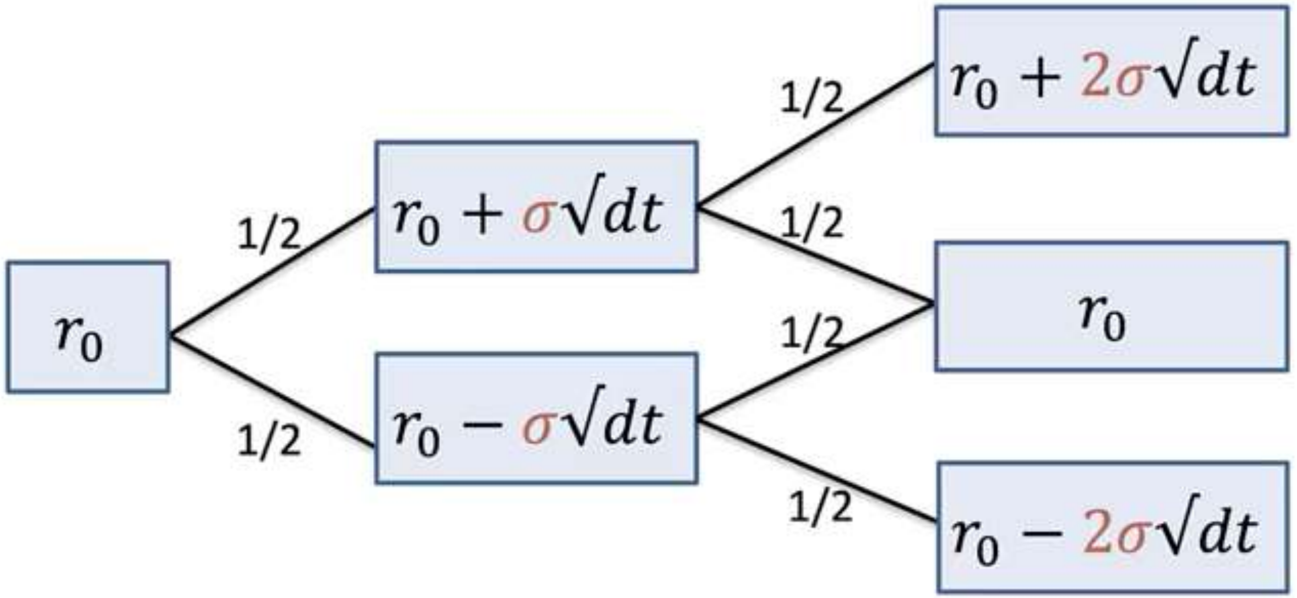
$$\sigma_{change\ in\ rate} = \sigma\sqrt{dt}$$

- For convenience the standard deviation of the rate of change is sometimes referred to as the **standard deviation of the rate**.

- In the previous example, the standard deviation of the rate is 1.15% × 0.2887 = 0.332% (or 33.2 basis points).

Interest Rate Tree without Drift

- It is possible to build a zero drift interest rate tree using a binomial model.



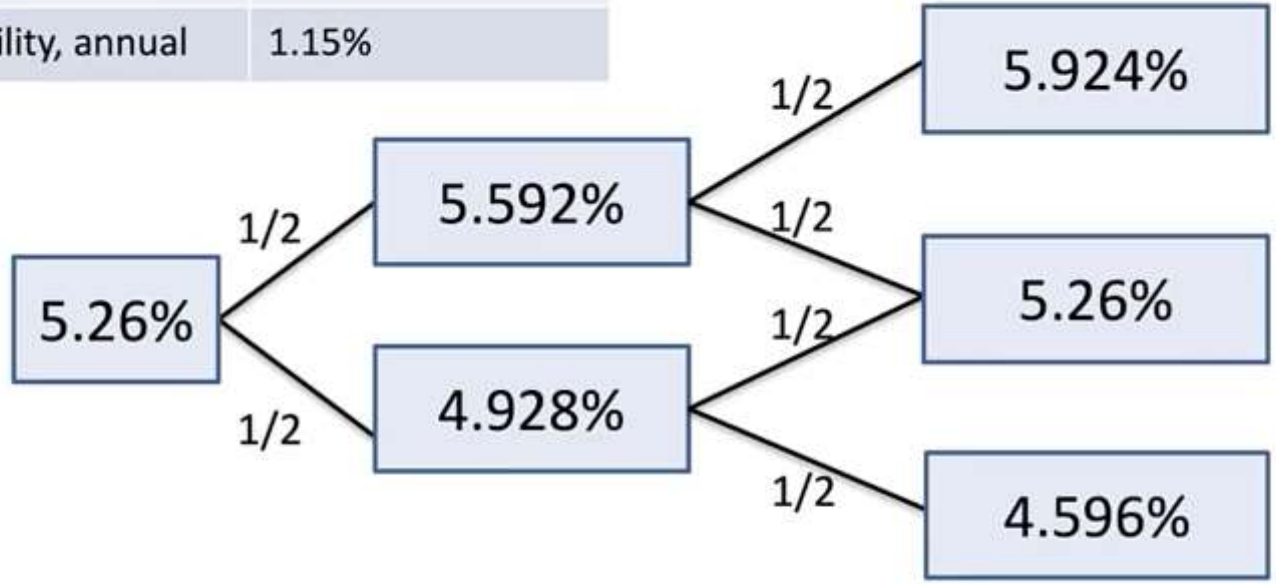
- Since drift is zero, rate **recombines** to current rate, r_0, at node [2,2].
- We can demonstrate how the expected change in the rate **(drift) is zero** as follows:

$$\text{Expected change in the rate} = E(dr) = 0.5\times\sigma\sqrt{dt} + 0.5\times-\sigma\sqrt{dt} = 0$$

Interest Rate Tree without Drift

Time	1/12
Initial short rate	5.26%
Drift, annual	N/A (zero)
Volatility, annual	1.15%

$$r_{t+1} = r_t + \sigma\sqrt{\frac{1}{12}}$$



- Top node in period 1 = 5.26% + 1.15% × 0.2887 = 5.592%
- Lower node in period 2 = 5.26% - (2 × 1.15% × 0.2887) = 4.596%

Model 2 - with Constant Drift

- Model 2 contains a **constant drift** and it is an extension to the Model 1.
  - The drift term is essentially a positive **risk premium associated with longer time horizons**.

Model 1	$dr = \sigma dw$
Model 2	$dr = \lambda dt + \sigma dw$

- Where
  - λ = drift
  - dt = **small time interval** (measured in years) (e.g., one month = 1/12, 2 months = 2/12, and so forth)
  - σ = **annual basis-point volatility** of rate changes
  - dw = **normally distributed random variable** with mean 0 and standard deviation $\sqrt{dt}$

Model 2 - with Constant Drift

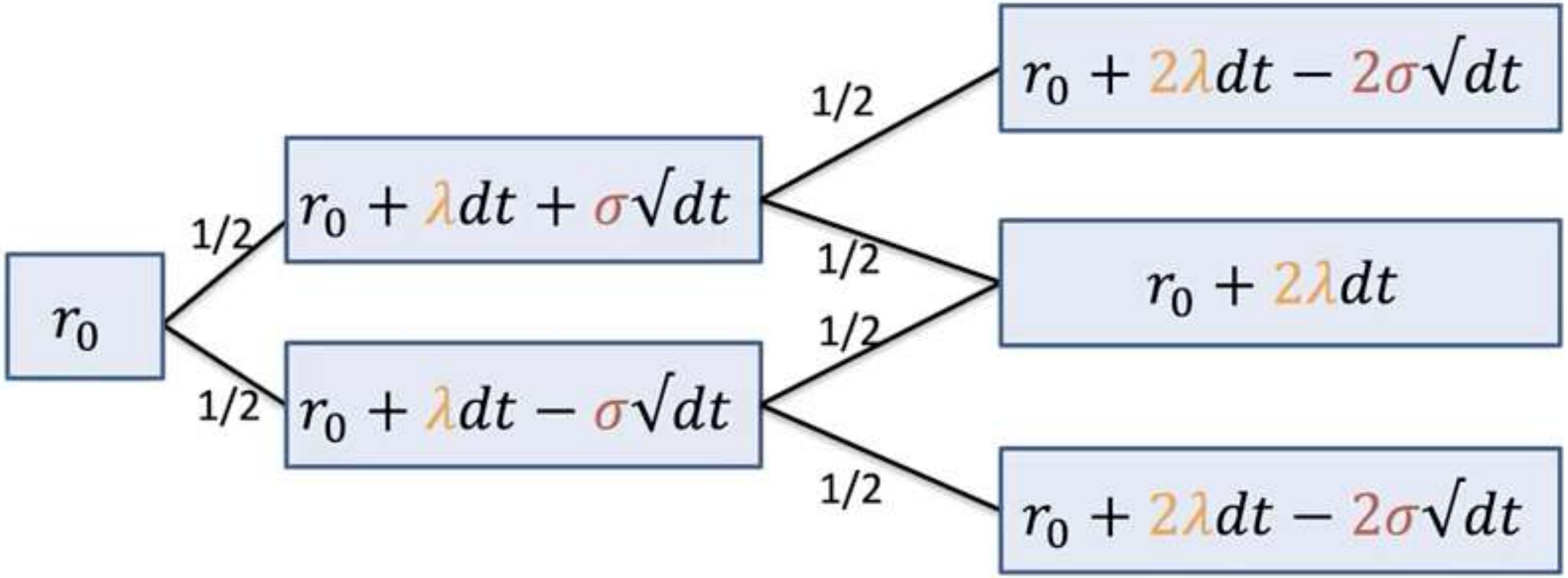
In our first example, we had:

- $dr = \sigma dw$
  - $= 1.15\% \times 0.25 = 0.2875\%$
- New short-term rate $= \ 5.26\% \ + \ 0.2875\% \ = \ \mathbf{5.55\%}$

If we now add an annual drift of 0.25%:

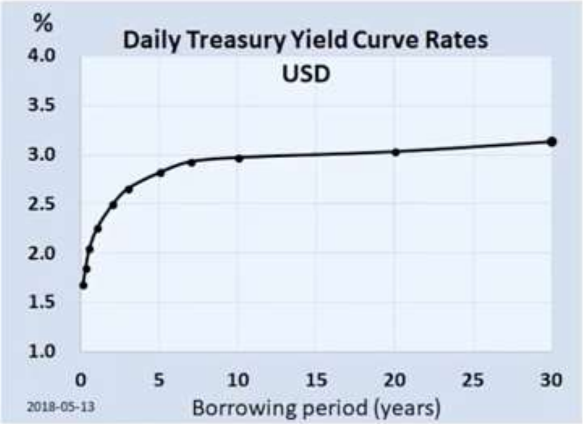
- The change in the short-term rate is given by:
  - $dr = \lambda dt + \sigma dw$
  - $= 0.25\% \times \frac{1}{12} + 1.15\% \times 0.25 = 0.3083\%$
- Since the short-term rate started at 5.26%, the short-term rate after a month is 5.5683%:
  - New short-term rate $= \ 5.26\% \ + \ 0.3083\% \ = \ 5.5683\%$
- The monthly drift is $0.25\% \times 1/12 \ = \ 0.0208\%$.
  - The 2.08 bps drift per month (0.0208%) represents any combination of expected changes in the short-term rate (i.e., **true drift**) and a **risk premium**.

Interest Rate Tree with Drift



Model 1 or Model 2?

- Model 2 is more effective than Model 1
  - Intuitively, the **drift term** accommodates the typically observed **upward-sloping nature** of the term structure.



- However, in the **long-term**, it is **difficult** to make a case for **rising expected rates**.

Ho – Lee Model

- Model 2 assumes that the drift (lambda) is **constant** from step to step along the tree.
  - The Ho-Lee Model assumes **that drift changes over time.**

Model 1	$dr = \sigma dw$
Model 2	$dr = \lambda dt + \sigma dw$
Ho-Lee Model	$dr = \lambda_t dt + \sigma dw$

- A drift that varies with time is called a ***time dependent*** **drift**.
  - For example, there might be an annualized drift of **10 basis points in month 1**, of **20 basis points in month 2**, and so on.

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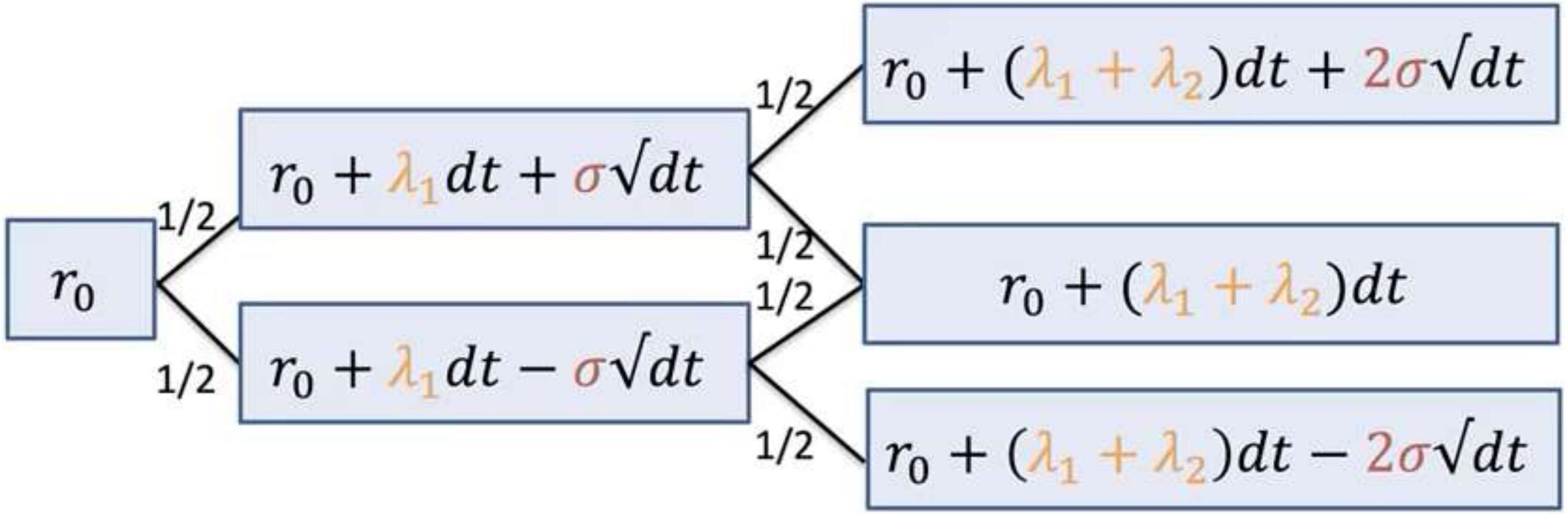
  - For example, there might be an annualized drift of **10 basis points in month 1**, of **20 basis points in month 2**, and so on.

Interest Tree under Ho-Lee Model



- Where λ_1 and λ_1 are estimated from observed **market prices.**
  - The drift can move in **any** direction — it can be negative or positive for a given time interval.

The Vasicek Model

- The Vasicek Model introduces **mean reversion** into the rate model.
    - When the **short-term rate is below** its long-run equilibrium value, the **drift is positive, driving the rate up** toward a long-run value.
    - And vice-versa.

Model 1	$dr = \sigma dw$
Model 2	$dr = \lambda dt + \sigma dw$
Ho-Lee Model	$dr = \lambda_t dt + \sigma dw$
Vasicek Model	$dr = k(\theta - r)dt + \sigma dw$

- Where:
    - k = a parameter that measures the speed of reversion adjustment
    - θ = long-run value of the short-term rate assuming risk neutrality
    - r = current interest rate level

The Vasicek Model

$$dr = k(\theta - r)dt + \sigma dw$$

- A **high *k*** will produce quicker (larger) **adjustments** than smaller values of *k*.
- Furthermore, the **greater** the **difference** between ***r* and *θ***, the **greater the expected change** in the short-term rate toward ***θ***.

- Under the assumption of risk-neutrality, the long-run value of the short-term rate can be approximated as:

$$\theta \approx r_l + \frac{\lambda}{k}$$

  - Where r_l is the long-run true rate of interest.

Interest Rate Tree under Vasicek Model

- Representing a Vasicek interest rate process with a tree is not quite straightforward because it leads to a **non-recombining tree**.
- Let's demonstrate the process assuming a starting rate of 6%.

Initial short rate	6.0%
dt (month)	0.0833
Drift, annual	0.4%
Drift, per month	0.0333%
Volatility, annual	1.3%
Theta, θ	13%
K	0.05

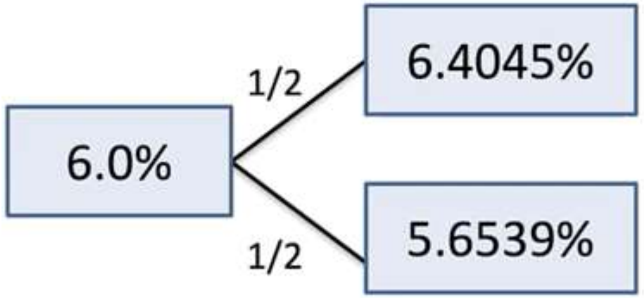
Interest Rate Tree under The Vasicek Model

Initial short rate	6.0%
dt (month)	0.0833
Drift, annual	0.4%
Drift, per month	0.0333%
Volatility, annual	1.3%
Theta, θ	13%
K	0.05

1. First Period Upper and Lower Node Calculations

$$dr = k(\theta - r)dt \pm \sigma dw$$

Calculation



$$6.0\% + 0.05(13\% - 6\%)\left(\frac{1}{12}\right) + \frac{1.3\%}{\sqrt{12}}$$

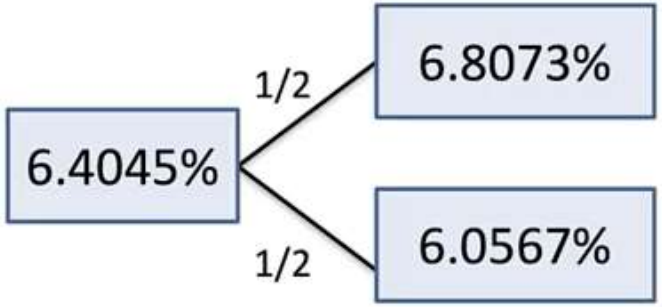
$$6.0\% + 0.05(13\% - 6\%)\left(\frac{1}{12}\right) - \frac{1.3\%}{\sqrt{12}}$$

Interest Rate Tree under Vasicek Model

Initial short rate	6.0%
dt (month)	0.0833
Drift, annual	0.4%
Drift, per month	0.0333%
Volatility, annual	1.3%
Theta, θ	13%
K	0.05

2. Second Period Upper Node Calculations

$$dr = k(\theta - r)dt \pm \sigma dw$$



Calculation

$$6.4045\% + 0.05(13\% - 6.4045\%)\left(\frac{1}{12}\right) + \frac{1.3\%}{\sqrt{12}}$$

$$6.4045\% + 0.05(13\% - 6.4045\%)\left(\frac{1}{12}\right) - \frac{1.3\%}{\sqrt{12}}$$

Interest Rate Tree under Vasicek Model

Initial short rate	6.0%
dt (month)	0.0833
Drift, annual	0.4%
Drift, per month	0.0333%
Volatility, annual	1.3%
Theta, θ	13%
K	0.05

3. Second Period Lower Node Calculations

$$dr = k(\theta - r)dt \pm \sigma dw$$

Calculation

5.6539%

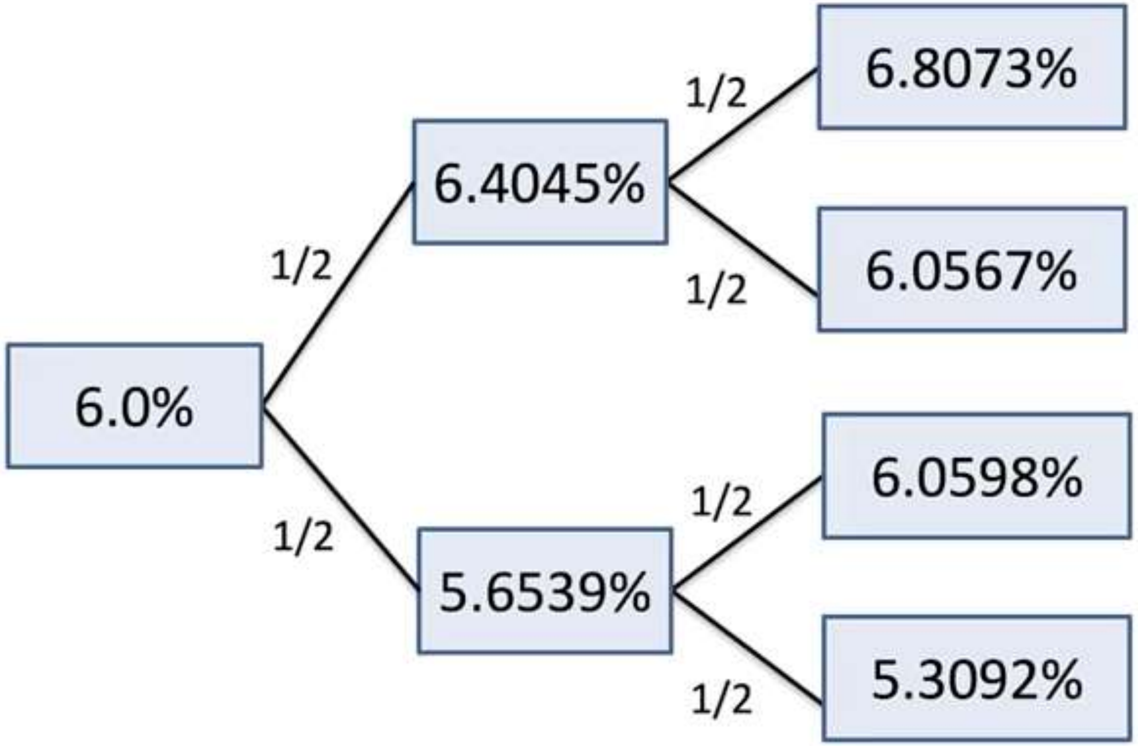
1/2 → 6.0598%

$$5.6539\% + 0.05(13\% - 5.6539\%)\left(\frac{1}{12}\right) + \frac{1.3\%}{\sqrt{12}}$$

1/2 → 5.3092%

$$5.6539\% + 0.05(13\% - 5.6539\%)\left(\frac{1}{12}\right) - \frac{1.3\%}{\sqrt{12}}$$

Interest Rate Tree under Vasicek Model

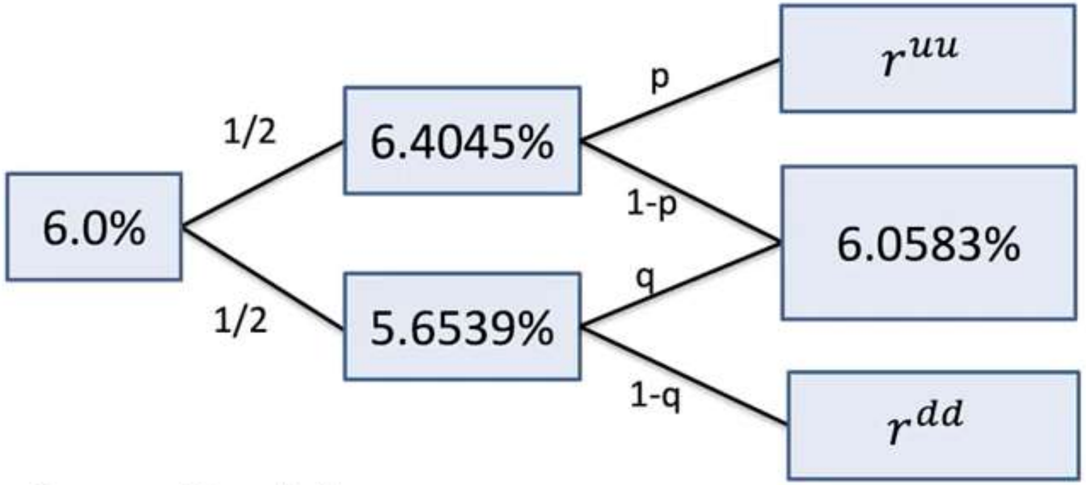


Interest Rate Tree under Vasicek Model

Step 1: Find the **average** of the two middle **nodes**.

$$Average = \frac{6.0567\% + 6.0598\%}{2} = 6.0583\%$$

Step 2: Do away with the 50% probability of up-down movements and replace them with (p, 1 - p) and (q, 1 - q):



Step 3: Solve for p, q, r^{uu}, and r^{dd}.

A Few Facts on Vasicek Model

Expected Rate in T Years

- The expectation of the rate in the Vasicek model after *T* years is given by:

$$r_0 e^{-kT} + \theta\left(1 - e^{-kT}\right)$$

Half-Life

- The mean-reverting parameter *k* does not intuitively describe the pace of mean-reversion.
  - A more intuitive quantity is the factor's *half-life*, defined as the time it takes the factor to progress half the distance toward its goal.

$$\text{Half-life} = \tau_{years} = \frac{ln(2)}{k}$$

Effectiveness of Vasicek Model

- The mean reversion parameter under the Vasicek model (1) improves the **specification of the term structure** and (2) produces a **specific term structure of volatility**.

- The Vasicek model will produce a term structure of volatility that is **declining**.
  - Particularly when we consider r_0 and θ calibrated to **match observed market prices**.

- As a result, the Vasicek model produces a term structure of volatility that is **declining**, implying that it **overstates short-term volatility** but **understates long-term volatility.**

- In contrast, Model 1 which has zero drift generates a **flat** volatility of interest rates across all maturities

Model with Time Dependent Volatility

- **Time-dependent drift** can be used to fit many **bond** or **swap rates**.
  - In the same way, a **time-dependent volatility function** can be used to fit many **option prices**.
- A simple model with a **time-dependent volatility function** might be written as follows:

$$\boxed{dr = \lambda(t)dt + \sigma(t)dw}$$

$$\boxed{dr = \lambda(t)dt + \sigma(t)dw}$$

- A closer look at the function above reveals that this model **augments** Model 1 and the Ho-Lee model.
  - The functional form of Model 1 (with zero drift), $dr = \sigma dw$, now includes **time-dependen**t drift and **time-dependent** volatility.
  - The Ho-Lee model, $dr = \lambda(t)dt + \sigma dw$, now includes **nonconstant volatility**.

Extension of Model 3

- A special case of time-dependent volatility (which we call **Model 3**) can be represented as follows:

$$dr = \lambda(t)dt + \sigma e^{-\alpha t}dw$$

- The volatility of the short rate **starts at the constant σ** and then **exponentially** declines to zero.

Extension of Model 3

Example

Current short term rate	3.00%
Drift, λ	0.24%
Annual volatility (with initial $\sigma = 1.3\%$)	$\sigma e^{-0.3t}$
dw	0.2

Question: Determine the change in the short-term rate after one month.

Solution

$$dr = \lambda(t)dt + \sigma e^{-\alpha t}dw$$
$$= 0.24\% \times \frac{1}{12} + 1.3\% \times e^{-0.3\left(\frac{1}{12}\right)} \times 0.2$$
$$= 0.27\%$$

- As noted, the volatility of the short rate starts at sigma, but over time, declines exponentially toward zero. We can illustrate this using data from the above example,
  - At t = 0, $e^{-0.3\times 0} = 1.0$; Volatility term $= \sigma e^{-\alpha t}dw = 1.3\% \times 1.0 \times dw$
  - At t = 5, $e^{-0.3\times 5} = 0.223$; Volatility term $= 1.3\% \times 0.223 \times dw$
  - At t = 10, $e^{-0.3\times 10} = 0.0498$; Volatility term $= 1.3\% \times 0.0498 \times dw$

Effectiveness of Time Dependent Volatility Model

- Time-dependent volatility models provide a useful tool for **pricing fixed income options** in situations where a precise **market price is not easily observable**.
  - The models provide a means of interpolating from known to unknown option prices.

- However, if the purpose of the model is to **value and hedge fixed income securities**, including options, then a **model with mean reversion** might be **preferred for two reasons**:
  1. While **mean reversion** is based on **economic intuitions**, time-dependent volatility relies on the difficult argument that the market has a **forecast of short-term volatility in the distant future**.
  2. The downward-sloping factor structure and term structure of volatility in mean reverting models capture the behavior of interest rate movements **better than parallel shifts** and a flat term structure of volatility.

Effectiveness of Time Dependent Volatility Model

- Time-dependent volatility models are also useful for pricing **multi-period derivatives** like caplets and floorlets.
  - In a **caplet**, the buyer receives payments at the end of each period if the interest rate **exceeds the agreed strike price**.
  - In a **floorlet**, the buyer receives payments at the end of each period if the interest rate falls **below the agreed strike price**.

- Time-dependent volatility models are, however, **criticized** because they forecast volatility **far out into the future**, which calls their long- to medium-term reliability into question.

The Cox-Ingersoll-Ross (CIR) Model

- In periods with high inflation, short-term interest rates are **usually high** and **inherently unstable** and, as a result, the basis-point volatility of the short rate tends to be **high**.
- When the short-term rate is **very low**, basis-point volatility is limited by the constraint that interest rates **cannot** decline much below **zero**.

- In essence, the **CIR model exhibits mean reversion** just as with the Vasicek model.

$$\boxed{dr = k(\theta - r)dt + \sigma\sqrt{r}dw}$$

- However, the CIR model multiplies volatility by the square root of the level of the interest rate.
  - Unlike the Vasicek model, the CIR model **does not allow for negative interest** rates because of the square root component.

The Cox-Ingersoll-Ross (CIR) Model

Example

Step, dt = 1/12	0.0833
Initial rate	6.00%
Volatility per annum, σ	1.3%
Long-run rate, θ	20%
Mean reversion adjustment, k	0.05

Question: Determine the change in the short-term rate **after one month**.

Solution

- $\mathrm{dr} = \mathrm{k}(\theta - \mathrm{r})\mathrm{dt} + \sigma\sqrt{\mathrm{r}}\mathrm{dw}$
    - $= 0.05(20\% - 6\%)\left(\frac{1}{12}\right) + 1.3\%\sqrt{6}\%\times 0.2$
    - $= 0.0583\% + 0.06369\% = 0.122\%$
- Therefore, the expected short-term rate after one month is **6.122%** (6% plus 0.122%).