Problems 2:
Binomial pricing

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Question 1

Let S be the price of stock at t, and suppose that at $t + \Delta t$ the stock can only change into two values: up $Su > S$ or down $Sd < S$ (here u and d represent the relative increase or decrease of the stock price). What are the probabilities of the up and down moves of the stock price, if the market is assumed to have no arbitrage opportunities?

Answer: *No arbitrage implies a specific pricing of an asset in the future; the expected future value should be equal to its current value taking into account a riskless investment of the equivallent ammount:*

$$\mathbb{E}\{S(t + \Delta t)\} = e^{r\Delta t} S(t)$$

If $S(t + \Delta t)$ can only take two values Su and Sd, then the expected value is:

$$\mathbb{E}\{S(t + \Delta t)\} = Su\lambda + Sd(1 - \lambda)$$

where λ and $1 - \lambda$ are the probabilities of Su and Sd respectively. This is the binomial model with one step $N = 1$. Therefore, combining the two equations above and excluding $S(t)$ gives

$$e^{r\Delta t} = u\lambda + d(1 - \lambda)$$

here r is riskless so thi is riskless prob

which gives the risk-neutral probability $\lambda = \frac{e^{r\Delta t} - d}{u - d}$.

Question 2

Let $S = £100$ be the stock price, which after time T can only change up $Su = £120$ or down $Sd = £80$. Compute the risk-neutral probabilities of up move Su and down move, assuming that the risk-free interest rate $r = 0$. How will these probabilities change, if $r > 0$?

Answer: *The relative changes of the stock price are* $u = Su/S = £120/£100 = 1.2$ *and* $d = Sd/S = £80/£100 = .8$*. Using the formula* $\lambda = \frac{e^{r\Delta t} - d}{u-d}$ *for risk-neutral probability and* $r = 0$, $\Delta t = 1$ *gives:*

$$\lambda = \frac{1 - 8/10}{12/10 - 8/10} = \frac{2/10}{4/10} = \frac{1}{2}$$

If $r > 0$*, then* $e^r > 1$*, and the probability of* Su *increases* $\lambda > 1/2$.

Question 3

Assuming one step binomial model, consider stock with spot price $S = \$80$, price after up move $Su = \$100$ and after down move $Sd = \$60$. Assuming riskless rate of return $r = 0$, price European put option with the strike price $K = \$90$ and expiring in $T = 1$ year. Describe how the value changes as a function of the strike price. What changes if $r > 0$? What if the stock pays dividents with $s > 0$ APR?

Answer: *Using* $u = \$100/\$80 = 1.25$ *and* $d = \$60/\$80 = .75$*, compute risk-neutral probability:*

$$\lambda = \frac{1 - 6/8}{10/8 - 6/8} = \frac{2/8}{4/8} = \frac{1}{2}$$

If S *goes up to* $Su = \$100$*, then the final payoff of the put option is* $V_u = \max[0, K - Su] = \max[0, 90 - 100] = 0$*; if* S *goes down to* $Su = \$60$*, then* $V_d = \max[0, K - Sd] = \max[0, 90 - 60] = 30$*. Thus, the value of the option is:*

$$V(t) = e^{-r}[V_u\lambda + V_d(1-\lambda)] = \$0\frac{1}{2} + \$30\frac{1}{2} = \$15$$

The value of the put changes monotonically with K *(i.e. increases if* K *increases and vice versa). The value is positive for* $K > Sd = \$60$*. If* $r > 0$*, then the value is discounted by the factor* $e^{-r} < 1$ *(i.e. descreses). If the stock pays dividents at* $s > 0$*, then the effective rate is the difference* $r - s$*, and so the value is discounted by* $e^{-(r-s)}$*. In particular, if* $s > r$*, then the value of the put increases.*

Question 4

Assuming one step binomial model and risk-free rate $r = 0$, price European call option on stock described in the previous question with the strike price $K = \$90$ and expiring in $T = 1/2$ years.

Answer: *The values $u = Su/S$, $d = Sd/S$ and the risk-neutral probability λ are the same as in the previous question. If S goes up to $Su = \$100$, then the final payoff of the call is $V_u = \max[0, Su - K] = \max[\$0, \$100 - \$90] = \$10$; if S goes down to $Su = \$60$, then $V_d = \max[0, Sd - K] = \max[\$0, \$60 - \$90] = \$0$. Thus, the value of the option is:*

$$V(t) = e^{-r}[V_u\lambda + V_d(1-\lambda)] = \$10\frac{1}{2} + \$0\frac{1}{2} = \$5$$

Question 5

Consider a replicating portfolio $V = SX + B$ for a stock assumed to follow a one step binomial model (i.e. at the expiration date T the stock price S can only move up to Su or down to Sd). Assuming no arbitrage, derive the value X of the stock quantity and B of a riskless investment. Derive the value V of an option in the replicating portfolio.

Answer: *Because S can only have two values at the expriration date, replication $V = SX + B$ implies that the derivative can also have only two values: $V_u = SuX + Be^{r(T-t)}$ or $V_d = SdX + Be^{r(T-t)}$. Subtracting one value from another gives the equation for X:*

$$V_u - V_d = SuX - SdX = S(u-d)X \quad \Longleftrightarrow \quad X = \frac{V_u - V_d}{S(u-d)}$$

Substituting X into $B = (V_u - SuX)e^{-r(T-t)}$ gives:

$$B = \frac{V_d u - V_u d}{(u-d)e^{r(T-t)}}$$

The value V of the option equals the value $SX + B$ of the replicating portfolio, and after substituting X and B into $V = SX + B$ we derive the equation:

$$V(t) = e^{-r(T-t)}[V_u\lambda + V_d(1-\lambda)]$$

where $\lambda = \frac{e^{r(T-t)} - d}{u-d}$. Observe that the equation above is the discounted expected value $\mathbb{E}\{V(T)\}$ of the final payoff, assuming risk-neutral probability.

Question 6

The one step Binomial pricing of an option is $V(t) = e^{-r(T-t)}[V_u\lambda + V_d(1-\lambda)]$. Derive the correspondong pricing formula, if the stock price can change twice during the period $T - t$ (i.e. the $N = 2$ step binomial model).

Answer: *After $(T-t)/2$ years the option's price changes from V into V_u or V_d. Then, after the second period $(T-t)/2$, the price changes either from V_u into V_{uu}, V_{ud} or from V_d into V_{du}, V_{dd}. Because $V_{ud} = V_{du}$, there are only three final values after two steps: V_{uu}, $V_{ud} = V_{du}$ and V_{dd}. Then we have*

$$
\begin{aligned}
V(t) &= e^{-r(T-t)/2}[V_u\lambda + V_d(1-\lambda)] \\
&= e^{-r(T-t)/2}\Big[\underbrace{\Big(e^{-r(T-t)/2}\Big[V_{uu}\lambda + V_{ud}(1-\lambda)]\Big)}_{V_u}\lambda \\
&\quad + \underbrace{\Big(e^{-r(T-t)/2}[V_{du}\lambda + V_{dd}(1-\lambda)]\Big)}_{V_d}(1-\lambda)\Big] \\
&= e^{-r(T-t)}\Big[V_{uu}\lambda^2 + 2V_{ud}\lambda(1-\lambda) + V_{dd}(1-\lambda)^2\Big]
\end{aligned}
$$

Note that because there are $N = 2$ steps, the risk-neutral probability λ is now computed for the period of $(T-t)/2$ years:

$$\lambda = \frac{e^{r\frac{(T-t)}{2}} - d}{u-d}$$

Question 7

Assume a $N = 2$ step binomial model for stock with spot price $S = £400$, one-step up price $Su = £550$ and down price $Sd = £350$. Price European call option with the strike price $K = £400$ expiring in $T = 1$ year, and assuming risk-free rate $r = 0$. What is the price of the corresponding put (i.e. with the same strie price K)?

Answer: *Compute u, d and λ:*

$$u = \frac{Su}{S} = \frac{55}{40}, \quad d = \frac{Sd}{S} = \frac{35}{40}, \quad \lambda = \frac{e^{r\frac{T-t}{2}} - d}{u-d} = \frac{1}{4}$$

For two-steps we have:

$$
\begin{aligned}
Suu &= £400 \cdot (55/40)^2 = £756.25 & V_{uu} &= \max[0, Suu - K] = £356.25 \\
Sud &= £400 \cdot (55/40)(35/40) = £481.25 & V_{ud} &= \max[0, Sud - K] = £81.25 \\
Sdd &= £400 \cdot (35/40)^2 = £306.25 & V_{dd} &= \max[0, Sdd - K] = £0
\end{aligned}
$$

The expected payoff is:

$$\mathbb{E}\{V(T)\} = V_{uu}\lambda^2 + 2V_{ud}\lambda(1-\lambda) + V_{dd}(1-\lambda)^2 = £52.73$$

The call price is $C(t) = e^{-r(T-t)}\mathbb{E}\{V(T)\}$, *and for* $r = 0$ *it is* £52.73. *The price of put can be computed using put-call parity:*

$$P(t) = e^{-r(T-t)}K - S(t) + C(t) = \pounds 400 - \pounds 400 + \pounds 52.73 = \pounds 52.73$$

Question 8

Assume a $N = 2$ step binomial model for stock with spot price $S = \pounds 5$, one-step up price $Su = \pounds 8$ and down price $Sd = \pounds 4$. Price European put option with the strike price $K = \pounds 6$ expiring in $T = 1$ year, and assuming risk-free rate $r = 0$. What is the price of the corresponding call (i.e. with the same strie price K)?

Answer: *Compute* u, d *and* λ*:*

$$u = \frac{Su}{S} = \frac{8}{5}\,, \quad d = \frac{Sd}{S} = \frac{4}{5}\,, \quad \lambda = \frac{e^{r\frac{T-t}{2}} - d}{u - d} = \frac{1}{4}$$

For two-steps we have:

$$\begin{aligned} Suu &= \pounds 5 \cdot (8/5)^2 = \pounds 12.80 & V_{uu} &= \max[0, K - Suu] = \pounds 0 \\ Sud &= \pounds 5 \cdot (8/5)(4/5) = \pounds 6.40 & V_{ud} &= \max[0, K - Sud] = \pounds 0 \\ Sdd &= \pounds 5 \cdot (4/5)^2 = \pounds 3.20 & V_{dd} &= \max[0, K - Sdd] = \pounds 2.80 \end{aligned}$$

The expected payoff is:

$$\mathbb{E}\{V(T)\} = V_{uu}\lambda^2 + 2V_{ud}\lambda(1-\lambda) + V_{dd}(1-\lambda)^2 = \pounds 1.575$$

The put price is $P(t) = e^{-r(T-t)}\mathbb{E}\{V(T)\}$, *and for* $r = 0$ *it is* £1.575. *The price of call can be computed using put-call parity:*

$$C(t) = S(t) - e^{-r(T-t)}K + P(t) = \pounds 5 - \pounds 6 + \pounds 1.575 = \pounds 0.575$$

Question 9

BTC traded at $S = \pounds 2,000$, and put options on BTC with strike price $K = \pounds 1,500$ expiring in $T = 1$ year are traded at £100 per put. Suppose that after $T = 1$ the price drops to $S(T) = \pounds 1,000$. Compute and compare profits and returns on three investments: 1) one BTC; 2) equivalent amount invested in puts; 3) the amount split equally between BTC and puts. What are the profits and returns, if the price increases to $S(T) = \pounds 4,000$? What are the expected profits and expected returns on the three investments? How can these expected values be estimated if $S(T) \in \{\pounds 1,000, \pounds 4,000\}$?

Answer: *For $S(T) = £1,000$, the profits are:*

$$\begin{aligned} S(T) - S(t) &= £1,000 - £2,000 = -£1,000 \\ 20[K - S(T) - P(t)] &= 20[£1,500 - £1,000 - £100] = £8,000 \\ 10[K - S(T) - P(t)] + \frac{1}{2}[S(T) - S(t)] &= 10[£1,500 - £1,000 - £100] - £500 = £3,500 \end{aligned}$$

The annual rates of return are -50%, 400% and 175%.

For $S(T) = £4,000$, we have:

$$\begin{aligned} S(T) - S(t) &= £4,000 - £2,000 = £2,000 \\ 20[0 - P(t)] &= 20[-£100] = -£2,000 \\ 10[0 - P(t)] + \frac{1}{2}[S(T) - S(t)] &= 10[-£100] + 0.5£2,000 = £0 \end{aligned}$$

The annual rates of return are 100%, -100% and 0%.

For the investment into stock the expected value is:

$$\mathbb{E}_P\{S(T) - S(t)\} = \int_0^\infty S(T)\, dP(S(T)) - S(t)$$

Similarly, for options this is $\mathbb{E}_P\{V(T) - V(t)\}$, where $V(T)$ is terminal payoff, and $V(t)$ is the option's price. For a mixture of stock and options this is $\mathbb{E}_P\{0.5[S(T) - S(t)] + 0.5[V(T) - V(t)\}$.

The binomial assumption for $S(T)$ means that there are only two probabilities $P(S(T)) \in \{\lambda, 1 - \lambda\}$. In the simplest case of $\lambda = 1/2$ the expected returns are given by the average values computed in the previous question:

$$\begin{aligned} \mathbb{E}\{S(T) - S(t)\} &= \frac{1}{2}[-£1,000 + £2,000] = £500 \\ \mathbb{E}\{20[\max[0, K - S(T)] - P(t)]\} &= \frac{1}{2}[£8,000 - £2,000] = £3,000 \\ \mathbb{E}\{10[\max[0, K - S(T)] - P(t)] + \frac{1}{2}[S(T) - S(t)]\} &= \frac{1}{2}[£3,500 + £0] = £1,750 \end{aligned}$$