Practitioner's introduction to measure theory

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Abstract

Here we provide a quick non-technical introduction to measure theory and give intuitive descriptions of concepts that will be useful in learning financial calculus.

Contents

1 Motivation

• Suppose X takes values uniformly randomly on the interval [0,1]. Then how to compute the following probabilities.

- $P(X = 0.5) = ?$
- $P(0.4 < X < 0.5) = ?$

- $P(X \in \mathbb{Q}) =$? Here $\mathbb{Q}$ is the set of rational numbers.

- Applications in mathematical finance.

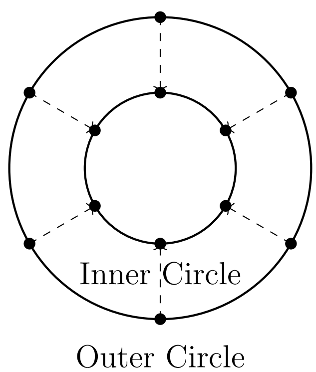
  - Equivalent martingale measure / risk-neutral measure.
  - T-Forward measure.
  - Change of measure via Radon-Nikodym derivative, Change of numeraire.
  - Girsanov's theorem.

2 What is Measure theory?

• As the name suggests, the theory is about measurements: **length**, **area**, **volume**. We want the defined measurement to agree with our intuitive understanding of the world:

- **additivity:** $length(A \cup B) = length(A) + length(B)$ for disjoint sets A and B.
- $length(A \setminus B) = length(A) - length(B)$ for sets $A \subset B$.

• **Countable vs Uncountable additivity:** Let us look at the two concentric circled drawn below. As we can see, it is possible to have a one-to-one correspondence between the points in the inner circle to that of the outer circle (*bijection*). A natural question would be, why are the lengths of the two circles not equal. See also the video at this link: Aristotle's Wheel Paradox.



One point to note is that, the number of points on the circle is not *countable*[1], it is *uncountable*[2]. This observations suggests that we cannot expect the measure of an uncountable union of pair-wise disjoint sets to be equal to the sum of the measure of the sets:

$$length\left(\cup_{r\in\mathbb{R}} X_r\right) \neq \sum_{r\in\mathbb{R}} length(X_r) \quad \text{for pair-wise disjoint sets } X_r \text{ in general.} \tag{1}$$

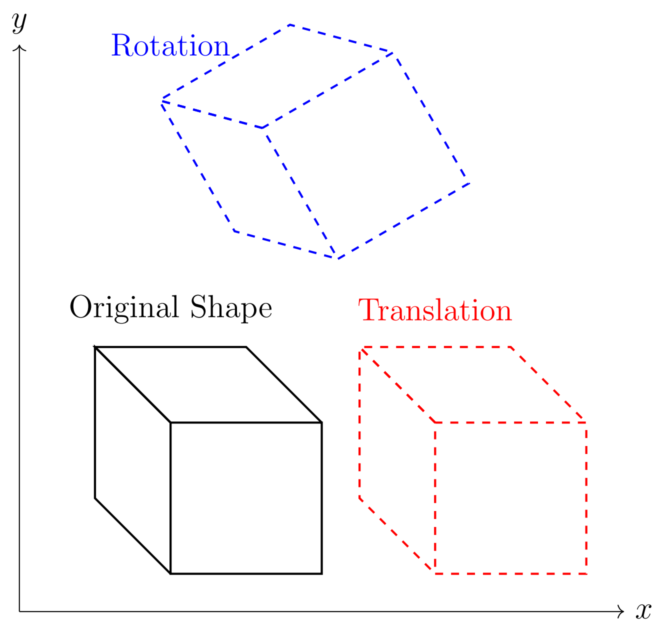
[1] *Countable infinity:* A set is countably infinite if we can match the elements 1-for-1 with the natural numbers.

[2] *Uncountable infinity:* We cannot match the elements 1-for-1 with 1, 2, 3, ... We will always have some left over no matter what scheme we use. Look up Cantor's diagonal argument.

Thus, the additive property of measures should hold only up to countably infinite elements:

$$\textbf{countable additivity:} \quad length\left(\cup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} length(X_i) \quad \text{for pair-wise disjoint sets } X_i. \tag{2}$$

• **Naive theory of measurement, define measure on all subsets:** First let us set up a naive theory of measurement on the three-dimensional closed unit cube $\Omega = [0,1] \times [0,1] \times [0,1]$. Closed means the boundary points such as (0,0,0) are included. We would 'naively' expect that, we can define volume for any subset of Ω. Let the function λ denote this volume measure: for any given subset $X \subset \Omega$, $\lambda(X)$ is the volume of X. As mentioned above, we expect it to retain the property $\lambda(X \cup Y) = \lambda(X) + \lambda(Y)$ for disjoint sets $X, Y \subset \Omega$. Generalizing this by induction, for pairwise disjoint sets $X_i \subset \Omega$, $\lambda\left(\cup_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} \lambda(X_i)$. We would expect this notion of volume to be translation and rotation invariant: The translated-red and (translated+rotated)-blue cubes below should have the same volume as the original.



• **Inconsistency in this naive theory, Banach–Tarski paradox:** A strong version of the paradox roughly states the following: Given any two three-dimensional bounded subsets A and B with non-empty interior, we can cut both of them into *finitely many* equal number of *congruent* pieces, $A = \cup_{i=1}^{k} A_i$ and $B = \cup_{i=1}^{k} B_i$ with $A_i \cap A_j = B_i \cap B_j = \emptyset$ for all $i \neq j$ such that $A_i \equiv B_i$. Here $A_i \equiv B_i$ means congruence in the geometrical sense. Thus, going by our previous notion of our previous volume measure,

$$\lambda(A) = \lambda\left(\cup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \lambda(A_i) = \sum_{i=1}^{k} \lambda(B_i) = \lambda\left(\cup_{i=1}^{k} B_i\right) = \lambda(B). \tag{3}$$

• In other words, if we try to define volume on all the subsets, while requiring it to respect certain intuitive properties (additivity, rotation & translation invariance), we end up proving that the volume of all subsets are equal, say ν. Appealing again to additivity, $\lambda(X \cup Y)\ \lambda(X) + \lambda(Y) \Rightarrow \nu = 2\nu \Rightarrow \nu = 0$ or ∞. Thus, it turns out to be a trivial measure: volume of all subsets is equal to zero or infinity.

• **Measure space:** As we see above, it is not possible to define measure (say, volume) on all subsets of a set Ω that we are interested in. Thus, we need to introduce Σ which is the set off all 'measurable' subsets of Ω. Moreover, if we can define measure of disjoint sets X and Y, we should also be able to define measure of $X \cup Y$. Similarly, if both U and V are measurable (we can define their measure) with $U \subset V$, then we can also define measure of $V \setminus \tilde{U}$. In other words,

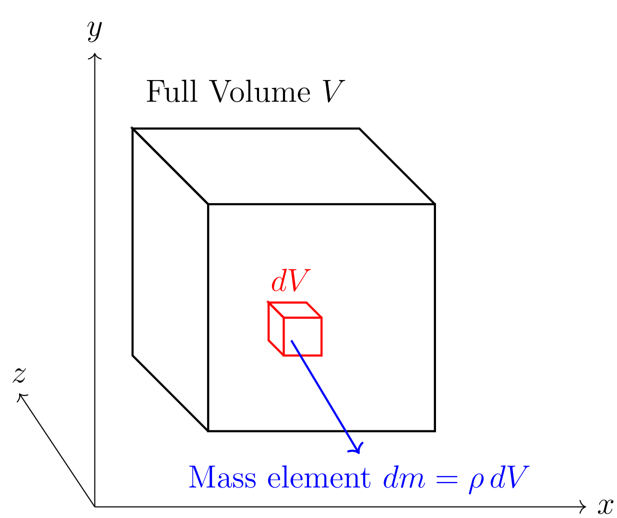
$$\begin{aligned} X, Y \in \Sigma \ \text{ with } \ X \cap Y = \emptyset \ &\Rightarrow \ X \cup Y \in \Sigma, \\ U, V \in \Sigma \ \text{ with } \ U \subset V \ &\Rightarrow \ V \setminus U \in \Sigma \end{aligned} \tag{4}$$

As Σ respects union/set-addition and intersection/set-subtraction, it is an algebra. It is called a σ-algebra. The pair (Ω, Σ) is called a *measurable space* as 'it can be measured'. Once we define the measure, say μ, the tuple (Ω, Σ, μ) is called a *measure space* as the measure has already been defined.

• **Practitioner's view:** As we see above, all notions from the *naive theory* are kept except for the fact that we can define measure, say volume, for all possible subsets. In practice, we do not need to worry about if a set is measurable or not. All sets that we will come across in industry settings will be measurable. We just need to respect certain intuitive properties such as *countable additivity* and we are good to go.

3 Change of measures and Radon-Nikodym derivative

• **Volume measure vs mass measure:** Given an existing volume measure $dV = dx \cdot dy \cdot dz$, we can always derive a new mass measure $dm = \rho(x, y, z)dV$.



It is intuitively clear that we can always find/there exists a density function $\rho = \frac{dm}{dV}$. However, $\frac{dV}{dm}$ may not exist. It boils down to the property that $dm > 0 \Rightarrow dV > 0$. But $dV > 0$ can still mean $dm = 0$ if that volume element is 'empty' and there is no mass. Formally, it may be stated as below.

• **Radon-Nikodym theorem:** The theorem guarantees the existence of a Radon-Nikodym density/derivative $\rho = \frac{d\mathbb{P}}{d\mathbb{Q}}$ when $\mathbb{P} \ll \mathbb{Q}$, i.e., if $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$. ρ is analogous to the density seen above, while $\mathbb{P}$ and $\mathbb{Q}$ are equivalent to mass and volume measures satisfying $d\mathbb{P} > 0 \Rightarrow d\mathbb{Q} > 0$.

• Radon-Nikodym theorem tells us when we can go from one measure to another. For example, we can always represent mass measure in terms of volume measure: $dm = \rho\, dV$, but not the other way. It finds application extensively in math finance, such as in *Girsanov's*.

4 Cumulative and Probability density functions

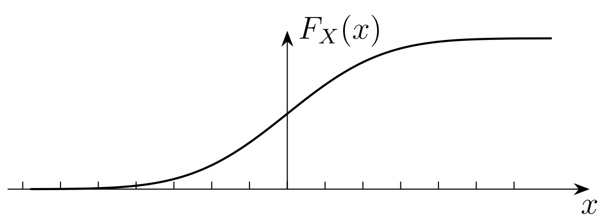
• The cumulative distribution function of a real-valued[3] random variable X is defined as

$$F_X(r) = P(X \leq r). \tag{5}$$

It is clear that $F_X(-\infty) = 0$ and $F_X(\infty) = 1$. Moreover, it has to be a monotonically increasing function of x. That is, if $x > y$, $F_X(x) \geq F_X(y)$. Moreover,

$$P(r_0 < X \leq r_1) = F_X(r_1) - F_X(r_2). \tag{6}$$

Do note the $<$ and $<=$, it becomes important when CDF becomes discontinuous. Below, we plot the CDF of a normal distribution with mean 0 and variance 1.



• The probability density function $f(x)$ is used to specify the probability of the random variable falling within a particular range of values, as opposed to taking on any one value. It becomes important in the continuous case where the random variable can take a continuous range of values, say on the interval [0,1] as opposed to taking discrete values, say $1, 2, 3, \cdots$.

• Roughly speaking, $P(r < X < r + dr) = f_X(r)\, dr$. More formally,

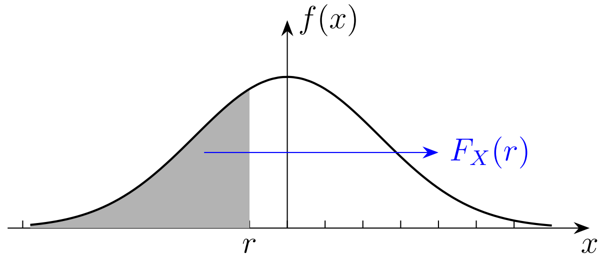
$$P(r_0 < X < r_1) = \int_{r_0}^{r_1} f_X(r)\, dr.$$

[3] X takes values between $-\infty$ and ∞. For example, X cannot be $1 + 5\,i$.

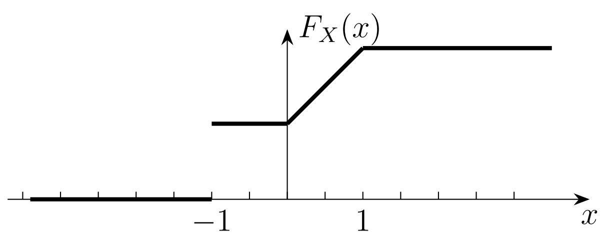
• If f_X is continuous at r,

$$f_X(r) = \frac{d}{dx} F_X(r).$$

• Below, we plot the probability density function (PDF) $f(x)$ of a normal distribution while indicating the cumulative distribution function (CDF) as area under the curve.



• Note that PDF need not always exist (unless we appeal to Dirac-Delta functions). For example, if the random variable takes the value -1 with probability half, and values uniformly randomly on the interval [0,1] with another probability half. As we see below, the CDF is not continuous and hence not differentiable at -1.



• **PDF under coordinate/variable transformation:** We will show an one-dimensional example. For more than one dimensions, the Jacobian becomes a useful tool to do this computation. Say, we have a random variable $Y = 2X$, and are asked to compute the PDF of Y. One way is to see that

$$\begin{aligned} F_Y(r) &= F_X\left(\frac{r}{2}\right) \\ \Rightarrow f_Y(r) &= \frac{d}{dr} F_Y(r) = \frac{d}{dr} F_X\left(\frac{r}{2}\right) = \frac{d(r/2)}{dr} \frac{d}{d(r/2)} F_X\left(\frac{r}{2}\right) = \frac{1}{2} f_X\left(\frac{r}{2}\right). \end{aligned} \tag{7}$$

We can also try to derive it from the first principles from the observation that $P(r < X < r+dr) = P(2r < Y < 2r + dr)$.

5 Concluding first half of motivation

• $\mathbf{P(X = 0.5)}$: Recall from Sec. 1 that X takes values uniformly randomly on the interval [0,1]. As there are infinitely many numbers, and they have equal probability, with the probabilities summing up to 1, $P(X = 0.5)$ should be equal to $1/\infty = 0$. A bit more formally, $\{0.5\}$ is a measure zero set on the interval [0,1] as its length-measure is 0. Thus, $P(X = 0.5)$ is equal to the PDF times the length-measure, which has to vanish.

• $\mathbf{P(0.4 < X < 0.5)}$: The PDF of the uniform distribution over interval [0,1] is a constant given by $f_X(r) = 1$ for $0 \leq r \leq 1$. Thus, $P(0.4 < X < 0.5)$ is the integral of the constant function from .4 to .5, giving the value 0.1.

• $\mathbf{P(X \in \mathbb{Q})}$: As we see above, $\{0.5\}$ is a measure zero set on the interval [0,1]. We also note that the set of rationals $\mathbb{Q}$ is countable. Thus, the length-measure of $\mathbb{Q}$ is the countable sum of 0's implying that $\mathbb{Q}$ is also of zero length-measure on the interval [0,1]. This implies that $P(X \in \mathbb{Q}) = 0$.

• Note that, the set of numbers $\{r : 0.4 < r < 0.5\}$ is uncountable. Thus, even though the sets $\{0.45\}$ etc are of measure zero, it **does not** imply that, their union $\{r : 0.4 < r < 0.5\}$ will also be of measure zero, as it is an uncountable union. There are both kinds: (a) **Cantor sets** are uncountable union of point sets and still of measure zero, (b) non-empty **Open intervals (a,b)** are uncountable union of point sets with length measure $b - a$.

References

[1] Marek Capiński and Ekkehard Kopp, *Measure, Integral and Probability*, Springer-Verlag, New York (1998).