Black Scholes Model

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Assumptions

The stock price follows the lognormal process with constant mean return and volatility.

The short selling of securities with full use of proceeds is permitted. what?

There are no transactions costs or taxes. All securities are perfectly divisible. what?

There are no dividends during the life of the derivative.

There are no riskless arbitrage opportunities.

Security trading is continuous.

The risk-free rate of interest, r, is constant and the same for all maturities.

Assumptions

Black Scholes PDE

Consider a derivative $C \equiv C(S,t)$.

(1) The stock S follows the SDE:

$$dS = \mu S dt + \sigma S dW.$$

(2) Ito's Lemma:

$$dG = \left(a.\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2} b^2 . \frac{\partial^2 G}{\partial x^2} \right) dt + b.\frac{\partial G}{\partial x} dW$$

where $dx = adt + bdW$

(3) Set $G \equiv C(S,t)$; $x \equiv S$; $a \equiv \mu S$; $b \equiv \sigma S$

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dW$$

BS PDE

Why Only This Portfolio?

Because this is the only combination (option + stock) where we can:

Cancel out the randomness (hedge the stochastic part)

End up with a deterministic return (so we can apply the risk-free rate assumption)

(4) Construct riskless portfolio Π

$\rightarrow$ 1 unit of derivative

$\rightarrow -\dfrac{\partial C}{\partial S}$ units of stock

$$\Pi = C - \frac{\partial C}{\partial S} S$$ imp

(5) Change in value of Π due to small change dS in S in time dt

$$d\Pi = dC - \frac{\partial C}{\partial S} dS$$

Stochastic

(6) In the portfolio Π, the diffusion terms cancel out. Hence, the portfolio is riskless. It will realize the riskless return over dt

$$d\Pi = r\Pi dt$$

You will reach here by putting the values

BS PDE

$$d\Pi = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \sigma S \frac{\partial C}{\partial S} dW -$$

$$\frac{\partial C}{\partial S}(\mu S dt + \sigma S dW) = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt$$

Now, the expression for $d\Pi$ above does not contain any stochastic term. Hence, the portfolio Π is riskfree and will generate the riskfree return r over the interval dt so that

$$\frac{1}{\Pi}\frac{d\Pi}{dt} = r \text{ or } d\Pi = r\Pi\, dt.$$

Assumptions

BS PDE

$$(7)\ \mathbf{d\Pi = r\Pi dt} \quad \mathbf{or} \quad dC - \frac{\partial C}{\partial S} dS = r\left(C - \frac{\partial C}{\partial S} S\right) dt$$

$$(8)\ dC = \frac{\partial C}{\partial S} dS + rCdt - r\frac{\partial C}{\partial S} Sdt$$

$$dC = \frac{\partial C}{\partial S}(\mu Sdt + \sigma SdW) + rCdt - r\frac{\partial C}{\partial S} Sdt$$

$$dC = \left(\mu S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \sigma S\frac{\partial C}{\partial S} dW$$

$$(9)\ \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

Equate both expressions for d(Pie) to get this

BS Solutions for Calls and Put Options

do we need to solve PDE?

$$c = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

$$p = K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

Start the derivation of distn of LnSt by sigma*Wt >> N(0,sigma sq T)
Now, use S(t) = S(0)exp((u-sigma2/2)T + sigmaB(t))

Prob of exercise of call option in risk neutral world $= P(S_T > K) = P(\ln S_T > \ln K)$.

But $\boxed{\ln S_T \xrightarrow{\text{distribution}} N\left[\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right]}$.

Imp

Now, $P\left(\ln S_T > \ln K\right) = 1 - P\left(\ln S_T < \ln K\right) = 1 - P\left\{z < \frac{\ln K - \left[\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}}\right\}$

$$= 1 - \Phi\left\{\frac{\ln K - \left[\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}}\right\} = \Phi\left\{-\left[\frac{\ln K - \left[\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}}\right]\right\} = \Phi\left(d_2\right)$$

DELTA OF BS CALL

$$c = S\Phi\left(d_1\right) - K^{-rT}\Phi\left(d_2\right)$$

$$\boxed{\frac{\partial c}{\partial S} = \Phi\left(d_1\right)} + S\Phi'\left(d_1\right) - Ke^{-rT}\Phi'\left(d_2\right)$$

$$\Phi'\left(d_1\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \frac{\partial d_1}{\partial S}$$

$$\frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S}\left\{ \frac{\ln S - \ln K + \left[\left(r - \frac{1}{2}\sigma^2\right)T\right]}{\sigma\sqrt{T}} \right\} = \frac{1}{\sigma S\sqrt{T}}$$

$$\Phi'(d_1) = \frac{1}{\sigma S\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}$$

phi d1 dash = phi d2 dash

$$S\Phi'(d_1) = \frac{1}{\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}$$

$$\Phi'(d_2) = \frac{1}{\sigma S\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2}$$

$$Ke^{-rT}\Phi'(d_2) = \frac{Ke^{-rT}}{\sigma S\sqrt{T}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2}$$

$$S\Phi'(d_1) - Ke^{-rT}\Phi'(d_2)$$

$$= \frac{1}{\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}d_1^2} - \frac{Ke^{-rT}}{S}e^{-\frac{1}{2}d_2^2}\right)$$

$$= \frac{1}{\sigma\sqrt{T}}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}d_1^2} - e^{\ln\left(\frac{K}{S}\right)-rT-\frac{1}{2}d_2^2}\right) = 0$$

(Refer next slide)

Hence, $\boxed{\frac{\partial c}{\partial S} = \Phi(d_1)}$

$$\ln\left(\frac{K}{S}\right)-rT-\frac{1}{2}d_2^2=\ln\left(\frac{K}{S}\right)-rT-\frac{1}{2}\left(d_1-\sigma\sqrt{T}\right)^2$$

$$=\ln\left(\frac{K}{S}\right)-rT-\frac{1}{2}d_1^2-\frac{1}{2}\sigma^2T+d_1\sigma\sqrt{T}$$

$$=-\ln\left(\frac{S}{K}\right)-\left(r+\frac{1}{2}\sigma^2\right)T-\frac{1}{2}d_1^2+d_1\sigma\sqrt{T}=-\frac{1}{2}d_1^2$$

Condition Expectation of Stock Price

S not e rT phi(d1)

$$E\left[S_T \middle| (S_T \geq K)\right] = E\left[e^{\ln S_T} \middle| \left(e^{\ln S_T} \geq e^{\ln K}\right)\right]$$

$$= E\left[e^{\xi} \middle| \left(e^{\xi} \geq e^{\ln K}\right)\right] = A \text{ where}$$

$$\boxed{\xi = \ln S_T \text{ is } N\left[\ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right]}$$

$$= N\left(\lambda, \sigma^2 T\right)$$

$$A = \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{\ln K}^{\infty} e^{\xi} e^{-\frac{(\xi-\lambda)^2}{2\sigma^2 T}} d\xi$$

Expectation nikalte wqt distn
Epsilon ka hi pdega

why limit starts from ln K

$$= \frac{e^{\left(\lambda + \frac{1}{2}\sigma^2 T\right)}}{\sqrt{2\pi\sigma^2 T}} \int_{\ln K}^{\infty} e^{-\frac{\left[\xi-\left(\lambda+\sigma^2 T\right)\right]^2}{2\sigma^2 T}} d\xi$$

how things going on?

$$= \frac{e^{(\ln S_0 + rT)}}{\sqrt{2\pi\sigma^2 T}} \int_{\ln K}^{\infty} e^{-\frac{\left[\xi-\left(\lambda+\sigma^2 T\right)\right]^2}{2\sigma^2 T}} d\xi$$

$$\text{Set } z = \frac{\xi - \left(\lambda + \sigma^2 T\right)}{\sigma\sqrt{T}} = \frac{\xi - \ln S_0 - \left(r - \frac{1}{2}\sigma^2\right)T - \sigma^2 T}{\sigma\sqrt{T}}$$

$$= \frac{\xi - \ln S_0 - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \text{ so that } dz = \frac{d\xi}{\sigma\sqrt{T}}$$

$$A = \frac{S_0 e^{rT}}{\sqrt{2\pi}} \int_{-d_1}^{\infty} e^{-\frac{z^2}{2}} dz \quad \text{where} \quad d_1 = \frac{\ln S_0 - \ln K + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$= S_0 e^{rT} \left[1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{z^2}{2}} dz \right] = S_0 e^{rT} \left[1 - \Phi\left(-d_1\right) \right] = S_0 e^{rT} \Phi\left(d_1\right)$$

Interpretation of BS Results

Let us assume that you write a call option. Then, you create an obligation to honour the call, if the option holder decides to exercise the option.

However, you can cover this risk exposure and maintain your position riskless by taking a long position in $\boldsymbol{\Delta}$ units of stock.

$$\Delta = \frac{\partial c}{\partial S} = \Phi(d_1)$$

Interpretation of BS Results

But there are two things here:

(i) K will be received on option maturity. Hence, it is the present value of K i.e. Ke^{-rT} that is relevant;

(ii) The probability of receiving K is equal to the probability that the option will be exercised $=N(d_2)$.

Hence, weighted cash inflow on writing the call is $Ke^{-rT}\Phi(d_2)$.

Thus, net value of call to writer: $-c=Ke^{-rT}\Phi(d_2) - S_0\Phi(d_1)$

Interpretation of BS Results

$$c = e^{-rT} E_Q\left[f\left(S_T\right)\right] = e^{-rT} E_Q\left[\max\left(S_T - K, 0\right)\right]$$

In the Black Scholes model:

$$c = e^{-rT}\left[e^{rT} S_0 \Phi\left(d_1\right) - K\Phi\left(d_2\right)\right]$$

First component of payoff = -K

However, it will be paid if the option finishes in the money

Interpretation of BS Results

Hence, we can write this component as a contingent payoff

$$C_T^1 = \begin{cases} -K & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

Now, expected value of C_T^1

$$E\left(C_T^1\right) = -K.P\left(S_T \geq K\right) + 0.P\left(S_T < K\right)$$

$$= -K.P\left(S_T \geq K\right) = -K.\Phi\left(d_2\right)$$

so that $-e^{-rT}K.\Phi\left(d_2\right) = \text{PV of } E\left(C_T^1\right)$

Interpretation of BS Results

S not e rT

Second component of payoff $= S_T$

However, it will be paid if the option finishes in the money

Hence, we can write this component as a contingent payoff

$$C_T^2 = \begin{cases} +S_T & \text{if } S_T \geq K \\ 0 & \text{otherwise} \end{cases}$$

Now, expected value of C_T^2

$$E\left(C_T^2\right) = E\left[S_T \middle| \left(S_T \geq K\right)\right] + 0.P\left(S_T < K\right) = E\left[S_T \middle| \left(S_T \geq K\right)\right]$$

$$= e^{rT} S_0 \Phi\left(d_1\right) \text{ so that } S_0 \Phi\left(d_1\right) = \text{PV of } E\left(C_T^2\right)$$

Interpretation of BS Results

$$
\begin{aligned}
c &= e^{-rT} E_Q\left[\max\left(S_T - K, 0\right)\right] \\
&= e^{-rT} E_Q\left[\left(S_T - K\right) \middle| S_T > K\right] \\
&= e^{-rT} \left\{E_Q\left[S_T \middle| S_T > K\right] - E_Q\left[K \middle| S_T > K\right]\right\} \\
&= e^{-rT} \left\{E_Q\left[S_T \middle| S_T > K\right] - K P_Q\left(S_T > K\right)\right\} \\
&= e^{-rT} \left[e^{rT} S_0 N\left(d_1\right) - K N\left(d_2\right)\right] \\
&= S_0 N\left(d_1\right) - K e^{-rT} N\left(d_2\right)
\end{aligned}
$$

Example

A stock price follows. geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is 38. What is the probability that a European call option on the stock with an exercise price of 40 and a maturity date in 6 months will be exercised?

The required probability is the probability of the stock price being above \$40 in six months time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim N\left(\ln 38+\left(0.16-\frac{0.35^2}{2}\right)0.5,\, 0.35^2\times 0.5\right) \text{ i.e.,}$$

$$\ln S_T \sim N\left(3.687,\, 0.247^2\right)$$

Since $\ln 40 = 3.689$, the required probability is

$$1-\Phi\left(\frac{3.689-3.687}{0.247}\right)=1-\Phi(0.008)$$

From normal distribution tables $\Phi(0.008)=0.5032$ so that the required probability is 0.4968

Example

What is the price of a European put option on a non-dividend-paying stock when the stock price is 69, the strike price is 70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is 6 months?

In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$

$$d_1 = \frac{\ln\left(\frac{69}{70}\right) + \left(0.05 + \frac{0.35^2}{2}\right) \times 0.5}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is:

$$p = 70e^{-0.05\times0.5}\Phi\left(0.0809\right) - 69\Phi\left(-0.1666\right)$$

$$= 70e^{-0.05} \times 0.5323 - 69 \times 0.4338$$

$$= 6.40$$

Factors That Affect Option Price

The instantaneous stock price S;

The term to maturity T-t;

$$C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

The riskfree rate r;

The stock volatility σ.

The strike price is not a variable as it is fixed by contract and remains constant over the life of the option.

Factors That Affect Option Price

$$\mathbf{c}(\Delta S, \Delta t, \Delta\sigma, \Delta r, \Delta q) = c_0 + \left(\frac{\partial c}{\partial S} = \Delta\right).\Delta S + \frac{1}{2}\left(\frac{\partial^2 c}{\partial S^2} = \Gamma\right).(\Delta S)^2$$

$$+ \left(\frac{\partial c}{\partial t} = \Theta\right).\Delta t + \left(\frac{\partial c}{\partial \sigma} = \nu\right).\Delta\sigma + \left(\frac{\partial c}{\partial r} = \rho\right)\Delta r + \left(\frac{\partial c}{\partial q} = \varphi\right)\Delta q$$

write down

$$
\begin{aligned}
c(\Delta S, \Delta t, \Delta\sigma, \Delta r, \Delta q) &= c_0 + \frac{\partial c}{\partial S}.\Delta S + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}.(\Delta S)^2 + \frac{\partial c}{\partial t}.\Delta t + \\
&\frac{\partial c}{\partial \sigma}.\Delta\sigma + \frac{\partial c}{\partial r}dr + \frac{\partial c}{\partial q}dq + \frac{1}{2}\frac{\partial^2 c}{\partial t^2}.(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 c}{\partial \sigma^2}.(\Delta\sigma)^2 \\
&+ \frac{\partial^2 c}{\partial t \partial S}.\Delta t\Delta S + \frac{\partial^2 c}{\partial t \partial \sigma}.\Delta t\Delta\sigma + \frac{\partial^2 c}{\partial S \partial \sigma}.\Delta S\Delta\sigma + \ldots \\
&= c_0 + \Delta.\Delta S + \frac{1}{2}\Gamma.(\Delta S)^2 + \Theta.\Delta t + \nu.\Delta\sigma + \rho.\Delta r + \varphi.\Delta q + \frac{1}{2}\text{Inertia}.(\Delta t)^2 \\
&+ \frac{1}{2}\text{Volga}.(\Delta\sigma)^2 + \text{Charm}.\Delta t\Delta S + \text{Veta}.\Delta t\Delta\sigma + \text{Vanna}.\Delta S\Delta\sigma + \ldots
\end{aligned}
$$

Option Greeks

df/ds

- Delta – The change in the value of an option given a one-point change in the price of the underlying assest. Delta is defined as the first derivative of f with respect to S.

df2/ds2

- Gamma – The change in the delta of an option given a one-point change in the price of the underlying asset. Gamma is defined as the second derivative of f with respect to S.

df/dt

- Theta – The change in the value of an option given a 1 day decrease in the option's time to maturity (t). Theta is often referred to as the decay rate of an option. It is defined as the first derivative of f with respect to t.

df/d sigma

- Vega – The change in the value of an option given a one-point change in the volatility. Vega is defined as the first derivative of f with respect to sigma.

df/dr

- Rho – The change in the value of an option given a one basis point change in the risk-free rate (r). Rho is defined as the first derivative of f with respect to r.