Black Scholes Model

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Option Greek Delta

Delta Hedging: Adjusting stock holdings to offset delta risk.

out of the money   in the money

- Call delta increases with increase in stock price from 0 for deep OTM calls to 1 for deep ITM calls.

- As expiration approaches, ITM call delta approaches 1 with increasing rapidity, OTM call delta approaches 0 with increasing rapidity. This is because time value erodes quickly as you approach maturity.

- Delta hedging provides immunity against price changes, but only in an infinitesimal region. For perfect hedging continuous rebalancing of portfolio is required. This is because delta value changes with every move of the stock price.

- However, if gamma is small, then the delta hedged portfolio is robust and rebalancing may be done at infrequent intervals.

Limitation: Only protects against small movements; large price jumps require gamma adjustment.

Its value won't change much if the stock price moves slightly (infinitesimally)

Delta neutrality only works for very small (infinitesimal) price changes

Why do we need a delta & gamma neutral portfolio?

Gamma comes only from options, not stocks.

Δ of a portfolio represents the rate of change in its value corresponding to an infinitesimal change in the stock price around a given value of the stock price.

All long options → positive gamma.

Short options → negative gamma.

It follows, then that that if a portfolio is neutral at the current stock price, then its value does not change in response to small changes in the stock price in the infinitesimal neighborhood of the current value of the stock price.

You have too much gamma (like +140 in the example)
You want to neutralize it (make total gamma = 0)

Option X has positive gamma (+0.7).
To reduce total gamma, you need to subtract some gamma.

The problem, however, is that this immunity is confined to infinitesimal changes i.e. very small changes in the stock price. In other words, if the jump in the stock price from the current value (at which the portfolio has been made Δ neutral) is large, this immunity breaks down and the portfolio value does undergo a change.

If you buy the option → you add its gamma.

If you short the option → you subtract its gamma.

Delta & Gamma neutral portfolio

FOR CALL:
in the money : S>K
at the money : Stock price ≈ strike price → breakeven zone
out of the m: S<K

Methods:

Create a gamma neutral portfolio using the options available without bothering about the delta.

Then, neutralize the delta of the portfolio to zero by an appropriate position in stock. This will not disturb the gamma since the stock has zero gamma.

Example

so gamma is made neutral by shorting options and delta is made neutral by shorting sstocks
because only options have gamma

one unit of option X has delta 0.60 and 0.70 gamma

A bank's position in options on a stock has a delta of 300 and a gamma of 140. An option X that is being traded has a delta of 0.60 and a gamma of 0.70. How to make the portfolio delta and gamma neutral?

Delta for a stock is always 1,-1

units to short = 140/0.70

- We first make the portfolio gamma neutral by shorting 200 units of option X.
- The delta of our portfolio is, now, 300-0.60*200=180
- Now, it will convert delta to zero, by shorting 180 units of the stock.
- Now, we have both delta and gamma neutrality.

delta for call 0 to 1
delta for put 0 to -1

Positive Gamma

To understand this phenomena, we proceed along the following:

(i) Construct a riskfree portfolio Π at t=0, $S=S_0$ consisting of:

(a) one unit of the derivative

(b) $-\Delta_0 = -\Delta\Big|_{S=S_0} = -\dfrac{\partial C}{\partial S}\Big|_{S=S_0}$ units of the stock.

The value of this portfolio is $\Pi_0 = C_0 - \dfrac{\partial C}{\partial S}\Big|_0 S_0 = C_0 - \Delta_0 S_0$ and the portfolio $\Delta_{p0}=0$.

(ii) Let the stock price make a movement dS at this point to S_1. The change in the value of portfolio Π due to this stock price movement is $d\Pi_0 = dC_0 - \Delta_0 dS$.

(iii) The portfolio Δ value will also change due to the change in the stock price. Let the new Δ_p be:

$$\Delta_{p1} = \Delta_{p0} + d\Delta = 0 + \frac{\partial \Delta_p}{\partial S}\Big|_0 dS = \Gamma_{p0} dS$$

(iv) To retain Δ neutrality of our portfolio Π at the new stock price S_1, we need to ***short*** a further $\Delta_{p1} = \Gamma_{p0} dS$ units of the stock. This shorting will be done at the current price of $S_1=S_0+dS$.

(v) If $\Gamma_0 > 0$, and if the stock price increases i.e. $dS>0$, this means increasing the short content of stock in Π i.e. selling stock. Thus, the net effect in that case would be that the investor would be selling stock after a price rise, thereby making a profit.

Positive Gamma

(vi) Similarly, if the stock price falls, i.e. $dS<0$, this means decreasing the short content of stock in Π i.e. buying stock. Thus, the net effect in that case would be that the investor would be buying stock after a price fall, thereby again, making a profit.

(vii) Let the stock price, now, make another movement $-dS'$ at this point to $S_2=S_1-dS'$. The change in the value of portfolio Π due to this stock price movement is $d\Pi_1 = dC_1 + \Delta_1 dS'$.

(viii) The portfolio Δ value will also change due to the change in the stock price. Let the new Δ be:
$$\Delta_{p2} = \Delta_1 + d\Delta' = 0 - \Gamma_{p0} dS' = -\Gamma_{p0} dS'$$

(ix) To retain Δ neutrality of our portfolio Π at the new stock price S_2, we need to long $\Delta_{p2} = \Gamma_{p0}dS'$ units of the stock. This buying will be done at the current price of $S_2=S_1-dS'$.

(x) Thus, in totality we have shorted $\Delta_{p1} = \Gamma_{p0}dS$ units of the stock at S_1 and bought $\Gamma_{p0}dS'$ units of the stock at S_2. If $dS=S_1-S_0>-dS'=S_1-S_2>0$ then clearly $S_1>S_2>S_0$. The net sale proceeds are $\Gamma_{p0}dS'^{*}S_1 - \Gamma_{p0}dS'^{*}S_2 = \Gamma_{p0}\ dS'^{*}(S_1-S_2)$. Thus, if $\Gamma_{p0}>0$, this amount is clearly positive because $dS, -dS'>0$.

Example

A bank's position in options on a stock has a delta of 0 (delta neutral) and a gamma of 800. The stock price is 90. After a short period of time, the price moves to 93.

Estimate the new delta.

What additional trade is necessary to keep the position delta neutral? The stock price again makes a move down to 91.

The bank again enters into a trade to make the portfolio delta neutral. Has the bank gained or lost money from the stock price movements?

~~So~~lution

After the stock price goes up to 93, the portfolio will lose its delta neutrality. Its new delta will be:

+ve -ve both

$\boxed{\text{new } \Delta = \text{old } \Delta + \Gamma * \text{Change in stock price:}}$ 0+800*3=2,400.

Hence, short further 2,400 stock to maintain delta neutrality. Therefore, I have shorted 2,400 units of stock at 93.

Now, stock price changes to 91. new Δ is: 0-800*2=-1,600.

Hence, to maintain delta neutrality I will need to buy 1,600 stock at 91.

Thus, my net profit booked: 1,600*(93-91)=3,200. sold stocks @1600*93 and bought back @91/unit

We are assuming that Γ has not changed because of the small price movements i.e. it remains at 800. In actual fact, it would change marginally.

Whether it’s a call or a put, the gamma is positive.
This means as the stock moves in your favor, delta moves to become more favorable too.
Example: You’re long a call. If the stock rises, delta increases → your option behaves more like the stock.
If stock falls, delta decreases → your losses are less severe.
Positive gamma helps you because it makes your position more favorable as the market moves.

Summary

When you're short an option (you sold it):
You have negative gamma.
That means delta moves against you as the market moves.
If stock rises, delta increases → you’re more exposed to losses.
If stock falls, delta decreases (in puts) → again more exposure.
Negative gamma hurts you because it increases your directional risk as the market move

Gamma measures the rate of change for delta with respect to the underlying asset's price.

All long options have positive gamma and all short options have negative gamma.

The gamma of a position tells us how much a 1 unit move in the underlying will change an option’s delta.

black scholes
In BSOPM, put and calls have the same gamma.
write down the formula nd see

The gamma is maximum for ATM calls and approaches 0 as the stock price decreases or increases. The gamma of both OTM and ITM calls approaches zero.

As expiration approaches, ATM call gamma increases very rapidly while gamma of OTM and ITM both approaches zero. If volatility increases,

When volatility is low, the gamma of At-The-Money options is high while the gamma for deeply into or out-of-the-money options approaches 0.

When volatility is high, gamma tends to be stable across all strike prices.

can ask reasoning from here

If gamma of a position is small, delta neutrality is robust and frequent rebalancing is not required.

Positive gamma can be used to book profits.

reasons in WA later

Theta (Θ)

Θ captures the impact of passage of time on the option price. The sensitivity of the derivative price to the passage of time is measured by Θ. Now, Θ can be measured with reference to the time elapsed since option inception or the time remaining till maturity. Usually the former is adopted. Thus, Θ of an option is the rate of change of its value with respect to the passage of time with all else remaining the same. Theta is sometimes referred to as the time decay of the portfolio.

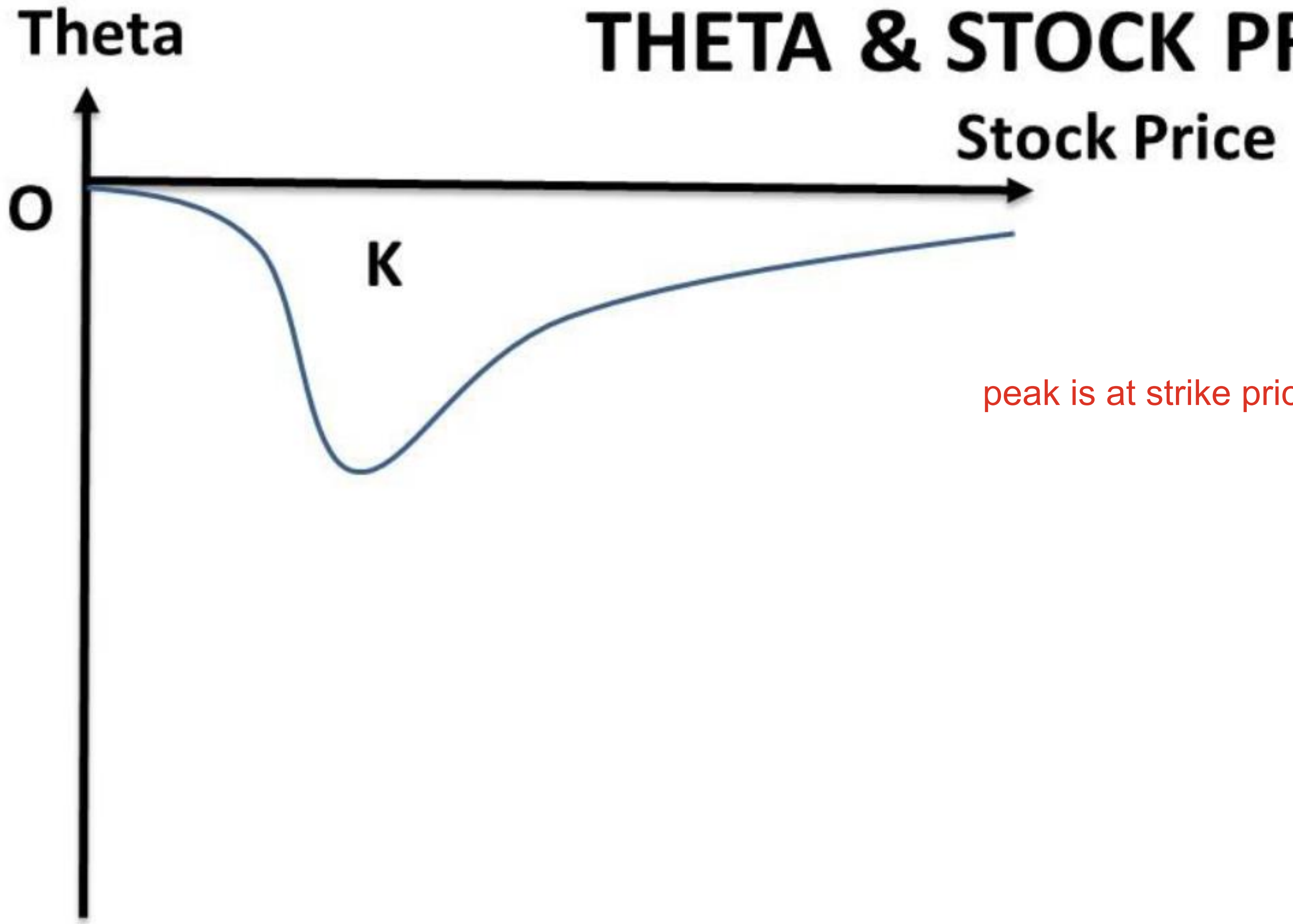
Θ is usually negative for a long option. This is because, as time passes with all else remaining the same, the option tends to become less valuable. Generally expressed as a negative number, the theta of an option reflects the amount by which the option's value will decrease every day. Θ is measured in terms is $/year or any other money units per time unit.

$$\Theta = \frac{\partial c}{\partial t} = S\mathbf{N}'(d_1)\frac{\partial d_1}{\partial t} - Ke^{-r(T-t)}\mathbf{N}'(d_2)\frac{\partial d_2}{\partial t} - rKe^{-r(T-t)}\mathbf{N}(d_2)$$

$$= -rKe^{-r(T-t)}\mathbf{N}(d_2) + S\mathbf{N}'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right) \text{since } S\mathbf{N}'(d_1) = Ke^{-r(T-t)}\mathbf{N}'(d_2)$$

$$= -rKe^{-r(T-t)}\mathbf{N}(d_2) + S\mathbf{N}'(d_1)\frac{\partial}{\partial t}\left(\sigma\sqrt{T-t}\right) = -rKe^{-r(T-t)}\mathbf{N}(d_2) - S\frac{\sigma}{2\sqrt{T-t}}\mathbf{N}'(d_1)$$

THETA & STOCK PRICE



peak is at strike price

Theta based strategies

Option sellers use theta to their advantage, collecting time decay every day.

Calendar spreads involve buying a longer-dated option (- Θ) and selling a nearer-dated option (+ Θ), thereby creating a net positive Θ that operates to their advantage.

Selling options with close expiration will give higher positive theta per day and buying longer expiration calls will give lower negative theta, yielding positive net time decay per day. This holds if the underlying does not move.

However, selling short dated calls will also create larger negative gamma compared to buying longer dated calls. Thus, the spread will have significant negative gamma. That means that a sharp move of the underlying will cause much higher loss. However, if the underlying doesn't move, then theta will kick off and the strategy will just earn money with every passing day.

Gamma – Theta Trade Off

- If you buy at the money options close to expiration, they will create a significant positive gamma portfolio. Thus, if a significant and quick price move of underlying occurs, such options with closer expiration will gain significantly.
- However, long positions in such short-dated options create high negative theta. Thus, they will undergo rapid time decay as maturity approaches.
- The trade-off is that if the underlying doesn't move, the negative theta will start to kick off much faster but if the stock price fluctuates significantly, then the positive gamma will capture large profits.

Vega

Vega is the first partial derivative of the option with respect to the volatility of the underlying asset.

Vega measures of an option's sensitivity to changes in the volatility of the underlying asset.

Volatility measures the amplitude and frequency at which option price moves up and down.

The higher the volatility, the higher the value of the option. Thus, if there is an increase in the volatility of the stock price, it will reflect as an increase in the call price as well whence vega of calls is positive.

All long options have positive vegas.

Vega

$$\upsilon = \frac{\partial c}{\partial \sigma} = S\mathbf{N}'(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-r(T-t)}\mathbf{N}'(d_1)\frac{\partial d_2}{\partial \sigma} = S\mathbf{N}'(d_1)\left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right) \; as \; S\mathbf{N}'(d_1) = Ke^{-r(T-t)}\mathbf{N}'(d_2)$$

$$Hence,\; \upsilon = \frac{\partial c}{\partial \sigma} = S\mathbf{N}'(d_1)\frac{\partial}{\partial \sigma}\left(\sigma\sqrt{T-t}\right) = S\sqrt{T-t}\mathbf{N}'(d_1) = Ke^{-r(T-t)}\sqrt{T-t}\mathbf{N}'(d_2)$$

$$\upsilon_c = \frac{\partial c}{\partial \sigma};\; \upsilon_p = \frac{\partial p}{\partial \sigma};\; \upsilon_c = S\mathbf{N}'(d_1)\sqrt{(T-t)}$$

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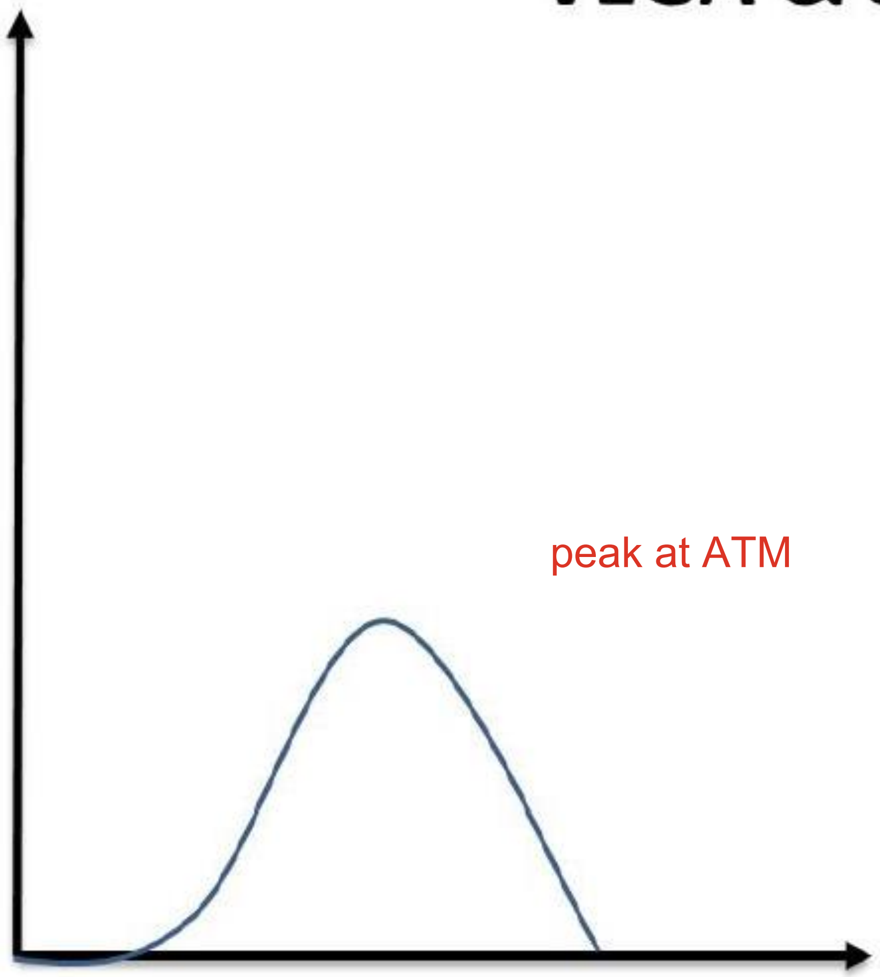
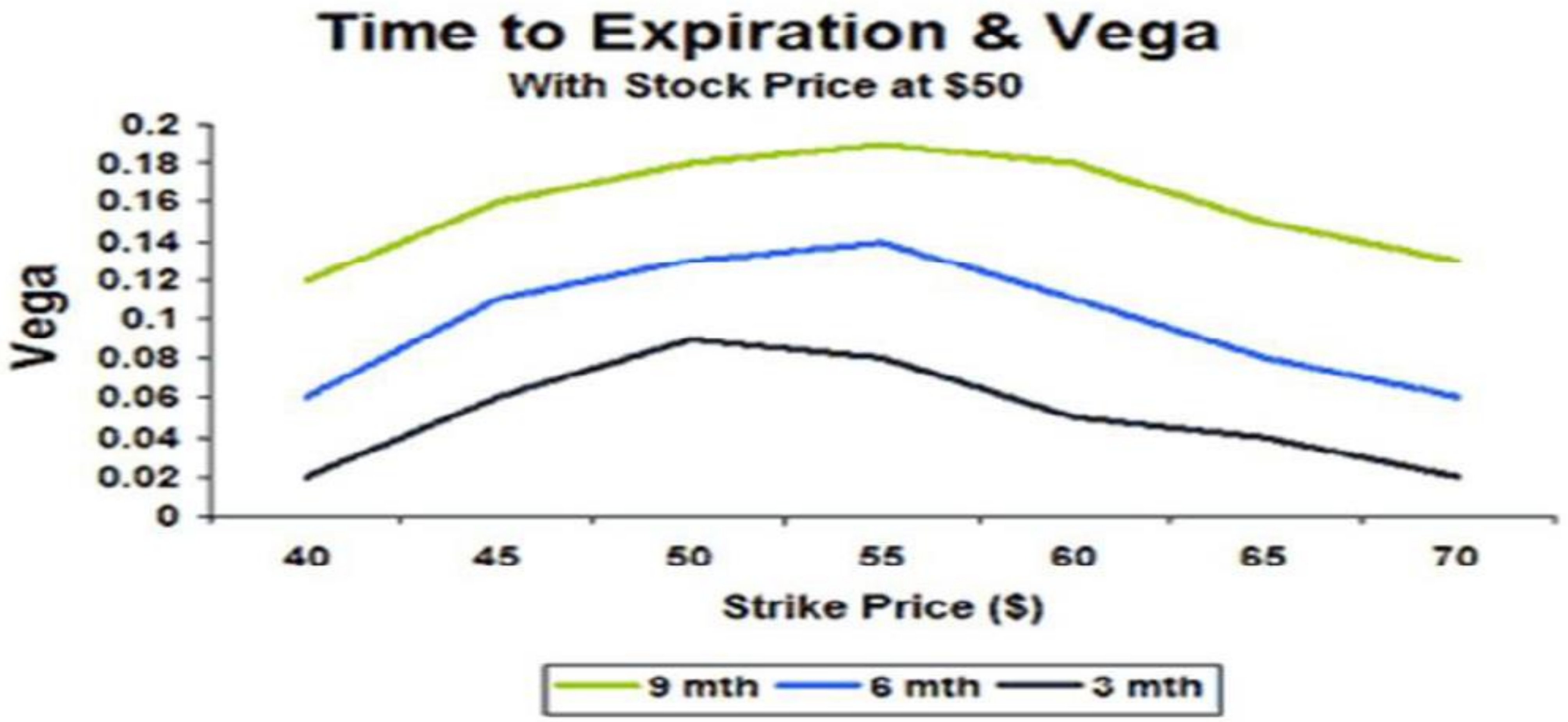
VEGA & STOCK PRICE



Vega is higher on ATM options than OTM or ITM options.

$$\boxed{\upsilon_c = S\Phi'(d_1)\sqrt{(T-t)}}$$


Time to Expiration & Vega
With Stock Price at $50
Vega
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
40
45
50
55
60
65
70
Strike Price ($)
9 mth
6 mth
3 mth

Vega

- Vega for long calls is positive, so when an investor buys calls, vega is his friend as his portfolio gains if the volatility rises.
- Vega is higher on ATM options than OTM or ITM options.
- Vega decreases as the option approaches expiration.

Example

Consider a portfolio that is Delta neutral, with a Gamma of -5,000 and a Vega of -8,000. The options shown in the table below can be traded. Calculate how the position can be made Gamma and Vega neutral by including positions in Option 1, 2 and the underlying stock.

	Delta	Gamma	Vega
Portfolio	0	-5000	-8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

Solution

w1 and w2 and are the quantities of Option 1 and Option 2

-5000 + 0.5w1 + 0.8w2 = 0

-8000 + 2.0w1 + 1.2w2 = 0

w1 = 400

w2 = 6000

Delta Neutral

400x0.6 +6000x 0.5 =3240