Financial Modelling With Python - Bond Markets

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Overview of Bond

- A bond is a contract between the issuer and the bondholder
- The issuer promises to pay the bondholder interest, if any, and principal on the remaining balance
- The following are major characteristics of bond
  - Par value of Face value
  - Maturity date / Redemption date
  - Coupon rate
  - Coupon date
  - Issue Price
- Zero coupon bond or Zero

Different Types of Bonds

- Corporate bonds
- Municipal bonds
- Government bonds
- Agency bonds
- Zero-coupon bonds
- Convertible bonds
- Callable bonds
- Puttable bond

Pros

- Bonds provide steady interest income to investors throughout the life of the bond
- Bonds are rated by credit rating agencies allowing investors to choose bonds from financially-stable issuers
- Although stock prices can fluctuate wildly over time, bonds usually have less price volatility risk
- Bond such as RBI / U.S. Treasuries are guaranteed by the government providing a safe return for investors

Cons

- Bond have credit risk meaning the issuer can default on making the interest payments or paying back the principal
- Bonds typically pay a lower rate of return than other investments such as equities
- Inflation risk can be an issue if prices rise by a faster rate than the interest rate on the bond
- If interest rates rise at a faster rate than the rate on a bond, investors lose out by holding the lower yielding security

Valuation of Bonds

Zero Coupon Bond

$$\boxed{PV = \frac{1}{(1+r)^n} F}$$

Time Value of Money

$$\mathrm{PV} = \frac{1}{(1+r)^n} FV$$

FV is the future value

PV is the present value

‘r’ is the annual rate of interest

‘n’ is the number of years

Valuation of Bonds

A level-coupon bond pays interest based on the coupon rate and the par value which is paid at maturity. If F denotes the par value and C denotes the coupon and n is the number of cash flows

$$\mathrm{PV} = \sum_{i=1}^{n} C \frac{1}{(1+r)^i} + \frac{F}{(1+r)^n}$$

Annuity

An ordinary annuity pays out the same money at the end of each year for n years

$$\mathrm{PV} = \sum_{i=1}^{n} C \frac{1}{(1+r)^i}$$

$$\mathrm{PV} = C \frac{1-(1+r)^{-n}}{r}$$

Valuation of Bonds

A level-coupon bond pays interest based on the coupon rate and the par value which is paid at maturity. If F denotes the par value and C denotes the coupon, n is the number of cash flows, m is the number of payments per year, and r is the annual interest rate compounded m times per annum

$$PV = \sum_{i=1}^{n} C \frac{1}{(1+\frac{r}{m})^i} + \frac{F}{(1+\frac{r}{m})^n}$$

Annuity

A general Annuity

verify???

$$PV = C \frac{1-(1+\frac{r}{m})^{-n}}{\frac{r}{m}} + \frac{F}{(1+\frac{r}{m})^n}$$

Valuation of Bonds

Consider a 20-years 7% bond with the coupon paid semi-annually. This means that a payment of Rs 100 * 0.07/2 = Rs 3.5 will be made every 6 months until maturity, and Rs 100 will be paid at maturity.

Assuming the risk free interest rate is 6%, Its price can be computed with:

$$PV = C\,\frac{1-(1+\frac{r}{m})^{-n}}{\frac{r}{m}} + \frac{F}{(1+\frac{r}{m})^{n}}$$

here n=nm

Years = 20

F (Face value)= 100

m = 2

n = 2*20 = 40

Coupon rate = 0.07, semi-annually

C = 100 * 0.07/2 = 3.5

r (interest rate) = 0.06

PV = 111.56

Valuation of Bonds

Years = 20

F (Face value)= 100

m = 2

n = 2*20 = 40

Coupon rate = 0.07, semi-annually

C = 100 * 0.07/2 = 3.5

r (interest rate) = 0.06

PV = 111.56

Years = 15

F (Face value)= 100

m = 2

n = 2*15 = 30

Coupon rate = 0.07, semi-annually

C = 100 * 0.07/2 = 3.5

r (interest rate) = 0.06

PV = 109.80

Valuation of Bonds

Years = 20

F (Face value)= 100

m = 2

n = 2*20 = 40

Coupon rate = 0.07, semi-annually

C = 100 * 0.07/2 = 3.5

r (interest rate) = 0.07

$\boxed{\text{PV = 100.00}}$

Years = 20

F (Face value)= 100

m = 2

n = 2*20 = 40

Coupon rate = 0.07, semi-annually

C = 100 * 0.07/2 = 3.5

r (interest rate) = 0.08

$\boxed{\text{PV = 90.10}}$

Yield of a Bond

Suppose you have bought a bond at Rs 90.10. The bond is a 20-years 7% bond with the coupon paid semi-annually.

what is The IRR?

IRR can be obtained, solving the following:

$$\boxed{PV = \sum_{i=1}^{n} C \frac{1}{(1+\frac{r}{m})^i} + \frac{F}{(1+\frac{r}{m})^n}}$$

Years = 20

PV = 90.10

F = 100

C = 3.5

m = 2

n = 20 * m = 40

The IRR is also called (in this case) yield of the bond.

Yield for a Portfolio of Bonds

Calculation for the yield to maturity for a portfolio of bonds is no different from that for a single bond. First, the cash flows of the individual bonds are combined. Then the yield is calculated based on the combined cash flow as if it were from a single bond.

Example: A bond portfolio consists of two zero-coupon bonds. The bonds are selling at Rs. 75 and Rs. 80, respectively. The term is exactly 3 years from now. To calculate the yield, we solve,

$$85 + 75 = \frac{100 + 100}{(1 + y)^3}$$

Annualized Yield $= \boxed{7.72\%}$

Price Volatility

The sensitivity of the percentage price change to changes in interest rates measures price volatility.

$$Price\ Volatility = -\frac{1}{P}\frac{\partial P}{\partial y} =$$

$$-\frac{(C/y)n-(C/y^2)((1+y)^{n+1}-(1+y))-nF}{(C/y)[(1+y)^{n+1}-(1+y)]+F(1+y)}$$

where n is the number of periods before maturity, y is the period yield, F is the par value, and C is the coupon payment per period.

Duration

The Macaulay duration, first proposed in 1938 by Macaulay, is defined as the weighted average of the times to an asset's cash flows. The weights are the cash lows' PVs divided by the asset's price.

$$\text{Macaulay Duration (MD)} = \frac{1}{P}\sum_{i=1}^{n} C_i \frac{1}{(1+y)^i} i$$

where n is the number of periods before maturity, y is the period yield, and C_i is the coupon payment at period i.

Duration

means when do u receive the payments/coupons

- The duration of any bond will be less than its maturity
- The lower a bond's coupon, the longer its duration
- A zero coupon bond's duration will be equal to its maturity

  In 0 cupon bond u get all cash flows at maturity only
- The longer a bond's maturity, the longer its duration

  obvious by formula

Duration

A general numerical formula for volatility is:

$$\boxed{\text{Effective Duration} = -\frac{(P_+ - P_-)}{P_0(y_+ - Y_-)}}$$

where P_- is the price if the yield is decreased by ∇y,

P_+ is the price if the yield is increased by ∇y,

P_0 is the initial price,

y is the initial yield,

$y_+ \equiv y + \nabla y$

$y_- \equiv y - \nabla y$,

and ∇y is sufficiently small.

Immunization

A portfolio is said to **immunize** a liability if its value at the horizon date covers the liability for small rate changes now.

For example, a Rs10,000 liability 5 years from now should be matched by a portfolio with an MD of 5 years and a future value of Rs10,000.

Assume that the liability is a certain L at time m and the current interest rate is y. We are looking for a portfolio such that

1. FV of the portfolio is L at the horizon m
2. $\frac{\partial FV}{\partial y} = 0$
3. FV is convex around y

Immunization

Condition (1) says the obligation is met.

Conditions (2) and (3) together mean that L is the portfolio's minimum FV at the horizon for small rate changes.

Immunization

Let $\mathrm{FV} \equiv (1+y)^m P$, where P is the PV of the portfolio.

$$\frac{\partial \mathrm{FV}}{\partial y} = m(1+y)^{m-1} P + (1+y)^m \frac{\partial P}{\partial y}$$

$$m = -(1+y)\,\frac{\partial P / P}{\partial y}$$

$$\mathrm{FV} = \sum_{i=1}^{n} \frac{C}{(1+y)^{i-m}} + \frac{F}{(1+y)^{n-m}}$$

$$\frac{\partial^2 \mathrm{FV}}{\partial y^2} = \sum_{i=1}^{n} \frac{(m-i)(m-i-1)\,C}{(1+y)^{i-m+2}} + \frac{(m-n)(m-n-1)\,F}{(1+y)^{n-m+2}} > 0$$

Immunization

generally we have more than 1 bonds to immunize the liability

If there is no single bond whose MD matches the horizon, a portfolio of two (or more) bonds, A and B, can be assembled by the solution of

$$1 = \omega_A + \omega_B,$$

Liabiity

$$D = \omega_A D_A + \omega_B D_B$$

D is duration

Convexity

The important notion of convexity is defined as

$$Convexity = \frac{1}{P}\frac{\partial^2 P}{\partial y^2}$$

It measures the curvature of the price/yield relation

- Greater convexity translates into greater price gains as interest rates fall
- Lessened price declines as interest rates rise

check the proofs

Convexity

$$\frac{1}{P}\left[\sum_{i=1}^{n} i(i+1)\frac{C}{(1+y)^{i+2}} + n(n+1)\frac{F}{(1+y)^{n+2}}\right]$$

$$= \frac{1}{P}\left\{\frac{2C}{y^3}\left[1 - \frac{1}{(1+y)^n}\right] - \frac{2Cn}{y^2(1+y)^{n+1}} + \frac{n(n+1)[F-(C/y)]}{(1+y)^{n+2}}\right\}$$

The important notion of convexity is defined as

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}$$

$$\text{duration (in years)} = \frac{\text{duration (in periods)}}{k}$$

Convexity

Let there be n kinds of bonds, with bond i having duration D_i and convexity C_i

There might be infinite possibilities to create a portfolio, which one to choose?

$$
\begin{aligned}
\text{maximize} \quad & \omega_1 C_1 + \omega_2 C_2 + \cdots + \omega_n C_n, \\
\text{subject to} \quad & 1 = \omega_1 + \omega_2 + \cdots + \omega_n, \\
& D = \omega_1 D_1 + \omega_2 D_2 + \cdots + \omega_n D_n, \\
& 0 \leq \omega_i \leq 1.
\end{aligned}
$$

Day Count Conventions

- A day-count convention is a standardized methodology for calculating the number of days between two dates.
- The calculation is important to bond traders because, when a bond is sold, the seller is entitled to some of the coupon payment.
- Or computing the present value (PV) when the next coupon payment is less than a full coupon period away.
- Among the most common conventions are 30/360, 30/365, actual/360, actual/365, and actual/actual.

Day Count Conventions

- Actual/actual uses the precise number of days in the month and the year.

The number of days between June 17, 2021, and October 1, 2021, is 106: 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

- The 30/360 convention is the simplest, as it assumes that each month has 30 days.

The number of days between June 17, 2021, and October 1, 2021, is 104: 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October.

In India day count convention for G-Secs is 30/360. (from RBI Publication)

Accrued Interest

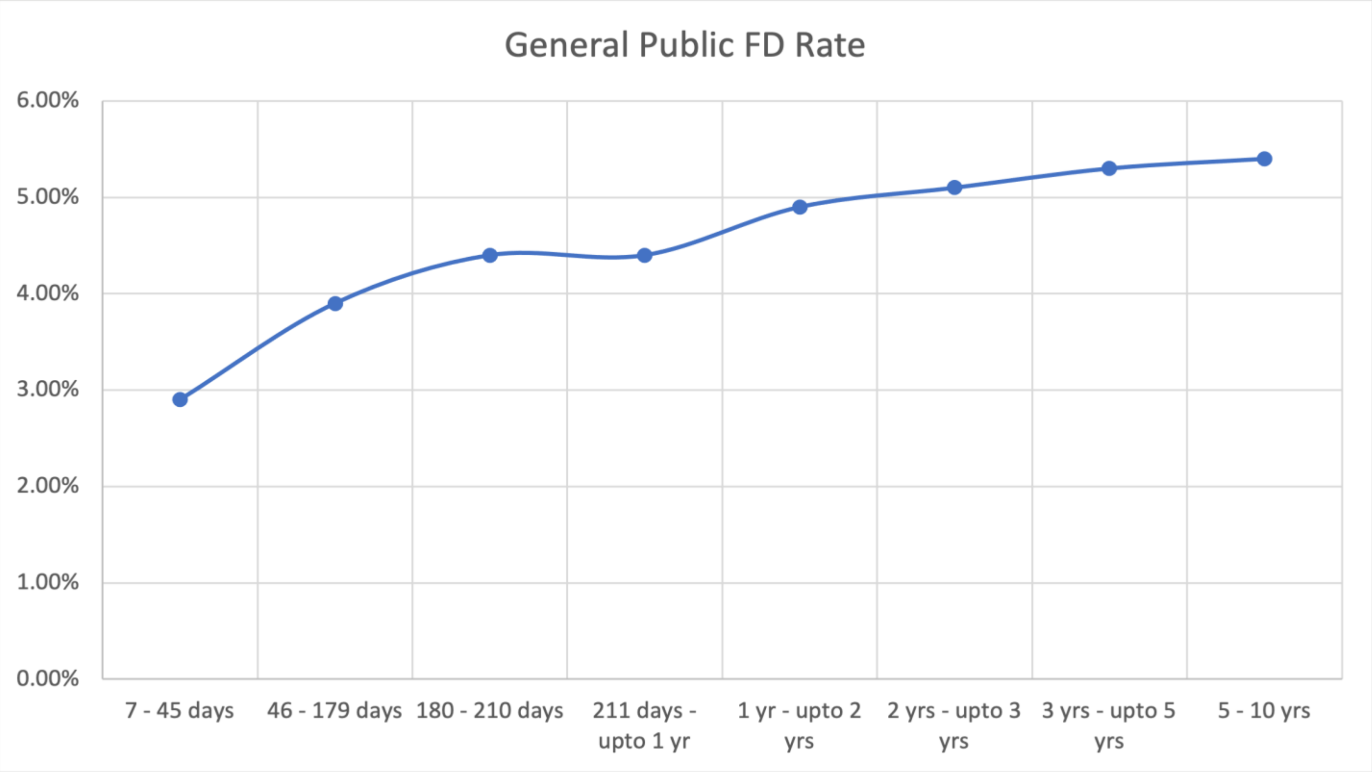
We have assumed that the next coupon payment date is exactly one period (6 months for bonds, for instance) from now. In reality, the settlement date may fall on any day between two coupon payment dates

For example, assume coupon is payable on the 20th of each month, and the accounting period is the end of each calendar month. The month of April will require an accrual of 10 days of interest, from the 21st to the 30th.

Topics Covered So Far …

- Time Value of Money
- Annuity
- Amortization
- Internal Rate of Return (IRR)
- Overview Bonds
- Valuation of bonds
- Duration of a bond
- Convexity of a bond

Interest Rate of SBI on FD



Tenure	General Public FD Rate
7 - 45 days	2.90%
46 - 179 days	3.90%
180 - 210 days	4.40%
211 days - upto 1 yr	4.40%
1 yr - upto 2 yrs	4.90%
2 yrs - upto 3 yrs	5.10%
3 yrs - upto 5 yrs	5.30%
5 - 10 yrs	5.40%

A Few Terms …

- Interest Rate
- Treasury Rates Govt Bond Rates
- LIBOR London Interbank Offered Rate
- Repo Rates The rate at which central banks lend short-term money to commercial banks in exchange for government securities
- Bond Yield
- Par Yield

The return an investor can expect from a bond, calculated as the annual interest payment divided by the bond's current market price. Different types include current yield, yield to maturity, and yield to call

Term Structure of Interest Rates

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- Term structure of interest rates, commonly known as the yield curve, depicts the interest rates of similar quality bonds at different maturities

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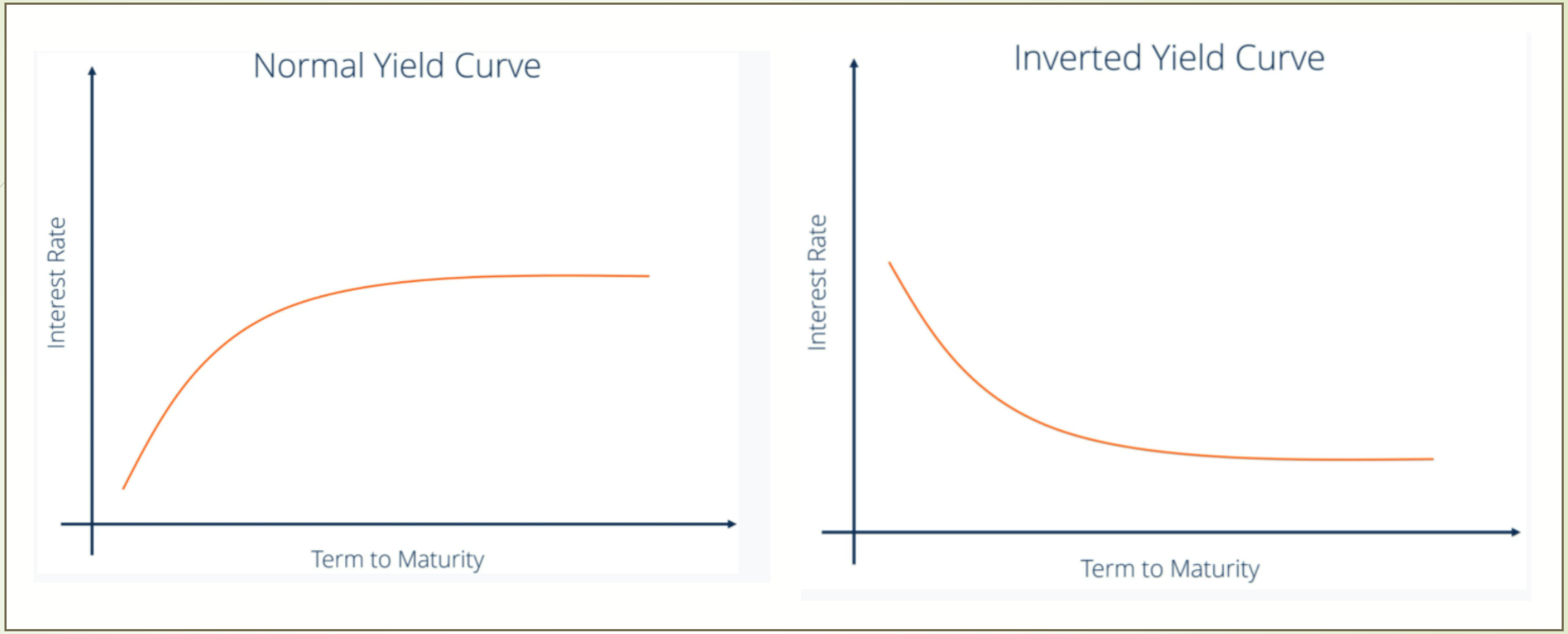
- The term structure of interest rates reflects expectations of market participants about future changes in interest rates and their assessment of monetary policy conditions

- One commonly used yield curve compares the three-month, two-year, five-year, 10-year and 30-year U.S. Treasury debt



Types of Yield Curve

Spot Rates

- The spot rate is the rate of return earned by a bond when it is bought and sold on the secondary market without collecting interest payments
- The spot interest rate for a zero-coupon bond is calculated the same way as the YTM for a zero-coupon bond
- A spot rate curve is a plot of spot rates against maturity. Its other names include spot yield curve and zero-coupon yield curve.

$$\mathrm{PV} = \sum_{i=1}^{n} C \frac{1}{(1+S_i)^i} + \frac{F}{(1+S_n)^n}$$

Spot Rates

Example:

Suppose the 1-year T-Bills (Zero coupon bond) has a yield of 8%. And the 2-year 10% T-note is trading at 90. Find the spot rates.

Solution:

Since 1-Year zero coupon bond has yield of 8%, so 1-Year spot rate is indeed 8%. $S_1 = 0.08$.

10% T Note → coupon value = 10 and par value = 100.

Cash flows are → 10 after 1 yr, 10+100 = 110 at 2nd yr

PV = 90

$$90 = \frac{10}{1.08} + \frac{110}{(1+S_2)^2}$$

$S_2 = 16.72\%$

Spot Rates

If we know S_1, S_2, …, S_{n-1} and Price of n-period coupon bond which is say P_n

The we can find S_n from the following:-

$$P_n = \sum_{i=1}^{n-1} \frac{C_i}{(1+S_i)^i} + \frac{C_n + F}{(1+S_n)^n}$$

Static Spread

Consider a risky corporate bond with the cash flows C_1, C_2, …, C_n and selling price P.

You also know the spot rates S_1, S_2, … S_n.

The price you obtained from the following equation

$$P^* = \sum_{i=1}^{n} \frac{C_i}{(1 + S_i)^i}$$

You will find that P* is different from P and also P < P*. Why?

Because riskiness must be compensated

Static Spread

The static spread is the amount ‘s’ by which the spot rate curve has to shift in parallel in order to price the bond correctly:

$$P = \sum_{i=1}^{n} \frac{C_i}{(1 + s + S_i)^i}$$

Term Structure of Credit Spreads

- A corporation is more likely to fail in, say, 10 years rather than in 1.

- One way to compute probability of default is

A lower corporate bond price means a higher perceived risk of default, hence a lower survival probability.

$$prob. of\ survive\ till\ 1\ period = \frac{\text{price of 1−period corporate zero}}{\text{price of 1−period Treasury zero}}$$

- probability the corporate bond survives past period n?

$$= \frac{\text{price of n−period corporate zero}}{\text{price of n−period Treasury zero}}$$

Term Structure of Credit Spreads

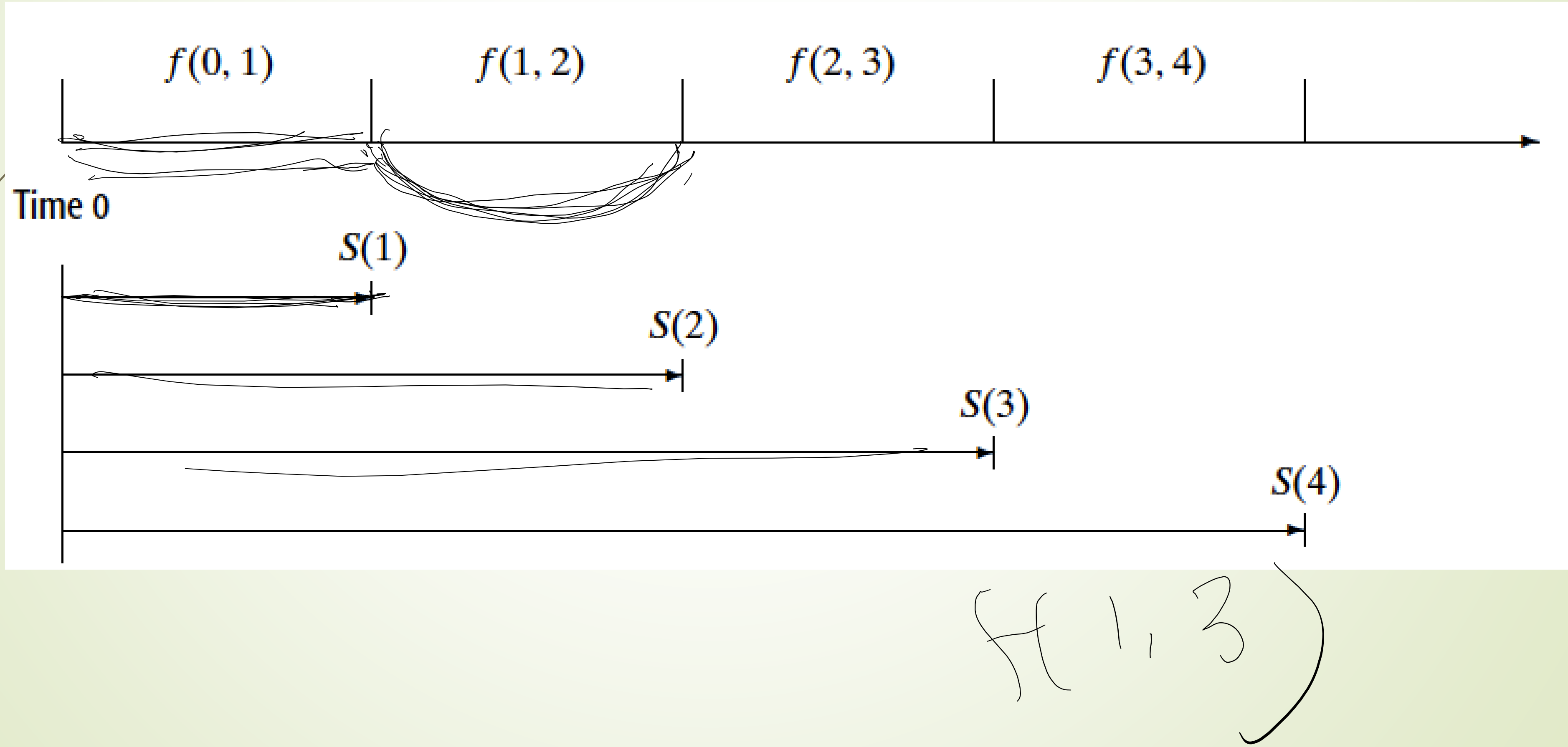
probability the corporate bond survives past period 2

= [1−probability of default (1 period)] * [1−forward probability of default (period 2)]

probability the corporate bond survives past period n

= [1−probability of default (n-1 periods)] * [1−forward probability of default (period n)]

Forward Rates



A forward rate is the interest rate agreed today for lending or borrowing money in the future

Forward Rates

d&lä/*l' gq rfc p_rc dmp rfc j_qr ncpgmb dpm k

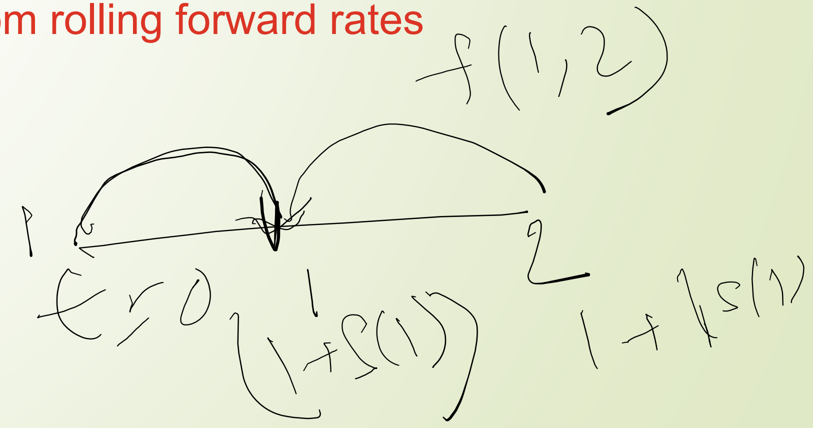
lä/ rm

Forward rates, spot rates can be derived from each other.

n

For example, the future value of Rs1 at time n can be derived in two ways. We can buy n-period zero-coupon bonds and receive $[\ 1+S(n)\ ]^n$ or we can buy one period zero-coupon bonds today and then a series of such bonds at the forward rates as they mature. The future value of this approach is $[\ 1+S(1)\ ][\ 1+f(1,2)\ ]\cdots[\ 1+f(n-1,n)\ ]$.

compounded return from spot rates equals the compounded return from rolling forward rates



Because they are identical,

$$S(n)=\{[\ \underline{1+S(1)}\ ][\ \underline{1+f(1,2)}\ ]\cdots[\ 1+f(n-1,n)\ ]\}^{1/n}-1.$$

here money invested was Rs. 1

Forward Rates

	Year			
	1	2	3	4
Spot Rate (%)	4.00	4.20	4.30	4.50

	Period		
	1 -> 2	2 -> 3	3 -> 4
Forward Rates(%)	4.40	4.50	5.10

$$1.045^4 \approx 1.040 \times 1.044 \times 1.045 \times 1.051 = 1.192$$

Forward Rates

	Year			
	1	2	3	4
Spot Rate (%)	4.00	4.20	4.30	4.50

	Period		
	1 -> 2	2 -> 3	3 -> 4
Forward Rates(%)	4.40	4.50	5.10

$$S(n) = \{[1 + S(1)][1 + f(1,2)] \cdots [1 + f(n-1,n)]\}^{1/n} - 1.$$

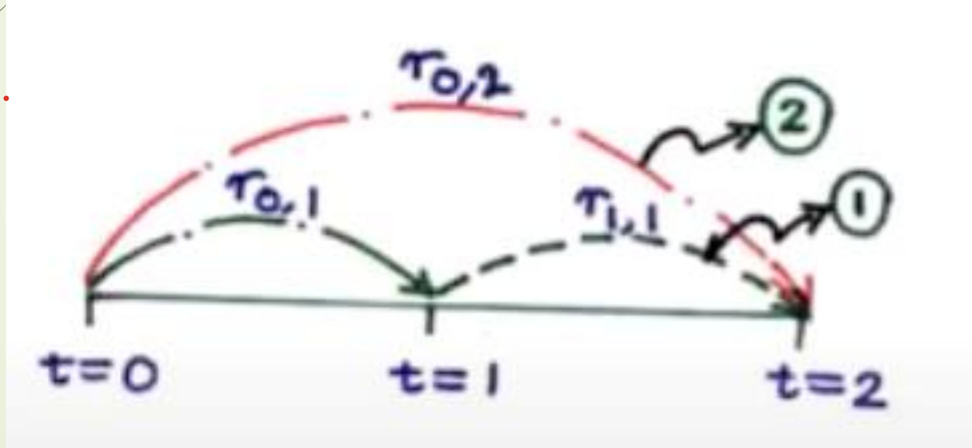
Theories of Term Structure

- Continues Compounding

po = present value of the bond

$$P_t = P_0 * e^{rt}$$

Long term IR is averafe of the short term IRs



$$e^{r_{0,1} + r_{1,1}} = e^{2r_{0,2}}$$

$$\Rightarrow r_{0,2} = \frac{r_{0,1} + r_{1,1}}{2}$$

r 0,2 spot rate
r 0,1 spot rate
r 1,1 forward rate

pt = future value at time t

Theories of Term Structure

The idea here is: when you invest over 2 years, your total return should equal the return of rolling over 1-year investments

Expected 1 year rate starting at time 1 in future

r 0,1 = 1 year IR starting from 0

$$r_{0,2} = \frac{r_{0,1} + E(r_{1,1})}{2}$$

$$r_{0,T} = \frac{r_{0,1} + E(r_{1,1}) + E(r_{2,1}) + \dots + E(r_{T-1,1})}{T}$$

The long-term rate

.
*

p
.*

$$E(r_{t,1}) = f_{0,t,t+1}$$

is the average of short-term rates, including future expectations.

Facts of the Term Structure of Interest Rates

1. Interest rates on bonds of different maturities move together over time
2. When short-term interest rates are low, yield curves are more likely to have an upward slope; when short-term rates are high, yield curves are more likely to slope downward and be inverted
3. Yield curves almost always slope upward

Theories to Explain the Facts

- ✓ Expectation theory explains the first two facts but not the third
- ✓ Segmented markets theory explains fact three but not the first two
- ✓ Liquidity premium theory combines the two theories to explain all three facts

limitation: it assumes investors are indifferent to maturity — but some investors prefer short-term bonds, while others prefer long-term

Expectations Theory

- Bond holders consider bonds with different maturities to be perfect substitutes
- The interest rate on a long-term bond will equal an average of the short-term interest rates that people expect to occur over the life of the long-term bond
- Buyers of bonds do not prefer bonds of one maturity over another; they will not hold any quantity of a bond if its expected return is less than that of another bond with a different maturity

Expectations Theory

$$r_{0,2} = \frac{r_1 + E(r_{1,1})}{2}$$

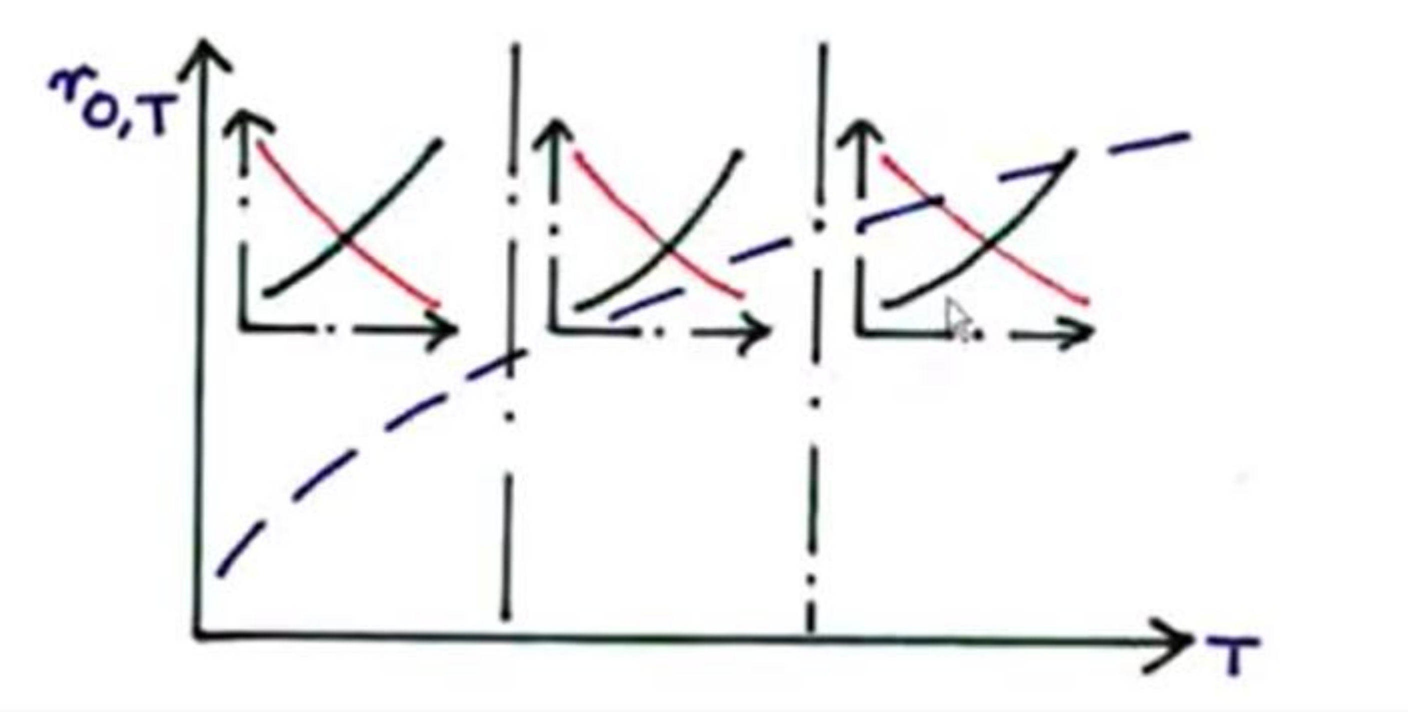
$$r_{0,T} = \frac{r_1 + E(r_{1,1}) + E(r_{2,1}) + \cdots + E(r_{T-1,1})}{T}$$

Expectations Theory

- Explains why interest rates on bonds with different maturities move together over time (fact 1)
- Explains why yield curves tend to slope up when short-term rates are low and slope down when short-term rates are high (fact 2)
- Cannot explain why yield curves usually slope upward (fact 3)

Segmented Markets Theory

Bonds of different maturities are not substituted at all



Segmented Markets Theory

- Bonds of different maturities are not substituted at all
- The interest rate for each bond with a different maturity is determined by the demand for and supply of that bond
- Investors have preferences for bonds of one maturity over another
- If investors generally prefer bonds with shorter maturities that have less interest-rate risk, then this explains why yield curves usually slope upward (fact 3)

Liquidity Premium Theories

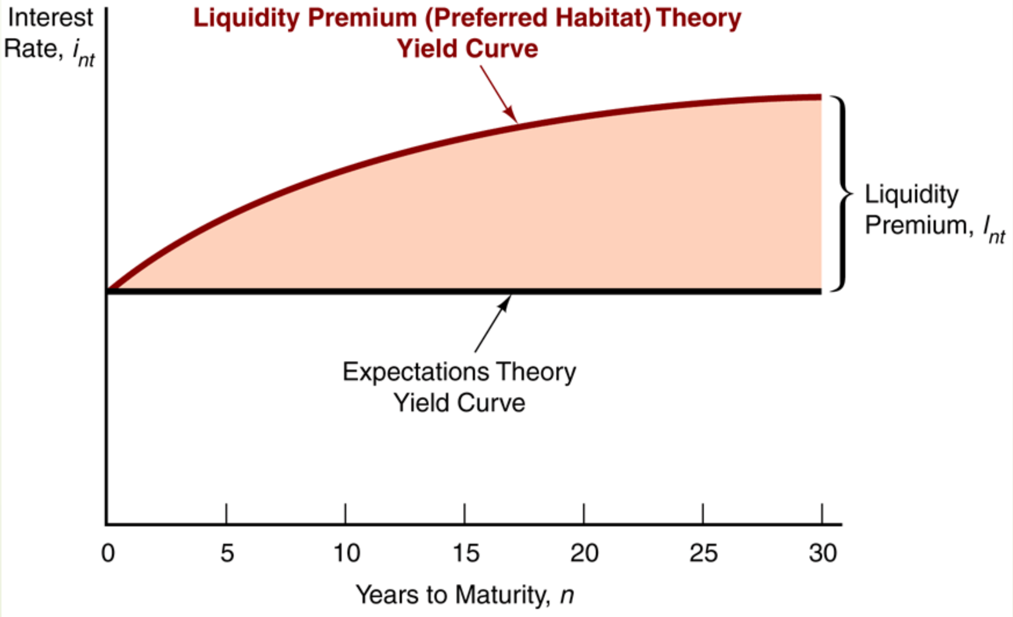
Bonds of different maturities are partial (not perfect) substitutes

The interest rate on a long-term bond will equal an average of short-term interest rates expected to occur over the life of the long-term bond plus a liquidity premium that responds to supply and demand conditions for that bond

Liquidity Premium Theories

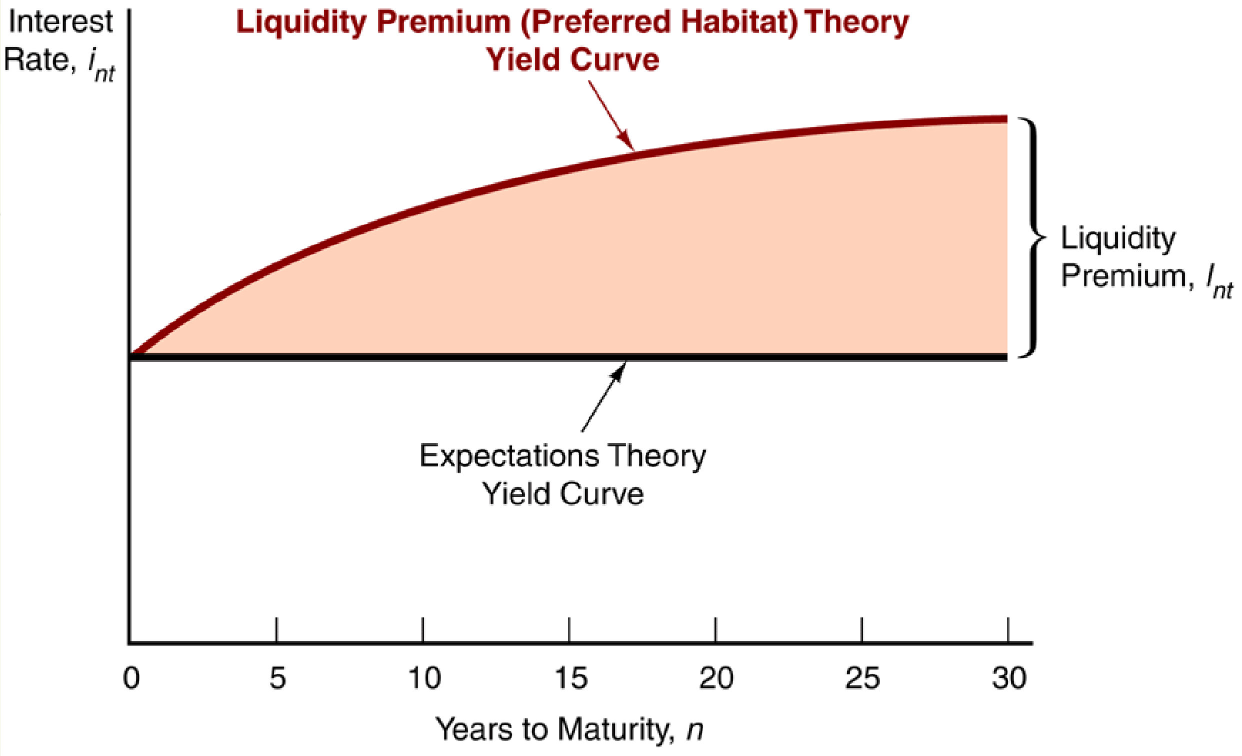
$$r_{0,T} = \frac{r_1 + E(r_{1,1}) + E(r_{2,1}) + \cdots + E(r_{T-1,1})}{T} + l_T$$

l_T is liquidity premium, >= 0 and raise with terms of maturity (T)



Liquidity Premium Theories

- Investors have a preference for bonds of one maturity over another
- They will be willing to buy bonds of different maturities only if they earn a somewhat higher expected return
- Investors are likely to prefer short-term bonds over longer-term bonds



Liquidity Premium Theories

Liquidity Premium Theories

- Interest rates on different maturity bonds move together over time; explained by the first term in the equation
- Yield curves tend to slope upward when short-term rates are low and to be inverted when short-term rates are high; explained by the liquidity premium term in the first case and by a low expected average in the second case
- Yield curves typically slope upward; explained by a larger liquidity premium as the term to maturity lengthens