A solution of Vasicek process

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October 27, 2024

Abstract

Here, we solve the so-called Vasicek stochastic differential equation which is used in mathematical finance literature as a model for interest rates.

• Vasicek model of interest rates uses an Ornstein–Uhlenbeck stochastic process to model the evolution. It assumes that interest rates follow a mean-reverting process, capturing the tendency for rates to return to a long-term average over time. The stochastic differential equation

$$dr(t) = a(\theta - r(t))dt + \sigma dW(t) \tag{1}$$

is a mean reverting process, see Fig. 1.

• Vasicek processes admit closed form solutions. However, before deriving the exact solution, let us try to explore some properties. For example, can we find expectation $E[r(t)]$ and variance $Var[r(t)]$ before solving the equation? This is useful, as in some cases, the SDE may not admit an exact solution, but the expectation might.

• **Expectation:** Let $f(t) = E[r(t)]$. This implies, by linearity of expectation,

$$df(t) = dE[r(t)] = E[r(t+dt)] - E[r(t)] = E[r(t+dt) - r(t)] = E[dr(t)]. \tag{2}$$

Taking expectation on both sides of Eq(1),

$$\begin{aligned} df(t) &= E\left[a(\theta - r(t))dt + \sigma dW(t)\right] \\ &= E\left[a(\theta - r(t))dt\right] + E\left[\sigma dW(t)\right] \\ &= aE\left[\theta - r(t)\right]\, dt + \sigma \cancel{E\left[dW(t)\right]} \\ &= a(\theta - f(t))dt. \end{aligned} \tag{3}$$

This implies,

$$\boxed{\frac{df(t)}{(\theta - f(t))} = a\, dt.}$$

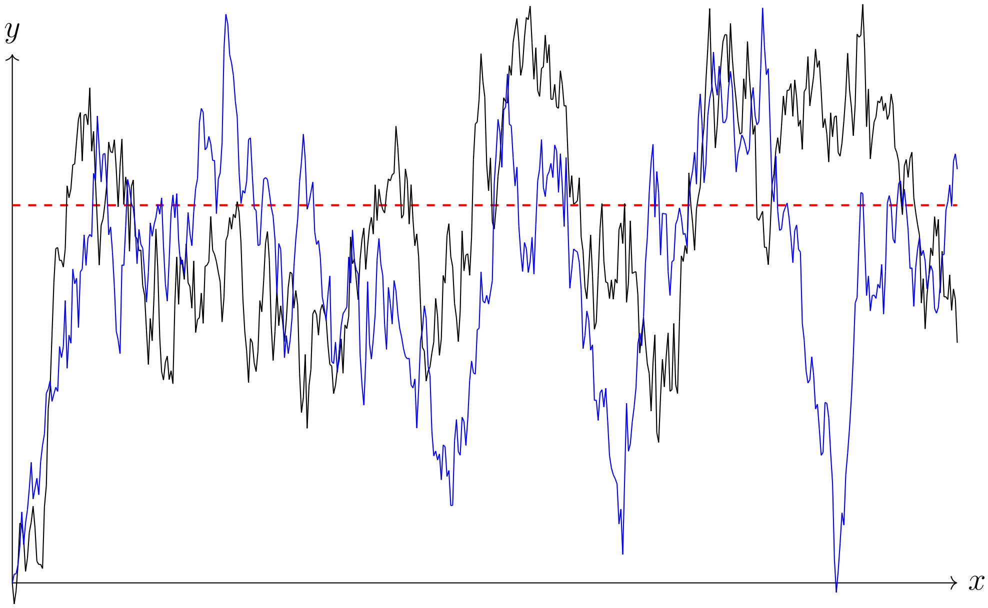

Figure 1: Instances of mean-reverting Vasicek processes. $a = 1, \theta = 5$ and $\sigma = .4$. Notice that the rate can become negative. This is one drawback of the model, necessitating further modifications.

Assuming $f(0) = r(0) < \theta$ and integrating both sides, we get

$$-\log\left(\frac{\theta - f(t)}{\theta - r(0)}\right) = at \qquad \text{How?}$$

implying

$$\begin{aligned} E[r(t)] = f(t) &= \theta - (\theta - r(0))e^{-at} \\ &= \theta\left(1 - e^{-at}\right) + r(0)e^{-at}. \end{aligned}$$

• As a sanity check, we notice that the expectation obeys the limiting behaviors

$$\lim_{t\to 0} E[r(t)] = r(0) \quad \text{and} \quad \lim_{t\to\infty} E[r(t)] = \theta$$

• **Variance:** To compute the variance, we essentially need to compute $E[r(t)^2]$ as we already know $E[r(t)]$. Let us denote $s(t) = r(t)^2$. Using Ito's lemma to differentiate, this implies,

$$\begin{aligned} ds &= 2r\, dr + dr \wedge dr \quad \left[\text{note that } dr \wedge dr = \sigma^2 dt\right] \quad \text{what is this ?} \\ &= 2ra(\theta - r)dt + 2r\sigma dW(t) + \sigma^2 dt. \end{aligned}$$

Taking expectations on both sides, we get

$$dE[s] = 2E[r]a\theta dt - 2aE[s]dt + \sigma^2 dt. \tag{4}$$

• The above ordinary differential equation has the form

$$dX = [h(t) - X]\ dt \Rightarrow dX + Xdt = h(t)\ dt. \tag{5}$$

revise and then come

We now wish to find an integrating factor $u(t)$ such that $u(t)dX + Xu(t)dt = d(U(t)X)$. Matching coefficients of dX, we see that $u(t) = U(t)$. Moreover, $du(t) = u(t)dt \Rightarrow u(t) = e^t$. Thus, the above equation becomes

$$\begin{aligned} e^t\ (dX + Xdt) &= e^t\ h(t)\ dt \\ \Rightarrow d\left(e^t X\right) &= e^t\ h(t)\ dt. \end{aligned} \tag{6}$$

It so happens that, for the vasicek process, the right hand side, $e^t\ h(t)\ dt$ happens to have a closed form solution, leading to a closed form solution for the variance $Var[r(t)]$. However, we will leave the algebra to the readers, :)

• **Closed form solution:** To solve the SDE and get a closed form answer for $r(t)$, let us rearrange the terms of the SDE a bit:

$$dr(t) + ar(t)dt = a\theta dt + \sigma dW(t)$$

We want to find an integrating faction F(t) such that

$$F(t) * (dr(t) + ar(t)dt) = d(G(t)r(t))$$

We can then write

$$d(G(t)r(t)) = a\theta F(t)dt + \sigma F(t)dW(t)$$

and solve this by integrating both sides. Let us first check whether such an integrating factor $F(t)$ exists.

$$d(G(t)r(t)) = r(t)dG(t) + G(t)dr(t) = F(t) * ar(t)dt + F(t) * dr(t)$$

Matching coefficients of $dr(t)$, $G(t) = F(t)$. Then matching coefficients of dt,

$$dF(t) = F(t)adt \Rightarrow F(t) = e^{at}.$$

So, we have

$$d\left(e^{at}r(t)\right) = a\theta e^{at}dt + \sigma e^{at}dW(t)$$

implying

$$r(t) = r(0)e^{-at} + \theta\left(1 - e^{-at}\right) + \sigma e^{-at}\int_0^t e^{as}dW(s). \tag{7}$$

• The problem thus reduces to finding the solution of an integral of the type

$$\int_0^t f(s)dW(s).$$

Note that, $\boxed{dW(s) \sim \mathcal{N}(0, ds)}$ is a normal distribution with mean 0 and variance ds. Thus,

$$\boxed{f(s)dW(s) \sim \mathcal{N}(0, f^2(s)ds).} \quad \textit{Why ?} \tag{8}$$

As the increments dW are independent of each other, after multiplication by a factor, they are still independent. Moreover, sum of independent normals is a normal distribution where the mean and the variance of the sum is sum of the means and variances of the individual distributions. Hence,

$$\boxed{\int_0^t f(s)dW(s) = \mathcal{N}\left(0, \int_0^t f(s)^2\ ds\right)}.$$