CSC165H1: Problem Set 1 Sample Solutions

Due January 25, 2017 before 10pm

Note: solutions are incomplete, and meant to be used as guidelines only. We encourage you to ask follow-up questions on the course forum or during office hours.

1. **[8 marks] Propositional formulas.** For each of the following propositional formulas, find the following two items:

   (i) The truth table for the formula. (You don't need to show your work for calculating the rows of the table.)

   (ii) A logically equivalent formula that only uses the $\neg$, $\wedge$, and $\vee$ operators; *no* $\Rightarrow$ or $\Leftrightarrow$. (You *should* show your work in arriving at your final result. Make sure you're reviewed the "extra instructions" for this problem set carefully.)

   (a) **[4 marks]** $(p \Leftrightarrow \neg q) \Rightarrow p$.

Solution

Truth table:

p	q	$(p \Leftrightarrow \neg q) \Rightarrow p$
False	False	True
False	True	False
True	False	True
True	True	True

Equivalent formula:

$$\neg\Big((p \wedge \neg q) \vee (\neg p \wedge q)\Big) \vee p$$
$$\Big(\neg(p \wedge \neg q) \wedge \neg(\neg p \wedge q)\Big) \vee p$$
$$\Big((\neg p \vee q) \wedge (p \vee \neg q)\Big) \vee p$$

Note: this actually can be simplified to $\neg q \vee p$, but this isn't required.

   (b) **[4 marks]** $p \wedge \neg r \Rightarrow q \vee r$.

Solution

Truth table:

p	q	r	$p \wedge \neg r \Rightarrow q \vee r$
False	False	False	True
False	False	True	True
False	True	False	True
False	True	True	True
True	False	False	False
True	False	True	True
True	True	False	True
True	True	True	True

Equivalent formula:

$$
\begin{aligned}
&p \wedge \neg r \Rightarrow q \vee r \\
&\neg(p \wedge \neg r) \vee (q \vee r) \\
&(\neg p \vee r) \vee (q \vee r) \\
&\neg p \vee r \vee q \vee r
\end{aligned}
$$

Note: we can simplify this to $\neg p \vee r \vee q$, i.e., $r \vee r$ can be simplified to r.

2. **[3 marks] Generalizing $\Leftrightarrow$.** Suppose we have four propositional variables p, q, r, and s. Write a propositional formula that is True when all four variables have the same truth value, and False otherwise. (This is similar in spirit to the $\Leftrightarrow$ operator.) Explain in a few brief sentences how you obtained your formula – "guess and check" is not an acceptable response!

Solution

$(p \wedge q \wedge r \wedge s) \vee (\neg p \wedge \neg q \wedge \neg r \wedge \neg s)$. The first parenthesized part is true if and only if all four variables are true, and the second part is true if and only if all four variables are false; these are the only two possibilities for the four variables having the same truth value.

Alternatively, $(p \Leftrightarrow q) \wedge (p \Leftrightarrow r) \wedge (p \Leftrightarrow s)$. This asserts that p has the same truth value as q, and has the same truth value as r, and has the same truth value as s. This can only happen when all four variables have the same truth value.

3. **[11 marks] Translating statements.** The teams in the National Hockey League (NHL) are grouped into divisions of seven or eight teams. Here are some sets and predicates used to model the NHL:

Symbol	**Definition**
T	the set of all teams (represented by city name)
D	the set of all divisions (represented by region name)
$Stanley(t)$	"team t has played in a Stanley Cup final," where $t \in T$
$Canada(t)$	"team t is in Canada," where $t \in T$
$BelongsTo(t, d)$	"team t is in division d," where $t \in T$ and $d \in D$

Using only these sets and predicates, **the standard propositional operators and quantifiers**,[1] and the equals symbol =, translate each of the following English statements into predicate logic.

Note: a division is just a name, and is *not* a set of teams.

(a) **[2 marks]** Every Canadian team has played in a Stanley Cup final.

Solution

$\forall t \in T,\ Canada(t) \Rightarrow Stanley(t)$.

(b) **[2 marks]** Team *toronto* is in division *atlantic*.

[1] Updated Jan 21.

Solution

$BelongsTo(toronto, atlantic)$

(c) **[2 marks]** There is a division that contains only non-Canadian teams.

Solution

$\exists d \in D,\ \forall t \in T,\ BelongsTo(t, d) \Rightarrow \neg Canada(t).$

(d) **[2 marks]** Each division has a team that has played in a Stanley Cup final.

Solution

$\forall d \in D,\ \exists t \in T,\ BelongsTo(t, d) \wedge Stanley(t).$

(e) **[3 marks]** Every team is in exactly one division. (Hint: use the equals symbol =.)

Solution

$\forall t \in T,\ \forall d_1, d_2 \in D,\ BelongsTo(t, d_1) \wedge BelongsTo(t, d_2) \Rightarrow d_1 = d_2 \wedge \big(\exists d \in D,\ BelongsTo(t, d)\big)$

4. **[9 marks] One-to-one functions.** So far, most of our predicates have had sets of numbers as their domains. But this is not always the case: we can define properties of any kind of object we want to study, including functions themselves!

   Let S and T be sets. We say that a function $f : S \to T$ is **one-to-one** if no two distinct inputs are mapped to the same output by f. For example, if $S = T = \mathbb{Z}$, the function $f_1(x) = x + 1$ is one-to-one, since every input x gets mapped to a distinct output. However, the function $f_2(x) = x^2$ is not one-to-one, since $f_2(1) = f_2(-1) = 1$.

   (a) **[2 marks]** Suppose we want to define a predicate $OneToOne(f)$, which expresses whether a given function $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. (Note that we are using a concrete domain and range of $\mathbb{Z}$ here.)

   $$OneToOne(f) : \underline{\hspace{6cm}}, \text{ where } f : \mathbb{Z} \to \mathbb{Z}.$$

   Using the language of predicate logic, show how to fill in the blank to define $OneToOne$. You may use an expression like "$f(x) =$ [something]" in your formula.[2]

   Solution

   $$OneToOne(f) : \forall x, y \in \mathbb{Z},\ f(x) = f(y) \Rightarrow x = y, \text{ where } f : \mathbb{Z} \to \mathbb{Z}.$$

   (b) **[3 marks]** We say that a function $f : S \to T$ is **two-to-one** if no *three* distinct inputs are mapped to the same output by f. Now $f_2(x) = x^2$ is two-to-one (when $S = T = \mathbb{Z}$), but the function $f_3(x) = x(x - 1)(x - 2)$ is not. Note that $f_1(x) = x + 1$ *is* two-to-one according to this definition.

   Suppose we want to define a predicate $TwoToOne(f)$, which expresses whether a given function $f : \mathbb{Z} \to \mathbb{Z}$ is two-to-one. Using the language of predicate logic, show how to fill in the blank to define $TwoToOne$.

   $$TwoToOne(f) : \underline{\hspace{6cm}}, \text{ where } f : \mathbb{Z} \to \mathbb{Z}.$$

   Solution

   $$TwoToOne(f) : \forall x, y, z \in \mathbb{Z},\ f(x) = f(y) \wedge f(x) = f(z) \Rightarrow x = y \vee x = z \vee y = z, \text{ where } f : \mathbb{Z} \to \mathbb{Z}.$$

   (c) **[1 mark]** Consider the following rather famous statement:

   "There does not exist a one-to-one function with domain $\mathbb{R}$ and range $\mathbb{N}$."

   Let F be the set of all functions with domain $\mathbb{R}$ and range $\mathbb{N}$. First, show how to express this statement in predicate logic by filling in the blank in the following sentence:

   $$\neg\Big(\exists f \in F,\ \underline{\hspace{6cm}}\Big).$$

[2] Recall that a legitimate formula is built up only from quantifiers (each ranging over a defined set), boolean operators ($\wedge$, $\vee$, $\neg$, etc.), and predefined functions and predicates.

Just as how when we write a^b, this is actually shorthand for an exponentiation function $exp(a, b)$, when we write $f(x)$ in a formula this is shorthand for a *function evaluation* operator $F(f, x)$. For the set A of all functions from $\mathbb{R}$ to $\mathbb{R}$, this operator is defined as $F : A \times \mathbb{R} \to \mathbb{R}$, where $F(f, x)$ is equal to $f(x)$. So while you can write a formula like $\forall x \in \mathbb{R},\ f(x) = 10$, this is actually shorthand for the more technically correct $\forall x \in \mathbb{R},\ F(f, x) = 10$.

You should reuse your work from part (a), but do not write "*OneToOne*" in your formula here. (In other words, copy over the definition of your predicate from part (a), except pay attention to the sets of numbers involved!)

Solution

$$\neg\Big(\exists f \in F,\ \forall x, y \in \mathbb{R},\ f(x) = f(y) \Rightarrow x = y\Big).$$

(d) **[3 marks]** Finally, use the negation rules to show how to simplify your expression for part (c) so that negations are applied only to predicates (and remember you can use the $\neq$ symbol).

Solution

Steps for simplifying the expression from part (c):

$$\neg\Big(\exists f \in F,\ \forall x, y \in \mathbb{R},\ f(x) = f(y) \Rightarrow x = y\Big)$$
$$\forall f \in F,\ \exists x, y \in \mathbb{R},\ \neg\Big(f(x) = f(y) \Rightarrow x = y\Big)$$
$$\forall f \in F,\ \exists x, y \in \mathbb{R},\ f(x) = f(y) \wedge x \neq y$$

5. **[9 marks] Working with infinity.** Sometimes when dealing with predicates over the natural numbers, we don't care so much about which numbers satisfy the given predicate, or even exactly how many numbers satisfy it. Instead, we care about whether *infinitely many* numbers satisfy the predicate. For example:[3]

"There are infinitely many primes."

We saw that for the natural numbers, we can express the idea of "infinitely many" by saying that for every natural number n_0, there is a number greater than n_0 that satisfies the predicate:

$$\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \land Prime(n)$$

You may use the $Prime$ **predicate in your solutions for all parts of this question.**

(a) **[3 marks]** Use the above idea to express the following statement in predicate logic. Briefly justify why your statement is correct.

"There are infinitely many numbers that are one less than a prime number."

Solution

$$\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \land \Big(\exists p \in \mathbb{N},\ n = p - 1 \land Prime(p)\Big)$$

or

$$\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \land Prime(n+1)$$

We leave as an exercise understanding and justifying the accuracy of these formulas.

(b) **[4 marks]** Express the following statement in predicate logic (remember to use the negation simplification rules). Briefly justify why your statement is correct.

"There are *finitely* many numbers that cannot be written as the sum of two primes."

Solution

$$\neg\Big(\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \land \big(\forall p_1, p_2 \in \mathbb{N},\ Prime(p_1) \land Prime(p_2) \Rightarrow n \neq p_1 + p_2\big)\Big)$$
$$\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \neg\Big(n > n_0 \land \big(\forall p_1, p_2 \in \mathbb{N},\ Prime(p_1) \land Prime(p_2) \Rightarrow n \neq p_1 + p_2\big)\Big)$$
$$\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, n \leq n_0 \lor \neg\Big(\forall p_1, p_2 \in \mathbb{N},\ Prime(p_1) \land Prime(p_2) \Rightarrow n \neq p_1 + p_2\Big)$$
$$\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, n \leq n_0 \lor \Big(\exists p_1, p_2 \in \mathbb{N},\ \neg\big(Prime(p_1) \land Prime(p_2) \Rightarrow n \neq p_1 + p_2\big)\Big)$$
$$\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, n \leq n_0 \lor \big(\exists p_1, p_2 \in \mathbb{N},\ Prime(p_1) \land Prime(p_2) \land n = p_1 + p_2\big)$$

(c) **[2 marks]** Here is another variation of combining predicates with infinity. Let $P : \mathbb{N} \to \{\text{True}, \text{False}\}$

[3] Course Notes, page 26.

be a predicate defined over the natural numbers. We say that P is **eventually true** if and only if there exists a natural number n_0 such that all natural numbers greater than n_0 satisfy P.

Use this idea to express the following statement in predicate logic (no justification is required here):

"Eventually, all natural numbers are prime."

Solution

$\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n > n_0 \Rightarrow Prime(n).$