# Properties of Big-Oh, Omega, and Theta

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Here are a bunch of useful properties of Big-Oh, Omega, and Theta. Of course, you can use these to prove other statements as well; for example, some of the properties we have covered in lecture and tutorial are special cases of these theorems.

**Note**: sometimes we'll allow you to prove statement using these properties as external facts, and sometimes we won't allow you to use these (or perhaps allow only a subset of them). Keep in mind that you should be able to *prove* almost all of these using just the definitions of Big-Oh, Omega, and Theta!

## Elementary functions

The following theorem tells us how to compare four different types of "primitive" functions: constant functions, logarithms, powers of $n$, and exponential functions.

**Theorem 1** (Elementary function hierarchy)**.** For all $a, b \in \mathbb{R}^+$, the following statements are true:

1. If $a > 1$ and $b > 1$, then $\log_a n \in \Theta(\log_b n)$.
2. If $a < b$, then $n^a \in \mathcal{O}(n^b)$ and $n^a \notin \Omega(n^b)$.
3. If $1 \le a < b$, then $a^n \in \mathcal{O}(b^n)$ and $a^n \notin \Omega(b^n)$.
4. If $a > 1$, then $1 \in \mathcal{O}(\log_a n)$ and $1 \notin \Omega(\log_a n)$.
5. If $a > 1$, then $\log_a n \in \mathcal{O}(n^b)$ and $\log_a n \notin \Omega(n^b)$.
6. If $b > 1$, then $n^a \in \mathcal{O}(b^n)$ and $n^a \notin \Omega(b^n)$.

## Basic properties

**Theorem 2** (Reflexivity)**.** For all $f : \mathbb{N} \to \mathbb{R}^{\ge 0}$, $f \in \Theta(f)$.

**Theorem 3** (Quasi-symmetry)**.** For all $f, g : \mathbb{N} \to \mathbb{R}^{\ge 0}$, $g \in \mathcal{O}(f)$ if and only if $f \in \Omega(g)$. (Note: as a consequence of this, $g \in \Theta(f)$ if and only if $f \in \Theta(g)$.)

**Theorem 4** (Transitivity)**.** For all $f, g, h : \mathbb{N} \to \mathbb{R}^{\ge 0}$, if $f \in \mathcal{O}(g)$ and $g \in \mathcal{O}(h)$, then $f \in \mathcal{O}(h)$. *Moreover*, the statement is still true if you replace Big-Oh with Omega, or if you replace Big-Oh with Theta.

# Operations on functions

**Theorem 5** (Sum of functions)**.** For all $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$, the following hold:

1. If $f \in \mathcal{O}(h)$ and $g \in \mathcal{O}(h)$, then $f + g \in \mathcal{O}(h)$.
2. If $f \in \Omega(h)$, then $f + g \in \Omega(h)$.
3. If $f \in \Theta(h)$ and $g \in \mathcal{O}(h)$, then $f + g \in \Theta(h)$.

**Theorem 6** (Multiplication by a constant)**.** For all $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and all $a \in \mathbb{R}^+$, $a \cdot f \in \Theta(f)$.

**Theorem 7** (Product of functions)**.** For all $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $g_1 \in \mathcal{O}(f_1)$ and $g_2 \in \mathcal{O}(f_2)$, then $g_1 \cdot g_2 \in \mathcal{O}(f_1 \cdot f_2)$. *Moreover*, the statement is still true if you replace Big-Oh with Omega, or if you replace Big-Oh with Theta.

**Theorem 8** (Floor and ceiling)**.** For all $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f(n)$ is eventually greater than or equal to 1, then $\lfloor f \rfloor \in \Theta(f)$ and $\lceil f \rceil \in \Theta(f)$.