CSC165H1: Problem Set 3

Due March 8, 2017 before 10pm

General instructions

Please read the following instructions carefully before starting the problem set. They contain important information about general problem set expectations, problem set submission instructions, and reminders of course policies.

- Your problem sets are graded on both correctness and clarity of communication. Solutions which are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.
- Each problem set may be completed in groups of up to three. If you are working in a group for this problem set, please consult `https://github.com/MarkUsProject/Markus/wiki/Student_Groups` for a brief explanation of how to create a group on MarkUs.

  Exception: Problem Set 0 must be completed individually.
- Solutions must be typeset electronically, and submitted as a PDF with the correct filename. **Hand-written submissions will receive a grade of ZERO.**

  The required filename for this problem set is **problem_set3.pdf**.
- Problem sets must be submitted online through MarkUs. If you haven't used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it! If you are working with a partner, you must form a group on MarkUs, and make one submission per group. "I didn't know how to use MarkUs" is not a valid excuse for submitting late work.
- Your submitted file should not be larger than 9MB. This may happen if you are using a word processing software like Microsoft Word; if it does, you should look into PDF compression tools to make your PDF smaller, although please make sure that your PDF is still legible before submitting!
- Submissions must be made *before* the due date on MarkUs. You may use *grace tokens* to extend the deadline; please see the Problem Set page for details on using grace tokens.
- The work you submit must be that of your group; you may not refer to or copy from the work of other groups, or external sources like websites or textbooks. You may, however, refer to any text from the Course Notes (or posted lecture notes), except when explicitly asked not to.

Additional instructions

- All problems in this problem set must be proved **using induction**. (Note that you can, of course, use other techniques like proof by contrapositive or by cases in addition to induction).
- For each proof, clearly define the predicate ($P(n)$) that is relevant for your induction proof, so that you're proving a statement with structure $\forall n \in \mathbb{N},\ P(n)$ or $\forall n \in \mathbb{N},\ n \geq \_\_ \Rightarrow P(n)$.
- You may not use forms of induction we have not covered in lecture.
- Please follow the same guidelines as Problem Set 2 for all proofs.

1. **[4 marks] A summation.** Prove the following statement:

$$\forall x \in \mathbb{R}^+,\ x \neq 1 \Rightarrow \left(\forall n \in \mathbb{N},\ \sum_{i=0}^{n-1} x^i = \frac{x^n - 1}{x - 1}\right)$$

2. **[4 marks] Fibonacci sequence.** The *Fibonacci sequence* is a sequence of natural numbers $F_1, F_2, F_3, \ldots$ defined as follows:

$$F_0 = 0,\ F_1 = 1, \quad \text{and for all } n \geq 2,\ F_n = F_{n-1} + F_{n-2}$$

The first few terms in the Fibonacci sequence are 0, 1, 1, 2, 3, 5, 8, 13, etc.

Prove that for every natural number n, $\gcd(F_n, F_{n+1}) = 1$. You may use the following two properties of the greatest common divisor in your proof:[1]

$$\forall n \in \mathbb{N},\ \gcd(0, n) = n \tag{Claim 1}$$

$$\forall n, a, b, p, q \in \mathbb{Z},\ n \mid a \wedge n \mid b \Rightarrow n \mid ap + bq \tag{Claim 2}$$

Hint: prove the contrapositive form of the induction step.

3. **[7 marks] Counting more subsets.** We saw in the tutorial in Week 6 how to use induction to prove statements about counting various types of subsets of a set. In this question, we'll apply the same idea to count the number of *pairs of disjoint subsets* of a set. Make sure to read the following definitions carefully so that you understand exactly what you're working with for this question!

Definition 1 (S_n)**.** Let $n \in \mathbb{N}$. We define the set $S_n = \{0, 1, \ldots, n-1\}$ (with $S_0 = \emptyset$). Note that $|S_n| = n$.

Definition 2 (DP_n)**.** Let $n \in \mathbb{N}$. We define the set of *disjoint subset pairs* of S_n as follows:

$$DP_n = \big\{(A, B) \mid A, B \subseteq S_n \text{ and } A \cap B = \emptyset\big\}$$

For example, for $n = 0$, $S_n = \emptyset$, and $DP_n = \{(\emptyset, \emptyset)\}$. For $n = 3$, $S_n = \{0, 1, 2\}$; some examples of elements of DP_3 are $(\{1\}, \{0, 2\})$ and $(\{0, 1, 2\}, \emptyset)$. Note that order matters in tuples, and so the pair $(\{0, 2\}, \{1\})$ is also in DP_3, and is considered *different* from $(\{1\}, \{0, 2\})$.

(a) **[2 marks]** Write down what DP_1 and DP_2 are, using set notation. Explicitly list all elements in each of these sets. **Make sure your PDF correctly shows curly braces, { and }, to represent sets.**

(b) **[5 marks]** Find a formula for the size of DP_n (in terms of n), and then prove that your formula is correct for all values of n using induction. That is, the statement you should complete and prove looks like:

$$\forall n \in \mathbb{N},\ |DP(n)| = \underline{\hspace{3cm}}$$

You may not use any statements about subsets from Tutorial 6 or the Course Notes; however, you can (and should) use the same ideas from the *proofs* of those statements.

[1] Note that you've actually proven both of these claims in previous weeks of this course!

4. **[5 marks] Pigeonhole principle.** The *pigeonhole principle* is a rather useful statement in mathematics that can often be used to prove the existence of certain surprising objects.

   The version of the pigeonhole principle we'll look at in this question is the following: "For all natural numbers n greater than 1, and all subsets S, T of $\mathbb{N}$ where $|S| = n$ and $|T| = n - 1$, there does not exist a one-to-one function from S to T."

   Formally we define the predicate $PHP(n)$ to be the following statement:

   $$\forall S, T \subseteq \mathbb{N},\ |S| = n \land |T| = n - 1 \Rightarrow \Big(\forall f : S \to T,\ \exists s_1, s_2 \in S,\ s_1 \neq s_2 \land f(s_1) = f(s_2)\Big)$$

   Prove that $\forall n \in \mathbb{N},\ n \geq 2 \Rightarrow PHP(n)$.