

LENTES GRAVITACIONALES EN ASTROFÍSICA Y COSMOLOGÍA

SEMANA - 14

PARTE III:

LENSING A NIVEL COSMOLÓGICO

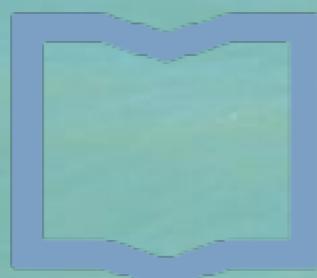
MARTÍN MAKLER

ICAS/IFICI/CONICET & UNSAM Y CBPF

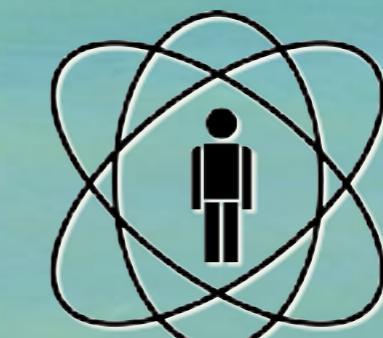
ICAS



CONICET



Instituto de
Ciencias Físicas
ICIFI-ECYT_UNSAM-CONICET



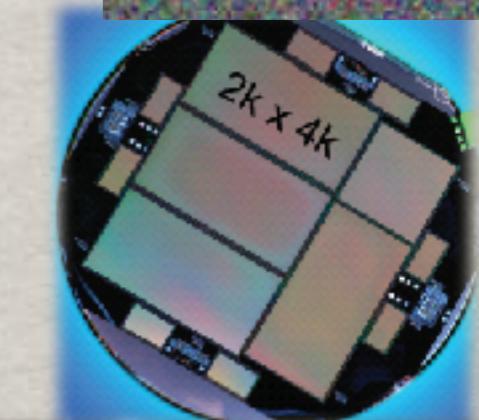
CBPF

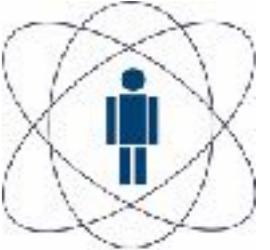
ESTRUCTURA GENERAL DE LA MATERIA

Parte III: lensing a nivel cosmológico

[perturbaciones, week lensing]

- Revisión de cosmología: formación y distribución de las de estructuras en el Universo
- Mas allá del plano único: lentes y estructura en gran escala
- Estadísticas de *lensing*
- Revisión de radiación cósmica de fondo
- Lensing* de la radiación cósmica de fondo





Recapitulando

Metrica de Friedmann con perturbación escalar

$$ds^2 = a^2(\tau) \left\{ (1 + 2\frac{\varphi}{c^2}) d\tau^2 - (1 - 2\frac{\varphi}{c^2}) [d\chi^2 + r^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)] \right\}$$

↑ geometria del fondo
(FLRW)

Propagación de la luz en el fondo homogéneo

$$\chi = \tau_0 - \tau$$

Lenteamento pela estrutura em grade escala: além da aproximação de lente fina

Métrica de Friedmann com perturbação escalar

$$ds^2 = a^2(\tau) \left\{ (1 + 2\frac{\varphi}{c^2}) d\tau^2 - (1 - 2\frac{\varphi}{c^2}) [d\chi^2 + r^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)] \right\}$$

Equação de geodésica

$$\frac{d^2x^i}{d\lambda^2} = -g^{ik} \left(\frac{\partial g_{kl}}{\partial x^m} - \frac{1}{2} \frac{\partial g_{lm}}{\partial x^k} \right) \frac{dx^m}{d\lambda} \frac{dx^l}{d\lambda}$$

Ângulo de desvio

$$\theta_\sigma(\chi) = \theta_\sigma^0 - \frac{2}{c^2} \int_0^\chi \frac{d\chi'}{r^2(\chi')} \int_0^{\chi'} d\chi'' \varphi_{,\sigma}(\vec{\theta}r(\chi''), \chi'')$$


Funções de lente

Ângulo de desvio

$$\theta_\sigma(\chi) = \theta_\sigma^0 - \frac{2}{c^2} \int_0^\chi \frac{d\chi'}{r^2(\chi')} \int_0^{\chi'} d\chi'' \varphi_{,\sigma}(\vec{\theta} r(\chi''), \chi'')$$

Se reduz a uma integral ao longo da trajetória do fóton


$$\theta_\sigma(\chi) = \theta_\sigma^0 - \frac{2}{c^2} \int_0^\chi d\chi' \varphi_{,\sigma}(\vec{\theta} r(\chi'), \chi') \frac{r(\chi - \chi')}{r(\chi)r(\chi')}$$

Potencial de lente

$$\Psi = \frac{2}{c^2 r(\chi)} \int_0^\chi d\chi' \varphi(\vec{\theta} r(\chi'), \chi') \frac{r(\chi - \chi')}{r(\chi')}$$

Convergência

$$\kappa(\vec{\theta}) = \frac{1}{2} \nabla_\theta^2 \Psi = \frac{1}{c^2} \int_0^\chi d\chi' \nabla_\perp^2 \varphi(\vec{x}_\perp, \chi') \frac{r(\chi - \chi') r(\chi')}{r(\chi)}$$

Convergencia

Usando la ecuación de Poisson para las perturbaciones

$$\nabla^2 \varphi(\vec{x}, \tau) = \frac{3H_0^2 \Omega_m}{2a(\tau)} \delta(\vec{x}, \tau)$$

Y promediando para una distribución no uniforme de fuentes

$$n(\chi) = N_{tot}^{-1} (dN/d\chi)$$

La convergencia se escribe como

$$\kappa(\vec{\theta}) = \frac{3H_0^2 \Omega_m}{2c^2} \int_0^{\chi_H} d\chi \delta(\vec{x}_\perp, \chi) \frac{g(\chi)}{a(\chi)}$$

donde

$$g(\chi) = r(\chi) \int_\chi^{\chi_H} d\chi' \frac{r(\chi - \chi') n(\chi')}{r(\chi')}$$

Función ventana
eficiencia de lentes

Correlaciones

$$\kappa(\vec{\theta}) = \frac{3H_0^2\Omega_m}{2c^2} \int_0^{\chi_H} d\chi \delta(\vec{x}_\perp, \chi) \frac{g(\chi)}{a(\chi)}$$

En el espacio de Fourier

$$\kappa(\vec{\ell}) = \int d^2\theta \kappa(\vec{\theta}) e^{i\vec{\ell}\cdot\vec{\theta}}$$

tenemos

$$\langle \kappa(\vec{\ell}) \kappa(\vec{\ell}') \rangle = (2\pi)^2 \delta^{(2)}(\vec{\ell} - \vec{\ell}') P_\kappa(\ell)$$

Espectro de potencias
de la estructura en gran escala:
dependencia cosmológica!

el espectro de potencias es dado por

$$P_\kappa(\ell) = \frac{9H_0^4\Omega_m^2}{4c^4} \int_0^{\chi_H} d\chi \frac{g^2(\chi)}{a^2(\chi)} P_\delta \left(\frac{\ell}{r(\chi)}, \chi \right)$$

Correlaciones

$$P_\kappa(\ell) = \frac{9H_0^4\Omega_m^2}{4c^4} \int_0^{\chi_H} d\chi \frac{g^2(\chi)}{a^2(\chi)} P_\delta \left(\frac{\ell}{r(\chi)}, \chi \right)$$

Espectro de potencias

$$\begin{aligned} P_\delta(k, \tau) &= D_+(k, \tau) P_k^{\text{lin}} \\ &= D_+(k, \tau) T(k) P_k \\ &= D_+(k, \tau) T(k) A_0 k^{n_s - 1} \end{aligned}$$

Espectro de potencias
de la estructura en gran escala:
dependencia cosmológica!

Angular x espacial en el espacio k

$$k_\sigma = \ell_\sigma / r(\chi)$$

Tiempo conforme

$$\chi = \tau_0 - \tau$$

Cosmic Shear

- Separating into redshift bins
- Weak lensing (statistical)

© **Sensitive to dark energy (geometry + growth factor)**

© **Less sensitive to baryonic physics**

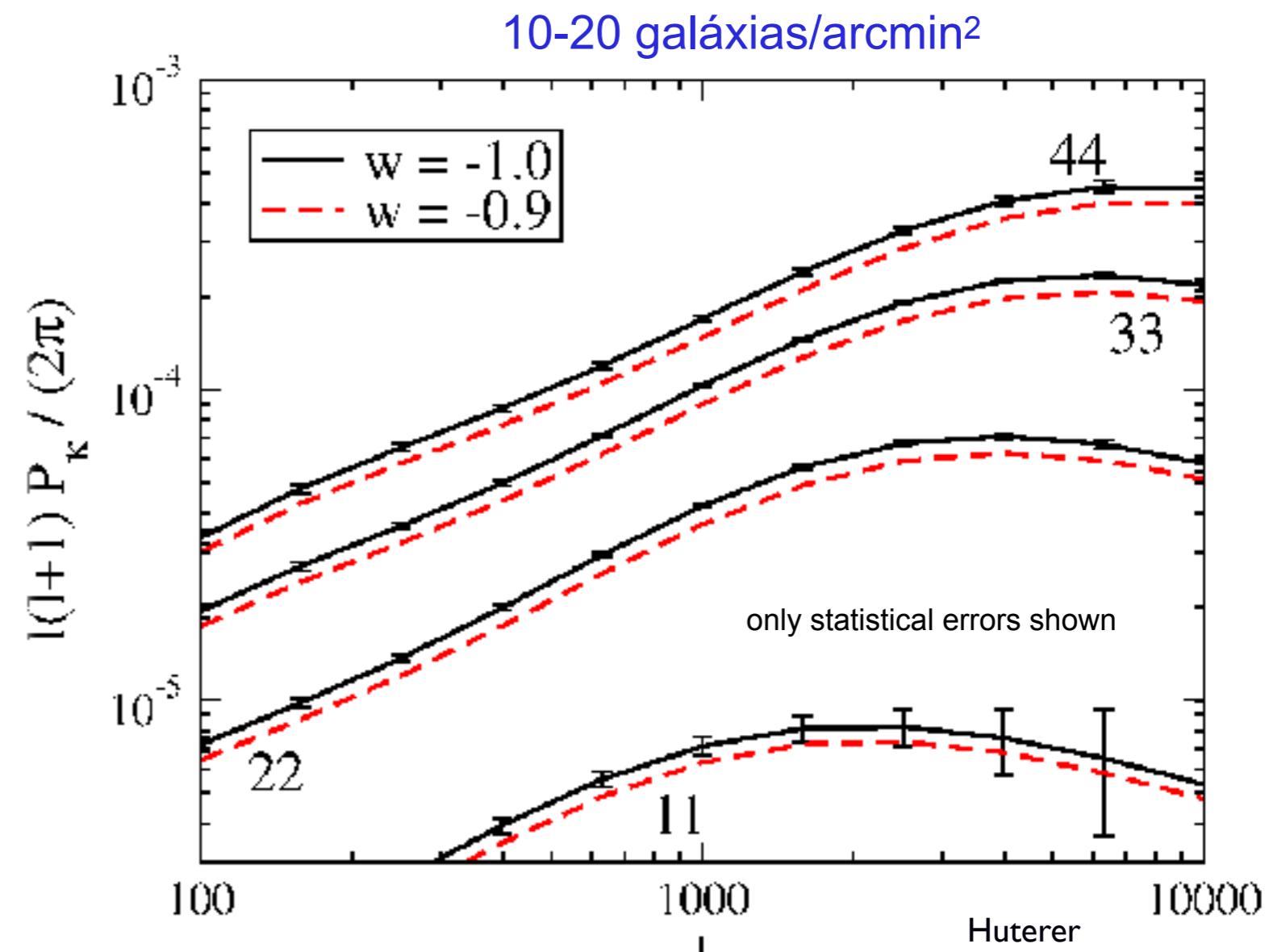
- large separations: linear
- Example: cosmic shear power spectrum on three photo-z slices (tomography)
- DES:

unique combination of area, seeing and depth

shapes of ~ 300 million galaxies with $\langle z \rangle = 0.7$

$$C_\ell^{r_a r_b} = \int dz \frac{H(z)}{D_A^2(z)} W_a(z) W_b(z) P^{s_a s_b}(k = \ell/D_A; z)$$

$$\Delta C_\ell = \sqrt{\frac{2}{(2\ell-1)f_{sky}}} \left(C_\ell + \frac{\sigma^2(\gamma_\ell)}{n_{eff}} \right)$$



Example: DES Y1 Results

► Data

- 1321 deg² of g,r,i,z imaging data
- 26 million source galaxies
- 650,000 luminous red galaxies (redMaGIC)

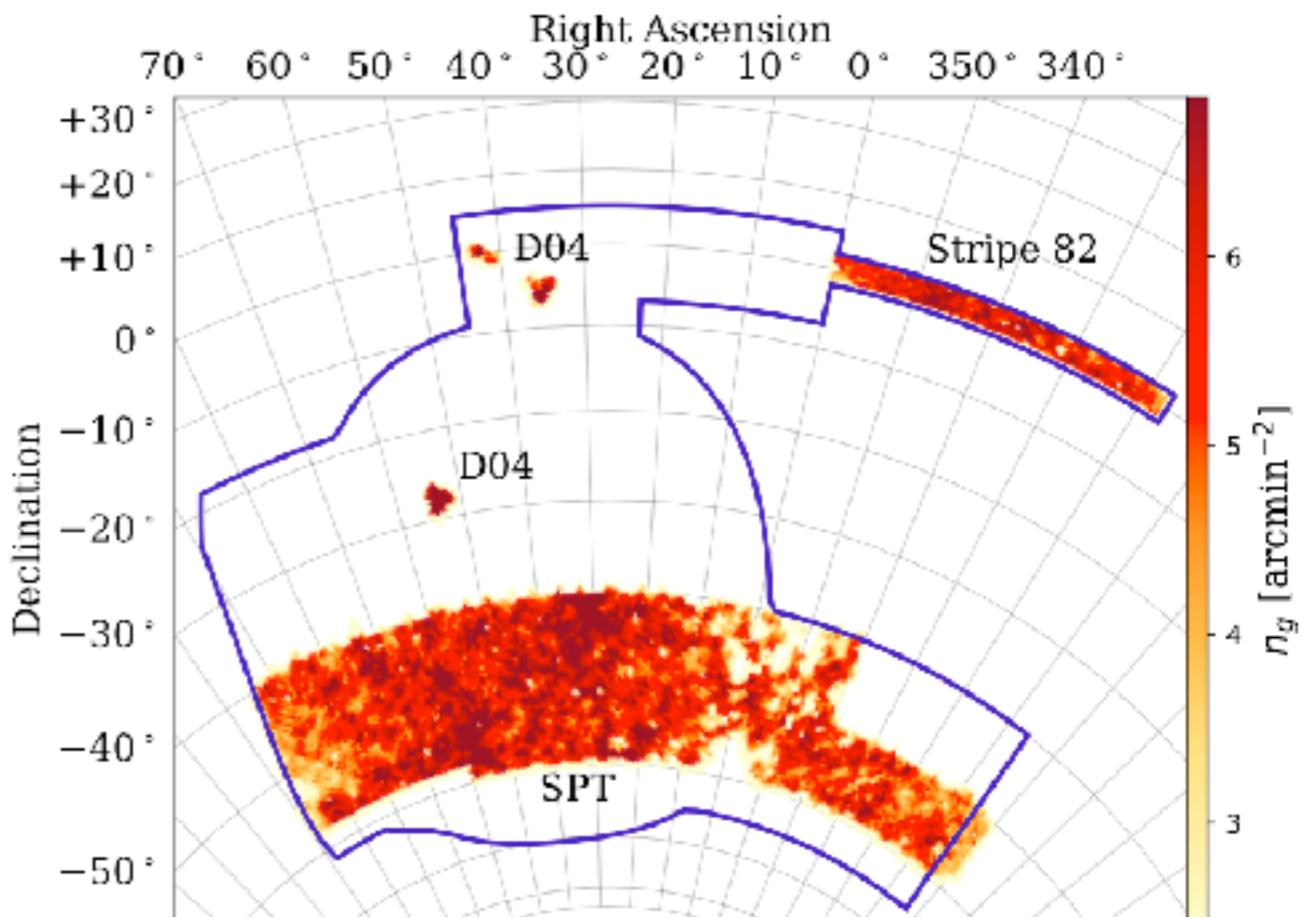
► Combined analysis of 3 two-point function

- the cosmic shear correlation function in four redshift bins
- the galaxy angular autocorrelation function of LRGs in five redshift bins
- the galaxy-shear cross-correlation of LRG positions and source galaxy shears.

► Dark Energy Survey Year 1 Results: Cosmological Constraints from Galaxies

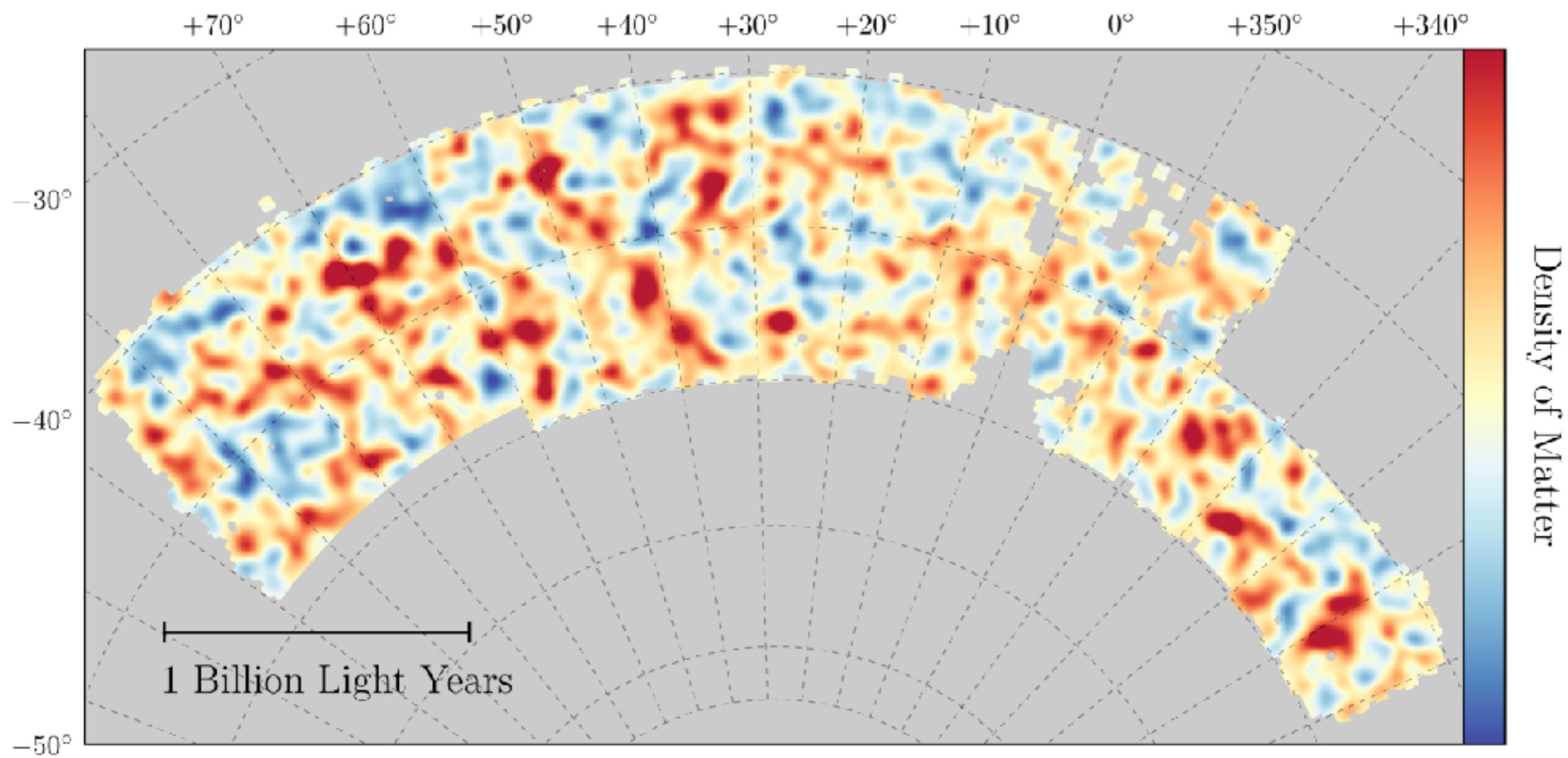
- 10 “supporting papers” jointly submitted
- Careful analysis of systematics
- Joint likelihoods

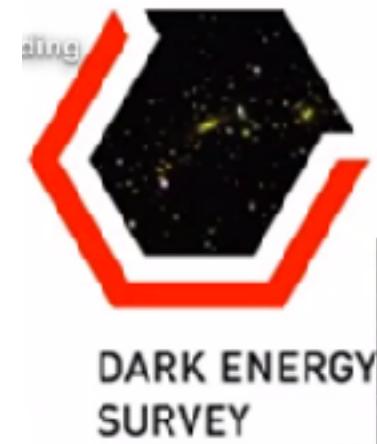
“[...] this paper presents the most stringent cosmological constraints from a galaxy imaging survey to date and, combined with external data, the most stringent constraints overall”.



Cosmic Shear Measurements

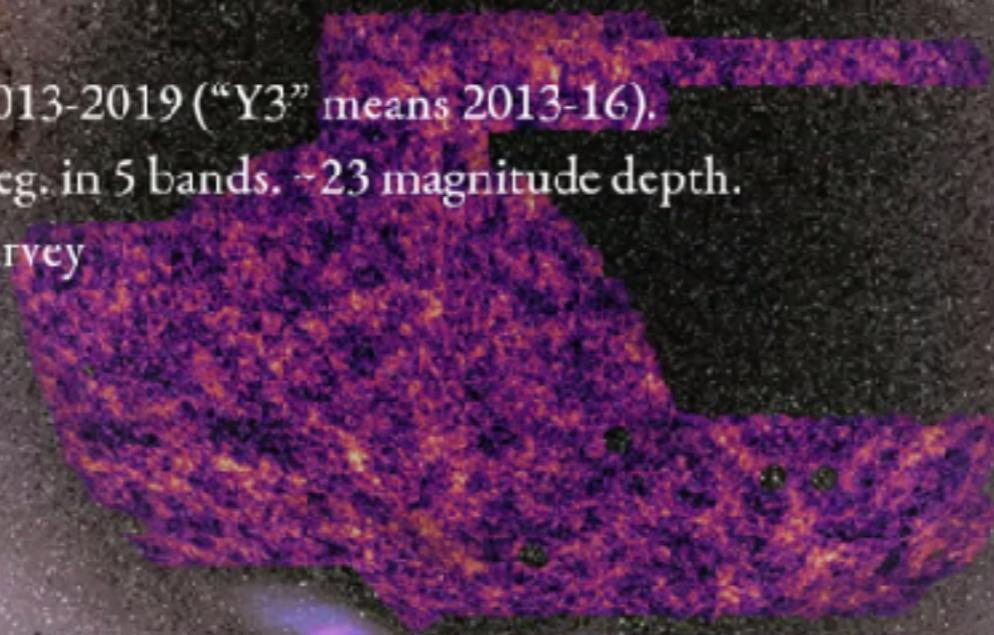
Dark Matter Map 2.0



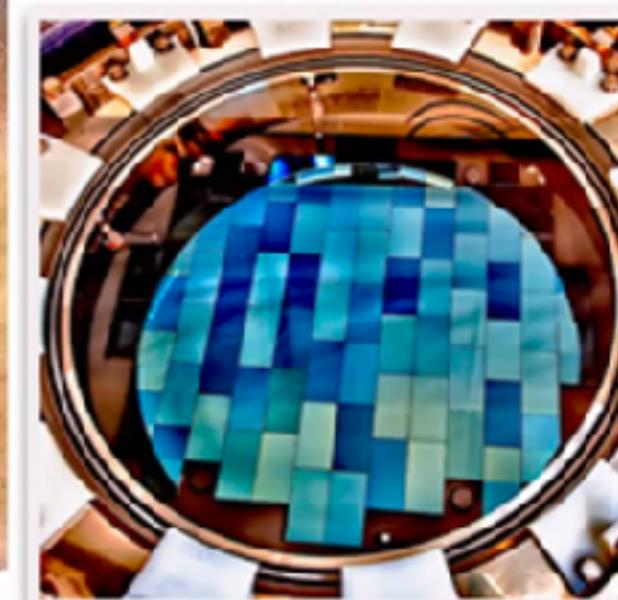
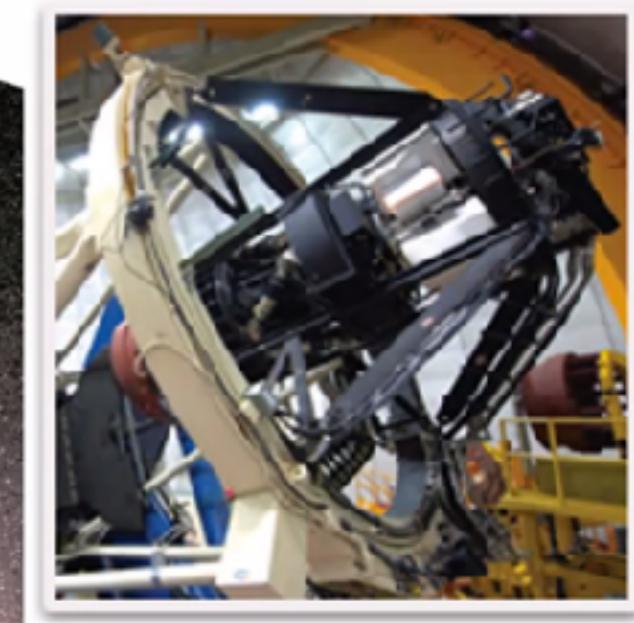


DARK ENERGY
SURVEY

- Led by Fermilab
- DECam: The 570 Megapixel camera for the Blanco 4m telescope in Chile.
- Survey Observations 2013-2019 (“Y3” means 2013-16).
- Wide field: 5000 sq. deg. in 5 bands, ~23 magnitude depth.
- 27 sq. deg. 5-yr SNIa survey

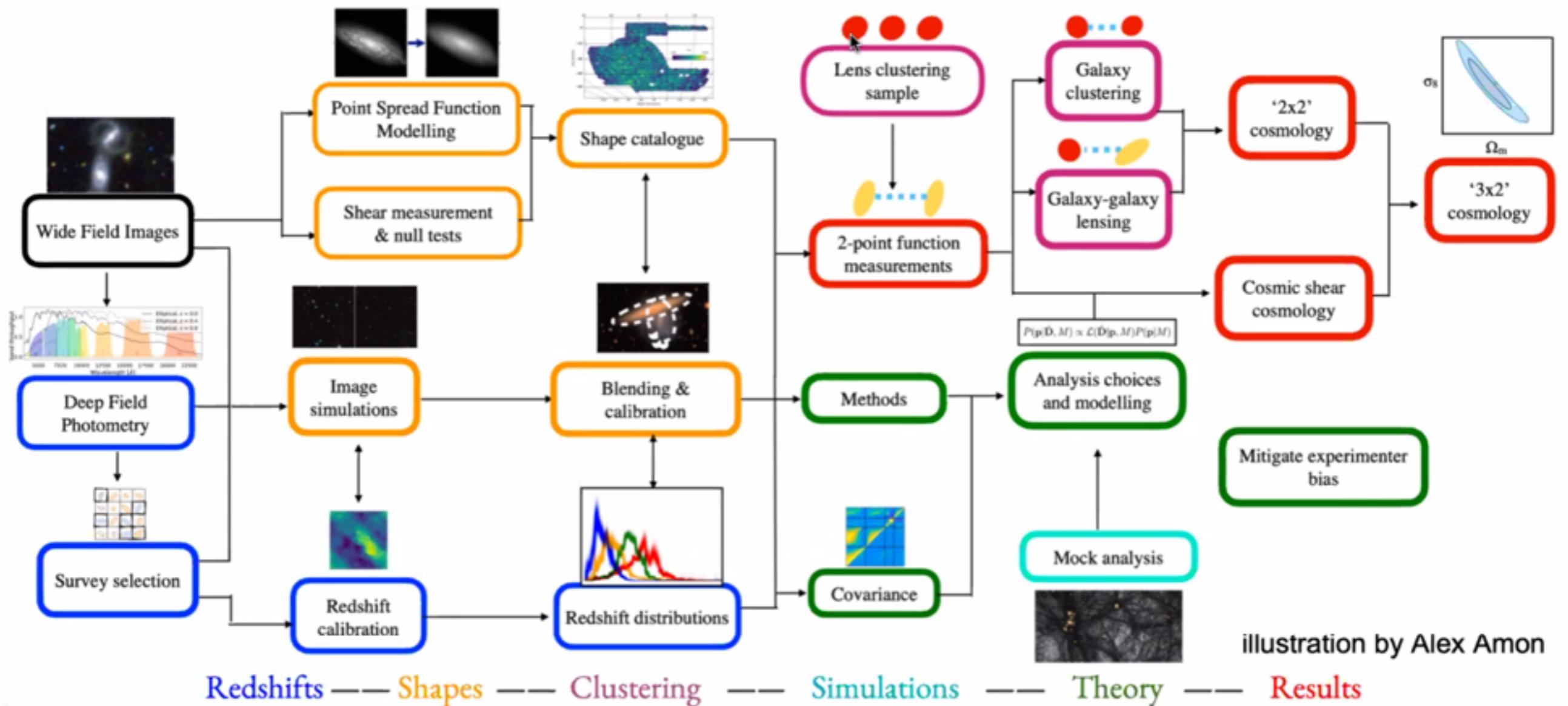


N. Jeffrey; Dark Energy Survey Collaboration



Dark Matter map from DES observations

DES Year 3: pixels to cosmology



Estimadores estadísticos

Dark Energy Survey Year 1 Results: Cosmological Constraints from Galaxy Clustering and Weak Lensing

Galaxy-galaxy lensing: $\gamma_t(\theta)$

Cosmic shear: $\xi_{\pm}(\theta)$

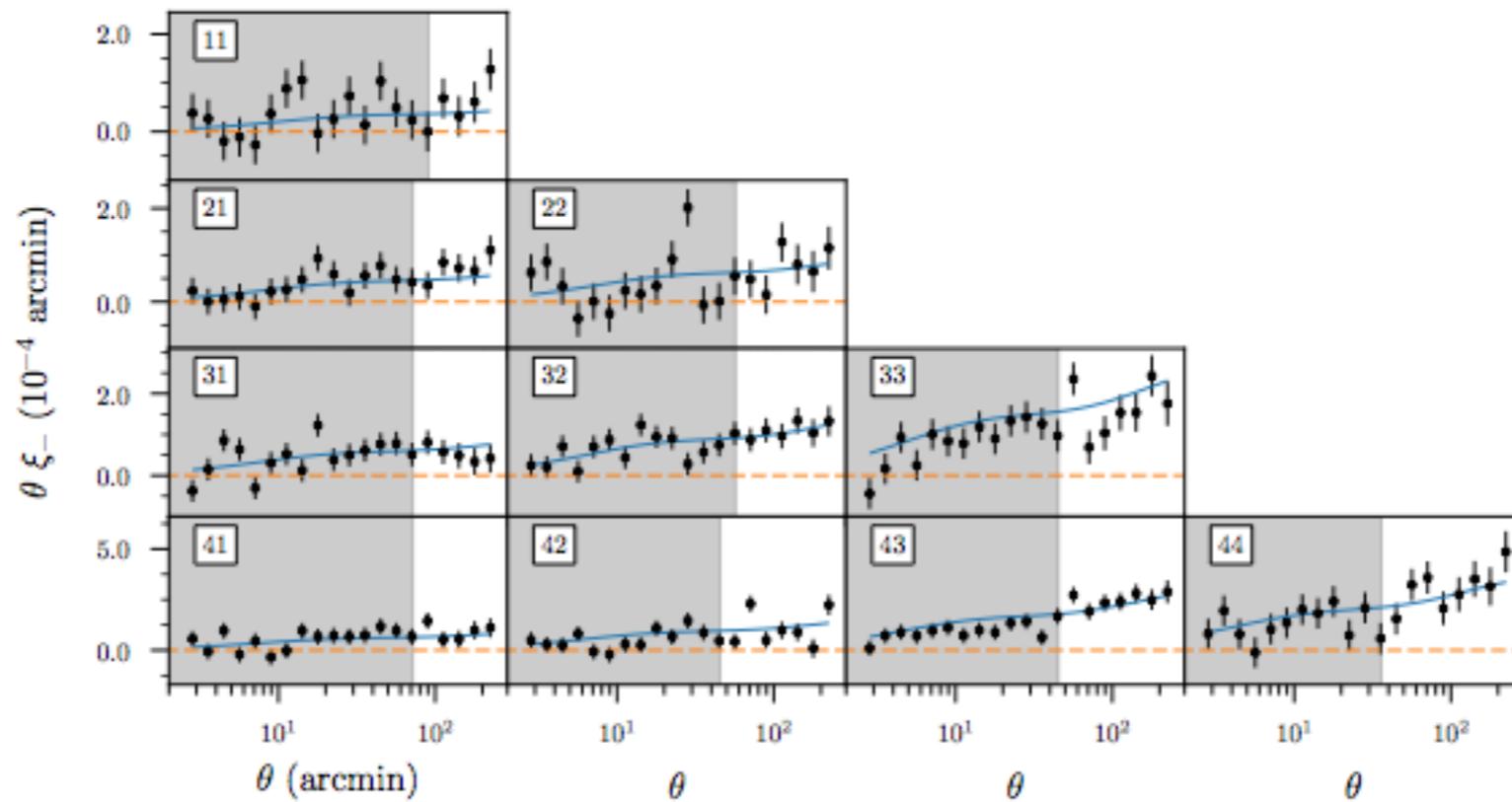
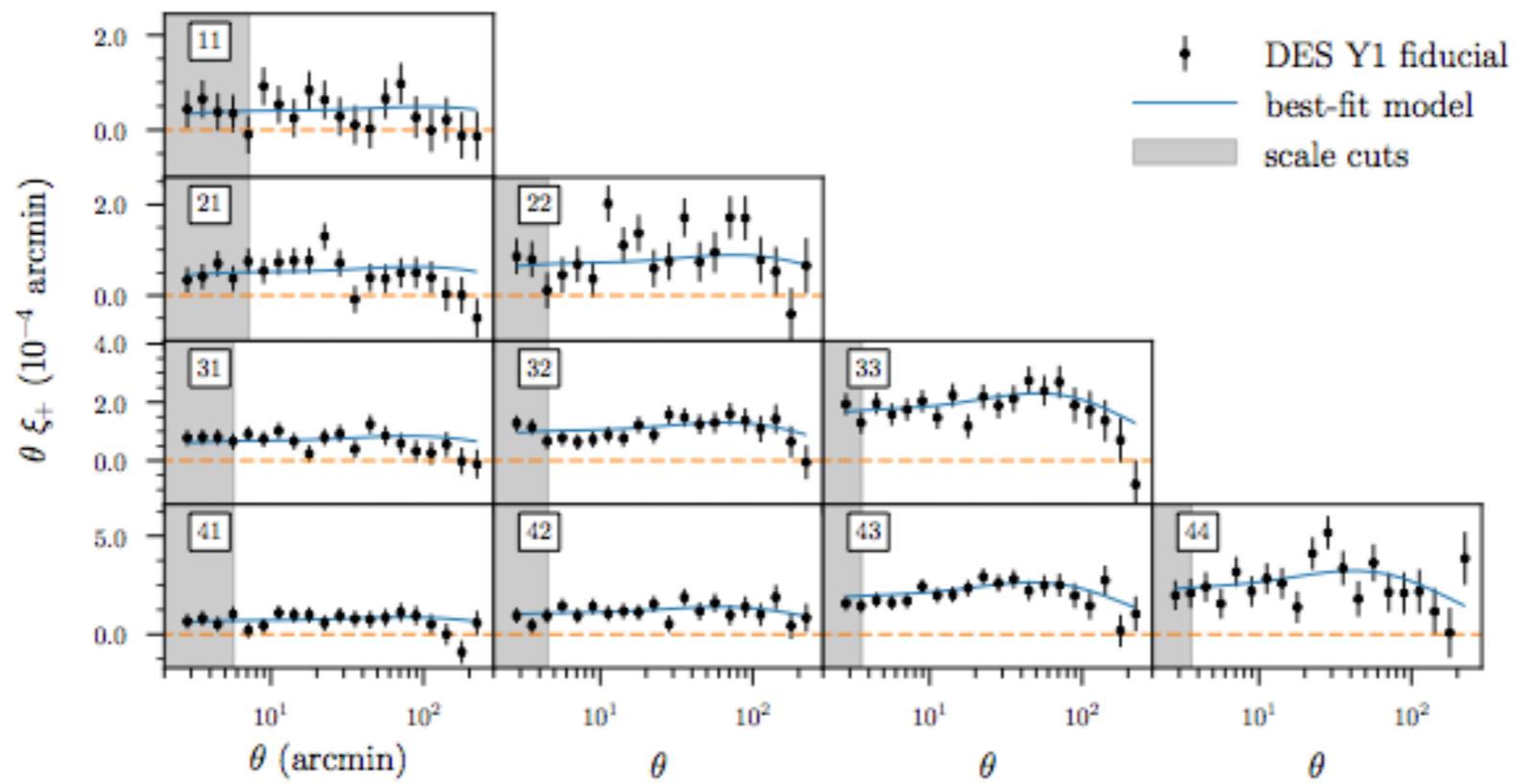
$$\xi_{+/-}^{ij}(\theta) = (1 + m^i)(1 + m^j) \int \frac{dl}{2\pi} l J_{0/4}(l\theta) \\ \int d\chi \frac{q_\kappa^i(\chi) q_\kappa^j(\chi)}{\chi^2} P_{\text{NL}} \left(\frac{l + 1/2}{\chi}, z(\chi) \right)$$

$$q_\kappa^i(\chi) = \frac{3H_0^2 \Omega_m}{2c^2} \frac{\chi}{a(\chi)} \int_\chi^{\chi_h} d\chi' \frac{n_\kappa^i(z(\chi')) dz/d\chi'}{\bar{n}_\kappa^i} \frac{\chi' - \chi}{\chi'}, \quad (\text{IV.3})$$

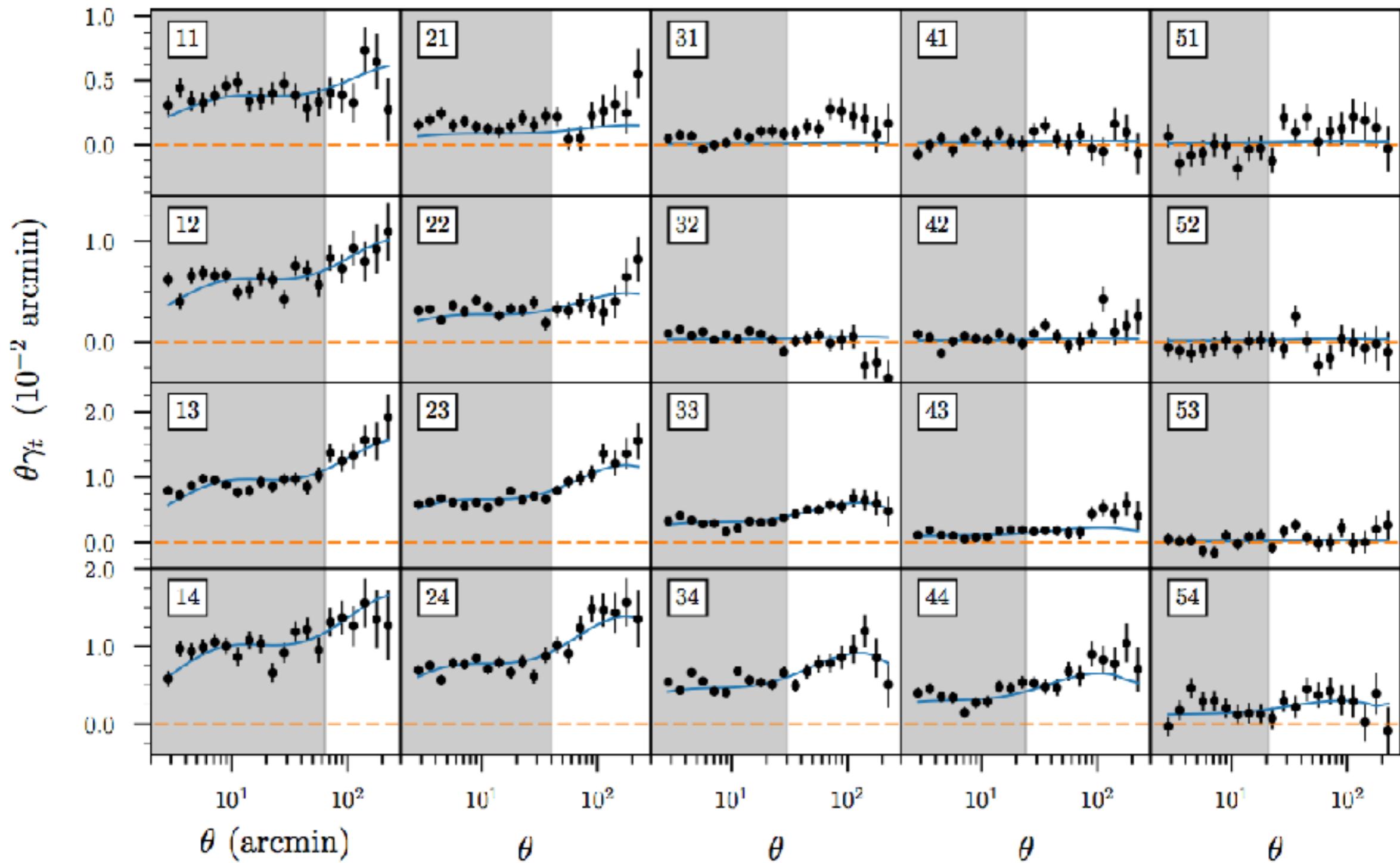
Galaxy Clustering: $w(\theta)$

$$w^i(\theta) = \int \frac{dl}{2\pi} l J_0(l\theta) \int d\chi \frac{q_{\delta_g}^i \left(\frac{l+1/2}{\chi}, \chi \right) q_{\delta_g}^j \left(\frac{l+1/2}{\chi}, \chi \right)}{\chi^2} \\ \times P_{\text{NL}} \left(\frac{l + 1/2}{\chi}, z(\chi) \right) \quad (\text{IV.4})$$

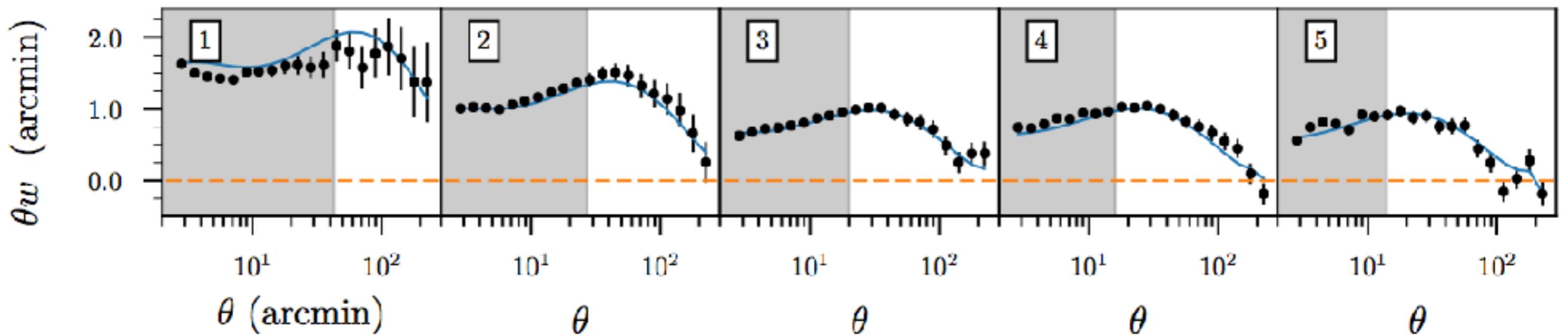
Weak Lensing Tomography



galaxy-shear correlation

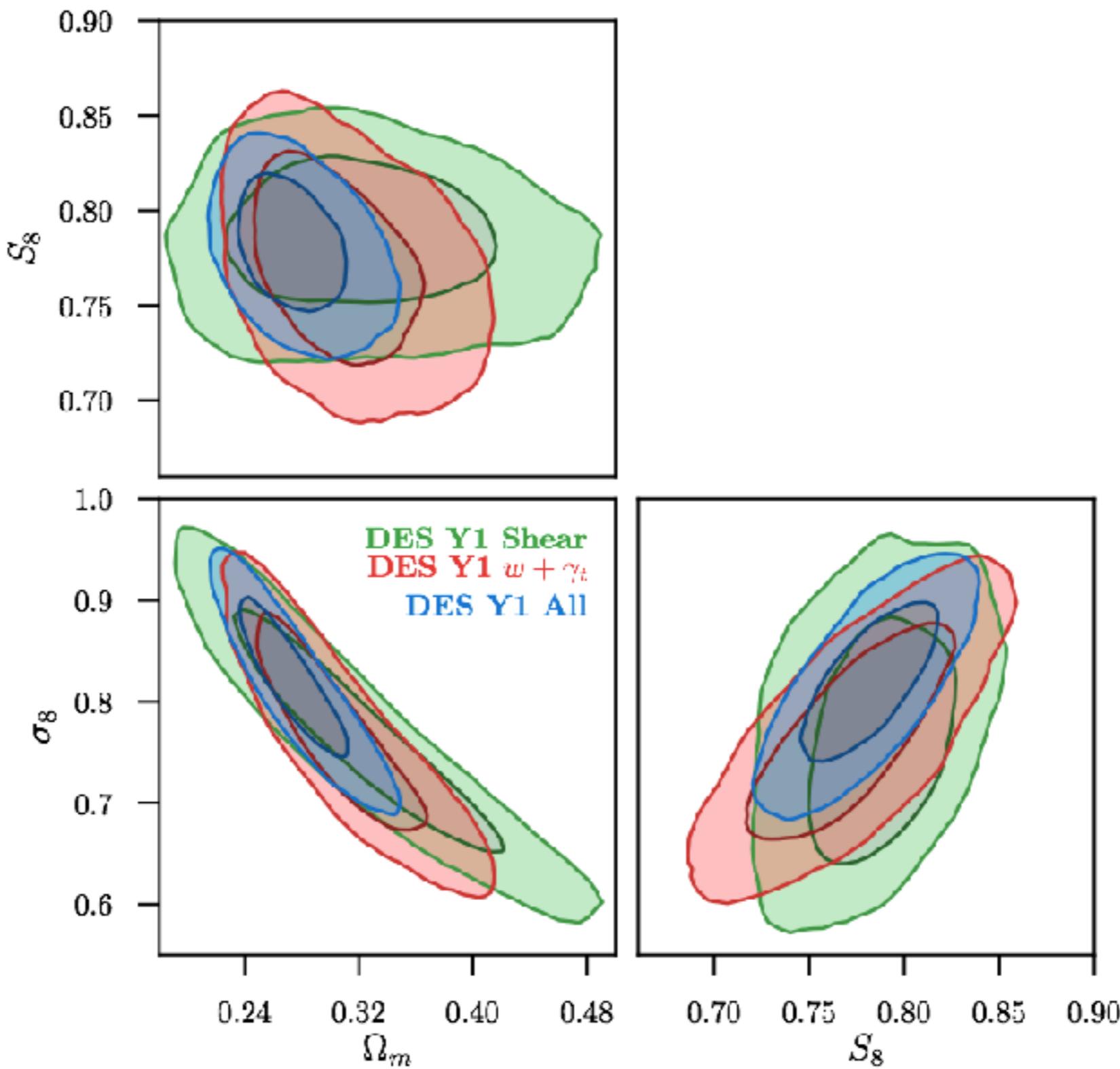


Angular Correlation Function

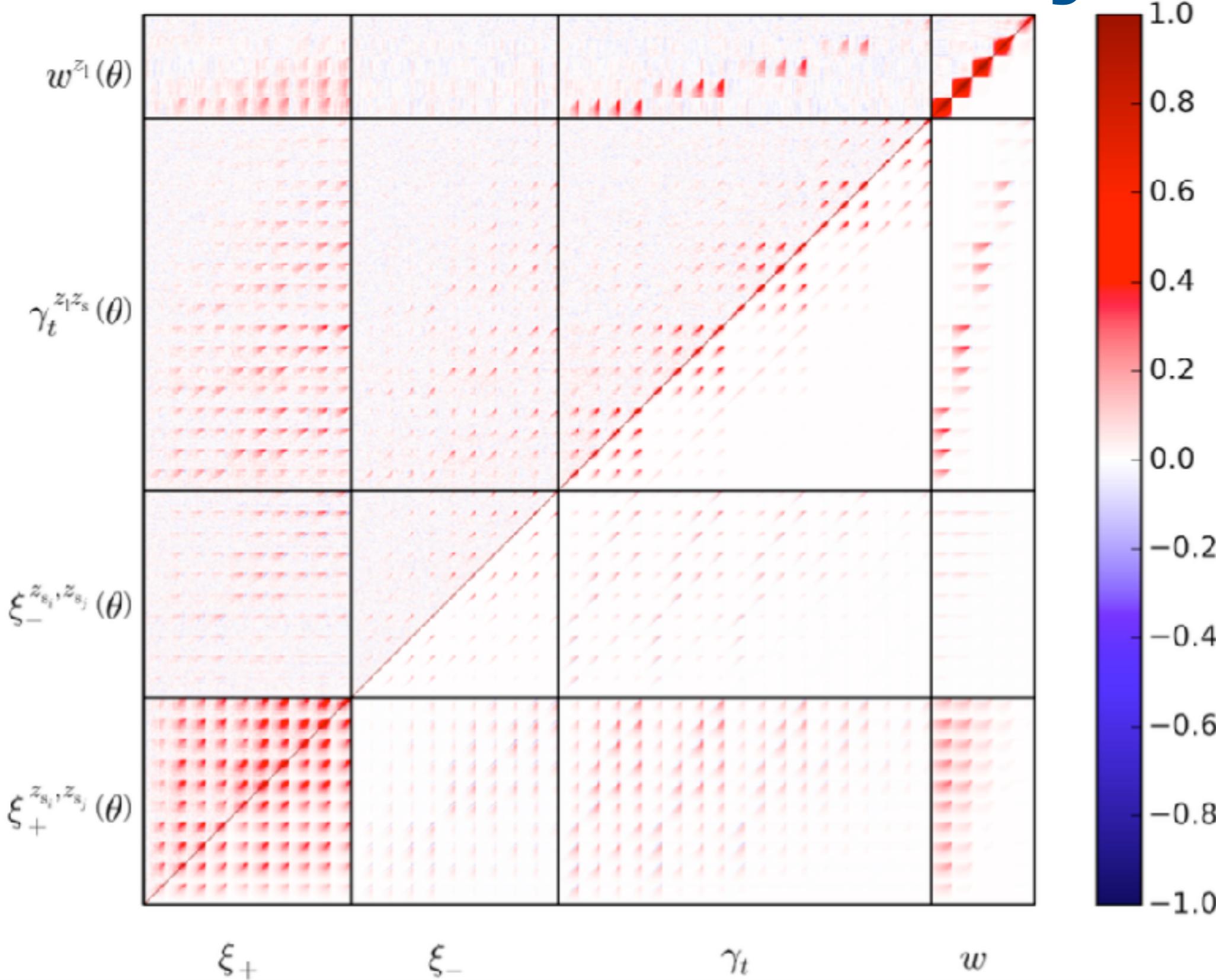


Scaled angular correlation function, $\theta w(\theta)$, of redMaGiC galaxies in five redshift bins

Consistency of the Probes



Combined Analysis



457×457
element
covariance
matrix!

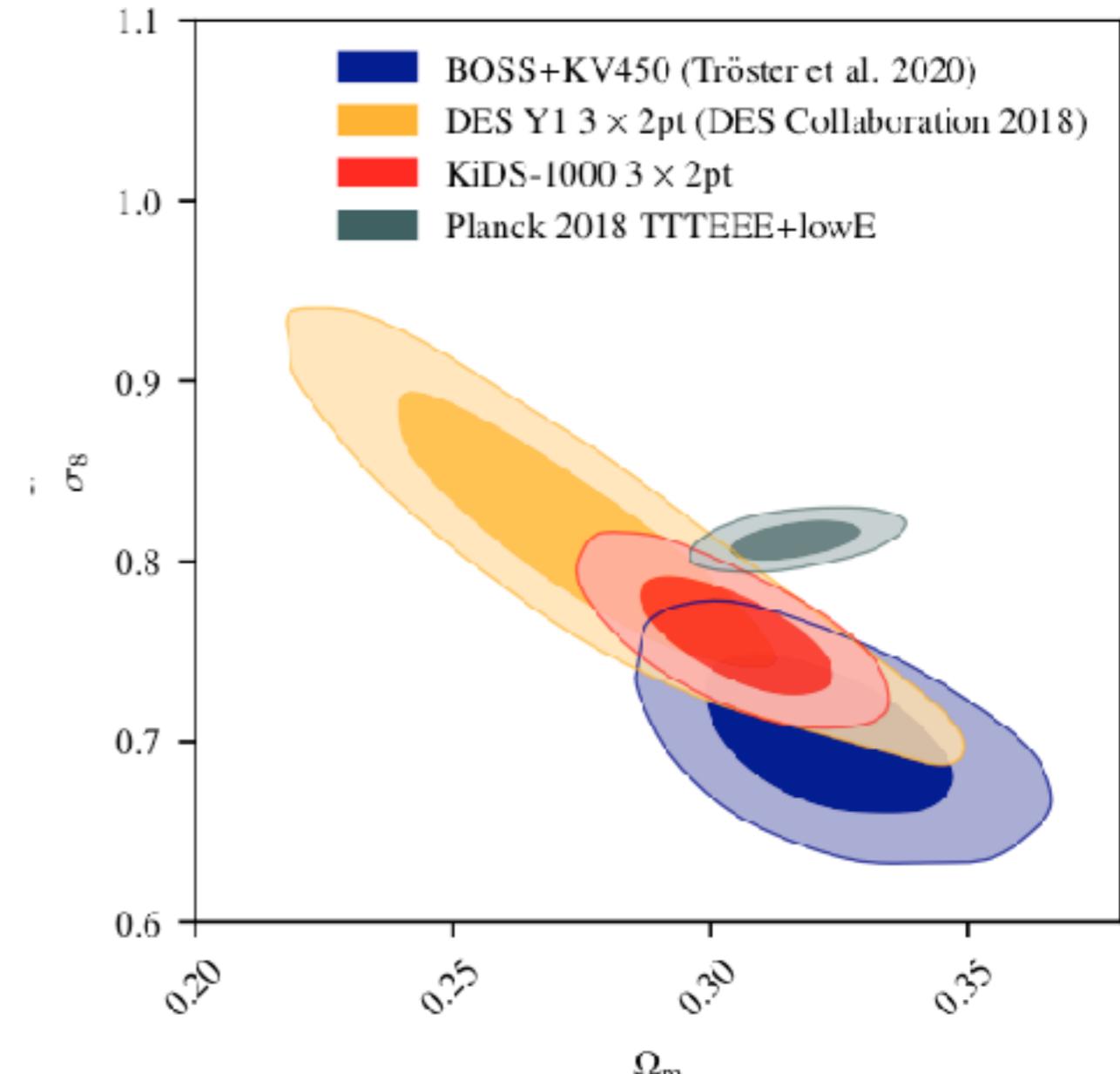
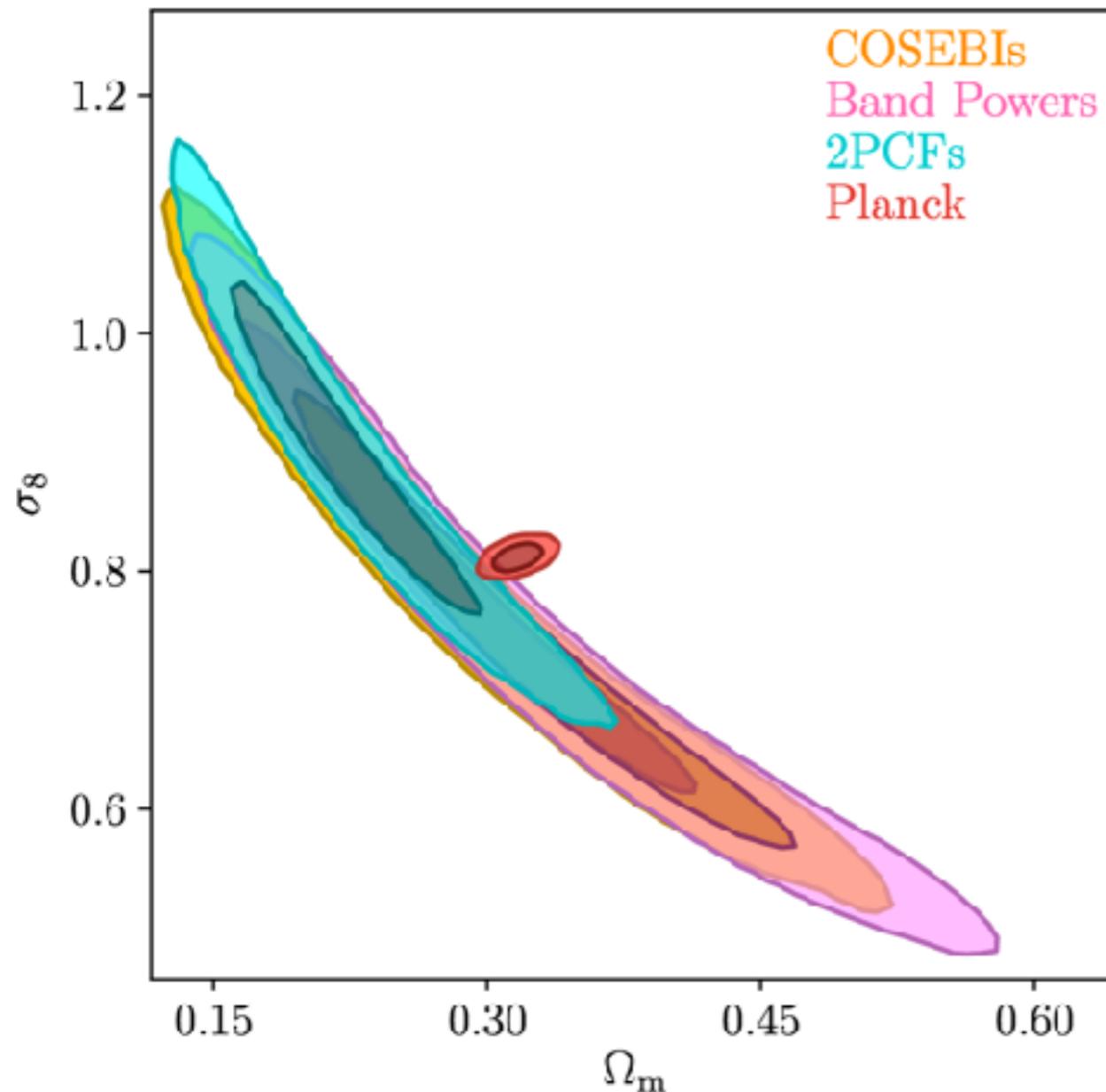
20 “nuisance
parameters”
+ 6/7
cosmological

KILO DEGREE SURVEY

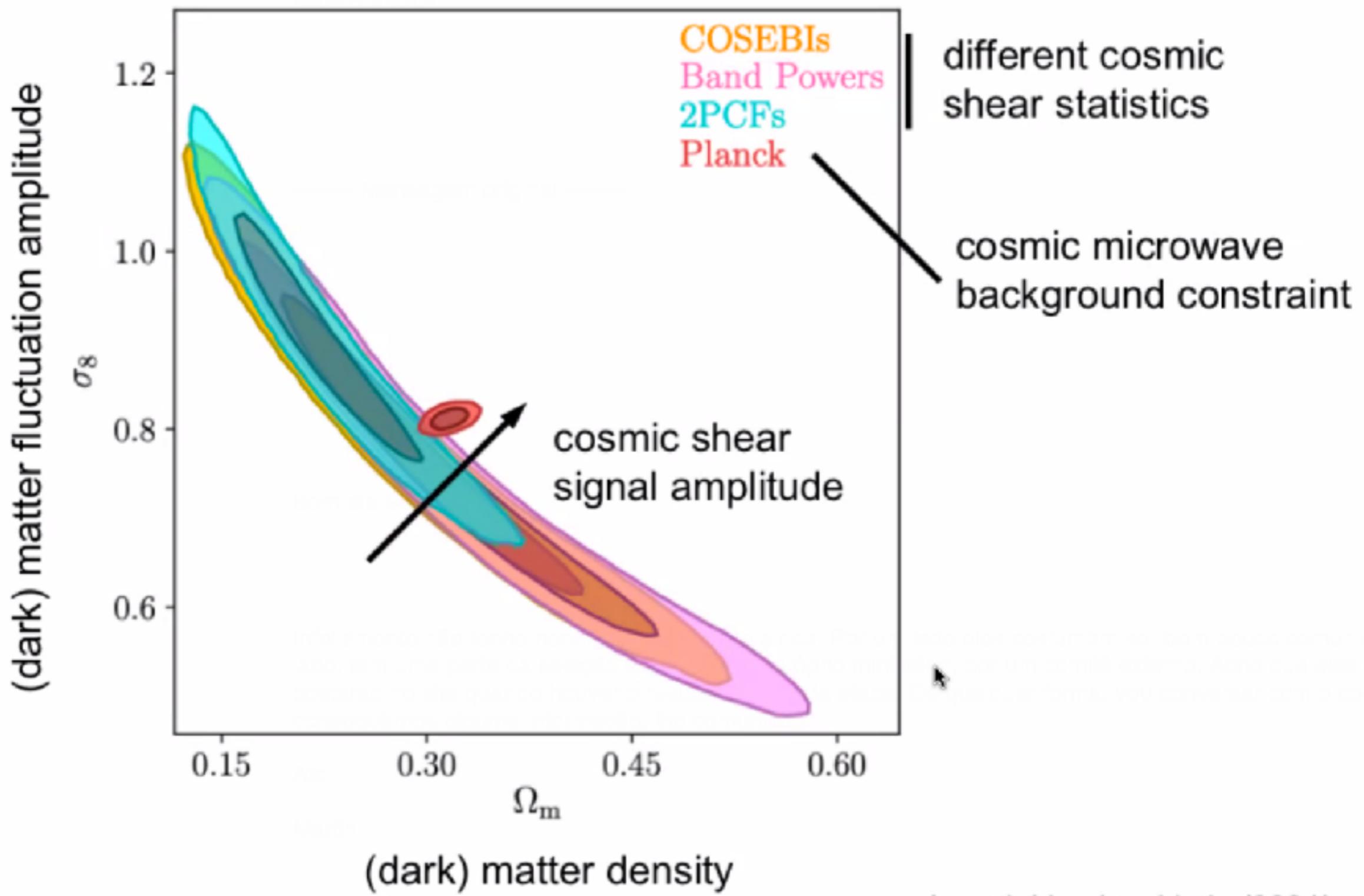
KiDS: 1000 sq-deg

http://kids.strw.leidenuniv.nl/KiDS-1000_preprints.php

Cosmological parameter constraints from tomographic weak gravitational lensing

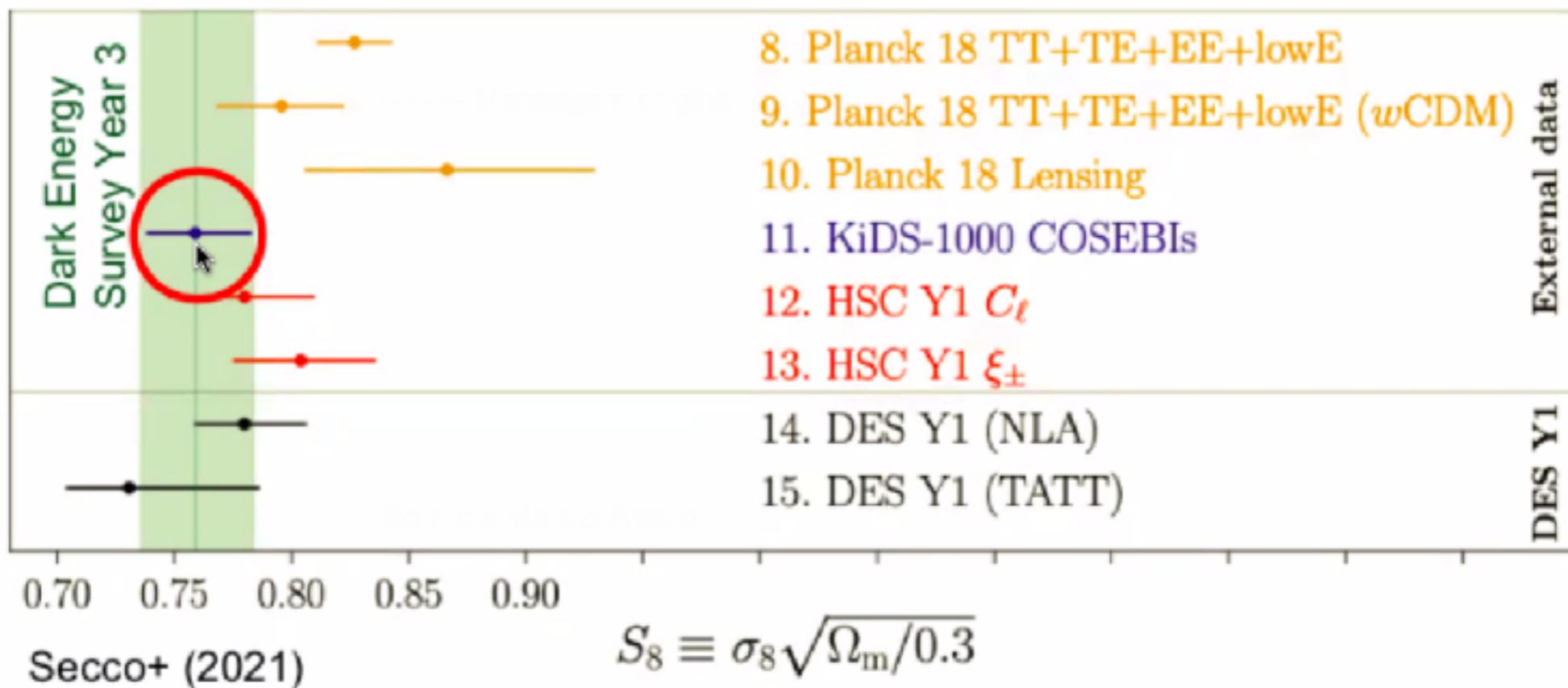


Cosmic shear cosmological constraints



Asgari, Lin, Joachimi+ (2021)

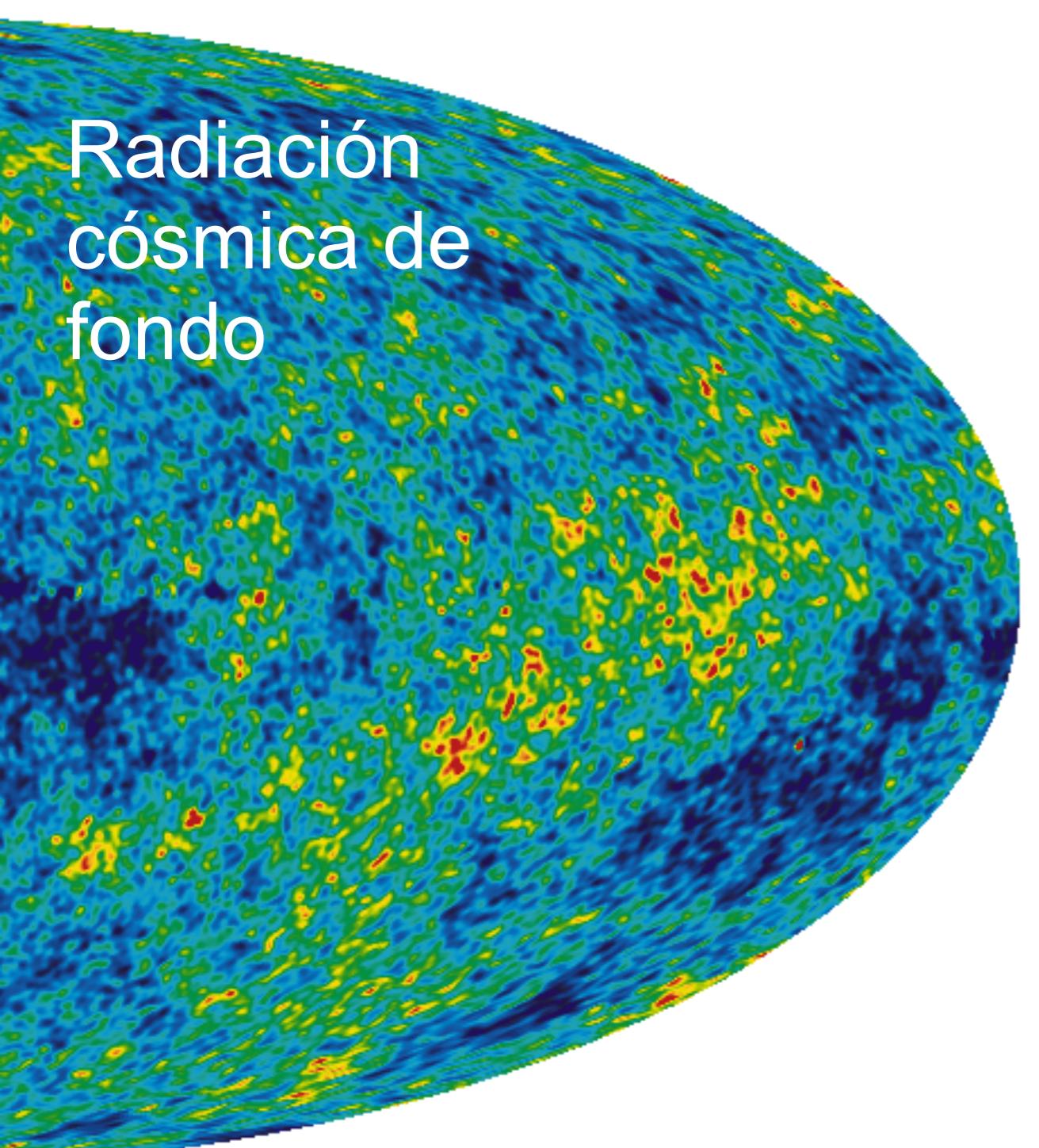
Consistency with DES



Cosmic shear agreement with nearly independent DES Year 3 results is excellent.

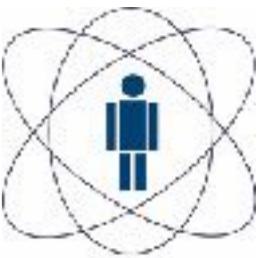
Problema con S_8 que se fue intensificando con el tiempo!

¿Cómo se formaron las estructuras?



Fluctuaciones $\sim 10^{-4}$

Fluctuaciones $\sim 10^3$

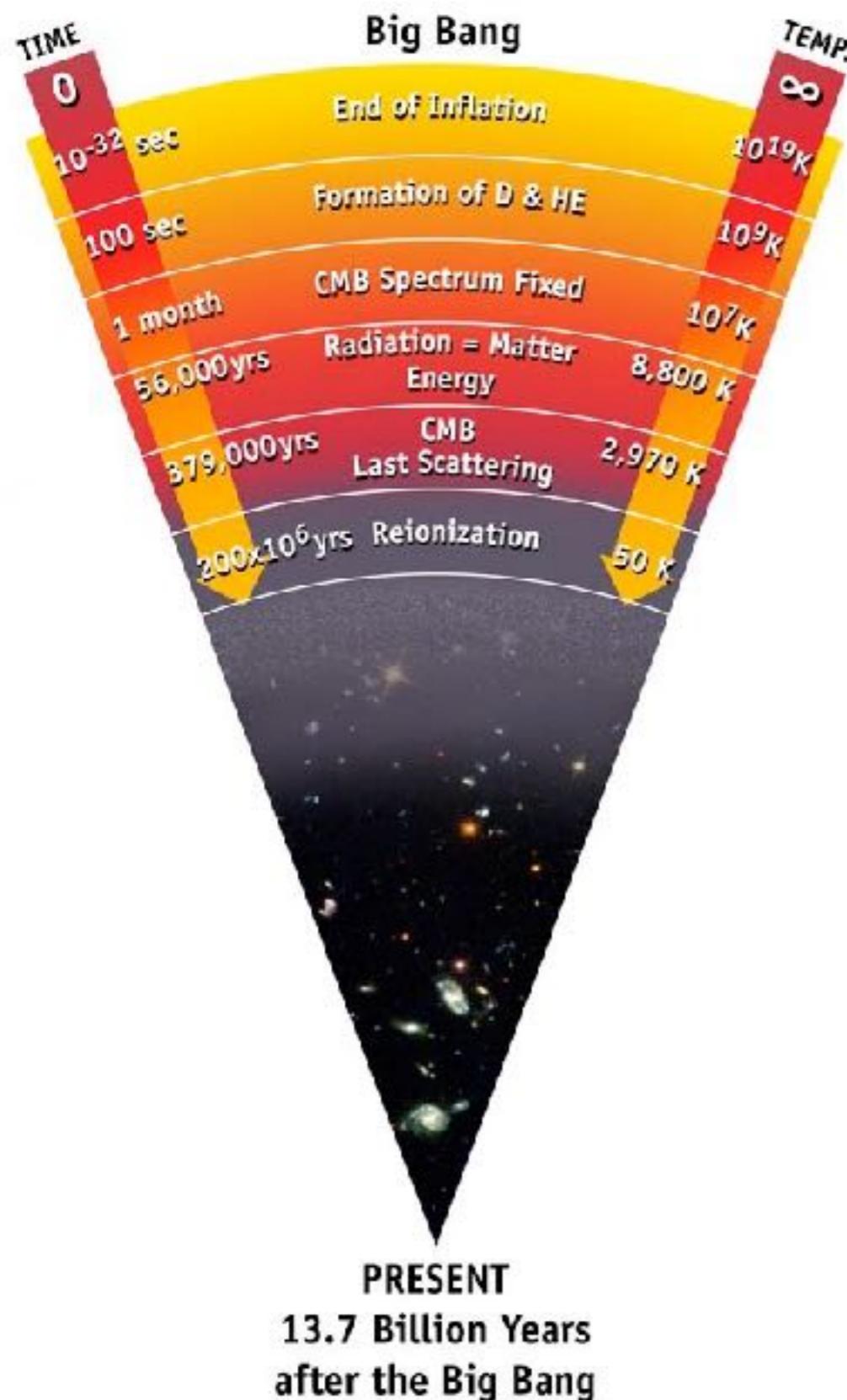


Some Milestones in the History of the Universe

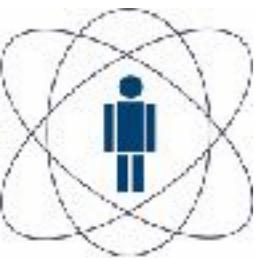
kT (radiation)	Event
2×10^{-4} eV	Today
10^{-3} eV	Galaxy formation
0.26 eV	H recombination (matter-radiation decoupling)
10 eV	Matter domination
300 keV	Formation of light elements (He^4 , He^3 , D e Li) (primordial nucleosynthesis)
0.5 MeV	End of leptonic era ($e^+ e^-$ annihilation)
100 MeV	End of hadronic era and beginning of leptonic era (hadronization, annihilation hadron anti-hadron)
1000 GeV	Electroweak phase transition
10^{15} GeV	Bariosynthesis? Great Unification?
10^{19} GeV	End of quantum era? Inflation?

↑ time

Endireitando a seta do tempo...



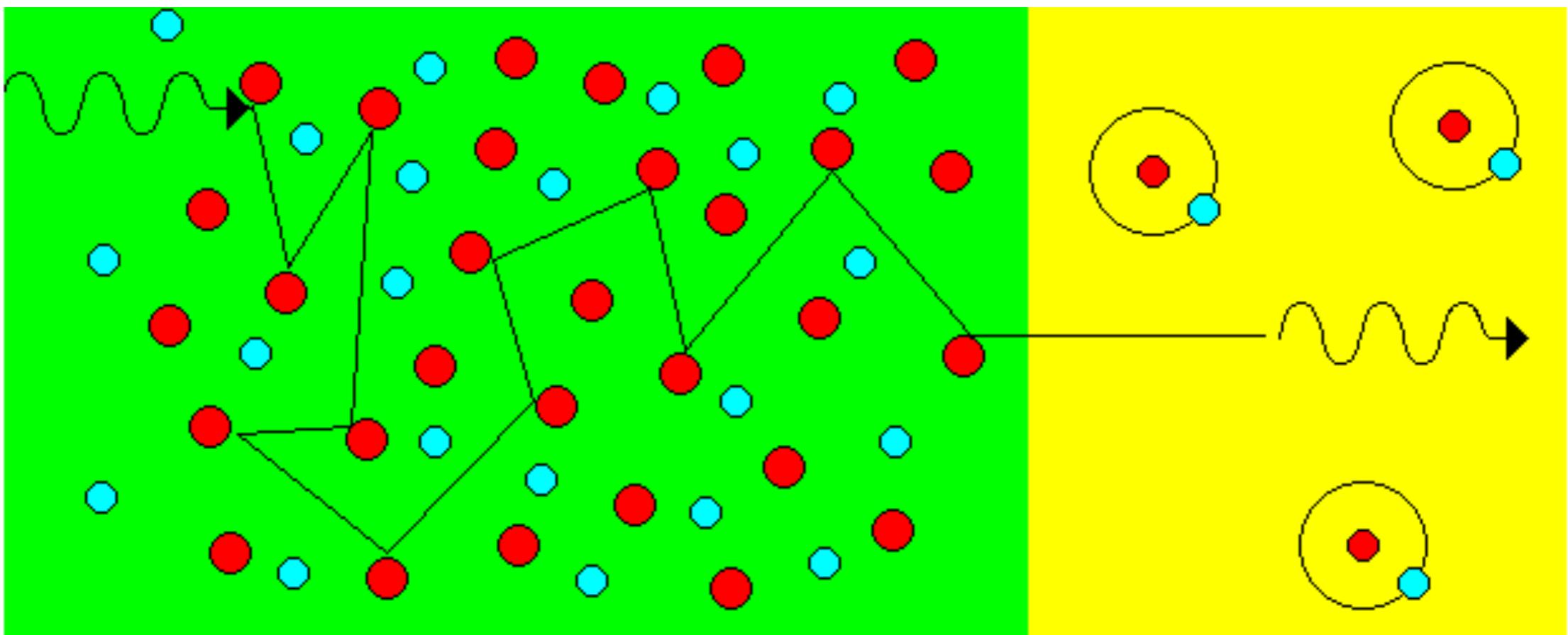
- Fase primordial densa e quente
→ “*Big-Bang*”
- Processos físicos:
 - ◎ inflação, PBH, quebras de simetria
 - ◎ bariogênese, leptogênese
 - ◎ Hadronização
 - ◎ Aniquilação matéria-anti-matéria...
 - ◎ Síntese dos elementos leves
 - ◎ Recombinação
 - ◎ Formação de estruturas
- Fósseis:
 - Ondas grav. primordiais, PBH
 - Fundo cósmico de neutrinos
 - Abundância primordial
 - Radiação cósmica de fundo
 - Estruturas no Universo



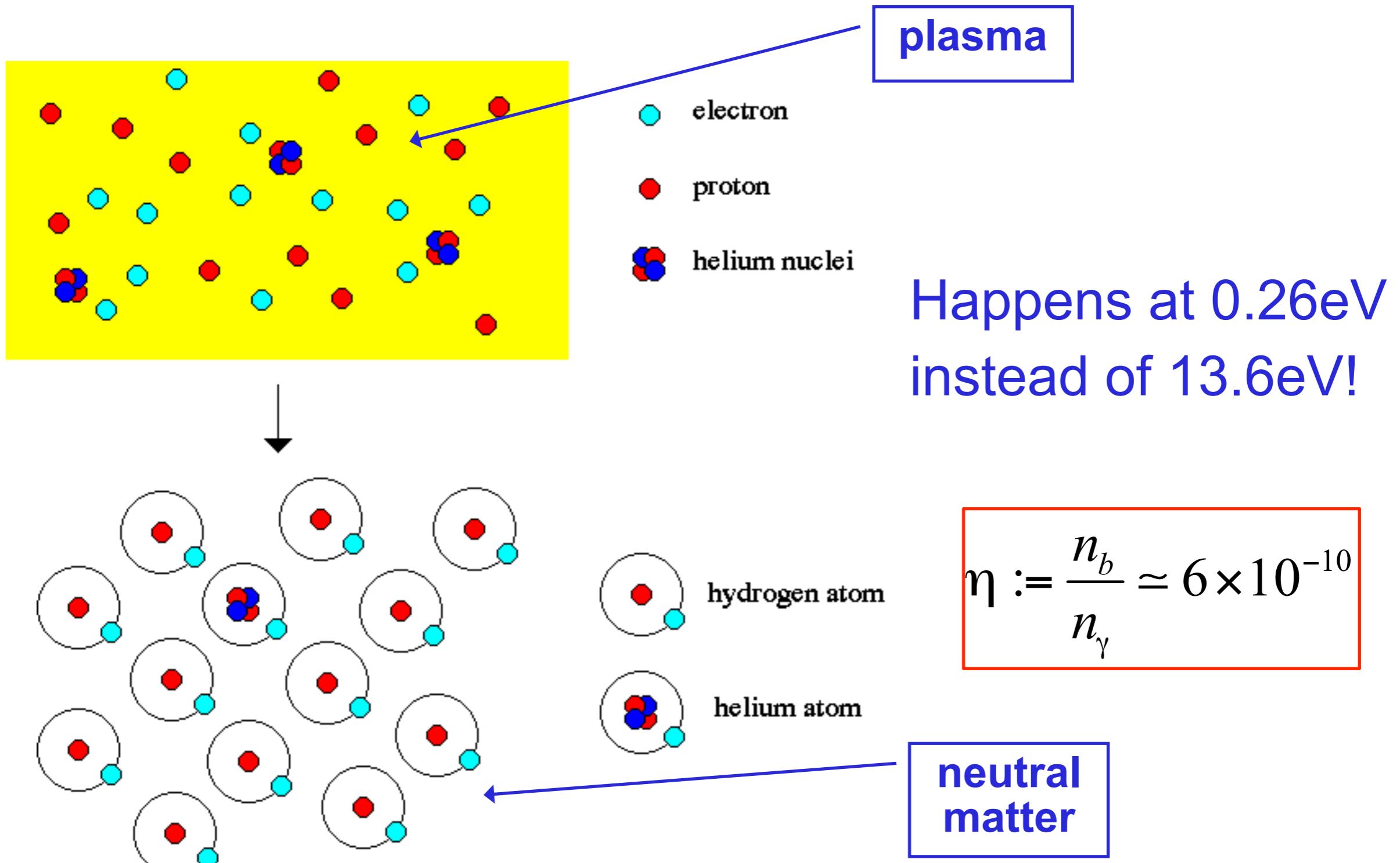
Recombination

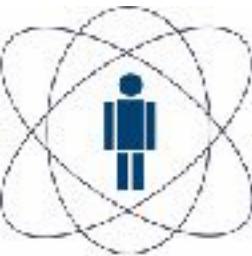
When the temperature drops below 3.000K
the electrons get bounded to the nuclei

- The Universe becomes transparent
- Light propagates freely



Recombination

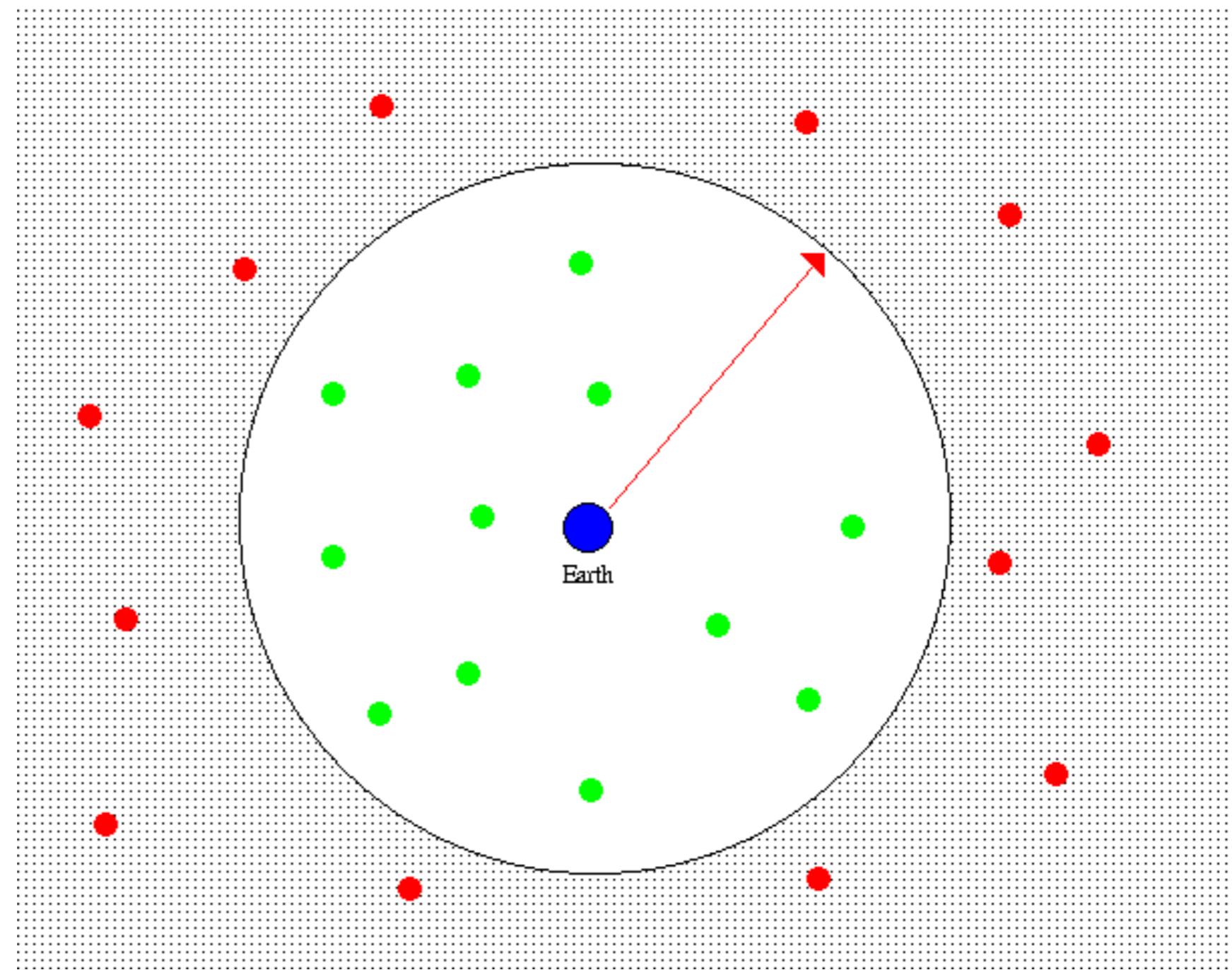


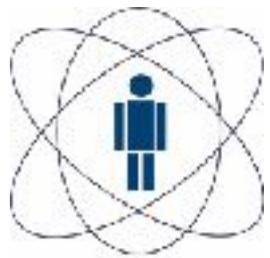


The Cosmic Microwave Background

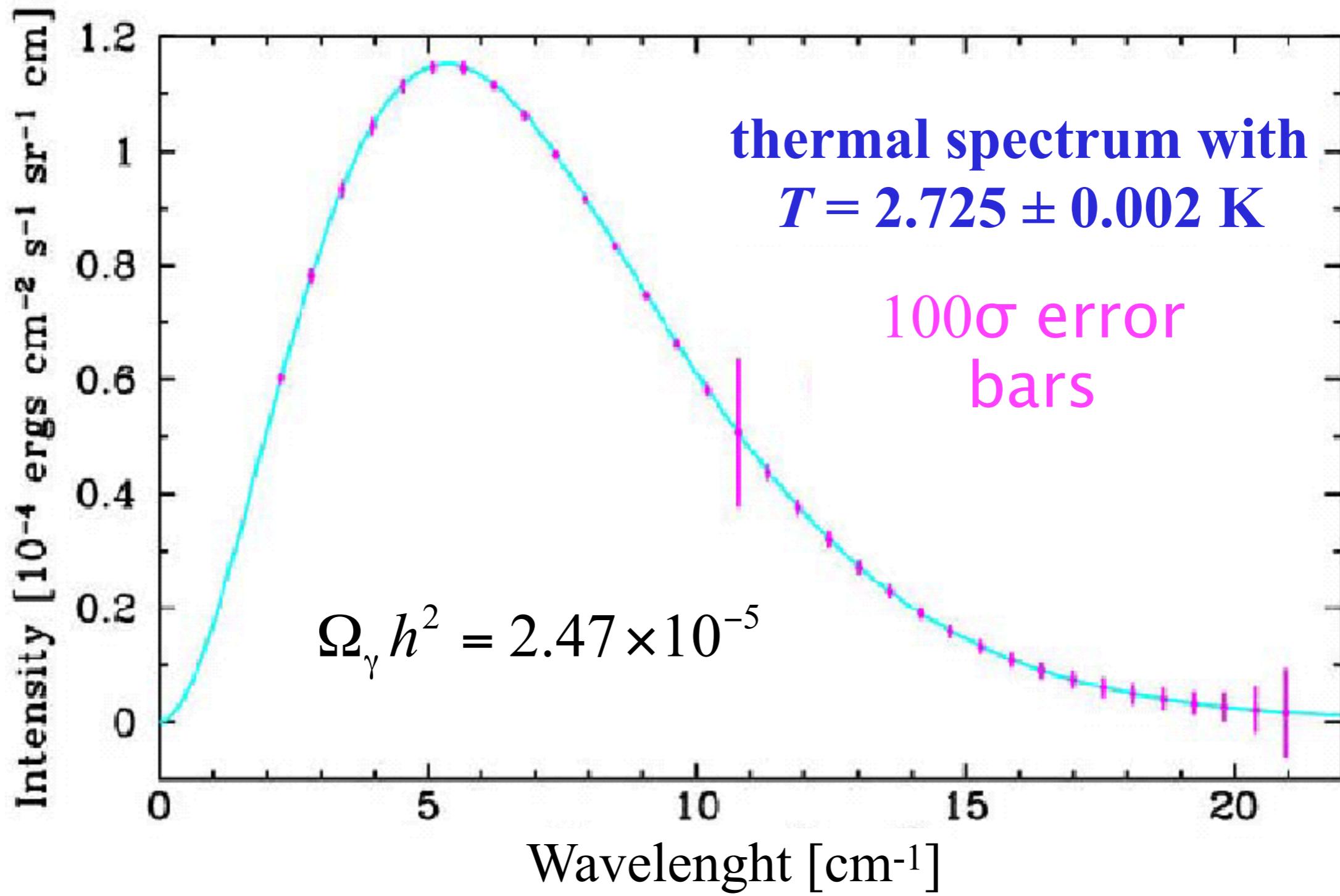
“Photosphere”
seen from our
galaxy

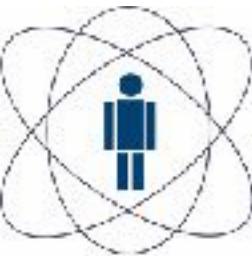
Expect to see:
Black body with
redshift
 $z \sim 1000$





Spectrum of the CMB





Cosmic Microwave Background I

Cosmic Microwave Background Radiation Map

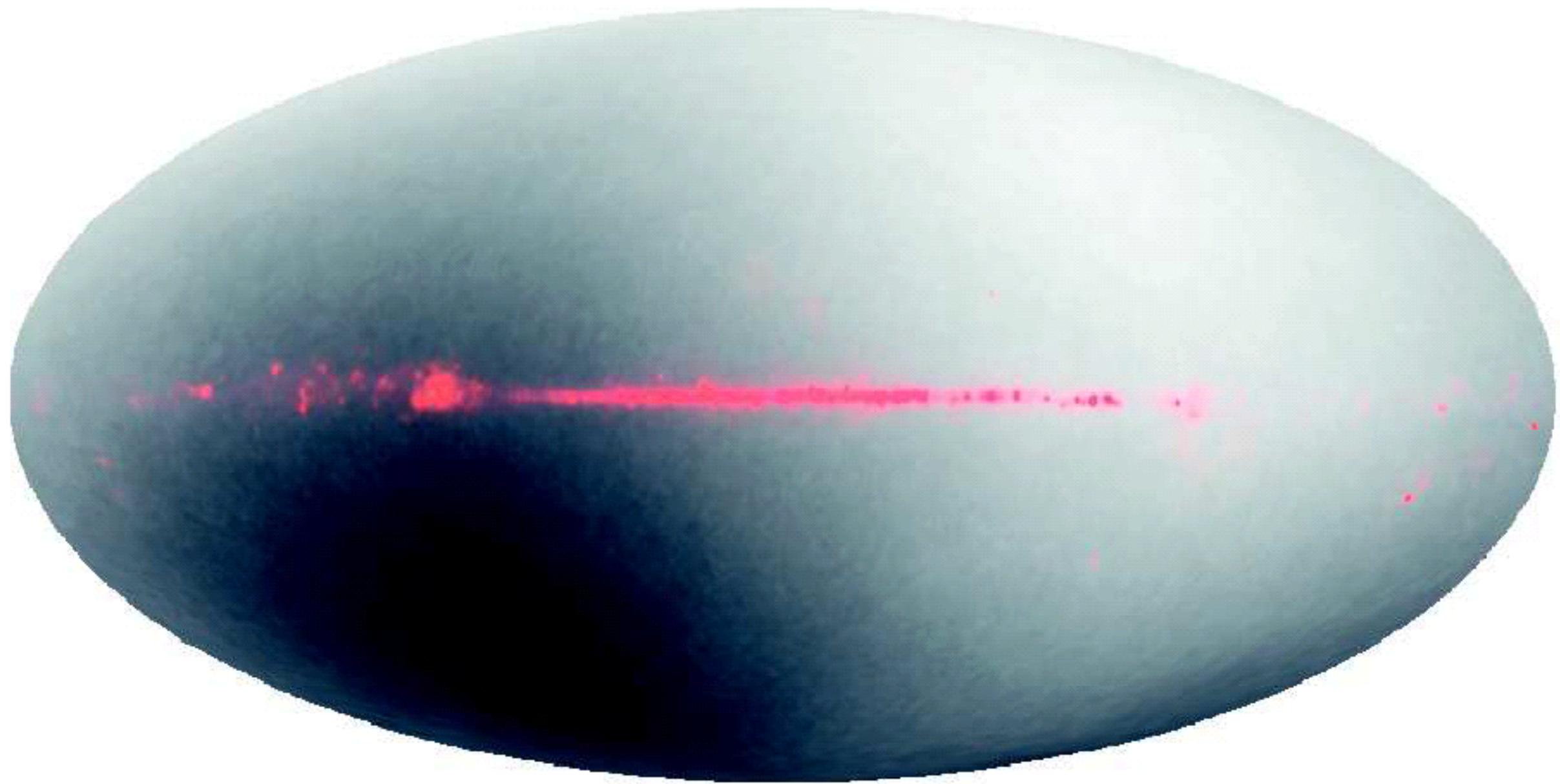


Contrast: 1x

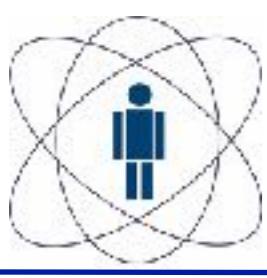


400x

- Epoch of matter and radiation decoupling (about 380.000 years after the “*Big-Bang*”).
- $T_0 = 2.725 \pm 0.002$. Redshift, $z = 1089$.
- Highly homogeneous primordial Universe
- Dipole: $\Delta T = 3.346 \pm 0.017$ mK $\rightarrow v_{\text{gal}} = 360$ Km/s



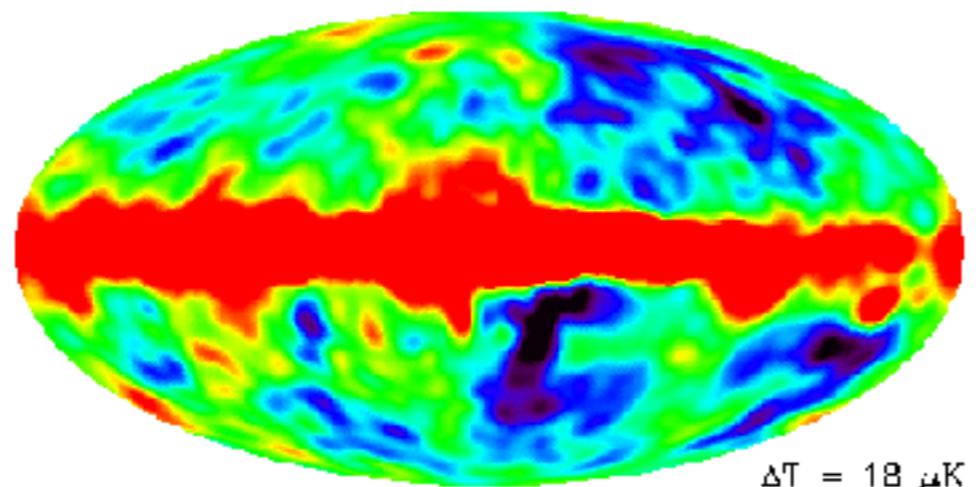
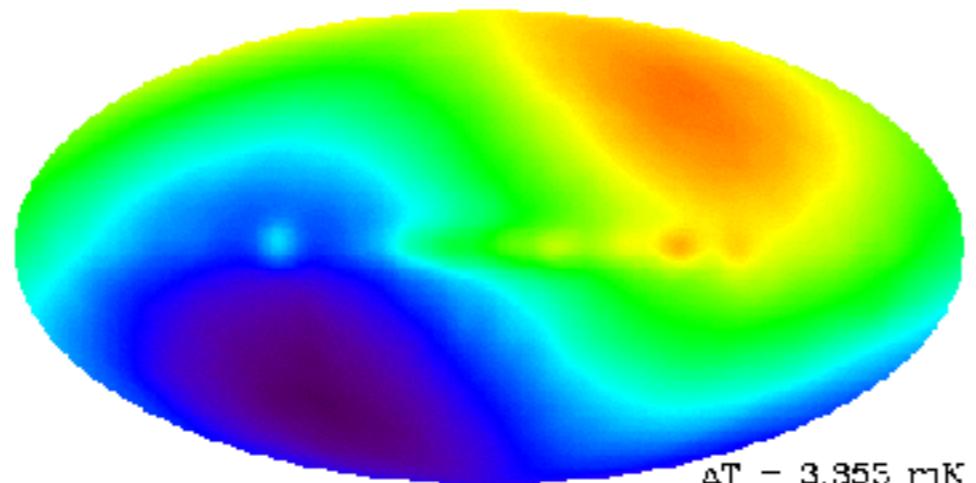
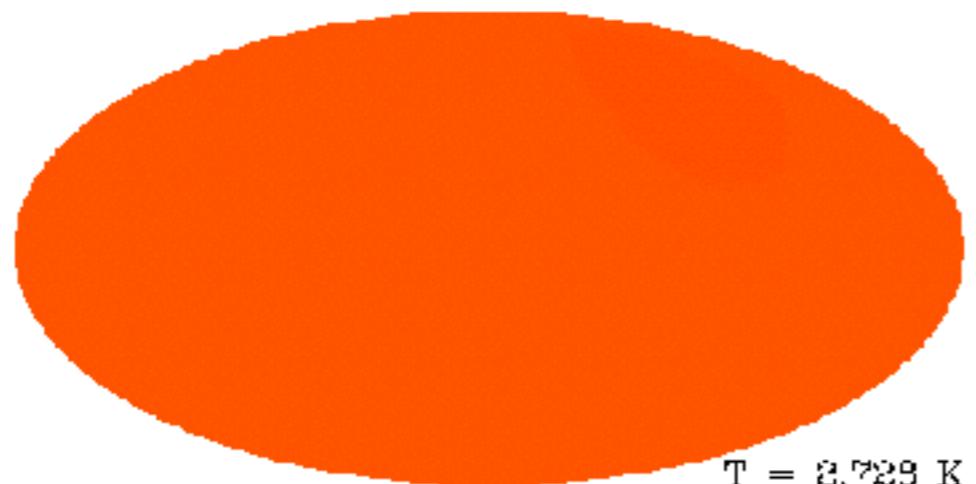
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The Perturbed Universe

CMB anisotropies and
Large-scale Structure

Anisotropies in the Cosmic Background



- $T_0 = 2.725 \pm 0.002$.
redshift, $z = 1089$
- The primordial Universe
was highly homogeneous
- Dipole:
 $\Delta T = 3.346 \pm 0.017 \text{ mK}$
 $\Rightarrow v_{\text{gal}} = 360 \text{ Km/s}$
- Temperature fluctuations:

$$\frac{\Delta T}{T} \approx 10^{-5}$$

Nobel prizes: 1978, 2006

- Principais resultados do satélite COBE:
 - Corpo negro: natureza térmica da RCF
 - Anisotropias → sementes das estruturas em grandes escalas

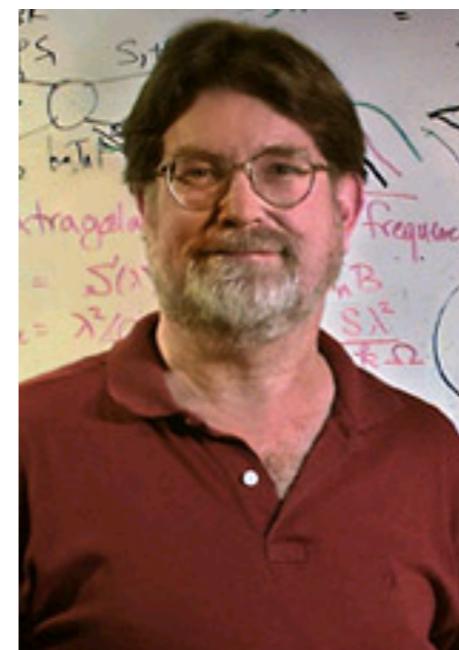


KUNG.
VETENSKAPS AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES



John C. Mather

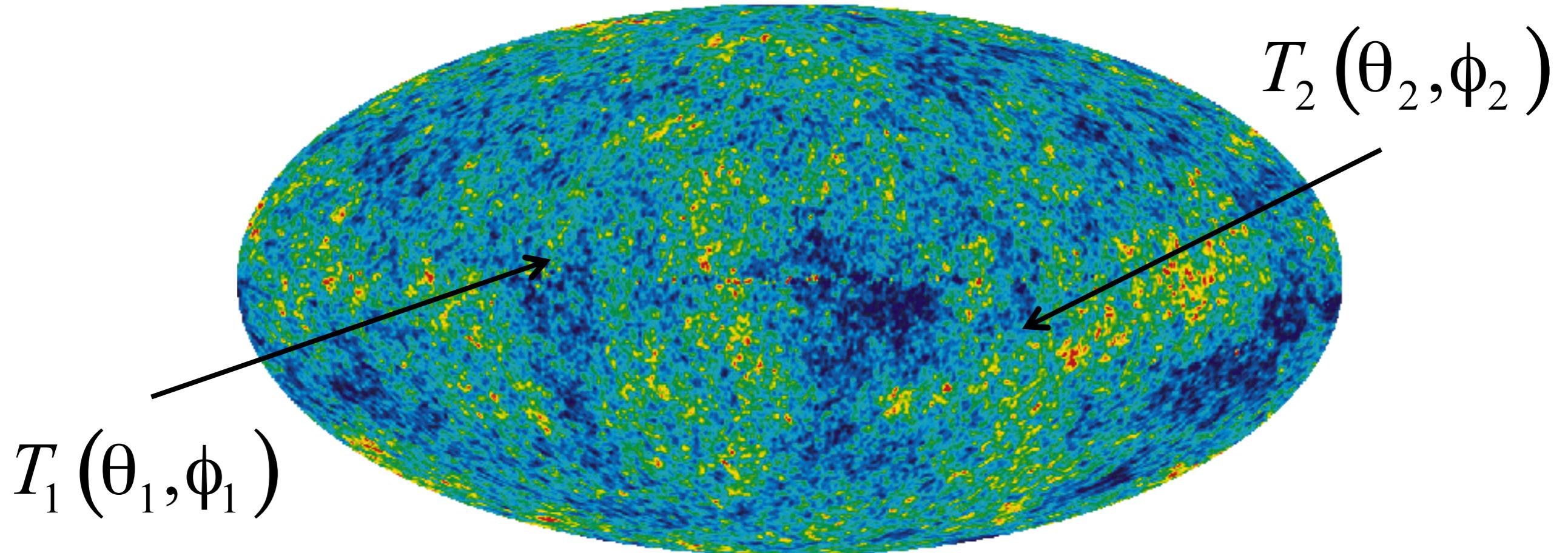
Prêmio Nobel
2006



George F. Smoot



Cosmic Microwave Background Power Spectrum

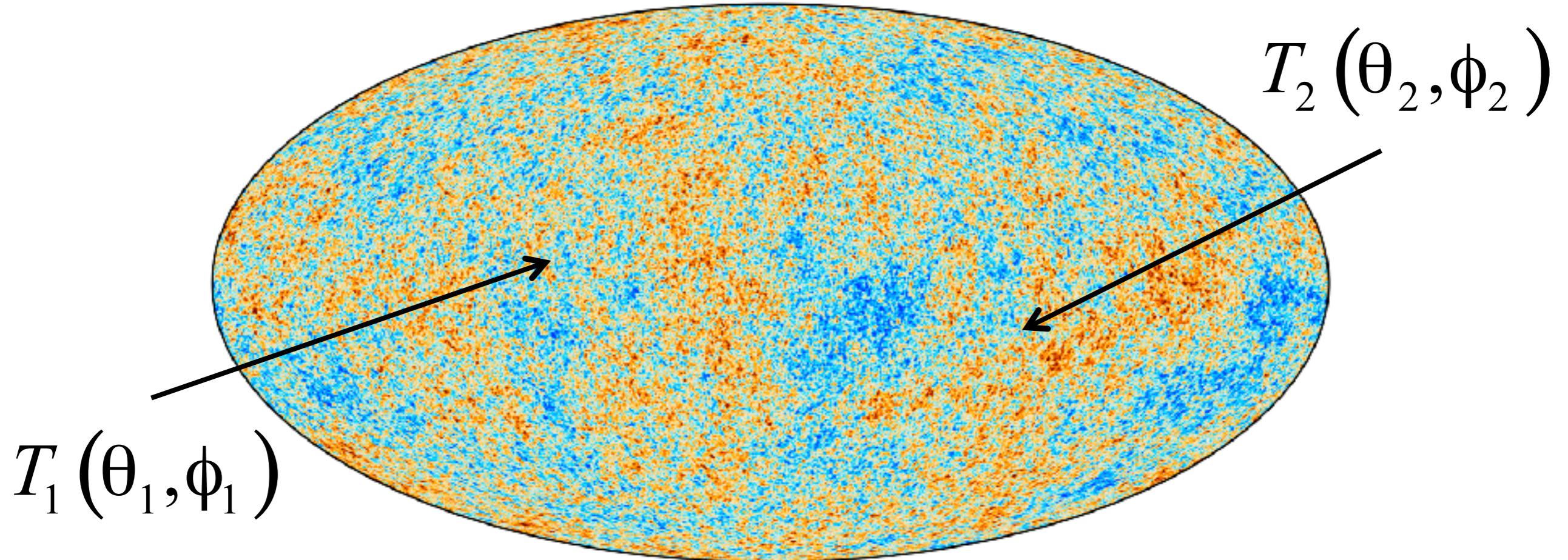


$$\langle T_1 T_2 \rangle = \sum a_{lm} Y_{lm}(\theta, \phi)$$

$$\left\langle |a_{lm}|^2 \right\rangle^{1/2} \equiv C_l$$

WMAP2008

Cosmic Microwave Background Power Spectrum



$$\langle T_1 T_2 \rangle = \sum a_{lm} Y_{lm}(\theta, \phi)$$

$$\left\langle |a_{lm}|^2 \right\rangle^{1/2} \equiv C_l$$

Planck 2015

Relativistic perturbation theory

- The starting point is the perturbed Robertson-Walker metric

$$\begin{aligned} ds^2 &= \left[g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} \right] dx^\mu dx^\nu \\ &= a^2(\tau) \left[-d\tau^2 + \gamma_{ij}(\vec{x}) dx^i dx^j + h_{\mu\nu}(\vec{x}, \tau) dx^\mu dx^\nu \right], \end{aligned}$$

- At the linear level modes decouple
- Scalar perturbations:

$$ds^2 = a^2(\tau) \left[-(1 + 2\Psi) d\tau^2 + (1 - 2\phi) \gamma_{ij} dx^i dx^j \right]$$

(for a perfect fluid $\Phi = \Psi$)

Results from the linear analysis

Need for Dark Matter

- Baryonic matter: can only cluster after decoupling
 $t_{\text{dec}} \sim 380.000$ years
and for $r > \lambda_J$ (Jeans length)
- Cold Dark Matter starts clustering at $t_{\text{eq}} \sim 56.000$ years (matter-radiation equality)
- Baryons follow the Dark Matter potential wells
- Silk damping decreases the amplitude of baryon perturbations

CMB Anisotropies

● Primary Anisotropies

$$\frac{\Delta T}{T} = \left[\Psi - \hat{r} \cdot \vec{v} + \frac{1}{3} \delta_m \right] \eta_{ls}$$

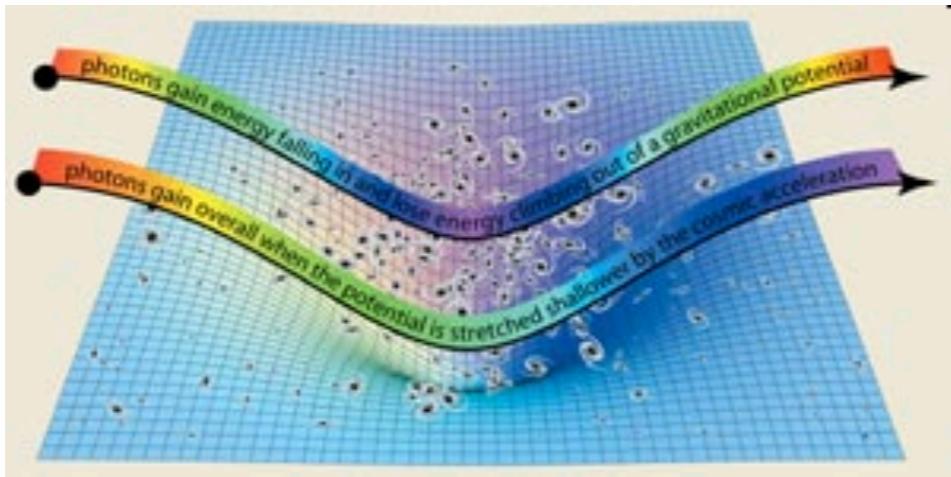
↑ ↑ ↘
Sachs-Wolfe Doppler Intrinsic temperature
effect effect fluctuation

- On large scales
→ Sachs-Wolfe plateau
- On small scales:
baryon acoustic oscillations

$$\delta_m \simeq 2\Psi$$

Integrated Sachs-Wolfe Effect

- Cumulative effect of gravitational shifts



<http://physicsworld.com/cws/article/print/19419>

$$\frac{\delta T_{ISW}}{T} = 2 \int_{\tau_{dec}}^{\tau_0} \frac{\partial \phi}{\partial \tau} d\tau$$

- Linear evolution, $\Omega_M = 1 \rightarrow \phi = \text{const.}$
- Late effect:
 - Correlation between the cosmic microwave background and large-scale structure!

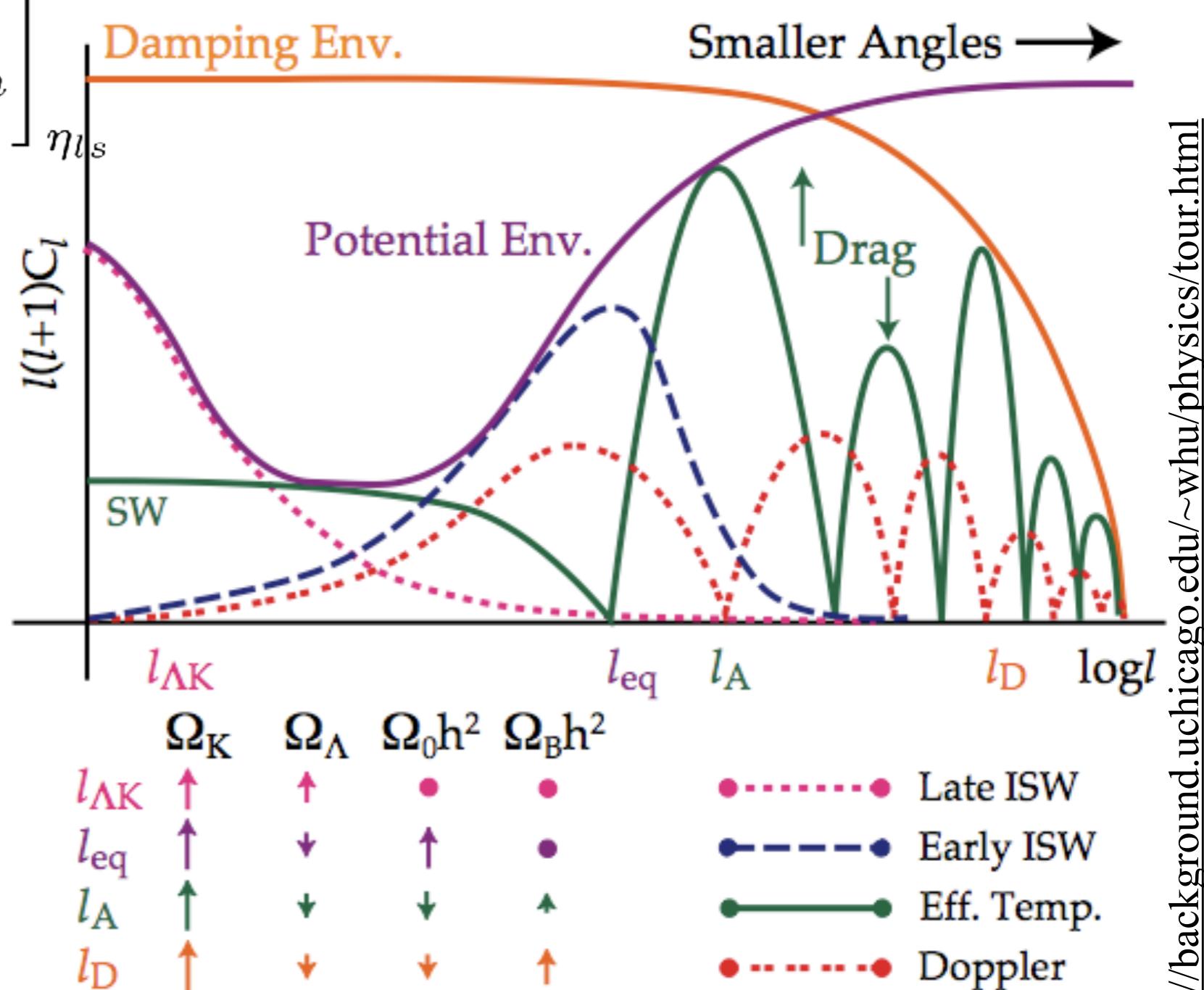


Dark Energy

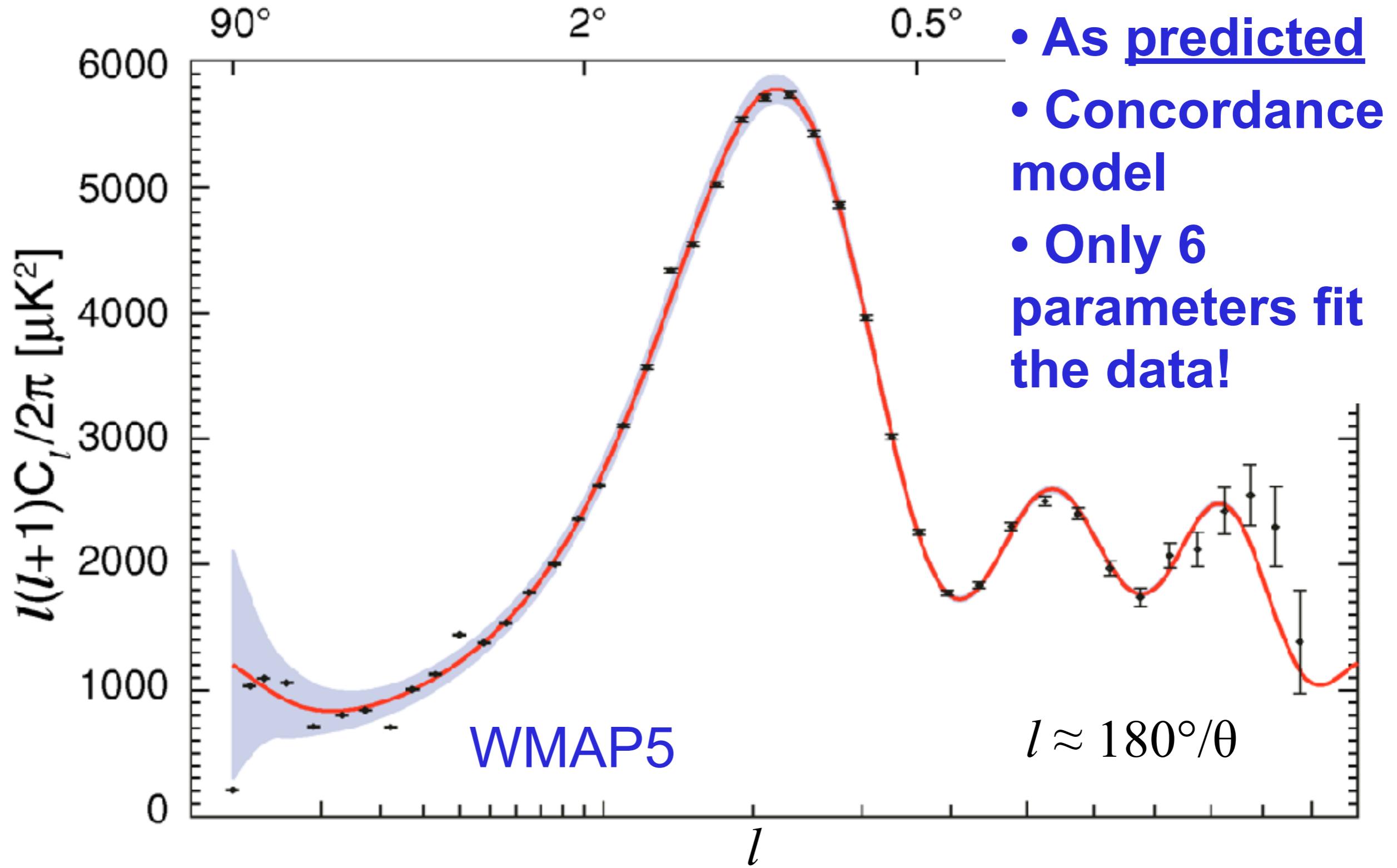
Combining the Contributions

$$\frac{\Delta T}{T} = \left[\Psi - \hat{r} \cdot \vec{v} + \frac{1}{3} \delta_m \right] + 2 \int_{\tau_{dec}}^{\tau_0} \frac{\partial \phi}{\partial \tau} d\tau$$

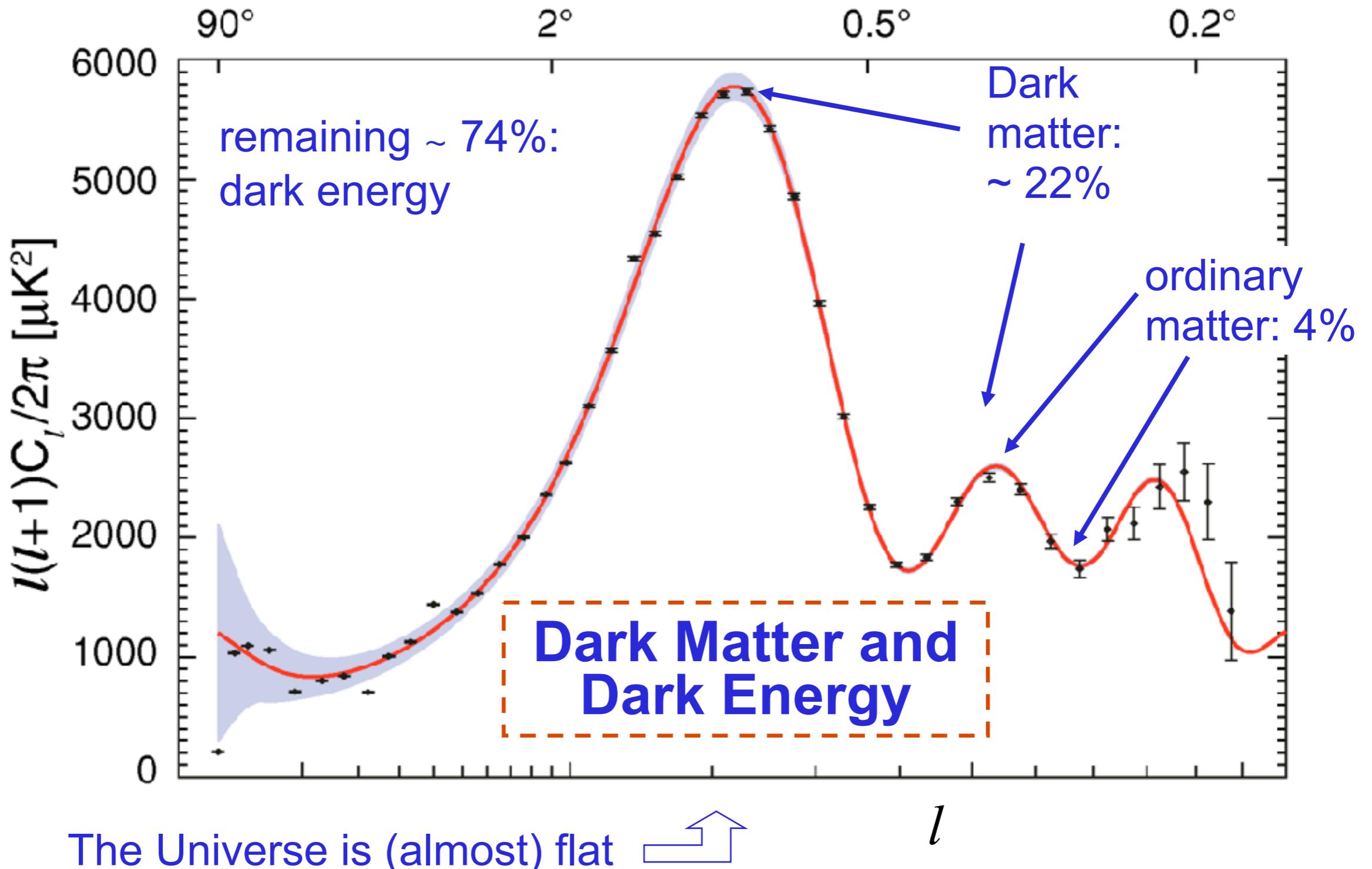
- Damping:
- Photon diffusion in hot regions (Silk damping)
- Finite thickness of last scattering surface



Angular Power Spectrum

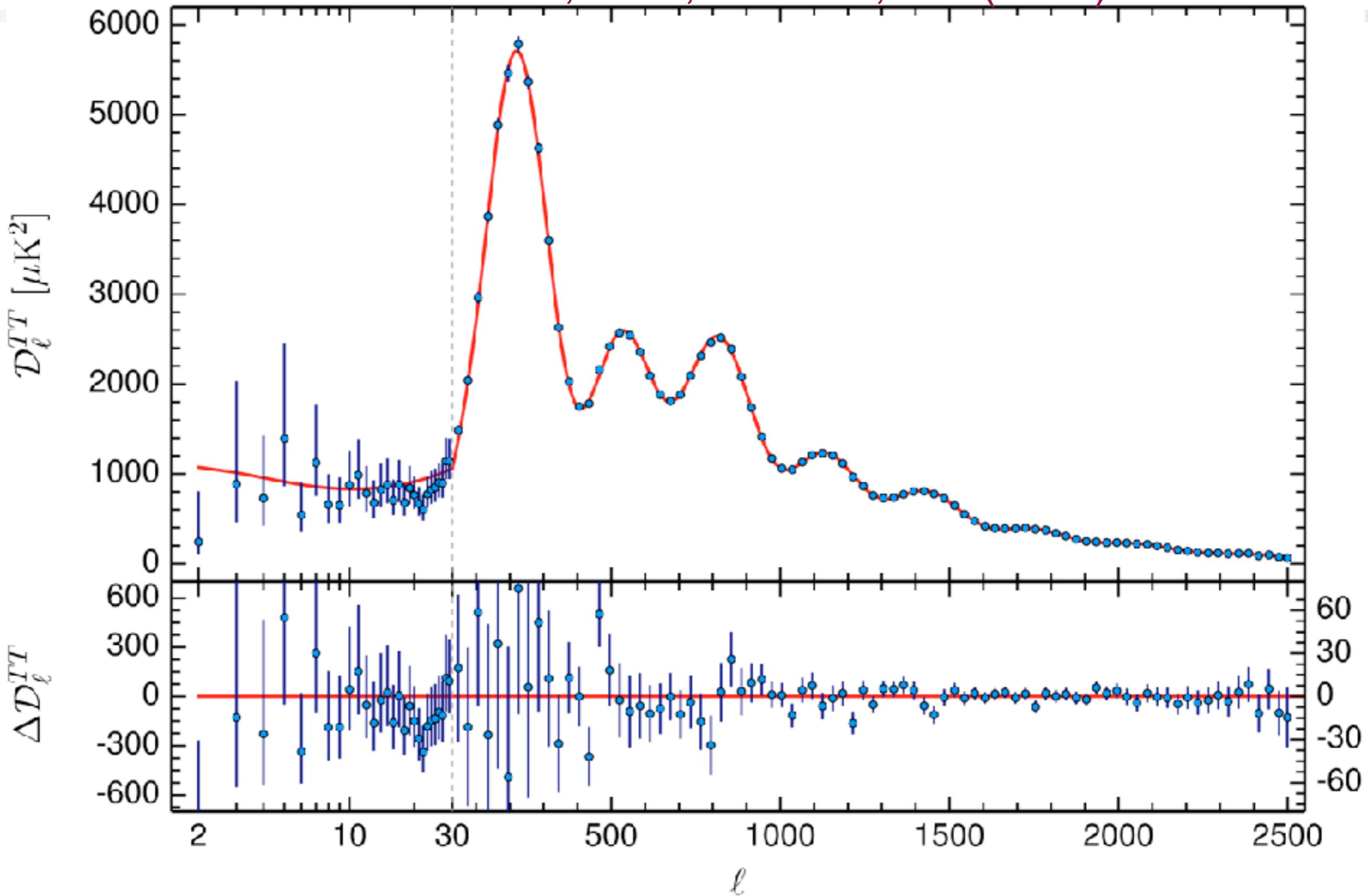


The holy grail of Cosmology



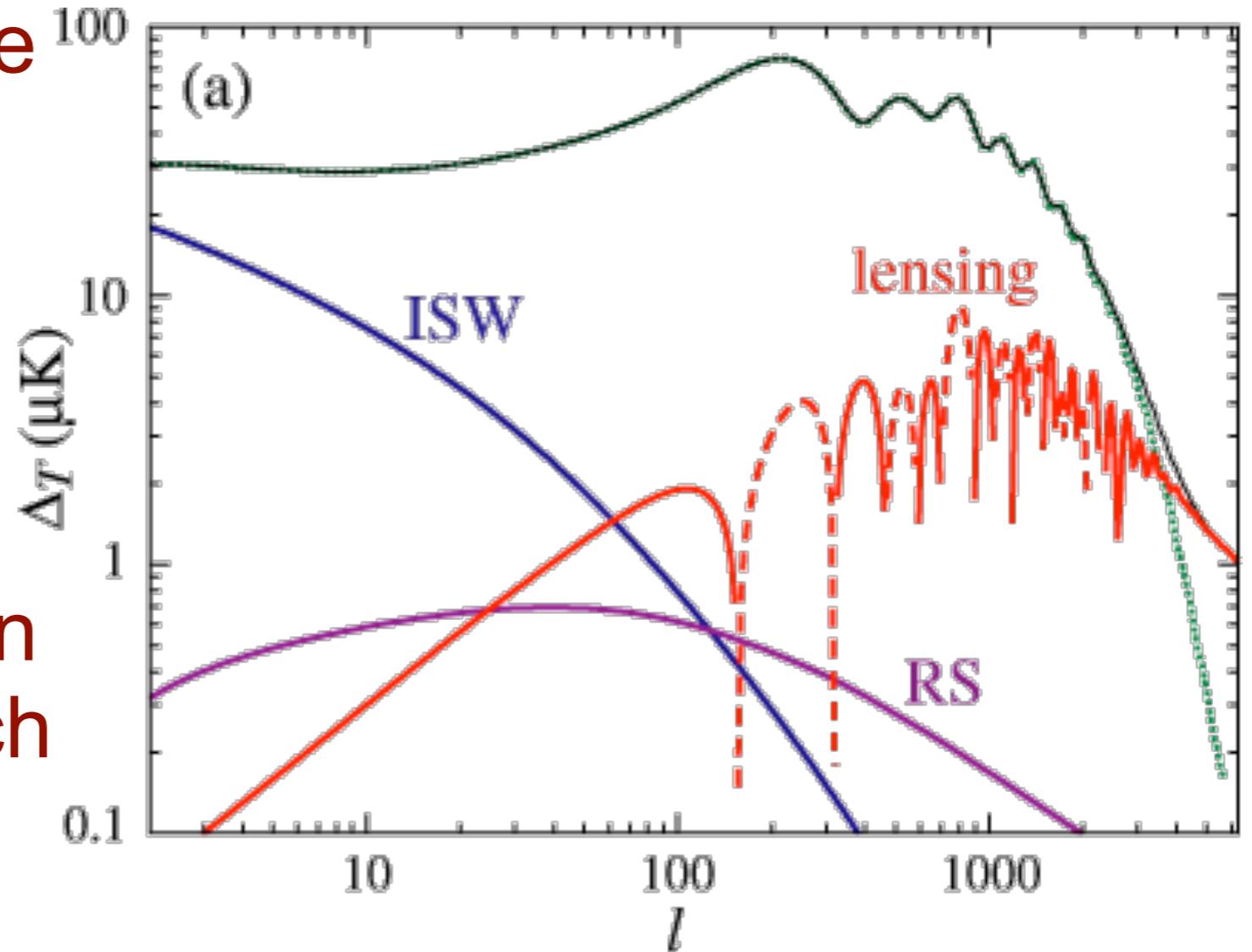
Planck 2015 power spectrum

Planck collaboration, 2016, A&A 594, A11 (2016)

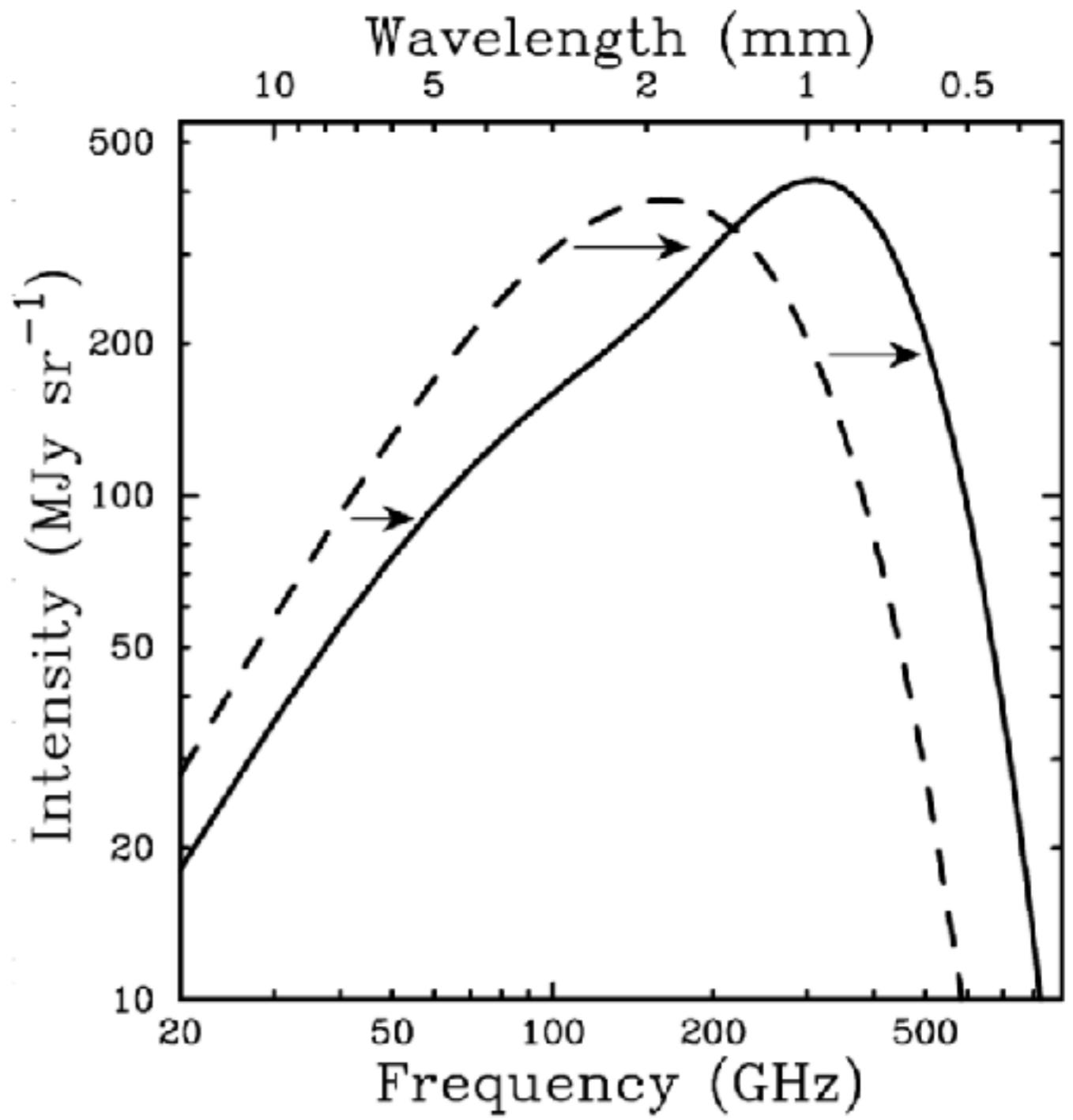
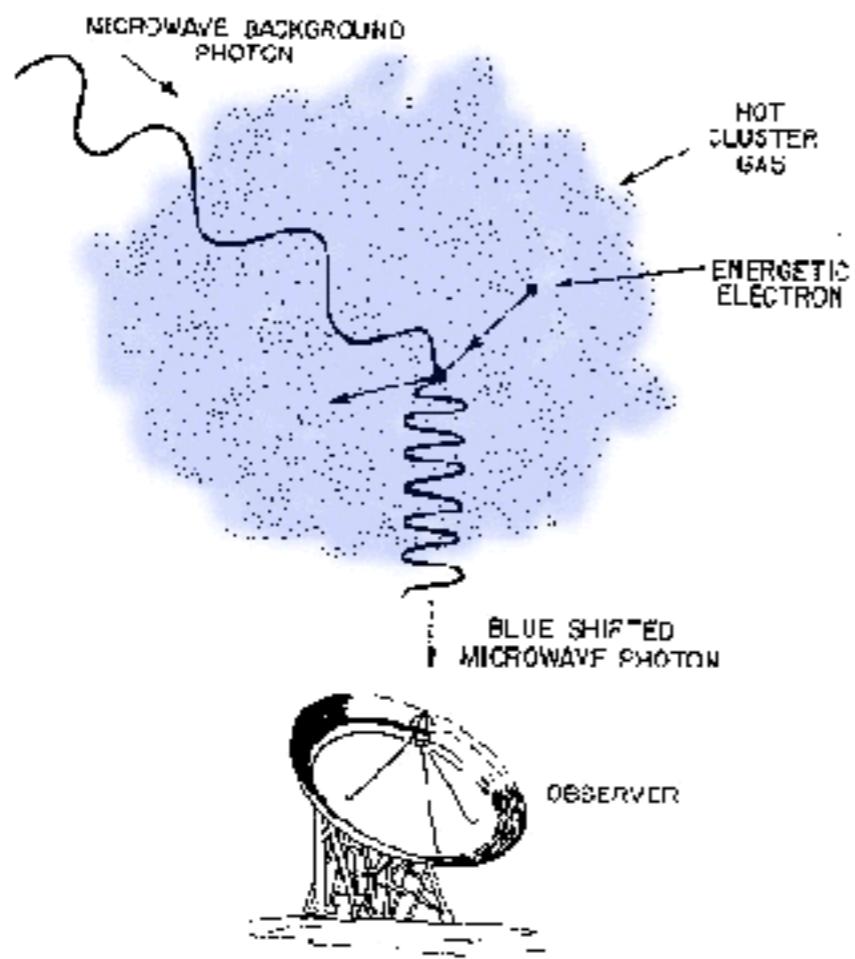


Secondary Anisotropies

- Integrated Sachs-Wolfe effect
- Gravitational Lensing
- Rees-Sciama effect
- Gravitational waves
- Scattering: reionization and Sunyaev Zel'dovich

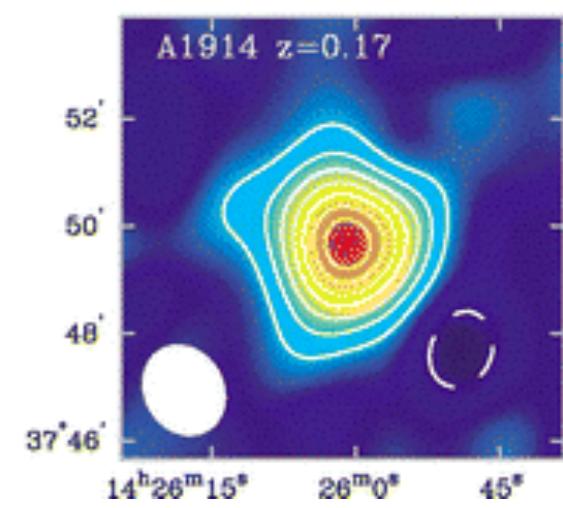
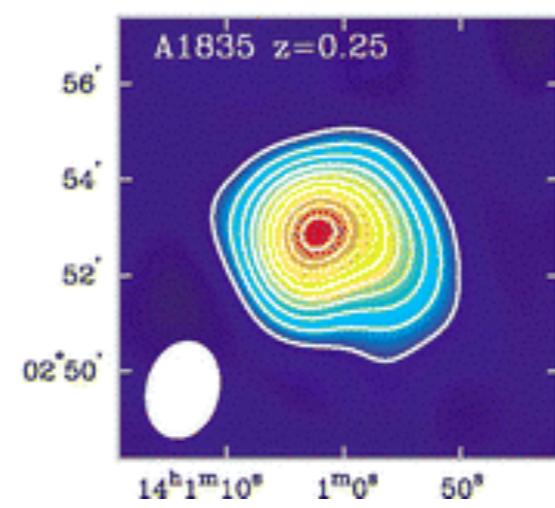
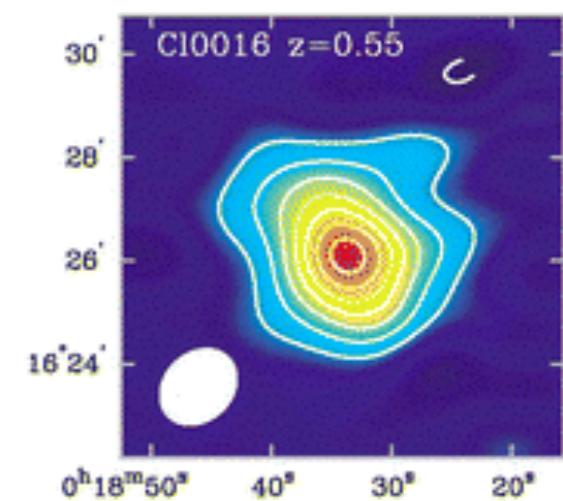
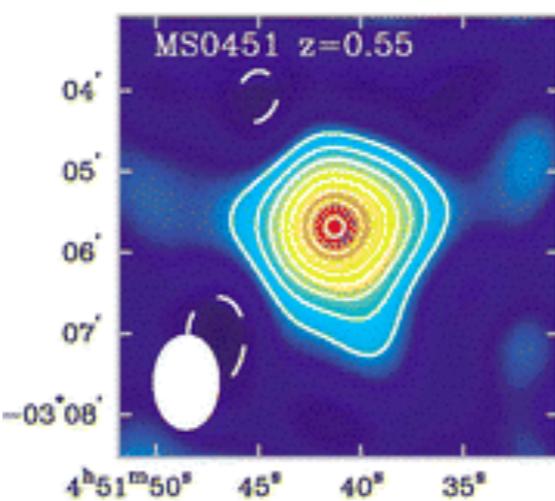
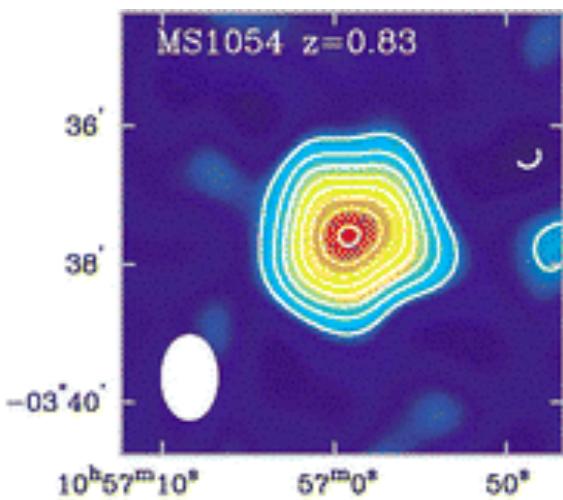
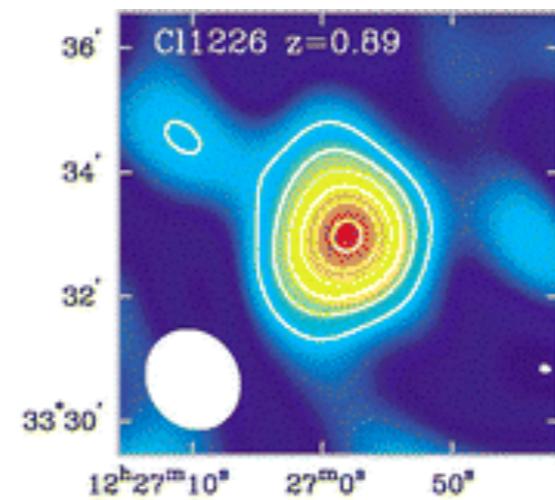
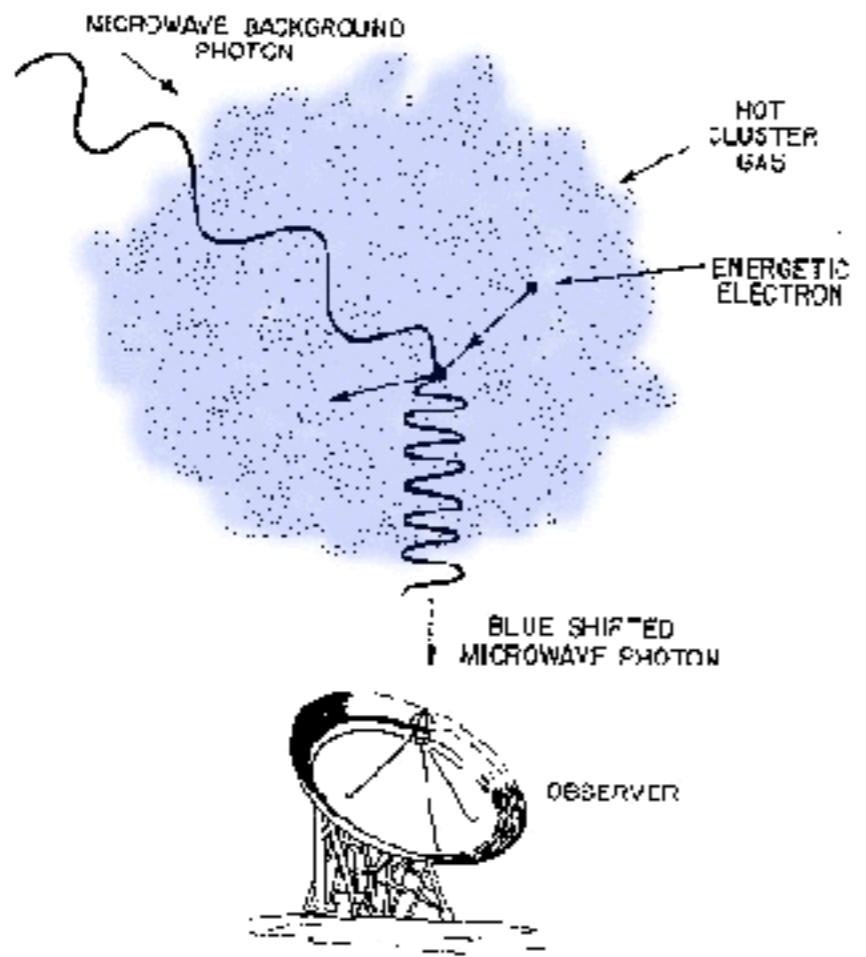
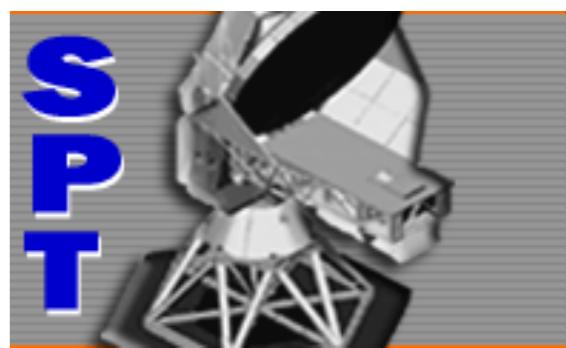


Sunyaev Zel'dovich Effect



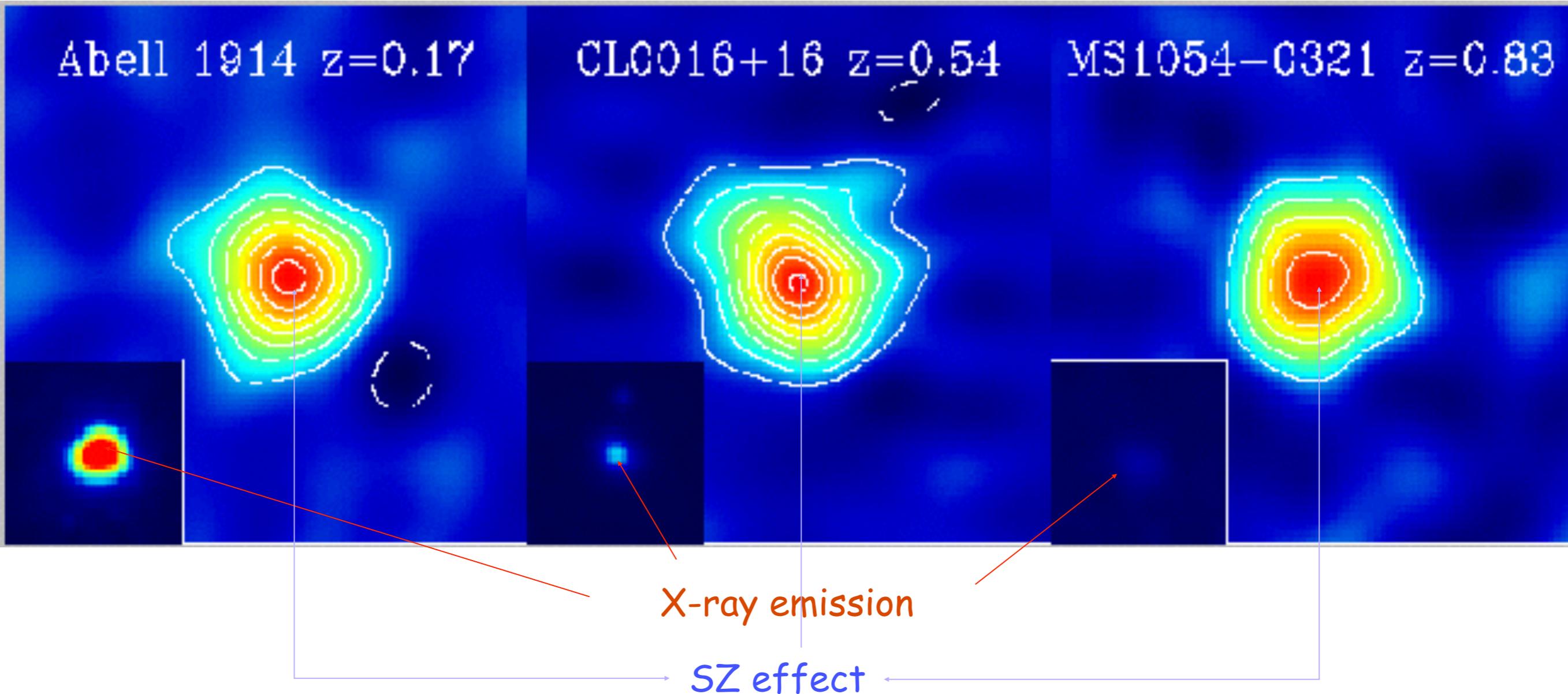
- Efeito Compton inverso
- Distorção da RCF em escalas de arcmin
- Independente de z

Sunyaev Zel'dovich Effect

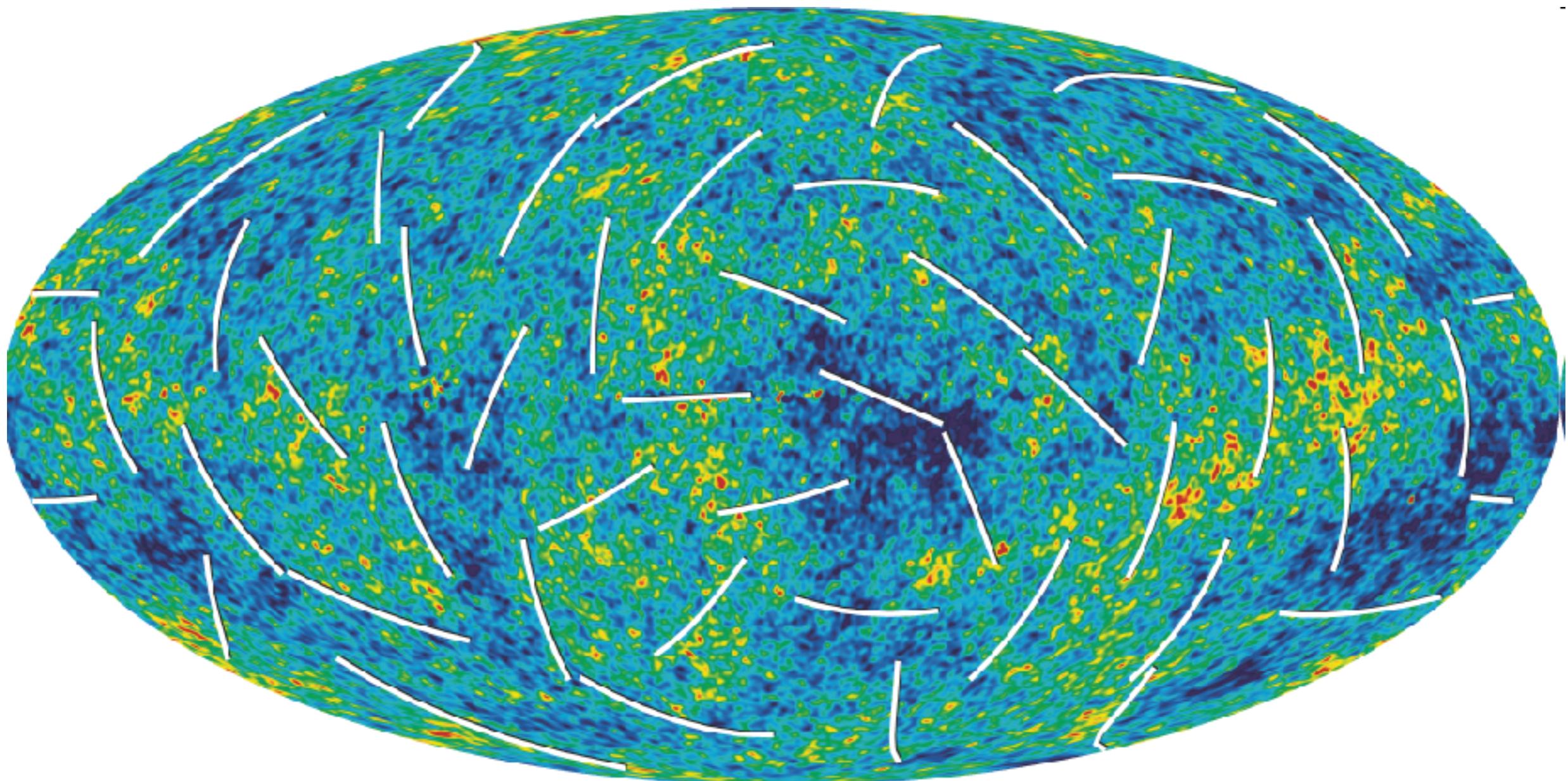


Advantage of Sunyaev Zel'dovich Effect

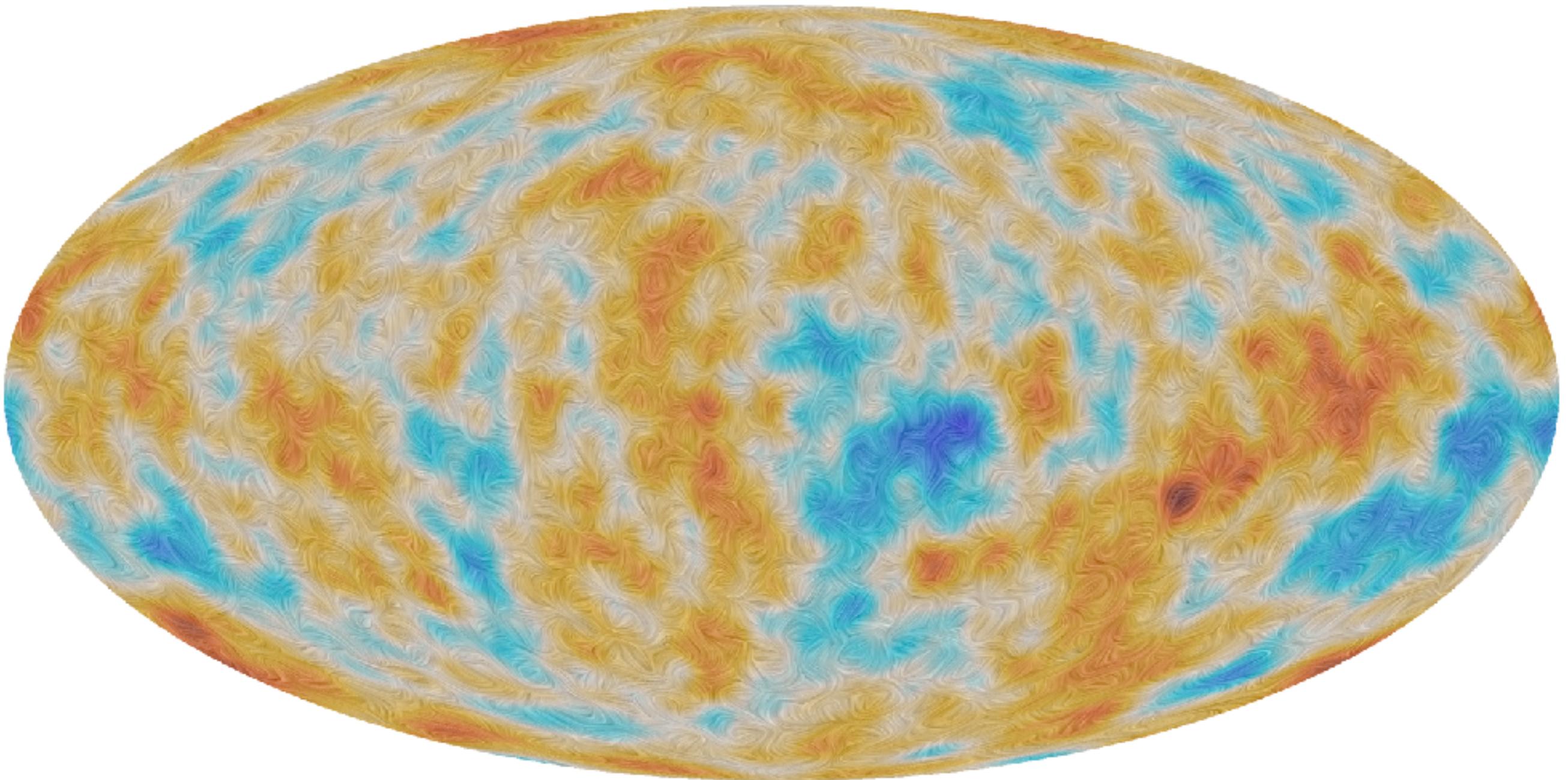
- X-ray flux drops as $(1 + z)^{-4}$
- Decrement is independent from z

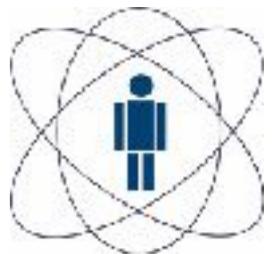


CMB Polarization

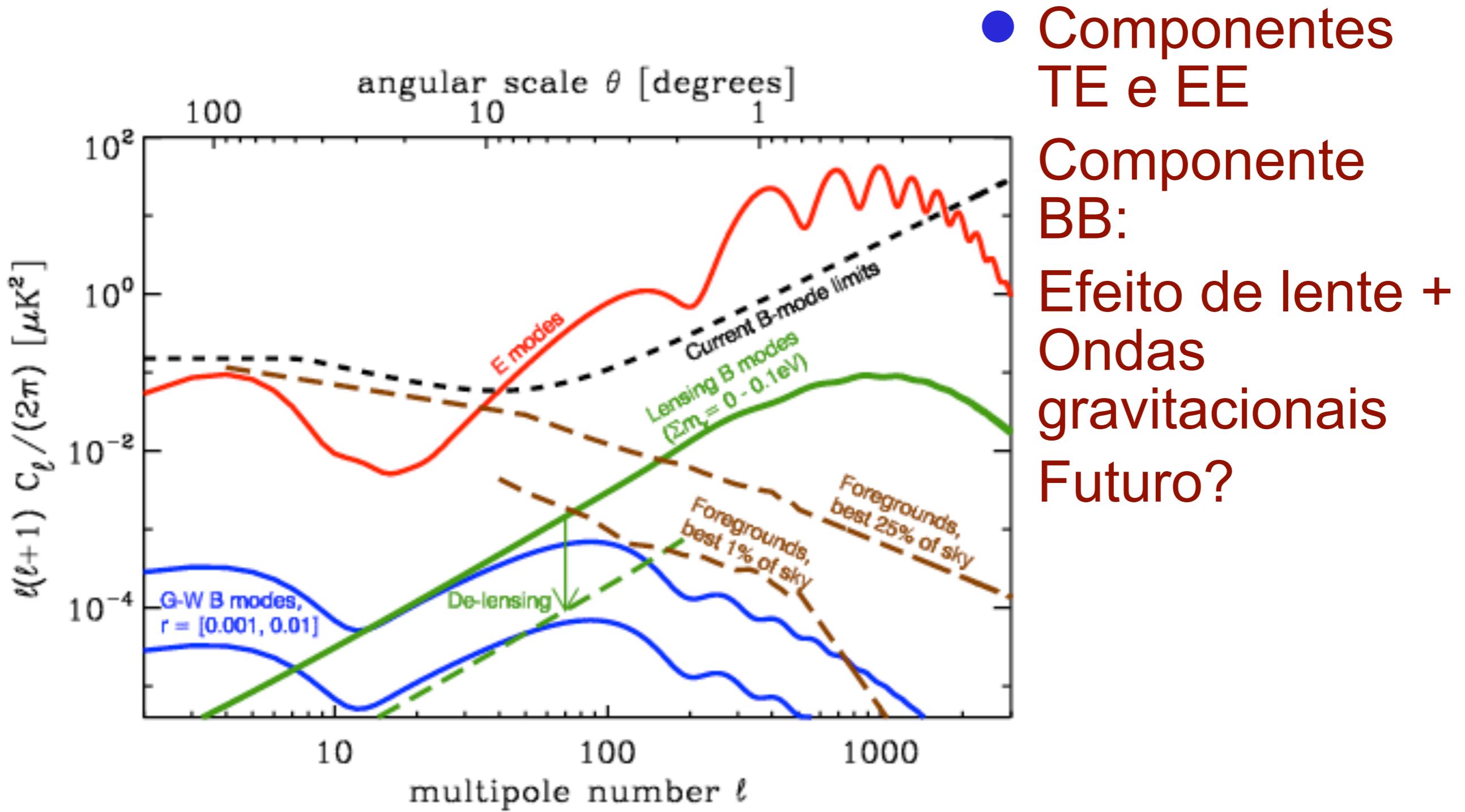


Planck Polarization Map

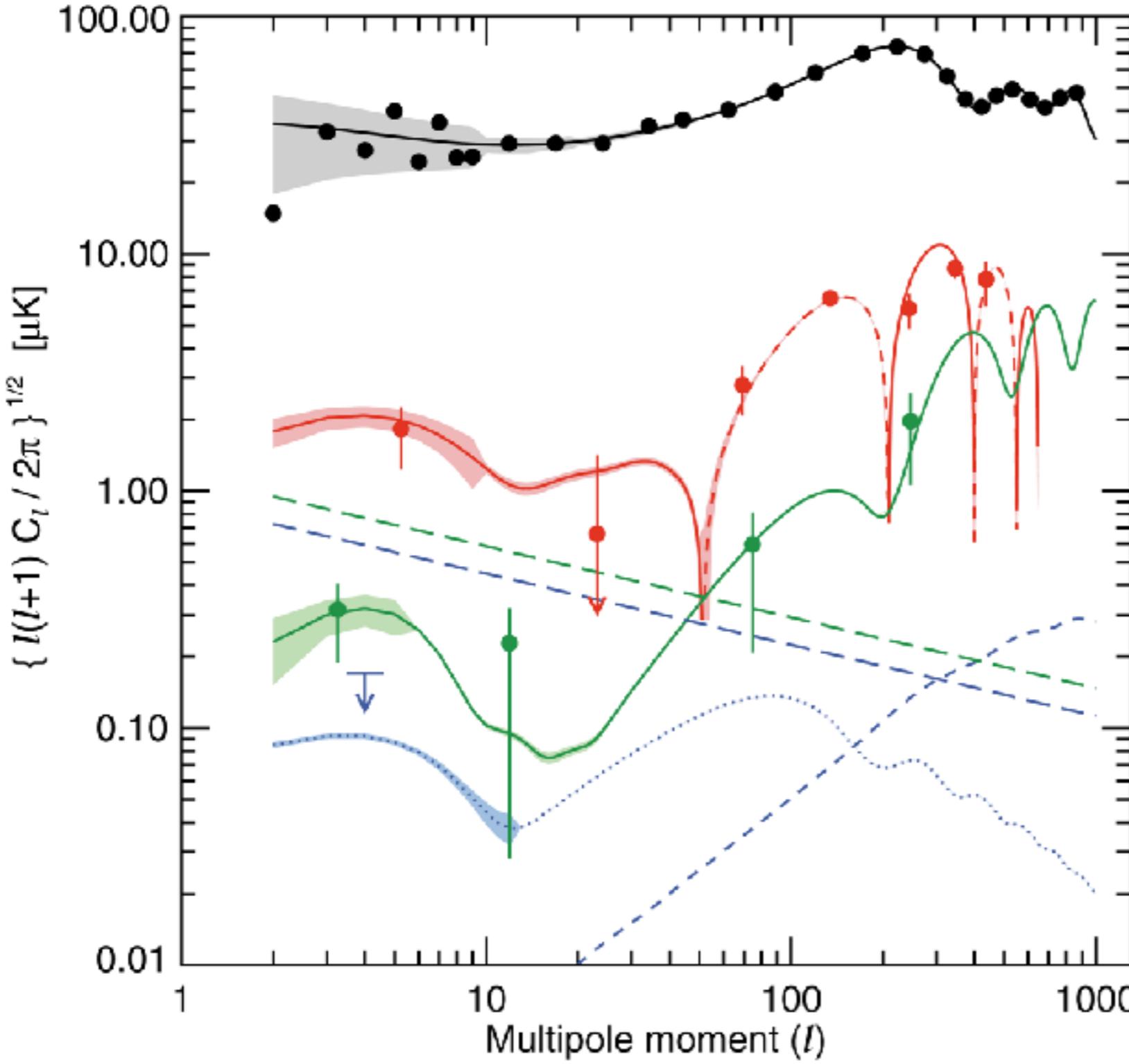




Espectro da Polarização

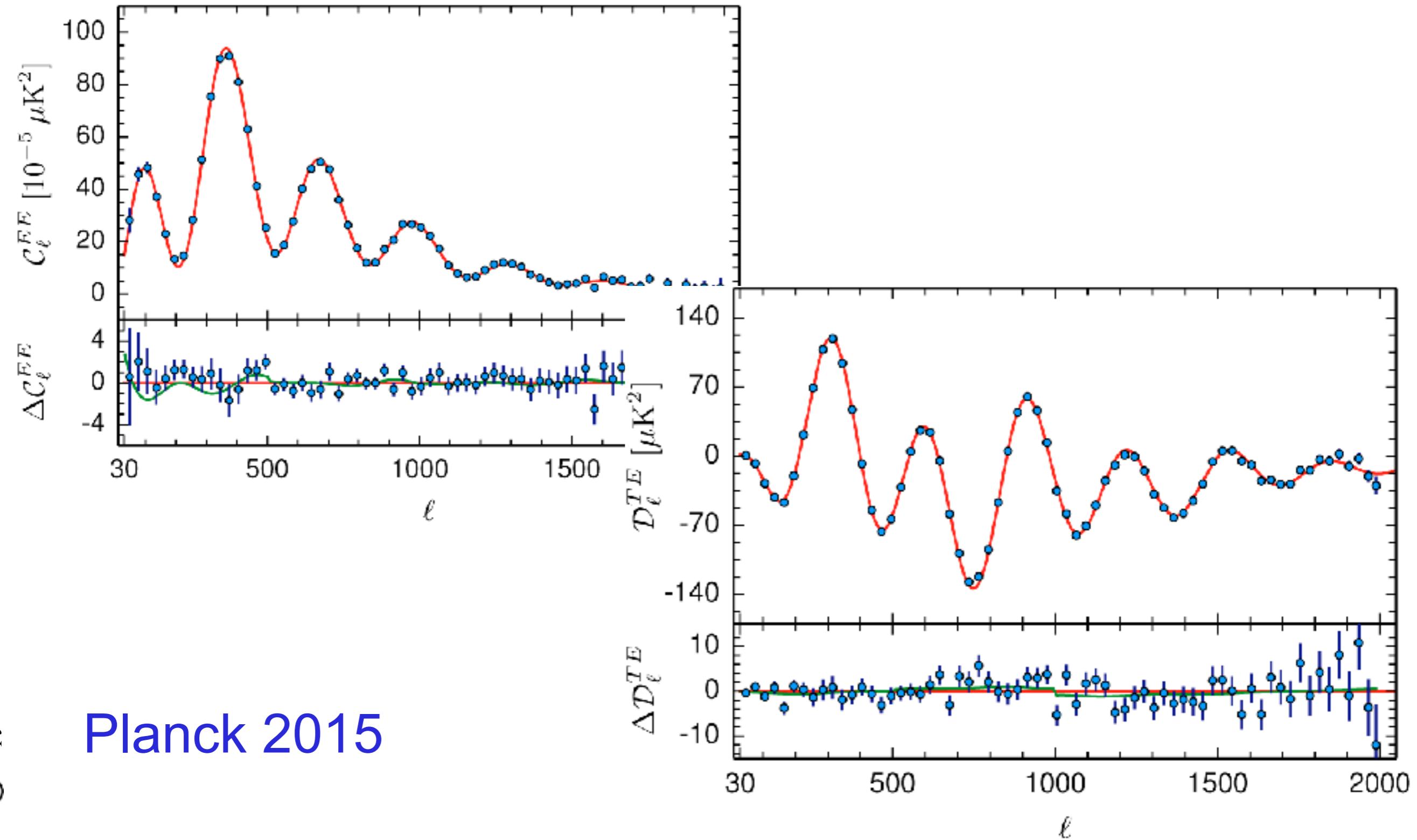


Polarization Spectrum



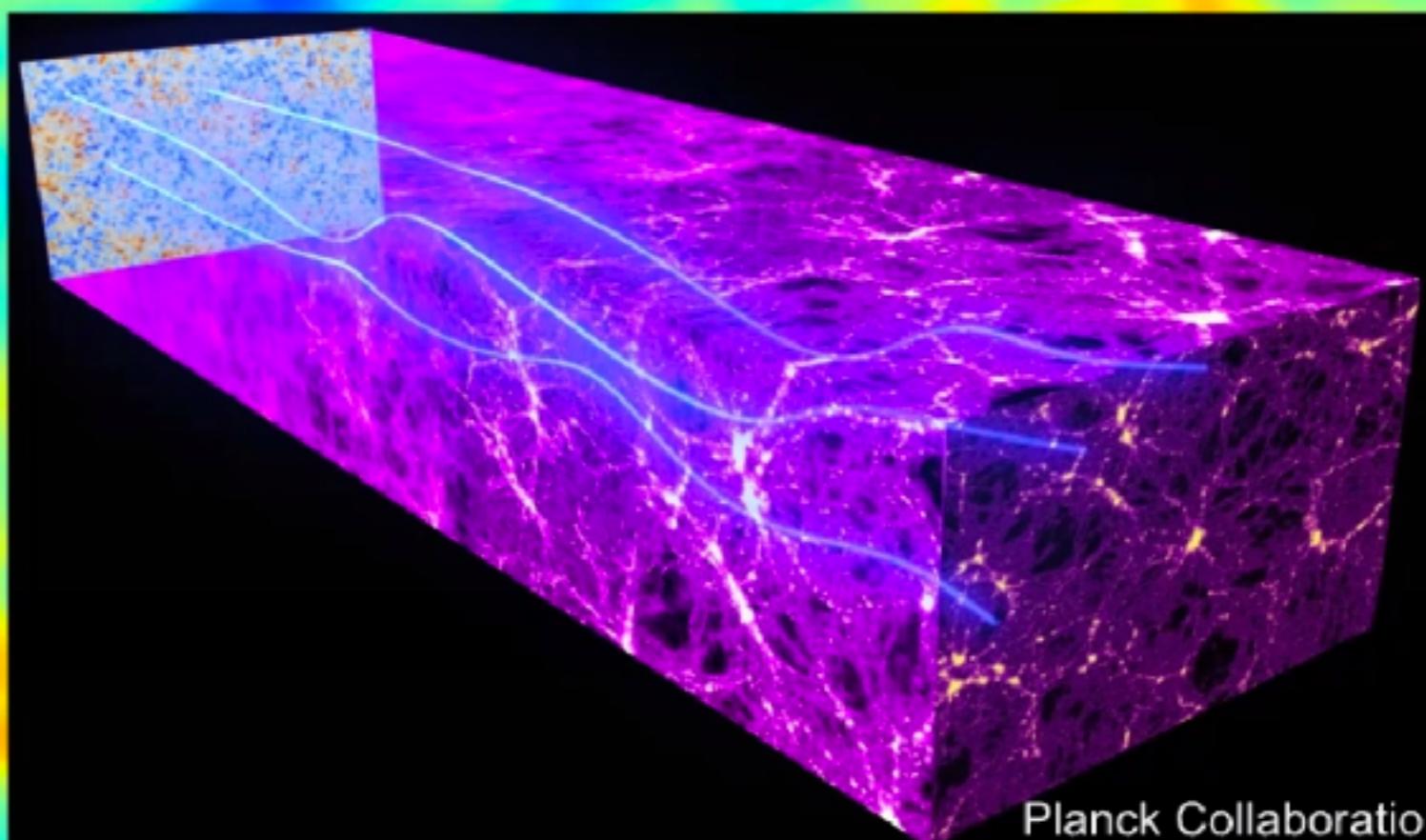
- TE and EE components
- BB component
- Gravitational waves

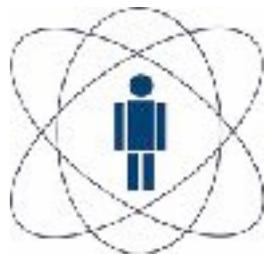
Polarization Power Spectra



CMB as a Backlight

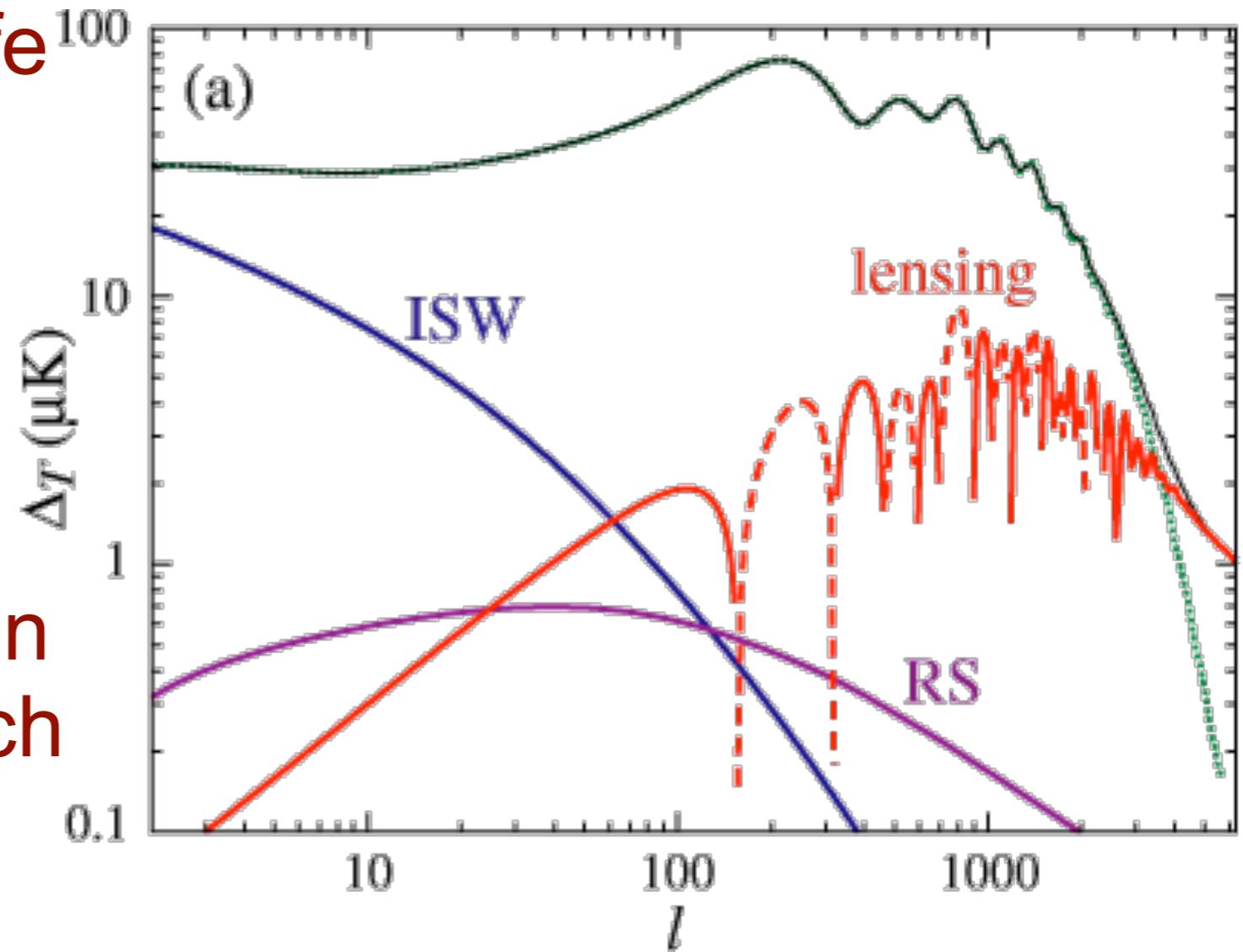
- Photons from the CMB are gravitationally deflected by structure
- Can reconstruct maps of the dark matter distribution
- Lensing probes the growth of structure → dark energy, Σm_ν
- Need improved temperature + polarization measurements on small angular scales across large areas of the sky



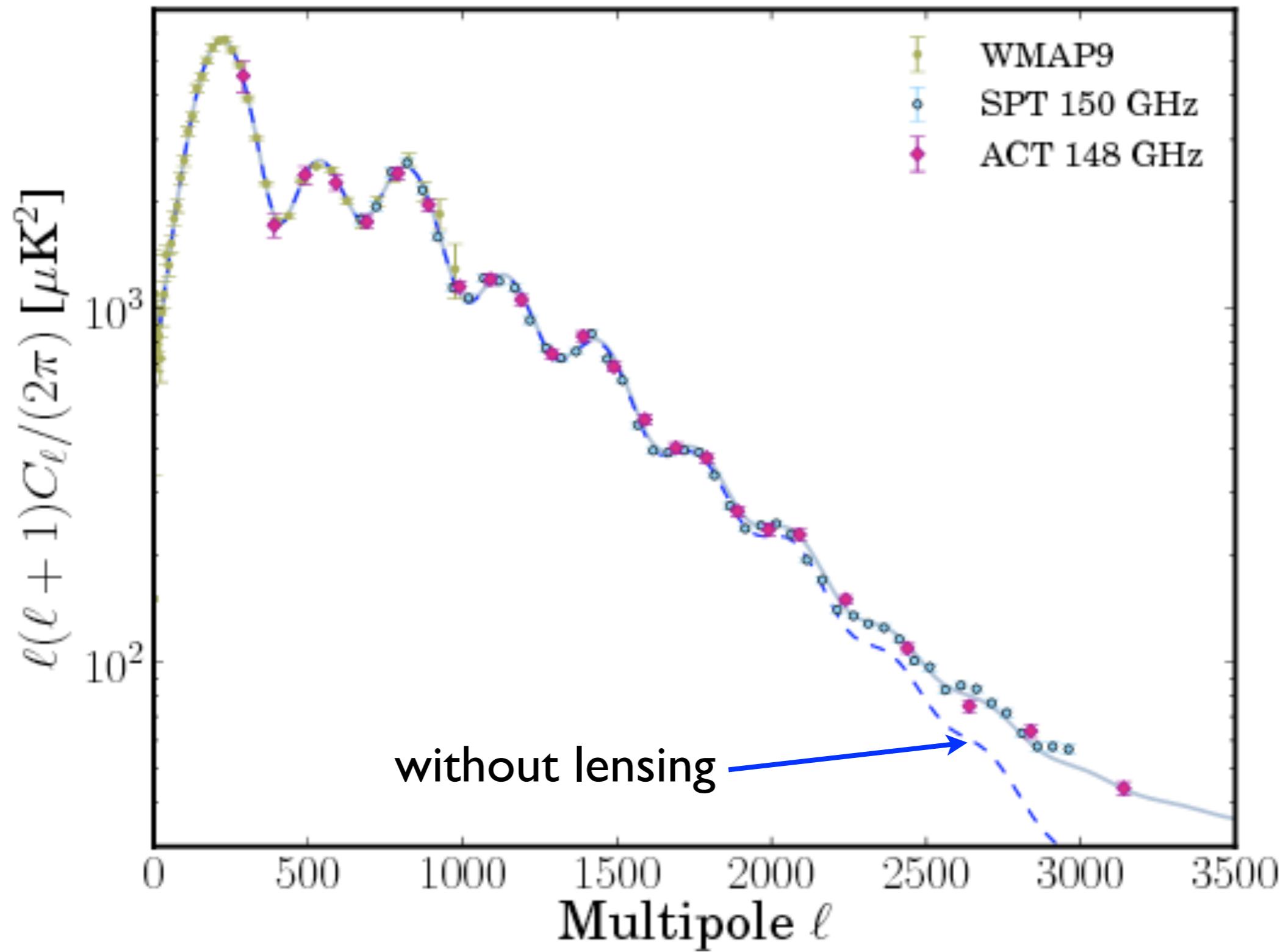


Secondary Anisotropies

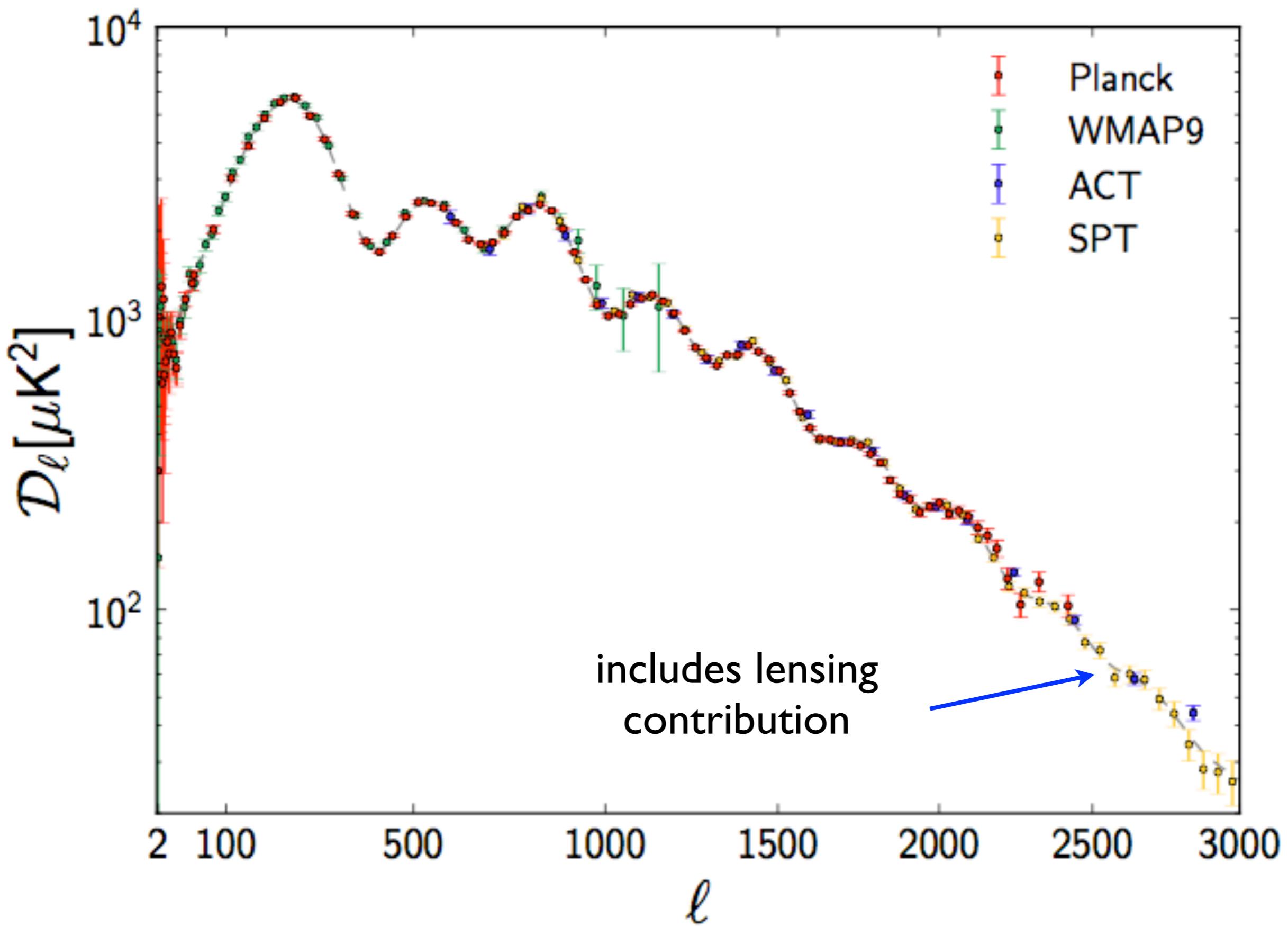
- Integrated Sachs-Wolfe effect
- Gravitational Lensing
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- Gravitational waves
- Scattering: reionization and Sunyaev Zel'dovich



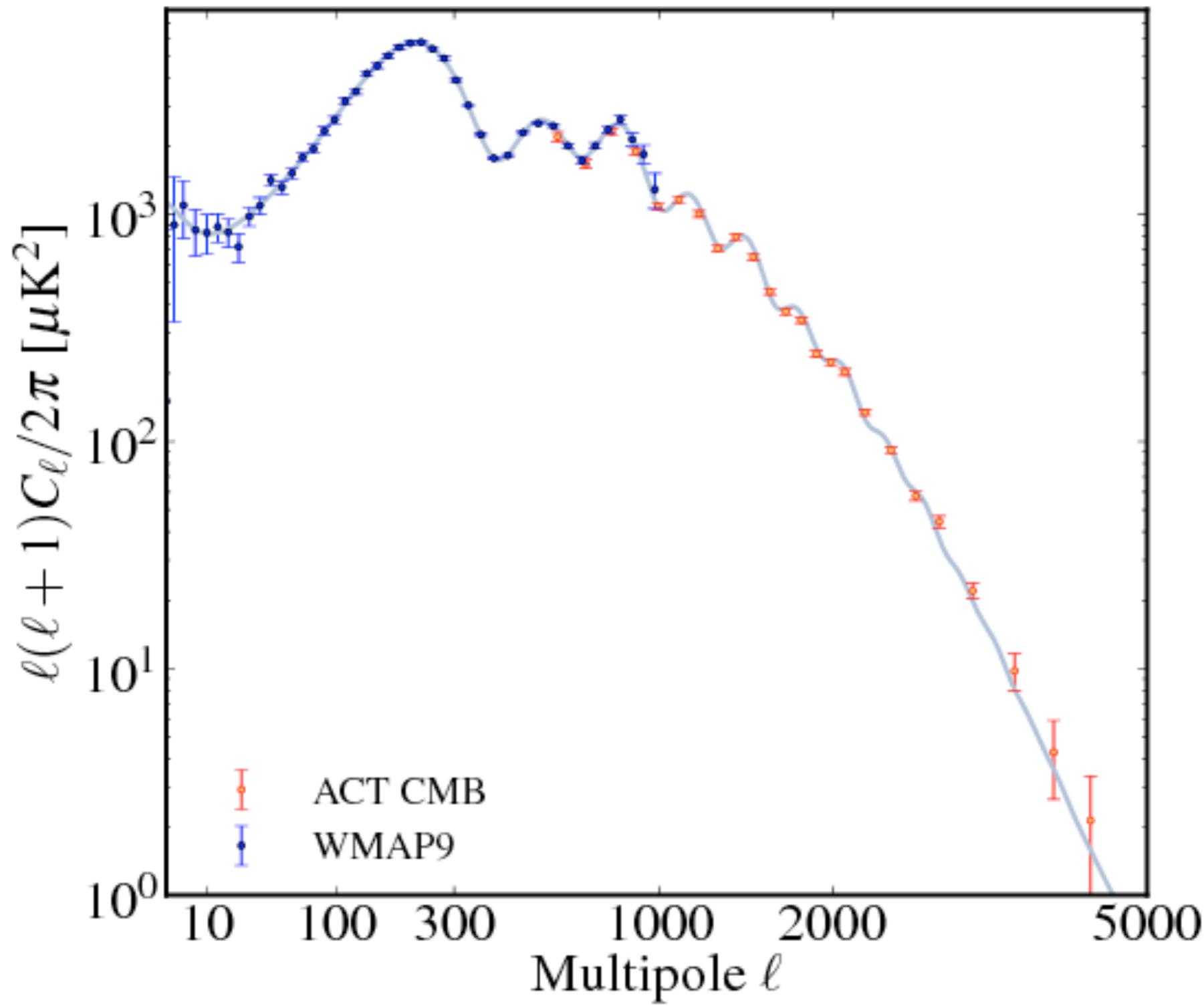
Small scale spectrum



Power spectrum @ 2013



Small scale spectrum



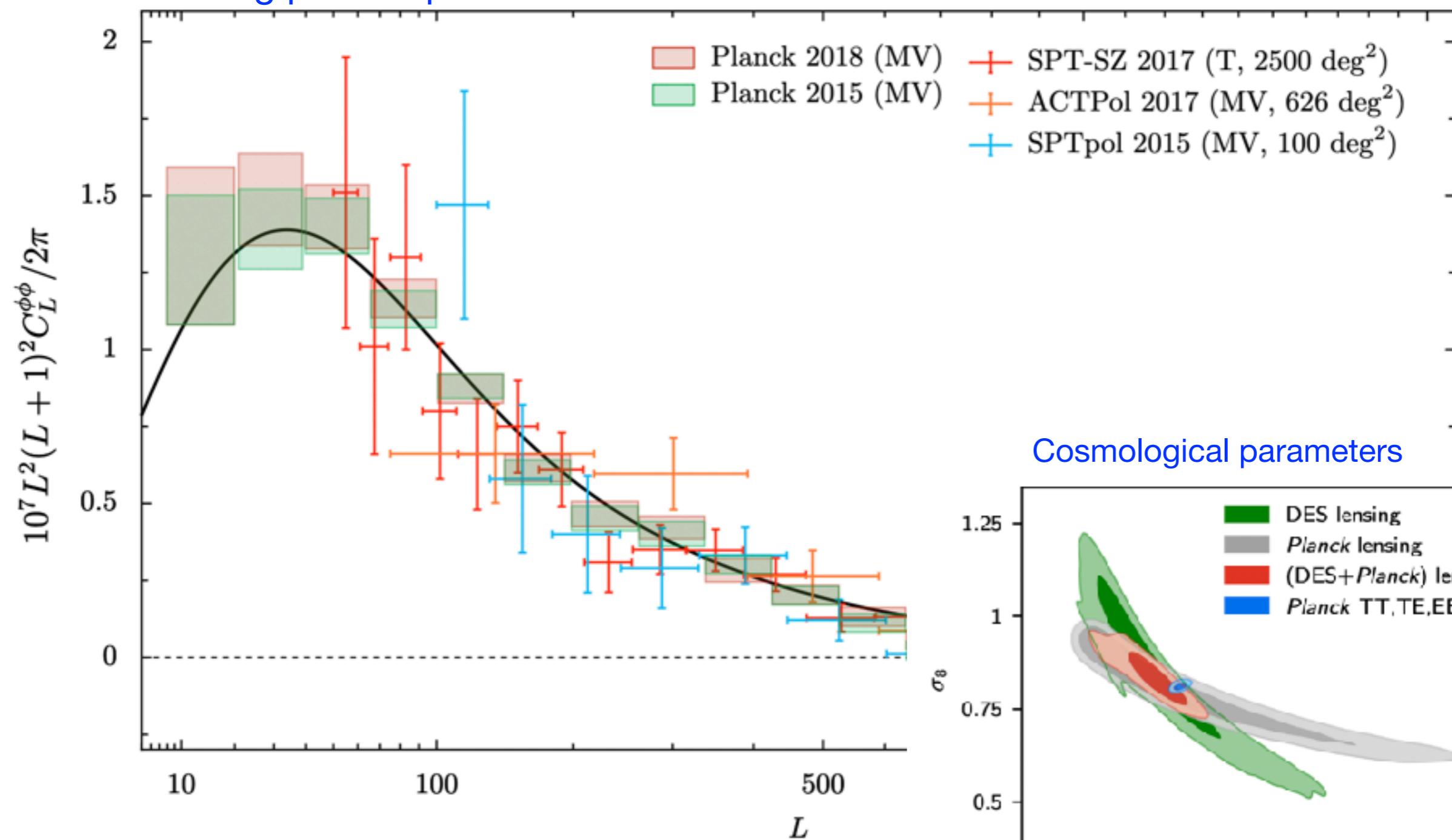
Planck 2018 results

VIII. Gravitational lensing

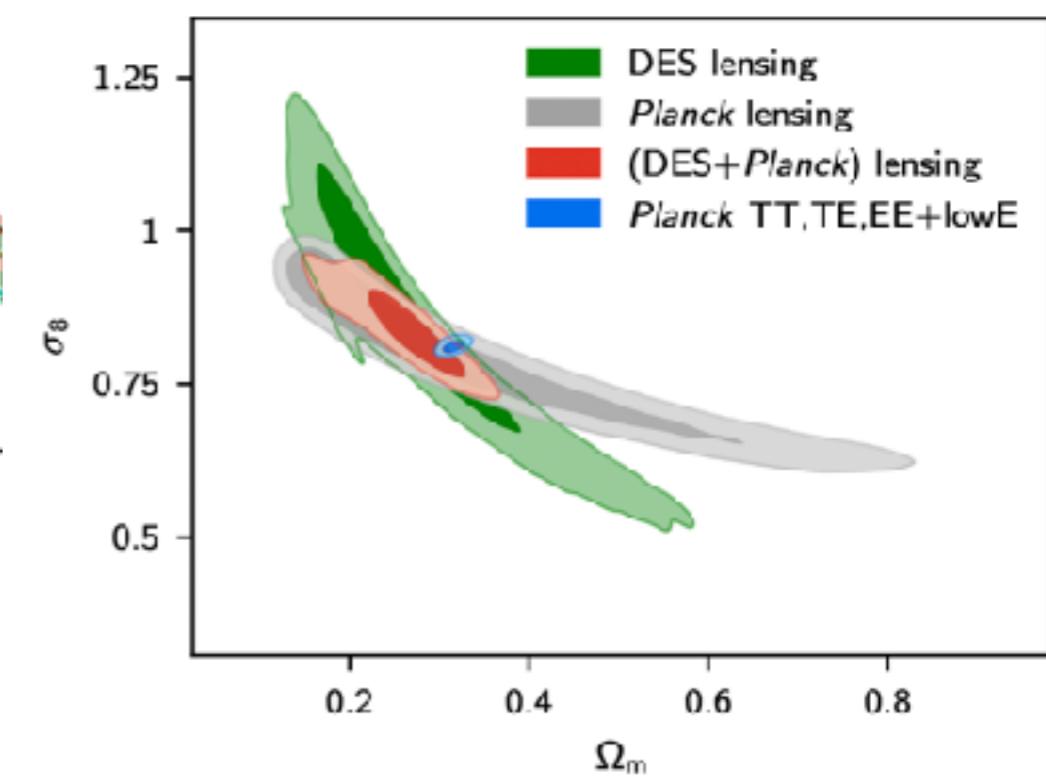
**Astronomy
&
Astrophysics**
Special issue

Planck 2018 results

Lensing power spectrum



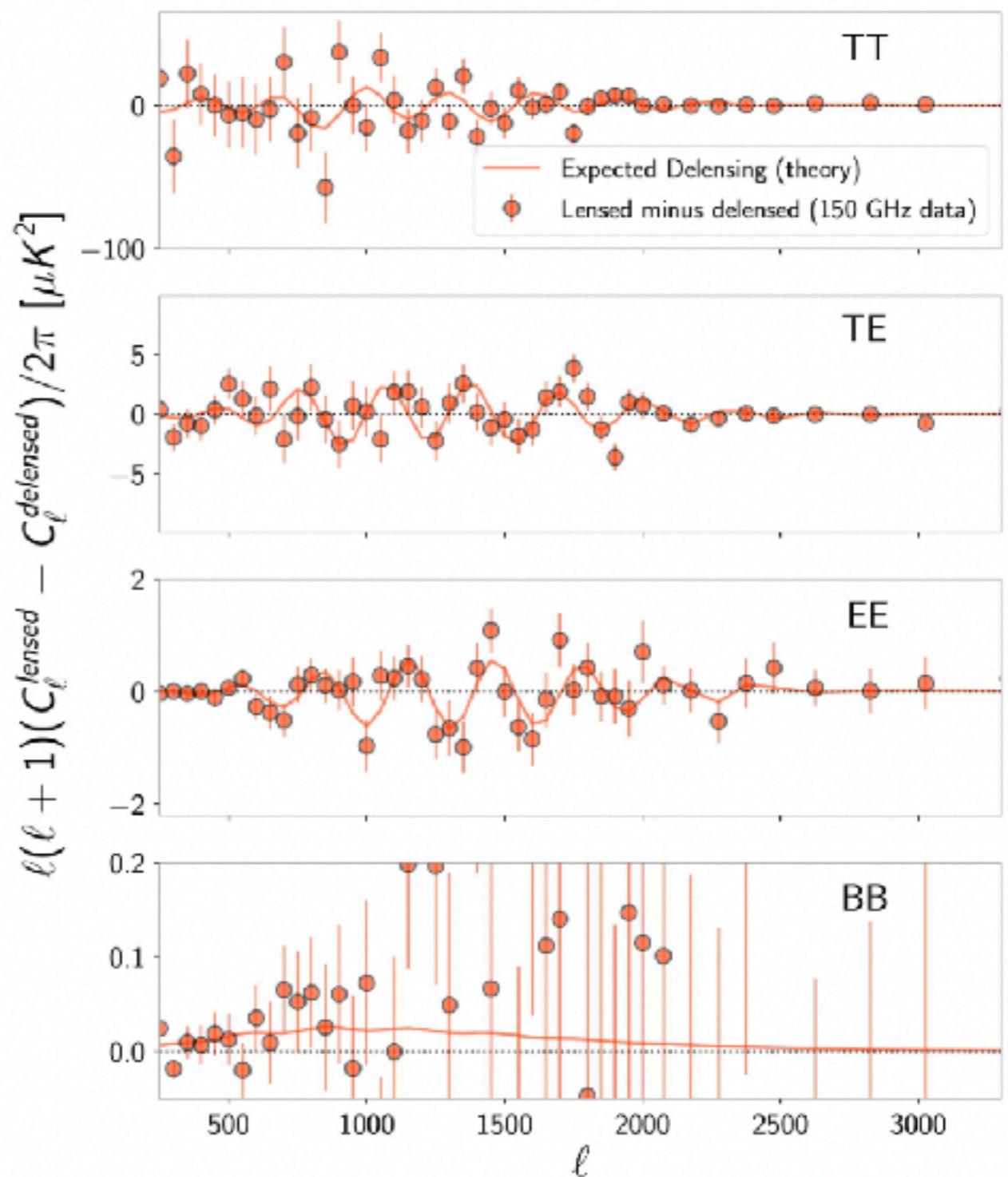
Cosmological parameters



Removiendo la señal de lentes de los espectros de potencias

The Atacama Cosmology Telescope:
delensed power spectra and
parameters

arXiv:2007.14405



Correlación entre grupos de galaxias y la convergencia medida en la CMB

Monthly Notices
ROYAL ASTRONOMICAL SOCIETY
MNRAS 511, 3548–3560 (2022)
Advance Access publication 2022 February 2

Cross-correlation of Planck cosmic microwave background lensing with DESI galaxy groups

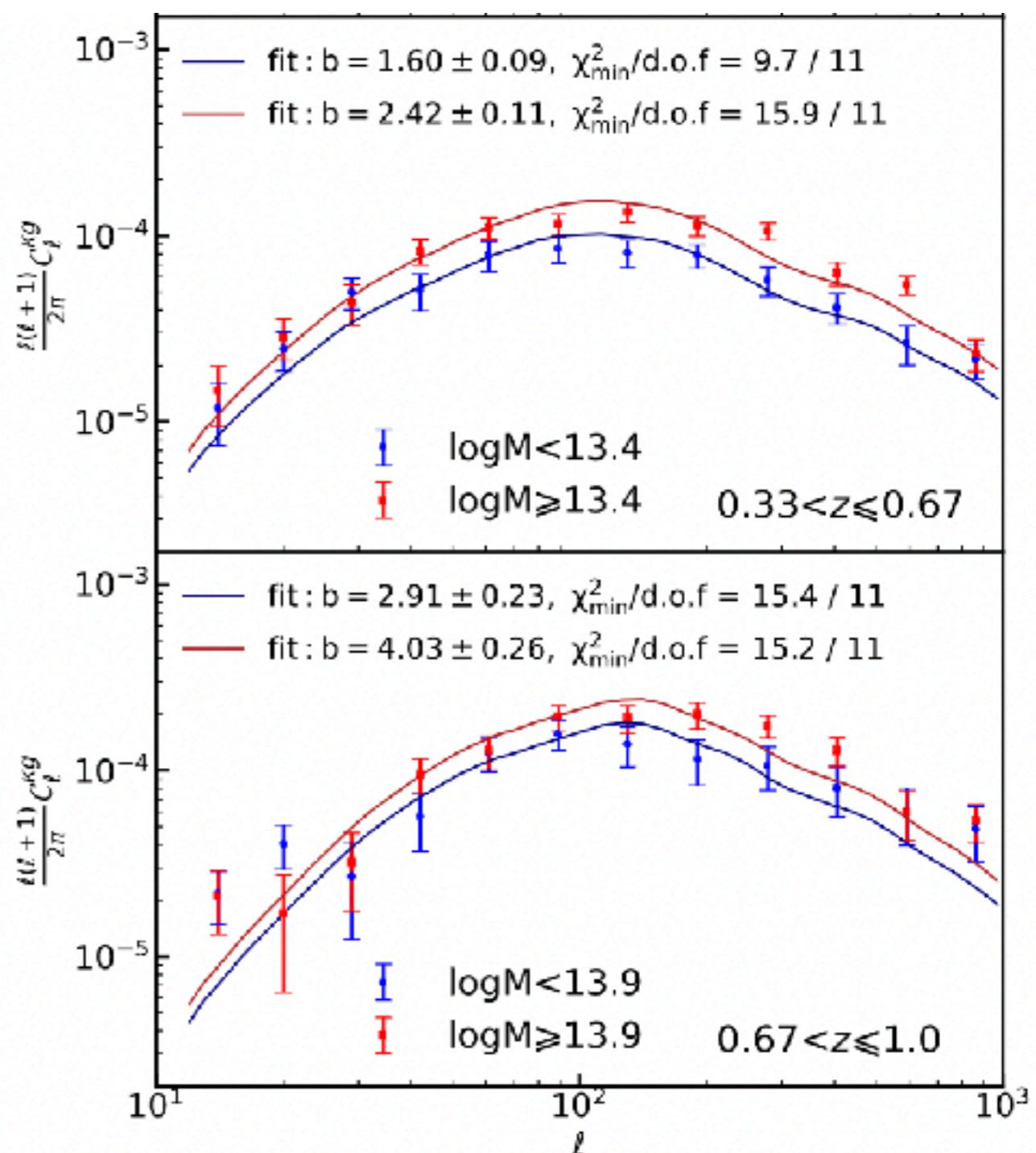
Galaxy/galaxy group overdensity δ_g and CMB lensing convergence κ are both projections of 3D density fields, expressed as line-of-sight integrals over their respective projection kernels. The angular cross-correlation power spectrum, adopting the Limber approximation (Limber 1953), is

$$C_\ell^{\kappa g} = \int d\chi W^\kappa(\chi) W^g(\chi) \frac{1}{\chi^2} P_{mg} \left(k = \frac{\ell + 1/2}{\chi}; z \right). \quad (1)$$

The Limber approximation is inaccurate for $\ell < 10$, but such very large-scale modes are excluded from our fitting anyway, due to poor S/N. The above expression assumes spatial flatness. Here, W^κ and W^g are the projection kernels for κ and the group number density fields, respectively.

$$W^g(z) = n(z) = \frac{c}{H(z)} W^g(\chi). \quad (2)$$

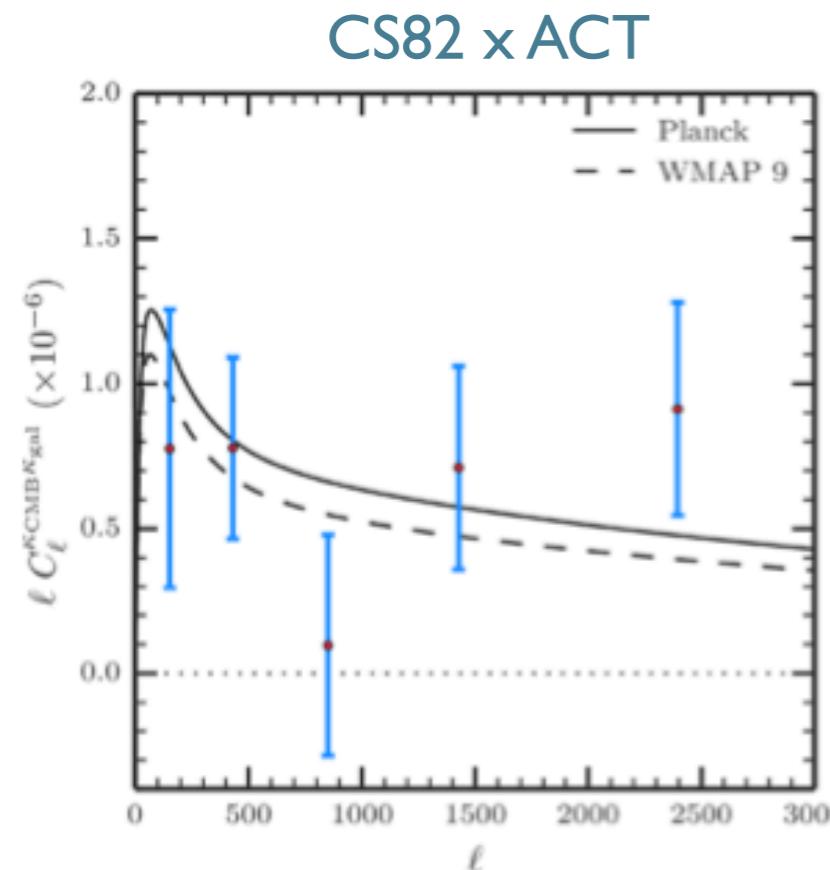
$$W^\kappa(z) = \frac{3}{2c} \Omega_{m0} \frac{H_0^2}{H(z)} (1+z) \frac{\chi(\chi_* - \chi)}{\chi_*} = \frac{c}{H(z)} W^\kappa(\chi). \quad (3)$$



**Correlation of CMB lensing
with LSS lensing in CS82**

EDITORS' SUGGESTION

First measurement of the cross-correlation of CMB lensing and galaxy lensing



The authors measure for the first time the cross-correlation between CMB lensing and galaxy lensing. This provides a robust test of the Λ CDM model on the largest cosmic scales, and offers powerful constraints on the evolution and nature

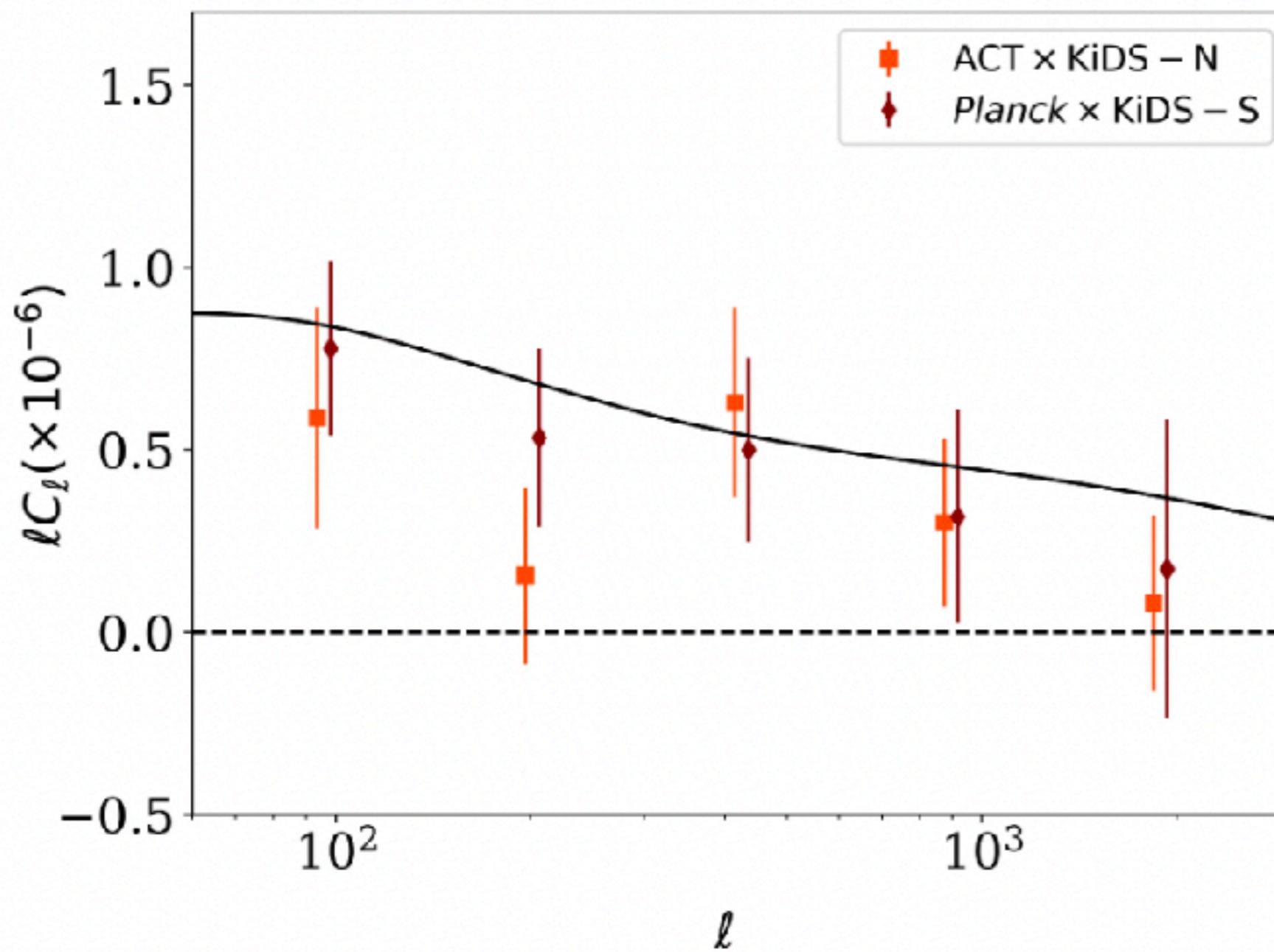
$$\kappa(\hat{\mathbf{n}}) = \int_0^\infty dz W^\kappa(z) \delta(\chi(z)\hat{\mathbf{n}}, z)$$

$$W^\kappa(z) = \frac{3}{2} \Omega_m H_\circ^2 \frac{(1+z)}{H(z)} \frac{\chi(z)}{c} \int_z^\infty dz_s p_s(z_s) \frac{\chi(z_s) - \chi(z)}{\chi(z_s)} .$$

$$W^{\kappa_{\text{CMB}}}(z) = \frac{3}{2} \Omega_m H_\circ^2 \frac{(1+z)}{H(z)} \frac{\chi(z)}{c} \left[\frac{\chi(z_\star) - \chi(z)}{\chi(z_\star)} \right]$$

$$C_\ell^{\kappa_{\text{CMB}} \kappa_{\text{gal}}} = \int_0^\infty \frac{dz}{c} \frac{H(z)}{\chi(z)^2} W^{\kappa_{\text{CMB}}} W^{\kappa_{\text{gal}}} P\left(k = \frac{\ell}{\chi}, z\right)$$

Strong detection of the CMB lensing and galaxy weak lensing cross-correlation from ACT-DR4, *Planck* Legacy, and KiDS-1000



arXiv:2011.11613

Cosmology from cross-correlation of ACT-DR4 CMB lensing and DES-Y3 cosmic shear

$$C_\ell^{\kappa\zeta\gamma_E} = \int_0^{z_H} dz \frac{H(z)}{\chi^2(z)c} W_\kappa^{\text{CMB}}(z) W_\zeta^\gamma(z) P_{\delta\delta}\left(k = \frac{\ell + 0.5}{\chi(z)}, z\right)$$

$$W_\kappa^{\text{CMB}}(z) = \frac{3H_0^2\Omega_{\text{m},0}}{2H(z)c} \frac{\chi(z)}{a(z)} \frac{\chi(z^*) - \chi(z)}{\chi(z^*)},$$

$$W_\zeta^\gamma(z) = \frac{3H_0^2\Omega_{\text{m},0}}{2H(z)c} \frac{\chi(z)}{a(z)} \int_z^{z_H} dz' n(z') \frac{\chi(z') - \chi(z)}{\chi(z')}$$

