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Probability and Stochastic Processes

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34.1 Axioms of Probability

34.1.1 Axioms of Probability

I. $P(A) \ge 0$, II. P(S) = 1, III. If AB = 0 then P(A + B) = P(A) + P(B) [. S = a set of elements of outcomes $\{\zeta_1\}$ of an experiment (certain event), 0 = empty set (impossible event). $\{\zeta_1\} = \text{elementary event}$ if $\{\zeta_1\}$ consists of a single element. A + B = union of events, AB = intersection of events, event = a subset of S, P(A) = probability of event A.

34.1.2 Corollaries of Probability

$$P(0), P(A) = 1 - P(\overline{A}) \le 1, (\overline{A} = \text{complement set of } A)$$

$$P(A + B) \neq P(A) + P(B), \quad P(A + B) = P(A) + P(B) - P(AB) \le P(A) + P(B)$$

Example

 $S = \{hh, ht, th, tt\}$ (tossing a coin twice), $A = \{heads \text{ at first tossing}\} = \{hh, ht\}$, $B = \{only \text{ one head came up}\} = \{ht, th\}$, $G = \{heads \text{ came up at least once}\} = \{hh, ht, th\}$, $D = \{tails \text{ at second tossing}\} = \{ht, tt\}$

34.2 Conditional Probabilities—Independent Events

34.2.1 Conditional Probabilities

$$P(A|M) = \frac{P(AM)}{P(M)} = \frac{\text{probability of event } AM}{\text{probability of event } M} = \text{conditional probability of } A \text{ given } M.$$

- 1. P(A|M) = 0 if AM = 0
- 2. $P(A|M) = \frac{P(A)}{P(M)} \ge P(A)$ if $AM = A \ (A \subset M)$
- 3. $P(A|M) = \frac{P(M)}{P(M)} = 1$ if $M \subset A$
- 4. P(A+B|M) = P(A|M) + P(B|M) if AB = 0

Example

$$P(f_i) = 1/6$$
, $i = 1, \dots 6$. $M = \{\text{odd}\} = \{f_1, f_3, f_5\}$, $A = \{f_1\}$, $AM = \{f_1\}$, $P(M) = 3/6$, $P(AM) = 1/6$, then $P(f_1|\text{even}) = P(AM)/P(M) = 1/3$

34.2.2 Total Probability

 $P(B) = P(B|A_1)P(A_1) + \dots + P(B|A_n)P(A_n)$ arbitrary event, $A_iA_j = 0$ $i \neq j = 1, 2, \dots n$, $A_1 + \dots + A_n = S = \text{certain event}$.

34.2.3 Baye's Theorem

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \dots + P(B|A_n)P(A_n)}$$

$$A_i A_j = 0$$
, $i \neq j = 1, 2, \dots n$, $A_1 + A_2 + \dots + A_n = S = \text{certain event}$, B=arbitrary

34.2.4 Independent Events

P(AB) = P(A)P(B) implies A and B are independent events.

34.2.5 Properties

1.
$$P(A|B) = P(A)$$

- 2. P(B|A) = P(B)
- 3. $P(A_1A_2 \cdots A_n) = P(A_1) \cdots P(A_n)$, $A_i = \text{ independent events}$
- 4. P(A + B) = P(A) + P(B) P(A)P(B)
- 5. $\overline{AB} = (\overline{A+B}), \ P(\overline{A+B}) = 1 P(A+B), \ P = (\overline{AB}) = P(\overline{A})P(\overline{B})$

If A nad B are independent. Overbar means complement set.

6. If
$$A_i, A_2, A_3$$
 are independent and A_1 is independent of A_2A_3 then
$$P(A_1A_2A_3) = P(A_1)P(A_2)P(A_3) = P(A_1)P(A_2A_3). \text{ Also } P[A_1(A_2+A_3)] = P(A_1A_2) + P(A_1A_3) - P(A_1A_2A_3) = P(A_1) \\ [P(A_2) + P(A_3) - P(A_2)P(A_3)] = P(A_1)P(A_2+A_3)$$

34.2.6
$$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

34.3 Compound (Combined, Experiments

34.3.1 $S=S_1 \times S_2 = Cartesian product$

Example

 $S_1 = \{1,2,3\}, S_2 = \{\text{heads, tails}\}, S = S_1 \times S_2 = \{(1 \text{ heads}), (1 \text{ tails}), (2 \text{ heads}), (2 \text{ tails}), (3 \text{ heads}), (3 \text{ tails})\}$

34.3.2 If
$$A_1 \subset S_1$$
, $A_2 \subset S_2$ then $A_1 \times A_2 = (A_1 \times S_2)(A_2 \times S_1)$ (see Figure 34.1)

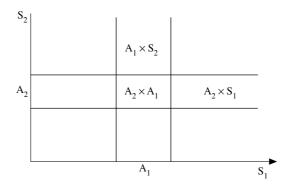


FIGURE 34.1

34.3.3 Probability in Compound Experiments

$$P(A_1) = P(A_1 \times S_2)$$
 where $\zeta_1 \in A_1$ and $\zeta_2 \in A_2$

34.3.4 Independent Compound Experiments

$$P(A_1 \times A_2) = P_1(A_1)P_2(A_2)$$

Example

P(heads) = p, P(tails) = q, p+q=1, $E = \text{experiment tossing the coin twice} = <math>E_1 \times E_2$ ($E_1 = \text{experiment of first tossing}$), $E_2 = \text{experiment of second tossing}$), $S_1 = \{h,t\}$ $P_1\{h\} = p$ $P_2\{t\} = q$, $E_2 = E_1 = \text{experiment of the second tossing}$, $S = S_1 \times S_2 = [hh,ht,th,tt]$, $P\{hh\} = P_1\{h\}P_2\{h\} = p^2 = \text{assume independence}$, $P\{ht\} = pq$, $P\{th\} = qp$, $P[t,t] = q^2$. For heads at the first tossing, $H_1 = \{hh,ht\}$ or $P(H_1) = P\{hh\} + P\{ht\} = p^2 + pq = p$

34.3.5 Sum of more Spaces

 $S = S_1 + S_2$, $S_1 =$ outcomes of experiment E_1 and $S_2 =$ outcomes of experiment E_2 . S = space of the experiment $E = E_1 + E_2$; $A = A_1 + A_2$ where A_1 and A_2 are events of E_1 and E_2 : $A_1 \subset S_1$, $A_2 \subset S_2$; $P(A) = P(A_1) + P(A_2)$.

34.3.6 Bernoulli Trials

P(A) = probability of event A, E×E×E×...×E = perform experiment n times = combined experiment.

$$p_n(k) = \binom{n}{k} p^k q^{n-k}$$
 = probability that events occurs k times in any order $P(A) = p$, $P(\overline{A}) = q$, $p+q=1$

Example

A fair die was rolled 5 times. $p_5(2) = \frac{5!}{(5-2)!2!} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{5-2} = \text{probability that "four" will come up twice.}$

Example

Two fair dice are tossed 10 times. What is the probability that the dice total seven points exactly four times?

Solution

Event
$$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}, P(B) = 6 \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{6} = p, P(\overline{8}) = 1 - p = \frac{5}{6}$$
. The probability of B occurring four times and \overline{B} six times is $\binom{10}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^6 = 0.0543$.

34.3.7

 $P\{k_1 \le k \le k_2\}$ = probability of success of A (event) will lie between k_1 and k_2

$$P\{k_1 \le k \le k_2\} = \sum_{k=k_1}^{k_2} p_n(k) = \sum_{k=k_1}^{k} \binom{n}{k} p^k q^{n-k}$$

1. Approximate value:
$$\sum_{k=k}^{k_2} \binom{n}{k} p^k q^{n-k} \cong \frac{1}{\sqrt{2\pi npq}} \sum_{k=k}^{k_2} e^{-(k-np)^2/2npq}, npq >> 1$$

34.3.8 DeMoivre-Laplace Theorem

$$p_n(k) = \binom{n}{k} p^k q^{n-k} \cong \frac{1}{\sqrt{2\pi npq}} e^{-(k-np)/2npq}, npq >> 1$$

34.3.9 Poisson Theorem

$$\frac{n!}{k!(n-k)!} p^k q^{n-k} \cong e^{-np} \frac{(np)^k}{k!} = e^{-a} \frac{a^k}{k!}, n \to \infty, p \to 0, np \to a$$

34.3.10 Random Points in Time

 $P\{k \text{ in } t_a\} \cong e^{-m_a l T} \frac{(nt | T)^k}{k!} = e^{-\lambda t_a} \frac{(\lambda t_a)^k}{k!}, t_2 - t_1 = t_a << T, \text{ n random points in (0,T)}, \ \lambda = n / T \text{ . If } n \to \infty, \quad T \to \infty, \quad n / T \to \lambda \text{ the approximation becomes equality.}$

1.
$$P\{\text{one in } t_a\} \cong e^{-\lambda t_a} \lambda t_a \cong \lambda t_a$$

2.
$$\lim_{t_a \to 0} \frac{P\{\text{one in } t_a\}}{t_a} = \lambda$$

3.
$$P(k \text{ in } (t_1, t_2)) = e^{-\int_{t_1}^{t_2} \lambda(t)dt} \frac{\left[\int_{t_1}^{t_2} \lambda(t)dt\right]^k}{k!}, \quad p = \int_{t_1}^{t_2} \alpha(t)dt, \quad n\alpha(t) = \lambda(t)$$

34.4 Random Variables

34.4.1 Random Variable

To every outcome ζ of any experiment we assign a number $X(\zeta) = x$. The function X, whose domain in the space S of all outcomes and its range is a set of numbers, is called a random variable (r.v.).

34.4.2 Distribution Function

 $F_x(x) = P\{X \le x\}$ defined on any number $-\infty < x < \infty$. $\{X \le x\}$ is an event for any real number x.

34.4.3 Properties of Distribution Function

1.
$$F(-\infty) = 0$$
, $F(+\infty) = 1$

2.
$$F(x_1) \le F(x_2)$$
 for $x_1 < x_2$

3.
$$F(x+) = F(x)$$
 continuous from the right

34.4.4 Density Function (or Frequency Function)

$$f(x) = \frac{dF(x)}{dx}; \quad f(x) = \lim_{\Delta x \to o} \frac{P\{x \le X \le x + \Delta x\}}{\Delta x}; \quad P\{X = x\} = 0 \text{ for continuous distribution function;}$$

$$f(x) = \sum_{i} p_{i} \delta(x - x_{i}) = \text{ density of discrete type, } \quad p_{i} = F(x_{i}) - F(x_{i} - x_{i}).$$

Example

Poisson distribution:
$$P\{X=k\} = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \dots, \ \lambda > 0$$
. Then $f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta(x-k)$.

Example

If X is normally distributed
$$\left(f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-m_x)^2/2\sigma^2}\right)$$
 with $m_x=1000$ and $\sigma=50$, then the probability

that X is between 900 and 1,050 is
$$P\{x_1 \le X \le x_2\} = \int_{-\infty}^{x_2} f(y) d(y) - \int_{-\infty}^{x_1} f(y) d(y) = \frac{1}{2} + \text{erf } \frac{x_2 - m_x}{\sigma}$$

$$-\frac{1}{2} - \operatorname{erf} \frac{x_1 - m_x}{\sigma} = \operatorname{erf} 1 + \operatorname{erf} 2 = 0.819 \text{ where error function of } x = \operatorname{erf} x = \frac{1}{\sqrt{2\pi}} \int_0^x \exp(-y^2/2) dy$$

34.4.5 Tables of Distribution Functions (see Table 34.1)

TABLE 34.1 Distribution and Related Quantities

Definitions

- 1. Distribution function (or cumulative distribution function [c.d.f.]): F(x) = probability that the variate takes values less than or equal to $x = P\{X \le x\} = \int_{-x}^{x} f(u) du$
- 2. Probability density function (p.d.f.):

$$f(x); P\{x_l < X \le x_u\} = \int_{x_l}^{x_u} f(x) dx; \ f(x) = \frac{dF(x)}{dx}$$

- 3. Probability function (discrete variates) $f(x) = \text{probability that the variate takes the value } x = P\{X = x\}$
- 4. Probability generating function (discrete variates):

$$P(t) = \sum_{x=0}^{\infty} t^x f(x), \quad f(x) = (1/x!) \left(\frac{\partial^x P(t)}{\partial t^x} \right)_{t=0}, \quad x = 0, 1, 2, \dots, X > 0$$

5. Moment generating function (m.f.g):

$$M(t) = \int_{-\infty}^{\infty} t^x f(x) dx. \quad M(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \dots + \frac{\mu_r' t^r}{r!} + \dots,$$

$$\mu_r' = r^{th} \text{ moment about the origin } = \int_{-\infty}^{\infty} x^r f(x) dx = \frac{\partial^r M(t)}{\partial t^r} \bigg|_{t=0}$$

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

- 6. Laplace transform of p.d.f.: $f^{L}(s) = \int_{0}^{\infty} e^{-sx} f(x) dx$, $X \ge 0$
- 7. Characteristic function : $\Phi(t) = \int_{e}^{\infty} \frac{i\pi}{f(x)} f(x) dx$, $\Phi_{X+Y}(t) = \Phi_X(t) \Phi_Y(t)$
- 8. Cumulant function: $K(t) = \log \Phi(t)$, $K_{X+Y}(t) = K_X(t) + K_Y(t)$
- 9. r^{th} cumulant: the coefficient of $(jt)^r/r!$ in the expansion of K(t)
- 10. rth moment about the origin:

$$\mu_r^{\odot} = \int_{-\infty}^{\infty} x^r f(x) dx = \left(\frac{\partial^r M(t)}{\partial t^r} \right) \Big|_{t=0} = (-j)^r \left(\frac{\partial^r \Phi(t)}{\partial t^r} \right) \Big|_{t=0}$$

- 11. Mean: $\mu = \text{ first moment about the origin } = \int_{-\infty}^{\infty} x f(x) dx = \mu_1'$
- 12. r^{th} moment about the mean: $\mu_r = \int_{-\infty}^{\infty} (x \mu)^r f(x) dx$
- 13. Variance: σ^2 second moment about the mean $=\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \mu_2$
- 14. Standard deviation: $\sigma = \sqrt{\sigma^2}$
- 15. Mean derivation: $\int_{-\infty}^{\infty} |x \mu| f(x) dx$
- 16. Mode: A fractile (value of r.v.) for which the p.d.f is a local maximum
- 17. Median: m =the fractile which is exceeded with probability 1/2.

18. Standardized rth moment about the mean:
$$\eta_r = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^r f(x) dx = \frac{\mu_r}{\sigma^r}$$

19. Coefficient of skewness: $\eta_3 = \mu_3 / \sigma^3$

20. Coefficint of kurtois: $\eta_4 = \mu_4 / \sigma^4$

21. Coefficient of variation: (standard deviation) / mean = σ / μ

22. Information content: $I = -\int_{0}^{\infty} f(x) \log_{2}(f(x)) dx$

23. rth factorial moment about the origin (discrete case):

$$\mu_{(r)}^{\circledcirc} = \sum_{x=0}^{\infty} f(x)x(x-1)\cdots(x-r+1), \quad X \geq 0, \quad \mu_{(r)}^{\circledcirc} = \left(\frac{\partial^r P(t)}{\partial t^r}\right)_{t=1}$$

24. rth factorial moment moment about the mean (discrete case):

$$\mu_{(r)} = \sum_{x=0}^{\infty} f(x-\mu)(x-\mu)(x-\mu-1)\cdots(x-\mu-r+1), \ X \ge 0$$

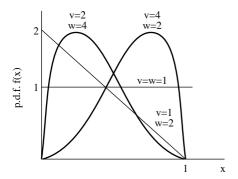
25. Relationships between moments:

$$\mu'_r = \sum_{i=0}^r \binom{r}{i} \mu_{r-i}(\mu'_1); \quad \mu_r = \sum_{i=0}^r \binom{r}{i} \mu'_{r-i}(-\mu'_1)^i, \quad \mu_0 = \mu'_0 = 1, \quad \mu_1 = 0$$

26. log is the natural logarithm

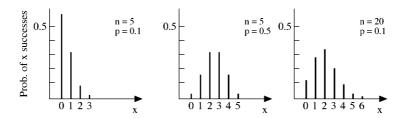
Distributions

1. Beta: p.d.f = $f(x) = x^{v-1}(1-x)^{w-1} / B(v,w)$ $0 \le x \le 1$, $B(v,w) = \text{beta function} = \int_0^1 u^{v-1}(1-u)^{w-1} du$; r^{th} moment about the origin $\prod_{i=0}^{r-1} (v+i)(v+w+i)$; mean = v/(v+w); variance = $vw/(v+w)^2(v+w+1)$; mode = (v-1)/(v+w+2), v>1, w>1; coefficient of skewness: $[2(w-v)(v+w+1)^{1/2}]/[(v+w+2)(vw)^{1/2}]$; coefficient of kurtois: ([3(v+w)(v+w+1)

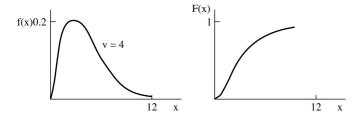


2. Binominal: n, p is the number of successes in n independent Bernoulli trials where the probability of success at each trial is p and the probability of failure is q = 1 - p, n = positive integer $0 . c.d.f <math>= F(x) = \sum_{i=0}^{x} {n \choose i} p^{i} q^{n-i}$, x = 1 - p

integer; p.d.f. $= f(x) = \binom{n}{x} p^x q^{n-x}$, x = integer; moment generating function: $[p \exp(t) + q]^n$; probability generating function: $(pt+q)^n$; characteristic function: $\Phi(t) = [p \exp(jt) + q]^n$. moments about the origin: mean=np, second = np(np+q), third = $np[(n-1)(n-2)p^2 + 3p(n-1) + 1]$; moment about the mean: variance = npq, third = npq[1+3pq(n-2)] standard deviation : $(npq)^{1/2}$; mode: $p(n+1)-1 \le x \le p(n+1)$; coefficient of skewness: $(q-p)/(npq)^{1/2}$; coefficient of kurtois: 3-(6/n)+(1/npq); factorial moments about the mean: second = npq, third = -2npq(1+q); coefficient of variation = $(q/np)^{1/2}$

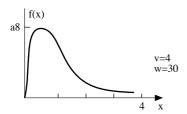


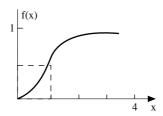
- **3. Cauchy:** p.d.f = $f(x) = 1/[\pi b[(x-a)/b]^2 + 1]]$, $\alpha = \text{shift parameter}$, b= ,scale parameter, $-\infty < x < \infty$; mode = a median = a
- **4. Chi-Squared:** p.d.f. $f(x) = [x^{(\nu-2)/2} \exp(-x/2)]/[2^{\nu/2}\Gamma(\nu/2)]$, ν (shape parameter) = degrees of freedom, $0 \le x < \infty$; moment generating function: $(1-2t)^{-\nu/2}$, t > 1/2; characteristic function: $\Phi(t) = (1-2jt)^{-\nu/2}$; cumulant function: $(-\nu/2)\log(1-2jt)$; r^{th} cumulant; $2^{r-1}\nu[(r-1)!]$; r^{th} moment about the origin: $2^r\prod_{i=0}^{r-1}[i+(\nu/2)]$; mean = ν ; variance: 2ν ;



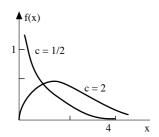
standard deviation $(2v)^{1/2}$; Laplace transform of the p.d.f: $(1+2s)^{-\nu/2}$

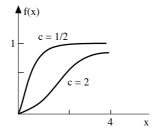
- **5. Discrete uniform:** $a \le x \le a+b-1$, x = integer, a = lower limit of the range, b = scale parameter; c.d. f = F(x) = (x-a+1)/b; p.d.f.= f(x) = 1/b; probability generating function: $(t^a t^{a=b})/(1-t)$; characteristic function: $\exp[j(a-1)t] = \sinh(jtb/2)\sinh(jt/2)/b$; mean: a + (b-1)/2; variance: $(b^2 1)/12$; coefficient of skewness 0; information content: $\log_2 b$.
- **6. Exponential:** $0 \le x < \infty$, b = scale parameter = mean, $\lambda = 1/b = \text{alternative parameter}$; c.d. $f = F(x) = 1 \exp(-x/b)$; p.d. $f = f(x) = (1/b)\exp(-x/b)$; moment generating function: 1/(1-bt), t > (1/b); Laplace transform of the p.d.f: 1/(1+bs); characteristic function: 1/(1-jbt); cumulant function: $-\log(1-jbt)$; r^{th} cumulant: $(r-1)!b^r$; r^{th} moment about the origin: $r!b^r$; mean: b: variance: b^2 ; standard deviation: b; mean deviation: b by b; b0 (b1) (b2); mode: b3; coefficient of skewness: b3; coefficient of variation: b3; information content: $\log_2(eb)$.
- **7. F-distribution:** $0 \le x < \infty$, v and $w = \text{positive integers} = \text{degrees of freedom: p.d.} f = f(x) = [\Gamma[\frac{1}{2}(v+w)](v/w)^{v/2}] \times \frac{(v-2)/2}{2} / [\Gamma(\frac{1}{2}v)\Gamma(\frac{1}{2}w)(1+xv/w)^{(v+w)/2}]$; rth moment about the origin: $[(w/v)^r\Gamma(\frac{1}{2}v+r)\Gamma(\frac{1}{2}w-r)/[\Gamma(\frac{1}{2}v)\Gamma(\frac{1}{2}w)]$, w > 2r; mean: w/(w-2), w > 2; variance: $[2w^2(v+w-2)]/[v(w-2)^2(w-4)]$, w > 4; mode [w(v-2)]/[v(w+2)], v > 1; coefficient of skewness: $[(2v+w-2)[8(w-4)]^{1/2}]/[(w-6)(v+w-2)^{1/2}]$, w > 6; coefficient of variation: $[[2(v+w-2)/[v9w-4)]]^{1/2}$, w > 4.



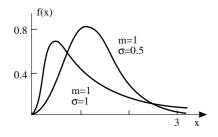


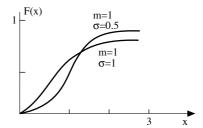
8. Gamma: $0 \le x < \infty$, b = scal e parameter > 0 (or $\lambda = 1/b$), c>0 shaper parameter; p.d.f = $f(x) = (x/b)^{c-1}$ [exp(-x/b)]/[$b\Gamma(c)$], $\Gamma(c) = \int_0^\infty \exp(-u)u^{c-1}du$; moment generating function: $(1-bt)^{-c}$, t > 1/b; Laplace fransform of the p.d.f.: $(1+bs)^{-c}$; characteristic function: $(1-jbt)^{-c}$; cumulant function: $-c \log 91 - jbt$); r^{th} cumulant: $(r-1)!cb^r$; r^{th} moment about the origin: $b^r \prod_{i=0}^{r-1} (c+i)$; mean: bc; variance: b^2c ; standard deviation: $b\sqrt{c}$; mode: b(c-1), $c \ge 1$; coefficient of skewness: $2c^{-1/2}$; coefficient of kurtosis: 3 + 6/c: coefficient of variation: $c^{-1/2}$





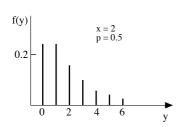
9. Logonormal: $0 \le x < \infty$, m = scale parameter = median > 0, $\mu = \text{mean of } \log X > 0$, $m = \exp \mu$, $\mu = \log m$, $\sigma = \text{shape parameter} = \text{standard deviation of } \log X$, $w = \exp(\sigma^2)$; p.d.f = $f(x) = [1/x\sigma(2\pi)^{1/2}] \exp[-[\log(x/m)]^2/2\sigma^2]$ rth moment about the origin: $m' \exp(r^2\sigma^2/2)$; mean: $m \exp(\sigma^2/2)$; variance: $m^2w(w-1)$: standard deviation: $m(w^2-w)^{1/2}$; mode m/w; median; m; coefficient of skewness: $(w+2)(w-1)^{1/2}$; coefficient of kurtosis: $w^4+2w^3+3w^2-3$; coefficient of variation: $(w-1)^{1/2}$.

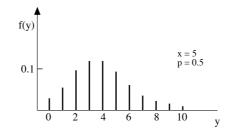




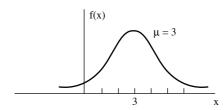
10. Negative bionomial: $y = \text{number of failures (integer)}, \ x = \text{number of failures before } x^{th} \text{ success in a sequence of Bernoulli trials; } p = \text{probability of success at each trial}, \ q = 1 - p, \ 0 \le y < \infty, \ 0 < p < 1; \ \text{c.d.f.} = \ F(y) = \sum_{i=1}^{y} \binom{x+i-1}{i} p^x q^i;$ p.d.f. = $f(y) = \binom{x+y-1}{y} p^x q^y$; moment generating function: $p^x (1-q\exp t)^{-x}$; probability generating function: $p^x (1-qt)^{-x}$; characteristic function: $p^x [1-q\exp(jt)]^{-x}$; cumulant function: $x \log(p) - x \log(1-q\exp t)$; Cumulants: first = xq/p, second = xq/p^2 , third = $xq(1+q)/p^3$, fourth = $xq(6q+p^2)/p^4$; mean: xq/p; Moments about the mean: variance = xq/p^2 , third = $xq(1+q)/p^3$, fourth = $(xq/p^4)(3xq+6q+p^2)$; standard deviation: $(xq)^{1/2}/p$; coefficient of skewness:

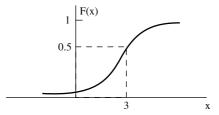
 $(1+q)(xq)^{-1/2}$; coefficient of kurtosis: $3+\frac{6}{x}+\frac{p^2}{xq}$; factorial moment generating function: $(1-q^t/p)^{-x}$ rth factorial moment about the origin: $(q/p)^r(x+r-1)^r$; coefficient of variaton: $(xq)^{-1/2}$.



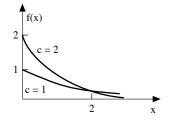


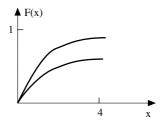
11. Normal: $-8 < x < \infty$, $\mu = \text{mean} = \text{location parameter}$, $\sigma = \text{standard deviation} = \text{scale parameter}$, $\sigma > 0$; p.d.f. = $f(x) = [1/\sigma(2\pi)^{1/2}] \exp[-(x-\mu)^2/2\sigma^2]$; moment generating function: $\exp(\mu t + \frac{1}{2}\sigma^2 t^2)$; characteristic function: $\exp(j\mu t - \frac{1}{2}\sigma^2 t^2)$; cumulant function: $j\mu t - \frac{1}{2}\sigma^2 t^2$; r^{th} cumulant: $K_2 = \sigma^2$, $K_r = 0$, r > 2; mean: μ r^{th} moment about the mean: $\mu_r = 0$ for r odd, $\mu_r = (\sigma^r r!)/[2^{r/2}[(r/2)!]]$ for r even; variance: σ^2 ; standard deviation: σ ; mean deviation: $\sigma(2/\pi)^{1/2}$; mode: μ ; median: μ ; coefficient of skewness: 0; coefficient of kurtosis: 3; information content: $\log_2[\sigma(2\pi e)^{1/2}]$





12. Pareto: $1 \le x < \infty$, c = shape parameter; $c.d.f. = F(x) = 1 - x^{-c}$; $p.d.f. = f(x) = cx^{-c-1}$; r^{th} moment about the origin: c/(c-r), c > r; mean: c/(c-1), c > 1; variance: $[c/(c-2)] - [c/(c-1)]^2$, c > 2; coefficient of variation: $(c-1)/[c(c-1)]^{1/2}$, c > 2.





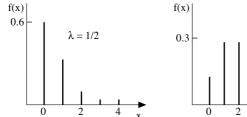
13. Pascal: n = number of teals, $n \ge 1$, x = the Bernoulli success parameter = the number of trials up to and including the x^{th} success, p = probability of success at each trial, $0 ; p.d.f. = <math>f(n) = \binom{n-1}{n-x} p^x q^{n-x}$; moment generating function: $p^x \exp(tx)/(1-q\exp t)^x$ probability generating function: $(pt)^x/(1-qt)^x$; characteristic function: $p^x \exp(jtx)/(1-q\exp(jt)^x)$; mean: x/p; variance: xq/p^2 ; standard deviation: $(xq)^{1/2}/p$; coefficient of variation: $(q/x)^{1/2}$.

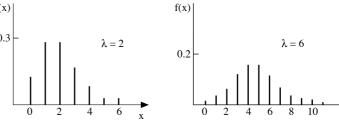
14. Poisson: $0 \le x < \infty, \lambda = \text{mean (a parameter)}; \text{ c.d.f.} = F(x) = \sum_{i=0}^{x} \lambda^{i} \exp(-\lambda)/i!; \text{ p.d.f.} = f(x) = \lambda^{x} \exp(-\lambda)/x!;$ moment generating function: $\exp[\lambda[\exp(t)-]];$ probability generating function: $\exp[-\lambda(1-t)];$ characteristic function:

 $\exp[\lambda[\exp(jt) - 1]]$; cumulant function: $\lambda[\exp(t) - 1] = \sum_{i=0}^{\infty} t^i / i!$; r^{th} cumulant: λ ; moment about the origin: mean= λ , second=

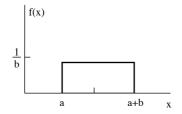
 $\lambda + \lambda^2; \text{ third= } \lambda[(\lambda+1)^2+\lambda], \text{fourth } = \lambda(\lambda^3+6\lambda^2+7\lambda+1); \\ r^{\text{th}} \text{ moment about the mean, } \mu_i: \lambda \sum_{i=0}^{r-2} \binom{r-1}{i} \mu_i, \\ r > 1, \\ \mu_0 = 1.$

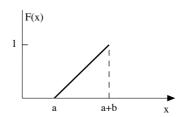
Moments about the mean: variance = λ , third= λ , fourth = $\lambda(1+3\lambda)$, fifth = $\lambda(1+10\lambda)$, sixth = $\lambda(1+25\lambda+15\lambda^2)$; standard deviation = $\lambda^{1/2}$; coefficient of skewness: $\lambda^{-1/2}$; coefficient of kurtosis: $3+1/\lambda$; factorial moments about the mean: second = λ , third = -2λ , fourth = $3\lambda(\lambda+2)$; coefficient of variation: $\lambda^{-1/2}$.



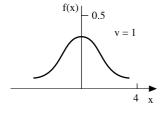


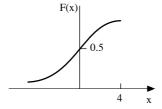
15. Rectangular: $a \le x \le a+b$, x= range, a = lower limit,b=scale parameter; c.d.f = F(x) = (x-a)/b; p.d.f. = f(x) = 1/b; moment generating function: $\exp(at)[\exp(bt)-1]/bt$; Laplace transform of the p.d.f: $\exp(-as)[1-\exp(-bs)]/bs$; characteristic function: $\exp(jat)[\exp(jbt)-1]/jbt$; mean: a+b/2; r^{th} moment about the mean: $\mu_r = 0$ for r odd, $\mu_r = (b/2)^r/(r+1)$ for r even; variance: $b^2/12$; standard deviation: $b/\sqrt{12}$; mean deviation b/4 ; median a+b/2 ; standardized r^{th} moment about the mean: $\mu_r = 0$ for r odd, $\mu_r = 3^{r/2}/(r+1)$ for r even; coefficient of skewness: 0; coefficient of kurtosis: 915; coefficient of variation: $b/[3^{1/2}(2a+b)]$; information content: $\log_2 b$.



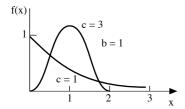


16. Student's: $-\infty < x < \infty$, v = shape parameter (degrees of freedom), $v \equiv$ positive integer; p.d.f. $= f(x) = [\Gamma[(v+1)/2] [1+(x^2/v)]^{-(v+1)/2}]/[(\pi v)^{1/2}\Gamma(v/2)]$; mean: 0; r^{th} moment about the mean: $\mu_r = 0$ for r odd, $\mu_r = [1 \cdot 3 \cdot 5 \cdots (r-1)v^{r/2}]/[(v-2)(v-4)\cdots(v-r)]$ for r even, r < v: variance: v/(v-2), v > 2; mean deviation: $v^{1/2}\Gamma(\frac{1}{2}(v-1)/\pi^{1/2}\Gamma(\frac{1}{2}v))$; mode: 0; coefficient of skewness and kurtosis: 0





17. Weibull: $0 \le x < \infty, b > 0$ scale parameter, c = shape aprameter c > 0; $c.d.f. = F(x) = 1 - \exp[-(x/b); \text{ p.d.f.} = f(x)] = (cx^{c-1}/b^c) \exp[-(x/b)^c]$; r^{th} moment about the origin: $b^r \Gamma[(c+r)/c]$; mean: $b\Gamma[(c+1)/c]$.



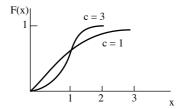


TABLE 34.2 Normal Distribution Tables.

 $f(x) = \text{distribution density} = (1/\sqrt{2\pi})e^{-x^2/2}, \quad F(x) = \text{cumulative distribution function} = \int_{-\infty}^{x} (1/\sqrt{2\pi})e^{-\gamma^2/2}d\tau$

 $f'(x) = -xf(x), f''(x) = (x^2 - 1)f(x), \quad F(-x) = 1 - F(x), \quad P\{-x < X < x\} = 2F(x) - 1$

х	F(x)	f(x)	f'(x)	f''(x)	х	F(x)	f(x)	f'(x)	f''(x)
.00	.5000	.3989	0000	3989	.50	.6915	.3521	1760	2641
.01	.5040	.3989	0000	3989	.51	.6950	.3503	1787	2592
.02	.5080	.3989	0080	3987	.52	.6985	.3485	1812	2543
.03	.5120	.3988	0120	3984	.53	.7019	.3467	1837	2493
.04	.5160	.3986	0159	3980	.54	.7054	.3448	1862	2443
.05	.5199	.3984	0199	3975	.55	.7088	.3429	1886	2392
.06	.5239	.3982	0239	3968	.56	.7123	.3410	1920	2341
.07	.5279	.3980	0279	3960	.57	.7157	.3391	1933	2289
.08	.5319	.3977	0318	3951	.58	.7190	.3372	1956	2238
.09	.5359	.3973	0358	3941	.59	.7224	.3352	1978	2185
.10	.5398	.3970	0199	3975	.60	.7257	.3332	1999	2133
.11	.5438	.3965	0239	3968	.61	.7291	.3312	2020	2080
.12	.5478	.3961	0279	3960	.62	.7324	.3292	2041	2027
.13	.5517	.3956	0318	3951	.63	.7357	.3271	2061	1973
.14	.5557	.3951	0358	3941	.64	.7389	.3251	2080	1919
.15	.5596	.3945	0592	3856	.65	.7422	.3230	2099	1865
.16	.5636	.3939	0630	3838	.66	.7454	.3209	2118	1811
.17	.5675	.3932	0668	3819	.67	.7486	.3187	2136	1757
.18	.5714	.3925	0707	3798	.68	.7517	.3166	2153	1702
.19	.5753	.3918	0744	3777	.69	.7549	.3144	2170	1647
.20	.5793	.3910	0782	3754	.70	.7580	.3132	2186	1593
.21	.5832	.3902	0820	3730	.71	.7611	.3101	2201	1538
.22	.5871	.3894	0857	3706	.72	.7642	.3079	2217	1483
.23	.5910	.3885	0894	3680	.73	.7673	.3056	2231	1428
.24	.5948	.3876	0930	3653	.74	.7704	.3034	2245	1373
.25	.5987	.3867	0967	3625	.75	.7734	.3011	2259	1318
.26	.6026	.3857	1003	3596	.76	.7764	.2989	2271	1262
.27	.6064	.3847	1039	3566	.77	.7794	.2966	2284	1207
.28	.6103	.3836	1074	3535	.78	.7823	.2943	2296	1153
.29	.6141	.3825	1109	3504	.79	.7852	.2920	2307	1098
.30	.6179	.3814	1144	3471	.80	.7881	.2897	2318	1043
.31	.6217	.3802	1179	3437	.81	.7910	.2874	2328	0988
.32	.6255	.3791	1213	3402	.82	.7939	.2850	2337	0934
.33	.6293	.3778	1247	.3367	.83	.7967	.2827	2346	0880
.34	.6331	.3765	1280	3330	.84	.7995	.2803	2355	0825
	.0001	.5705	1200	.5550	.07	.1773	.2005		.0023

TABLE 34.2 Normal Distribution Tables. (continued)

 $f(x) = \text{ distribution density} = (1/\sqrt{2\pi})e^{-x^2/2}, \quad F(x) = \text{ cumulative distribution function} = \int_{-\infty}^{x} (1/\sqrt{2\pi})e^{-\gamma^2/2}d\tau \ ,$

 $f'(x) = -xf(x), f''(x) = (x^2 - 1)f(x), \quad F(-x) = 1 - F(x), \quad P\{-x < X < x\} = 2F(x) - 1$

•									
х	F(x)	f(x)	f'(x)	f''(x)	х	F(x)	f(x)	f'(x)	f''(x)
.35	.6368	.3752	1313	3293	.85	.8023	.2780	2363	0771
.36	.6406	.3739	1346	3255	.86	.8051	.2756	2370	0718
.37	.6443	.3725	1378	3216	.87	.8078	.2732	2377	0664
.38	.6480	.3712	1410	3176	.88	.8106	.2709	2384	0611
.39	.6517	.3697	1442	3135	.89	.8133	.2685	2389	0558
.40	.6554	.3683	1473	3094	.90	.8159	.2661	2395	0506
.41	.6591	.3668	1504	3015	.91	.8186	.2637	2400	0453
.42	.6628	.3653	1534	3008	.92	.8212	.2613	2404	0401
.43	.6664	.3637	1564	2965	.93	.8238	.2589	2408	0350
.44	.6700	.3621	1593	2920	.94	.8264	.2565	2411	0299
.45	.6736	.3605	1622	2875	.95	.8289	3.2541	2414	0248
.46	.6772	.3589	1651	2830	.96	.8315	.2516	2416	0197
.47	.6808	.3572	1679	2783	.97	.8340	.2492	2417	0147
.48	.6844	.3555	1707	2736	.98	.8365	.2468	2419	0098
.49	.6879	.3538	1734	2689	.99	.8389	.2444	2420	0049
.50	.6915	.3521	1760	2641	1.00	.8413	.2420	2420	0000
1.00	.8413	.2420	2420	.0000	1.50	.9332	.1295	1943	.1619
1.01	.8438	.2396	2420	.0048	1.51	.9345	.1276	1927	.1633
1.02	.8461	.2371	2419	.0096	1.52	.9357	.1257	1910	.1647
1.03	.8485	.2371	2418	.0143	1.53	.9370	.1238	1894	.1660
1.04	.8508	.2323	2416	.0190	1.54	.9382	.1219	1877	.1672
1.05	.8531	.2299	2414	.0236	1.55	.9394	.1200	1860	.1683
1.06	.8554	.2275	2411	.0281	1.56	.9406	.1182	1843	.1694
1.07	.8577	.2251	2408	.0326	1.57	.9418	.1163	1826	.1704
1.08	.8599	.2227	2405	.0371	1.58	.9429	.1145	1809	.1714
1.09	.8621	.2203	2401	.0414	1.59	.9441	.1127	1792	.1722
1.10	.8643	.2176	2396	.0458	1.60	.9452	.1109	1775	.1730
1.11	.8665	.2155	2392	.0500	1.61	.9463	.1092	1757	.1738
1.12	.8686	.2131	2386	.0542	1.62	.9474	.1074	1740	.1745
1.13	.8708	.2107	2381	.0583	1.63	.9484	.1057	1723	.1751
1.14	.8729	.1083	2375	.0624	1.64	.9495	.1040	1705	.1757
1.15	.8749	.2059	2368	.0664	1.65	.9505	.1023	1687	.1762
1.16	.8770	.2036	2361	.0704	1.66	.9515	.1006	1670	.1766
1.17	.8790	.2012	1354	.0742	1.67	.9525	.0989	1652	.1770
1.18	.8810	.1989	2347	.0780	1.68	.9535	.0973	1634	.1773
1.19	.8830	.1965	2339	.0818	1.69	.9545	.0957	1617	.1776
1.20	.8849	.1942	2330	.0854	1.70	.9554	.0940	1599	.1778
1.21	.8869	.1919	2322	.0890	1.71	.9564	.0925	1581	.1779
1.22	.8888	.1895	2312	.0926	1.72	.9573	.0909	1563	.1780
1.23	.8907	.1872	2303	.0960	1.73	.9582	.0893	1546	.1780
1.24	.8925	.1849	2293	.0994	1.74	.9591	.0878	1528	.1780
1.25	.8944	.1826	2283	.1027	1.75	.9599	.0863	1510	.1780
1.26	.8962	.1804	2273	.1060	1.76	.9608	.0848	1492	.1778
1.27	.8980	.1781	2262	.1092	1.77	.9616	.0833	1474	.1777
	.0700	.1,01	0_	.1072	1.//	.5010	.0055	.17/7	.1,77

TABLE 34.2 Normal Distribution Tables. (continued)

 $f(x) = \text{ distribution density} = (1/\sqrt{2\pi})e^{-x^2/2}, \quad F(x) = \text{ cumulative distribution function} = \int_{-\infty}^{x} (1/\sqrt{2\pi})e^{-\gamma^2/2}d\tau \ ,$

 $f'(x) = -xf(x), f''(x) = (x^2 - 1)f(x), \quad F(-x) = 1 - F(x), \quad P\{-x < X < x\} = 2F(x) - 1$

x	F(x)	f(x)	f'(x)	f''(x)	x	F(x)	f(x)	f'(x)	f"(x)
1.28	.8997	.1758	2251	.1123	1.78	.9625	.0818	1457	.1774
1.29	.9015	.1736	2240	.1153	1.79	.9633	.0804	1439	.1772
1.30	.9032	.1714	2228	.1182	1.80	.9641	.0790	1421	.1769
1.31	.9049	.1691	2204	.1211	1.81	.9649	.0775	1403	.1765
1.32	.9066	.1669	2204	.1239	1.82	.9556	.0761	1386	.1761
1.33	.9082	.1647	2191	.1267	1.83	.9664	.0748	1368	.1756
1.34	.9099	.1626	2178	.1293	1.84	.9671	.0734	1351	.1751
1.35	.9115	.1604	2165	.1319	1.85	.9678	.0721	1333	.1746
1.36	.9131	.1582	2152	.1344	1.86	.9686	.0707	1316	.1740
1.37	.9147	.1561	2138	.1369	1.87	.9693	.0694	1298	.1734
1.38	.9162	.1539	2125	.1392	1.88	.9699	.0681	1281	.1727
1.39	.9177	.1518	2110	.1415	1.89	.9706	.0689	1264	.1720
1.40	.9192	.1479	2096	.1437	1.90	.9713	.0656	1247	.1713
1.41	.9207	.1476	2082	.1459	1.91	.9719	.0644	1230	.1705
1.42	.9222	.1456	2067	.1480	1.92	.9726	.0632	1213	.1697
1.43	.9236	.1435	2052	.1500	1.93	.9732	.0620	1196	.1688
1.44	.9251	.1415	2037	.1519	1.94	.9738	.0608	1179	.1679
1.77	.5251	.1415	.2037	.1317	1.54	.5750		.1175	.1075
1.45	.9265	.1394	2022	.1537	1.95	.9744	.0596	1162	.1670
1.46	.9279	.1374	2006	.1555	1.96	.9750	.0584	1145	.1661
1.47	.9306	.1354	1991	.1572	1.97	.9756	.0573	1129	.1651
1.48	.9306	.1334	1975	1588	1.98	.9761	0.562	1112	.1641
1.49	.9319	.1315	1959	.1604	1.99	.9767	.0551	1096	.1630
1.50	.9332	.1295	1943	.1619	2.00	.9772	.0540	1080	.1622
2.00	.9773	.0540	1080	.1620	2.50	.9938	.0175	0438	.0920
2.01	.9778	.0529	1064	.1609	2.51	.9940	.0171	0429	.0906
2.02	.9783	.0519	1048	.1598	2.52	.9941	.0167	0420	.0892
2.03	.9788	.0508	1032	.1586	2.53	.9943	.0163	0411	.0868
2.04	.9793	.0498	1016	.1575	2.54	.9945	.0158	0403	.0878
2.05	.9798	.0488	1000	.1563	2.55	.9946	.0155	0394	.0850
2.06	.9803	.0478	0985	.1550	2.56	.9948	.0151	0386	.0836
2.07	.9809	.0468	0969	.1538	2.57	.9949	.0147	0377	.0823
2.08	.9812	.0459	0954	.1526	2.58	.9951	.0143	0369	.0809
2.09	.9817	.0449	0939	.1513	2.59	.9952	.0139	0361	.0796
2.10	.9821	.0440	0924	.1500	2.60	.9953	.0136	0353	.0782
2.11	.9826	.0431	0909	.1487	2.61	.9955	.0132	.0345	.0769
2.12	.9830	.0422	0894	.1474	2.62	.9956	.0129	0338	.0756
2.13	.9834	.0413	0879	.1460	2.63	.9957	.0126	0330	.0743
2.14	.9838	.0404	0865	.1446	2.64	.9959	.0122	0323	.0730
2.15	.9842	.0396	0850	.1433	2.65	.9960	.0119	0316	.0717
2.16	.9846	.0387	0836	.1419	2.66	.9961	.0119	0310	.0705
2.17	.9850	.0379	0822	.1419	2.67	.9962	.0113	0309	.0692
2.17	.9854	.0379	0822	.1403	2.68	.9962	.0113	0302	.0692
2.19	.9857	.0363	0794	.1377	2.69	.9964	.0110	0293	.0668
2.17	.7031	.0303	0/74	.1377	2.07	.990 4	.0107	0200	.0000

TABLE 34.2 Normal Distribution Tables. (continued)

 $f(x) = \text{distribution density} = (1/\sqrt{2\pi})e^{-x^2/2}, \quad F(x) = \text{cumulative distribution function} = \int_{-\infty}^{x} (1/\sqrt{2\pi})e^{-y^2/2}d\tau$

 $f'(x) = -xf(x), f''(x) = (x^2 - 1)f(x), F(-x) = 1 - F(x), P\{-x < X < x\} = 2F(x) - 1$

) (50)	$\mathcal{A}_{j}(\mathcal{A}), j$	(30) (30	1)) (x),	$\Gamma(-x) - 1 - \Gamma(x),$	$\Gamma\{-x < x < x\}$	21 (3)	•		
х	F(x)	f(x)	f'(x)	f''(x)	x	F(x)	f(x)	f'(x)	f''(x)
2.20	.9861	.0355	0780	.1362	2.70	.9965	.0104	0281	.0656
2.21	.9864	.0347	0767	.1348	2.71	.9966	.0101	0275	.0644
2.22	.9868	.0339	0754	.1333	2.72	.9967	.0099	0269	.0632
2.23	.9871	.0332	0740	.1319	2.73	.9968	.0096	0262	.0620
2.24	.9875	.0325	0727	.1304	2.74	.9969	.0093	0256	.0608
2.25	.9868	.0317	0714	.1289	2.75	.9970	.0091	0250	.0597
2.26	.9881	.0310	0701	.1275	2.76	.9971	.0088	0244	.0585
2.27	.9884	.0303	0689	.1260	2.77	.9972	.0086	0238	.0574
2.28	.9887	.0297	0676	.1245	2.78	.9973	.0084	0233	.0563
2.29	.9890	.0290	0664	.1230	2.79	.9974	.0081	0227	.0562
2.30	.9893	.0283	0652	.1215	2.80	.9974	.0079	0222	.0541
2.31	.9896	.0277	0639	.1200	2.81	.9975	.0077	0216	.0531
2.32	.9898	.0270	0628	.1185	2.82	.9976	.0075	0211	.0520
2.33	.9901	.0264	0616	.1170	2.83	.9977	.0073	0206	.0510
2.34	.9904	.2058	0604	.1155	2.84	.9977	.0071	0201	.0500
2.35	.9906	.0252	0593	.1141	2.85	.9978	.0069	0196	.0490
2.36	.9909	.0246	0581	.1126	2.86	.9979	.0067	0191	.0480
2.37	.9911	.0241	0570	.1111	2.87	.9979	.0065	0186	.0470
2.38	.9913	.0235	0559	.1096	2.88	.9980	.0063	0182	.0460
2.39	.9916	.0229	0548	.1081	2.89	.9981	.0061	0177	.0451
2.40	.9918	.0224	0538	.1066	2.90	.9981	.0060	0173	.0441
2.41	.9920	.0219	0527	.1051	2.91	.9982	.0058	0168	.0432
2.42	.9922	.0213	0516	.1036	2.92	.9982	.0056	0164	.0423
2.43	.9925	.0208	0506	.1022	2.93	.9983	.0055	0160	.0414
2.44	.9927	.0203	0496	.1007	2.94	.9984	.0053	0156	.0405
2.45	.9929	.0198	0486	.0992	2.95	.9984	.0051	0152	.0396
2.46	.9931	.0194	0476	.0978	2.96	.9985	.0050	0148	.0388
2.47	.9932	.0189	0467	.0963	2.97	.9985	.0048	0144	.0379
2.48	.9934	.0184	0457	.0949	2.98	.9986	.0047	0140	.0371
2.49	.9936	.0180	0448	.0935	2.99	.9986	.0046	0137	.0363
2.50	.9938	.0175	0438	.0920	3.00	.9987	.0044	0133	.0355
3.00	.9987	.0044	0133	.0355	3.50	.9998	.0009	0031	.0098
3.05	.9989	.0038	0116	.0316	3.55	.9998	.0007	0026	.0085
3.10	.9990	.0033	0101	.0281	3.60	.9998	.0006	0022	.0073
3.15	.9992	.0028	0088	.0249	3.65	.9999	.0005	0019	.0063
3.20	.9993	.0024	0076	.0220	3.70	.9999	.0004	0016	.0054
3.25	.9994	.0020	0066	.0194	3.75	.9999	.0004	0013	.0046
3.30	.9995	.0017	0057	.0170	3.80	.9999	.0003	0011	.0039
3.35	.9996	.0015	0049	.0149	3.85	.9999	.0002	0009	.0033
3.40	.9997	.0012	0042	.0130	3.90	1.0000	.0002	0008	.0028
3.45	.9997	.0010	0036	.0113	3.95	1.0000	.0002	0006	.0024
3.50	.9998	.0009	0031	.0098	4.00	1.0000	.0001	0005	.0020

TABLE 34.3 Student t-Distribution Table

$$f(x) = \int_{-\infty}^{x} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi} \Gamma 9n/2} \left(1 + \frac{y^2}{n}\right)^{-(n+1)/2} dy$$

n= number of degrees of freedom, numbers give x of distribution, e.g., for n=6 and F=0.975, x=2.447, F(-x)=1-F(x)

n \ F	.60	.75	.90	.95	.975	.99	.995	.9995
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	.289	.816	1.886	2.920	4.303	6.965	9.925	31.598
3	.277	.765	1.638	2.353	3.182	4.541	5.841	12.924
4	.271	.741	1.533	2.132	2.776	3.747	4.604	8.610
5	.267	.727	1.476	2.015	.2.571	3.365	4.032	6.869
6	.265	.718	1.440	1.943	2.447	3.143	3.707	5.959
7	.263	.711	1.415	1.895	2.365	2.998	3.499	5.408
8	.262	.706	1.397	1.860	2.306	2.896	3.555	5.041
9	.261	.703	1.383	1.833	2.262	2.821	3.250	4.781
10	.260	.700	1.372	1.812	2.228	2.764	3.169	4.587
11	.260	.697	1.363	1.796	2.201	2.718	3.106	4.437
12	.259	.695	1.356	1.782	2.179	2.681	3.055	4.318
13	.259	.694	1.350	1.771	2.160	2.650	3.012	4.221
14	.258	.692	1.345	1.761	2.145	2.624	2.977	4.140
15	.258	.691	1.341	1.753	2.131	2.602	2.947	4.073
16	.258	.690	1.337	1.746	2.120	2.583	2.921	4.015
17	.257	.689	1.333	1.740	2.110	2.567	2.898	3.965
18	.257	.688	1.330	1.734	2.101	2.552	2.878	3.922
19	.257	.688	1.328	1.729	2.093	2.539	2.861	3.883
20	.257	.687	1.325	1.725	2.086	2.528	2.845	3.850
21	.257	.686	1.323	1.721	2.080	2.518	2.831	3.819
22	.256	.686	1.321	1.717	2.074	2.508	2.819	3.792
23	.256	.685	1.319	1.714	2.069	2.500	2.807	3.767
24	.256	.685	1.318	1.711	2.064	2.492	2.797	3.745
25	.256	.684	1.316	1.708	2.060	2.485	2.787	3.725
26	.256	.684	1.315	1.706	2.056	2.479	2.779	3.707
27	.256	.684	1.314	1.703	2.052	2.473	2.771	3.690
28	.256	.683	1.313	1.701	2.048	2.467	2.763	3.674
29	.256	.683	1.311	1.699	2.045	2.462	2.756	3.659
30	.256	.683	1.310	1.697	2.042	2.457	2.750	3.646
40	.255	.681	1.303	1.684	2.201	2.423	2.704	3.551
60	.254	.679	1.296	1.671	2.000	2.390	2.660	3.460
120	.254	.677	1.289	1.658	1.980	2.358	2.617	3.373
∞	.253	.674	1.282	1.645	1.960	2.326	2.576	3.291

34.4.6 Conditional Distribution

$$F_x(x|M) = P\{X \le x|M\} = \frac{P\{X \le x, M\}}{P\{M\}},$$

 $\{X \le x, M\} = \text{ event of all outcomes } \zeta \text{ such that } X(\zeta) \le x \text{ and } \zeta \in M$

1.
$$F(\infty|M) = 1, F(-\infty|M) = 0$$

2.
$$F(x_2|M) - F(x_1|M) = P\{x_1 < X \le x_2|M\} = \frac{P\{x_1 < X \le x_2, M\}}{P\{M\}}$$

TABLE 34.4 The Chi-Squared Distribution

$$F(x) = \int_{0}^{x} \frac{1}{2^{n/2} F(n/2)} y^{(n-2)/2} e^{-y/2} dy$$

n = number of degrees of freedom

$n\F$.005	,010	.025	.050	.100	.250	.500	.750	.900	.950	.975	.990	.995
1	.0000393	.000157	.000982	.00393	0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	6.25	7.38	9.21	10.6
3	.0717	.115	.216	.352	.584	.584	2.37	4.11	6.25	7.81	9.35	11.3	12.8
4	.207	.297	.484	.711	1.06	1.06	1.92	3.36	5.39	9.49	11.1	13.3	14.9
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7
6	.676	.872	1.24	2.20	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14,8	18.5	21.0	23.3	26.2	28.3
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	2.3	25.0	27.5	30.6	32.8
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4
22	8.64	9.94	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7

TABLE 34.5 The F-Distribution

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma(r_1 + r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2)-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1 + (r_1x/r_2)]^{(r_1 + r_2)/2}} dx$$

 $P{F \le f} = 0.95$

$r_2 \ r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	262.22
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.74	5.72	5.69
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.71	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38
13	4.67	3.81	3.41	3.18	3.03	2.92	1.83	2.77	2.71	2.67	2.60	2.53	2.46	2,41	2.38	2.34	2.30
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.37	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.34	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.32	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.30	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84

25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79
28	1.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma[(r_1 + r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2)-1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1 + (r_1x/r_2)]^{(r_1+r_2)/2}} dx \equiv F \text{ distribution}$$

 $P\{F \le f\} = 0.975$

$r_2 \ r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.6	968.6	976.7	984.9	993.1	997.2	1001	1006	1010
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.85	8.66	8.56	8.51	8.46	8.41	8.36
5	1.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96
7	807	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78
9	5.21	2.71	2.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45
10	6.94	5.46	4.83	4.47	4.24	2.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45
17	6.04	4.62	4.01	3.66	3.44	3.28	3.26	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27

TABLE 34.5 The F-Distribution

$$F(f) = p\{F \le f\} = \int_{0}^{f} \frac{\Gamma(r_1 + r_2)/2 \left[(r_1 / r_2)^{r_1 / 2} x^{(r_1 / 2) - 1} \right]}{\Gamma(r_1 / 2) \Gamma(r_2 / 2) \left[1 + (r_1 x / r_2) \right]^{(r_1 + r_2) / 2}} dx$$

 $P\{F \le f\} = 0.95$

$r_2 \ r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14
23	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	.234	2.23	2.17	2.11	2.05	1.98
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67

$$F(f) = p\{F \le f\} = \int_0^f \frac{\Gamma[(r_1 + r_2)/2](r_1/r_2)^{r_1/2} x^{(r_1/2) - 1}}{\Gamma(r_1/2)\Gamma(r_2/2)[1 + (r_1x/r_2)]^{(r_1 + r_2)/2}} dx$$

 $P\{F \le f\} = 0.99$

$r_2 \ r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022	6056	6106	6157	6209	6235	6261	6287	6313
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82

8 9	11.26 10.56	8.65 8.02	7.59 6.99	7.01 6.42	6.63 6.06	6.37 5.80	6.18 5.61	6.03 5.47	5.91 5.35	5.81 5.26	5.67 5.11	5.52 4.96	5.36 4.81	5.28 4.73	5.20 4.65	5.12 4.57	5.03 4.48
10	10.04	7.56	6,55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.225	4.10	4.10	4.02	3.94	3.86
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.96	3.23	3.08	3.00	2.92	2.84	2.75
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.32	3.43	3.80	3.15	3.00	2.92	2.824	2.76	2.67
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55
22	7.95	5.72	4.82	4.31	3.99	3.76	2.59	3.45	3.35	3.26	2.12	2.98	2.83	2.75	2.67	2.58	2.83
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.75
24	7.82	5.61	4.72	4.22	3.90	3.67	3.77	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.67
25	7.77	5.57	4.68	4.28	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	2.18	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	1.52	2.44	2.35	2.26
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	.3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84
F(f)	$= p\{F \le f\} = 0$	$\int_{0}^{f} \frac{\Gamma[(r_{1} + \frac{1}{2})\Gamma]}{\Gamma(r_{1} / 2)\Gamma}$	$(r_2)/2](r_1/2)$	$\frac{(r_2)^{r_1/2} x^{(r_1/2)}}{(r_1 x / r_2)]^{(r_1+1)}}$	$\frac{1}{1+r_2}$ dx												
	$\{f\} = 0.995$																
$r_2 \ r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
1	16211 2	20000 2	21615 2	22500 2	3056 2	3437 2	3715 2	23925 2	4091 2	24224 2	24426	24630	24836	24920	25044	25148	25253

TABLE 34.5 The F-Distribution (continued)

$$F(f) = p\{F \le f\} = \int_{0}^{f} \frac{\Gamma(r_1 + r_2)/2 \left[(r_1/r_2)^{r_1/2} x^{(r_1/2)-1} \right]}{\Gamma(r_1/2)\Gamma(r_2/2) \left[1 + (r_1x/r_2) \right]^{(r_1 + r_2)/2}} dx$$

 $P\{F \le f\} = 0.995$

$r_2 \ r_1$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.08	42.78	42.62	42.47	42.31	42.15
4	31.33	26.28	24.26	23.15	22.46	21.87	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.65	7.53	7.42	7.31
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.24	5.05	4.86	4.76	4.65	4.55	4.44
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12
13	11.37	8.198	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.17	4.07	3.97	3.87
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.96	3.86	3.76	3.66
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.64	3.54	3.44	3.33
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3,61	3.51	3.41	3.31	3.21
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.40	3.30	3.20	3.10
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.31	3.21	3.11	3.00
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.15	3.05	2.95	2.84
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.08	2.98	2.88	2.77
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.02	2.92	2.82	2.72
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66

25 26	9.48 9.41	6.60 6.54	5.46 5.41	4.84 4.79	4.43 4.38	4.15 4.10	3.94 3.89	3.78 3.73	3.64 3.60	3.54 3.49	3.37 3.33	3.20 3.15	3.01 2.97	2.92 2.87	2.82 2.77	2.72 2.67	2.61 2.84
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.83	2.73	2.63	2.77
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.79	2.69	2.59	2.71
29	9.23	6.40	5.28	4.66	4.26	3.98	3.77	3.61	6.48	3.48	3.21	3.04	2.86	2.76	2.66	2.56	2.66
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	2.95	2.78	2.60	2.50	2.40	2.30	2.18
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96

TABLE 34.6 The Poisson function

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

x\λ	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0	3.5	4	5	6	7	8	9	10
0	0.9048	0.6703	0.5488	0.4493	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0067	0.0025	0.0009	0.0003	0.0001	0.0000
1	0.0905	0.2681	0.3293	0.3595	0.3679	0.3347	0.2707	0.2052	0.1494	0.1057	0.0733	0.0337	0.0149	0.0064	0.0027	0.0011	0.0005
2	0.0045	0.0536	0.0988	0.1438	0.1839	0.2510	0.2707	0.2565	0.2240	0.1850	0.1465	0.0842	0.0446	0.0223	0.0107	0.0050	0.0023
3	0.0002	0.0072	0.0198	0.0383	0.0613	0.1255	0.1804	0.2138	0.2240	0.2158	0.1954	0.1404	0.0892	0.0521	0.0286	0.0150	0.0076
4		0.0007	0.0030	0.0077	0.0153	0.0471	0.0902	0.1336	0.1680	0.1888	0.1954	0.1755	0.1339	0.0912	0.0573	0.0337	0.0189
5		0.0001	0.0004	0.0012	0.0031	0.0141	0.0361	0.0668	0.1008	0.1322	0.1563	0.1755	0.1606	0.1277	0.0916	0.0607	0.0378
6		İ		0.0002	0.0005	0.0035	0.0120	0.0278	0.0504	0.0771	0.1042	0.1462	0.1606	0.1490	0.1221	0.0911	0.0631
7			j		0.0001	0.0008	0.0034	0.0099	0.0216	0.0385	0.0595	0.1044	0.0137	0.1490	0.1396	0.1171	0.0901
8		1				0.0001	0.0009	0.0031	0.0081	0.0169	0.0298	0.0653	0.1033	0.1304	0.1396	0.1318	0.1126
9							0.0002	0.0009	0.0027	0.0066	0.0132	0.0363	0.0688	0.1014	0.1241	0.1318	0.1251
10		İ					ŀ	0.0002	0.0008	0.0023	0.0053	0.0181	0.0413	0.0710	0.0993	0.1186	0.1251
11		1	1		!				0.0002	0.0007	0.0019	0.0082	0.0225	0.0452	0.0722	0.0970	0.1137
12			Į.						0.0001	0.0002	0.0006	0.0034	0.0113	0.0264	0.0481	0.0728	0.0948
13			J							0.0001	0.0002	0.0013	0.0052	0.0142	0.0296	0.0504	0.0729
14]								0.0001	0.0005	0.0022	0.0071	0.0169	0.0324	0.0521
15			1						ĺ	}		0.0002	0.0009	0.0033	0.0090	0.0194	0.0347
16]	ļ			ļ		İ		j		ļ	0.0003	0.0014	0.0045	0.0109	0.0217
17													0.0001	0.0006	0.0021	0.0058	0.0128
18	ŀ										ļ			0.0002	0.0009	0.0029	0.0071
19	ŀ	ļ]										ļ	0.0001	0.0004	0.0014	0.0037
20			1										!		0.0002	0.0006	0.0019
21												1			0.0001	0.0003	0.0009
22		Į.						ļ								0.0001	0.0004
23			1									ļ					0.0002
24]										0.0001
			1	1				ļ			}						
L	L	L	<u> </u>	L	L	L	1	ł		L		L	L		<u> </u>		L

TABLE 34.7 The Poisson Distribution

$$F(x) = \sum_{k=0}^{x} \frac{e^{-\lambda} \lambda^{k}}{k!}$$

x\λ	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.5	3.0	3.5	4.0
0	0.8187	0.6703	0.5488	0.4493	0.3679	0.3012	0.2466	0.2019	0.1653	0.1353	0.0821	0.0489	0.0302	0.0183
1	0.9825	0.9384	0.8781	0.8088	0.7358	0.6626	0.5918	0.5249	0.4268	0.4060	0.2873	0.1991	0.1359	0.0916
2	0.9989	0.9921	0.9769	0.9526	0.9197	0.8795	0.8335	0.7834	0.7306	0.6767	0.5438	0.4232	0.3208	0.2381
3	0.9999	0.9992	0.9966	0.9909	0.9810	0.9662	0.9463	0.9212	0.8913	0.8571	0.7576	0.6472	0.5366	0.4335
4	1.0000	0.9999	0.9996	0.9986	0.9963	0.9923	0.9857	0.9763	0.9636	0.9473	0.8912	0.8153	0.7254	0.6288
5		1.0000	1.0000	0.9998	0.9994	0.9985	0.9968	0.9940	0.9896	0.9834	0.9580	0.9161	0.8576	0.7851
6			1	1.0000	0.9999	0.9997	0.9994	0.9987	0.9974	0.9955	0.9858	0.9665	0.9347	0.8893
7				Ì	1.0000	1.0000	0.9999	0.9997	0.9994	0.9989	0.9958	0.9881	0.9733	0.9489
8				l			1.0000	1.0000	0.9999	0.9998	0.9989	0.9962	0.9901	0.9786
9					ł			1	1.0000	1.0000	0.9997	0.9989	0.9967	0.9919
10											0.9999	0.9997	0.9990	0.9972
11						İ					1.0000	0.9999	0.9997	0.9991
12					1							1.0000	0.9999	0.9997
13					-	1							1.0000	0.9999
14										1				1.0000
		Ì											ļ	
							<u></u>		<u> </u>			<u> </u>		

34.4.7 Conditional Density

$$f(x|M) = \frac{dF(x|M)}{dx} = \lim_{\Delta x \to 0} \frac{P\{x \le X \le x + \Delta x | M\}}{\Delta x}$$

$$\int_{-\infty}^{\infty} f(x|M)dx = F(\infty|M) = 1$$

Example

 $X(f_i) = 10i$, $i = 1, \dots 6$ where $f_i = \text{face of a die. } M = \{f_2, f_4, f_6\} = \text{even event. For } x \ge 60, \{X \le x, M\}$

 $P\{M\} = (2/6)/(3/6) = 2/3$; for $20 \le x < 40, \{X \le x, M\} = \{f_2\}, F(x|M) = P\{f_2\}/P\{M\} = (1/6)(3/6)$ = 1/3; for x < 20, $\{X \le x, M\} = 0$ and F(x|M) = 0.

34.4.8 Total Probability

 $F(x) = F(x|A_1)P(A_1) + F(x|A_2)P(A_2) + \dots + F(x|A_n)P(A_n)P(A_n)$, A_i 's are mutually exclusive and their sum is equal to the certain event **S**.

34.5 Function of One Random Variable (r.v.)

34.5.1 Random Variable (Definition)

To every experimental outcome ζ we assign a number $X(\zeta)$. The domain of X is the space S, and its range is the set I_X of the real numbers $X(\zeta)$..

34.5.2 Function of r.v.

$$Y = g(X) = g[X(\zeta)]$$

34.5.3 Distribution Function of Y (see 34.5.2)

$$F_{y}(y) = P\{Y \le y\} = P\{g(X) \le y\} = P\{X \in I_{y}\}$$

Note: To find $F_y(y)$ for a given y we must find that set I_y and the probability that X is in I_y . Refer to Figure 34.2:If $y \ge k$ then $g(x) \le y$ for any x. Hence $\{Y \le y\} = \text{certain event and } F_y(y) = P\{Y \le y\} = 1$. If $y = y_1$, then $g(x) \le y_1$ for $x \le x_1$ and, hence, $F_y(y_1) = P\{Y \le y_1\} = P\{X \le x_1\} = F_x(x_1)$ (x_1 depends on y_1), If $y = y_2$, then $g(x) = y_2$ has three solutions x_2', x_2'', x_2''' : $g(x_2') = g(x_2''') = g(x_2''') = y_2$ and from Figure 34.2 $g(x) \le y_2$ if $x \le x_2^{\oplus}$ or $x_2'' \le x \le x_2'''$ and hence, $F_y(y_2) = P\{X \le x_2'\} + P\{x_2'' \le x \le x_2'''\} = F_X(x_2') + F_X(x_2''') - F_X(x_2''')$. If $y < \ell$ no value of x produces $g(x) \le y$ and the event $\{Y \le y\}$ has zero probability: $F_y(y) = 0$.

Example

 $Y=1/X^2$. If y>0, there are two solutions: $x_1=-\sqrt{y}, x_2=1/\sqrt{y}.g(x) \le y$ if $x \le x_1$ or $x \ge x_2$ and thus

$$F_{y}(y) = P\{Y \le y\} = P\{X \le -1/\sqrt{y}\} + P\{X \ge 1/\sqrt{y}\} = F_{x}(-1/\sqrt{y}) + 1 - F_{x}(1/\sqrt{y}).$$

if y < 0, no x will produce $g(x) \le y$ and, hence, $F_{y}(y) = 0$.

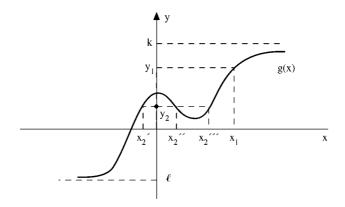


FIGURE 34.2

34.5.4 Density Function of Y=g(X) in Terms of $f_X(x)$ of X

1) Solve y=g(x) for x in terms of y. If x_1, x_2, \dots, x_n are all its real roots, then $y=g(x_1)=\dots=g(x_n)=\dots$, then $f_Y(y)=\frac{f_X(x_1)}{|g'(x_1)|}+\dots+\frac{f_X(x_n)}{|g'(x_n)|}+\dots$, g'(x)=dg(x)/dx. If y=g(x) has no real roots then $f_Y(y)=0$.

Example 1

$$g(x) = aX + b$$
 and $x = (y - b)/a$ for every y , $g'(x) = a$ and hence $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right)$

Example 2

 $g(X) = aX^2$ with the r.v. $y = ax^2$, a > 0. If y < 0 roots are imaginary and $f_y(y) = 0$. If y > 0 then $x_1 = \sqrt{y/a}$ and $x_2 = -\sqrt{y/a}$. Since $g'(x_1) = 2ax_1 = 2\sqrt{ay}$ and $g'(x_2) = 2ax_2 = 2\sqrt{ay}$, then $f_y(y) = \frac{1}{2\sqrt{ay}} \left[f_x\left(\sqrt{\frac{y}{a}}\right) + f_x\left(-\sqrt{\frac{y}{a}}\right) \right] u(y)$, u(y) = unit step function.

Example 3

 $Y = a\sin(X+\theta), \ a>0 \ . \ \text{If} \quad |y| < a \ \text{ then} \quad y = a\sin(x+\theta) \quad \text{has infinitely many solutions} \quad x_n = \sin^{-1}\frac{y}{a} - \theta,$ $n = \cdots, -1, 0, 1, \cdots . dg(x_n) / \ dx = a\cos(x_n + \theta) = \sqrt{a^2 - y^2} \quad \text{and from 34.5.4} \quad f_Y(y) = \left(1/\sqrt{a^2 - y^2}\right) \sum_{n = -\infty}^{\infty} f_X(x_n),$ $|y| < a \ . \ \text{For} \quad |y| > 0 \quad \text{there exist no solutions, and} \quad f_Y(y) = 0 \ .;$

Example 4

 $Y = be^{-aX}u(X), a > 0, b > 0$. If y < 0 or y > b then the equation $y = b\exp(-ax)u(x)$ has no solution, and hence $f_Y(y) = 0$. If 0 < y < b, then $x = -(1/a)\ln(y/b).g@x) = -abe^{-ax} = -ay$ and $f_Y(y) = f_X(-(1/a)\ln(y/b))/ay, 0 < y < b$.

34.5.5 Conditional Density of Y=g(x)

$$f_Y(y|M) = \frac{f_X(x_1|M)}{|g'(x_1)|} + \dots + \frac{f_X(x_n|M)}{|g'(x_n)|} + \dots$$

Example

$$Y = aX^2, a > 0, X \ge 0, f_x(x|X \ge 0) = \frac{f_X(x)}{1 - F_X(0)}u(x) \text{ (see 34.5.4 Example 2), and hence } f_Y(y|X \ge 0) = \frac{1}{1 - F_X(0)}u(x) = \frac{f_X(x)}{1 - F_X(0)}u(x).$$

$$\left[f(x|X \ge t) = f(x)/\{1 - F(t)\} = /\int_t^\infty f(x)dx, x \ge t \right]$$

34.5.6 Expected Value

$$E\{X\} = \int_{-\infty}^{\infty} xf(x)dx \text{ continuous r.v.}$$

$$E\{X\} = \sum_{n} x_{n} P\{X = x_{n}\} = \sum_{n} x_{n} p_{n} \text{ discrete r.v.}$$

34.5.7 Expected Value of a Function g(X)

$$E\{Y = g(X)\} = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx \quad \text{continuous r.v.}$$

$$E\{g(X)\} = \sum_{k} g(x_k) P\{X = x_k\}$$
 discrete type of r.v.

34.5.8 Conditional Expected Value

$$E\{X|M\} = \int_{-\infty}^{\infty} x f(x|M) dx$$
 continuous r.v.

$$E\{X|M\} = \sum_{n} x_n P\{X = x_n | M\} \text{ discrete r.v.}$$

34.5.9 Variance

$$\sigma^2 = E\{(X - \mu)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \text{ continuous r.v.}$$

$$\sigma^2 = \sum_n (x_n - \mu)^2 P\{X = x_n\} \text{ discrete r.v.}$$

$$\sigma^2 = E\{X^2\} - E^2\{X\}$$

Example

 $P{X = k} = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, \dots = \text{Poisson distribution.}$

$$E(X) = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^{k}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^{k}}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k}}{k!}.$$

but

$$\frac{d}{d\lambda}e^{\lambda} = \frac{d}{d\lambda}\sum_{k=1}^{\infty}k\frac{\lambda^{k-1}}{k!} = \frac{1}{\lambda}\sum_{k=1}^{\infty}k\frac{\lambda^{k}}{k!} = e^{\lambda}$$

or

$$\lambda = e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!}$$

and hence, $E\{X\} = \lambda$.

34.5.10 Moments About the Origin

$$\mu'_{k} = E\{X^{k}\} = \int_{-\infty}^{\infty} x^{k} f(x) dx = \sum_{r=0}^{k} {k \choose r} \mu^{r} \mu_{k-r}, \ \mu'_{1} = \mu = E\{X\}, \ \mu'_{0} = 1$$

34.5.11 Central Moments

$$\mu_k = E\{X^k\} = \int_{-\infty}^{\infty} (x - \mu)^k f(x) dx = E\left\{\sum_{r=0}^k \binom{k}{r} (-1)^r \mu^r X^{k-r}\right\} = \sum_{r=0}^k \binom{k}{r} (-1)^r \mu^r \mu'_{k-r}$$

$$\mu_0 = \mu'_0 = 1$$
, $\mu_1 = \mu'_1 - \mu = 0$, $\mu_2 = \mu'_2 - 2\mu\mu'_1 + \mu^2 = \mu'_2 - \mu^2$, $\mu_3 = \mu'_3 - 3\mu\mu'_2 + 3\mu^2\mu'_1 - \mu^3 = \mu'_3 - 3\mu\mu'_2 + 2\mu^3$

34.5.12 Absolute Moments

$$M_{k} = E\{|X|^{k}\} = \int_{-\infty}^{\infty} |x| f(x) dx$$

34.5.13 Generalized Moments

$$_{a}\mu'_{k} = E\{(X-a)^{k}\}, \quad _{a}M'_{k} = E\{|X-a|^{k}\}$$

Example 1

$$E\{X^{2n}\} = \frac{1}{2a} \int_{-a}^{a} x^{2n} dx = \frac{a^{2n}}{2n+1}, \, \sigma^2 = E\{x^2\} = \frac{a^2}{3}$$

for X uniformly distributed in (-a,a).

Example 2

$$E\{X^n\} = \frac{a^{b+1}}{\Gamma(b+1)} \int_0^\infty x^n x^b e^{-ax} dx = \frac{a^{b+1} \Gamma(b+n+1)}{a^{b+n+1} \Gamma(b+1)}$$

for a gamma density $f(x) = \{a^{b+1} / \Gamma(b+1)\} x^b e^{-ax} u(x), u(x) = \text{unit step function.}$

34.5.14 Tchebycheff Inequality

 $P\{|X - \mu| \ge k\sigma\} \le \frac{1}{k^2}, \ \mu = E\{X\}.$ Regardless of the shape of f(x), $P\{\mu - \varepsilon < X < \mu + \varepsilon\} \ge 1 - \frac{\sigma^2}{\varepsilon^2}$ Generalizations:

1. If
$$f_y(y) = 0$$
 then $P\{Y \ge \alpha\} \le \frac{E\{Y\}}{\alpha}$, $\alpha > 0$

2.
$$P\{|X-\alpha|^n \ge \varepsilon^n\} \le \frac{E\{|X-\alpha|^n\}}{\varepsilon^n}$$

34.5.15 Characteristic Function

$$\Phi(\omega) = E\{e^{j\omega x}\} = \int_{-\infty}^{\infty} f(x)dx \text{ for continuous r.v.}$$

$$\Phi(\omega) = \sum_{k} e^{j\omega x_{k}} P\{X = x_{k}\} \text{ for discrete type r.v.}$$

$$\Phi(0) = 1, \quad |\Phi(\omega)| \le 1$$

Example 1

$$\Phi(\omega) = E\{e^{j\omega Y}\} = E\{e^{j\omega(aX+b)}\} = e^{j\omega b}E\{e^{j\omega aX}\}, \ \text{if} \ Y = aX+b$$

Example 2

$$P\{X=k\} = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, \dots \text{ Poisson distribution } \Phi(\omega) = e^{-\lambda} \sum_{k=0}^{\infty} e^{j\omega k} \frac{\lambda^k}{k!} = e^{\lambda} (e^{j\omega} - 1)$$

34.5.16 Second Characteristic Function

$$\Psi(\omega) = \ln \Phi(\omega)$$

34.5.17 Inverse of the Characteristic Function

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\omega) e^{-j\omega x} d\omega$$

34.5.18 Moment Theorem and Characteristic Function

$$\frac{d^n \Phi(0)}{d\omega^n} = j^n \mu'_{n \odot} \quad E\{X^n\}$$

34.5.19 Convolution and Characteristic Function

 $\Phi(\omega) = \Phi_1(\omega) \Phi_2(\omega) \text{ , where } \Phi_1(\omega) \text{ and } \Phi_2(\omega) \text{ are the characteristic functions of the density functions } f_1(x) \text{ and } f_2(x) \text{ . } \Phi(\omega) = E\{e^{j\omega(X_1+X_2)}\} \text{ and } f(x) = f_1(x)*f_2(x) \text{ where * indicates convolution.}$

34.5.20 Characteristic Function of Normal r.v.

$$\Phi(\omega) = \exp(j\mu\omega - \frac{1}{2}\sigma^2\omega^2)$$

34.6 Two Random Variables

34.6.1 Joint Distribution Function

$$\begin{split} F_{xy}(xy) &= P\{X \leq x, Y \leq y\}, \quad F_{xy}(x,\infty) = F_x(x), F_{xy}(\infty,y) = F_y(y), \\ F_{xy}(\infty,\infty) &= 1, F_{xy}(-\infty,y) = 0, F_{xy}(x,-\infty) = 0 \end{split}$$

34.6.2 Joint Density Function

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}, f(x) = \int_{-\infty}^{\infty} f(x,y) dy, \quad f_y(y) = \int_{-\infty}^{\infty} f(x,y) dx$$

34.6.3 Conditional Distribution Function

$$\begin{split} F_{y}(y|M) &= P\{Y \leq y|M\} = \frac{P\{Y \leq y,M\}}{P\{M\}}, \ F\{y|X \leq x\} = \frac{P\{X \leq x,Y \leq y\}}{P\{X \leq x\}} = \frac{F_{xy}(x,y)}{F_{x}(x)} \\ F_{y}(y|X \leq a,Y \leq b) &= \frac{P\{X \leq a,Y \leq b,Y \leq y\}}{P\{X \leq a,Y \leq b\}} = \begin{cases} 1 & y \geq b \\ F_{xy}(a,y)/F_{xy}(a,b) & y < b \end{cases} \end{split}$$

34.6.4 Conditional Density Function

$$f_{y}(y|X \le x) = \frac{\partial F_{xy}(x,y)/\partial y}{F_{x}(x)} = \frac{\int_{-\infty}^{x} f_{xy}(\xi,y)d\xi}{\int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{xy}(\xi,y)d\xi dy}, \quad f_{y}(y|x < X \le x_{2} = \frac{\int_{x_{1}}^{x_{2}} f_{xy}(x,y)dx}{\Gamma_{x}(x_{2}) - F_{x}(x_{1})},$$

$$f_{y}(y|X = x) = \frac{f_{xy}(x,y)}{f_{x}(x)}$$

34.6.5 Baye's Theorem

$$f_y(y|X = x) = \frac{f_x(x|Y = y)f_y(y)}{f_x(x)}$$

34.6.6 Joint Conditional Distribution

$$F_{xy}(x, y | a < X \le b) = \frac{P\{X \le x, Y \le y, a < Y \le b\}}{P\{a < X \le b\}} = \begin{cases} \frac{F_{xy}(b, y) - F_{xy}(a, y)}{F_x(b) - F_x(a)} & x > b \\ \frac{F_{xy}(x, y) - F_{xy}(a, y)}{F_x(b) - F_x(a)} & a < x \le b \end{cases}$$

$$0 \qquad x \le a$$

34.6.7 Conditional Expected Value

$$E\{g(Y)|X=x\} = \int_{-\infty}^{\infty} g(y)f_{y}(y|X=x)dy = \frac{\int_{-\infty}^{\infty} g(y)f_{xy}(x,y)dy}{\int_{-\infty}^{\infty} f_{xy}(x,y)dy}, \quad E\{E\{Y|X\}\} = E\{Y\}$$

34.6.8 Independent r.v.

$$F_{xy}(x, y) = F_x(x)F_y(y); f_{xy}(x, y) = f(x)f(y); f_y(y|x) = f_y(y); f_x(x|y) = f_x(x)$$

34.6.9 Jointly Normal r.v.

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left[-\frac{1}{2(1-r^2)} \left[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2r(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \right] \right]$$

 $E\{X\} = \mu_1, E\{Y\} = \mu_2, \sigma_x = \sigma_1, \sigma_y = \sigma_2.$ If $r = 0, f(x, y) = f_x(x)f_y(y) \equiv \text{ independent. } |r| < 1, r = \text{correlation coefficient.}$

Conditional Densities

$$f_{y}(y|X=x) = \frac{1}{\sigma_{2}\sqrt{2\pi(1-r^{2})}} \exp\left[-\frac{1}{2\sigma^{2}(1-r^{2})} \left[y - \mu_{2} - \frac{r\sigma_{2}}{\sigma_{1}}(x - \mu_{1})\right]^{2}\right]$$

$$E\{Y|X=x\} = \mu_2 + \frac{r\sigma_2}{\sigma_1}(x-\mu_1), \quad \sigma_{y|x=x} = \sigma_2\sqrt{1-r^2}$$

If
$$\mu_1 = \mu_2 = 0$$
 then $E\{Y^2 | X = x\} = \sigma_2^2 (1 - r^2) + \frac{r^2 \sigma_2^2}{\sigma_1^2} x^2$

34.7 Functions of Two Random Variables

34.7.1 Definitions

 $Z = g(X,Y) = g[X(\zeta),Y(\zeta)], \quad F_z(z) = P\{Z \le z\}, \quad D_z = \text{region of xy-plane such that} \quad g(x,y) \le z, \quad \{Z \le z\}$ $= \{(X,Y) \in D_z\}$

34.7.2 Distribution Function

$$f_z(z)dz = P\{z < Z \le z + dz\} = \iint_{\Delta D_z} f_{xy}(x, y) dxdy$$

34.7.3 Density Function

$$f_z(z)dz = P\{z < Z \le z + dz\} = \iint_{\Delta D_z} f_{xy}(x, y) dxdy$$

Example 1

$$Z = X + Y, x + y \le z, F_z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{xy}(x, y) dx dy, \frac{dF_z(z)}{dz} = f_z(z) = \int_{-\infty}^{\infty} f_{xy}(z - y, y) dy.$$
 If the r.v. are inde-

pendent then $f_{xy}(x,y) = f_x(x)f_y(y)$ and hence $f_z(z) = \int_{-\infty}^{\infty} f_x(z-y)f_y(y)dy = \int_{-\infty}^{\infty} f_x(x)f_y(z-x)dx = f_x(z) * f_y(z) = \text{convolution of densities.}$

Example 2

$$Z = X^2 + Y^2$$
, if $z > 0$ so then $x^2 + y^2 \le z = \text{circle}$ with radius \sqrt{z} , $F_z(z) = \iint_{x^2 + y^2 \le z} f(x, y) dx dy$, if $z < 0$,

$$F_z(z) = 0. \ f_{xy}(x,y) = (1/2\pi\sigma^2) \exp[-(x^2+y^2)/2\sigma^2 \quad \text{then} \quad F_z(z) = \frac{1}{2\pi\sigma^2} \int_0^z 2\pi r e^{-r^2/2\sigma^2} dr = 1 - e^{-z/2\sigma^2},$$

$$z > 0$$
 and $f_z(z) = \frac{1}{2\sigma^2} e^{-z/2\sigma^2}, z \ge 0$

Example 3

$$\begin{split} f_{xy}(x,y) &= (1/2\pi\sigma^2) \exp[-(x^2+y^2)/2\sigma^2], \ Z = +\sqrt{X^2+Y^2}, F_z(z) = \frac{1}{2\pi\sigma^2} \int_0^z 2\pi r e^{-r^2/2\sigma^2} dr = 1 - e^{-z^2/2\sigma^2}, \\ z &> 0, \ f_z(z) = (z/\sigma^2) \exp(-z^2/2\sigma^20, \ z > 0 \equiv \text{ Rayleigh distributed}, \quad E\{Z\} = \sigma\sqrt{\pi/2}, \ E\{Z^2\} = 2\sigma^2, \sigma_z^2 \\ &= (2 - (\pi/2))\sigma^2 \end{split}$$

Example 4

If
$$f_{xy}(x,y) = f_{xy}(-x,-y)$$
 then $F_z(z) = 2\int_0^\infty \int_{-\infty}^{yz} f_{xy}(x,y) dx dy$, $f_z(z) = 2\int_0^\infty y f_{xy}(zy,y) dy$. Then for $f_{xy}(x,y) = (1/[2\pi\sigma_1\sigma_2\sqrt{1-r^2}]) \exp\left[-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_1^2}\right)\right]$ then $f_z(z)$ of $Z = X/Y$ is

$$f_z(z) = \left[2/(2\pi\sigma_1\sigma_2\sqrt{1-r^2})\right] \int_0^\infty y \exp\left[-\frac{y^2}{2(1-r^2)}\left[\frac{z^2}{\sigma_1^2} - \frac{2rz}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right]\right] dy.$$

But $\int_0^\infty y \exp[-y^2/2a^2] dy = a^2 \int_0^\infty e^{-w} dw = a^2$ and hence

$$f_z(z) = [(\sqrt{1 - r^2} \sigma_1 \sigma_2 / \pi] / [\sigma_2^2 (z - r\sigma_1 / \sigma_2)^2 + \sigma_1^2 (1 - r^2)].$$

If $\mu_2 = \mu_2 = 0$ then $f_z(z)$ is Cauchy density.

34.8 Two Functions of Two Random Variables

34.8.1 Definitions

 $Z = g(X,Y), W = h(X,Y), \ D_{zw} \text{ region of the xy plane such that} \quad g(x,y) \le z \text{ and } h(x,y) \le w,$ $\{Z \le z, W \le w\} = \{(X,Y) \in D_{xy}\}, \ F_{zw}(z,w) = \iint_{D_{zw}} f_{xy}(x,y) dx dy$

34.8.2 Density Function $f_{zw}(z, w)$

$$f_{zw}(z,w) = \frac{f_{xy}(x_1,y_1)}{\left|J(x_1,y_1)\right|} + \dots + \frac{f_{xy}(x_n,y_{n1})}{\left|J(x_n,y_n)\right|} + \dots, z = g(x_i,y_i), w = h(x_i,y_i) \text{ where } (x_i,y_i) \text{ are solutions. if } x_i = \frac{f_{xy}(x_1,y_1)}{\left|J(x_1,y_1)\right|} + \dots + \frac{f_{xy}(x_n,y_n)}{\left|J(x_n,y_n)\right|} + \dots + \frac{f_{xy}(x_n,y_n)}{\left$$

there are no real solutions for certain values of (z, w) then $f_{zw}(z, w) = 0$.

Jacobian of transformation

$$J(x,y) = \begin{vmatrix} \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \\ \frac{\partial h(x,y)}{\partial x} & \frac{\partial h(x,y)}{\partial y} \end{vmatrix}$$

Example 1

If z = ax + by, w = cx + dy then $x = a_1z + b_1y$, $y = c_1z + d_1w$, where a_1, b_1, c_1 and d_1 are functions of a,b,c, and d.

$$J(x,y) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, f_{zw} = (z,w) = 1/[|ad - bc|]f_{xy}(a_1z + b_1w, c_1z + d_1w)$$

Example 2

 $z=+\sqrt{x^2+y^2}$, w=x/y. If z>0 then the system has two solutions: $x_1=zw/\sqrt{1+w^2}$, $y_1=z/\sqrt{1+w^2}$ and $x_2=-x_1,y_2=-y_1$ for any w.

$$J(x,y) = \begin{vmatrix} x/\sqrt{x^2 + y^2} & y/\sqrt{x^2 + y^2} \\ 1/y & -x/y^2 \end{vmatrix} = (1+w^2)/(-z)$$

and from 34.8.2

$$f_{zw}(z,w) = \left[z/(1+w^2)\right] \left[f_{xy} \left(\frac{zw}{\sqrt{1+w^2}}, \frac{z}{\sqrt{1+w^2}} \right) + f_{xy} \left(\frac{-zw}{\sqrt{1+w^2}}, \frac{-z}{\sqrt{1+w^2}} \right) \right].$$

If z < 0, $f_{zw}(z, w) = 0$.

34.8.3 Auxiliary Variable

If z = g(x, y) we can introduce an auxiliary function w = x or w = y. $f_z(z) = \int_{-\infty}^{\infty} f_{zw}(z, w) dw$.

Example

If z = xy set auxiliary function w = x. The system has solutions

$$x = w, y = z/w.$$
 $J(x,y) = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -w$

and, hence,

$$f_{zw}(z,w) = (1/|w|)f_{xy}(w,z/w)$$
 and $f_z(z) = \int_{-\infty}^{\infty} (1/|w|)f_{xy}(w,z/w)dw$.

34.8.4 Functions of Independent r.v.'s

If X and Y are independent then Z = g(X) and W = h(Y) are independent and

$$f_{zw}(z, w) = \frac{f_x(x_1)}{|g'(x_1)|} \frac{f_y(y_1)}{|h'(y_1)|}$$

since

$$J(x,y) = \begin{vmatrix} g'(x) & 0\\ 0 & h'(x) \end{vmatrix} = g'(x)h'(x)$$

34.9 Expected Value, Moments, and Characteristic Function of Two Random Variables

34.9.1 Expected Value

$$E\{g(X,Y)\} = \iint_{-\infty}^{\infty} g(x,y)f(x,y)dxdy;$$

$$E\{z\} = \int_{-\infty}^{\infty} z f_z(z) dz \text{ if } z = g(x, y); \ E\{g(X, Y)\} = \sum_{k, n} g(x_k, y_n) p_{kn},$$

$$P{X = x_k, Y = y_n} = p_{kn}$$
 discrete case r.v.

34.9.2 Conditional Expected Values

$$E\{g(X,Y|M)\} = \iint_{-\infty}^{\infty} g(x,y)f(x,y|M)dxdy;$$

$$E\{g(X,Y|X=x\} = \int_{-\infty}^{\infty} g(x,y)f(x,y)dy / f_x(x) = \int_{-\infty}^{\infty} g(x,y)f(y|X=x)dy$$

34.9.3 Moments

$$\mu'_{kr} = E\{X^k Y^r\} = \int_{-\infty}^{\infty} x^k y^r f(x, y) dx dy, \mu'_{11} = R_{xy} = E\{XY\}$$

$$\mu_{kr} = E\{(X - \mu_x)^k (Y - \mu_y)^r\} = \int_{-\infty}^{\infty} (x - \mu_x)^k (y - \mu_y)^r f(x, y) dx dy$$

$$\mu_{20} = \sigma_x^2, \ \mu_{02} = \sigma_y^2, \ \mu = \mu_1'$$

34.9.4 Covariance

$$\mu_{11} = E\{(X - \mu_x)(Y - \mu_y)\} = E\{XY\} - \mu_x E\{Y\} - \mu_y E\{X\} + \mu_x \mu_y$$

34.9.5 Correlation Coefficient

$$r = E\{(X - \mu_x)(Y - \mu_y)\} / \sqrt{E\{(X - \mu_x)^2 E\{(Y - \mu_y)^2\}} = \mu'_{11} / \sigma_x \sigma_y$$

$$\mu_{11}^{\prime 2} \le \mu_{20}^{\prime} \mu_{02}^{\prime}, \mu_{11}^{\prime 2} \le \mu_{20} \mu_{02}, \ |r| = |\mu_{11}^{\prime 2}| / \sqrt{\mu_{20} \mu_{02}} \le 1$$

34.9.6 Uncorrelated r.v.'s

$$E\{XY\} = E\{X\}E\{Y\}$$

34.9.7 Orthogonal r.v.'s

$$E\{XY\} = 0$$

34.9.8 Independent r.v.'s

$$f(x,y) = f_x(x)f_y(y)$$

Note:

1. If X and Y are independent, g(X) and h(Y) are independent or

$$E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$$

2. If X and Y are uncorrelated, then

a.
$$E\{(X - \mu_x)(Y - \mu_y)\} = 0, r = 0$$

b.
$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2$$

c.
$$E\{(X+Y)^2\} = E\{X^2\} + E\{Y^2\}$$

d. $E\{g(X)h(Y)\} \neq E\{g(X)\}E\{h(Y)\}$ in general

34.9.9 Joint Characteristic Function

$$\Phi_{xy}(\omega_1,\omega_2) = E\{e^{j(\omega_1x+\omega_2y)} = \int_{-\infty}^{\infty} f_{xy}(x,y)e^{j(\omega_1x+\omega_2y)}dxdy, \Psi_{xy}(\omega_1,\omega_2) = \ln \Phi_{xy}(\omega_1,\omega_2)$$

$$f_{xy}(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} e^{-j(\omega_1 x + \omega_2 y)} \Psi_{xy}(\omega_1 \omega_2) d\omega_1 d\omega_2$$

$$\Phi_{x}(\omega) = E\{e^{j\omega X}\} = \Phi_{xy}(\omega,0), \ \Phi_{y}(\omega) = \Phi_{xy}(0,\omega)$$

Example

$$\Phi_z(\omega) = E\{e^{j\omega z}\} = E\{e^{j(a\omega X + b\omega Y)}\} = \Phi_{xy}(a\omega, b\omega) \text{ if } Z = aX + bY.$$

$$\Phi_{xy}(\omega_1,\omega_2) = \Phi_{x}(\omega_1)\Phi_{y}(\omega_2)$$

if X and Y are independent.

34.9.10 Moment Theorem

$$\frac{\partial^k \partial^r \Phi(0,0)}{\partial \omega_1^k \partial \omega_2^r} = j^{(k+r)} \mu'_{kr}$$

34.9.11 Series Expansion of $\Phi(\omega_1, \omega_2)$

$$\begin{split} &\Phi(\omega_{1},\omega_{2}) = 1 + jE\{X\}\omega_{1} + jE\{Y\}\omega_{2} - \frac{1}{2}\{X^{2}\}\omega_{1}^{2} \\ &- \frac{1}{2}E\{Y^{2}\}\omega_{2}^{2} - E\{XY\}\omega_{1}\omega_{2} + \dots + \frac{1}{4!}\binom{4}{2}E\{X^{2}Y^{2}\}\omega_{1}^{2}\omega_{2}^{2} + \dots, \\ &\Psi(\omega_{1},\omega_{2}) = \ln\Phi(\omega_{1},\omega_{2}) = j\mu_{x}\omega_{1} + j\mu_{x}\omega_{2} - \frac{1}{2}\sigma_{x}^{2}\omega_{1}^{2} - r\sigma_{x}\sigma_{x}\omega_{1}\omega_{2} - \frac{1}{2}\sigma_{x}^{2}\omega_{2}^{2} + \dots \end{split}$$

34.10 Mean Square Estimation of R.V.'s

34.10.1 Mean Square Estimation of r.v.'s

- a. a minimizes $E\{X-a\}^2$ if $a=E\{X\}=\mu_x$
- b. The function $g(X) = E\{Y|X\} = \text{regression curve minimizes}$

$$E\{[Y - g(X)]^{2}\} = \int_{-\infty}^{\infty} [y - g(x)]^{2} f(x, y) dx dy$$

c. $a = \frac{r\sigma_y}{\sigma_x}$ and $b = E\{Y\} - aE\{X\}$ minimize the m.s. error

$$e = E\{[Y - (aX + b)^2]\} = \int_{-\infty}^{\infty} [y - (ax - b)^2] f(x, y) dx dy$$

$$e_m = \text{minimum error} = \sigma_y^2 (1 - r^3), \quad r = \text{correlation coefficient of X and Y}.$$

d. If $E\{X\}=E\{Y\}=0$ the constant a that minimizes the m.s. error $e=E\{(y-ax)^2\}$ is such that $E\{(Y-aX)X\}=0$ (orthogonality principle) and the minimum m.s. error is: $e_m=E\{(Y-aX)Y\}$ $a=E\{XY\}/E\{X^2\}$ and hence $e_m=E\{Y^2\}-\frac{E^2\{XY\}}{E\{X^2\}}$, also $e_m=E\{Y^2\}-E\{(aX)^2\}$ $e_m\geq E\{[Y-E\{Y|X\}]^2\}$

34.11 Normal Random Variables

34.11.1 Jointly Normal

If $e\{X\} = E\{Y\} = 0$ the normal joint density is:

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma\sqrt{1-r^2}} \exp\left[-\frac{1}{2(1-r^2)} \left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right], \quad E\{X^2\} = \sigma_1^2, \quad E\{Y^2\} = \sigma_2^2$$

34.11.2 Conditional Density

$$f(y|x) = \frac{1}{\sigma_2 \sqrt{2\pi(1-r^2)}} \exp\left[-\frac{1}{2\sigma_2^2(1-r^2)} \left(y - \frac{r\sigma_2}{\sigma_1}x\right)^2\right],$$

$$E\{Y|X\} = \frac{r\sigma_2}{\sigma_1}X, \quad E\{Y^2|X\} = \sigma_2^2(1-r^2) + \frac{r^2\sigma_2^2}{\sigma_1^2}x^2$$

$$E\{XY\} = r\sigma_1\sigma_2$$
, $E\{X^2Y^2\} = \sigma_1^2\sigma_2^2 + 2r^2\sigma_1^2\sigma_2^2$

34.11.3 Mean Value

$$E\{(X - \mu_x)(Y - \mu_y)\} = r\sigma_1\sigma_2$$

34.11.4 Linear Transformations

$$Z = aX + bY$$
, $W = cX + dY$.

If X and Y are jointly normal with zero mean then

$$\sigma_z^2 = E\{Z^2\} = E\{(aX + bY)^2\} = a^2\sigma_x^2 + b^2\sigma_y^2 + 2abr_{xy}\sigma_x\sigma_y$$

$$\sigma_w^2 = E\{W^2\} = c^2 \sigma_x^2 + d^2 \sigma_y^2 + 2c dr_{xy} \sigma_x \sigma_y,$$

$$r_{zw}\sigma_z\sigma_w = E\{ZW\} = ac\sigma_x^2 + bd\sigma_y^2 + (ad + bc)r_{xy}\sigma_x\sigma_y$$

34.12 Characteristic Functions of Two Normal Random Variables

34.12.1 Characteristic Function

 $\Phi(\omega_1,\omega_2) = E\{\exp[j(\omega_1X + \omega_2Y]\} = \exp[-\frac{1}{2}(\sigma_1^2\omega_1^2 + 2r\sigma_1\sigma_2\omega_1\omega_2 + \sigma_2^2\omega_2^2)] \quad \text{for} \quad E\{X\} = E\{Y\} = 0,$ and X and Y jointly normal.

34.12.2 Characteristic Function with Means

 $\Phi(\omega_1, \omega_2) = \exp[j(\omega_1 \mu_x + \omega_2 \mu_y)] \exp\{-\frac{1}{2}\mu_{20}\omega_1^2 + 2\mu_{11}\omega_1\omega_2 + \mu_{02}\omega_2^2\}, \ \mu_{ij} = \text{ joint moments about the means.}$

34.13 Price Theorem for Two R.V's

34.13.1 Price Theorem

If X and Y are jointly normal with

$$\mu_{11} = E\{(X - \mu_x)(Y - \mu_y)\} = E\{XY\} - E\{X\}E\{Y\},$$

then,

$$E\{g(X,Y)\} = \iint_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

a. If
$$\mu_{11} = 0$$
 (r.v.'s independent) $E\{X^kY^r\} = E\{X^k\}E\{Y^r\}$

b.
$$E\{X^kY^r\} = kr \int_0^{\mu_{11}} E\{X^{k-1}Y^{r-1}\} d\mu_{11} + E\{X^k\} E\{Y^r\}$$

c.
$$E\{X^2Y^2\} = 4\int_0^{\mu_{11}} E\{XY\} d\mu_{11} + E\{X^2\} E\{Y^2\} = 4\int_0^{\mu_{11}} (\mu_{11} + E\{X\} E\{Y\}) d\mu_{11}$$

 $+ E\{X^2\} E\{Y^2\} = 2\mu_{11}^2 + 4\mu_{11} E\{X\} E\{Y\} + E\{X^2\} E\{Y^2\}$

34.14 Sequences of Random Variables

34.14.1 Definitions

31.14.1.1 Definitions

n real r.v. $X_1, X_2, \dots, X_n; F(x_1, x_2, \dots, x_n) = P\{X_1 \le x_1, \dots, X_n \le x_n\} = \text{distribution function}; f(x_1, \dots, x_n) = \partial^n F / \partial x_1, \dots, \partial x_n = \text{density function}.$

31.14.1.2 Marginal Densities

 $F(x_1,x_3) = F(x_1,\infty,x_3,\infty) = \text{marginal distribution for a sequence of four r.v.}; f(x_1,x_3) = \iint_{-\infty} f(x_1,x_2,x_3,x_4) dx_2 dx_4 \text{ marginal density.}$

31.14.1.3 Functions of r.v.'s

$$Y_1 = g_1(X_1, \dots, X_n), \dots, \quad Y_n = g_n(X_1, \dots, X_n),$$

$$f_{y_1,\dots,y_n}(y_1,\dots,y_n) = f(x_1,x_2,\dots,x_n)/|J(x_1,\dots,x_n)|, \quad J(x_1,\dots,x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial g_n}{\partial x_1} & \dots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}$$

31.14.1.4 Conditional Densities

$$f(x_1, \dots, x_k | x_{k+1}, \dots, x_n) = f(x_1, \dots, x_k, \dots, x_n) / f(x_{k+1}, \dots, x_n).$$

Example

$$f(x_1 | x_2, x_3) = f(x_1, x_2, x_3) / f(x_2, x_3), \quad F(x_1 | x_2, x_3) = \int_{-\infty}^{x_1} f(\xi_1, x_2, x_3) d\xi_1 / f(x_2, x_3)$$

34.14.1.5 Chain Rule

$$f(x_1,\dots,x_n) = f(x_n|x_{n-1},\dots,x_1)\cdots f(x_2|x_1)f(x_1)$$

34.14.1.6 Removal Rule

$$f(x_1|x_3) = \int_{-\infty}^{\infty} f(x_1, x_2|x_3) dx_2,$$

$$f(x_1|x_4) = \int_{-\infty}^{\infty} f(x_1|x_2, x_3, x_4) f(x_2, x_3|x_4) dx_2 dx_3, \quad f(x_1|x_3) = \int_{-\infty}^{\infty} f(x_1|x_2, x_3) f(x_2|x_3) dx_2 dx_3$$

34.14.1.7 Independent r.v.

$$F(x_1, \dots, x_n) = F(x_1) \dots F(x_n); f(x_1, \dots, x_n) = f(x_1) \dots f(x_n)$$

$$f(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = f(x_1, \dots, x_k) f(x_{k+1}, \dots, x_n)$$

if X_1, \dots, X_k are independent of X_{k+1}, \dots, X_n

34.14.2 Mean, Moments, Characteristic Function

34.14.2.1 Expected Value

$$E\{g(X_1,\dots,X_n)\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1,\dots,x_n) f(x_1,\dots,x_n) dx_1 \cdots dx_n$$

34.14.2.2 Conditional Expected Values

$$E\{X_1 | x_2, \dots, x_n\} = \int_{-\infty}^{\infty} x_1 f(x_1 | x_2, \dots, x_n) dx_1 = \int_{-\infty}^{\infty} x_1 f(x_1, \dots, x_n) dx_1 / f(x_2, \dots, x_n)$$

a.
$$E\{E\{X_1|X_2,\dots,X_n\}\}=E\{X_1\}$$

b.
$$E\{X_1X_2|X_3\} = E\{E\{X_1X_2|X_2,X_3\} = E\{X_2E\{X_1|X_2,X_3\}|X_3\}$$

c. $E\{X_1|x_2,\dots,x_n\}=E\{X_1\}$ if X_1 is independent from the remaining r.v.'s

34.14.2.3 Uncorrelated r.v.'s

 X_1, \dots, X_n are uncorrelated if the covariance of any two of them is zero, $E\{X_i, X_i\} = E\{X_i\}E\{X_i\}$ for $i \neq j$

34.14.2.4 Orthogonal r.v.'s

$$E\{X_iX_i\} = 0$$
 for any $i \neq j$

34.14.2.5 Variance of Uncorrelated r.v.'s

$$\sigma_{x_1 + \dots + x_n}^2 = \sigma_{x_1}^2 + \dots + \sigma_{x_n}^2, \ \sigma_z^2 = E\{|Z - E\{Z\}|^2\} \text{ if } Z = X + jY \equiv \text{ complex r.v.}, \ E\{Z_i Z_j^*\} = E\{Z_i\} E\{Z_j^*\} = \text{uncorrelated r.v.'s } i \neq j, \ E\{Z_i Z_j^*\} = 0 \equiv \text{orthogonal},$$

$$f(x_1, y_1, x_2, y_2) = f(x_1, y_1) f(x_2, y_2)$$
 if $Z_1 = X_1 + jY_1$ and $Z_2 = X_2 + jY_2$

are independent,

 $E\{Z_{i}Z_{2}^{*}\} = E\{Z_{1}\}E\{Z_{2}^{*}\} = \int_{-\infty}^{\infty} z_{1}f(x_{1}, y_{1})dx_{1}dy_{1} \int_{-\infty}^{\infty} z_{2}^{*}f(x_{2}, y_{2})dx_{2}dy_{2} = \text{uncorrelated if they are independent}$

34.14.2.6 Characteristic Functions

$$\Phi(\omega_1,\dots,\omega_n) = E\{e^{j(\omega_1 X + \dots + \omega_n X_n)}\},\,$$

$$\frac{\partial^{r} \Phi(0,0)}{\Phi \omega_{1}^{k_{1}}, \dots, \partial \omega_{n}^{k_{n}}} = j^{r} \mu'_{k_{1} \dots k_{n}}, \ \mu'_{k_{1} \dots k_{n}} = E\{X_{1}^{k_{1}} \dots X_{n}^{k_{n}}\}, \ r = k_{1} + k_{2} + \dots + k_{n}$$

for dependent variables

34.14.2.7 Characteristic Functions for Independent Variables

$$E\{e^{j\omega_1X_1}\cdots e^{j\omega_nX_n}\}=E\{e^{j\omega_1X_1}\}\cdots E\{e^{j\omega_nX_n}\}$$

Example

$$Z = X_1 + X_2 + X_3$$
, $\Phi_z(\omega) = E\{e^{j\omega(X_1 + X_2 + X_3)}\} = \Phi_1(\omega)\Phi_2(\omega)\Phi_3(\omega)$. Hence,

$$f_z(z)$$
 = density function = $f_1(z) * f_2(z) * f_3(z)$

34.14.2.8 Sample Mean

$$\overline{X} = (X_1 + \dots + X_n)/n, \ \overline{X} =$$

34.14.2.9 Sample Variance

$$\overline{S} = [(X_1 - \overline{X})^2 + \dots + (X_n - \overline{X})^2]/n = \sum_{i=1}^n \frac{X_i^2}{n} - \overline{X}^2, \ \overline{S} = \text{random variable}$$

34.14.2.10 Statistic

A function of one or more variables that does not depend upon any unknown parameter.

Example

 $Y = \sum_{i=1}^{n} X_k \equiv \text{ is a statistic}; \ Y = (X_1 - \mu)/\sigma \equiv \text{ is not a statistic unless } \mu \text{ and } \sigma \text{ are known: the sample mean is a statistic.}$

34.14.2.11 Random Sums

 $Y = \sum_{k=1}^{n} X_k$ = random sum; $E\{Y\} = \mu E\{n\}$ if n is an r.v. of discrete type and X_i 's are independent of n and $E\{X_k\} = \mu$; $E\{Y^2\} = \mu^2 E\{n^2\} + \sigma^2 E(n)$ if the r.v. X_k are uncorrelated with the same variance σ^2 .

34.14.3 Normal Random Variables

34.14.3.1 Density Function

 $f(x_1,\dots,x_n) = (1/[(2\pi)^{n/2}\sigma^n])\exp[-(x_1^2+\dots+x_n^2)/2\sigma^2]$ where the r.v. X_i are normal, independent with same variance σ^2 .

34.14.3.2 Density Eunction of the Sample Mean

$$f_{\bar{x}}(x) = (1/\sqrt{2\pi\sigma^2/n}) \exp(-nx^2/2\sigma^2)$$
 (see also 34.14.3.1)

34.14.3.3 Density Function of $\chi = [X_1 + ... + X_n]^{1/2}$ (see also 34.14.3.1)

$$f_x(\chi) = \frac{2}{2^{n/2} \sigma^n \Gamma(n/2)} \chi^{n-1} \exp(-\chi^2/2\sigma^2) u(\chi), n = n \text{ degrees of freedpm}, u(\chi) = \text{unit step function}.$$

34.14.3.4 Density Function of $Y = \chi^2 = X_1^2 + \dots + X_n^2$ (see also **34.14.3.1**)

$$f_y(y) = \frac{1}{2^{n/2} \sigma^n \Gamma(n/2)} y^{(n-2)/2} \exp(-ns/2\sigma^2) u(y), \ u(y) = \text{unit step function}$$

34.14.3.5 Density Function of the Variance \overline{S} (see also 34.14.2.9 and 34.14.3.1)

$$f_{\bar{s}}(s) = \frac{1}{2^{(n-2)/2} (\sigma/\sqrt{n})^{n-1} \Gamma[(n-1)/2]} s^{(n-3)/2} \exp(-ns/2\sigma^2) u(s)$$

34.14.3.6 Characteristic Function

$$\Phi(\omega_{1}, \omega_{2}) = E\{e^{j(\omega_{1}X_{1} + \omega_{2}X_{2})}\} = \exp[-\frac{1}{2}(\sigma_{1}^{2}\omega_{1}^{2} + r\sigma_{1}\sigma_{2}\omega_{1}\omega_{2} + r\sigma_{1}\sigma_{2}\omega_{2}\omega_{1} + \sigma_{2}^{2}\omega_{2}^{2})]$$

$$\sigma_{1}^{2} = \mu_{11}^{\prime}, \ \sigma_{2}^{2} = \mu_{22}^{\prime}, \ r\sigma_{1}\sigma_{2} = \mu_{12}^{\prime}$$

34.14.3.7 Matrix Form of Density Function

$$f(x_1,\dots,x_2) = \exp(-\frac{1}{2}x^T \mu'^{-1}x) / \sqrt{(2\pi)^n |\mu'|}$$

$$\mu' = \begin{bmatrix} \mu'_{11} & \mu'_{12} & \cdots & \mu'_{1n} \\ \vdots & & \vdots \\ \mu'_{n1} & \mu'_{n2} & \cdots & \mu'_{nn} \end{bmatrix}, \quad \mu'^{-1} \text{ inverse of } \mu', \quad x = [x_1, \dots, x_n]^T, |\mu'| = \text{determinant}$$

34.14.3.8 Characteristic Function in Matrix Form

$$\Phi(\omega_1, \dots, \omega_n) = \exp(-\frac{1}{2}\omega^T \mu' \omega), \ \omega = [\omega_1, \dots, \omega_n]^T, \ \mu' \equiv \text{see } 34.14.3.7$$

34.14.4 Convergence Concepts, Central Limit Theorem

34.14.4.1 Chebyshev Inequality (see 34.5.14)

 $P\{\mu - \varepsilon < X < \mu + \varepsilon\} \ge 1 - \frac{\sigma^2}{\varepsilon^2} \cong 1 \text{ for } \sigma << \varepsilon \text{ .Hence the probability of the event } \{|X - \mu| < \varepsilon\} \text{ is close to 1 If the observed value of X is } X(\zeta) \text{ of an experiment, then } \mu - \varepsilon < X(\zeta) < \mu + \varepsilon \text{ or } X(\zeta) - \varepsilon < \mu < X(\zeta) + \varepsilon \text{ which estimates the mean.}$

34.14.4.2 Limiting Distribution

If $\lim_{n\to\infty} F_n(y) = F(y)$ for every point y at which F(y) is continuous, then the r.v. Y_n is said to have a limiting distribution F(y).

34.14.4.3 Stochastic Convergence

The r.v. Y_n converges stochastically to the constant c if and only if, for every $\varepsilon > 0$, $\lim_{n \to \infty} P\{|Y_n - c| < \varepsilon\} = 1$.

Example

$$P\{\left|\overline{X}_n - \mu\right| \ge \varepsilon\} = P\left\{\left|\overline{X}_n - \mu\right| \ge \frac{k\sigma}{\sqrt{n}}\right\} \le \frac{\sigma^2}{n\varepsilon^2} \quad (k = \varepsilon\sqrt{n}/\sigma)$$

the last inequality is due to Chebyshev inequality, $\overline{X}_n =$ mean of random sample of size n from a distribution with mean μ and variance σ^2 , the mean and variance of \overline{X}_n are μ and σ^2/n . Hence,

 $\lim_{n\to\infty} P\{|\overline{X}_n - \mu| \ge \varepsilon\} \le \lim_{n\to\infty} \frac{\sigma^2}{n\varepsilon^2} = 0 \text{ which implies that } \overline{X}_n \text{ converges stochastically to } \mu \text{ if } \sigma^2 \text{ is finite.}$

34.14.4.4 Convergence in Probability

 $\lim_{n\to\infty} P\{|Y_n-c|<\varepsilon\}=1$ implies Y_n converges to c in probability (same as stochastic convergence).

34.14.4.5 Convergence with Probability One

 $P\{\lim_{n\to\infty} Y_n = c\} = 1$ implies Y_n converges to c with probability one.

34.14.4.6 Limiting Distribution and Moment-Generating Function

If $\lim_{n \to \infty} M(t;n) = M(t)$ then Y_n has a limiting distribution with distribution function F(Y). M(t;n) = m moment generating function of Y_n in -h < t < h for all n; M(t) = m.g.f. of Y with d.f. F(Y) in $|t| \le h_1 < h$ (see Table 34.1).

Example

 Y_n 's have binomial distribution f(n,p). Let $\mu = np$ be the same for every n, that is, $p = \mu/n$. M(t,n) = np

$$E\{e^{tY_n}\} = [(1-p) + pe^t]^n = \left[1 + \frac{\mu(e^t - 1)}{n}\right]^n \text{ for all real values of t. Hence, } \lim_{n \to \infty} M(t;n) = \exp[\mu(e^t - 1)]$$
 for all t .

Note: From advanced calculus

$$\lim_{n\to\infty} \left[1 + \frac{b}{n} + \frac{\psi(n)}{n}\right]^{cn} = \lim_{n\to\infty} \left(1 + \frac{b}{n}\right)^{cn} = e^{bc}, \ \lim_{n\to\infty} \psi(n) = 0,$$

c and b do not depend upon n. Hence, Y_n has a limiting Poisson distribution with mean μ since the moment generating function of Poisson distribution is $\exp[\mu(e^t - 1)]$.

34.14.4.7 Central Limit Theorem

 $Y_n = \left(\sum_{i=1}^n X_i\right) / \sqrt{n\sigma}$ has a limiting distribution that is normal with mean zero and variance 1.

 X_1, X_2, \cdots, X_n is a random sample with mean μ and variance σ^2 .

Example

Let X_1, X_2, \cdots, X_n are r.v. from binomial distribution with $\mu = p$ and $\sigma^2 = p(1-p)$. to find the $P\{Y = 48, 49, 50, 51, 52\}$ is equivalent to finding the $P\{47.5 < Y < 52.5\}$. But $(Y_n - np)/\sqrt{np(1-p)}$ where $Y_n = \overline{X}_n = X_1 + X_2 + \cdots + X_n$ has a limiting distribution that is normal with mean zero and variance

1. If n = 100 and p = 1/2 then np = 50 and $\sqrt{np(1-p)} = 5$ and hence $P_r\{47.5 < Y < 52.5\} = P\left\{\frac{47.5 - 50}{5} < \frac{Y - 50}{5} < \frac{52.5 - 50}{5}\right\} = P\left\{-0.5 < \frac{Y - 50}{5} < 0.5\right\}$. But (Y - 50)/5 has a limiting distribution that is normal with mean zero and variance one. From tables (see 34.2), we find that the probability (Y - 50)/5 between -0.5 and 0.5 is 0.383.

34.14.4.8 Limiting Distribution of W_n=U_n/V_n

If $\lim_{n\to\infty} F_n(u) = F(u)$ and if V_n converges stochastically (see 34.14.4.3) to 1, then the limiting distribution of the r.v. $W_n = U_n/V_n$ is the same as that of U_n (the distribution F(w)).

Example

Let the distribution be $N(\mu, \sigma^2)$ (normal with mean μ and variance σ^2) and a sample from it with mean \overline{X}_n and variance S_n^2 . \overline{X}_n converges stochastically to μ and S_n^2 converges stochastically to σ^2 . Since S_n/σ converges stochastically to 1 (see 34.14.4.9), then $W_n = \overline{X}_n/(S_n/\sigma)$ has the same distribution as the limiting distribution of \overline{X}_n .

34.14.4.9 LiIniting Distribution of U_n/c

If $\lim_{n\to\infty} F_n(u) = F(u)$ and U_n and U_n converges stochastically (see 34.14.4.3) to $c\neq 0$ then U_n/c converges stochastically to 1.

34.15 General Concepts of Stochastic Processes

34.15.1 Introduction

34.15.1.1 The X(t) Real Function (see Figure 34.3)

X(t) ($X(t,\zeta)$) represents four different things:

- 1. A family of time functions (t variable, ζ variable)
- 2. A single time function (t variable, ζ fixed)
- 3. An r.v. (t fixed, ζ variable)
- 4. A single number (t, ζ fixed)

($\zeta \equiv$ outcomes of an experiment forming the space S, certain subsets of S are events, and probably probability of these events.)

34.15.1.2 Distribution Function (first order) F(x;t)

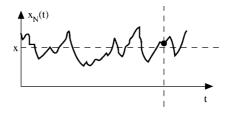
 $F(x;t) = P\{X(t) \le x\}$. The event $\{X(t) \le x\}$ consists of all outcomes ζ such that at specified t, the functions X(t) of the process do not exceed the given number x.

34.15.1.3 Density Function

$$f(x,t) = \frac{\partial F(x,t)}{\partial x}$$

34.15.1.4 Distribution Function (second order)

$$F(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$



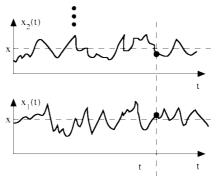


FIGURE 34.3

34.15.1.5 Density Function (second order)

$$f(x_1, x_2; t_1, t_2) = \frac{\partial F(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

Note:
$$F(x_1,\infty;t_1,t_2) = F(x_1;t_1), f(x_1;t_1) = \int_{-\infty}^{\infty} f(x_1,x_2;t_1,t_2) dx_2$$

34.15.1.6 Conditional Density

$$f(x_1,t_1|X_2(t_2)=x_2) = f(x_1,x_2;t_1,t_2)/f(x_2;t_2)$$

34.15.1.7 Mean

$$\mu(t) = \mu'(t) = E\{X(t)\} = \int\limits_{-\infty}^{\infty} x f(x;t) dx$$

34.15.1.8 Autocorrelation

$$R(t_1, t_2) = E\{X(t_1)X(t_2)\} \int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; t_1, t_2) dx_1 dx_2$$

34.15.1.9 Autocovariance

$$C(t_1, t_2) = E\{[X(t_1) - \mu(t_1)][X(t_2) - \mu(t_2)]R(t_1, t_2) - \mu(t_1)\mu(t_2)$$

34.15.1.10 Variance

$$\sigma_{x(t)}^2 = C(t,t) = R(t,t) - \mu^2(t)$$

34.15.1.11 Distribution Function (nth order)

$$F(x_1, \dots, x_n; t_1, \dots, t_n) = P\{X(t_1) \le x_1, \dots, X(t_n) \le x_n\}$$

34.15.1.12 Density Function (nth order)

$$f(x_1,\dots,x_n;t_1,\dots,t_n) = \frac{\partial F(x_1,\dots,x_n;t_1,\dots,t_n)}{\partial x_1,\partial x_2,\dots,\partial x_n}$$

34.15.1.13 Complex Process

Z(t) = X(t) + jY(t) is a family of complex functions and is statistically determined in terms of the two dimensional processes X(t) and Y(t).

34.15.1.14 Mean of Complex Process

 $\mu_{x}(t) = E\{X(t)\}, X(t) \equiv \text{real or complex}$

34.15.1.15 Autocorrelation of Complex Process

 $R(t_1,t_2) = E\{X(t_1)X^*(t_2)\},$ * indicates conjugation.

34.15.1.16 Autocovariance of Complex Process

$$C(t_1, t_2) = E\{[X(t_1) - \mu(t_1)][X^*(t_2) - \mu^*(t_2)]\} = R(t_1, t_2) - \mu(t_1)\mu^*(t_2)$$

34.15.1.17 Crosscorrelation of Complex Processes

$$R_{xy}(t_1, t_2) = E\{X(t_1)Y^*(t_2)\}$$

34.15.1.18 Crosscovariance of Complex Processes

$$C_{xy}(t_1, t_2) = R_{xy}(t_1, t_2) - \mu_x(t_1)\mu_y^*(t_2)$$

34.15.1.19 Uncorrelated Complex Processes

$$R_{xy}(t_1, t_2) = \mu_x(t_1)\mu_y^*(t_2), \quad C_{xy}(t_1, t_2) = 0$$

34.15.1.20 Orthogonal

$$R_{xy}(t_1, t_2) = 0$$

34.15.1.21 Independent Processes X(t) and Y(t)

The processes are independent if the group $X(t_1), \dots, X(t_n)$ is independent of the group $Y(t_1'), \dots, Y(t_n')$ for any $t_1, \dots, t_{\nu}, t_1', \dots, t_n'$

34.15.1.22 Normal Processes

A process X(t) is normal if r.v. $X(t_1), \dots, X(t_n)$ are jointly normal for any n.

Example

 $X(t) = A\cos at + B\sin at$, A and B = independent normal r.v. with $E\{A\} = E\{B\} = 0$ and $E\{A^2\} = E\{B^2\} = \sigma^2$ and a = constant. Hence $E\{X(t)\} = E\{A\}\cos at + E\{B\}\sin at = 0$, $R(t_1, t_2) = E\{(A\cos at_1 + B\sin at_1)(A\cos at_2 + B\sin at_2)\} = E\{A^2\cos at_1\cos at_2 + E\{B^2\}\sin at_1\sin at_2 = \sigma^2\cos \omega(t_2 - t_1)$ which implies that X(t) has mean value zero and variance σ^2 and,hence $f(x,t) = (1/\sigma\sqrt{2\pi})\exp(-x^2/2\sigma^2)$. But $r = R(t_1,t_2)/\sqrt{R(t_1,t_1)R(t_2,t_2)} = \cos a(t_1 - t_2)$ and from (34.11.1) $f(x_1,x_2;t_1,t_2) = [1/(2\pi\sigma^2)/(1-\cos^2 a\tau)]\exp[-(x_1^2 - 2x_1x_2\cos a\tau + x_2^2)/(2\sigma^2(1-\cos^2 a\tau))]$ where $\tau = t_1 - t_2$.

Note: f(x;t) is independent of t and $f(x_1,x_2;t_1,t_2)$ depends only on $t_1-t_2=$ time difference.

34.16 Stationary Processes

34.16.1 Strict and Wide Sense Stationary

34.16.1.1 Strict Sense

X(t) and $X(t+\varepsilon)$ have the same statistics for all shifts ε .

34.16.1.2 Jointly Stationary

X(t),Y(t) the same joint statistics as $X(t+\varepsilon)$, $Y(t+\varepsilon)$ for all shifts ε .

34.16.1.3 Complex Process

Z(t) = X(t) + jY(t) is stationary if X(t) and Y(t) are jointly stationary.

34.16.1.4 Density Function of the nth Order

 $f(x_1, \dots, x_n; t_1, \dots, t_n) = f(x_1, \dots, x_2; t_1 + \varepsilon, \dots, t_n + \varepsilon)$ for all shifts ε .

- 1. f(x,t) = f(x) = independent of time,
- 2. $E\{X(t)\} = \mu =$
- 3. $f(x_1, x_2; t_1, t_2) = f(x_1, x_2; \tau), \ \tau = t_1 t_2 \equiv \text{ joint density of } X(t + \tau) \text{ and } X(t)$
- 4. $R(\tau) = E\{X(t+\tau)X(t)\} = R(-\tau),$
- 5. $R_{xy}(\tau) = E\{X(t+\tau)Y(t)\}$

34.16.1.5 Wide Sense Stationary

X(t) is wide sense stationary if $E\{X(t)\} = \mu = \text{constant}$ and $E\{X(t+\tau)X(t)\} = R(\tau)$.

Note: A wide sense process may not be a strict one. The converse is always true. A strict sense normal process is also a wide one.

Example

 $X(t) = \cos(at + \varphi)$, φ is an r.v. and α is a constant. Then $E\{X(t)\} = E\{\cos(at + \varphi)\} = E\{\cos at \cos \varphi\}$ $-\{\sin at \sin \varphi\} = \cos at E\{\cos \varphi\} - \sin at E\{\sin \varphi\}$ and to have a stationary process X(t) we must have the mean independent of t or equivalently

$$E\{\cos \varphi\} = E\{\sin \varphi\} = 0. \ R(t+\tau,t) = E\{\cos(a(t+\tau) + \varphi)\cos(at + \varphi)\}$$
$$= \frac{1}{2}\cos a\tau + \frac{1}{2}E\{\cos(2at + a\tau + 2\varphi)\}.$$

For the autocorrelation to be independent of t, we must have $E\{\cos 2\varphi\} = E\{\sin 2\varphi\} = 0$. Hence $R(\tau) = \frac{1}{2}\cos a\tau$ and the process is wide stationary.

34.17 Stochastic Processes and Linear Deterministic Systems

34.17.1 Memoryless and Time-Invariant System

34.17.1.1 Output System

$$Y(t) = g[X(t)]$$

34.17.1.2 Mean of Output

$$E\{Y(t)\} = \int_{-\infty}^{\infty} g(x) f_x(x;t) dx$$

34.17.1.3 Autocorrelation of Output

$$E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1)g(x_2)f_x(x_1x_2;t_1,t_2)dx_1dx_2, x_i \text{ is the r.v. of the process } X(t) \text{ at } t = t_i.$$

34.17.1.4 Output Density Function

$$f_y(y_1, \dots, y_k; t_1, \dots, t_k) = f_x(x_1, \dots, x_k; t_1, \dots, t_k) / [|g'(x_1)| \dots |g'(x_k)|]$$

$$y_1 = g(x_1), \dots, y_k = g(y_k), \text{ prime indicates derivative.}$$

Example

If

$$Y(t) = \begin{cases} 1 & X(t) \le x \\ 0 & X(t) > x \end{cases}$$

then $P\{Y(t) = 1\} = P\{X(t) \le x\}$, $P\{Y(t) = 0\} = P\{X(t) \ge x\}$ and hence, $E\{Y(t)\} = 1 \cdot P\{X(t) \le x\} + 0 \cdot P\{X(t) \ge x\} = F_x(x)$

34.17.2 Linear System

34.17.2.1 Linear Operator

 $Y(t) = L[X(t)], L[a_1X_1(t) + a_2X_2(t)] = a_1L[X_1(t)] + a_2L[X_2(t)], L[AX(t)] = AL[X(t)], A \equiv \text{ random variable}, Y(t+\tau) = L[X(t+\tau)] \text{ implies L (system) is time invariant, L = transformation (system)}.$

34.17.2.2 Linear Transformation

$$E\{Y(t)\} = L[E\{X(t)\}]$$

34.17.2.3 Autocorrelation

$$\begin{split} R_{xy}(t_1,t_2) &= L_{t_2}[R_{xx}(t_1,t_2)] = L_{t_2}[E\{X(t_1)X(t_2)\}]; \ R_{yy}(t_1,t_2) = L_{t_1}[R_{xy}(t_1,t_2)] = L_{t_1}[E\{X(t_1)Y(t_2)\}]; \\ E\{Y(t_1)Y(t_2)Y(t_3)] &= L_{t_1}L_{t_2}L_{t_3}[E\{X(t_1)X(t_2)X(t_3)\}] \ \text{ which means to evaluate the left-hand side we must know the mean of the right-hand side for every } t_1,t_2,t_3. \end{split}$$

34.17.3 Stochastic Continuity and Differentiation

34.17.3.1 Continuity in the m.s. Sense

If $E\{[X(t+\tau)-X(T)]^2\}\to 0$ as $\tau\to 0$ the X(t) is continuous in the mean square (m.s.) sense.

34.17.3.2 Continuity with Probability 1

 $\lim X(t+\varepsilon) = X(t), \ \varepsilon \to 0$ for all outcomes

34.17.3.3 Continuity in the m.s.

$$E\{[X(t+\tau)-X(t)]^2\} \to 0 \text{ with } \tau \to 0$$

34.17.3.4 Continuity in the Mean

$$E\{X(t+\tau)\} \rightarrow E\{X(t)\}, \ \tau \rightarrow 0$$

34.17.3.5 Continuity of Stationary Process

 $E\{[X(t+\tau)-X(t)]^2\}=2[R(0)-R(\tau)]$ The process is continuous if its autocorrelation function is continuous at $\tau=0$.

34.17.3.6 Differentiable Stationary Process

A stationary process X(t) is differentiable in the m.s. sense if its autocorrelation $R(\tau)$ has derivatives of order up to two.

34.17.3.7 Autocorrelation of Derivatives

$$R_{xx'}(\tau) = -\frac{dR_{xx}(\tau)}{d\tau}, \ R_{x'x'}(\tau) = \frac{dR_{xx'}(\tau)}{d\tau},$$

$$R_{x'x'}(\tau) = -\frac{d^2 R_{xx}(\tau)}{d\tau^2}, \ E\{[X'(t)]^2\} = R_{x'x'}(0) = -\frac{d^2 R_{xx}(0)}{d\tau^2}.$$

Primes indicate differentiation.

34.17.3.8 Taylor Series

If

$$R(\tau) = \sum_{n=0}^{\infty} R^{(n)}(0) \frac{\tau^{n}}{n!}$$

then $X^{(n)}(t)$ exists in the m.s. sense and

$$X(t+\tau) = \sum_{n=0}^{\infty} X^{(n)}(t) \frac{\tau^n}{n!}$$

where (n) in exponent means n^{th} derivative and $R(\tau)$ must have derivatives of any order.

34.17.4 Stochastic Integrals, Averages, and Ergoticity

34.17.4.1 Integral of a Process

 $S = \int_{0}^{b} X(t)dt$ = and defines a number $S(\zeta)$ for each outcome.

34.17.4.2 M.S. Limit of a Sum

$$\lim E\left\{ \left[S - \sum_{i=1}^{n} X(t_i) \Delta t_i \right]^2 \right\} = o \text{ for } \Delta t_i \to 0$$

34.17.4.3 Mean Value of Integrals

 $E\{S\} = \int_a^b E\{X(t)\}dt = \int_a^b \mu(t)dt \text{ (see 34.17.4.1) since the integral can be equated to a sum.}$

34.17.4.4 The Mean of S²

$$E\{S^2\} = \int_a^b \int_a^b E\{X(t_1)X(t_2)\}dt_1dt_2 = \int_a^b \int_a^b r(t_1,t_2)dt_1dt_2$$

34.17.4.5 Variance of S

$$\sigma_s^2 = \int_a^b \int_a^b [R(t_1, t_2) - \mu(t_1)\mu(t_2)dt_1dt_2 = \int_a^b \int_a^b C(t_1, t_2)dt_1dt_2;$$

$$\sigma_s^2 = \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T} \right) C(\tau) d\tau$$

for a = -T and b = T; if X(t) is real $C(\tau)$ is even and

$$\sigma_s^2 = \frac{1}{T} \int_0^{2T} (1 - \frac{\tau}{2T}) C(\tau) d\tau = \frac{1}{T} \int_0^{2T} (1 - \frac{\tau}{2T}) [R(\tau) - \mu^2] d\tau$$

34.17.4.6 Ergotic Process

A process X(t) is ergotic if all its statistics can be determined from a single function (realization) $X(t,\zeta)$ of the process.

34.17.4.7 Ergoticity of the Mean

$$\lim_{T \to \infty} \overline{X}_T(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)dt = E\{X(t)\} = \mu = \text{ constant (stationary process) iff}$$

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{2T} \left(1 - \frac{\tau}{2T} \right) [R(\tau) - \mu^{2}] d\tau = 0$$

Example

$$E\{X(t)\}, R(\tau) = \exp(-\alpha |t|), E\{S\} = \frac{1}{2T} \int_{T}^{T} E\{(t)\}dt0,$$

$$\sigma_s^2 = \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau}{2T} \right) \exp(-\alpha \tau) d\tau = \frac{1}{\alpha T} - \frac{1 - e^{-2\alpha T}}{2\alpha^2 T^2}$$

which approaches zero as $T \to \infty$ and hence X(t) is ergotic in the mean.

34.17.4.8 Ergoticity of the Autocorrelation

$$\lim_{T\to\infty}R_T(\lambda)=\lim_{T\to\infty}\frac{1}{2T}\int\limits_{-T}^TX(t+\lambda)X(t)dt=E\{X(t+\lambda)X(t)\}=R(\lambda) \text{ iff }$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{0}^{2T}\left(1-\frac{\tau}{2T}\right)\left[R_{yy}(\tau)-R^{2}(\lambda)\right]d\tau=0$$

where $Y(t) = X(t + \lambda)X(t)$, $E\{Y(t)\} = E\{X(t + \lambda)X(t)\} = R(\lambda)$

34.18 Correlation and Power Spectrum of Stationary Processes

34.18.1 Correlation and Covariance

34.18.1.1 Correlation

$$R(\tau) = E\{X(t+\tau)X^*(t)\} = R_x(\tau), * indicates conjugation; R(-\tau) = R^*(\tau)$$

34.18.1.2 Crosscorrelation

$$R_{xy}(\tau) = E\{X(t+\tau)Y^*(t)\} = R_{yx}^*(-\tau)$$

34.18.1.3 Auto and Crosscovariance

$$C(\tau) = E\{[X(t+\tau) - \mu^*]\} = R(\tau) - |\mu| \equiv \text{ covariance};$$

$$C_{xy}(\tau) = E\{[X(t+\tau) - \mu_x][Y^*(t) - \mu_y^*]\} = R_{xy}(\tau) - \mu_x \mu_y^* \equiv \text{crosscovariance}$$

34.18.1.4 Properties of Correlation

1.
$$R_{yy}(\tau) = R_{yy}(\tau) + R_{yy}(\tau) + R_{yy}(\tau) + R_{yy}(\tau)$$
 if $Z(t) = X(t) + Y(t)$

2.
$$R_{ww}(\tau) = R_{xx}(\tau)R_{yy}(\tau)$$
 if $W(t) = X(t)Y(t)$ and $X(t)$ is independent of $Y(t)$

3.
$$R(0) = E\{|X(t)|^2 \ge 0$$

4.
$$E\{[X(t+\tau) \pm X(t)]^2\} = 2[R(0) \pm R(\tau)]$$

5.
$$|R(\tau)| \le R(0)$$
, $R(\tau)$ is maximum at the origin,

6.
$$E\{[X(t+\tau) + aY^*(t)]^2 = R_{xx}(0) + 2aR_{xy}(\tau) + a^2R_{yy}(0), X(t) \text{ and } Y(t) \text{ are real processes.}$$

7.
$$2|R_{xy}(\tau)| \le R_{xx}(0) + R_{yy}(0)$$

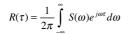
34.18.2 Power Spectrum of Stationary Processes

34.18.2.1 Power Spectrum (spectral density; see Table 34.8.)

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \equiv \text{ real, since } R(-\tau) = R^*(\tau);$$

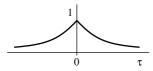
$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} d\omega = \int_{-\infty}^{\infty} S(f) e^{j2\pi f\tau} df; R(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega = E\{|X(t)|^2\};$$

if X(t) is real implies $R(\tau)$ is real and even and, hence, $S(-\omega) = S(\omega) \equiv \text{even}$, $R(\tau)$ correlation

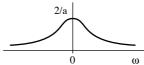


$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

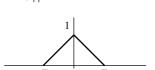
 $e^{-a|t|}$



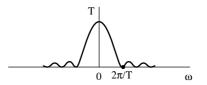
$$\frac{2_a}{a^2 + \omega^2}$$



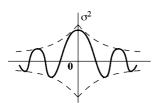
$$1 - \frac{|\tau|}{T}, \ 0 \le \tau \le T$$
$$0, \ |t| > 0$$

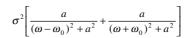


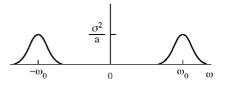
$$\frac{4\sin^2(\omega T/2)}{T\omega^2}$$



$$\sigma^2 e^{-a|t|} \cos \omega_0 \tau$$

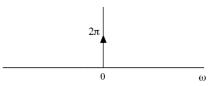






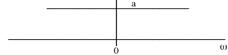




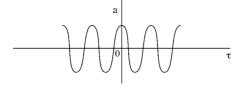


 $a\delta(\tau)$





 $a\cos\omega_0\tau$



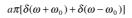




TABLE 34.8 (continued)

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega \tau} d\omega$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega \tau} d\tau$$

$$A\sigma^{2} a \frac{a^{2} + \omega_{0}^{2}}{(\omega^{2} - \omega_{0}^{2} - a^{2})^{2} + 4a^{2}\omega^{2}}$$

$$A\sigma^{2} a \frac{a^{2} + \omega_{0}^{2}}{(\omega^{2} - \omega_{0}^{2} - a^{2})^{2} + 4a^{2}\omega^{2}}$$

$$A\sigma^{2} a \frac{a^{2} + \omega_{0}^{2}}{(\omega^{2} - \omega_{0}^{2} - a^{2})^{2} + 4a^{2}\omega^{2}}$$

$$A\sigma^{2} a \frac{a^{2} + \omega_{0}^{2}}{(\omega^{2} - \omega_{0}^{2} - a^{2})^{2} + 4a^{2}\omega^{2}}$$

$$A\sigma^{2} a \frac{a^{2} + \omega_{0}^{2}}{(\omega^{2} - \omega_{0}^{2} - a^{2})^{2} + 4a^{2}\omega^{2}}$$

$$\sigma^{2} e^{-a^{2}e^{-a^{2}r^{2}}} \cos \omega_{0} \tau$$

34.18.2.2 Cross-power Spectrum

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-j\omega\tau}d\tau = S_{yx}^{*}(\omega), R_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{j\omega\tau}d\omega;$$

$$R_{xy}(0) = \frac{1}{2\pi} \int_{0}^{\infty} S_{xy}(\omega) d\omega = E\{X(t)Y^{*}(t)\}$$

34.18.2.3 Orthogonal Processes

$$R_{xy}(\tau) = 0, \ S_{xy}(\omega) = 0, \ R_{x+y}(\tau) = R_x(\tau) + R_y(\tau), \ S_{x+y}(\omega) = S_x(\omega) + S_y(\omega) \text{ (see 34.14.2.4)}$$

34.18.2.4 Relationships Between Processes (see Table 34.9)

TABLE 34.9

X(t)	$R(\tau) = E\{X(t+\tau)X(t)\}$	$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau$
aX(t)	$\left a\right ^{2}R(\tau)$	$ a ^2 S(\omega)$
$\frac{dX(t)}{dt}$	$-rac{d^2R(au)}{d au^2}$	$\omega^2 S(\omega)$
$\frac{d^n X(t)}{dt^n}$	$(-1)^n rac{d^{2n}R(au)}{d au^{2n}}$	$\omega^{2n}S(\omega)$

$$X(t) \qquad R(\tau) = E\{X(t+\tau)X(t)\} \qquad S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau$$

$$X(t)e^{\pm j\omega_0 t} \qquad e^{\pm j\omega_0 t}R(\tau) \qquad S(\omega \mp \omega_0)$$

$$b_0 \frac{d^m X(t)}{dt^m} + b_1 \frac{d^{m-1}X(t)}{dt^{m-1}} \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega\tau} \left| b_0(j\omega)^m + b_1(j\omega)^{m-1} + \dots + b_m \right|^2 S(\omega) d\omega$$

$$b_0 \frac{d^2 X(t)}{dt^2} + b_1 \frac{dX(t)}{dt} + b_2 X(t) \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega\tau} \left| b_0(j\omega)^2 + b_1(j\omega) + b_2 \right| S(\omega) d\omega \qquad |b_0(j\omega) + b_1(j\omega) + b_2| S(\omega)$$

34.18.2.5 Power Spectrum as Time Average (ergoticity)

$$\lim_{T\to\infty} E\{S_T(\omega)\} = S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \text{ where } S_T(\omega) = \frac{1}{2T} \left| \int_{-T}^T X(t)e^{-j\omega t}dt \right|^2 \text{ and } \int_{-\infty}^{\infty} |\tau R(\tau)|d\tau < \infty.$$

Example

 $R(\tau) = e^{-2\lambda|t|} \equiv$ autocorrelation of random telegraph signal,

$$S(\omega) = \int_{-\infty}^{\infty} e^{-2\lambda |t|} e^{-j\omega t} d\tau = \int_{-\infty}^{0} e^{2\lambda \tau} e^{-j\omega t} d\tau + \int_{0}^{\infty} e^{-2\lambda \tau} e^{-j\omega t} d\tau = \frac{4\lambda}{4\lambda^2 + \omega^2}$$

Example

If
$$X(t) = \sum_{i=1}^{n} A_i e^{j\omega_i t}$$
 then $R(\tau) = \sum_{i=1}^{n} \sigma_i^2 e^{j\omega_i \tau}$ where $E\{A_i^2\} = \sigma_i^2$ and, hence,

$$S(\omega) = \sum_{i=1}^{n} \sigma_{i}^{2} \int_{-\infty}^{\infty} e^{-j(\omega - \omega_{i})} d\tau = 2\pi \sum_{i=1}^{n} \sigma_{i}^{2} \delta(\omega - \omega_{i})$$

34.18.3 Linear Systems and Stationary Processes

34.18.3.1 Transfer Function

 $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$; h(t) = impulse response of a linear and time invariant system. h(t) is produced if delta function is the input to the system.

34.18.3.2 Mean

$$E\{Y(t)\} = \mu_y = \int_{-\infty}^{\infty} E\{X(t-\alpha)h(\alpha)d\alpha = \mu_x \int_{-\infty}^{\infty} h(\alpha)d\alpha = \mu_x H(0), Y(t) \equiv \text{ output of the system, } X(t) \equiv \text{ input of the system, } h(t) = \text{impulse response of the system}$$

34.18.3.3 Cross-Correlation (outputs—inputs)

$$E\{Y(t)X^*(t-\tau)\} = R_{yx}(\tau) = \int_{-\infty}^{\infty} E\{X(t-\alpha)X^*(t-\tau)\}h(\alpha)d\alpha$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau - \alpha)h(\alpha)d\alpha = R_{xx}(\tau) * h(\tau)$$

34.18.3.4 Autocorrelation of Input

$$E\{X(t-\alpha)X^*(t-\tau)\} = R_{xx}[t-\alpha-(t-\tau)] = R_{xx}(\tau-\alpha)$$

34.18.3.5 Autocorrelation of Output

$$R_{yy}(\tau) = E\{Y(t+\tau)Y^{*}(t)\} = \int_{-\infty}^{\infty} E\{Y(t+\tau)X^{*}(t-\alpha)\}h^{*}(\alpha)d\alpha = R_{yx}(\tau) * h^{*}(-\tau);$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau), \ R_{yy}(\tau) = R_{xy}(\tau) * h(\tau), \ R_{yy}(\tau) = R_{xx}(\tau) * h^*(-\tau) * h(\tau)$$

(see 34.18.3.2 and 34.18.3.3)

34.18.3.6 White Noise Input

 $X(t) = \text{ white noise, } R_{xx}(\tau) = \delta(\tau), \ h(t) = 0 \text{ for } t < 0 \text{ (casual system), then } R_{xy}(\tau) = h(-\tau) = 0 \text{ for } \tau > 0 \text{ which implies } Y(t) \text{ and } X(t) \text{ are orthogonal.}$

34.18.3.7 Power Spectrum

 $S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$ and $S_{yy}(\omega) = S_{xy}(\omega)H(\omega)$ and hence, $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2 F\{h^*(-t)\} = H^*(\omega)$, F stands for Fourier transform)

34.18.3.8 Positiveness of Power Spectrum

 $S(\omega) \ge 0$

Example

Y(t) = dX(t)/dt, $H(\omega) = j\omega \equiv$ transfer function of a differentiation, and hence, 34.18.3.7 gives $S_{xx'}(\omega) = S_{xx}(\omega)(-j\omega)$ and $S_{x'x'}(\omega) = \omega^2 S_{xx}(\omega)$. Thus $R_{xx'}(\tau) = F^{-1}\{S_{xx}(\omega)(-j\omega)\} = -\frac{dR_{xx}(\tau)}{d\tau}$ and $R_{x'x'}(\tau) = F^{-1}\{\omega^2 S_{xx}(\omega)\} = -\frac{d^2R_{xx}(\tau)}{d\tau^2}$ where F^{-1} where F-I stands for inverse Fourier transform.

34.18.3.9 Multiple Terminals Spectra (see Figures 34.4 and 34.5)

$$\begin{split} R_{y_1 y_2}(\tau) &= \int_{-\infty}^{\infty} R_{x_1 y_2}(t - \alpha) h_1(\alpha) d\alpha = R_{x_1 y_2}(\tau) * h_1(\tau) = \int_{-\infty}^{\infty} R_{x_1 x_2}(\tau + \beta) h_2^*(\beta) d\beta \\ &= R_{x_1 x_2}(\tau) * h_2^*(-\tau); S_{y_1 y_2}(\omega) = S_{y_1 y_2}(\omega) H_1(\omega); S_{x_1 y_2}(\omega) = S_{x_1 x_2}(\omega) H_2^*(\omega); \end{split}$$

$$S_{y_1y_2}(\omega) + S_{x_1x_2}(\omega)H_1(\omega)H_2^*(\omega)$$

$$X_1(t) \longrightarrow \begin{bmatrix} h_1(t) & Y_1(t) & X_2(t) & h_2(t) \\ H_1(\omega) & & & H_2(\omega) \end{bmatrix}$$

FIGURE 34.4

$$\begin{array}{c|c} S_{x1x2}(\omega) \\ \hline R_{x1x2}(\tau) \end{array} \hspace{-0.5cm} \begin{array}{c|c} h_2^*(-\tau) \\ \hline H_2^*(\omega) \end{array} \hspace{-0.5cm} \begin{array}{c|c} S_{x1x1}(\omega)H_2^*(\omega) \\ \hline R_{x1x2}(\tau) \end{array} \hspace{-0.5cm} \begin{array}{c|c} h_l(\tau) \\ \hline H_l(\omega) \end{array} \hspace{-0.5cm} \begin{array}{c|c} S_{x1x2}(\omega)H_l(\omega)H_2^*(\omega) \\ \hline R_{y1y2}(\tau) \end{array} \hspace{-0.5cm} \begin{array}{c|c} \end{array}$$

FIGURE 34.5

Note: If $H_1(\omega)H_2(\omega) = 0 \equiv$ disjoint systems,, $S_{y_1y_2}(\omega) = 0$; if also $E\{X(t)\} = 0$, $Y_1(t)$ and $Y_2(t)$ are uncorrelated

Example (see Figure 34.6)

$$H_1(\omega) = \frac{a+jw}{2a+jw}, \ H_2(\omega) = \frac{a}{2a+j\omega}, \ R_x(\tau) = e^{-2\lambda|\tau|} \equiv \text{ random telegraph signal}, \ S_x(\omega) = \frac{4\lambda}{4\lambda^2+\omega^2},$$

$$S_{\nu_{1}}(\omega) = \frac{4\lambda}{4\lambda^{2} + \omega^{2}} \frac{a^{2} + \omega^{2}}{4a^{2} + \omega^{2}} @S_{\nu_{2}}(\omega) = \frac{4\lambda}{4\lambda^{2} + \omega^{2}} \frac{a^{2}}{4a^{2} + \omega^{2}}, S_{\nu_{1}\nu_{2}}(\omega) = \frac{4\lambda}{4\lambda^{2} + \omega^{2}} \frac{a + j\omega}{2a + j\omega} \frac{a}{2a - j\omega}$$

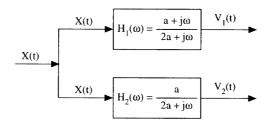


FIGURE 34.6

34.18.3.10 Stochastic Differential Equations

1.
$$a_n Y^{(n)}(t) + \dots + a_0 Y(t) = X(t)$$
, $E\{Y(t)\} = \frac{1}{2} E\{X(t)\}$ (see (34.18.3.2)

$$H(\omega) = \frac{1}{a_{x}(i\omega)^{n} + \dots + a_{0}}, S_{y}(\omega) = S_{x}(\omega)|H(\omega)|^{2}$$

2.
$$a_n Y^{(n)}(t) + \dots + a_0 Y(t) = b_m X^{(m)} + \dots + b_0 X(t)$$

$$S_{y}(\omega) = \frac{\left|b_{m}(j\omega)^{m} + \dots + b_{0}\right|^{2}}{\left|a_{n}(j\omega)^{n} + \dots + a_{0}\right|} S_{x}(\omega)$$

Exponent in parenthesis indicates derivatives.

Example

$$Y^{(2)}(t) + 2hY^{(1)}(t) + k^2Y(t) = X(t) \equiv \text{ white noise process, then } R_x(t) = \sigma^2\delta(\tau) \text{ and } S_x(\omega) = \sigma^2,$$

$$S_{y}(\omega) = \frac{1}{\left|(j\omega)^{2} + 2hj\omega + k^{2}\right|^{2}} S_{x}(\omega) = \frac{\sigma^{2}}{(\omega^{2} - k^{2})^{2} + 4h^{2}\omega^{2}} = \frac{4\gamma^{2}\alpha(\alpha^{2} + \beta^{2})}{\left[\omega^{2} - (\beta^{2} + \alpha^{2})\right]^{2} + 4\alpha^{2}\omega^{2}}$$

where
$$\alpha = h$$
, $\beta = \sqrt{k^2 - h^2}$ and $\gamma^2 = \sigma^2/(2hk^2)$. From Table 34.8 $R_y(\tau) = \gamma^2 e^{-a|\tau|} \left(\cos \beta \tau + \frac{\alpha}{\beta} \sin \beta |\tau|\right)$

34.18.3.11 Hilbert Transform

$$\mathsf{F}\!\left\{\frac{1}{\pi t}\right\} = -j\,\mathsf{sgn}(\omega), \ \ \hat{X}(t) = \frac{1}{\pi}\int_{-\tau}^{\infty} \frac{X(t)}{t-\tau}d\tau = X(t) * \frac{1}{\pi t} \equiv \mathsf{Hilbert transform}, \ \ \mathsf{F}\!\left\{\hat{X}(t)\right\} = -j\,\mathsf{sgn}(\omega) \mathsf{F}\!\left\{X(t)\right\}$$

$$=H(\omega)\mathsf{F}\{X(\omega)\}S_{x}(\omega)=S_{x}(\omega)\big|H(\omega)\big|^{2}=S_{x}(\omega),\ S_{xx}(\omega)=S_{x}(\omega)H(\omega)=\begin{cases} -jS_{x}(\omega) & \omega>0\\ jS_{x}(\omega) & \omega<0 \end{cases}$$

(see 34.18.3.7),
$$R_{\bar{x}}(\tau) = R_{x}(\tau)$$
, $R_{xx}(\tau) = R_{xx}(\tau) * h(\tau) = \frac{1}{\pi} \int_{-\pi}^{\infty} \frac{R_{x}(\xi)}{\tau - \xi} d\xi = \hat{R}_{x}(\tau), R_{xx}(-\tau) = -R_{xx}(\tau)$,

$$R_{xx}(0) = 0, R_{xx}(\tau) = R_{xx}(-\tau) - R_{xx}(\tau)$$

34.18.3.12 Analytic Signal

$$Z(t) = X(t) + j \hat{X}(t), R_{z}(\tau) = 2[R_{y}(\tau) + jR_{y}(\tau)] = 2[R_{y}(\tau) + j \hat{R}_{x}(\tau)],$$

$$S_z(\omega) = 2[S_x(\omega) + jS_{xx}(\omega)] = \begin{cases} 4S_x(\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases}, X(t) = \text{Re}[Z(t)], R_x(\tau) = \frac{1}{2}\text{Re}\{R_z(\tau)\}$$

34.18.3.13 Periodic Stochastic Process

 $R(\tau + T) = R(\tau)$ then X(t) is periodic in the m.s. sense.

$$R(\tau) = \sum_{n = -\infty}^{\infty} \alpha_n e^{jn\omega_0 \tau}, \ \omega_0 = 2\pi/T, \ \alpha_n = \frac{1}{T} \int_{0}^{T} R(\tau) e^{-jn\omega_0 \tau} d\tau, \ S(\omega) = F\{R(\tau)\}$$

$$=2\pi\sum^{\infty}\alpha_n\delta(\omega-n\omega_0)$$

34.18.3.14 Periodic Processes in Linear System (see 34.18.3.13)

$$S_{y}(\omega) = 2\pi |H(\omega)|^{2} \sum_{n=-\infty}^{\infty} \alpha_{n} \delta(\omega - n\omega_{0}) = 2\pi \sum_{n=-\infty}^{\infty} \alpha_{n} |H(n\omega_{0})|^{2} \delta(\omega - n\omega_{0}) \equiv \text{ output spectra density}$$

$$R_{yy}(\tau) = \mathsf{F}^{-1}\{S_y(\omega)\} = \sum_{n=-\infty}^{\infty} \alpha_n \big| H(n\omega_0) \big|^2 e^{jn\omega_0\tau} \equiv \text{periodic}$$

34.18.3.15 Fourier Series

$$X(t) = \sum_{n = -\infty}^{\infty} A_n e^{jn\omega_o}, A_n = \frac{1}{T} \int_0^T X(u) e^{-jn\omega_0 u} du \equiv \text{uncorrelated (and orthogonal) r.v.},$$

$$E\{an\} = \begin{cases} E[X(t)\} & n = 0 \\ 0 & n \neq 0 \end{cases} E\{A_k A_n^*\} = \begin{cases} \alpha_n & k = n \\ 0 & k \neq n \end{cases}, \ \alpha_n \equiv \text{ coefficient of } R(\tau)$$

(see 34.18.3.13)

34.18.4 Band-Limited Processes

34.18.4.1 Lowpass Process

$$S_x(\omega) = 0, \ |\omega| > \omega_N, \ R_x^{(n)}(\tau) = \frac{1}{2\pi} \int_{-\omega_N}^{\omega_N} (j\omega)^n S_x(\omega) e^{j\omega\tau} d\omega < \infty,$$

$$X(t+\tau) = \sum_{n=0}^{\infty} X^{(n)}(t) \frac{\tau^n}{n!}$$
 (see 17.3.8).

34.18.4.2 Sampling Theorem

$$R_x(\tau) = \sum_{n=1}^{\infty} R(nT) \frac{\sin(\omega_N \tau - n\pi)}{\omega_N \tau - n\pi}, \ \omega_N = \frac{\pi}{T}, \text{ if } S_x(\omega)$$

is bandlimited (see 34.18.4.1);

$$R_x(\tau - a) = \sum_{n = -\infty}^{\infty} R_x(nT - a) \frac{\sin(\omega_N \tau - n\pi)}{\omega_N \tau - n\pi}$$

or

$$R_{\scriptscriptstyle X}(\tau) = \sum_{n=-\infty}^{\infty} \, R_{\scriptscriptstyle X}(nT-a) \frac{\sin(\omega_{\scriptscriptstyle N}(\tau+a)-n\pi)}{\omega_{\scriptscriptstyle N}(\tau+a)-n\pi}; \, X(t) = \sum_{n=-\infty}^{\infty} \, X(nT) \frac{\sin(\omega_{\scriptscriptstyle N}t-n\pi)}{\omega_{\scriptscriptstyle N}t-n\pi}, \, \, \omega_{\scriptscriptstyle N} = \pi/T$$

If X(t) is bandlimited.

34.18.4.3 Bandpass Process

The narrow-band (quasi-monochromatic $\omega_N << \omega_0$) process $W(t) = X(t) \cos \omega_0 t + Y(t) \sin \omega_0 t$ is widesense stationary iff $E\{X(t)\} = E\{Y(t)\} = 0$ and $R_x(\tau) = R_y(\tau)$ and $R_{xy}(\tau) = -R_{yx}(\tau)$ then $R_w(\tau) = R_x(\tau) \cos \omega_0 \tau + R_{yx}(\tau) \sin \omega_0 \tau$.

34.19 Linear Mean-Square Estimation

34.19.1 Definitions

34.19.1.1 Estimate

 $\hat{G}(t) \equiv \text{ estimate of } G(t)$

34.19.1.2 Mean Square (m.s.) Error

$$e = E\{\left|G(t) - \hat{G}(t)\right|^2\}$$

34.19.1.3 Minimum m.s. Error, Linear Operation on Data

 $\hat{G}(t) = L\{X(t)\}, L = \text{linear operation}$

Example

Find a linear operator L such that $e = E\{[G(t) - L\{X(\xi)\}]^2\}$, $\xi \in I$ is minimum. It turns out that $G(t) - L\{X(\xi)\}$ is orthogonal to $X(\xi)$ for every ξ in I in I and if there is an optimum solution, then L is this optimum. Hence, if $E\{[G(t) - L\{X(\xi)\}]X(\xi_i)\} = 0$, $\xi_i \in I$, then the m.s. is minimum and is given by $e_m = E\{[G(t) - L\{X(\xi)\}]G(t)\}$.

34.19.2 Orthogonality in Linear m.s. Estimation

34.19.2.1 Orthogonality

 $E\{X_iX_j^*\}=0$ implies X_i and X_j are orthogonal r.v. If $E\{X_iZ^*\}=0$ for $i=1,2,\cdots,n$ then $E\{a_1X_1+\cdots+a_nX_n\}Z^*\}=0$, a_i 's are constatus.

34.19.2.2 m.s. Error Estimation

The m.s. error $e = E\{|S_0 - (a_1S_1 + \dots + a_nS_n|^2)\}$ is minimum if we find the constants a_i such that the error $S_0 - (a_1S_1 + \dots + a_nS_n)$ is orthogonal to the data S_1, \dots, S_n or $E\{[S_0 - (a_1S_1 + \dots + a_nS_n)]S_i^*\} = 0$, $i = 1, 2, \dots, n$ or equivalently $E\{[S_0^* - (a_1^*S_1^* + \dots + a_n^*S_n^*)]S_i^*\} = 0$,

Example

To estimate $S(t+\tau)$ given S(t) we use 34.19.2.2 $E\{[S(t+\tau)-aS(t)]S(t)\}=0$ or $R(\tau)-aR(0)=0$ or

Mean Square error (34.19.2.2)

To estimate

$$e = E\{[S(t+\tau) - aS(t)]^2\} = E\{[S(t+\tau) - aS(t)][S(t+\tau) - aS(t)]\}$$

= $E\{[S(t+\tau) - aS(t)]S(t+\tau)\} = R(0) - aR(\tau) = R(0) - R^2(\tau) - R^2(\tau) / R(0)$

since $S \equiv$ data is orthogonal to the error <picture>). (This is a prediction problem.)

Example

To estimate S(t) in terms of X(t) we obtain (34.19.2.2) $E\{[S(t) - aX(t)]X(t)\} = 0$ or $R_{sx}(0) - aR_x(0) = 0$ or $a = R_{sx}(0)/R_x(0)$. Mean square error: $e = E\{[S(t) - aX(t)]S(t)\} = R_s(0) - aR_{sx}(0) = R_s(0) - R_{sx}^2(0)/R_s(0)$. If X(t) = S(t) + n(t) and n(t) = noise is orthogonal S(t), then $R_{sx}(\tau) = R_s(\tau)$, $R_x(\tau) = R_s(\tau) + R_n(\tau)$ and $a = R_s(0)/[R_s(0) + R_n(0)]$ and $e = R_s(0)/[R_s(0) + R_n(0)]$. (This is a filtering problem.)

Examples

To estimate S(t) in terms of S(0) and $S(t_0)$ we set the error to be orthogonal to the data S(0) and $S(t_0)$. Hence, $E\{[S(t) - a_1S(0) - a_2S(t_0)S(0)\} = 0$ and $E\{[S(t) - a_1S(0) - a_2S(t_0)]S(t_0)\} = 0$. Therefore, we

obtain the systems $R(t) = a_1 R(0) + a_2 R(t_0)$ and $R(t_0 - t) = a_1 R(t_0) + a_2 R(0)$ and solve it for the unknown a_1 and a_2 The m.s. error is $e = E\{[S(t) - a_1 S(0) - a_2 S(t_0)]S(t)\} = R(0) - [a_1 R(t) + a_2 R(t_0 - t)]$. (This is an interpolation problem.)

34.20 The Filtering Problem for Stationary Processes

34.20.1 Wiener Theory

34.20.1.1 Wiener Integral Equation

$$R_{gx}(t-\xi) = \int_{a}^{b} R_{x}(\alpha-\xi)h(t,\alpha)d\alpha, \ a \le \xi \le b.$$

If $h(t,\xi)$ is found to satisfy the integral equation for the process G(t) and X(t), then the m.s. error is given by (orthogonality principle):

$$e = E\left\{ G(t) - \int_{a}^{b} X(\alpha)h(t,\alpha)d\alpha \right\} G(t) = R_{8}(0) - \int_{a}^{b} R_{gx}(t-\alpha)h(t,\alpha)d\alpha$$

34.20.1.2 Wiener Integral Equation for Stationary Processes

$$R_{gx}(\tau) = \int_{a}^{b} R_{x}(\tau - \alpha)h(\alpha)d\alpha$$

34.20.1.3 Solution of 34.20.1.2

$$S_{gx}(\omega) = S_x(\omega)H(\omega)$$
 or $H(\omega) = S_{gx}(\omega)/S_x(\omega)$

Example

$$X(t) = S(t) + N(t)$$
, $X(t) \equiv$ given in $-\infty < t < \infty$, $N(t) \equiv$ noise, $G(t) = S(t)$ and

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} R_x(\tau - \alpha)h(\alpha)d\alpha, S_{xx}(\omega) = S_x(\omega)H(\omega), e = \text{m.s. error}$$

$$= E\{[S(t) - \int_{-\infty}^{\infty} X(t - \alpha)h(\alpha)d\alpha]S(t)\} = R_s(0) - \int_{-\infty}^{\infty} R_{sx}(\alpha)h(\alpha)d\alpha.$$

If $f(\tau) = R_s(\tau) - R_{sx}(-\tau) * h(\tau)$ then e = f(0). But

$$F(\omega) = S_s(\omega) - S_{sx}(-\omega)H(\omega) = S_s(\omega) - \frac{S_{sx}(-\omega)S_{sx}(\omega)}{S_{r}(\omega)}$$

and from inverse formula with $\omega = 0$

$$e = f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[S_s(\omega) - \frac{S_{sx}(-\omega)S_{sx}(\omega)}{S_x(\omega)} \right] d\omega.$$

34.20.1.4 Signal and Noise Uncorrelated

 $S_{sn}(\omega) = 0, \ S_{x}(\omega) = S_{s}(\omega) + S_{n}(\omega), \ S_{sx}(\omega) = S_{s}(\omega), \ H(\omega) = S_{s}(\omega)/[S_{s}(\omega) + S_{n}(\omega)],$

$$e = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{S_s(\omega) S_n(\omega)}{S_s(\omega) + S_n(\omega)} d\omega$$

Example

If $R_s(\tau) = Ae^{-|\tau|}\cos\omega_0\tau$, $\omega_0 >> 1$ and $S_n(\omega) = N$, $S_s(\omega) = A/[1 + (\omega - \omega_0)^2]$ for $\omega > 0$, $H(\omega) = A/[A + N + N(\omega - \omega_0)^2]$, $\omega > 0$,

$$e = \frac{1}{\pi} \int_{-\infty}^{\infty} ANd\omega / [A + N + N(\omega - \omega_0)^2] = A/[1 + A/N]^{1/2}, \ E\{S^2(t)\} = R_s(0) = A.$$

If N>>A the filtering does not improve the estimation of S(t), e = A. If A = 3N then $e = R_s(0)/2$.

34.20.1.5 Wiener-Hopf Equation

$$R(\tau + \lambda) = \int_{0}^{\infty} R(\tau - \alpha)h(\alpha)d\alpha, \ \tau \ge 0$$

$$e = \text{m.s. error} = E\{[S(\tau + \lambda) - \int_{0}^{\infty} S(\tau - \alpha)h(\alpha)d\alpha]S(\tau + \lambda)\} = R(0) - \int_{0}^{\infty} R(-\lambda - \alpha)h(\alpha)d\alpha$$

Solution to Wiener-Hopf Equation $S(\omega) = \text{is a rational function of } \omega$.

Step 1

$$S(\omega)\Big|_{\omega=p/j} = \frac{A(p^2)}{B(p^2)} = F(p)F(-p), \ F(p) = \frac{C(p)}{D(p)}, \ F(-p) = \frac{C(-p_i)}{D(-p)}\Big|_{\omega=p/j}$$

C and D contain all the roots of $A(p^2)$ and $B(p^2)$

Step 2

$$D(p_i) = 0$$
, Re $p_i \le 0$, $F(p) = \frac{C(p)}{D(p)} = \frac{a_1}{p - p_1} + \dots + \frac{a_n}{p - p_n}$, $a_i = \frac{C(p_i)}{D'(p_i)}$

Step 3

$$F_1(p) = \frac{a_1 e^{p_1 \lambda}}{p - p_1} + \dots + \frac{a_n e^{p_n \lambda}}{p - p_n} = \frac{C_1(p)}{D(p)}$$

 λ = the constant in Wiener-Hopf equation $R(\tau + \lambda)$, $C_1(p_i) = C(p_i)e^{p_i\lambda}$.

Step 4

$$H(p) = \frac{F_1(p)}{F(p)} = \frac{C_1(p)}{C(p)}$$

Example

$$R(\tau) = Ae^{-|t|}, S(\omega) = 2A/(1+\omega^20.$$

Step 1

$$S(p/j) = \frac{2A}{1-p^2} = \frac{\sqrt{2A}}{1+p} \frac{\sqrt{2A}}{1-p}$$

Step 2

$$F(p) = \frac{C(p)}{D(p)} = \frac{\sqrt{2A}}{1+p}, p_1 = -1.$$

Step 3

$$F_1(p) = \frac{\sqrt{2A}}{1+p}e^{-\lambda}$$

Step 4

$$H(p) = \frac{F_1(p)}{F(p)} = e^{-\lambda}h(t) = e^{-\lambda}\delta(t).$$

Estimate $S(t + \lambda) \sim e^{-\lambda}$ $S(t) \equiv$ prediction.

$$e = E\{[S(t+\lambda) - e^{-\lambda}S(t)]S(t+\lambda)\} = R(0) - e^{-\lambda}R(\lambda) = A(1 - e^{-2\lambda}0.$$

34.21 Harmonic Analysis

34.21.1 Series Expansion

34.21.1.1 $R(\tau) = \text{periodic } (R(\tau + T) = R(\tau))$:

$$X(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega_0 t}, \ \omega_0 = 2\pi/T,$$

$$A_n = \frac{1}{T} \int_{-T/2}^{T/2} X(t)e^{-jn\omega_0 t} dt, \ E\{A_n A_n^*\} = 0, \ n \neq 0 \text{ orthogonal},$$

$$R(\tau) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 \tau}, \ S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} \alpha_n \delta(\omega - n\omega_0), \ E\{|A_n|^2\} = \alpha_n$$

34.21.1.2 Karhunen-Loeve Expansion

If the expansion

$$X(t) \cong \sum_{n=1}^{\infty} B_n \varphi_n(t), |t| < \frac{T}{2}$$

has orthogonal coefficients $E\{B_nB_m^*\}=0,\ n\neq m$, then φ_n must satisfy the integral equation

$$\int_{-T/2}^{T/2} r(t_1, t_2) \varphi(t_2) dt_2 = \lambda \varphi(t_1), \ |t_1| < \frac{T}{2}$$

for term $\lambda = \lambda_n$ of λ and the variance of B_n must be equal to $\lambda_n : E\{|B_n|^2\} = \lambda_n$

34.21.1.3 If 34.21.1.2 is satisfied
$$E\{|X(t) - X(t)|^2\} = R(t,t) - \sum_{n=1}^{\infty} \lambda_n |\varphi_n(t)|^2, |t| < \frac{T}{2}$$

34.21.2 Fourier Transforms

34.21.2.1 Fourier Transforms

$$X(\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt, \int_{-\infty}^{\infty} |X(t)|^2 dt < \infty,$$

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \ E\{X(\omega)\} = \int_{-\infty}^{\infty} E\{X(t)\} e^{-j\omega t} dt$$

34.21.2.2 Autocorrelation of $X(\omega)$

$$R(t_1t_2) = E\{X(t_1)X^*(t_2)\}\$$

$$\Gamma(\omega_{1},\omega_{2})=\int\int\limits_{-\infty}^{\infty}\,R(t_{1},t_{2})e^{-j(\omega_{1}t_{1}-\omega_{2}t_{2})}dt_{1}dt_{2}=E\{X(\omega_{1})X^{*}(\omega_{2})\};$$

$$R(t_1, t_2) \leftarrow \xrightarrow{\mathsf{F}_{t_1}} E\{X(\omega_1)X^*(t_2)\} \leftarrow \xrightarrow{\mathsf{F}_{t_2}} E\{X(\omega_1)X^*(\omega_2)\}$$

34.21.2.3 Linear Systems

$$Y(\omega) = \text{ output } = X(\omega)H(\omega); \ E\{Y_1(\omega)Y_2^*(\omega)\} = E\{X_1(\omega)H_1(\omega)X_2(\omega)H_2(\omega)\} = E\{X_1(\omega_1)X_2^*(\omega_2)H_1(\omega_1)H_2^*(\omega_2); \ \Gamma_{y_1,y_2}(\omega_1,\omega_2) = \Gamma_{x_1,x_2}(\omega_1,\omega_2)H_1(\omega_1)H_2^*(\omega_2)$$

34.22 Markoff Sequences and Processes

34.22.1 Definitions

34.22.1.1 Definition

 $F(x_n|x_{n-1},\dots,x_1) = F(x_n|x_{n-1})$. For continuous r. v. type $f(x_n|x_{n-1},\dots,x_1) = f(x_n|x_{n-1})$. Also

$$f(x_1, x_2, \dots, x_n) = f(x_n | x_{n-1}) = f(x_{n-1} | x_{n-2}) \dots f(x_2 | x_1) f(x_1)$$

$$f(x_n|x_{n-1},\dots,x) = f(x_1,x_2,\dots,x_n)/f(x_1,x_2,\dots,x_{n-1}) = f(x_n|x_{n-1})$$

34.22.1.2 Homogeneous

 $f(x_n|x_{n-1}) \equiv \text{independent of n}$

34.22.1.3 Stationary

 $f_{x_n}(x) \equiv$ the sequence is homogeneous and the r.v. X_n have the same density.

34.22.1.4 Chapman-Kalmogoroff Equation

$$f(x_n|x_s) = \int_{-\infty}^{\infty} f(x_n|x_r) f(x_r|x_s) dx_r$$
 where $n > r > s$ are any integers.

34.22.2 Markoff Chains

34.22.2.1 Markoff Chains

$$P\{X_n = a_{i_n} | X_{n-1} = a_{i_{n-1}}, \dots, X_1 = a_{i_1}\} = P\{X_n a_{i_n} | X_{n-1} = a_{i_{n-1}}\} X_n \equiv \text{Markoff chain}$$

34.22.2.2 Conditional Densities

$$p_i(n) = P\{X_n = a_i\}; \ P_{ij}(n,s) = P\{X_n = a_i \big| X_s = a_j\}, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_i p_i(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n,s) p_j(s); \sum_j P_{ij}(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n) = 1, n > s; \\ p_i(n) = \sum_j P_{ij}(n) = 1, n > s; \\ p_i(n) = 1, n > s; \\ p_i(n) = 1, n > s; \\ p_i(n) = 1, n >$$

$$\sum_{j} p_{ij}(n,s) = 1; P_{ij}(n,s) = \sum_{k} P_{ik}(n,r)P_{kj}(r,s), \text{ n > r >s discrete case Chapman-Kolmogoroff equation.}$$

34.22.2.3 Matrices

P(n,s) = square matrix with $P_{ij}(n,s)$ elements; p(n) column matrix with elements $p_i(n)$; P(n) = P(n,s)p(s); P(n,s) = P(n,r)P(r,s)

34.22.2.4 Homogeneous Chains

The conditional probabilities $P_{ij}(n,s)$ depend only on the difference n-s, $P_{ij}(n-s)$. Matrix P(n-s) with $P(1)=\Pi$. With P(n-s)=1 and P(n)=1 and

34.22.3 Markoff Processes

34.22.3.1 Markoff Process

 $P\{X(t_n) \le x_n \big| X(t_{n-1}), \dots, X(t_1)\} = P\{X(t_n) \le x_n \big| X(t_{n-1})\}$ for every n and $t_1 < t_2 \dots < t_n$. Note:

- 1. For $t_1 < t_2$, $P\{X(t_1) \le x_1 | X(t) \text{ for all } t \ge t_2\} = P\{X(t_1) \le x_1 | X(t_2)\}$
- 2. If $X(t_2) X(t_1)$ is independent of X(t) for every $t \le t_1$ ($t_1 < t_2$) the process X(t) is Markoff.

- 3. $E\{X(t_n)|X(t_{n-1}),\dots,X(t_1)\}=E\{X(t_n)|X(t_{n-1})\}$
- 4. $R(t_3,t_2)R(t_2,t_1) = R(t_3,t_1)r(t_2,t_2)$ for every $t_3 > t_2 > t_1$ and if X(t) is normal Markoff with zero mean.

34.22.3.2 Continuous Process (Chapman-Kolmogoroff equation)

$$p(x,t;x_0,t_0) = \int_{-\infty}^{\infty} p(x,t;x_1,t_1)p(x_1,t_1;x_0,t_0)dx_1 \text{ if } t > t_1 > t_0. \ p(x,t;x_0,t_0) = f_{x(t)}(x|X(t_0) = x_0) \text{ for } t \ge t_0 \equiv \text{conditional density of } X(t)$$

34.22.3.3 Conditional Mean and Variance

$$E\{X(t)\big|X(t_0)=x_0=a(x_0,t,t_0); \qquad E[[X(t)-a(x_0,t,t_0)]^2\big|X(t_0)=x_0\}=b(x_0,t,t_0); \qquad a(x_0,t,t_0)=\sum_{n=0}^{\infty}xp(x,t;x_0,t_0)dx, \ b(x_0,t,t_0)=\sum_{n=0}^{\infty}(x-a)^2p(x,t;x_0,t_0)dx; \ a(x_0,t_0,t_0)=x_0, \ b(x_0,t_0,t_0)=0$$

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