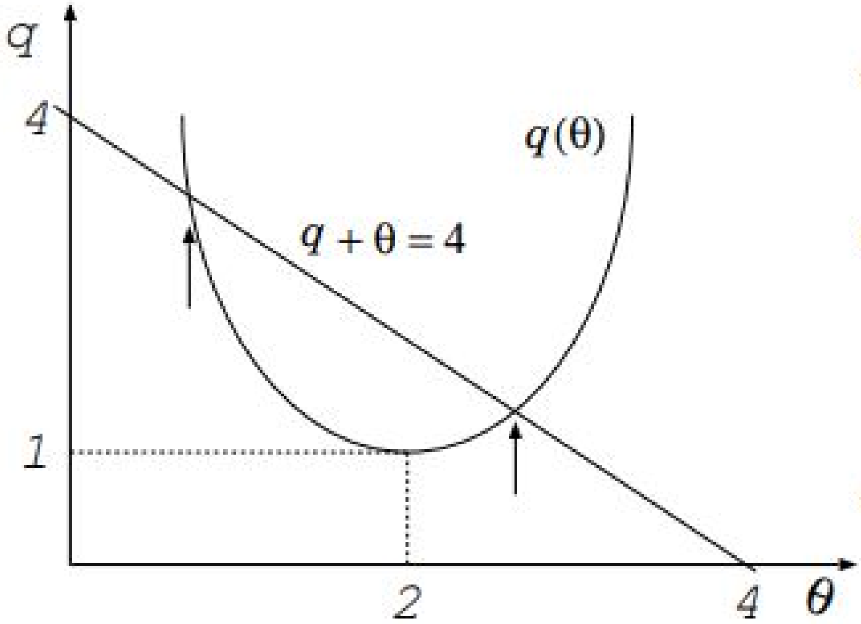
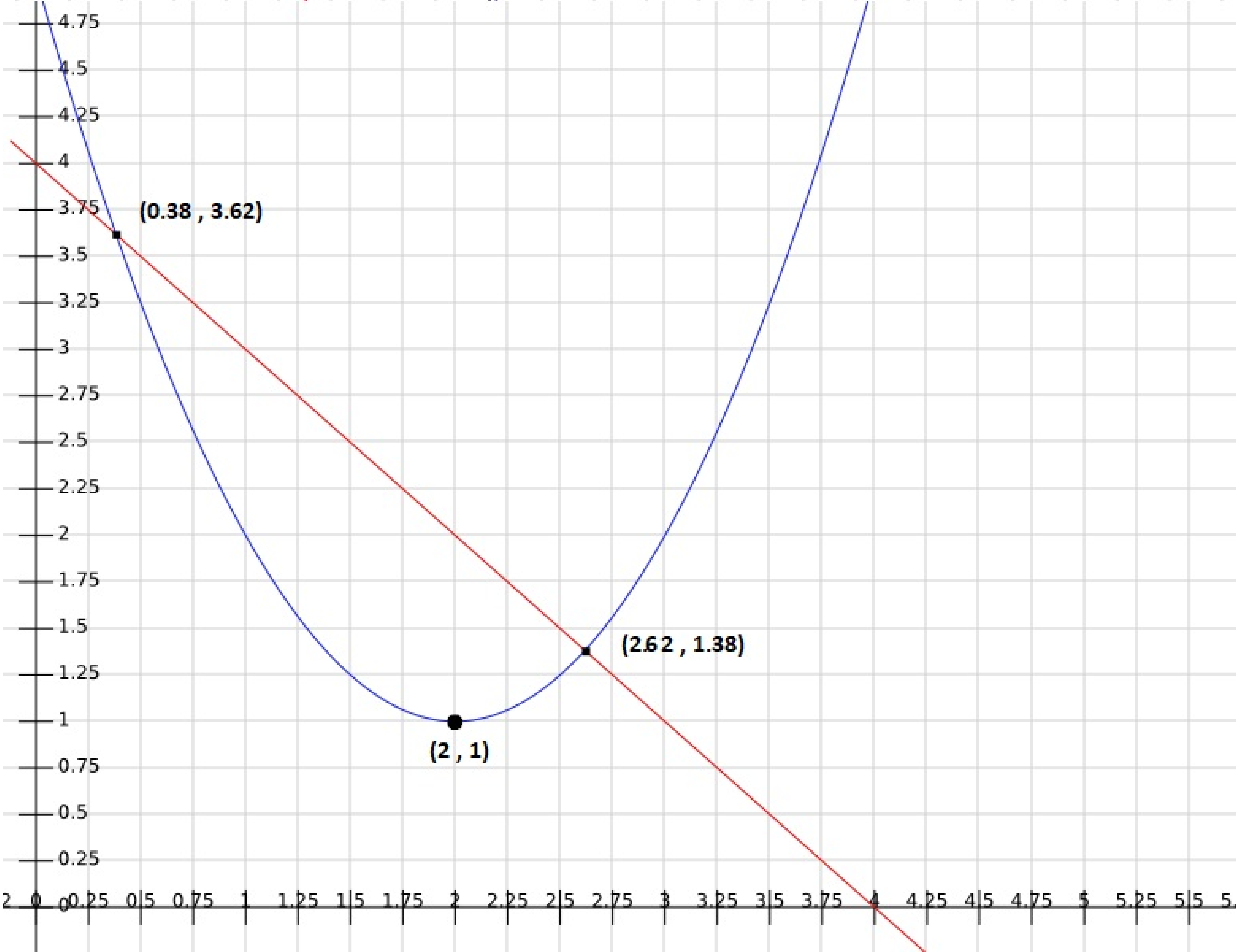
Optimización con restricciones: ejemplo simple 2



- Dado: $q(\theta) = 1 + (\theta - 2)^2$,
- Calcular: $\theta^\star = \underset{\theta:\, q+\theta=4}{\arg\min}\, q(\theta)$

  (restricción de igualdad: $q + \theta - 4 = 0$)
- Solución: ??

3.4. (p.3.16, 0.5 puntos) Ejercicio b) de la página indicada: minimizar una función con una condición de igualdad.


(0.38 , 3.62)
(2.62 , 1.38)
(2 , 1)

minimizar $1 + (\theta - 2)^2$

restricción $1 + (\theta - 2)^2 + \theta = 4$

Como solo hay una restricción y se trata de una igualdad, podemos saber el valor de θ directamente

$1 + (\theta - 2)^2 + \theta = 4 \Rightarrow 1 + \theta^2 + 4 - 4\theta + \theta = 4 \Rightarrow \theta^2 - 3\theta + 1 = 0$

$$\frac{3 \pm \sqrt{9-4}}{2} \quad \frac{3+\sqrt{5}}{2} = 2{,}61 \qquad \frac{3-\sqrt{5}}{2} = 0{,}381$$

Solo en esos dos casos se cumple la restricción, veamos cual mininiza:

$$1 + \left(\frac{3+\sqrt{5}}{2} - 2\right)^2 = 1{,}38197 \qquad 1 + \left(\frac{3-\sqrt{5}}{2} - 2\right)^2 = 3{,}6180$$

Mínimo que cumple la restricción

$$\theta = \frac{3+\sqrt{5}}{2}$$

Multiplicadores Lagrange

$$\Lambda(\theta,\beta) = 1 + (\theta-2)^2 + \beta(1 + (\theta-2)^2 + \theta - 4)$$

$$\Lambda(\theta,\beta) = 1 + (\theta-2)^2 + \beta((\theta-2)^2 + \theta - 3)$$

$$\frac{\partial \Lambda(\theta,\beta)}{\partial \theta} = 0 + 2(\theta-2) + \beta(2(\theta-2) + 1 - 0) = 0$$

$$\Rightarrow \quad 2\theta - 4 + 2\theta\beta - 4\beta + \beta = 0$$

$$\Rightarrow \quad \theta(2+2\beta) = 4 + 3\beta \qquad \Rightarrow \quad \theta^*(\beta) = \frac{4+3\beta}{2+2\beta}$$

$$\Lambda_D(\beta) = \Lambda(\theta^*(\beta),\beta) = 1 + \left(\frac{4+3\beta}{2+2\beta} - 2\right)^2 + \beta\left(\left(\frac{4+3\beta}{2+2\beta} - 2\right)^2 + \left(\frac{4+3\beta}{2+2\beta}\right) - 3\right)$$

$$= 1 + \left(\frac{4+3\beta}{2+2\beta} - 2\right)^2 + \beta\left(\left(\frac{4+3\beta}{2+2\beta} - 2\right)^2\right) + \beta\left(\frac{4+3\beta}{2+2\beta}\right) - 3\beta$$

$$\frac{d\Lambda_D}{d\beta} = 0 + 2\left(\frac{4+3\beta}{2+2\beta} - 2\right)\left(-\frac{1}{2(\beta+1)^2}\right) + \left(\frac{4+3\beta}{2+2\beta} - 2\right)^2 + \frac{\beta^2}{(2+2\beta)(\beta+1)^2} + \left(\frac{4+3\beta}{2+2\beta}\right) - \frac{\beta}{2(\beta+1)^2} - 3 = 0$$

$\boxed{\text{* Desarrollo derivadas y simplificación en el anexo}}$

$$\beta^* = -\frac{5 \pm \sqrt{5}}{5}$$

Conociendo β^*, podemos sustituir en $\theta^*(\beta^*)$

$$\theta^*(\beta) = \frac{4+3\beta}{2+2\beta} \qquad \beta^* = -\frac{5 \pm \sqrt{5}}{5}$$

<u>Solución 1</u>

$$\theta^*(\beta^*) = \frac{4+3\left(-\frac{5+\sqrt{5}}{5}\right)}{2+2\left(-\frac{5+\sqrt{5}}{5}\right)}$$

$$\theta^*(\beta^*) = \frac{4+3\cdot(-1,447)}{2+2\cdot(-1,447)}$$

$$\boxed{\theta^*(\beta^*) = 0,3814}$$

<u>Solución 2</u>

$$\theta^*(\beta^*) = \frac{4+3\left(\frac{5-\sqrt{5}}{5}\right)}{2+2\left(\frac{5-\sqrt{5}}{5}\right)}$$

$$\theta^*(\beta^*) = \frac{4+3\cdot(-0,5528)}{2+2\cdot(-0,5528)}$$

$$\boxed{\theta^*(\beta^*) = 2,6180}$$

Averiguar cuál hace mínimo el valor de la función objetivo:

$$1+(0,3814-2)^2 = 3,6198$$

$$1+(2,6180-2)^2 = \underline{1,38197}$$

→ mínimo

$$\theta = \frac{3+\sqrt{5}}{2} = 2,6180$$

ANEXO 1: Desarrollo derivada $\frac{d\Lambda_D}{dB}$

Función a derivar: $1 + \left(\frac{4+3\beta}{2+2\beta} - 2\right)^2 + \beta\left(\left(\frac{4+3\beta}{2+2\beta} - 2\right)^2\right) + \beta\left(\frac{4+3\beta}{2+2\beta}\right) - 3\beta$

$$\frac{d}{d\beta}(1) = 0$$

$$\frac{d}{d\beta}\left(\left(\frac{4+3\beta}{2+2\beta} - 2\right)^2\right) = \text{derivada\_exponente} \cdot \text{derivada interior}$$

$$\text{derivada\_exponente} = 2\left(\frac{4+3\beta}{2+2\beta} - 2\right)$$

$$\text{derivada\_interior} = \frac{d}{d\beta}\left(\frac{4+3\beta}{2+2\beta} - 2\right) = \frac{d}{d\beta}\left(\frac{4+3\beta}{2+2\beta}\right) - \cancelto{0}{\frac{d}{d\beta}(2)}$$

$$\frac{d}{d\beta}\left(\frac{4+3\beta}{2+2\beta}\right)$$

$$\boxed{\text{Regla Cociente } \left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}}$$

$$\frac{d}{d\beta}\left(\frac{4+3\beta}{2+2\beta}\right) = \frac{3 \cdot (2+2\beta) - (2 \cdot (4+3\beta))}{(2+2\beta)^2}$$

$$= \frac{6 + \cancelto{0}{6\beta} - 8 - \cancelto{0}{6\beta}}{4(\beta+1)^2}$$

$$= -\frac{1}{2(\beta+1)^2}$$

$$\frac{d}{d\beta}\left(\left(\left(\frac{4+3\beta}{2+2\beta}\right) - 2\right)^2\right) = 2\left(\frac{4+3\beta}{2+2\beta} - 2\right)\left(\frac{-1}{2(\beta+1)^2}\right)$$

$$\frac{d}{d\beta}\left(\beta\cdot\left(\left(\frac{4+3\beta}{2+2\beta}-2\right)^2\right)\right)=\frac{d}{d\beta}(\beta)\cdot\left(\frac{4+3\beta}{2+2\beta}\right)^2+\beta\cdot\underbrace{\frac{d}{d\beta}\left(\left(\frac{4+3\beta}{2+2\beta}\right)^2\right)}_{\text{calculada antes}}$$

$$=1\cdot\left(\frac{4+3\beta}{2+2\beta}\right)^2+\beta\cdot\left(\underbrace{2\left(\frac{4+3\beta}{2+2\beta}-2\right)\left(-\frac{1}{2(\beta+1)^2}\right)}_{\text{SIMPLIFICAMOS}}\right)$$

SIMPLIFICAMOS

$$-\frac{2\left(\frac{4+3\beta}{2+2\beta}-2\right)}{1}\cdot\left(\frac{1}{2(\beta+1)^2}\right)=-\frac{\cancel{2}\left(\frac{4+3\beta}{2+2\beta}-2\right)}{\cancel{2}(\beta+1)^2}$$

$$=-\frac{\frac{4+3\beta-4-4\beta}{2+2\beta}}{(\beta+1)^2}=-\frac{\frac{-\beta}{2+2\beta}}{(\beta+1)^2}=-\frac{-\beta}{(2+2\beta)(\beta+1)^2}$$

$$\frac{d}{d\beta}\left(\beta\cdot\left(\left(\frac{4+3\beta}{2+2\beta}-2\right)^2\right)\right)=\left(\frac{4+3\beta}{2+2\beta}\right)^2+\beta\left(\frac{\beta}{(2+2\beta)(\beta+1)^2}\right)$$

$$\frac{d}{d\beta}\left(\frac{4+3\beta}{2+2\beta}\right) = -\frac{1}{2(\beta+1)^2}$$

$$\frac{d}{d\beta}\left(\beta\left(\frac{4+3\beta}{2+2\beta}\right)\right) = 1\left(\frac{4+3\beta}{2+2\beta}\right) + \beta\left(\frac{-1}{2(\beta+1)^2}\right)$$

$$\frac{d}{d\beta}(-3\beta) = -3$$

$$\boxed{\text{SIMPLIFICAR } \frac{d\Lambda_p}{d\beta}}$$

$$2\left(\frac{4+3\beta}{2+2\beta}-2\right)\left(-\frac{1}{2(\beta+1)^2}\right) = \left(\frac{\beta}{(2+2\beta)(\beta+1)^2}\right)$$

$$\left(\frac{4+3\beta}{2+2\beta}-2\right)^2 + \frac{\beta^2}{(2+2\beta)(\beta+1)^2} = \frac{\beta^2 + \left(\frac{4+3\beta}{2+2\beta}-2\right)^2(2+2\beta)(\beta+1)^2}{(2+2\beta)(\beta+1)^2}$$

$$= \frac{\beta^2 + \left(\frac{-15\beta^2-32\beta-16}{4(\beta+1)^2}+4\right)(2+2\beta)(\beta+1)^2}{(2+2\beta)(\beta+1)^2}$$

$$= \frac{\beta^2 + \left(\frac{-15\beta^2-32\beta-16}{4(\beta+1)^2}+4\right)(2\beta^2+2+4\beta+2\beta^3+2\beta+4\beta^2)}{(2+2\beta)(\beta+1)^2}$$

$$\Rightarrow \frac{\beta^3+3\beta^2}{2(2+2\beta)(\beta+1)^2}$$

$$\frac{4+3\beta}{2+2\beta} + \left(-\frac{\beta}{2(\beta+1)^2}\right) = \frac{3\beta^2+6\beta+4}{2(\beta+1)^2}$$

$$\frac{d\Lambda_D}{d\beta} = \frac{\beta}{(2+2\beta)(\beta+1)^2} + \frac{\beta^3+3\beta^2}{2(2+2\beta)(\beta+1)^2} + \frac{3\beta^2+6\beta+4}{2(\beta+1)^2} - 3$$

$$= \frac{\beta\cdot 2}{4(\beta+1)^3} + \frac{\beta^3+3\beta^2}{4(\beta+1)^3} + \frac{(3\beta^2+6\beta+4)(2\cdot(\beta+1))}{4(\beta+1)^3} - 3$$

$$= \frac{\beta^3+3\beta^2+2\beta + (\beta+1)\cdot 2(3\beta^2+6\beta+4)}{4(\beta+1)^3} - 3$$

$$= \frac{\beta(\beta+1)(\beta+2) + (\beta+1)\cdot 2(3\beta^2+6\beta+4)}{4(\beta+1)^3} - 3$$

$$= \frac{\cancel{(\beta+1)}\left(\beta(\beta+2) + 2(3\beta^2+6\beta+4)\right)}{4(\beta+1)^3} - 3$$

$$= \frac{\beta^2+2\beta + 6\beta^2+12\beta+8}{4(\beta+1)^2} - 3$$

$$= \frac{(7\beta^2+14\beta+8)}{4(\beta+1)^2} - 3$$

SIMPLIFICAR

$$\frac{7\beta^2 + 14\beta + 8}{4(\beta+1)^2} - 3 = 0$$

$$7\beta^2 + 14\beta + 8 \quad - 12(\beta+1)^2 = 0$$

$$7\beta^2 + 14\beta + 8 \quad - 12\beta^2 - 12 - 24\beta = 0$$

$$+5\beta^2 + 10\beta + 4 \quad = 0$$

$$-\frac{10 \pm \sqrt{100 - 80}}{10} = -\frac{10 \pm \sqrt{2^2 \cdot 5}}{10} =$$

$$-\frac{10 \pm 2\sqrt{5}}{10} = -\frac{5 \pm \sqrt{5}}{5}$$