URBAN EDGE

LA Assignment

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1. Substituting points in eq^n.

$$A + B + C = 1$$
$$A + 2B + 4C = -1$$
$$A + 3B + 9C = 1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Perform GE,

$$[A\ b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$\Rightarrow A + B + C = 1$; $B + 3C = -2$; $2C = 4$

Solving, we get

$A = 7$

$B = -8$

$C = 2$

$\Rightarrow \boxed{y = 2x^2 - 8x + 7}$

2. $$A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix} \quad \begin{bmatrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 + 5R_1 \\ R_4 = R_4 - 5R_1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix} \quad \begin{bmatrix} R_3 = R_3 + 2R_2 \\ R_4 = R_4 + 2R_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad [R_4 = R_4 - 3R_3]$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} = LU$$

3. $T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$

i) Standard basis of R^3 - $(1,0,0), (0,1,0), (0,0,1)$

$$\Rightarrow T(1,0,0) = (1,0,1)$$
$$T(0,1,0) = (2,1,1)$$
$$T(0,0,1) = (-1,1,-2)$$

$$\Rightarrow \text{LT matrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \underline{\underline{T}}$$

ii) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} = R \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$

$= L \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ Basis $(C(T)) = (1,0,1), (2,1,1)$
i.e, $C(T)$ is a line in R^3

We have, x & y are pivot variables, z is free

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ y-3x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ Basis $(N(T)) = (3,-1,0)$
i.e, $N(T)$ is a point in R^3

Basis $(C(T')) = (1,2,-1), (0,1,1)$
i.e, $C(T')$ is a line in R^3

Convert T' to RR echelon form,

$T' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow z$ is free
$\therefore x = -z$
$y = z$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ Basis $(N(T')) = (-1,1,1)$
i.e, $N(T')$ is a point in R^3

iii) $|T - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(1-\lambda)(-2-\lambda)-1\} - 2(-1) + (-1)(-1+\lambda) = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2+\lambda-3) + 2 + 1 - \lambda = 0$$

$$\Rightarrow -\lambda^3 + 4\lambda - 3 + 2 + 1 - \lambda = 0$$

$$\Rightarrow \lambda^3 - 3\lambda = 0$$

$$\lambda(\lambda^2 - 3) = 0$$

$$\Rightarrow \lambda = 0 \quad \text{or} \quad \lambda^2 = 3$$

$$\text{i.e., } \lambda = \pm\sqrt{3}$$

For $\lambda = 0$, $Tx = \lambda x = Tx = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow (x, y, z) = k(3, -1, -1)$$

For $\lambda = -\sqrt{3}$, $Tx = \lambda x$ =

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \frac{x}{3+\sqrt{3}} = \frac{-y}{1+\sqrt{3}} = \frac{z}{4+2\sqrt{3}} = k$$

$$\Rightarrow (x, y, z) = k(3+\sqrt{3}, -1-\sqrt{3}, 4+2\sqrt{3})$$

For $\lambda = \sqrt{3}$, $Tx = \lambda x$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & 1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & 2-\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \frac{x}{3-\sqrt{3}} = \frac{-y}{1-\sqrt{3}} = \frac{z}{4-2\sqrt{3}} = k$$

$$\Rightarrow (x, y, z) = k(3-\sqrt{3}, -1+\sqrt{3}, 4-2\sqrt{3})$$

(ii) Let q_1, q_2, q_3 be orthogonal vectors spanning $C(T)$

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}}(1,0,1) = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$q_2 = \frac{B}{\|B\|}, \quad B = b - (q_1^T b) q_1$$

$$= (2,1,1) - \left(\frac{3}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{1}{2}, 1, \frac{-1}{2}\right)\left(\frac{2}{\sqrt{6}}\right)$$

$$q_3^T q_1 = q_3^T q_2 = 0$$

$$\Rightarrow \frac{x}{\sqrt{2}} + \frac{z}{\sqrt{2}} = 0$$

$$\frac{x}{\sqrt{6}} + \frac{2y}{\sqrt{6}} - \frac{z}{\sqrt{6}} = 0$$

$$\Rightarrow q_3 = (-z, z, z) = (-1, 1, 1)$$

$$\Rightarrow T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3/\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \\ 0 & 0 & 0 \end{bmatrix} = \underline{QR}$$

4. Data:

x	-4	1	2	3
y	4	6	10	8

Let $A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$

LSQ soln: $A^TA\hat{x} = A^Tb$

$\Rightarrow \hat{x} = (A^TA)^{-1}A^Tb$

$$= \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix}^{-1} \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$= \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 28 \\ 34 \end{bmatrix}$$

$$= \frac{1}{116} \begin{bmatrix} 772 \\ 80 \end{bmatrix} = \begin{bmatrix} 6.6552 \\ 0.6897 \end{bmatrix}$$

$\therefore$ Eqn is, $\boxed{y = 6.6552 + 0.6897x}$

5. We have,

Normal vector $v = (1, 1, 3, 0, 4)$

$$\Rightarrow Q = \frac{vv^T}{v^Tv} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 1 & 1 & 3 & 0 & 4 \\ 1 & 1 & 3 & 0 & 4 \\ 3 & 3 & 9 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 12 & 0 & 16 \end{bmatrix}$$

Since $P + Q = I \Rightarrow P = I - Q$

$$= \frac{1}{27}\begin{bmatrix} 26 & -1 & -3 & 0 & -4 \\ 26 & -1 & -3 & 0 & -4 \\ -3 & -3 & 18 & 0 & -12 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & -4 & -12 & 0 & 11 \end{bmatrix}$$

6. $A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$

We have,

$A_1 = a$

$A_2 = \begin{vmatrix} a & 2 \\ 2 & a \end{vmatrix} = a^2 - 4$

$A_3 = \begin{vmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{vmatrix} = a(a^2-4) - 2(2a-4) + 2(4-2a)$
$= (a-2)^2(a+4)$

For A to be +ve definite,

$A_1, A_2, A_3 > 0$

$\Rightarrow a > 0 \quad - ①$

$a^2 - 4 > 0$

$\Rightarrow a^2 > 4$

$\Rightarrow a < -2, \; a > 2 \quad - ②$

$(a-2)^2(a+4) > 0$

$\Rightarrow a > -4, \; a > 2 \quad - ③$

From ①, ②, ③ we get

$\boxed{a > 2}$

7. $f(x) = x^T Bx = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

Let $B = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$

$\Rightarrow x^T Bx = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= \begin{bmatrix} ax_1 + dx_2 + ex_3 & dx_1 + bx_2 + fx_3 & ex_1 + fx_2 + cx_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= ax_1^2 + dx_1x_2 + ex_1x_3 + dx_1x_2 + bx_2^2 + fx_2x_3 + ax_3 + fx_2x_3$

$= ax_1^2 + bx_2^2 + cx_3^2 + 2dx_1x^2 + 2ex_1x_3 + 2fx_2x_3$

Comparing the coeffecients,

$a = 2$; $b = 2$; $c = 2$; $d = -1$; $e = 0$; $f = -1$

$\Rightarrow B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

8. $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 31 & -27 \\ -27 & 9 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix}$$

We have,

$|AA^T - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 10-\lambda & 20 & 20 \\ -20 & 40-\lambda & 40 \\ -20 & 40 & 40-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2(\lambda - 90) = 0$

i.e, $\lambda = 0$ or $\lambda = 90$

When $\lambda = 0$,

$$\begin{bmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x - 2y - 2z = 0$

$-2x + 4y + 4z = 0$

Let $x = 1$, $y = 1 \Rightarrow z = -1/2$

$\therefore v = (2, 2, -1)$

$\hat{v} = (2/3, 2/3, -1/3)$

When $\lambda = 90$,

$$\begin{bmatrix} -80 & -20 & -20 \\ 20 & -50 & 40 \\ -20 & 40 & -50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving,

$$\frac{x}{-1800} = \frac{-y}{3600} = \frac{z}{3600} = k$$

$\Rightarrow v = (-1, 2, 2)$

$\hat{v} = (-1/3, 2/3, 2/3)$

$$\therefore V = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$A^T A$ has the same eigenvalues as AA^T

When $\lambda = 0$,

$$\begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v = (-1, 3)$, $\hat{v} = \left(\frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

When $\lambda = 90$,

$$\begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v = (3, 1)$, $\hat{v} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$

$$\Rightarrow V = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 7/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} \Rightarrow V^T = \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$\therefore \; A = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$= \underline{U \Sigma V^T}$$