optimisation.

Taylor's Theorem:

$$f(x+a) = f(x) + f'(x)\,a + \dots + \frac{f^m(x)\,.a^m}{m!} + \frac{f^{(m+1)}(c)}{(m+1)!}\,a^{m+1}.$$

finding c is task in many scenarios.

$c \in (x, x+a)$

$f: R^n \to R$ Gradient vector.

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

Stationary point of f $\nabla f(\bar{x}) = 0$

Hessian matrix $\to n \times n$

$f: R^n \to R$

$$\left\{ (\nabla^2 f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} \right\}$$

Hessian → func^n partially diff. w.r.t every other variable.

In sab ki ek common matrix.

Taylor for higher dim.

$$f(x+p) = f(x) + \langle \nabla f(x+lp), p \rangle \quad \text{for some } l \in (0,1).$$

$$f(x+p) = f(x) + \langle \nabla f(x), p \rangle + \frac{1}{2} \langle \nabla^2 f(x+tp)p, p \rangle.$$

#.) $f(x+p) = f(x) + \langle \nabla f(x), p \rangle + \frac{1}{2} \langle \nabla^2 f(x+tp)\, p, p \rangle.$

$t \in (0,1)$

x_*. local minimizer in an open neighbourhood. $\{\nabla f(x_*) = 0\}$

#. $f(x) = x^3$ suppose not true Take $p = -\nabla f(x_*)$

at $x=0$ $\rightarrow$ $\nabla f(x_*) \neq 0$ $\rightarrow$ observe that $\langle \nabla f(x_*), p \rangle = -\|\nabla f(x_*)\|^2 < 0$

- ∇f cont. at x^*.
  ↳ $\exists\ T>0$ such that $\langle \nabla f(x_* + sp), p \rangle < 0$.
  for all $s \in [0,T]$.

#. $f(x_* + t_* p) = f(x_*) + t_* \underbrace{\langle \nabla f(x_* + s_* p), p \rangle}_{(-ve)}$ for some s_* & t_*.
$s_* \in (0, t_*)$

{ thus $f(x_* + t_* p) < f(x_*)$
local minimizer }

#.) $\langle Bv, v \rangle \geq 0$ (semidefinite) for all $v \in \mathbb{R}^n \setminus \{0\}$

$\langle Bv, v \rangle > 0$
↳ Definite for all $v \in \mathbb{R}^n \setminus \{0\}$.

#. x_* is a local minimizer of f and if f is twice continously diffble in open neighbourhood of x_*.
$\nabla^2 f(x_*)$ is +ve semidefinite.

#. $\nabla f(x_*) = 0$ and $\nabla^2 f(x_*)$ is positive definite. Then x_* is strict local minimzer of a.

sufficient but not necessary.
↳ Ex.) $f(x) = x^4$ at 0.

$\nabla^2 f$ is cont. at x_* as it is +ve def. at x_*.

$\exists\ r > 0$, such that $\nabla^2 f(z)$ is +ve definite for $\forall\ z \in B_r(x_*)$

$\rightarrow$ for $s \in (0,1)$ $\quad f(x_* + p) = f(x_*) + \langle \cancel{\nabla f(x_*)}^{\,0}, p \rangle$

$$+ \frac{1^2}{2} \langle (\nabla^2 f(x_* + sp) p, p \rangle.$$

#. $f(x_*) + \frac{1}{2} \langle \nabla^2 f(x_* + sp) p, p \rangle.$

as +ve defiteness implies $\nabla^2 f$ in neighbourhood of x_*.

★ Notion of Convexivity

set $\Omega \subset \mathbb{R}^n$ is said to be convex.

$x, y \in \Omega \Rightarrow \alpha x + (1-\alpha) y \in \Omega \quad \forall \alpha \in [0,1]$

$\downarrow$

Geometric Interpret $\rightarrow$ one can join any two points.

Strictly convex, when Inequality follows for $x \neq y$.

#. local minimizer x_* of f is a global maximizer of f.

$\Rightarrow \exists \; y \in \mathbb{R}^n \rightarrow f(y) < f(x_*)$.

#. $x = \lambda y + (1-\lambda) x_*$ for some $\lambda \in (0,1]$.

By convexity of f, we have, $f(x) \leq \lambda f(y) + (1-\lambda) f(x_*) < f(x_*)$.

#. Not Every convex function, but a convex function with lower bound needs to have an minimizer.

#. f, g convex $(\mathbb{R}^n \rightarrow \mathbb{R}) \Rightarrow$ then so $f+g$.

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ $\quad \mu \geq 0$ then μf

obv.

#. diff. function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, If and only if

$\langle \nabla f(x), (y-x) \rangle$ for every $x, y \in \mathbb{R}^n$.

$f(y) \geq f(x) + \langle \nabla f(x), (y-x) \rangle$ for every $y \in$

\# Corollary

Let $f: \mathbb{R}^n \to \mathbb{R}$ be a convex func, then x_* is global minimum if $(\nabla f(x_*) = 0)$.

\# Simply, $f(y) \geq f(x_*) + \langle \nabla f(x_*), (y - x_*) \rangle$

$= f(x_*)$

↳ So, hence proved It is global minimizer.

\# Going Reverse, If Ineq. for diff. func. Exist ⇒ Convex function.

aim: $f(\alpha w + (1-\alpha) z) \leq \alpha f(w) + (1-\alpha) f(z)$

Take $x = \alpha w + (1-\alpha) z$ then $w - x = (1-\alpha)(w - z)$

and $z - x = (z - w)\alpha$.

So, $f(w) \geq f(x) + \langle \nabla f(x), (w - x) \rangle$ {for w & x} —①

\# applying for z & x. $f(z) \geq f(x) + \langle \nabla f(x), (z - x) \rangle$. —②

By doing $\alpha \times (1) + (1-\alpha) \times (2)$.

\# $\alpha f(w) \geq \alpha f(x) + (\alpha \langle \nabla f(x), (w - x) \rangle)$

$(1-\alpha) f(z) \geq (1-\alpha) f(x) + ((1-\alpha) \langle \nabla f(x), (z - x) \rangle)$ This goes to zero.

$\Rightarrow \alpha f(w) + (1-\alpha) f(z) \geq f(x) +$

\# next objective, $f(y) \geq f(x) + \langle \nabla f(x), (y - x) \rangle$ for every $x, y \in \mathbb{R}^n$.

for every diff'ble convex fn.

★ Take $x \neq y$ in $\mathbb{R}^n$. Convexity of f implies →

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y). \quad — ①$$

Take $h = y-x$, rewriting ①.

$$f\left(x+\frac{h}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(x+h). \quad — ②$$

$$\Rightarrow f(x+h) - f(x) \geq \frac{f(x+h/2) - f(x)}{2^{-1}}.$$

#. On the same note and patterns →.

$$★ \quad f\left(x+\frac{h}{4}\right) = f\left(\frac{x}{2} + \frac{1}{2}\left(x+\frac{h}{2}\right)\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f\left(x+\frac{h}{2}\right).$$

$$f\left(x+\frac{h}{2}\right) - f(x) \geq \frac{f(x+h/4) - f(x)}{2^{-1}}.$$

Thus, $$f(x+h) - f(x) \geq \frac{f(x+h/4) - f(x)}{2^{-2}}.$$

By Inducing a recursive pattern ∴

$$f(x+h) - f(x) \geq \frac{f(x+2^{-k}h) - f(x)}{2^{-k}} \quad \forall k \in \mathbb{N}.$$

for convex diffable fn.s.

#.

↳ $g : \mathbb{R}^n \to \mathbb{R}$

↓

directional derivative at pt. x & dir p.

$$\lim \frac{g(x+sp) - g(x)}{}$$

$s \to 0$ $\quad$ $s.$

#. If g is diff'ble, then it's directional derivative $\langle \nabla g(x), p \rangle$.

For. Recursive Inequality.

$$f(x+h) - f(x) \geq \frac{f(x+2^{-k}h) - f(x)}{2^{-k}} \quad \forall\, k \in \mathbb{N}$$

as $k \to \infty$. $\quad f(y) - f(x) \geq \langle \nabla f(x), h \rangle = \langle \nabla f(x), (y-x) \rangle.$