GENERAL TOPOLOGY, SPRING 2025

PROBLEM SHEET 1

Due date : Not for submission.

1. List of problems

(1) Let X be a nonempty set and $\mathcal{A}$ is a collection of subsets of X. Let $\mathcal{A}' :=$ The collection of all intersections of finitely many members of $\mathcal{A} \cup \{\emptyset\} \cup \{X\}$. Prove that the following collection $\mathcal{B}$ forms a topology on X.

$$\mathcal{B} := \text{The set of all unions of members of } \mathcal{A}'.$$

(2) For each of the following collection $\mathcal{A}$, describe $\mathcal{A}'$ as denoted above.
- (A) $X = \mathbb{R}, \mathcal{A} = \{[0,1]\}$;
- (B) $X = \mathbb{R}, \mathcal{A} = \{[0,1], [3,4], [5,6)\}$;
- (C) $X = \mathbb{R}^2, \mathcal{A} = \{B((0,0), 1/n) : n \geq 1\}$;
- (D) $X = \mathbb{R}^2, \mathcal{A} = \{\text{The set of all finite subsets of } \mathbb{R}^2\}$;
- (E) $X = \mathbb{R}^2, \mathcal{A} = \{\text{The set of all infinite subsets of } \mathbb{R}^2\}$;
- (F) $X = \mathbb{R}^2, \mathcal{A} = \{A \subset \mathbb{R}^2 : |\mathbb{R}^2 \setminus A| \leq 10\}$;
- (G) $X = \mathbb{R}^2, \mathcal{A} = \{A \subset \mathbb{R}^2 : |\mathbb{R}^2 \setminus A| < \infty\}$;
- (H) $X = \mathbb{R}^2, \mathcal{A} = \{A \subset \mathbb{R}^2 : \mathbb{R}^2 \setminus A \text{ is countable}\}$;

(3) Recall the definition of an open set in $\mathbb{R}^3$ from calculus. Show that the collection of open sets indeed forms a topology on $\mathbb{R}^3$.

(4) Repeat the above problem for a general metric space. Does the triangle inequality play a part in the proof? If so, where?

Date: January 9, 2025.