Quiz 6

Theorem ★ : If $G \geq H$, iff Left wins the game $G-H$ playing second.

1.) $G=H$ iff $G-H \in P$

Proof.) ① By Theorem ★. $G \geq H$, iff. $o_R(G-H)=L$. Similarly, $H \geq G$ iff $o_R(H-G)=L$. Since $H-G=-(G-H)$, we have :

$$o_R(H-G)=o_L(G-H)$$

Thus, $G=H$ implies $o_R(G=H)=L$ and $o_L(G-H)=L$, which means $G-H \in P$. Conversely, if $G-H \in P$, then $o_R(G-H)=L$, implying $G \geq H$ and $H \geq G$,

so $G=H$.

Proof.2) Claim: $G>H \iff G-H \in L$

If $G>H$, then $G \geq H$ and $H \ngeq G$. By theorem ⊛.,

$$o_R(G-H)=L \text{ and } o_R(H-G) \neq L.$$

Since $H-G=-(G-H)$, $o_R(H-G)=o_L(G-H)$. Thus $o_L(G-H) \neq L$, $\Rightarrow$ $G-H \in L$.

Conversely, If $G-H \in L$, then $o_R(G-H)=L$ and $o_L(G-H)=L$, so $G \geq H$ and $H \ngeq G$, meaning $G>H$.

3.) Claim $G<H \iff G-H \in R$

- This is symmetric to part 2. If $G<H$, then $H>G$, so $H-G \in L$. Since, $G-H=-(H-G)$, $G-H \in R$. Conversely, if $G-H \in R$, then $H-G \in L$, so $H>G$, meaning $G < H$.

4.) $G <> H \iff G - H \in \mathcal{N}$

Proof: If $G <> H$, then $G \not\geq H$ and $H \not\geq G$. By Theorem ★

$$O_R(G-H) \neq \mathcal{L} \quad \text{and} \quad O_R(H-G) \neq \mathcal{L}$$

Since $H-G = -(G-H)$, $O_R(H-G) = O_L(G-H)$. Thus $O_L(G-H) \neq \mathcal{L}$, so $G-H \in \mathcal{N}$

Conversely, If $G-H \in \mathcal{N}$, then $O_R(G-H) \neq \mathcal{L}$ and $O_L(G-H) \neq \mathcal{L}$, so $G \not\geq H$ and $H \not\geq G$, meaning $G <> H$