#. Stuff around $\mathbb{C}$ is a complete field!

$\Rightarrow$ If ass. $\{z_n\}_{n\geq 1}$ is a Cauchy Seq $\to$ $Re(z_n)$ are also a Cauch.

$\to$ $Im(z_n)$ " " " "

$\lim_{n\to\infty} Re(z) = a$ $\lim_{n\to\infty} Im(z) = b$

$\{\lim_{n\to\infty} z_n = a + ib\}$

#. Generally, Infinity series $\to$ seq^n of partial sums $\{S_n z\} \to$ (2) conv

If $(f_n(z)) \to$ Seq of funcs conv.

★ Uniform convergence,

Given $\{f_n\}_{n\geq 1}$, $f_n : \Omega \to \mathbb{C}$, the series $\sum_{n\geq 1} f(z)$ $|f_n(z) - f_m(z)| < \varepsilon \ \forall \cancel{n,m}$ $z \in \Omega$.

General $\to$

$\downarrow$

Power: $\sum_{n=0}^{\infty} a_n (z - z_0)^n$

#. $\{\sum_{n\geq 0} a_n z^n\} \to \exists R \in \mathbb{R}$ $0 \leq R \leq \infty$

converges $|z| < R$ diverges $|z| \geq R$.

#. $$\left\{ R = \frac{1}{\limsup |a_n|^{1/n}} \right\}$$

(1) Taking the case, $|z| < R$ Take some $\rho \in R$ $|z| < \rho < R$

$\left(\frac{|z|}{\rho}\right)^n$

$\frac{1}{\rho} > \frac{1}{R} \to (\limsup |a_n|^{1/n})$ // $\}$ //

★ $\frac{1}{\rho} > \limsup |a_n|^{1/n} \Rightarrow \left\{ \frac{|z|}{\rho} > \limsup |z| |a_n|^{1/n} \right\}$ $\to$ (1) //

$\exists N$ such that, $n > N \Rightarrow \left[\left(\frac{|z|}{\rho}\right)^n > |z|^n a_n \right]$ for $\{|z| < R\}$

#. We have

$$S = \sum_{n\geq 0} a_n z^n = \sum_{n=0}^{N} a_n z^n + \sum_{n=N+1}^{\infty} a_n z^n \leq \underbrace{\sum_{n=0}^{N} a_n z^n}_{= S' \text{ maybe}} + \underbrace{\sum_{n=N+1}^{n} \left|\frac{z}{\rho}\right|^n}_{< \infty}$$

as $|z| < |\rho| < R$

$$\frac{|z|}{|\rho|} < 1$$

For uniform Convergence $\to \{|z| \leq \rho < R\} \Rightarrow \rho' : \quad \rho < \rho' < R$

Cauchy seq^n, $\{S_n(z)\}$, $S_n(z) = \sum_{k=0}^{n} a_k z^k$ converges uniformly.

#.) $|S_n(z) - S_m(z)| < \varepsilon \qquad n, m > N$

Say $n > m$, $|S_n(z) - S_m(z)| = \left|\sum_{k=m+1}^{n} a_k z^k\right| \leq \sum_{k=m+1}^{n} |a_k z^k|$ $\boxed{\star}$

$\therefore \to$ By all the def^n we have done till now, $\exists N$, such that, $k \geq N$

$$|a_k z^k| < \left(\frac{|z|}{\rho}\right)^k \leq \left(\frac{|z|}{\rho'}\right)^k$$

$$\Rightarrow \left(\frac{|z|}{\rho}\right)^k \leq \left(\frac{\rho}{\rho'}\right)^k \Rightarrow \underbrace{\sum_{n=1}^{N} |a_n z^n| + \sum_{n=N+1}^{\infty} |a_n z^n|}_{\text{Every thing } \leq \left(\frac{\rho}{\rho'}\right)^N}$$

More Explaination

For simple divergence : Weistrass - M test.

$\star$ Result: $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{1/n} \leq \limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$

#. $f(z) = \sum_{n\geq 0} a_n z^n$ converges $\to |z| < R$

$$f'(z) = \sum_{n\geq 1} n a_n z^{n-1} \to \text{Checking It's diff.ble.}$$

Goal $f_1(z) = f'(z)$

$|z| < R$.

Say R_1 = Radius of convergence

$$\left\{ R_1^{-1} = \limsup |(n+1)\, a_{n+1}|^{1/n} \right\}$$

$$= \limsup (n+1)^{1/n} \limsup \left(|a_{n+1}|^{1/n+1} \right)^{\frac{(n+1)}{n}}$$

for $z \cdot f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n+1}$.

Radius of conv. $= \left(\limsup_{n\to\infty} |n a_n|^{1/n} \right)^{-1}$

$\limsup |n|^{1/n} \cdot \limsup |a_n|^{1/n}$

$\cdot \to \sum_{n\geq 0} (n+1)\, a_{n+1} z^n$

$$z f_1(z) = \sum_{n\geq 1} n a_n z^n = a_1 z + 2a_2 z^2 \to \geq z(a_1 + 2a_2 + \dots)$$

↳ converges for $z \leq R$

$$= z \sum_{n\geq 1} n a_n z^{n-1}$$

$g : \mathbb{C} \to \mathbb{C}$ is a complex funcⁿ. $\sum_{n\geq 0} a_n z^n$ & $\sum_{n\geq 0} a_n g(z) \cdot z^n$

$$\left\{ g(z) \sum_{n\geq 0} a_n z^n. \right\}$$

★ $g_N(z) = \sum_{n=1}^{N} b_n z^n$.

$f'(z) = f_1(z)$ Say z_0, $|z_0| < R$. $f(z) = \sum_{n\geq 0} a_n z^n$.

$$= \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N}^{\infty} a_n z^n}_{R_N(z)}$$

$$\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0)$$

$\left[\frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) \right] + \left[S_N'(z_0) - f_1(z_0) \right] + \left[\frac{R_N(z) - R_N(z_0)}{z - z_0} \right]$ //.

$\to 0$

$+ a_n z^n$

- $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$

$0 + a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$