② (X, τ)

How τ is generated from metric d on X.

① We can separate any two open sets, by triangle Inequality.

② If (X, τ) is coming from a metric, It must be that given any two distinct points $p, q \in X$. $\exists\ A, B \in X. \Rightarrow p \in A,\ q \in B$, and $A \cap B = \Phi$.

③ $|X| > 1$. $\tau = \{\Phi, X, A\}$

Definitions :→ If (X, τ) is a topological space,

* Open Set : Each member of τ is open set.
* Closed Set : the complement $A^c = X \setminus A$ is closed set, for any member A of τ.

This generalises → Closed Sets are closed under finite union and arbitrary intersection

$$\cap A_\alpha^c = (\cup A_\alpha)^c \quad \text{and} \quad \bigcup_{i=1}^{N} A_i^c = \left(\bigcap_{i=1}^{n} A_i\right)^c$$

Basis of Topology :→ Suppose $X \neq \Phi$. $\mathcal{B} \to$ basis. $X = \bigcup_{B \in \mathcal{B}} B$

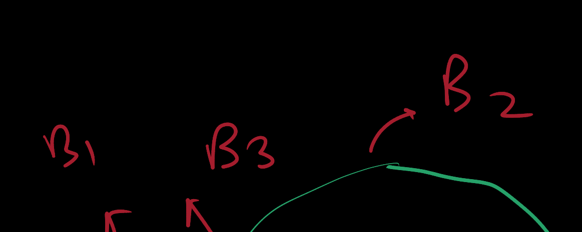
* If $x \in X$. $B_1 \cap B_2$, $B_1, B_2 \in \mathcal{B}$, $\Rightarrow \exists\ B_3 \to x \in B_3 \supseteq B_1 \cap B_2$.

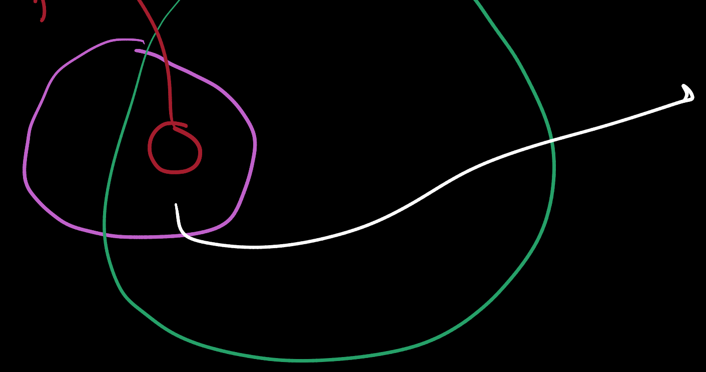
$$\tau = \{\text{all possible union of members of } \mathcal{B}\}.$$

Thus a set A is open in X if for Each $p \in A$ if $\exists\ B \in \mathcal{B}$ So that $p \in B \subseteq A$.

Remark. : $p \in B_1 \cap B_2 \Rightarrow \exists\ B_3 \in \mathcal{B} \to p \in B_3 \subseteq B_1 \cap B_2$

In some Cases this is not a Ball.





$B_1 \cap B_2$ → not a Ball

But $B_1, B_2 \in \tau$.

#. $(X, \tau) = (\mathbb{R}$, usual topology on open sets).

$\mathcal{B}$ = (Intervals $(b-a)$ of finite length $b-a$).

⊙ $X = \mathbb{R}^d$. τ = open sets in $\mathbb{R}^d$, τ = open sets in $\mathbb{R}^d$, given by Euclidean

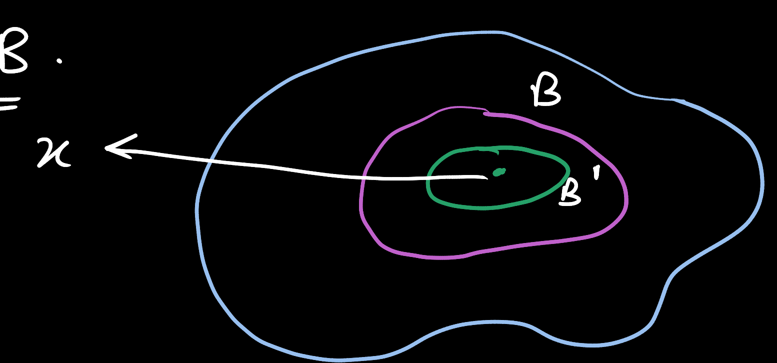
dist^n $d(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^2\right)^{1/2}$ $\mathcal{B}$ = all balls of finite radius.

#. $X = \mathbb{R}^2$ $\mathcal{B}$ = collection of all finite length rectangles.

#. Let $\mathcal{B}, \mathcal{B}'$ are two basis for topology τ, τ' on X. Then, the following are Equivalent.

★ τ' is finer than τ.

★ $\forall x \in X$ and any $B \in \mathcal{B}$, $\exists$ $B' \in \mathcal{B}'$ ⇒ $x \in B' \subseteq B$.



#. Proof Ⓐ ⇒ Ⓑ

① τ' is finer than τ

② $\mathcal{B}'$ is also a basis

③ Given $x \in B \in \mathcal{B}'$ → $\exists$ $B' \in \mathcal{B}'$ ⇒ $x \in B' \subseteq B$

Ⓑ ⇒ Ⓐ ① $\mathcal{B}' \subseteq \tau'$ (Each element in τ is union from $\mathcal{B}$)

② Ⓑ Says for each $B \in \mathcal{B}' \Rightarrow \exists$ $B'_\alpha \in \mathcal{B}'$?

$\left[B = \bigcup_\alpha B'_\alpha \right]$

Damn!

#. Two topologies, τ, τ' are same, If and only If they have same basis

Proof :- Need or one way to show is, that, Basis of τ is also basis for τ', then one can directly claim.

Ex. $X = \mathbb{R}^2$, τ = open sets in $\mathbb{R}^2$ $\left[d(x,y) = \left(\sum_{i=1}^{2} |x_i - y_i|^2 \right)^{1/2} \right]$

Thus all $\leftarrow$ Balls of radius

topologies are of this kind.

Remark: Topologies τ_p introduced, given by the metric d_p on $\mathbb{R}^d$.

$$\Rightarrow d_p(x,y) = \begin{cases} \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{1/p}, & 1 \le p < \infty \\ \max_{i=1,\dots,d} |x_i - y_i| & p = \infty \end{cases}$$

$\boxed{\text{are same Topology}}$

#. Basis in topo, diff. from the ones in linear algebra. In L.A., a vector can be uniquely presented but here multiple ways to write an open set as union of other open sets.

and number of elements in Basis is not fixed.

#. Let X be an non-empty set. Given any collection of subsets $\mathcal{A}$ covering X, i.e. $\left[X = \bigcup_{A \in \mathcal{A}} A \right]$

$\mathcal{A} \to$ subbasis wrt topology,

τ = all possible union members of $\mathcal{B}$

$\mathcal{B}$:= all intersection of finitely many elements in $\mathcal{A}$.

#. $X = \mathbb{R}^d$, τ = topology generated by balls of finite radius,

$\mathcal{A}$ = all balls of radius $(1/n)$ for some $n \in \mathbb{N}$.

$= \{ B(x, 1/n) : x \in \mathbb{R}^d, n \ge 1 \}$.

(X, τ) as above, $X = \mathbb{R}^2$. and

$\{ \dots : x \ne 0, n \ge 1 \}$

$\mathcal{A} = \{B(x, 1/n) : x \neq 0,\ n \geq 1\}$

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