$D(g \circ f)(a) = Dg(f(a)) \cdot Df(a)$

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by the equations

$$f(x) = f(x^1, x^2) = (e^{2x^1+x^2},\ 3x^2 - \cos x^1,\ (x^1)^2 + x^2 + 2),$$
$$g(y) = g(y^1, y^2, y^3) = (3y^1 + 2y^2 + (y^3)^2,\ (y^1)^2 - y^3 + 1).$$

If $F(x) = g \circ f(x)$, find $DF(0)$ and if $G(y) = f \circ g(y)$, find $DG(0)$.

$G(y) = f(g(y))$

$DG(0) = Df(g(0)) \cdot Dg(0)$

$g(0) = (0, 1)$ $\quad f(0) = (1, -1, 2)$.

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{x_1} & \frac{\partial f_1}{x_2} \\ \frac{\partial f^2}{x_1} & \frac{\partial f^2}{x_2} \\ \frac{\partial f^3}{x_1} & \frac{\partial f^3}{x_2} \end{bmatrix}$$

$$Df(0) = \begin{bmatrix} 2 & 1 \\ \sin 0 & 3 \\ 0 & 1 \end{bmatrix} = \begin{matrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{matrix}$$

$F(x) = g(f(x))$

$DF(0) = Dg(f(0)) \cdot Df(0)$.

$$Df(0) = \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$$

$g(y) = (3y^1 + 2y^2 + (y^3)^2,\ (y^1)^2 - y^3 + 1)$

$f(0) = (1, -1, 2)$

$$Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{bmatrix}$$

$$Dg(1, -1, 2) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix}$$

So, $\begin{bmatrix} 3 & 2 & 4 \\ 2 & -1 & 0 \end{bmatrix} \Big[$

2) $f(tx) = t^m f(x)$

$R^m \to R$

$f'(tx) \cdot t = t^m f'(x) \Rightarrow f'(tx) = t^{m-1} f'(x)$

$f^n(tx) = t^{m-n} f^n(x)$

- $g(t) = f(tx)$

$g(t) = t^m f(x)$

$g'(t) = m t^{m-1} f(x)$

$g'(1) = m f(x)$

alternatively: $g'(t) = \sum_{i=1}^{t} \cdot \frac{\partial(tx_i)}{dt} = \sum_{i=1}^{n} D_i f(tx) \cdot x_i$

$g'(1) = \sum_{i=1}^{n} D_i f(x) \cdot x_i$

$\sum^{n} D_i f(x) \cdot x_i = m f(x)$

$i=1$

\# $f(x) = \left\{ \begin{array}{l} f: R^2 \to R \\ f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{array} \right.$

$e^{\frac{-1}{x^2}}$ $\qquad e^t = \sum_{t \geq 0} \frac{t^n}{n!}$

$\Rightarrow \frac{\left(\frac{-1}{x^2}\right)^n}{n!}$

\# $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ $\qquad Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$

mistake here
Rember It

$2 \to 1$
(2×1)

$\frac{y(x^2+y^2) - xy(2x)}{(x^2+y^2)^2} = x^2y + y^3 - 2x^2y = \frac{y^3 - x^2y}{(x^2+y^2)^2}$ ①

②

$\frac{x(x^2+y^2) - xy(2y)}{(x^2+y^2)^2} = \frac{x^3 + xy^2 - 2xy^2}{(x^2+y^2)^2} = \frac{x^3 - xy^2}{(x^2+y^2)^2}$ ②

$\to \lim_{y \to 0}$ $\frac{\frac{(x+h) \cdot y}{x^2+y^2} - f(x,y)}{h} = \frac{\frac{xy}{(x^2+y^2)} + \frac{hy}{(x^2+y^2)} - \frac{xy}{(x^2+y^2)}}{h} = \frac{y}{x^2+y^2}$

mistake here
Rember It

$\frac{0}{0^+} = 0$

$f'(x)$

(does not Exist) $\to \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{\frac{h \cdot 0}{h^2+0^2} - 0}{h} = \underbrace{\lim_{h \to 0} \frac{0}{h}}_{} = 0$

wrong

\# They do Exist. $D_1 f(0,0)$. and $D_2 f(0,0)$

① So, $\frac{f(h,k) - f(0,0) - D_1(f(0,0)) - D_2(f(0,0))}{\sqrt{h^2+k^2}} = \frac{\frac{hk}{h^2+k^2}}{\sqrt{h^2+k^2}} = \frac{hk}{(h^2+k^2)^{3/2}}$

$=$ $= \infty$

Try approaching through y axis $\to k \to 0 \lim \frac{0}{\ldots}$

Try approaching through x axis $\cdot \to$ $\lim_{h\to 0}\frac{}{h^3}$ =

along $y=x$ $\quad \frac{h^2}{(2h^2)^{3/2}} = \frac{1}{2^{3/2}\cdot h} = \infty$ // does not exist

along $y = x$. $\quad \cancel{f(h,k) - f(0,0)} / \cancel{h^2+k^2}$ $\quad \lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \frac{x^2}{2x^2} \neq \frac{1}{2}$ But It should be 0

5) $f(x,y) = \varphi(ax+by)$, where $a,b \in \mathbb{R}$ and φ of class C^q in some open set containing 0. Show Taylor's formula about $(0,0) \to$

$$f(x,y) = \sum_{m=0}^{q-1} \frac{\varphi^{(m)}(0)}{m!} \sum_{j=0}^{m} \binom{m}{j} (ax)^j (by)^{m-j} + R_q(x,y)$$

Sol) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = f(a) + f'(a)\cdot(x-a) + R_2(x)$

Taylor's theorem for functions of several variables.

$$f(x) = \underbrace{f(a) + \sum_{i=1}^{n} D_i f(a)\cdot(x^i - a^i)}_{} + R_2(x)$$

$\phi(t) = f(a+th)$ $\qquad$ Proof) Chain Rule $(\phi = f\cdot\theta)$, where $\theta: R \to R^n$.

$\phi'(t) = Df(a+th)\cdot t.$ $\qquad$ $\{\theta(t) = a+th\}$ //

$\phi = f\cdot(\theta)$.

$\phi' = D(f(\theta))\cdot D(\theta)$. $\quad \{D(f(\theta(t)))\cdot D(\theta(t)).\}$

$\cancel{\phi'}$ = $\boxed{\theta(t) = a + th}$

$D(\theta(t)) = h.$ $\qquad$ thus $\quad D(f(a+th))\cdot h$

$f(x,y) = \varphi(ax+by) \Rightarrow$ For one variable $\underbrace{\sum \frac{\varphi^{n-1}(a)}{(n-1)!}\cdot(b-a)^{(n-1)}}_{}$

$(ax+by)^m$ $\quad$ n^{th} term? $= {}^mC_n\,(ax)^n\cdot(by)^{m-n}$ $\qquad$ $\sum_{n=0}^{q-1} \frac{\varphi^n(a)\,t^n}{n!} + R_q(t)$

Taking $t = ax+by$ and substituting it in the Equation $\rightarrow$

#. $\sum_{m=0}^{g-1} \frac{\varphi^m(0)}{m!} \cdot \sum_{m=0}^{g-1} \binom{m}{j} (ax)^j (by)^{m-j} + R_g(ax+by)$

#. 3) $f(x) = e^{-1/x^2}$ (0)

$\lim_{h\to 0} \frac{e^{-\frac{1}{(x+h)^2}}}{h} \Rightarrow \frac{e^{-1/h^2}}{h} = \frac{e^{-\frac{1}{h^2}}}{h} =$ let $h = \frac{1}{t}$ $h\to 0$ $t \to \infty$

at $x\to 0$

$\lim_{t\to\infty} t e^{-t^2} = \frac{t}{e^{t^2}} = \frac{\infty}{\infty}$ $\frac{1}{2te^{t^2}} = 0$

By Induction $f^k(0) = 0$ $\forall\, k \leq n$. For $f^{(n+1)} = 0$. consider

$f^{(n+1)}(0) = \frac{f^n(0+h) - f^n(0)}{h}$. $= \lim_{h\to 0} \frac{f^n(h)}{h}$

as $e^t = \sum \frac{t^n}{n!}$ $e^{-1/x^2} = 1 + \frac{(-1/x^2)}{1} + \frac{(-1/x^2)^2}{2!} + \ldots$

Thus

$\lim_{h\to 0} \frac{P_n(1/h)\, e^{-1/h^2}}{h}$

$f^n(x) \rightarrow$ Polynomial in x.

we can state the polynomial as

$f\left[P_n\left(\frac{1}{x}\right) \cdot e^{-1/x^2}\right]$

$h \to \frac{1}{t}$ | $h\to 0 \Rightarrow t\to\infty$

$\lim_{t\to\infty} \frac{P(t)\cdot e^{-t^2}}{\frac{1}{t}} = \boxed{t\cdot P(t) \cdot e^{-t^2}}$

as any polynomial in R^*, e^{-t^2} decays way for more faster. So limit goes to zero.