Optimization

Problem Sheet 2

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1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex on $\mathbb{R}^n$ if for all $x, y \in \mathbb{R}^n$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \qquad \text{for all } \alpha \in [0,1].$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be concave on $\mathbb{R}^n$ if $-f$ is convex on $\mathbb{R}^n$.

(1a) Determine whether the following function $f : \mathbb{R}^3 \to \mathbb{R}$ is convex or concave on $\mathbb{R}^3$.

$$f(x) := \log\left(e^{x_1} + e^{x_2} + e^{x_3}\right) \qquad \text{for } \; x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Justify your answer. Here, by log we mean logarithm to the base e.

(1b) Determine whether the function $g : [0, \infty)^3 \to \mathbb{R}$, defined below, is convex or concave on $\mathbb{R}^3$.

$$g(x) := \left(\prod_{i=1}^{3} x_i\right)^{\frac{1}{3}} \qquad \text{for } \; x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Justify your answer.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function on $\mathbb{R}$. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that

$$(b-a)f(x) - (b-x)f(a) - (x-a)f(b) \leq 0 \qquad \text{for all } x \in [a,b].$$

3. Recall that the Taylor series of a function $f : \mathbb{R} \to \mathbb{R}$ centered at a point $a \in \mathbb{R}$ is given by

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Define the function $f : (-1, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by

$$f(x) := \sin^{-1} x \qquad \text{for } x \in (-1, 1).$$

Let b_6 denote the coefficient of x^6 in the Taylor series of $(f(x))^2$ centered at the point $x = 0$. Find the value of b_6.

4. Let $a \in \mathbb{R}$ be a parameter. Let $G_a : \mathbb{R} \to \mathbb{R}$ be a linear function such that

$$G_a(-1) = 1 \qquad \text{and} \qquad G_a(a) = a^2.$$

Find the value of a such that the following integral is minimized:

$$\int_{-1}^{1} \left(x^2 - G_a(x)\right)^2 \, \mathrm{d}x.$$

5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be bounded if there exists a constant $M > 0$ such that

$$|f(x)| \leq M \qquad \text{for all } x \in \mathbb{R}.$$

Determine whether the following statement is **TRUE** or **FALSE**:

Every bounded real-valued convex function on $\mathbb{R}$ is a constant function.

If you declare the above statement to be **TRUE**, then you have to justify your answer. On the other hand, if you declare the above statement to be **FALSE**, then you have to provide an example of a non-constant convex function on $\mathbb{R}$ which is bounded.

6. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be bounded from below if there exists a constant $M \in \mathbb{R}$ such that

$$f(x) \geq M \qquad \text{for all } x \in \mathbb{R}.$$

Determine whether the following statement is **TRUE** or **FALSE**:

Every convex function which is bounded from below must have a global minimizer in $\mathbb{R}$.

If you declare the above statement to be **TRUE**, then you have to justify your answer. On the other hand, if you declare the above statement to be **FALSE**, then you have to provide an example of a convex function on $\mathbb{R}$ which is bounded from below but doesn't have a global minimizer in $\mathbb{R}$.

7. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be strictly convex on $\mathbb{R}^n$ if for all $x, y \in \mathbb{R}^n$, $x \neq y$,

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y) \qquad \text{for } \alpha \in (0,1).$$

Determine whether the following statement is **TRUE** or **FALSE**:

There exists a strictly convex function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a) = f(b) < f(x) \qquad \textit{for all } x \in \mathbb{R} \setminus \{a, b\}$$

for some distinct $a, b \in \mathbb{R}$.

If you declare the above statement to be **TRUE**, then you have to find a strictly convex function having the aforementioned property. On the other hand, if you declare the above statement to be **FALSE**, then you have to justify your answer.

8. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex on $\mathbb{R}^n$ if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \quad \text{for all } x, y \in \mathbb{R}^n,\ \alpha \in [0,1].$$

Let $\mathcal{S}$ denote the set of all global minimizers of f, i.e.

$$\mathcal{S} := \Big\{ x \in \mathbb{R}^n \text{ such that } f(x) \leq f(z) \text{ for all } z \in \mathbb{R}^n \Big\}.$$

Show that $\mathcal{S}$ is a convex set in $\mathbb{R}^n$.

9. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex on $\mathbb{R}^n$ if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \quad \text{for all } x, y \in \mathbb{R}^n,\ \alpha \in [0,1].$$

   Using the above definition of convexity, show that the following function is convex

$$f(x) := \langle Ax, x \rangle \quad \text{for } x \in \mathbb{R}^n,$$

   where A is a $n \times n$ symmetric positive semidefinite matrix.

10. It is given that a function $f : \mathbb{R}^n \to \mathbb{R}$ is such that both f and $-f$ are convex on $\mathbb{R}^n$. Then, show that f must be of the form

$$f(x) = \langle b, x \rangle + c \qquad \text{for } x \in \mathbb{R},$$

   where $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

11. Let $n \geq 4$ and consider the following $n \times n$ symmetric tridiagonal matrix:

$$A := \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

   Show that the above matrix is positive definite. More precisely, show that

$$\langle Ax, x \rangle > 0 \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\}$$

12. Determine whether the following statement is **TRUE** or **FALSE**:

   If $f, g : \mathbb{R} \to \mathbb{R}$ are convex functions on $\mathbb{R}$, then the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) := f(x) - g(x) \qquad \textit{for } x \in \mathbb{R}$$

   is also convex on $\mathbb{R}$.

   If you declare the above statement to be **TRUE**, then you have to justify your answer. On the other hand, if you declare the above statement to be **FALSE**, then you have to provide examples of two convex functions on $\mathbb{R}$ such that their difference fails to be convex on $\mathbb{R}$.

13. Let $n \geq 3$ and let $m \geq 4$. Suppose $x_* \in \mathbb{R}^n$ is such that

$$\|b - Ax_*\|^2 \leq \|b - Ay\|^2 \qquad \text{for all } y \in \mathbb{R}^n,$$

   where A is a given $m \times n$ matrix and $b \in \mathbb{R}^m$ is a given vector. Show that x_* satisfies the following equation:

$$A^\top A x_* = A^\top b.$$

   Here $A^\top$ denotes the transpose of A.

14. Determine whether the following statement is **TRUE** or **FALSE**:

    If $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are convex functions on $\mathbb{R}$, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by

    $$f(x) := \max\Big\{f_1(x), f_2(x)\Big\} \qquad \textit{for } x \in \mathbb{R}$$

    is also convex on $\mathbb{R}$.

    If you declare the above statement to be **TRUE**, then you have to justify your answer. On the other hand, if you declare the above statement to be **FALSE**, then you have to provide examples of two convex functions f_1 and f_2 on $\mathbb{R}$ such that the function $f := \max\{f_1, f_2\}$ fails to be convex on $\mathbb{R}$.

15. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

    $$f(x, y) := 5x^2 - xy + 5y^2 - 11x + 11y + 11 \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

    (15a) Find the critical point(s) of the function f in $\mathbb{R}^2$.

    (15b) Does the above defined function f have a point of global minimum in $\mathbb{R}^2$? Justify your answer.

16. Compute the expression for the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as

    $$g(x) = f(x) + \langle \nabla f(x), p\rangle + \frac{1}{2}\left\langle \nabla^2 f(x)p, p\right\rangle \qquad \text{for } x \in \mathbb{R}^2,$$

    where the point $p = (1, 0)$ and the function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

    $$f(x) = e^{x_1 - x_2} + e^{x_1 + x_2} + x_1 + x_2 + 1 \qquad \text{for } x = (x_1, x_2) \in \mathbb{R}^2.$$

17. Show that a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ is convex if and only if

    $$\int_0^1 f(x + \alpha(y - x))\,\mathrm{d}\alpha \le \frac{f(x) + f(y)}{2} \qquad \text{for every } x, y \in \mathbb{R}^3.$$

18. A function $f : \mathbb{R} \to (0, \infty)$ is said to be log-convex on $\mathbb{R}$ if $\log f$ is convex on $\mathbb{R}$.

    Let $f : \mathbb{R} \to (0, \infty)$ be a twice continuously differentiable log-convex function.

    Show that

    $$(f'(x))^2 \le f(x)f''(x) \qquad \text{for all } x \in \mathbb{R}.$$

19. A function $f : \mathbb{R} \to (0, \infty)$ is said to be log-convex on $\mathbb{R}$ if $\log f$ is convex on $\mathbb{R}$.

    Determine whether the following statement is **TRUE** or **FALSE**:

    The sum of any two log-convex functions is always log-convex.

    If you declare the above statement to be **TRUE**, then you have to justify your answer. On the other hand, if you declare the above statement to be **FALSE**, then you have to provide examples of two log-convex functions on $\mathbb{R}$ such that their sum fails to be log-convex on $\mathbb{R}$.

20. A function $f : \mathbb{R} \to (0, \infty)$ is said to be log-concave on $\mathbb{R}$ if $\log f$ is concave on $\mathbb{R}$.

    Determine whether the following statement is **TRUE** or **FALSE**:

    The sum of any two log-concave functions is always log-concave.

    If you declare the above statement to be **TRUE**, then you have to justify your answer. On the other hand, if you declare the above statement to be **FALSE**, then you have to provide examples of two log-concave functions on $\mathbb{R}$ such that their sum fails to be log-concave on $\mathbb{R}$.

21. Recall that

$$|x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}$$

(21a) Fix $a_1 \in \mathbb{R}$. Find the global minimizer(s) of $|x - a_1|$ on $\mathbb{R}$.
Is this function convex on $\mathbb{R}$?

(21b) Fix $a_1, a_2 \in \mathbb{R}$ such that $a_1 < a_2$. Find the global minimizer(s) of the function

$$|x - a_1| + |x - a_2|$$

on $\mathbb{R}$.
Is this function convex on $\mathbb{R}$?

(21c) Fix $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 < a_2 < a_3$. Find the global minimizer(s) of the function

$$|x - a_1| + |x - a_2| + |x - a_3|$$

on $\mathbb{R}$.
Is this function convex on $\mathbb{R}$?

(21d) Taking cues from your answers to the above three sub-questions, determine the global minimizer(s) of the function

$$\sum_{i=1}^{m} |x - a_i|$$

where $a_1, a_2, \ldots, a_m \in \mathbb{R}$ and $a_1 < a_2 < \cdots < a_m$ with $m > 3$ being a natural number. Does your answer depend on the parity of m?
