Optimization

Problem Sheet 1

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1. Let $a < b$ be two real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. It is given that f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the interval $[a, b]$. It is further given that $f^{(n)}$ is differentiable on (a, b). Define a function $p : [a, b] \to \mathbb{R}$ as follows:

$$p(x) := f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \text{for } x \in [a, b].$$

(1a) Show that
$$p^{(k)}(a) = f^{(k)}(a) \quad \text{for } k = 0, 1, \ldots, n.$$

(1b) Define a function $F : [a, b] \to \mathbb{R}$ as follows:
$$F(x) := f(x) - p(x) - K(x-a)^{n+1} \qquad \text{for } x \in [a, b],$$
for some constant K. Find the value of the constant K which ensures $F(b) = 0$.

(1c) Show that there exists a point $c \in (a, b)$ such that
$$F^{(n+1)}(c) = 0.$$
Hint: Employ Rolle's theorem on F followed by the application of Rolle's theorem on F', F'' and so on until $F^{(n)}$.

(1d) Deduce from your answers for Questions **(1b)** and **(1c)** that
$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

2. Let $n \geq 2$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. Pick two point $x, p \in \mathbb{R}^n$. Define $g : [0, 1] \to \mathbb{R}$ as follows:

$$g(s) := f(x + sp) \qquad \text{for } s \in [0, 1].$$

(2a) Show that
$$g'(s) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x + sp)p_i = \langle \nabla f(x + sp), p \rangle \qquad \text{for } s \in (0, 1).$$

(2b) Show that
$$g''(s) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x + sp)p_j p_i = \left\langle \nabla^2 f(x + sp)p, p \right\rangle \qquad \text{for } s \in (0, 1).$$

(2c) Single variable Taylor's theorem applied to the function g says that there exist $\tau, \ell \in (0,1)$ such that

$$g(1) = g(0) + g'(0) + \frac{g''(\tau)}{2}$$

and

$$g(1) = g(0) + g'(\ell).$$

Hence deduce the following expressions:

$$\begin{aligned} f(x+p) &= f(p) + \langle \nabla f(x+\ell p), p \rangle\,, \\ f(x+p) &= f(p) + \langle \nabla f(x), p \rangle + \frac{1}{2}\left\langle \nabla^2 f(x+s_* p)p, p \right\rangle. \end{aligned}$$

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x) := 100(x_2 - x_1^2)^2 + (1-x_1)^2 \qquad \text{for } x := (x_1, x_2) \in \mathbb{R}^2.$$

(3a) Compute the gradient $\nabla f(x)$.

(3b) Determine the point(s) where the gradient from Question **(3a)** vanishes.

(3c) Compute the Hessian $\nabla^2 f(x)$.

(3d) Are the stationary point(s) found in Question **(3b)** minimizer(s) of f?

4. Let $b \in \mathbb{R}^n$ be a given vector. Find the gradient and Hessian of the following function:

$$f(x) := \langle b, x \rangle = \sum_{i=1}^{n} b_i x_i \qquad \text{for } x \in \mathbb{R}^n.$$

5. Let $a, b \in \mathbb{R}^n$ be two given vectors. Find the gradient and Hessian of the following function:

$$g(x) := \langle a, x \rangle \, \langle b, x \rangle = \left(\sum_{i=1}^{n} a_i x_i \right) \left(\sum_{i=1}^{n} b_i x_i \right) \qquad \text{for } x \in \mathbb{R}^n.$$

6. Consider the function

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2 (x_1^2 - x_2^2)}{(x_1^2 + x_2^2)} & \text{if } (x_1, x_2) \neq (0,0), \\ 0 & \text{if } (x_1, x_2) = (0,0). \end{cases}$$

Compute the Hessian of f at the point $(0,0)$.
