Verification of Toffoli Circuit Equivalence

The tomography circuit

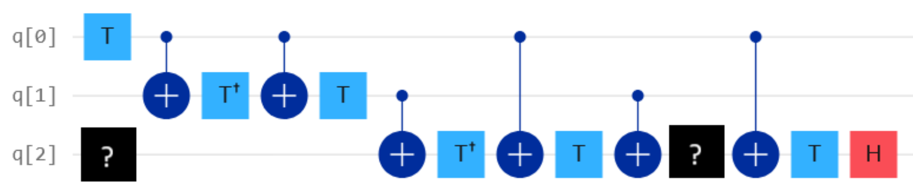

Figure 1: Toffoli Equivalent Circuit

The left unknown gate is considered as U_A and the left one U_B. The full operation on $|q_2\rangle$ is:

$$U_{\text{total}} = H \cdot T \cdot \text{CNOT}_{0,2} \cdot U_B \cdot \text{CNOT}_{1,2} \cdot T \cdot \text{CNOT}_{0,2} \cdot T^\dagger \cdot \text{CNOT}_{1,2} \cdot U_A$$

Upon which we apply the 4 possible control combinations.

Possible Control Combinations

Controls $|00\rangle$

All CNOTs are inactive (they act as the Identity I).

$$U_{00} = H \cdot T \cdot I \cdot U_B \cdot I \cdot T \cdot I \cdot T^\dagger \cdot I \cdot U_A$$
$$U_{00} = HTU_B(TT^\dagger)U_A = HTU_BIU_A = HTU_BU_A$$

Controls $|01\rangle$

CNOTs from q_0 are inactive (I). CNOTs from q_1 are active (X).

$$U_{01} = H \cdot T \cdot I \cdot U_B \cdot X \cdot T \cdot I \cdot T^\dagger \cdot X \cdot U_A$$
$$U_{01} = HTU_B(XTT^\dagger X)U_A$$
$$U_{01} = HTU_B(XIX)U_A = HTU_B(X^2)U_A$$
$$U_{01} = HTU_BIU_A = HTU_BU_A$$

Controls $|10\rangle$

CNOTs from q_0 are active (X). CNOTs from q_1 are inactive (I).

$$U_{10} = H \cdot T \cdot X \cdot U_B \cdot I \cdot T \cdot X \cdot T^\dagger \cdot I \cdot U_A$$
$$U_{10} = HTXU_BTXT^\dagger U_A$$

Controls $|11\rangle$

All CNOTs are active (X). The requirement is $U_{11} = \alpha X$.

$$U_{11} = H \cdot T \cdot X \cdot U_B \cdot X \cdot T \cdot X \cdot T^\dagger \cdot X \cdot U_A$$

Observation

From equations of $|00\rangle$ & $|01\rangle$, we can observe that the equation holds perfectly if:

$$U_A = H \qquad U_B = T^\dagger$$

We will substitute this into each control case.

Verification

Controls $|00\rangle$

Substitute $U_A = H$ and $U_B = T^\dagger$:

$$U_{00} = HT(T^\dagger)(H)$$

$$U_{00} = H(TT^\dagger)H = HIH = H^2$$
$$U_{00} = I \quad (\textbf{Correct})$$

Controls $|01\rangle$

This is identical to Case 1, so:

$$U_{01} = I \quad (\textbf{Correct})$$

Controls $|10\rangle$

Substitute $U_A = H$ and $U_B = T^\dagger$:

$$U_{10} = HTX(T^\dagger)TXT^\dagger(H)$$

$$U_{10} = HTX(T^\dagger T)XT^\dagger H = HTX(I)XT^\dagger H$$

$$U_{10} = HT(X^2)T^\dagger H = HT(I)T^\dagger H$$

$$U_{10} = H(TT^\dagger)H = HIH = H^2$$

$$U_{10} = I \quad (\textbf{Correct})$$

Controls $|11\rangle$

Substitute $U_A = H$ and $U_B = T^\dagger$:

$$U_{11} = HTX(T^\dagger)XTXT^\dagger X(H)$$

Let's analyze the large central matrix $M = TXT^\dagger XTXT^\dagger X$:

$$M = T \cdot (XT^\dagger X) \cdot T \cdot (XT^\dagger X)$$

We use the identities $XT^\dagger X = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$. Let $A = XT^\dagger X$. Since T and A are diagonal, they commute: $TA = AT$.

$$M = TATA = T^2A^2$$

$$T^2 = S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad A^2 = \begin{pmatrix} (e^{-i\pi/4})^2 & 0 \\ 0 & 1^2 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$$

$$M = T^2A^2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$M = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -iZ$$

Now substitute M back into the equation for U_{11}:

$$U_{11} = H \cdot M \cdot H = H(-iZ)H$$

$$U_{11} = -i(HZH)$$

Using the identity $HZH = X$:

$$U_{11} = -iX \quad (\textbf{Correct})$$

This is equivalent to X with a global phase of $\alpha = -i$.

Conclusion

All four control cases produce the required operations (I, I, I, and αX). Therefore, the solution is correct:

- **First U3 Gate:** $U_A = H = U3(\pi/2, 0, \pi)$
- **Second U3 Gate:** $U_B = T^\dagger = U3(0, x, \phi + \lambda - \pi/4)$

where choosing $\phi = 0$ for simplicity, we get,

- **First U3 Gate:** $U_A = H = U3(\pi/2, 0, \pi)$
- **Second U3 Gate:** $U_B = T^\dagger = U3(0, 0, -\pi/4)$