Multivariate Normal Distribution

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1D : $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

vectors

$\boxed{\text{transpose}}$

k-Dim: $f_{\bar{x}}(x_1, \dots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \exp\left[-\frac{1}{2}(\bar{x}-\bar{\mu})^T \Sigma^{-1} (\bar{x}-\bar{\mu})\right]$, where

$\sqrt{(\bar{x}-\bar{\mu}) \Sigma^{-1} (\bar{x}-\bar{\mu})}$ is the Mahalanobis distance

k=2

$\boxed{\text{YOUR TASK}}$

2-DIM Case:

$$(*) \quad f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right],$$

where $\bar{\mu} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$, $\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$

TASK: Prove that the k-DIM Gaussian reduces to (*) when k = 2.

$\rho = \rho_{xy} = \frac{\operatorname{cov}(X,Y)}{\sigma_x\sigma_y}$

NB! $\bar{x} = \vec{x} = \mathbf{x}$

2-Dim case

$$f(x,y) = \frac{1}{\sqrt{(2\pi)^k \, |\underset{1}{\Sigma}|}} \exp\underbrace{\left[-\frac{1}{2} \left(\bar{x} - \bar{\mu}_y \right)^T \cdot \underset{2}{\Sigma}^{-1} \left(\bar{x} - \bar{\mu} \right) \right]}_{3}$$

Determinant of covariance matrix:

1 $$|\Sigma| = \begin{vmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho \sigma_y \sigma_x & \sigma_y^2 \end{vmatrix} = \sigma_x^2 \sigma_y^2 - \rho^2 \sigma_x^2 \sigma_y^2$$

$$= \underline{\sigma_x^2 \sigma_y^2 (1 - \rho^2)}$$

Inverse of the co.var. matrix:

2 $$\Sigma^{-1} = \frac{1}{\mathrm{Det}(\Sigma)} \Sigma' = \underline{\frac{1}{\sigma_x^2 \sigma_y^2 (1-\rho^2)} \begin{bmatrix} \sigma_y^2 & -\rho \sigma_x \sigma_y \\ -\rho \sigma_y \sigma_x & \sigma_x^2 \end{bmatrix}}$$

Calculating exp. part only:

3 $$-\frac{1}{2} \begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix}^T \cdot \Sigma^{-1} \begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix}$$

$$-\frac{1}{2} \begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix}^T \cdot \underbrace{\frac{1}{\sigma_x^2 \sigma_y^2 (1-\rho^2)} \begin{bmatrix} \sigma_y^2 & -\rho \sigma_x \sigma_y \\ -\rho \sigma_y \sigma_x & \sigma_x^2 \end{bmatrix}}_{\Sigma^{-1}} \begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix}$$

$$-\frac{1}{2}\begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix} \cdot \frac{1}{\sigma_x^2\sigma_y^2(1-\rho^2)} \begin{bmatrix} \sigma_y^2(x-\mu_x)-(\rho\sigma_x\sigma_y)(y-\mu_y) \\ \rho\sigma_y\sigma_x(x-\mu_x)+\sigma_x^2(y-\mu_y) \end{bmatrix}$$

multiply

minus

$$-\frac{1}{2\sigma_x^2\sigma_y^2(1-\rho^2)}\left[\sigma_y^2(x-\mu_x)^2-\cancel{(\rho\sigma_x\sigma_y)(y-\mu_y)(x-\mu_x)}+\cancel{\rho\sigma_y\sigma_x(x-\mu_x)(y-\mu_y)} + \sigma_x^2(y-\mu_y)^2\right]$$

$$=\frac{1}{2\sigma_x^2\sigma_y^2(1-\rho^2)}\left[\sigma_y^2(x-\mu_x)^2+\sigma_x^2(y-\mu_y)^2-2\rho\sigma_x\sigma_y(x-\mu_x)(y-\mu_y)\right]$$

$$=\frac{1}{2(1-\rho^2)}\left[\frac{\cancel{\sigma_y^2}(x-\mu_x)^2}{\sigma_x^2\cancel{\sigma_y^2}}+\frac{\cancel{\sigma_x^2}(y-\mu_y)^2}{\sigma_y^2\cancel{\sigma_x^2}}-\frac{2\rho\cancel{\sigma_x}\cancel{\sigma_y}(x-\mu_x)(y-\mu_y)}{\sigma_x^{\cancel{2}}\sigma_y^{\cancel{2}}}\right]$$

(3)

$$=\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]$$

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left[\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{y-\mu_y^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right.\right.$$

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{(1-\rho^2}} \exp\left[\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_x{}^2}{\sigma_x{}^2} + \frac{y-\mu_y{}^2}{\sigma_y{}^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right)$$