Signals and Systems: Module 3

Suggested Reading: S&S 1.3.2, 1.3.3, 1.4.1

Discrete Time Complex Exponential

In the last module, we discussed the complex exponential in continuous time (CT). We will start off here by studying the discrete-time (DT) version, which has important similarities and differences.

The definition of a DT complex exponential signal is as follows:

$$x[n] = C\alpha^n,$$

where C, α are complex numbers. Notice here that the base is α, rather than e as in the CT case, which is common for DT signals. We can readily express the DT case in a form that is more analogous to CT case:

$$x[n] = Ce^{\beta n}, \qquad \text{where } \alpha = e^{\beta}.$$

Importantly, in this substitution, note that β must in general be a complex number. Even for a purely real α, if $\alpha < 0$, then β must be complex, as we need a complex exponential for e^{β} to take negative values.

To study this signal, similar to the CT case, we write

$$\begin{aligned} C &= |C|e^{j\phi} \text{ (polar form)} \\ \beta &= \sigma + j\omega \text{ (rectangular form)} \end{aligned}$$

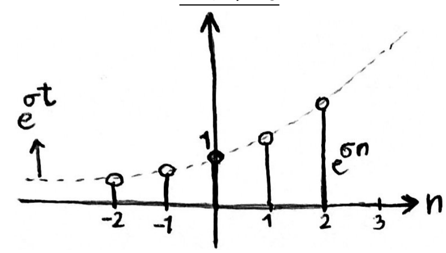
and we decompose it into the product of three terms:

$$x[n] = \underbrace{|C|}_{\text{Term 1}} \cdot \underbrace{e^{\sigma n}}_{\text{Term 2}} \cdot \underbrace{e^{j(\omega n + \phi)}}_{\text{Term 3}}.$$

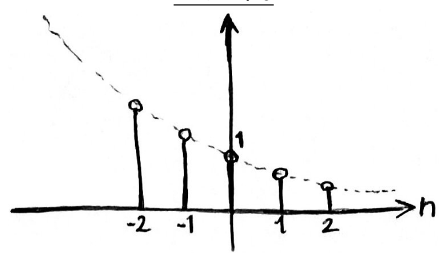
- **Term 1**: $x_1[n] = |C|$ simply scales the signal, as we saw in the CT case.

- **Term 2**: $x_2[n] = e^{\sigma n}$ is the discrete version of the real exponential signal we saw in the CT case. This provides the DT envelope for $x[n]$. Below, we give plots of $e^{\sigma n}$ for different ranges of σ. We show the CT signal $e^{\sigma t}$ (dotted curve) for auxiliary purposes to visualize the exponential envelope being discretized at each integer n.

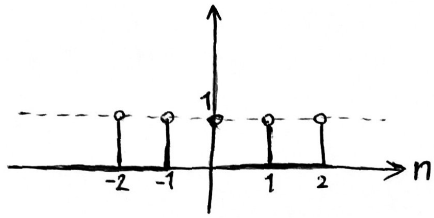
If $\sigma > 0$



If $\sigma < 0$



If $\sigma = 0$

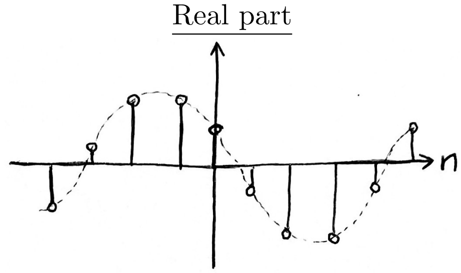


- **Term 3**: $x_3[n] = e^{j(\omega n+\phi)}$ is a complex exponential signal, whose real and imaginary parts are:

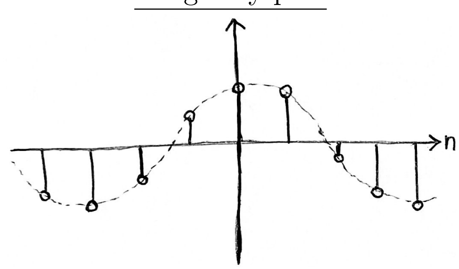
$$e^{j(\omega n+\phi)} = \underbrace{\cos(\omega n+\phi)}_{\text{real}} + j\underbrace{\sin(\omega n+\phi)}_{\text{imaginary}}$$

We can visualize the real and imaginary parts in separate plots below. Again, we tend to rely on the CT version of the signals (dotted curves) as an auxiliary to visualize the trend in the DT signals.

Real part



Imaginary part



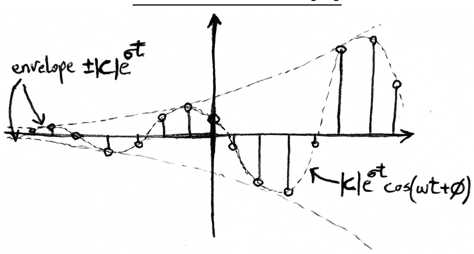
Since n can only take integer values, these plots are implicitly assuming ω is not "too large." Specifically, consecutive values of n, i.e., from n to $n+1$, change the input argument of the sinusoids by ω. So, if $\omega = \pi$, for example, we would only have two datapoints per period 2π of the continuous signal. The plot above is showing 8 values of n in each 2π, so we must have $\omega \approx \pi/4$.

The DT complex exponential signal $x[n]$ is then given as the multiplication of the three terms. The real and imaginary parts of $x[n]$, $x_{Re}[n]$ and $x_{Im}[n]$ are thus:

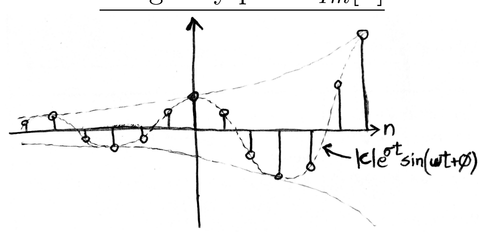
$$x_{Re}[n] = |C|e^{\sigma n}\cos(\omega n + \phi),$$
$$x_{Im}[n] = |C|e^{\sigma n}\sin(\omega n + \phi),$$

where $x[n] = x_{Re}[n] + jx_{Im}[n]$. We can plot these components as follows, assuming $\sigma > 0$:

Real part $x_{Re}[n]$



Imaginary part $x_{Im}[n]$



Note that we use two CT auxiliary functions to help plot this DT signal, both of which were components of our plot in the CT exponential case:

1. The exponential envelope $\pm|C|e^{\sigma t}$.

2. The CT complex exponential $|C|e^{\sigma t}e^{j(\omega t+\phi)}$, whose magnitude is guided by the exponential envelope.

The DT signal $x[n]$ is then obtained as **discrete samplings** of the CT complex exponential, i.e., for integer values of t.

While there are clearly many similarities between CT and DT signals in general, there are also several important differences. We are now at a juncture to explore some of those, particularly for the case of the periodic complex exponential signal. Recall that for the CT version $x(t) = e^{j\omega t}$, we previously identified the following properties:

- **Periodicity.** The fundamental frequency of $x(t)$ is ω, and $x(t)$ is periodic for any ω.
- **Rate of oscillation.** The larger the magnitude of ω, the higher the rate of oscillation is in the signal.

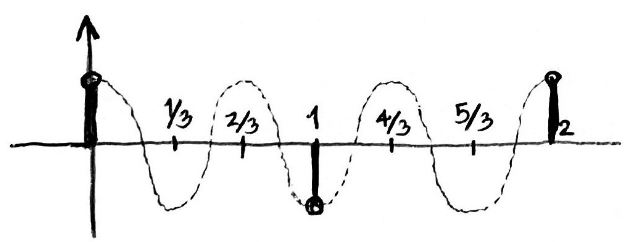
We will now search for the DT versions of both these properties. As we will see, there are definite differences between the DT and CT counterparts.

Periodicity in Discrete Time

Suppose we want to compute the fundamental period of $x[n] = \cos(3\pi n)$. If we were to consider the CT version of this signal,

$$x(t) = \cos(3\pi t) \Rightarrow \text{fund. per.} = T_0 = \frac{2\pi}{3\pi} = \frac{2}{3}$$

This means the auxiliary curve on the plot has period $\frac{2}{3}$:



But, $x[n]$ only samples the curve at integer points. As a result, the fundamental period of $\cos(3\pi n)$ is not 2/3. Instead, it is the **least common multiple** (LCM) of the fundamental period of the CT signal ($\frac{2}{3}$) and the sampling period (1). Therefore, the fundamental period of $\cos(3\pi n)$ is $N_0 = \text{LCM}\left(\frac{2}{3}, 1\right) = 2$, which is not the same as the CT counterpart.

This shows how to obtain the fundamental period of a single DT complex exponential signals. In general, for $x[n] = e^{j\omega_0 n}$, the fundamental period is

$$N_0 = \text{LCM}\left(T_0 = \frac{2\pi}{\omega_0}, 1\right)$$

How about a summation of signals?

Example 1. *Determine the fundamental period of the DT signal*

$$x[n] = e^{j(\frac{2\pi}{3})n} + e^{j(\frac{3\pi}{4})n}$$

Ans: We first determine the fundamental periods of each exponential.

① *Determine the fundamental period of* $e^{j(\frac{2\pi}{3})n}$

Approach 1:	*The fundamental period of the CT signal* $e^{j(\frac{2\pi}{3})t}$ *is* 3. *Thus, the fundamental period of the DT signal* $e^{j(\frac{2\pi}{3})n}$ *is* $\mathrm{LCM}\,(3,1) = 3$. *Sometimes, they match.*
Approach 2: (intuition)	*The angle* $\frac{2\pi}{3}n$ *must be incremented by a multiple of* 2π *for the values of this exponential to repeat. For the angle to be incremented by a single multiple of* 2π, n *must be incremented by* 3.

② *Determine the fundamental period of* $e^{j(\frac{3\pi}{4})n}$

Approach 1:	*The fundamental period of* $e^{j(\frac{3\pi}{4})t}$ *is* $\frac{2\pi}{(\frac{3\pi}{4})} = \frac{8}{3}$. *The fundamental period of* $e^{j(\frac{3\pi}{4})n}$ *is then* $\mathrm{LCM}\,(\frac{8}{3}, 1) = 8$.
Approach 2: (intuition)	*Incrementing the angle* $(\frac{3\pi}{4})n$ *by* 2π *would require* n *to be incremented by* $\frac{8}{3}$, *which is impossible. Incrementing the angle* $(\frac{3\pi}{4})n$ *by* 4π *would require* n *to be incremented by* $\frac{16}{3}$, *which is also impossible. But incrementing the angle* $(\frac{3\pi}{4})n$ *by* 6π *is possible: it requires* n *to be incremented by* 8*! Thus, the fundamental period of* $e^{j(\frac{3\pi}{4})n}$ *is* 8.

For the entire signal $x[n]$ *to repeat, each of the terms must go through an integer number if its own fundamental period. The smallest increment of* n *that accomplishes this is* $\mathrm{LCM}\,(3, 8) = 24$. *Thus, the fundamental period of the DT* $x[n]$ *is* $N_0 = 24$, *which in this case happens to be the same as the corresponding CT signal, i.e.,* $T_0 = 24$.

Thus, in general, suppose $x[n] = x_1[n]+x_2[n]$, where x_1 and x_2 are periodic. Once we have found the fundamental periods of x_1 and x_2, to be say N_1 and N_2, we find the fundamental period of $x[n]$ using the same procedure as in CT: $N = \mathrm{LCM}\,(N_1, N_2)$. The difference from CT lies in the determination of N_1 and N_2.

Sometimes, a DT complex exponential is not periodic, even though its CT version will always be!

Example 2. *Is e^{jn} periodic?*

Ans: The CT signal $e^{jt} = \cos t + j \sin t$ has period 2π. If e^{jn} were to repeat, then its period must be $\mathrm{LCM}\,(2\pi, 1)$ for the 2π to get realigned with integers. As 2π is irrational, $\mathrm{LCM}\,(2\pi, 1)$ does not exist. Therefore, e^{jn} is aperiodic.

So, if the fundamental period of a CT complex exponential $x(t)$ is irrational, then its DT version $x[n]$ will be aperiodic, since the values of $x(t)$ at integer sample points $t = n$ never repeat. On the other hand, if the CT periodic complex exponential has a *rational* fundamental period, its DT version is periodic, and vice versa.

Based on the above examples, we conclude the following regarding periodicity of DT signals:

- The fundamental period N_0 of a DT signal is always an integer. In fact, this is true for all periodic DT signals, not limited to the DT complex exponential.
- The fundamental frequency of a DT signal is always

  $$\omega_0 = \frac{2\pi}{N_0} = \frac{2\pi}{\text{integer}}.$$

  To determine ω_0, we typically need to find N_0 first.

Rate of Oscillation

For the signal $x[n] = e^{j\omega n}$, consider what happens when we add 2π to ω, to get a signal with frequency $\omega + 2\pi$:

$$e^{j(\omega+2\pi)n} = e^{j2\pi n} e^{j\omega n} = e^{j\omega n},$$

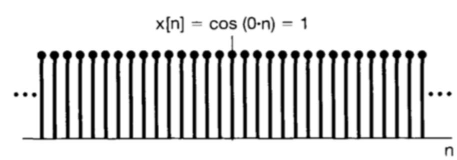
since $e^{j2\pi n} = 1$ for any integer n. In other words, the exponential at frequency $\omega + 2\pi$ is no different than the one at ω. This is very different than the continuous-time case, where signals $e^{j\omega t}$ are all distinct for different ω. For discrete-time complex exponentials, we actually need only consider a frequency interval of length 2π in which to choose ω; in most cases, we will stick to $0 \leq \omega < 2\pi$ or $-\pi \leq \omega < \pi$.

So, increasing ω in $e^{j\omega n}$ does not continually increase the rate of oscillation. Rather, over the interval $0 \leq \omega < 2\pi$, as we increase from ω, the rate of oscillation will increase until $\omega = \pi$, and then it will actually *decrease* until $\omega = 2\pi$, at which point it returns to 0. Signals with lower rates of oscillation have values of ω near 0 and 2π, while higher rates of oscillation are located near $\pm\pi$.

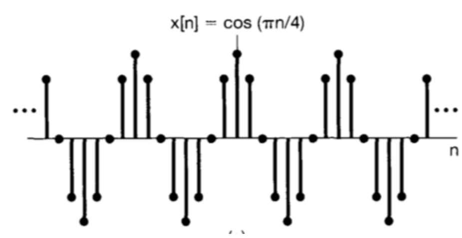
In the figure below, we show how the frequency of variation changes for different values of ω, using $x[n] = \cos(\omega n)$. The comparison between $\omega = \pi/4$ and $\omega = 7\pi/4$ here demonstrates a very interesting and useful property: *the rate of oscillation is perfectly symmetric about* $\omega = \pi$, i.e.,

$$\omega = \pi + \pi x \quad \text{and} \quad \omega = \pi - \pi x$$

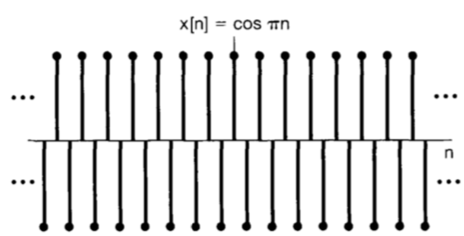
give the same oscillation rate for any $0 \leq x \leq 1$. The difference in $e^{j\omega n}$ between the two frequencies $\omega = \pi + \pi x$ and $\omega = \pi - \pi x$ lies purely in the sign of the imaginary component $\sin(\omega n)$.



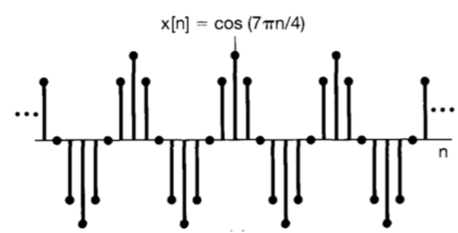
(a) $\omega_0 = 0$



(b) $\omega_0 = \pi/4$



(c) $\omega_0 = \pi$



(d) $\omega_0 = 7\pi/4$

Note also that when $\omega = \pi$ or any odd multiple of π, $e^{j\pi n} = (e^{j\pi})^n = (-1)^n$.

DT Harmonically Related Complex Exponentials

Similar to in CT, we can define a set of DT harmonically related complex exponentials (HRCEs). The set of HRCEs with a common period N is defined as

$$x_k[n] = e^{jk(\frac{2\pi}{N})n}, \qquad k = 0, \pm 1, \pm 2, \cdots \text{ (any integers).}$$

But in discrete-time, not all of these exponentials will be distinct. In particular, given an integer N, there are exactly N distinct HRCEs. To see this, note that for an arbitrary integer k,

$$x_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n} = e^{jk(\frac{2\pi}{N})n} \cdot e^{j2\pi kn} = x_k[n],$$

which implies there are only N distinct periodic exponentials in the set. We therefore write our set of distinct HRCEs as

$$x_0[n] = 1, x_1[n] = e^{j(\frac{2\pi}{N})n}, x_2[n] = e^{j(\frac{4\pi}{N})n}, \cdots, x_{N-1} = e^{j\frac{2\pi(N-1)}{N}n},$$

where the fundamental period and fundamental frequency of signal $x_k[n]$ (except for $k = 0$) are

$$N_k = \text{LCM}(N/k, 1) \quad \text{and} \quad \omega_k = 2\pi/N_k,$$

respectively. When N is even, N_k is going to be minimized for $k = N/2$, giving $N_k = 2$ and the maximum fundamental frequency π.

Recall that for CT HRCEs $x_k(t) = e^{jk\omega_0 t}$, there are infinite number of signals in the family given ω_0. This is another major difference between CT and DT signals.

The *DT HRCE Example for N=6* supplementary PDF on Brightspace does a deep dive into an HRCE family.

Summary of Differences

The following table summarizes the key differences of uniqueness and periodicity between the CT $e^{j\omega_0 t}$ and DT $e^{j\omega_0 n}$ signals. m and N must both be integers and should be reduced to their simplest form. Above all, these differences stem from the fact that t is continuous, while n is discrete.

Property	$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Uniqueness	Each ω_0 is a distinct signal	Only distinct for ω_0 in a range of 2π
Periodicity	Periodic for any choice of ω_0	Periodic only if $\omega_0 = 2\pi m/N$
Fund. freq.	ω_0	ω_0/m
Fund. per.	$\omega_0 = 0$: undefined $\omega_0 \neq 0$: $2\pi/\omega_0$	$\omega_0 = 0$: undefined $\omega_0 \neq 0$: $\mathrm{LCM}\,(2\pi/\omega_0, 1)$

Example 3. *Consider the DT signals $x[n] = \cos(8\pi n/31)$ and $x[n] = \cos(n/6)$. Are these periodic? If so, what are their fundamental frequencies? How about the DT versions $x(t) = \cos(8\pi t/31)$ and $x(t) = \cos(t/6)$?*

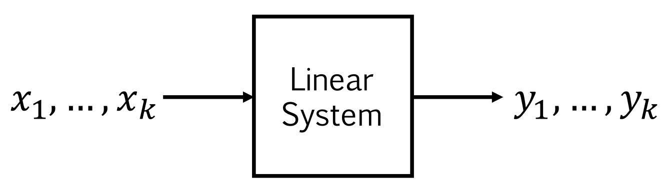
Ans: First of all, both CT versions are $x(t)$ are periodic, because these are CT sinusoids. The fundamental frequencies for $x(t) = \cos(8\pi t/31)$ and $x(t) = \cos(t/6)$ are $\omega_0 = 8\pi/31$ and $\omega_0 = 1/6$, and the fundamental periods are $T_0 = 31/4$ and $T_0 = 12\pi$.

Now, consider $x[n] = \cos(8\pi n/31)$. If we want to work in terms of frequency, we can try to write the CT $\omega_0 = 8\pi/31 = 2\pi m/N$ for the smallest integers m, N. Then, the fundamental frequency of the DT version is $2\pi/N$, and the fundamental period is N. Since the fraction here is already reduced, with both $m = 4$ and $N = 31$ being integers, $x[n]$ is periodic with fundamental frequency $\omega_0 = 2\pi/31$ and period $N_0 = 4 \times 31/4 = 31$.

As for $x[n] = \cos(n/6)$, we would need $\omega_0 = 1/6 = 2\pi m/N$, or $m/N = 1/12\pi$ for integers m and N. Since π is irrational, this is not possible, and hence $x[n]$ is aperiodic.

Why are we interested in HRCEs?

In this course, we will take particular interest in linear systems. For a linear system, given some "test signals" $x_1, x_2, \cdots, x_k$,



we know that a new input signal x linearly represented using the test signals as $x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$ will generate an output $y = \alpha_1 y_1 + \alpha_2 y_2 +$

$\cdots + \alpha_k y_k$.

HRCEs are good candidates for test signals when the input signal x is periodic. In CT, this is the set of signals $x_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \cdots$. As we will see when we study Fourier series, any periodic input signal $x(t)$ with a fundamental frequency ω_0 can be represented as $x(t) = \sum_{k=-\infty}^{\infty} \alpha_k e^{jk\omega_0 t}$ for some constants α_k. We can therefore determine $y(t)$ by looking at the impact the system has on each HRCE $e^{jk\omega_0 t}$. In general, we are considering an infinite number of such component signals, as k can go to infinity.

In DT, on the other hand, the set of test signals is finite: $x_k[n] = e^{jk\frac{2\pi}{N}n}, k = 0, \cdots, N-1$. As we will see, any periodic input signal $x[n]$ with fundamental period N can be represented as $x[n] = \sum_{k=0}^{N-1} \alpha_k e^{jk\frac{2\pi}{N}n}$. We can therefore determine $y[n]$ by looking at the impact the system has on each HRCE $e^{jk\frac{2\pi}{N}n}$.

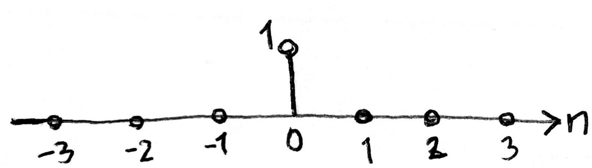
Discrete Time Unit Impulse and Unit Step Signals

We will now study two other basic signals – the unit impulse and step functions – that are also of considerable importance in signal and system analysis. We start here by looking at them in DT, and will consider the CT cases next.

One of the simplest DT signals is the discrete-time **unit impulse** (or **unit sample**), which is defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

and visualized as follows:

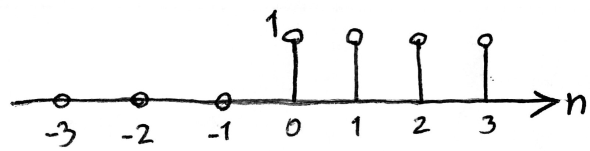


In other words, $\delta[n]$ is 1 at exactly $n = 0$ and 0 everywhere else. $\delta[n]$ is sometimes referred to as the **Kronecker delta** (after Leopold Kronecker).

A second basic DT signal is the discrete-time **unit step**, denoted by $u[n]$

and defined by

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$



In other words, it "steps" up to 1 at $n = 0$, and is 0 before that.

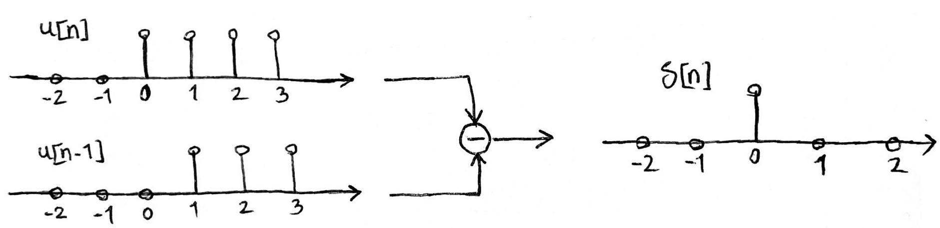
Why are we interested in studying the unit impulse and unit step signals? Similar to HRCEs, they provide another family of signals that are a convenient choice for test signals in linear systems (usually for aperiodic signals).

Properties of $\delta[n]$ and $u[n]$

There is a close relationship between the discrete-time unit impulse and unit step. In particular, the discrete-time unit impulse is the **first difference** of the discrete-time step:

$$\delta[n] = u[n] - u[n-1].$$

Visually:



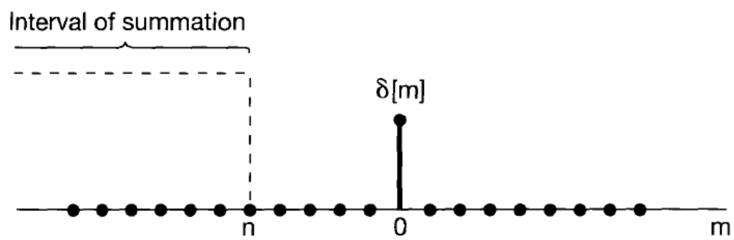
Conversely, the discrete-time unit step is the running sum of the unit sample:

$$u[n] = \sum_{m=-\infty}^{n} \delta[m].$$

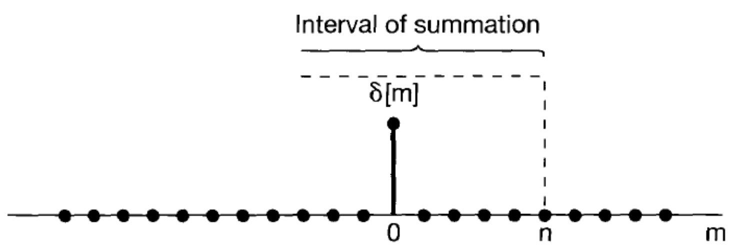
There are two useful interpretations of these relationships:

- **Interpretation 1.** The above equation is illustrated graphically in the figure below. Since the only nonzero value of the unit sample is at the point at which its argument is zero, we see from the figure that the running sum in the equation above is 0 for $n < 0$ and 1 for $n \geq 0$.

  $\underline{n < 0}$: $u[n] = 0$

  $\underline{n \geq 0}$: $u[n] = 1$

- **Interpretation 2.** By changing the variable of summation in the expression from m to $k = n - m$, the original summation

  $$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

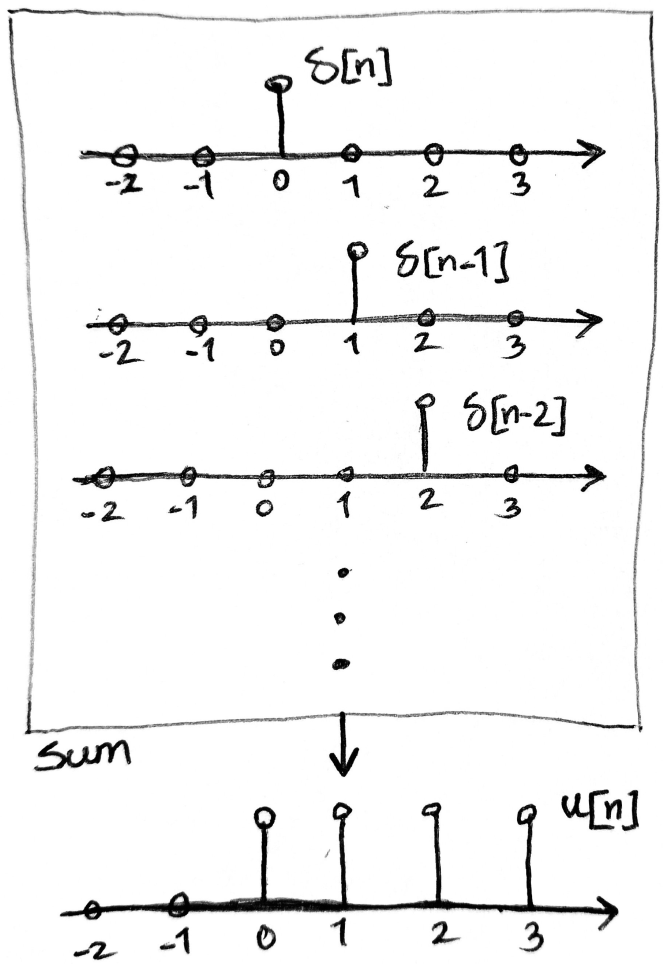
  becomes

  $$u[n] = \sum_{k=\infty}^{0} \delta[n-k]$$

  or equivalently,

  $$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

This indicates that $u[n]$ is the superposition of delayed impulses, as visualized below. More generally, the interpretation of signals as superpositions of (weighted) impulse signals will be useful to us later when we look to find outputs of systems in terms of these test inputs.



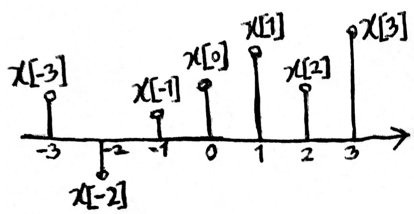
Sampling Property

The unit impulse sequence can be used to **sample** the value of a signal at $n = 0$. In particular, since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n] = \begin{cases} x[0] & n = 0 \\ 0 & n \neq 0 \end{cases}$$

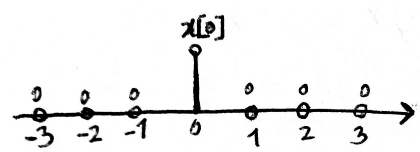
Visually:

$x[n]$

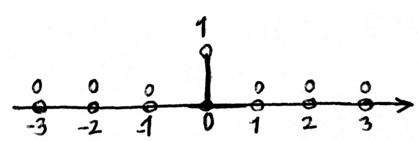


$x[n]\delta[n]$

$\rightarrow$



$\delta[n]$



This is the sampling of $x[n]$ at $n = 0$.

More generally, if we consider a shifted unit sample $\delta[n - n_0]$, then

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0] = \begin{cases} x[n_0] & n = n_0 \\ 0 & n \neq n_0 \end{cases}$$

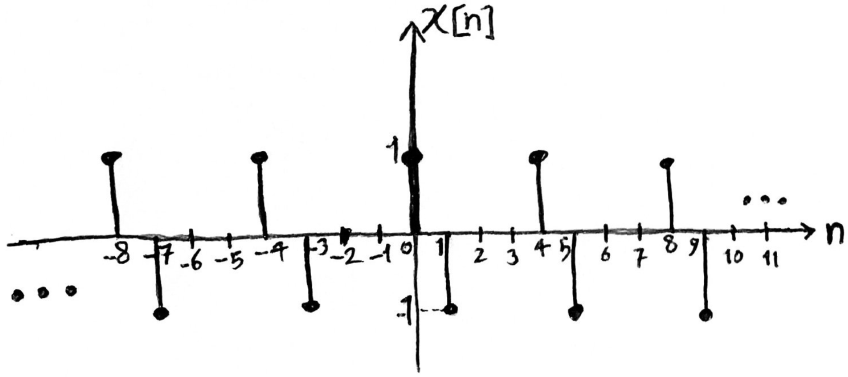
Therefore, we can decompose $x[n]$ as a weighted sum of unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k].$$

As mentioned above, these shifted unit impulses $\delta[n-k]$ are good test signals for linear systems.

Example 4. *Sketch the signal* $x[n] = \sum_{k=-\infty}^{\infty} \{\delta[n - 4k] - \delta[n - 4k - 1]\}$. *Is* $x[n]$ *periodic?*

Ans:



From the sketch, we see that the signal is periodic with a period $N = 4$. We can also show it mathematically as follows:

$$\begin{aligned} x[n+N] &= \sum_{k=-\infty}^{\infty} \{\delta[n+N-4k] - \delta[n+N-4k-1]\} \\ &= \sum_{k=-\infty-N/4}^{\infty-N/4} \{\delta[n+N-4(k+N/4)] - \delta[n+N-4(k+N/4)-1]\} \\ &= \sum_{k=-\infty}^{\infty} \{\delta[n-4k] - \delta[n-4k-1]\} = x[n] \end{aligned}$$

in which after writing the representation of $x[n+N]$, we add/subtract an integer from k to see if we can get back to the original representation of $x[n]$. The value to add which removes the effect of N is $N/4$. Therefore, the only condition for periodicity that we have is for $N/4$ to be an integer. This means that the fundamental period of the signal is 4.

Example 5. *Decompose the signal $x[n] = (1/2)^{-n}\{u[n+4] - u[n+1]\}$ in terms of its even and odd components.*

Ans: Based on the relationship between $u[n]$ and $\delta[n]$, we have

$$\begin{aligned} x[n] &= \left(\frac{1}{2}\right)^{-n} \{\delta[n+4] + \delta[n+3] + \delta[n+2]\} \\ &= (1/16)\delta[n+4] + (1/8)\delta[n+3] + (1/4)\delta[n+2] \end{aligned}$$

We want to express $x[n] = x_{even}[n] + x_{odd}[n]$. For the even component,

$$\begin{aligned}
x_{even}[n] &= \frac{1}{2}\left(x[n] + x[-n]\right) \\
&= \frac{1}{2}\big((1/2)^{-n}\left\{\delta[n+4] + \delta[n+3] + \delta[n+2]\right\} \\
&\qquad + (1/2)^{n}\left\{\delta[-n+4] + \delta[-n+3] + \delta[-n+2]\right\}\big) \\
&= (1/32)\delta[n+4] + (1/16)\delta[n+3] + (1/8)\delta[n+2] \\
&\qquad + (1/32)\delta[n-4] + (1/16)\delta[n-3] + (1/8)\delta[n-2]
\end{aligned}$$

where in the last line, we used the fact that $\delta[n_0 - n] = \delta[n - n_0]$. We can also write this in terms of the unit step as:

$$x_{even}[n] = (1/2)^{-n+1}\left\{u[n+4] - u[n+1]\right\} + (1/2)^{n+1}\left\{u[n-2] - u[n-5]\right\}.$$

Next, we obtain $x_{odd}[n]$ in a similar way:

$$\begin{aligned}
x_{odd}[n] &= \frac{1}{2}\left(x[n] - x[-n]\right) \\
&= \frac{1}{2}\big((1/2)^{-n}\left\{\delta[n+4] + \delta[n+3] + \delta[n+2]\right\} \\
&\qquad - (1/2)^{n}\left\{\delta[-n+4] + \delta[-n+3] + \delta[-n+2]\right\}\big) \\
&= (1/32)\delta[n+4] + (1/16)\delta[n+3] + (1/8)\delta[n+2] \\
&\qquad - (1/32)\delta[n-4] - (1/16)\delta[n-3] - (1/8)\delta[n-2] \\
&= (1/2)^{-n+1}\left\{u[n+4] - u[n+1]\right\} - (1/2)^{n+1}\left\{u[n-2] - u[n-5]\right\}
\end{aligned}$$

The following are plots of $x[n]$ and its even-odd decomposition: