Signals and Systems: Module 6

Suggested Reading: S&S 3.1-3.5

If there was one theme from the previous module, it would be that the input-output relationship of LTI systems can be fully specified by the impulse response. In this lecture, we will begin our exploration of alternative representations of LTI systems and signals that give us another extremely useful tool for analysis. These so-called **Fourier methods** – bearing the name of the 18th century mathematician Jean Baptise Joseph Fourier – are based on the concept of frequency, i.e., that time-varying signals can be decomposed into frequency contents.

Complex Exponentials and LTI Systems

Fourier series – and Fourier analysis generally – is based on complex exponential signals. To motivate the use of such signals, we will first consider the response of LTI systems to these signals. We will find that complex exponentials are, mathematically, **eigenfunctions** of LTI systems, meaning that the output when such a signal is applied is simply the input scaled by a constant.

In particular, consider an LTI system with an impulse response $h(t)$, and suppose we apply the periodic complex exponential signal $x(t) = e^{j\omega t}$. By the convolution integral, the output $y(t)$ is

$$\begin{aligned} y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} e^{j\omega\tau} h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{jwt}e^{-j\omega\tau}d\tau = e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \end{aligned}$$

since t is a constant with respect to the integration. So, the response is of the form

$$y(t) = H(j\omega)e^{j\omega t} \qquad \text{where} \qquad H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

as long as the integral converges. Importantly, notice that $x(t)$ is changed *exactly* by $H(j\omega)$, which depends *only* on ω. Since $H(j\omega)$ is in general complex, we can write it more explicitly in terms of magnitude and phase:

$$y(t) = e^{j\omega t}\left(|H(j\omega)| \cdot e^{j\angle H(j\omega)}\right)$$

It follows that $y(t)$ will be of the same frequency ω as $x(t)$, with the only change being its amplitude (amplified by $|H(j\omega)|$) and phase (shifted by $\angle H(j\omega)$).

Example 1. *Consider an LTI system with impulse response $h(t) = e^{-t}u(t)$. What is the output $y(t)$ for an input $x(t) = \cos(t)$?*

Ans: We know that $\cos(t) = \frac{1}{2}(e^{jt} + e^{-jt})$. Since it is an LTI system, we can consider the responses to $x(t) = e^{j\omega t}$ for $\omega = 1, -1$. We have

$$\begin{aligned} H(j\omega) &= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-\tau}e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-(1+j\omega)\tau}d\tau \\ &= \frac{1}{1+j\omega} = \frac{1-j\omega}{1+\omega^2} \end{aligned}$$

So, when $x_1(t) = e^{jt}$, the output is

$$y_1(t) = e^{jt}H(j\omega)|_{\omega=1} = e^{jt}\left(\frac{1-j}{2}\right) = e^{jt}\cdot\left(\frac{1}{\sqrt{2}}e^{j(-\frac{\pi}{4})}\right)$$

Likewise, for $x_2(t) = e^{-jt}$, the output is

$$y_2(t) = e^{-jt}H(j\omega)|_{\omega=-1} = e^{-jt}\left(\frac{1+j}{2}\right) = e^{-jt}\cdot\left(\frac{1}{\sqrt{2}}e^{j(\frac{\pi}{4})}\right)$$

Since the system is linear, by superposition, we have $y(t) = \frac{1}{2}(y_1(t) + y_2(t))$. Therefore,

$$y(t) = \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\left(e^{j(t-\frac{\pi}{4})} + e^{-j(t-\frac{\pi}{4})}\right) = \frac{1}{\sqrt{2}}\cos\left(t - \frac{\pi}{4}\right)$$

Example 1 suggests that due to Euler's identity, sinusoidal signals will also be eigenfunctions of LTI systems. In other words, when a sinusoid is input to an LTI system, will the output be a version of this sinusoid with a new amplitude and phase, but at the same frequency? The answer is *yes, but only when the impulse response is a real signal.* If $h(t)$ has imaginary components, $H(j\omega)$ may not have a symmetric part for the frequencies $\pm\omega$.

Example 2. *For the same impulse response in Example 1, find $y(t)$ when $x(t) = \cos(\sqrt{3}t)$.*

Ans: In this case, $\omega = \sqrt{3}$, and $h(t)$ is real, so

$$H(j\omega)\big|_{\omega=\sqrt{3}} = \frac{1 - j\sqrt{3}}{1+3} = \frac{1}{2}e^{j(-\frac{\pi}{3})} \to y(t) = \frac{1}{2}\cos(\sqrt{3}t - \frac{\pi}{3})$$

Compared to Example 1, we see that the attenuation at $\omega = \sqrt{3}$ is stronger than the attenuation at $\omega = 1$. The lower frequency components are stronger in the output. This is consistent with the fact that the system we are considering here is a lowpass filter.

More generally, suppose we can represent the input $x(t)$ to an LTI system as a linear combination of complex exponentials. In other words,

$$x(t) = \sum_k a_k e^{j\omega_k t} = a_1 e^{j\omega_1 t} + a_2 e^{j\omega_2 t} + \cdots$$

From the eigenfunction property of LTI systems, we know that $a_k e^{j\omega_k t} \to a_k H(j\omega_k) e^{j\omega_k t}$ for each k. Combined with the superposition property, it follows that the output will be

$$y(t) = \sum_k a_k H(j\omega_k) e^{j\omega_k t}$$

Representation of signals as linear combinations of complex exponentials, therefore, leads to a convenient expression for the response of an LTI system. With such a representation, we only need to determine $H(s_k)$, which unlike convolution does not require us to carry out the mathematical processes of time reversal and shifting. We will next study a technique that builds such a representation for *any* periodic signal.

Fourier Series in Continuous-Time

We have seen before that the complex exponential $x(t) = e^{j\omega_0 t}$ is periodic with fundamental frequency ω_0 and period $T = 2\pi/\omega_0$. We also defined the family of HRCEs for ω_0 as

$$x_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}, \qquad k = 0, \pm 1, \pm 2, ...$$

each of which has a fundamental frequency that is a multiple of ω_0, and importantly, each of which is periodic in T.

Consider an arbitrary CT signal $x(t)$ that is periodic with fundamental period T and frequency ω_0. The **Fourier series representation** of $x(t)$ expresses it in terms of the HRCEs as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_0 + a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t} + a_2 e^{2j\omega_0 t} + a_{-2} e^{-2j\omega_0 t} + \cdots$$

where $a_0, a_1, a_{-1}, a_2, a_{-2}, \ldots$ are the corresponding **Fourier series coefficients**. In the name "Fourier series representation," Fourier comes from the French mathematician. "Series" comes from the fact that we are taking a summation over HRCEs. "Representation" indicates that the original $x(t)$ is equivalently expressed by the series, except possibly at points of discontinuity. You have seen other series representations before, such as Taylor's expansion. The Fourier series representation is particularly useful for signals and systems because it is expressed in terms of our HRCE "test" signals, each a different term in the summation:

- The term for $k = 0$ is a constant, and so a_0 gives the DC component of the signal.
- The terms $k = \pm 1$ both have fundamental frequency of ω_0 and are collectively referred to as **fundamental components** or **first harmonic components**. a_1 and a_{-1} are the weights on these components.
- More generally, the components for $k = \pm K$ are the K**th harmonic components**, with coefficients a_K and a_{-K}.

Determining the Fourier Series Coefficients

Now, how to compute the a_k based on $x(t)$? We use the following general expression:

$$a_k = \frac{1}{T} \int_{\langle T \rangle} x(t) e^{-jk\omega_0 t} dt \ \ \forall k$$

where $\langle T \rangle$ is any time interval of length T. In other words, we can integrate over any single period of $x(t)$: $[0, T]$, $[-T/2, T/2]$, etc.

It is useful to understand where this formula comes from too. Multiplying the Fourier series expansion by $e^{-jn\omega_0 t}$ for an integer n and integrating from 0 to T, we have

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} \left[a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \right] dt$$

Interchanging the order of integration and summation, and applying Euler's formula, the right hand side becomes

$$\sum_{k=-\infty}^{\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t} dt\right] = \sum_{k=-\infty}^{\infty} a_k \left[\int_0^T \cos(k-n)\omega_0 t \; dt + j\int_0^T \sin(k-n)\omega_0 t \; dt\right]$$

This further simplifies quite nicely. Since $\cos m\omega_0 t$ and $\sin m\omega_0 t$ are periodic in T for any non-zero integer m, these integrals will be 0 except when $k = n$. When $k = n$, the integrand is 1, and so the full integral is T. Formally, we say that the HRCEs $x_k(t) = e^{jk\omega_0 t}$ are a set of **orthogonal functions** over the interval $[0, T]$.

What we are left with, then, is $\int_0^T x(t)e^{-jn\omega_0 t}dt = a_k T$, which leads to our initial formula: $a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$. Note also that in deriving this formula, we did not make use of the interval $[0, T]$ in particular, only that its length is T. It therefore suffices to integrate over *any* interval of length T to find a_k, because the HRCEs are orthogonal over any such interval.

The formula for a_k is sometimes called the Fourier series **analysis** equation, while the resulting expansion for $x(t)$ is called the Fourier series **synthesis** equation: it is the synthesis of $x(t)$ from the weighted sum of the harmonic components.

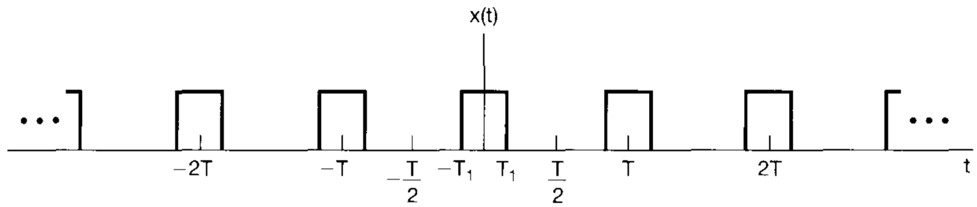
Example 3. *Find the Fourier series expansion of the **periodic square wave** with period T, which is defined over one period as*

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

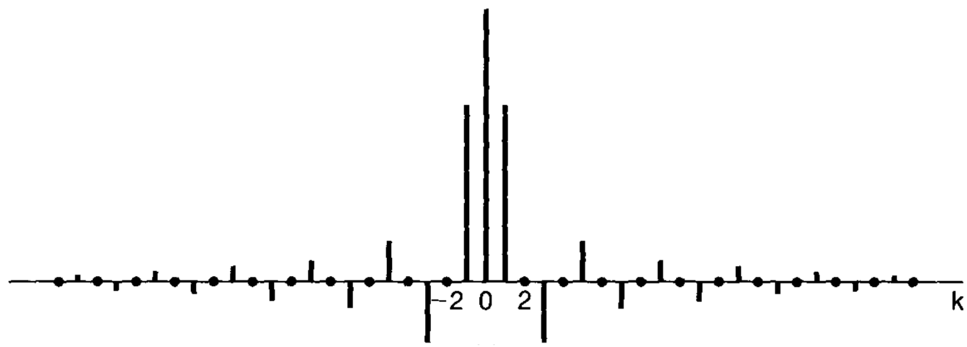
This is a signal we will encounter a number of times throughout the course. It is graphed in Figure 1(a).

Ans: While any interval of length T will suffice for integration, choosing $[-T/2, T/2]$ symmetric about the origin is most convenient in this case. The formula for a_k is

$$\begin{aligned} a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} \\ &= -\frac{1}{jk\omega_0 T}\left(e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right) = \frac{1}{k\omega_0 T}\left(\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{j}\right) \\ &= \frac{2}{k\omega_0 T}\sin(k\omega_0 T_1) = \frac{1}{\pi k}\sin(k\omega_0 T_1) \end{aligned}$$



(a) Square wave signal $x(t)$.



(b) a_k when $T_1 = T/4$, i.e., the signal is half on and half off.

Figure 1: Periodic square wave signal $x(t)$ and its Fourier coefficients a_k in Example 2.

where the last step follows since $\omega_0 T = 2\pi$. This formula is only valid for $k \neq 0$, though; for $k = 0$, we have to solve the integral separately:

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot dt = \frac{2T_1}{T}$$

Thus, the Fourier series expansion is

$$x(t) = \frac{2T_1}{T} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\sin(k\omega_0 T_1)}{k\pi} e^{j2\pi kt/T}$$

*It is often useful to plot the Fourier coefficients as a function of k, which gives us the **spectrum** of the signal $x(t)$. Since the a_k are purely real in this example, we only need one graph to plot the spectrum, whereas more generally we would need two graphs (magnitude/phase or real/imaginary).*

Suppose $T_1 = T/4$, so that $x(t)$ is a square wave that is "on" for half of T and "off" for the other half. In this case, the coefficients become

$$a_0 = \frac{1}{2} \qquad a_k = \frac{\sin(k\pi/2)}{k\pi}, k \neq 0$$

For a_k, note that $\sin(k\pi/2) = 0$ for k even and ± 1 for k odd. Therefore $a_1 = a_{-1} = 1/\pi$, $a_2 = a_{-2} = 0$, $a_3 = a_{-3} = -1/3\pi$, $a_4 = a_{-4} = 0$, $a_5 = a_{-5} = 1/5\pi$, and so on. This is plotted in Figure 1(b).

In certain cases, we can find the Fourier series coefficients by inspection or by manipulating Euler's identity rather than by direct computation through the formula. This tends to happen when the signal is given as a sum of sinusoids.

Example 4. *Find the Fourier series expansion of $x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$. Plot the corresponding frequency spectrum.*

Ans: Rather than computing a_k, we can apply Euler's formula here and convert the sinusoids to complex exponentials. We have

$$x(t) = 1+\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)+\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)+\frac{1}{2}\left(e^{j2\omega_0 t+j\pi/4} + e^{-j2\omega_0 t-j\pi/4}\right)$$

which we re-arrange to fit the Fourier series representation:

$$x(t) = 1+\left(1+\frac{1}{2j}\right)e^{j\omega_0 t}+\left(1-\frac{1}{2j}\right)e^{-j\omega_0 t}+\left(\frac{1}{2}e^{j\pi/4}\right)e^{2j\omega_0 t}+\left(\frac{1}{2}e^{-j\pi/4}\right)e^{-2j\omega_0 t}$$

From this, we can identify

$$a_0 = 1,\ a_1 = 1-\frac{j}{2},\ a_{-1} = 1+\frac{j}{2},\ a_2 = \frac{1}{2\sqrt{2}}+\frac{j}{2\sqrt{2}},\ a_{-2} = \frac{1}{2\sqrt{2}}-\frac{j}{2\sqrt{2}},$$

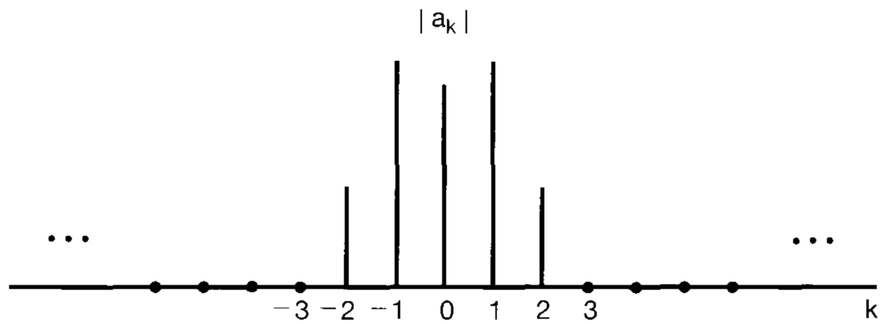
$$\text{and } a_k = 0 \text{ for } |k| > 2$$

Since the a_k are complex, we now need both a magnitude and a phase plot; the magnitude and phase for each a_k is obtained as:

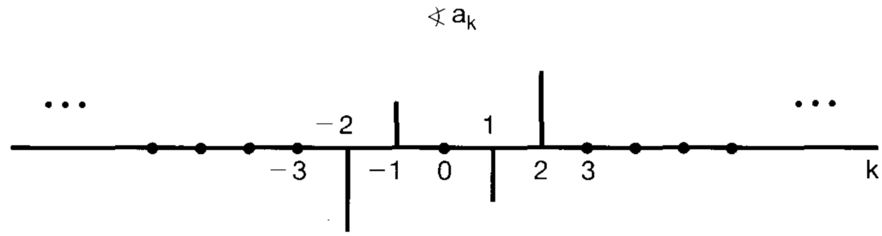
$$a_k = \alpha_k + j\beta_k \quad \rightarrow \quad |a_k| = \sqrt{\alpha_k^2+\beta_k^2}, \quad \theta_k = \arctan\beta_k/\alpha_k$$

So at $k = -2$, for example, $|a_{-2}| = \sqrt{2}/(2\sqrt{2}) = 1/2$ and $\theta_{-2} = \arctan(-1/1) = -\pi/4$. In solving for θ_k, we must preserve the signs of β_k and α_k after division to determine the right quadrant. We typically report $\theta_k \in (-\pi, \pi]$.

The resulting plot of a_k is given in Figure 2. Notice that the magnitude plot is an even function, while the phase plot is an odd function. This is the result of a_k and a_{-k} being complex conjugates, and will hold for any real signal $x(t)$.



(a) Magnitude plot $|a_k|$



(b) Phase plot θ_k

Figure 2: Plot of the complex coefficients a_k for the signal in Example 4.

When $x(t)$ is real, we can think of an alternative form of the Fourier series in terms of sinusoids, as Example 2 seems to indicate. To obtain this, we arrange the summation by grouping terms k and $-k$ as

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right]$$

In order for $x(t)$ to be real, we must have $a_{-k} = a_k^\star$ so that the imaginary components at frequency $k\omega_0$ will cancel. This can be verified by comparing the formulas for a_k and a_{-k}. In other words,

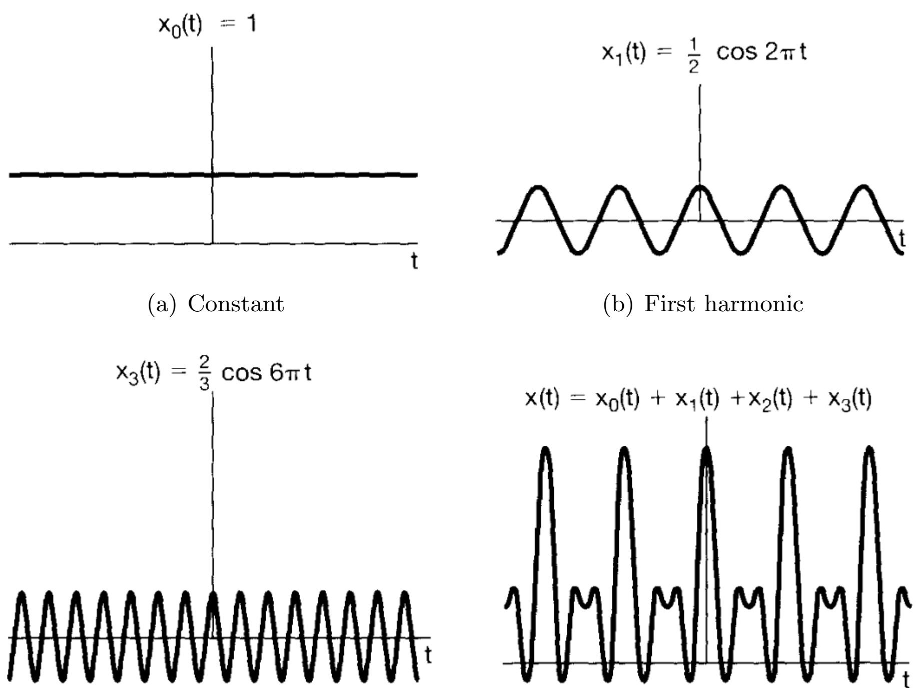
$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^\star e^{-jk\omega_0 t} \right] = a_0 + 2 \sum_{k=1}^{\infty} \mathrm{Re}\{a_k e^{jk\omega_0 t}\}$$

If we then express a_k in polar form as $a_k = A_k e^{j\theta_k}$, then by Euler's identity, since $\mathrm{Re}\{e^{j\theta}\} = \cos\theta$, the Fourier series of a real signal $x(t)$ can be expressed as

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \theta_k\right)$$

The complex exponential form is most convenient for our purposes, though, so we will use it almost exclusively.

Example 5. *Suppose we represent a periodic signal as* $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$, *where* $a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, $a_3 = a_{-3} = \frac{1}{3}$. *What are the harmonic components, expressed as sinusoids?*



(a) Constant

(b) First harmonic

(c) Third harmonic

(d) Signal

Figure 3: Constructing the signal in Example 1 as a linear combination of harmonically-related sinusoids.

There are three harmonic components, i.e., pairs of exponentials with the same fundamental frequency. Collecting them,

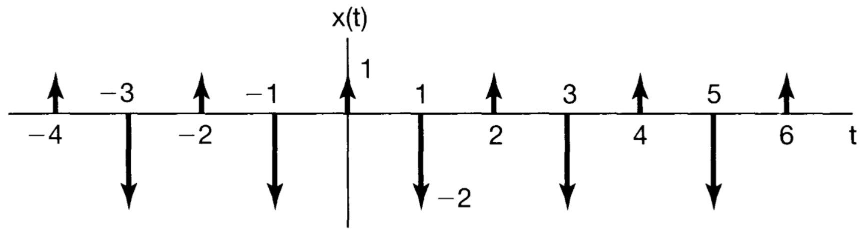
$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t})$$

Since $a_{-k} = a_k^\star$ *for all* k, $x(t)$ *is real. We can apply Euler's formula to represent* $x(t)$ *as a sum of cosines:*

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \cos(4\pi t) + \frac{2}{3}\cos(6\pi t)$$

We can see that the fundamental frequency is $\omega_0 = 2\pi$. The kth harmonic component is the term for $\omega_k = |k|\omega_0$. Therefore, the last three terms here are the first, second, and third harmonic components, respectively. In Figure 3, we plot a_0 in (a), the first and third harmonic components in (b) and (c), and the resultant signal in (d).

Example 6. *Find the Fourier series representation of the following signal $x(t)$ consisting of evenly spaced impulses with weights 1 and -2:*



Ans: As the first step, we find ω_0 and T. We see $T = 2$ and thus $\omega_0 = 2\pi/2 = \pi$. Then, we apply our formula for determining the coefficients a_k:

$$a_k = \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t}dt$$

A convenient interval for integration here is one that is shifted off of integers so we are clearly integrating over two impulses. For instance, we may chose $[-0.5, 1.5]$:

$$\begin{aligned}
a_k &= \frac{1}{2}\int_{-0.5}^{1.5} \left(\delta(t) - 2\delta(t-1)\right) e^{-jk\pi t}dt \\
&= \frac{1}{2}\int_{-0.5}^{1.5} \delta(t)e^{-jk\pi t}dt - \int_{-0.5}^{1.5} \delta(t-1)e^{-jk\pi t}dt = \frac{1}{2}e^{-jk\pi\cdot 0} - e^{-jk\pi\cdot 1} \\
&= \frac{1}{2} - e^{-jk\pi} = \frac{1}{2} - (-1)^k
\end{aligned}$$

This expression a_k is defined for all values of k: the DC component is $a_0 = -1/2$. Thus,

$$x(t) = \sum_{k=-\infty}^{\infty} \left(1/2 - (-1)^k\right) e^{j\pi kt}$$

Convergence of CT Fourier Series

If the Fourier series of a signal $x(t)$ exists, then $x(t)$ must be periodic. On the other hand, periodicity is not an *absolute* guarantee that the Fourier series exists. It is true, however, that Fourier series can be used to represent an extremely large class of periodic signals, including most if not all of those of interest in practice.

What characteristics must a periodic signal $x(t)$ satisfy in order for the coefficients a_k to be defined, and further, for $x(t)$ to be equal to its Fourier series representation? While we will not have time to study the answers to this question in detail, it turns out that as long as $x(t)$ satisfies three conditions, equality will hold except at isolated values of t for which $x(t)$ is discontinuous. These conditions are as follows:

1. Over any period, $x(t)$ must be absolutely integrable, i.e., $\int_T |x(t)|dt < \infty$. This guarantees that each a_k will be finite.
2. In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.
3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Virtually every signal of practical value will satisfy these conditions. As long as they hold true, the **partial sums** of the Fourier series

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

will converge to $x(t)$ as $N \to \infty$ at every point t of continuity. At a point of discontinuity, $x_N(t)$ will converge to the average of the values of $x(t)$ on the sides of the discontinuity.

Even as N becomes very large, though, $x_N(t)$ will always exhibit noticeable overshooting and undershooting behavior near points of discontinuity. This is known as **Gibbs phenomenon**, which is shown in Figure 3 for the square wave signal of Example 3.

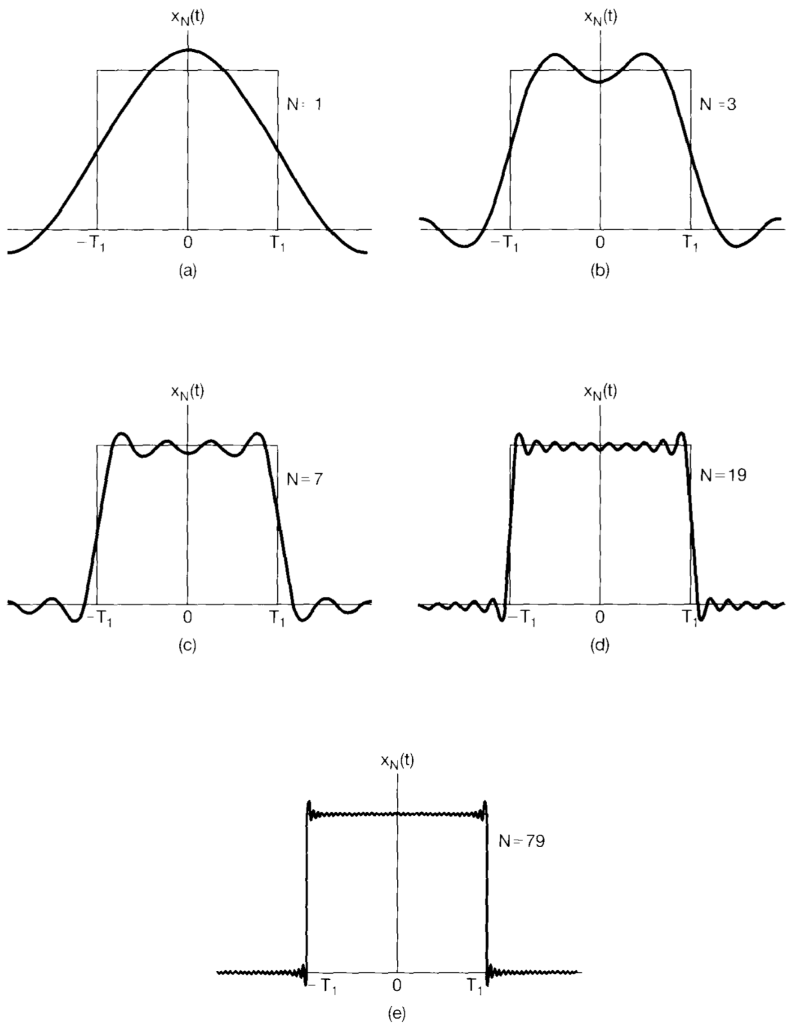


Figure 4: Illustration of Gibbs phenomenon for a square wave, with the partial sums $x_N(t)$ depicted for increasing N.

Properties of Continuous-Time Fourier Series

Fourier series representations have a number of important properties that can be leveraged to make analysis and computation easier. Most proofs follow from straightforward manipulations of the analysis equation $a_k = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt$. In presenting these properties, we will use the notation

$$x(t) \xleftrightarrow{FS} a_k$$

to signify the pairing of a periodic signal $x(t)$ with its Fourier series coefficients a_k.

Linearity. Suppose $x(t)$ and $y(t)$ are both periodic with period T. If $x(t) \xleftrightarrow{FS} a_k$ and $y(t) \xleftrightarrow{FS} b_k$, then the linear combination $z(t) = Ax(t) + By(t)$ is also periodic in T, and

$$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k$$

Proof: The Fourier series coefficients of $z(t)$ can be obtained via:

$$c_k = \frac{1}{T}\int_{<T>} z(t)e^{-jk\omega_0 t}dt.$$

As $z(t) = Ax(t) + By(t)$, based on the linearity of integration, we have

$$c_k = \frac{A}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt + \frac{B}{T}\int_{<T>} y(t)e^{-jk\omega_0 t}dt = Aa_k + Bb_k.$$

Time shifting. When a time shift is applied to $x(t)$, the period T is preserved, and the resulting coefficients are multiplied by a complex exponential. Specifically, if $x(t) \xleftrightarrow{FS} a_k$, then

$$x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0}a_k$$

One consequence of this is that the magnitudes of the coefficients in a time shift remain unaltered, i.e., $|b_k| = |a_k|$.

Proof: Let b_k be the Fourier series coefficients of $x(t-t_0)$. We have

$$b_k = \frac{1}{T}\int_{<T>} x(t-t_0)e^{-jk\omega_0 t}dt.$$

Setting $\tau = t - t_0$, the above integration becomes:

$$b_k = \frac{1}{T}\int_{<T>} x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = \frac{e^{-jk\omega_0 t_0}}{T}\int_{<T>} x(\tau)e^{-jk\omega_0\tau}d\tau = e^{-jk\omega_0 t_0}a_k.$$

Time reversal. When a signal undergoes a time reversal, the indices of the coefficients reverse. More specifically, if $x(t) \xleftrightarrow{FS} a_k$, then

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

As a result, if a periodic signal $x(t)$ is even, then its Fourier coefficients a_k must be even. Similarly, if $x(t)$ is odd, then a_k must be odd.

Time scaling. When a signal undergoes time scaling, the Fourier coefficients will not change, but the fundamental frequency and hence the Fourier series itself will change. In particular, if $x(t) \xleftrightarrow{FS} a_k$, then

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

i.e., the new fundamental frequency is $\alpha\omega_0$.

We leave the proof of time reversal and time scaling to you. It uses the same trick of changing integration variables as the proof of time shifting.

The above properties are mostly about the time transformation of signals. We can leverage the above properties to obtain the Fourier series coefficients of an unknown signal if we know it is a time transformation of another signal with known Fourier series representation. We illustrate this through two examples next.

Example 7. *Consider the signal $g(t)$ given in Figure 5(a). What is its Fourier series representation?*

Ans: We could apply the Fourier series formula directly. But we have already (in Example 3) calculated the coefficients for a similar periodic square wave, repeated again in Figure 5(b). In particular, the coefficients of $x(t)$ are

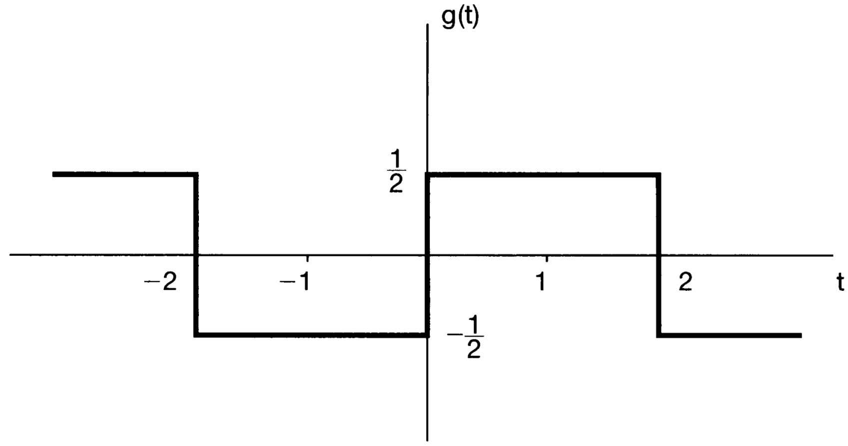
$$a_0 = \frac{2T_1}{T} \qquad a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad k \neq 0$$

Now, we need a relationship between $g(t)$ and $x(t)$. First, note that $g(t)$ has period $T = 4$, and thus $\omega_0 = 2\pi/4 = \pi/2$. Since $g(t)$ is $1/2$ 50% of the time and $-1/2$ the other 50%, $T_1 = T/4 = 1$. Now, to obtain $g(t)$ from $x(t)$, we need to undergo a delay by 1 (so that the signal switches at $t = 0$ and an offset by $-1/2$, i.e.,

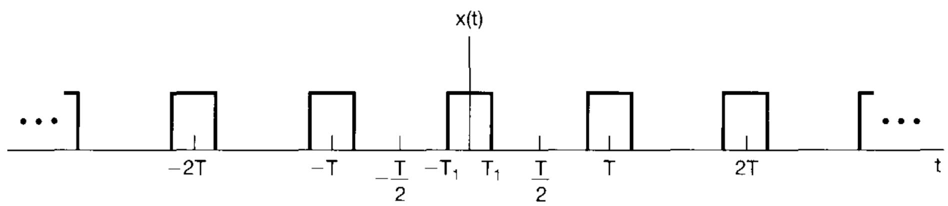
$$g(t) = x(t-1) - \frac{1}{2}$$

From the time-shifting property, a delay by t_0 causes a multiplication of a_k by $e^{-jk\omega_0 t_0}$ for $k \neq 0$. The subtraction by $1/2$ simply causes a_0 to decrease by this amount. Calling the coefficients of $g(t)$ b_k, then, we have

$$b_0 = \frac{1}{2} - \frac{1}{2} = 0 \qquad\qquad b_k = \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2} \quad k \neq 0$$



(a) Signal $g(t)$



(b) Signal $x(t)$

Figure 5: Square wave $g(t)$ for Example 1 and the original $x(t)$ for which we derived the Fourier series in Example 3.

Thus,

$$g(t) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2} e^{jk\pi t/2} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\sin(k\pi/2)}{k\pi} e^{jk\pi(t-1)/2}$$

Example 8. *Determine the Fourier series coefficients for the signal $y(t)$ given in Figure 8.*

Ans: We observe that the given signal $y(t)$ contains two rectangular pulses, one from $t = 0$ to $t = 1$ with a value of 1, and another from $t = 2$ to $t = 2.5$ with a value of 0.5. Therefore, similar to Example 7, we first start by using the coefficients of a periodic square wave given in Figure 1(b). Once again, for $x(t)$, we have

$$a_0 = \frac{2T_1}{T} \qquad a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \quad k \neq 0$$

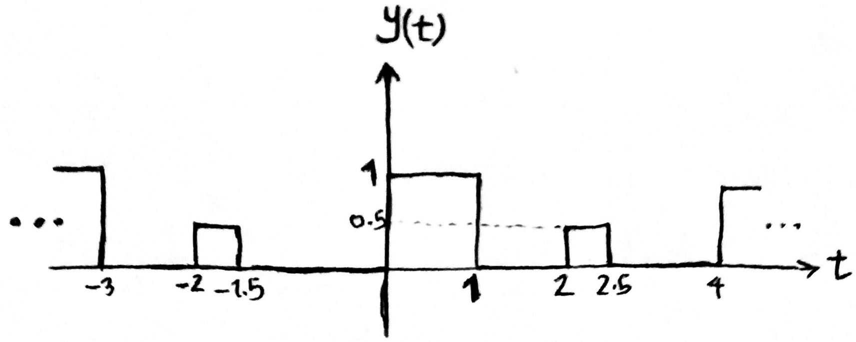


Figure 6: Signal for Example 8.

As in Example 7, the period of $y(t)$ *here is* $T = 4$*, and thus* $w_0 = 2\pi/T = \pi/2$*. However, the values of* T_1 *are different for each of the rectangular pulses, since they have different width in the time-domain. Let us define the periodic signal capturing the pulse from* $t = 0$ *to* $t = 1$ *as* $f(t)$*, and the one from* $t = 2$ *to* $t = 2.5$ *as* $h(t)$*. It follows that for* $f(t)$*,* $T_1 = T/8 = 0.5$ *since it is on for* 25% *of the period, and for* $h(t)$*,* $T_1 = T/16 = 0.25$ *since it is on for* 12.5% *of the period.*

Now, we can write out the relationship between $y(t)$ *with* $f(t)$ *and* $g(t)$ *as*

$$y(t) = f(t - 0.5) + 0.5h(t - 2.25)$$

We define the Fourier coefficients of $f(t)$*,* $h(t)$ *and* $y(t)$ *as* b_k*,* c_k *and* d_k*, respectively. We have*

$$\begin{aligned} b_0 &= \frac{1}{4}, \qquad b_k = \frac{\sin(k\pi/4)}{k\pi} \quad k \neq 0 \\ c_0 &= \frac{1}{8}, \qquad c_k = \frac{\sin(k\pi/8)}{k\pi} \quad k \neq 0 \end{aligned}$$

In order to find d_k*, we first use the time-shifting property to get the Fourier coefficients of* $f(t - 0.5)$ *as* $b_k e^{-j0.5kw_0}$*, and for* $h(t - 2.25)$ *as* $c_k e^{-j2.25kw_0}$*. Next, using the linearity property, we obtain*

$$d_k = e^{-j0.5kw_0} b_k + 0.5e^{-j2.25kw_0} c_k$$

It follows that

$$d_0 = b_0 + 0.5c_0 = \frac{1}{4} + 0.5\frac{1}{8} = \frac{5}{16}$$

$$d_k = \frac{\sin(k\pi/4)}{k\pi} e^{-jk\pi/4} + 0.5\frac{\sin(k\pi/8)}{k\pi} e^{-jk9\pi/8}, \ k \neq 0$$

There are a number of other useful Fourier series properties as well.

Multiplication. Suppose $x(t) \stackrel{FS}{\longleftrightarrow} a_k$ and $y(t) \stackrel{FS}{\longleftrightarrow} b_k$ both with period T. The product $x(t)y(t)$ will also be periodic with period T, and its Fourier series coefficients will be

$$x(t)y(t) \stackrel{FS}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Proof: We form the product as

$$x(t)y(t) = \left(\sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t} \right) \left(\sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t} \right)$$

By distributing the product on the right side, we have

$$x(t)y(t) = \sum_{l=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_l a_n e^{j(l+n)\omega_0 t}.$$

If we choose $k = l + n$, then the above equation becomes

$$x(t)y(t) = \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{\infty} a_l a_{k-l} \right) e^{jk\omega_0 t}.$$

This suggests

$$h_k = \sum_{l=-\infty}^{\infty} a_l a_{k-l}.$$

This property is indicative of a common theme we will see in the rest of the course: multiplication in the time domain corresponds to convolution in the frequency domain, and vice versa.

Conjugation. When we take the complex conjugate of a signal, its coefficients undergo conjugation and time reversal. That is, if $x(t) \stackrel{FS}{\longleftrightarrow} a_k$, then

$$x^{\star}(t) \stackrel{FS}{\longleftrightarrow} a^{\star}_{-k}$$

If $x(t)$ is real, then $x(t) = x^{\star}(t)$, so it follows that its coefficients are **conjugate symmetric**: $a_{-k} = a^{\star}_k$. This means that the magnitude graph $|a_k|$ even (i.e., $|a_k| = |a_{-k}|$) and the phase graph θ_k is odd (i.e., $\theta_k = -\theta_{-k}$).

Proof: Letting b_k be the Fourier series coefficients of $x^\star(t)$, then

$$b_k = \frac{1}{T}\int_{<T>} x^\star(t)e^{-jk\omega_0 t}dt = \left(\frac{1}{T}\int_{<T>} x(t)e^{jk\omega_0 t}dt\right)^\star = (a_{-k})^\star.$$

Differentiation. For a periodic signal $x(t) \overset{FS}{\longleftrightarrow} a_k$ with period T and fundamental frequency $\omega_0 = 2\pi/T$, the Fourier series coefficients b_k for the derivative signal $\frac{dx(t)}{dt}$ are expressed as follows:

$$\frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} b_k = jk\omega_0 a_k = jk\frac{2\pi}{T}a_k.$$

If we "reverse" this process, we find that the integral signal $\int_{-\infty}^{t} x(\tau)d\tau$ has the following Fourier series coefficients c_k:

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{FS}{\longleftrightarrow} c_k = \frac{1}{jk\omega_0}a_k = \frac{1}{jk(2\pi/T)}a_k,$$

for $k \neq 0$.

Proof: As the Fourier series representation of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

the derivative of $x(t)$ can be expressed as

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} \frac{d}{dt}\left(a_k e^{jk\omega_0 t}\right) = \sum_{k=-\infty}^{\infty} a_k(jk\omega_0)e^{jk\omega_0 t}.$$

Thus, $b_k = jk\omega_0 a_k$.

Similarly, the Fourier series of $x(t)$ becomes the following under integration:

$$\int_{-\infty}^{t} x(\tau)d\tau = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{t} e^{jk\omega_0 \tau}d\tau = \sum_{k=-\infty}^{\infty} \frac{1}{jk\omega_0}a_k e^{jk\omega_0 t}, k \neq 0.$$

Thus, $c_k = \frac{1}{jk\omega_0}a_k, k \neq 0$.

Parseval's relation. The power and energy in a periodic signal is contained in its Fourier series coefficients. In particular, the average power can be obtained as

$$\frac{1}{T}\int_{<T>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

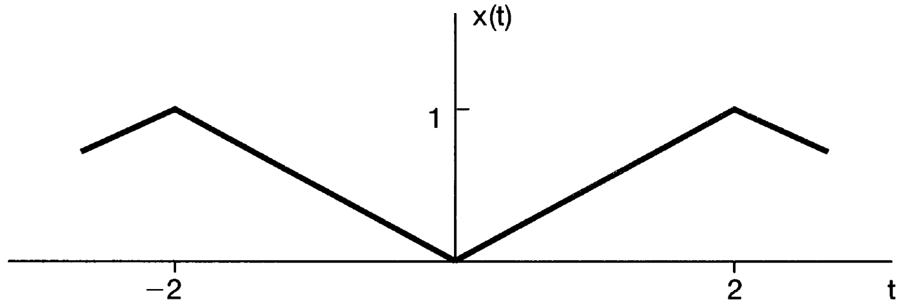


Figure 7: Triangular wave signal in Example 9.

This is known as **Parseval's relation** for continuous-time periodic signals.

Proof. Here we provide a proof based on intuition, with some hand-waving. Note that

$$\frac{1}{T}\int_{<T>} \left|e^{jk\omega_0 t}\right|^2 dt = 1,$$

since the average power of any periodic complex exponential $e^{jk\omega_0 t}$ is 1. Therefore,

$$\frac{1}{T}\int_{<T>} \left|a_k \cdot e^{jk\omega_0 t}\right|^2 dt = |a_k|^2,$$

since we can factor a_k outside of the integral. Since $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ contains many HRCEs, intuitively, the average power of $x(t)$ should be the sum of the average powers of each HRCE. Parseval's relationship is sometimes referred as the "power conservation law."

A full table of these and other properties for the continuous-time Fourier series is given in Table 3.1 of the textbook.

Example 9. *Consider the triangular wave signal $x(t)$ with period $T = 4$ shown in Figure 7. What are its Fourier series coefficients?*

Ans: The derivative of this signal is the signal $g(t)$ in Example 7. Denoting $g(t) \overset{FS}{\longleftrightarrow} d_k$ and $x(t) \overset{FS}{\longleftrightarrow} e_k$, the differentiation property indicates that

$$d_k = jk(\pi/2)e_k$$

since the fundamental frequency is $\omega_0 = 2\pi/T = \pi/2$. This equation can be used to express e_k in terms of d_k, except when $k = 0$:

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2},\ k \neq 0.$$

For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

$$e_0 = \frac{1}{2}.$$

The next example is a challenging one which will help to integrate all of these properties together and how to draw conclusions from them.

Example 10. *Suppose we are given the following facts about a signal $x(t)$:*

1. *$x(t)$ is a real signal.*
2. *$x(t)$ is periodic with period $T = 4$, and it has Fourier series coefficients a_k.*
3. *$a_k = 0$ for $|k| > 1$.*
4. *The signal with Fourier coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd.*
5. *$\frac{1}{4}\int_{<4>} |x(t)|^2 dt = 1/2$.*

Show that this information is sufficient to determine the signal $x(t)$ to within a sign factor (i.e., x vs. $-x$).

Ans: According to Fact 3, $x(t)$ has at most three nonzero Fourier series coefficients, at a_0, a_1, and a_{-1}, Then, since $x(t)$ has fundamental frequency $\omega_0 = 2\pi/4 = \pi/2$ (Fact 2), it follows that

$$x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1} e^{j\pi t/2}.$$

Since $x(t)$ is real (Fact 1), we can conclude that a_0 is real and $a_1 = a^{\star}_{-1}$. Consequently,

$$x(t) = a_0 + a_1 e^{j\pi t/2} + \left(a_1 e^{j\pi t/2}\right)^* = a_0 + 2Re\{a_1 e^{j\pi t/2}\}.$$

Let us now determine the signal corresponding to the Fourier coefficients b_k given in Fact 4:

- *Using the time-reversal property, we note that a_{-k} corresponds to the signal $x(-t)$.*

- *Also, the time-shift property in the table indicates that multiplication of the kth Fourier coefficient by* $e^{-jk\pi/2} = e^{-jk\omega_0 \cdot 1}$ *corresponds to the underlying signal being shifted by 1 to the right (i.e., having* t *replaced by* $t-1$*).*

We conclude that the coefficients b_k *correspond to the signal* $x(-(t-1)) = x(-t+1)$*. According to Fact 4, this signal must be odd, which means* $b_k = -b_{-k}$*. Since* $x(t)$ *is real,* $x(-t+1)$ *must also be real, which means* $b_k = b^{\star}_{-k}$*. Thus,* b_k *is purely imaginary and odd. Thus,* $b_0 = 0$ *and* $b_{-1} = -b_1$*.*

Since time-reversal and time-shift operations cannot change the average power per period, Fact 5 holds even if $x(t)$ *is replaced by* $x(-t+1)$*. That is,*

$$\frac{1}{4}\int_{<4>} |x(-t+1)|^2 dt = \frac{1}{2}.$$

We can now use Parseval's relation to conclude that

$$|b_1|^2 + |b_{-1}|^2 = 1/2.$$

Substituting $b_1 = -b_{-1}$ *in this equation, we obtain* $|b_1| = 1/2$*. Since* b_1 *is also known to be purely imaginary, it must be either* $j/2$ *or* $-j/2$*.*

Now, we can translate these conditions on b_0 *and* b_1 *into equivalent statements on* a_0 *and* a_1 *from Fact 4:*

- *For* $k = 0$*, since* $b_0 = 0$*, Fact 4 implies that* $a_0 = 0$*.*
- *With* $k = 1$*, this condition implies that* $a_1 = e^{-j\pi/2} b_{-1} = -jb_{-1} = jb_1$*.*

Thus if we take $b_1 = j/2$*, then* $a_1 = -1/2$*, and*

$$x(t) = -\cos(\pi t/2).$$

Alternatively, if we take $b_1 = -j/2$*, then* $a_1 = 1/2$*, and*

$$x(t) = \cos(\pi t/2).$$