Signals and Systems: Module 2

Suggested Reading: S&S 1.2, 1.3.1

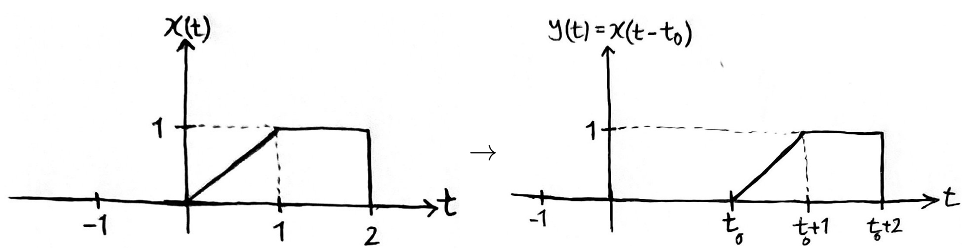
Transformations in Time

Much of this course focuses on transforming signals from one form to another (i.e., by a system). We will now look at a few elementary transformations of the independent variable, i.e., the time axis. These transformations will be useful in defining further classifications of signals too.

(1) Time Shift

The first type of time transformation is a shift in time. For a CT signal $x(t)$, a **time shift** is written as $x(t - t_0)$ for a shift of t_0. Note the minus sign here, which is typically assumed by convention.

Visually, consider the following signal $x(t)$ and its delayed counterpart $y(t) = x(t - t_0)$:



The visualization above is assuming $t_0 > 0$:

- When $t_0 > 0$, we say the signal is **shifted to the right**, or **delayed by** t_0 (seconds). Things happen later in $y(t)$ than in $x(t)$ when $t_0 > 0$.

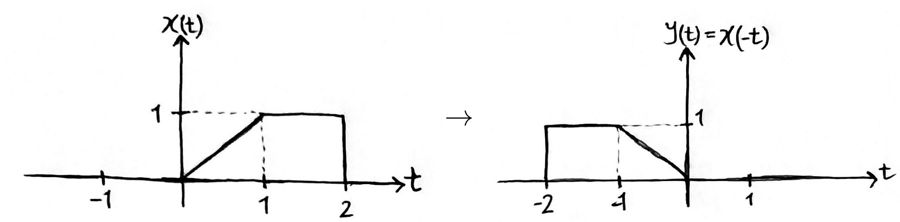
- When $t_0 < 0$, the signal is **shifted to the left**, or **advanced by** t_0 (seconds). Things happen sooner in $y(t)$ than in $x(t)$ when $t_0 < 0$.

For a DT signal $x[n]$, the delay is written as $x[n - n_0]$ for an integer n_0, with the same properties.

(2) Time Reversal

For a CT signal $x(t)$, a **time reversal** is written as $x(-t)$. This corresponds to taking the signal and playing it from finish to start rather than start to finish, i.e., backwards playback.

Visually, time reversal is a reflection over the vertical axis, e.g.,



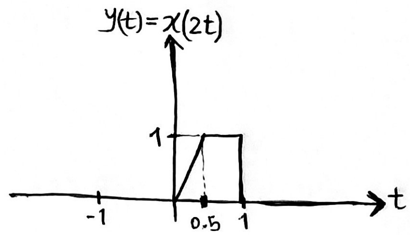
For a DT signal $x[n]$, a time reversal is written as $x[-n]$, and works the same way.

(3) Time Scaling

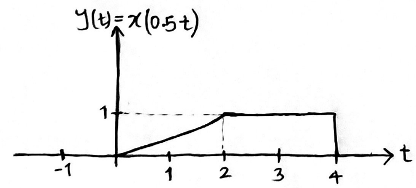
For a CT signal $x(t)$, let $y(t) = x(\alpha t)$, where $\alpha > 0$. α is the **time scaling** factor:

- When $\alpha > 1$, $y(t)$ is a shorter timescale (faster, sped-up) version of $x(t)$.
- When $0 < \alpha < 1$, $y(t)$ is a longer timescale (slowed down) version of $x(t)$.

For the signal used above, we visualize $\alpha = 0.5$ and $\alpha = 2$ as follows:



faster playback



slower playback

With discrete-time signals, time-scaling becomes a bit more tricky. The argument to $x[\alpha n]$, αn, must always be an integer, which means α must be an integer. In this case, $y[n] = x[\alpha n]$ will lose information compared to $x[n]$. For example, if $y[n] = x[5n]$, then $y[n]$ only contains values of $x[n]$ where n is a multiple of 5, i.e., $n = 0, \pm 5, \pm 10, ...$

(4) Composite Transformation

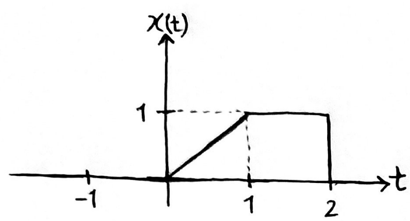
We also have situations where there are simultaneous transformations of multiple types, e.g., both time shift and scaling. In general, for a CT signal $x(t)$, we may wish to obtain $x(\alpha t + \beta)$ for given constants α, β. There are many potential ways to do this, but we must be careful of the order in which each operation is applied.

One way is to apply the operations in this order:

1. Time shift the signal according to β.
2. Time scale according to $|\alpha|$. If $|\alpha| < 1$ it is linearly stretched, if $|\alpha| > 1$ it is compressed.
3. Time reverse according to α. If $\alpha < 0$, it is then reverse in time, else there is no reversal.

Example 1. *Consider the $x(t)$ signal from the previous cases. Obtain a plot of $y(t) = 2 \cdot x(2 - 3t)$.*

Ans: We can break this down through the steps outlined above:



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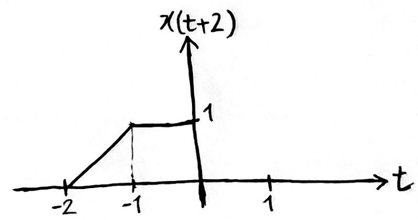
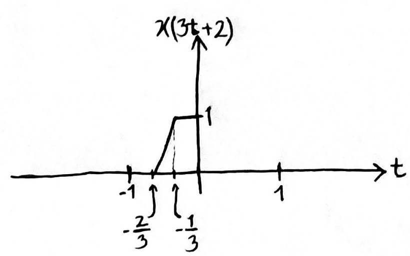


Advance by $t = 2$.

↙



Compress by $\alpha = 3$.

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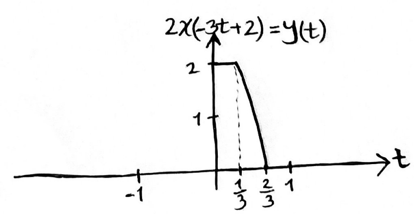


Time reversal (and y-axis).

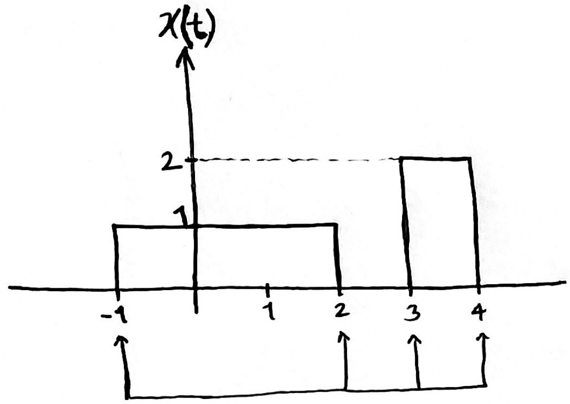
The scaling factor on the y-axis (in this case, 2) can be applied at any step. In this case it is shown in the last step.

Regardless of the method you use to plot composite transformations, you should always consider plugging in several values of t to verify the result. Even before you start plotting, plugging in values can help you decide the final shape of the signal to expect. Good values of t to check are "edge" points before the signal changes.

For instance, to verify our plot in Example 1, we can check the values of $y(t)$ in terms of $x(t)$ at the transition points $t = 0, 1/3, 2/3$ in $y(t)$:

- $y(0) = 2 \cdot x(2 - 3 \cdot 0) = 2x(2) = 2$, consistent with the plot of $y(t)$.
- $y(\frac{1}{3}) = 2 \cdot x(2 - 3 \cdot \frac{1}{3}) = 2x(1) = 2$, which is also also consistent.
- $y(\frac{2}{3}) = 2 \cdot x(2 - 3 \cdot \frac{2}{3}) = 2x(0) = 0$, which again is true.

Example 2. *Consider the following CT signal $x(t)$:*

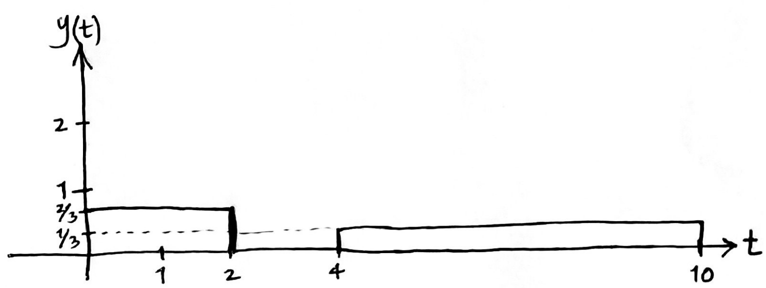


For a time transformation $y(t) = \frac{1}{3}x(4 - 0.5t)$, determine where the four indicated values of t in $x(t)$ should end up in $y(t)$. Then, apply the time transformation rules and make sure it is consistent.

Ans: For the transition points $t = -1, 2, 3, 4$ outlined in $x(t)$:

$$\begin{aligned}
t = -1 \quad &\rightarrow \quad 4 - 0.5t = -1 \quad \rightarrow \quad t = 10 \textit{ in } y(t) \\
t = 2 \quad &\rightarrow \quad 4 - 0.5t = 2 \quad \rightarrow \quad t = 4 \textit{ in } y(t) \\
t = 3 \quad &\rightarrow \quad 4 - 0.5t = 3 \quad \rightarrow \quad t = 2 \textit{ in } y(t) \\
t = 4 \quad &\rightarrow \quad 4 - 0.5t = 4 \quad \rightarrow \quad t = 0 \textit{ in } y(t)
\end{aligned}$$

Applying the time transformation rules for $y(t) = \frac{1}{3}x(-0.5t + 4)$, we get:



The transition points appear at $t = 10, 4, 2, 0$ as we expect.

You should also check out the *Transformations in Time Handout* on Brightspace for practice visually matching a signal $x(t)$ to different time transformations.

More Signal Classifications

Equipped with time transformations, we now return to our discussion on different signal classifications. Recall our first two were (1) CT vs. DT and (2) power and energy.

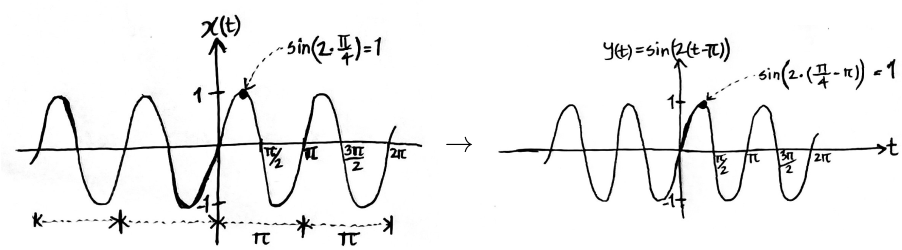
Classification #3: By the period (symmetry under time shift)

Another important signal classification is whether the signal is **periodic** or not, and if so, what the periodicity is. A signal which is not periodic is said to be **aperiodic**.

A periodic continuous-time signal $x(t)$ has the property that for some $T \neq 0$, $x(t) = x(t+T)$ for all values of t. In this case, we say $x(t)$ **is periodic with period** T. In other words, $x(t)$ is unchanged for a time shift of T.

For a discrete time signal, analogously, $x[n]$ **is periodic with period** N if $x[n] = x[n+N]$, for an integer $N \neq 0$.

Sinusoids are the canonical case of periodic signals. In fact, a major part of this course will be looking at how we can express *any* periodic signal in terms of combinations of sinusoids. For example, consider $x(t) = \sin(2t)$, which has period $T = \pi$:

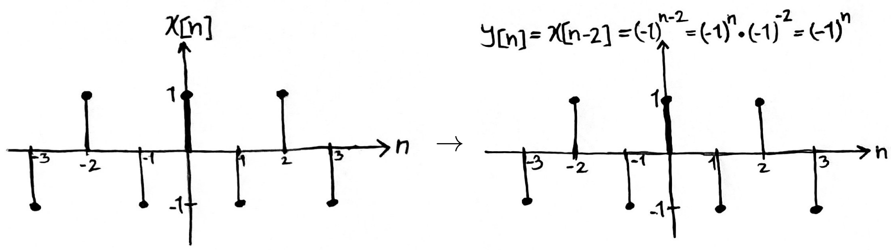


This repeats every $T = \pi$. To the right, we plot $y(t) = x(t-\pi)$.

We see that $x(t) = x(t-\pi)$, so $x(t)$ is periodic with period π.

More generally, a sinusoid signal $x(t) = \sin(\omega_0 t + \phi)$ has a period $T = 2\pi/\omega_0$. Note that this conclusion is only true for CT signals. For DT, the conclusion will be different. A DT sinusoid signal might even be aperiodic! Keep that in mind, and we will talk about this next week.

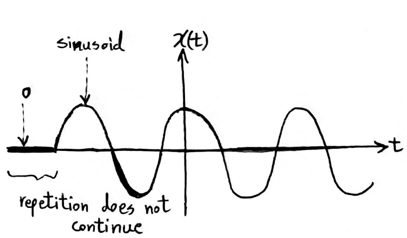
In discrete time, consider for example the alternating signal $x[n] = (-1)^n$. This has period $N = 2$:

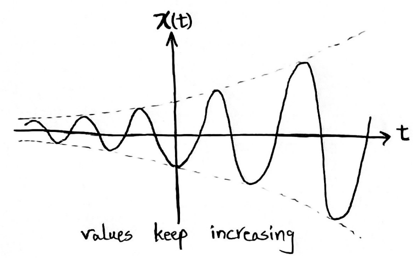


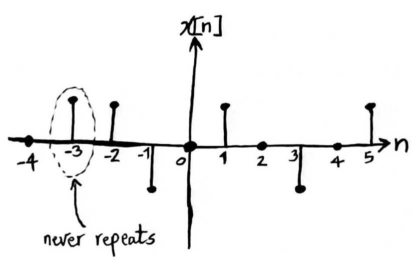
This repeats every $N = 2$. To the right, we plot $y[n] = x[n-2]$.

$x[n] = x[n-2]$, so $x[n]$ is periodic with period $N = 2$.

On the other hand, the following are examples of aperiodic signals:







From these figures and equations, we can deduce that if $x(t)$ is periodic with period T, it is periodic with period mT for any integer m. For example, with $x(t) = \sin(2t)$, we have $\pi, 2\pi, 3\pi, \cdots$ all as periods. Similarly, if $x[n]$ is periodic with period N, it is periodic with period mN for any integer m.

The **fundamental period** is the smallest period of a periodic signal $x(t)$

(or $x[n]$). We often denote this T_0 (or N_0).

Let's consider two important questions related to periodicity:

- If $x(t) = x_{Re}(t) + jx_{Im}(t)$ is a periodic complex signal, then both $x_{Re}(t)$ and $x_{Im}(t)$ are periodic real signals. Why?

$$x(t) = x(t-T) \to x_{Re}(t) + jx_{Im}(t) = x_{Re}(t-T) + jx_{Im}(t-T)$$
$$\to x_{Re}(t) = x_{Re}(t-T),\ x_{Im}(t) = x_{Im}(t-T) \text{ are periodic}$$

- Is the sum of two periodic signals $x_1(t)$ and $x_2(t)$, $x_1(t) + x_2(t)$ periodic? Often, but not always.

  Suppose the fundamental period of x_1 is T_1 and the fundamental period of x_2 is T_2. Consider the **least common multiple** $T = \text{LCM}(T_1, T_2)$ of T_1 and T_2, defined as

  the minimum T such that $T = k_1T_1 = k_2T_2$ for some integers k_1, k_2.

  If T exists, then the sum is periodic, and this is the fundamental period of $x_1(t) + x_2(t)$. If it does not exist, then $x_1 + x_2$ is aperiodic.

Example 3. *What is the fundamental period of $x(t) = \cos(3t) + \sin(2t)$?*

Ans: We know that a sinusoid $\sin(\omega_0 t + \phi)$ has a fundamental period $T_0 = \frac{2\pi}{\omega_0}$. So, $T_1 = \frac{2\pi}{3}$ and $T_2 = \pi$ for the two sinusoids comprising $x(t)$. The LCM is 2π, as

$$\frac{2\pi}{3} \times 3 = \pi \times 2 = 2\pi$$

Thus, the fundamental period of $x(t)$ is 2π.

Example 4. *Is $x(t) = \cos(3t) + \sin(\pi t)$ periodic?*

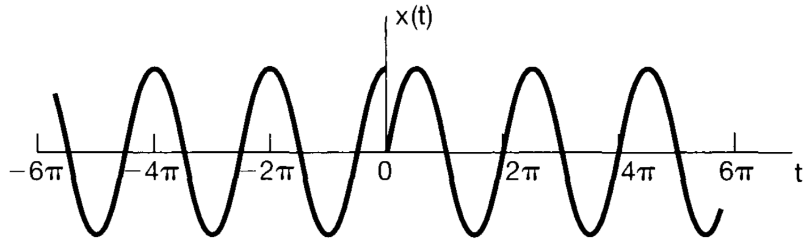
Ans: Here, $T_1 = \frac{2\pi}{3}$ and $T_2 = 2$. There are no integers such that $k_1 \cdot T_1 = k_2 \cdot T_2$, so the LCM does not exist and $x(t)$ is aperiodic.

Example 5. *Is the following signal periodic? If so, what is its fundamental period?*

$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}$$

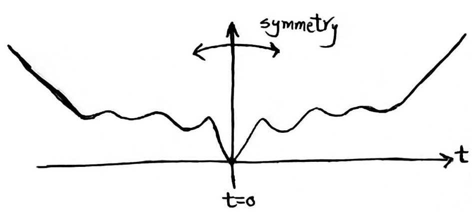
Ans: We know that $\cos(t) = \cos(t + 2\pi)$ and $\sin(t) = \sin(t + 2\pi)$, i.e., each of these functions have periods $T_0 = 2\pi$. So, if we were concerned with *them separately, then the answer would be yes. However, by the way $x(t)$ is constructed, we see that there is a discontinuity at $t = 0$ that does not repeat:* $\lim_{t\to 0^-} x(t) = 1$ *and* $\lim_{t\to 0^+} x(t) = 0$*, while* $\lim_{t\to 2\pi^-} x(t) = 0 = \lim_{t\to 2\pi^+} x(t)$*, for example. Since every feature in the shape of a periodic signal must recur periodically, we therefore conclude that $x(t)$ is not periodic.*

The discontinuity and non-periodicity can also be seen from the plot of $x(t)$ in the figure below.

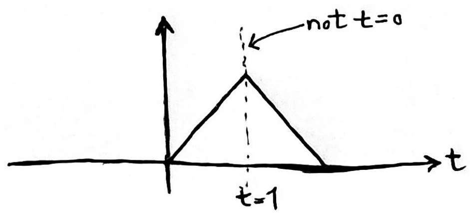


Classification #4: Even and odd signals (symmetry under time reversal)

A continuous time signal $x(t)$ is **even** if $x(t) = x(-t)$, meaning its time reversed version is identical. For discrete time, analogously, the condition is $x[n] = x[-n]$. Visually:



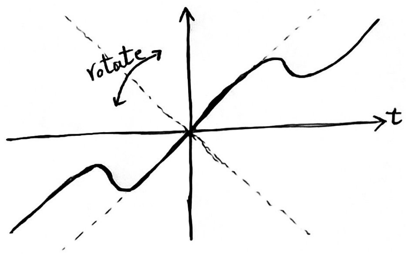
even



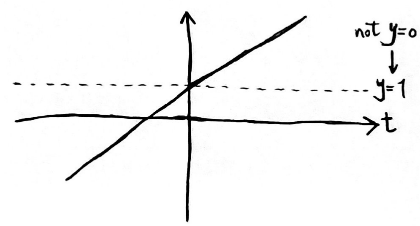
not even

In other words, a signal is even if it is a "mirror image" of itself around the vertical axis $t = 0$.

On the other hand, a signal is **odd** if $x(t) = -x(-t)$ (or $x[n] = -x[-n]$), meaning its time reversed version gives the negative of itself. An odd signal has symmetry around the origin. Visually:



odd



not odd

For an odd signal, we must have $x(0) = 0$ and $x[0] = 0$, since the equations require $x(0) = -x(0)$ and $x[-0] = -x[0]$.

Many functions/signals are *neither* even nor odd.

Example 6. *As an exercise, consider whether the following signals are even, odd, or neither:*

– *CT examples:* $x(t) =$

• $\cos(t)$ *Even*	• $\sin(t)$ *Odd*	• $\cos(t + \overbrace{\pi/4}^{\textit{offset } 45^\circ})$ *Neither*
• $\sin(t + \overbrace{\pi/2}^{90^\circ \textit{ advance}})$ *Even (same as* $\cos(t)$*)*	• $\lvert t\rvert$ *Even*	• t^2 *Even*
• t^3 *Odd*	• e^t *Neither*	• $\overbrace{e^{\lvert t\rvert}}^{\textit{even}}\overbrace{\sin(t)}^{\textit{odd}}$ *Odd (even · odd = odd)*

– *DT examples:* $x[n] =$

• $\sin(\pi n)$ *Both! (always* 0*)*	• $\cos(\pi n)$ *Even (*$= (-1)^n$*)*	• $(-1)^{n+1} = \overbrace{(-1)^n}^{\textit{even}} \cdot \overbrace{(-1)}^{\textit{even}}$ *Even (even · even = even)*

As illustrated in the above example, if a signal $x(t)$ is a product of signals, i.e., $x(t) = x_1(t)x_2(t)$, we can determine even/odd of $x(t)$ by looking at $x_1(t)$ and $x_2(t)$. These rules work like the products of positive and negative numbers: two evens make $x(t)$ even, one even and one odd make $x(t)$ odd, and two odds make $x(t)$ even.

Any signal can be written as a sum of an even signal and an odd signal, which we call **even-odd decomposition**:

$$x(t) = x_{even}(t) + x_{odd}(t), \qquad x[n] = x_{even}[n] + x_{odd}[n]$$

In particular, suppose we decompose $x(t)$ as

$$x(t) = \underbrace{\frac{x(t) + x(-t)}{2}}_{x_1(t)} + \underbrace{\frac{x(t) - x(-t)}{2}}_{x_2(t)} \quad \text{(the } x(-t) \text{ is added and subtracted)}$$

- $x_1(t)$ is even because

$$x_1(-t) = \frac{x(-t) + x(-(-t))}{2} = \frac{x(t) + x(-t)}{2} = x_1(t)$$

- $x_2(t)$ is odd because

$$x_2(-t) = \frac{x(-t) - x(-(-t))}{2} = \frac{x(-t) - x(t)}{2} = -x_2(t)$$

Even-odd decompositions are useful in the analysis of linear systems. If we use even and odd signals as our "test signals", then by the superposition property, the output of a new signal which is a combination of these even and odd components can be determined as a sum of individual outputs.

Example 7. *Find the even-odd decomposition of*

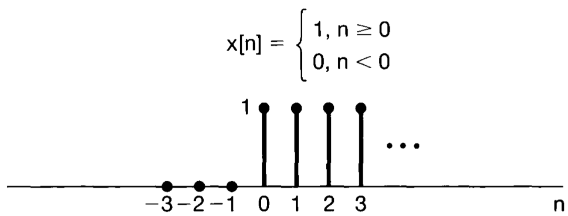
$$x[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Ans: Using our formulas for x_1 (even) and x_2 (odd) from above, we have that

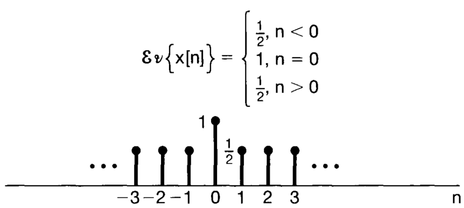
$$x_{even}[n] = \frac{1}{2}\left(x[n] + x[-n]\right) = \begin{cases} \frac{1}{2} & n > 0 \\ 1 & n = 0 \\ \frac{1}{2} & n < 0 \end{cases}$$

$$x_{odd}[n] = \frac{1}{2}\left(x[n] - x[-n]\right) = \begin{cases} \frac{1}{2} & n > 0 \\ 0 & n = 0 \\ -\frac{1}{2} & n < 0 \end{cases}$$

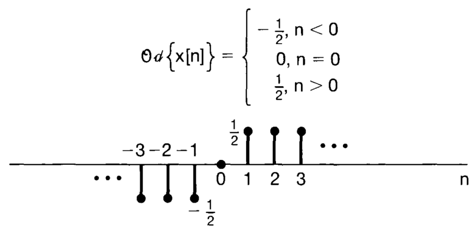
Plots of the signal and its decomposition are given below:



$x[n]$



$x_{even}[n]$



$x_{odd}[n]$

Classification #5: Complex exponential signals

We now introduce a special class of signals which is used in practice to construct many other signals. We will study it first in continuous time.

The CT **complex exponential signal** is of the form

$$x(t) = C \cdot e^{\alpha t}, \text{ where } C \text{ and } \alpha \text{ are in general complex numbers.}$$

To study this signal, we write

$$C = |C|e^{j\phi} \text{ (polar form), and } \alpha = \sigma + j\omega \text{ (rectangular form)}$$

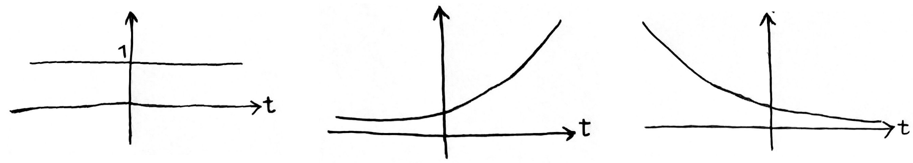
Then,

$$x(t) = |C|e^{j\phi} \cdot e^{(\sigma+j\omega)t} = \underbrace{|C|}_{\text{Term 1}} \cdot \underbrace{e^{\sigma t}}_{\text{Term 2}} \cdot \underbrace{e^{j(\omega t+\phi)}}_{\text{Term 3}}$$

Let us study the terms as separate signals and then put them together.

- **Term 1:** $|C|$ simply scales the signal.
- **Term 2:** $e^{\sigma t}$ is a **real exponential signal**. There are three possible types of behavior, depending on σ:

$\sigma = 0$ $\qquad\qquad$ $\sigma > 0$ $\qquad\qquad$ $\sigma < 0$



$\sigma < 0$ corresponds to an exponential decay for $t > 0$, which is common in describing transient responses in electrical circuits. $\sigma > 0$ corresponds to an exponential growth for $t > 0$, which tends to imply instability.

- **Term 3:** $e^{j(\omega t+\phi)}$ is a **periodic complex exponential**. Specifically,

$$e^{j(\omega t+\phi)} = \cos(\omega t + \phi) + j\sin(\omega t + \phi)$$

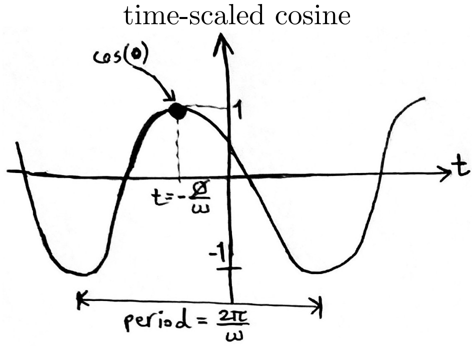
is periodic with fundamental period $2\pi/\omega$. To see this, we can consider the impact of a time shift $T = 2\pi/\omega$ using some properties of complex numbers:

$$e^{j\omega(t-\frac{2\pi}{\omega})+j\phi} = e^{j(\omega t-2\pi+\phi)} = \underbrace{e^{-j2\pi}}_{1} e^{j\omega t}e^{j\phi} = e^{j(\omega t+\phi)}$$

So, how do we plot $e^{j(\omega t+\phi)}$? We can consider the real and imaginary parts in separate plots:

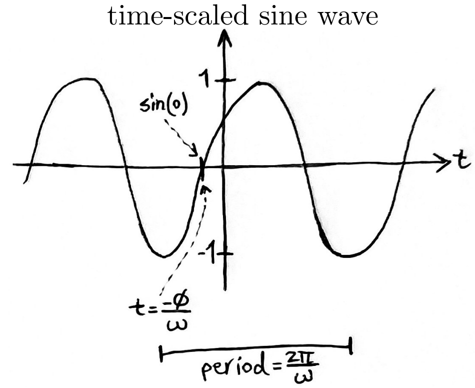
$$e^{j(\omega t+\phi)} = \underbrace{\cos(\omega t+\phi)}_{\text{Real}} + j\,\underbrace{\sin(\omega t+\phi)}_{\text{Imaginary}}$$

Real part: Time-shifted and time-scaled cosine



$\cos(\omega t+\phi) = \mathrm{Re}\{e^{j(\omega t+\phi)}\}$

Imaginary part: Time-shifted and time-scaled sine wave



$\sin(\omega t+\phi) = \mathrm{Im}\{e^{j(\omega t+\phi)}\}$

Both sinusoids have a phase shift of ϕ and a time scaling of ω.

In this signal, ω (omega) is called the **fundamental frequency**, which is related to the fundamental period as $\omega = 2\pi/T$. A smaller ω gives a slower oscillation, and a larger ω gives a faster oscillation. $\omega = 0$ gives a constant signal. ω is measured in radians/sec.

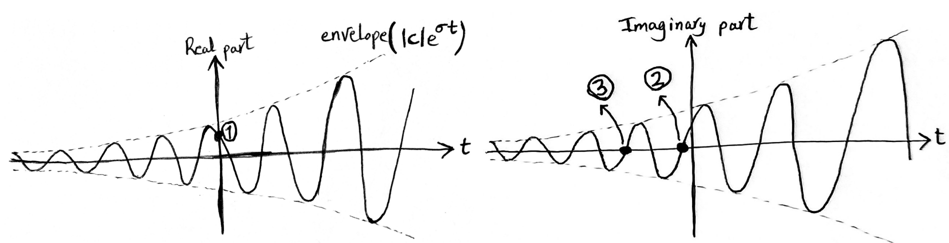
ϕ is called the **phase**, since it changes the angle of each sinusoidal component. ϕ is measured in radians. The impact of ϕ on the time axis is $\Delta t = \phi/\omega$ due to the time scaling by ω.

Note also that the real and imaginary parts of $x(t)$ are offset by $\Delta t = \frac{\pi}{2\omega}$. This makes sense because of the relationship between sin and cos, which are 90° apart.

Now, we combine the three terms together:

$$x(t) = \underbrace{|C|e^{\sigma t}}_{\substack{\text{multiplies both real}\\ \text{and imaginary parts}}} \cdot e^{j(\omega t + \phi)}$$

The shape of $x(t)$ for $t > 0$ vs. $t < 0$ will depend on the value of σ. For $\sigma > 0$, we will get the following behavior:



The envelope of the waveform, visualized here by the dashed lines, is $\pm|C|e^{\sigma t}$. This gives the oscillation magnitude at each value of t.

To gain some insight, let's find the coordinates of points ①, ② and ③ in the image above:

① is the value of the real part of $x(t)$ when $t = 0$:

$$|C|e^{\sigma 0} \cos(\omega \cdot 0 + \phi) = |C|\cos(\phi) \rightarrow (0, |C|\cos(\phi))$$

② is the time to the immediate left of $t = 0$ where the imaginary part is 0. This means

$$\sin(\omega t + \phi) = 0, \text{ which is at } t = -\frac{\phi}{\omega} \rightarrow \left(-\frac{\phi}{\omega}, 0\right)$$

③ is the third zero crossing point to the left of the imaginary axis. $\sin(\omega t + \phi)$ has a period of $\frac{2\pi}{\omega}$, so this is

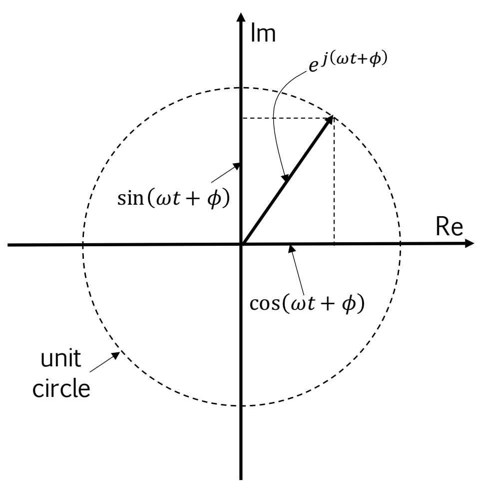
$$t = -\frac{\phi}{\omega} - \frac{2\pi}{\omega} \rightarrow \left(-\frac{\phi}{\omega} - \frac{2\pi}{\omega}, 0\right)$$

When $\sigma < 0$, the pattern in the above image is reversed for $t > 0$ vs. $t < 0$. Sinusoidal signals multiplied by decaying exponentials are commonly referred to as **damped sinusoids**. We see them when we consider the transient responses of many physical systems.

Instantaneous power. Consider the instantaneous power of $x(t) = |C|e^{\sigma t}e^{j(\omega t+\phi)}$. By definition,

$$P(t) = |x(t)|^2 = |C|^2 \left|e^{\sigma t}\right|^2 \left|e^{j(\omega t+\phi)}\right|^2$$

What is $|e^{j(\omega t+\phi)}|$? We can look at $e^{j(\omega t+\phi)}$ in the complex plane:



$e^{j\theta}$ is always on the unit circle, for any θ. Thus, $\left|e^{j(\omega t+\phi)}\right|^2 = 1$ for all ω, t, ϕ. So,

$$P(t) = |C|^2 \cdot |e^{\sigma t}|^2 = |C|^2 e^{2\sigma t}$$

Average power. Periodic signals provide important examples of signals with infinite total energy and finite average power. The general CT complex exponential signals is not periodic. However, recall from our discussion of the complex exponential signal $x(t)$ that if $\sigma = 0$, we do have a periodic signal:

$$x(t) = Ce^{j(\omega t+\phi)}, \text{ with } T = \frac{2\pi}{\omega}.$$

We can see that the total energy is infinite, since

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} C^2 \, dt = \infty$$

On the other hand, to determine the average power $P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$, we only have to integrate over one period, since the behavior will be the same over any period. In general, for a *periodic* signal $x(t)$,

$$P_\infty = \frac{1}{T} \int_{<T>} |x(t)|^2 \, dt, \text{ where } <T> \text{ is any period.}$$

So, for the periodic complex exponential $x(t)$,

$$P_\infty = \frac{1}{T} \int_0^T C^2 \, dt = C^2$$

HRCEs. Periodic complex exponential signals will be central to this course, as they are fundamental building blocks for many other signals. In particular, we will often consider sets of **harmonically related complex exponentials** (HRCEs), which are sets of periodic exponentials that all share a common period T_0.

A set of continuous-time HRCEs is a family of signals expressed as:

$$x_k(t) = e^{jk\omega_0 t}; k = 0, \pm 1, \pm 2, \cdots \text{(any integer)}$$

all of which have fundamental frequencies that are integer multiples of a single positive frequency ω_0. All of these signals are periodic. The fundamental frequency of $x_k(t)$ is $\omega_k = |k|\omega_0$, and its fundamental period is $T_k = \frac{2\pi}{|k|\omega} = \frac{T_0}{|k|}$.

Note that for any fundamental frequency ω_0, there are infinitely many CT HRCEs. We will draw upon this concept to form a useful set of "test inputs" for linear (and time-invariant) systems.

As an aside, the use of the term "harmonic" here is consistent with its use in music. For instance, on a stringed instrument, patterns of vibration can be described as a superposition of harmonically related periodic exponentials.

Example 8. *What is the magnitude of the signal* $x(t) = e^{j2t} + e^{j5t}$*?*

Ans: While we could expand both exponentials directly as sinusoids, combine the real and imaginary parts, and take the magnitude from there, we will obtain a cleaner solution by factoring out an exponential first. To do this, take the average frequency $(2 + 5)/2 = 3.5$*, and rewrite* $x(t)$ *using Euler's formula as*

$$x(t) = e^{j3.5t}\left(e^{-j1.5t} + e^{j1.5t}\right) = 2e^{j3.5t}\cos(1.5t)$$

Since the magnitude of a product is the product of the magnitudes, we can readily obtain

$$|x(t)| = 2\left|e^{j3.5t}\right|\left|\cos(1.5t)\right| = 2\left|\cos(1.5t)\right|$$

since $\left|e^{j\beta t}\right| = 1$ *for any number* β*.*