Signals and Systems: Module 4

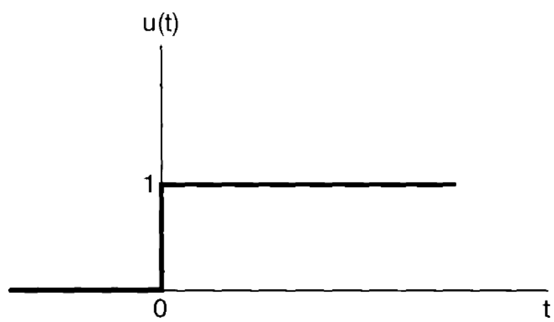
Suggested Reading: S&S 1.4.2, 1.5, 1.6

CT Unit Step and Unit Impulse Signal

The second half of the last module covered the unit step and unit impulse signals in discrete time (DT). We start here by covering their analogous versions in continuous time (CT).

The CT unit step signal, $u(t)$, is defined as:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



Different from $u[n]$ in DT which is defined at $n = 0$, $u(t)$ is discontinuous at $t = 0$. The CT unit step is sometimes called the **Heaviside step function** (after Oliver Heaviside).

The CT unit impulse signal $\delta(t)$ is related to CT unit step $u(t)$ in a manner analogous to the relationship between the DT unit impulse $\delta[n]$ and unit step $u[n]$. Recall that the DT unit impulse function is related to the unit step function via:

$$\delta[n] = u[n] - u[n-1]$$

For CT, the relationship becomes a derivative instead:

$$\delta(t) = \frac{du(t)}{dt}$$

But since $u(t)$ is not continuous and not differentiable at $t = 0$, what does the above equation exactly mean?

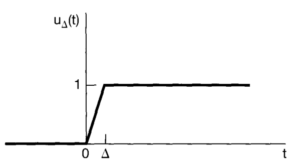
In contrast to the DT case, there is some formal difficulty with this equation as a representation of the unit impulse function. We can, however, interpret $\delta(t) = \frac{du(t)}{dt}$ by considering an approximation to the unit step, $u_\Delta(t)$, which rises from 0 to 1 in a short time interval Δ, as illustrated in Figure 1a. The unit step $u(t)$ changes values instantaneously, and thus can be thought of as an idealization of $u_\Delta(t)$ for infinitesimally small Δ. Formally,

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

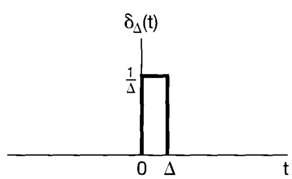
Let us now consider the derivative

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

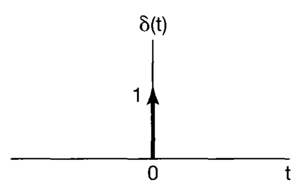
as shown in Figure 1b. By definition, $\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$.



(a) Continuous approximation to unit step, $u_\Delta(t)$.



(b) Derivative of $u_\Delta(t)$.



(c) Continuous-time unit impulse $\delta(t)$.

Figure 1

The area under $\delta_\Delta(t)$ is always 1, and therefore the area under the "infinite spike" $\delta(t)$ is also 1. $\delta(t)$ is visualized in Figure 1c: the arrow at $t = 0$ indicates an impulse at this point. The height of the arrow and the "1" next to it are for the *area* of the impulse, rather than the value of $\delta(t)$.

Thus, the integral of the unit impulse over all time is 1:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

In fact, this integral is 1 as long as the range of integration includes $\tau = 0$:

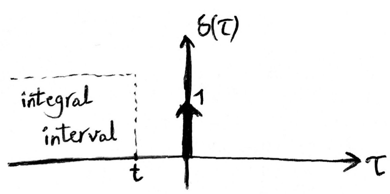
$$\int_{-a}^{b} \delta(t)\, dt = 1, \quad a, b \geq 0$$

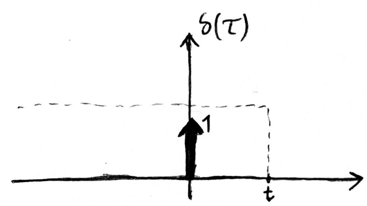
More generally, the indefinite integral up to time t gives the unit step function:

$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

where we use τ as the variable of integration to avoid confusion with t in the integral limit. In the plot below, we visualize this integral for $t < 0$ (before we hit the impulse) and $t > 0$ (after we hit the impulse):

$\underline{t < 0}$: $u(t) = 0$ $\qquad\qquad$ $\underline{t > 0}$: $u(t) = 1$





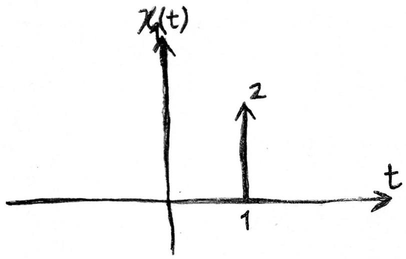
More generally, a scaled impulse $k\delta(t)$ for a constant k will have an area k:

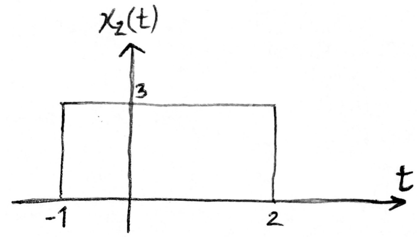
$$\int_{-\infty}^{\infty} k\delta(\tau)\, d\tau = k$$
$$\int_{-\infty}^{t} k\delta(\tau)\, d\tau = ku(t)$$

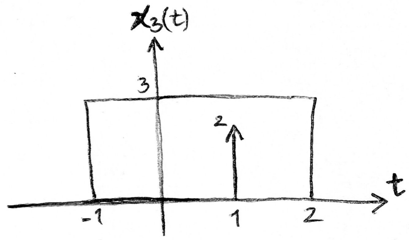
The continuous-time unit impulse is sometimes called the **Dirac Delta** (after Paul Dirac).

Example 1. *Plot* $x_1(t) = 2\delta(t-1)$, $x_2(t) = 3u(t+1) - 3u(t-2)$, $x_3(t) = x_1(t) + x_2(t)$.

Ans:





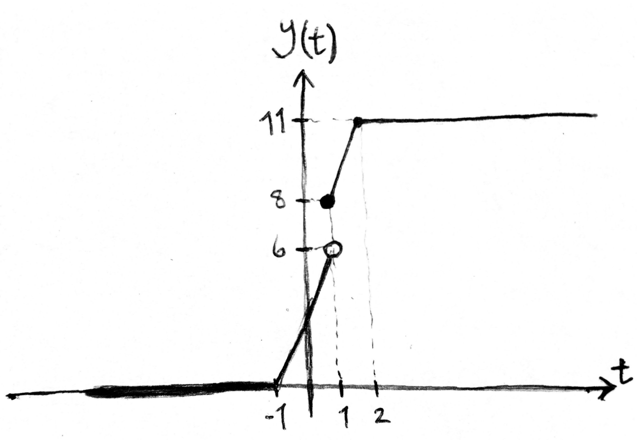


Example 2. *As a follow-up on the previous example, plot* $y(t) = \int_{-\infty}^{t} x_3(\tau)\, d\tau$.

Ans: We need to consider a few different intervals of t *separately:*

1	$t < -1$	$y(t) = 0$
2	$-1 \le t < 1$	$y(t) = \int_{-1}^{t} 3\, d\tau = 3(t+1) = 3t + 3$
3	$1 \le t < 2$	$y(t) = \int_{-1}^{t} 3\, d\tau + \textit{area of impulse} = 3t + 5$
4	$2 \le t$	$y(t) = \int_{-1}^{2} 3\, d\tau + \textit{area of impulse} = 9 + 2 = 11$

The corresponding plot is as follows:



As we can see from this example, integrating across the impulse creates a sudden change in function values.

Properties of $u(t)$ and $\delta(t)$

We will now investigate several important properties of the CT unit step and impulse signals.

– **Conversions between $u(t)$ and $\delta(t)$**

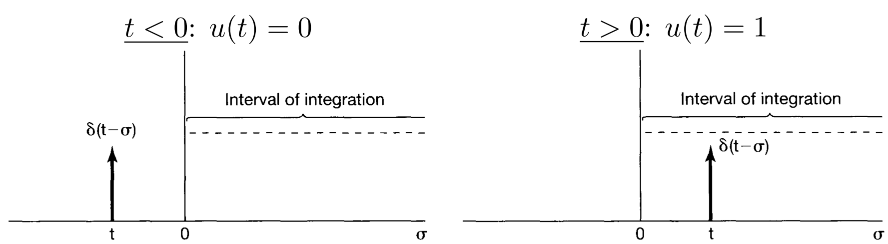
① $\delta(t) = \frac{du(t)}{dt}$

② $u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$

③ Changing the integration variable to $\sigma = t - \tau$ in the above integral gives

$$u(t) = \int_{+\infty}^{0} \delta(t-\sigma)\,(-d\sigma) = \int_{0}^{+\infty} \delta(t-\sigma)\, d\sigma$$

Note that $\delta(t-\sigma)$ is a function of σ, as it is inside the integral where the integration variable is σ. The graphical interpretation of this form of the relationship between $u(t)$ and $\delta(t)$ is given in the figure below. Since the area of $\delta(t-\sigma)$ is concentrated at the point $\sigma = t$, we again see that the integral in the equation above is 0 for $t < 0$ and 1 for $t > 0$.



This type of graphical interpretation of the behavior of the unit impulse under integration will be extremely useful later when we discuss convolution.

– **Integration of $\delta(t)$**

We already showed that

$$\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$$

Another form of the integration of the unit impulse function is

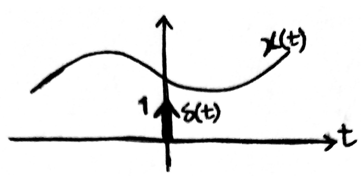
$$\int_{-\infty}^{\infty} \underbrace{\delta(t-\sigma)}_{\substack{\sigma \text{ is the variable,} \\ \text{the impulse is} \\ \text{at } \sigma = t.}} \, d\sigma = 1(t)$$

where $\delta(t-\sigma)$ is a unit impulse function of variable σ whose impulse is located at $\sigma = t$. Notice the output is a signal $1(t)$ with variable t, and the signal value is always 1.

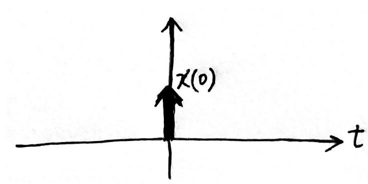
– **Sampling property**

As with the DT impulse, the CT impulse has a very important sampling property. In particular, for a number of reasons it will be important to consider the product of an impulse with a CT signal $x(t)$ under consideration:

$$x(t) \cdot \delta(t) = x(0) \cdot \delta(t)$$



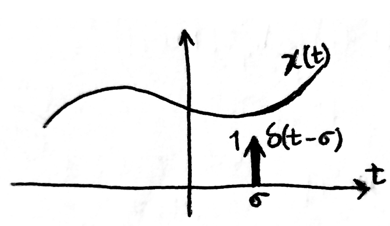
$\Rightarrow$



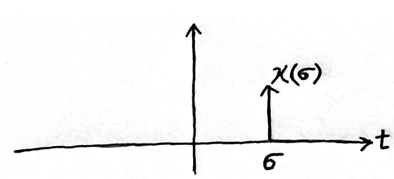
More generally,

$$\underbrace{x(t)}_{\text{signal}} \cdot \underbrace{\delta(t-\sigma)}_{\text{signal}} = \underbrace{x(\sigma)}_{\text{coeff}} \cdot \underbrace{\delta(t-\sigma)}_{\text{signal}}$$

where "coeff" stands for coefficient, as $x(\sigma)$ becomes the scaling factor on the signal $\delta(t-\sigma)$ at $t = \sigma$:



$\Rightarrow$



The value of $x(t)$ at $t = \sigma$ is in effect "stored" in the area of the impulse $\delta(t-\sigma)$.

– **Decomposing $x(t)$ as an integral of weighted, shifted impulses**

It will also be useful to write $x(t)$ in terms of $\delta(t)$ as follows:

$$\begin{aligned} x(t) &= x(t) \cdot 1(t) \\ &= x(t) \int_{-\infty}^{\infty} \delta(t-\sigma)\, d\sigma, \\ &= \int_{-\infty}^{\infty} x(t)\delta(t-\sigma)\, d\sigma \end{aligned}$$

where $1(t)$ is the constant 1 signal defined above. By the sampling property, $x(t)\delta(t-\sigma) = x(\sigma)\delta(t-\sigma)$. Therefore, we have

$$x(t) = \int_{-\infty}^{\infty} \underbrace{x(\sigma)}_{\text{coeff}} \underbrace{\delta(t-\sigma)}_{\text{signal}}\, d\sigma$$

where we see that $x(t)$ is contained in the weight of the impulse at $\sigma = t$. In comparison, for DT signals, we saw last time that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Similar to DT impulse functions, CT impulses can also be very useful test signals for linear systems. Letting $x_\sigma(t) = \delta(t-\sigma)$, then a new signal $x(t)$ can be represented as

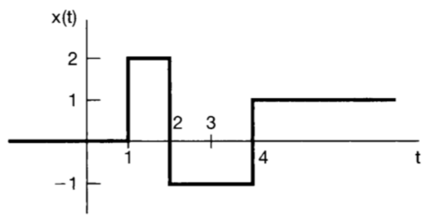
$$x(t) = \int_{-\infty}^{\infty} \underbrace{x(\sigma)}_{\text{coeff}} \underbrace{x_\sigma(t)}_{\text{test signals}}\, d\sigma$$

For linear systems, when inputting $x_\sigma(t)$ gives $y_\sigma(t)$, the output of the new input signal would be

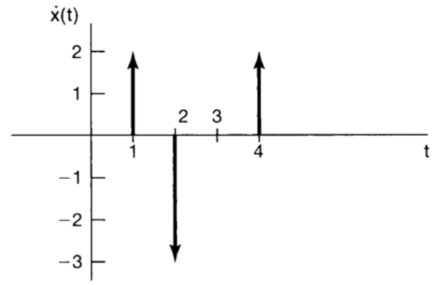
$$y(t) = \int_{-\infty}^{\infty} x(\sigma) y_\sigma(t)\, d\sigma$$

In this class, the test signals we will use will be either HRCEs (for periodic signals) or shifted unit impulses (for periodic signals).

Example 3. *Consider the signal $x(t)$ in Figure 7a. What is its mathematical formula? What is its derivative $\dot{x}(t)$?*



(a) Signal $x(t)$



(b) Derivative $\dot{x}(t)$

Figure 7: Signal $x(t)$ and its derivative $\dot{x}(t)$ in Example 3.

We can write $x(t)$ as the sum of three delayed, weighted unit step functions:

$$x(t) = 2u(t-1) - 3u(t-2) + 2u(t-4)$$

At $t = 2.5$, for example, this gives

$$x(2.5) = 2u(1.5) - 3u(0.5) + 2u(-1.5) = 2 - 3 + 0 = -1$$

The derivative $\dot{x}(t)$ is going to be 0 at all points except the three discontinuous points: $t = 1, 2, 4$. At these points, we need to have impulses with areas equal to the difference between the limits to the left and the right. At $t = 2$, for example, it should be -3 (going from 2 to -1). The result is shown in Figure 7b. We can also directly take the derivative of $x(t)$, since $\delta(t) = \frac{du(t)}{dt}$:

$$\dot{x}(t) = 2\delta(t-1) - 3\delta(t-2) + 2\delta(t-4)$$

We can readily check that $x(t) = \int_{-\infty}^{t} \dot{x}(\tau)d\tau$ for all values of t. At $t = 1.5$, for example, we get

$$2\int_{-\infty}^{1.5} \delta(\tau-1)d\tau - 3\int_{-\infty}^{1.5} \delta(\tau-2)d\tau + 2\int_{-\infty}^{1.5} \delta(\tau-4)d\tau = 2\times1 - 3\times0 + 2\times0 = 2$$

Interconnection of Systems

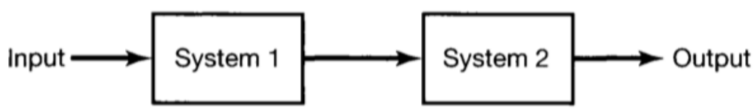
Equipped with an understanding of fundamental signals and their properties, we will now jump back to our study of systems.

To understand the impact of a system, it is often useful to view it as an interconnection of several subsystems. We may break down your cable TV system, for example, as an interconnection of a set-top box, a television, and several cables. By viewing a system this way, we can use our understanding of the component systems and how they are interconnected to analyze the impact on an input signal piece-by-piece. We can also construct a system to perform a specific function by incorporating several subsystems.

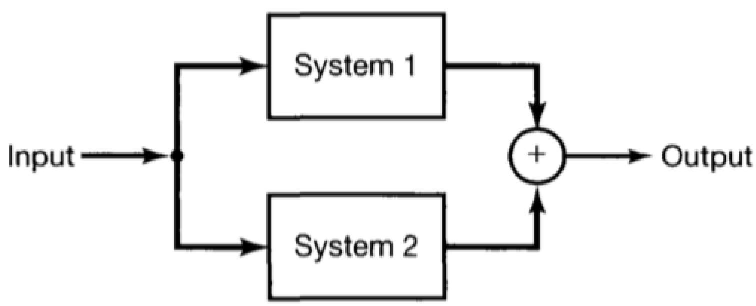
There are three fundamental interconnections that we will consider in this course:

(1) Series. In a **series** interconnection, systems operate on the output of other systems. This is shown for two systems in Figure 8a: the output of System 1 is the input to System 2, and the overall system transforms an input by processing it first by System 1 and then by System 2. A series interconnection is sometimes also called a **cascade** interconnection.

(2) Parallel. In a **parallel** interconnection, the same input signal is applied to each system, and the results are then combined to form the final output. Figure 8b shows this for two systems: the $\oplus$ symbol denotes addition, so that the output is the sum of the component outputs.



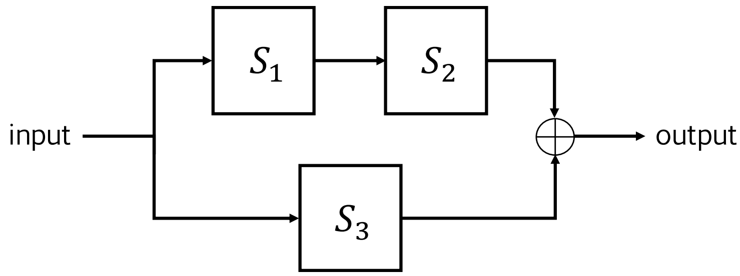
(a) Series



(b) Parallel

Figure 8: Two systems with (a) series and (b) parallel interconnections.

We can also readily combine series and parallel interconnections. In the following system, S_3 is connected in parallel with the series connection of S_1 and S_2:



(3) Feedback. In a **feedback** interconnection, the output of a system is fed back and combined with the input. Figure 9 gives a common case: the output of System 1 is the input to System 2, while the output of System 2 is fed back and added to the external input to produce the full input to System 1. Feedback arises in many applications, from thermostats (i.e., keeping the temperature at a desired level) to cruise control (i.e., regulating the velocity of the vehicle).

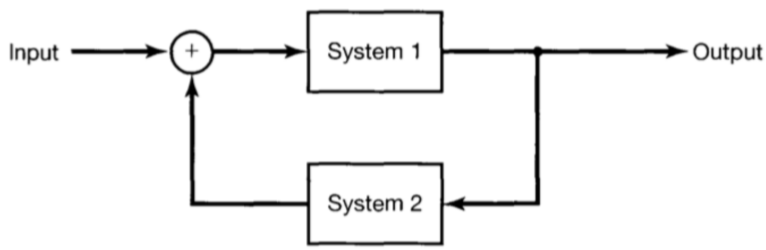
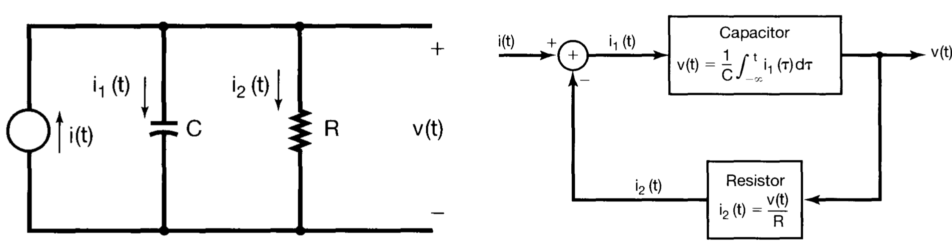


Figure 9: Two systems in a feedback interconnection.

Example 4. *Consider the electrical circuit of a resistor R and a capacitor C in Figure 10a. How can we represent this as a feedback interconnection?*



(a) Electrical circuit

(b) Feedback interconnection

Figure 10

Ans: The current $i(t)$, the input of the system, splits into $i_1(t)$ and $i_2(t)$ that run through C and R. The voltage $v(t)$, the system output, must be the same across both C and R. We can view this as a feedback system where the capacitor converts from $i_1(t)$ to $v(t)$, and the resistor in turn converts from $v(t)$ to $i_2(t)$, with $i_2(t)$ being subtracted from $i(t)$ to make $i_1(t)$.

Properties of Systems

Recall in Module 1 we formulated what it means for a system to be linear. Here we will introduce and discuss a number of other basic properties of systems. These properties have important physical interpretations and relatively simple mathematical descriptions.

(1) Memory vs. Memoryless

A system is **memoryless** if its output at a given time is dependent only on its input at that same time. Formally, we say $y(t)$ (or $y[n]$) at time t (or n) depends only on the instantaneous value of $x(t)$ (or $x[n]$) at time t (or n).

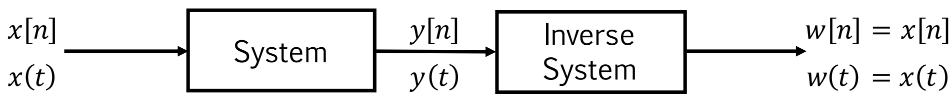
A system that is not memoryless is said to be **with memory**, i.e., the output is dependent on either past or future values of the input.

Example 5. *Classify whether the following signals are memoryless or with memory.*

- $y(t) = x(t)$ — *Memoryless, identity system*
- $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ — *With memory, e.g. capacitor* $\begin{cases} \textit{input: current} \\ \textit{output: voltage} \end{cases}$
- $y(t) = x(t - t_0)$ — *With memory, delay system (if $t_0 > 0$)*
- $y[n] = \sum_{k=-\infty}^{n} x[k] = y[n-1] + x[n]$ — *With memory, accumulator*
- $y[n] = x[n] - x[n-1]$ — *With memory, first difference*

(2) Invertible vs. Non-Invertible

A system is said to be **invertible** if distinct inputs lead to distinct outputs. In other words, there must be an **inverse system** that can be cascaded with the system to recover the original input:



Mathematically, a system is invertible if and only if for any two inputs $x_1(t), x_2(t)$ where $x_1(t) \neq x_2(t)$, we have outputs $y_1(t), y_2(t)$ where $y_1(t) \neq y_2(t)$. In other words, we need to be able to find a unique expression of the input $x(t)$ in terms of the output $y(t)$, which defines the inverse system.

On the contrary, to show a system is not invertible, we only need one counter example such that $x_1(t) \neq x_2(t)$ produces $y_1(t) = y_2(t)$.

Example 6. *The following are examples of invertible systems:*

CT: $y(t) = 2x(t)$
$w(t) = \frac{1}{2}y(t) = x(t)$. <u>*invertible*</u>

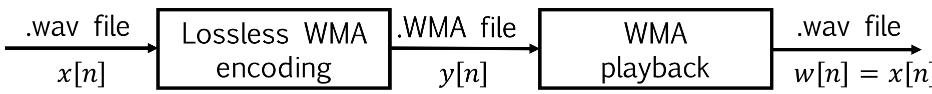
DT: *An accumulator*
$y[n] = \sum_{k=-\infty}^{n} x[k] = y[n-1] + x[n]$
$w[n] = y[n] - y[n-1] = x[n]$. <u>*invertible*</u>

Example 7. *The following are examples of non-invertible systems:*

CT: $y(t) = x^2(t)$
$x_1(t) = 1$ *and* $x_2(t) = -1$ *will both produce the same output* $y(t) = 1$. <u>*non-invertible*</u>

DT: $y[n] = 5$
the output does not depend on the input. we cannot recover $x[n]$. <u>*non-invertible*</u>

One application for which invertibility is important is **encoding**. In an encoding system for communications, a signal that we wish to transmit is first input to a system known as an encoder, and then is transmitted in its encoded form. When it arrives at the receiver, it is then inputted into a decoder. For lossless coding, the encoder must be invertible, i.e., the decoder should be able to exactly recover the original input signal:



(3) Causal vs. Non-Causal

A **causal** system is one for which the output at any given time depends only on the input from the present and the past. This property is often referred to as **non-anticipative**, since the system output does not anticipate future values of the input.

Example 8. *Accumulators are causal:*

$$y[n] = \sum_{k=-\infty}^{n} x[k] = y[n-1] + x[n]$$

Capacitors are also causal:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$

Any memoryless system will be causal by definition:

$$e.g.\ y(t) = x(t)$$

But, the following two systems are not causal because the output depends on the input from the "future":

$$y[n] = x[n] - x[n+1]$$
$$y(t) = x(t+1)$$

Non-causal seems like a strange property at first glance: how could the output be dependent on the future? These systems are actually of great practical importance, especially when we are processing data that has been recorded previously. If we want to determine a slowly varying trend, for example, we may decide to apply a moving average system:

$$y[n] = \frac{1}{2M+1}\sum_{k=-M}^{M} x[n-k]$$

which will have the effect of smoothing out fluctuations around the trend. This system is non-causal. Non-causality can also arise when the independent variable is not time, such as in image processing.

Example 9. *Is the system $y[n] = x[-n]$ causal? How about $y(t) = x(t)\cos(t+1)$?*

Ans: Our gut instinct is to conclude that the first system is causal and the second is not. But we need to consider each carefully.

For $y[n] = x[-n]$, if $n > 0$ then the system is only dependent on past input values. But we need to consider all values of n: if $n < 0$, then the argument to x will be positive while that to y is negative. Hence this system is not causal.

$y(t) = x(t)\cos(t+1)$ illustrates another important issue: we have to carefully isolate the effect of the input on the output. Even though $\cos(t+1)$ has $t+1 > t$, this is not part of the input $x(t)$. What we have here is the input multiplied by a number that varies with time. Since only the current value of $x(t)$ impacts $y(t)$, the system is causal, and in fact memoryless.

(4) Stable vs. Unstable

A system is said to be **stable** if bounded inputs will always lead to non-diverging outputs. Formally, a stable system is one for which

$$|x(t)| < B \qquad \text{implies that} \qquad |y(t)| < M$$

for some constants B and M: if the input is bounded by B for all time, then there exists a bound M on the output for all time as well. We often refer to this as **BIBO** – bounded-input bounded-output – stability.

Real life examples. A chain reaction would constitute an unstable system, since the size of the reaction would grow without bound even for just a small input. Even a bank account that accrues interest and experiences no withdrawals is technically an unstable system, since the amount will grow each month without bound over an infinite time horizon. Maybe banks should use this as a marketing strategy!

There are also many real-world examples of stable systems. Negative feedback is a way to enforce stability, with the output being used to correct or calibrate for differences in the input. Stability can also result from the dissipation of energy: for example, an RC circuit is stable due to the resistor in the circuit dissipating energy.

Example 10. *Consider the following DT systems. Are they stable?*

- $S_1 : y[n] = \frac{1}{2M+1}\sum_{k=-M}^{M} x[n-k]$

  Ans: This is a moving average system. Assume $|x[n]| < B$*. then*

$$\begin{aligned} |y[n]| &= \frac{1}{2M+1}\left|\sum_{k=-M}^{M} x[n-k]\right| \\ &< \frac{1}{2M+1}\sum_{k=-M}^{M} |x[n-k]| \\ &< \frac{1}{2M+1}(2M+1)B = B. \end{aligned}$$

  Since $|y[n]|$ *is bounded,* S_1 *is* *<u>stable</u>.*

- $S_2 : y[n] = \sum_{k=-\infty}^{n} x[k]$

  Ans: This is an accumulator system. $y[n]$ *is the sum of an infinite number of* $x[n]$*. So even if* $x[n]$ *is bounded,* $y[n]$ *can diverge. Therefore* S_2 *is* *<u>unstable</u>.*

Example 11. *Consider the following CT systems. Are they stable?*

- $S_1 : y(t) = tx(t)$

  Ans: We have $|y(t)| = |t||x(t)|$*. So, even if* $|x(t)|$ *is bounded,* t *will go to infinity, and so will* $|y(t)|$*. Thus,* S_1 *is* *<u>unstable</u>.*

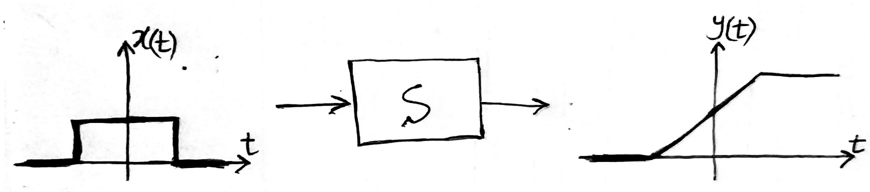
- $S_2 : y(t) = e^{x(t)}$

  Ans: If $|x(t)| < B$*, then* $|y(t)| < e^{B}$ *is bounded. Thus,* S_2 *is* *<u>stable</u>.*

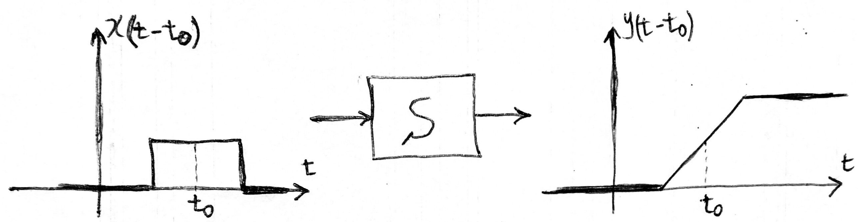
(5) Time Invariant vs. Time Varying

A system is said to be **time invariant** if its behavior and characteristics are fixed over time. In other words, applying some input at a particular time will give the same output as if we applied that same input at a different time.

That is, for any time-invariant system, if input $x(t)$ produces output $y(t)$, i.e.,

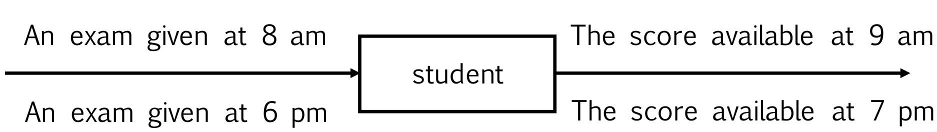


then a time-shifted input $x(t-t_0)$ will produce an output $y(t-t_0)$, i.e.,



A system is **time-varying** if it is not time invariant.

Example 12. *As a lighthearted example, consider that the exam score of a student may be time-varying. In particular, if the exam is given to a student at 8am, it will result in a different score than that of the same exam given to the same student at 6pm (less tired!).*



How to check whether a system is time-invariant? We test whether a time shift in the input signal results in an identical time shift in the output signal. Formally, for a discrete-time system, time-invariance holds if and only if

$$x[n] \to y_1[n] \qquad \text{implies} \qquad x[n-n_0] \to y_2[n] = y_1[n-n_0]$$

and for a continuous-time system,

$$x(t) \to y_1(t) \qquad \text{implies} \qquad x(t-t_0) \to y_2(t) = y_1(t-t_0)$$

Example 13. *Consider the systems* $S_1 : y(t) = \sin[x(t)]$ *and* $S_2 : y[n] = nx[n]$*. Are they time-invariant?*

Ans: For S_1*, we have* $x(t) \to y_1(t) = \sin[x(t)]$*. Time-shifting this output, we get* $y_1(t - t_0) = \sin[x(t - t_0)]$*, and when we input a time-shifted signal, we get* $x(t - t_0) \to y_2(t) = \sin[x(t - t_0)] = y_1(t - t_0)$*. Therefore, this system is* <u>*time-invariant*</u>*.*

For S_2*,* $x[n] \to y_1[n] = nx[n]$*. Time-shifting the output* $y_1[n]$*, we have* $y_1[n - n_0] = (n - n_0)x[n - n_0]$*. When we input the time-shifted* $x[n - n_0]$*, we get* $y_2[n] = x[n - n_0] \to nx[n - n_0]$*, which is not the same as* $y_1[n - n_0]$*. Since time-shifting the input does not cause an equivalent time shift in the output, the system is* <u>*time-varying*</u>*.*

Example 14. *Is the system* $y(t) = x(2t)$ *time-invariant?*

Ans: Intuitively, any time-scaling system will not be time-invariant, since scaling a time-shifted input will have a different effect than time-shifting the scaled output. In this case, an input $x(t)$ *produces an output* $y_1(t) = x(2t)$*, and when we shift this output, we get* $y_1(t - t_0) = x(2(t - t_0)) = x(2t - 2t_0)$*. On the other hand, suppose we first shift the input to* $x(t - t_0)$*. Applying this to the system produces* $y_2(t) = x(2t - t_0)$*. We see that* $y_1(t - t_0) \neq y_2(t)$*, and thus the system is* <u>*time-varying*</u>*.*

Another way to prove this – and more generally, to show any system does not possess a particular property – is to construct a counter-example. Consider the input $x(t) = u(t + 2) - u(t - 2)$*. This will lead to the output* $y_1(t) = u(2t + 2) - u(2t - 2) = u(t + 1) - u(t - 1)$*, which when time shifted gives* $y_1(t - t_0) = u(t - t_0 + 1) - u(t - t_0 - 1)$*. If we time-shift the input, on the other hand, we get* $x(t - t_0) = u(t - t_0 + 2) - u(t - t_0 - 2)$*, and then the output is* $y_2(t) = x(2t - t_0) = u(2t - t_0 + 2) - u(2t - t_0 - 2)$*, which is not the same as* $y(t - t_0)$*. Graphs of these two different outputs for* $t_0 = 2$ *are shown in Figure 11, where* y_1 *and* y_2 *correspond to the outputs from non-time-shifted and time-shifted inputs, respectively.*

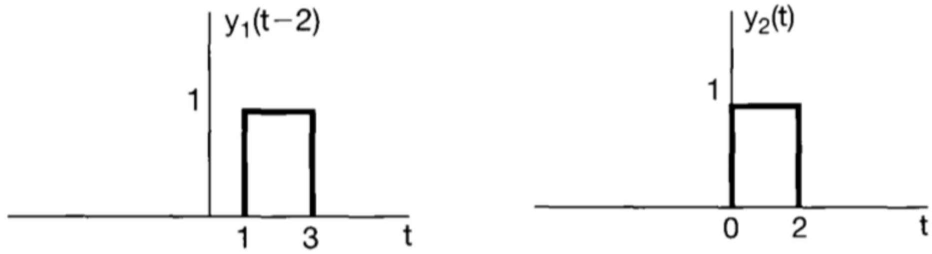


Figure 11: Results from (left) time-shifting the output and (right) applying a time-shifted input to the system in Example 14.