Signals and Systems: Module 8

Suggested Reading: S&S 4.1-4.3

We have now learned how to analyze a periodic signal in terms of its frequency contents. The so-called Fourier series is based on linear combinations of harmonically-related complex exponentials. For the next few modules, we will extend these concepts to signals that are *not* periodic, using the Fourier transform. As we will see, the key idea is to define the exponential "building blocks" as being infinitesimally close in frequency. We start here with the continuous-time case, and move to discrete-time later.

Continuous-Time Fourier Transform

Consider a signal $x(t)$ that has a finite duration: say $x(t) = 0$ if $|t| > T_1$, as shown in Figure 1(a). If we want to analyze the frequency contents of $x(t)$, we cannot use the Fourier series directly, since $x(t)$ is aperiodic. From $x(t)$, though, suppose we construct the periodic signal $\tilde{x}(t)$ in Figure 1(b) for which the finite component of $x(t)$ is contained in one period T. As T becomes larger, $\tilde{x}(t)$ will become closer to $x(t)$.

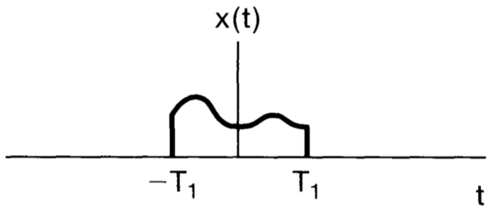
Fourier series of $\tilde{x}(t)$. The Fourier series of $\tilde{x}(t)$, by definition, can be obtained as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

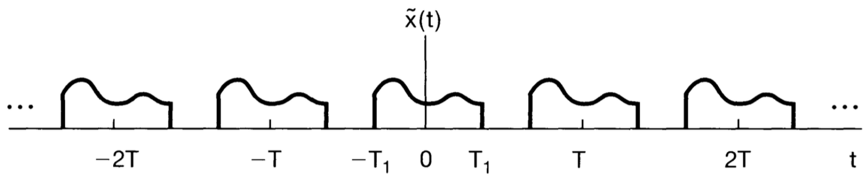
where $\omega_0 = 2\pi/T$. Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$ and $x(t) = 0$ otherwise, the equation for a_k can be written as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt$$

Now, let's consider Ta_k as a function of k. We can consider the envelope of Ta_k as as a function of ω, a continuous variable, instead of $k\omega_0$. The



(a) Finite energy signal $x(t)$



(b) Periodic $\tilde{x}(t)$ constructed from $x(t)$

Figure 1: Constructing a periodic version $\tilde{x}(t)$ of an aperiodic $x(t)$ for analysis.

coefficients a_k can be obtained at discrete values of the envelope, i.e., for $\omega = k\omega_0$:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \qquad a_k = \frac{1}{T}X(jk\omega_0)$$

Through synthesis, then, we can express $\tilde{x}(t)$ in terms of the envelope $X(j\omega)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

Fourier transform of $x(t)$. As we said, as $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$. The synthesis for $\tilde{x}(t)$ above will therefore be a representation of $x(t)$. What happens to the equation? As $T \to \infty$, $\omega_0 \to 0$, so it becomes an integral over ω as opposed to a summation over $k\omega_0$. In particular,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega, \quad \text{where} \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

The function $X(j\omega)$ is referred to as the **Fourier transform** or **Fourier integral** of $x(t)$, while the equation for $x(t)$ is the **inverse Fourier transform**. These expressions correspond to the analysis (transform) and synthesis (inverse) steps, respectively. We sometimes write this in abbreviated form using functions as:

$$x(t) = \mathcal{F}^{-1}(X(j\omega)), \qquad X(j\omega) = \mathcal{F}(x(t))$$

where $\mathcal{F}$ denotes the Fourier transform, and $\mathcal{F}^{-1}$ represents the inverse Fourier transform. We often abbreviate Continuous-Time Fourier Transform as **CTFT**.

Through the Fourier series, we saw that a periodic signal is comprised of periodic complex exponentials occurring at a discrete set of harmonically related frequencies $k\omega_0$. An aperiodic signal, by contrast, is comprised of periodic complex exponentials occurring at a continuum of frequencies ω. The function $X(j\omega)$ is thus commonly referred to as the **frequency spectrum** of an aperiodic signal $x(t)$, whereas the set of coefficients $\{a_k\}$ is the spectrum for a periodic signal.

Convergence. In deriving the Fourier transform, we assumed that $x(t)$ was of finite duration. More generally, the Fourier transform can be applied to an extremely broad class of finite or infinite-duration signals, but not all signals.

Suppose we transform $x(t)$ to $X(j\omega)$, and then apply the inverse transform $\mathcal{F}^{-1}$ to $X(j\omega)$. Call the recovered signal $\hat{x}(t)$. If the following three conditions on $x(t)$ hold, then $\hat{x}(t)$ will be a valid representation of $x(t)$:

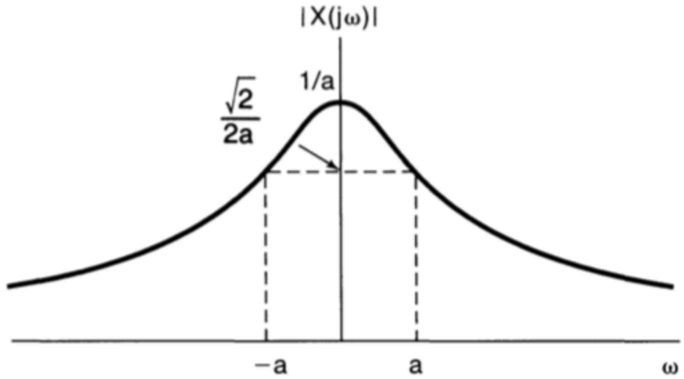
1. $x(t)$ is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |x(t)| dt < \infty$.
2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval, and these discontinuities are finite.

Therefore, absolutely integrable signals that are continuous or that have a finite number of discontinuities have Fourier transforms. Note these conditions are very similar to those we had for the Fourier series.

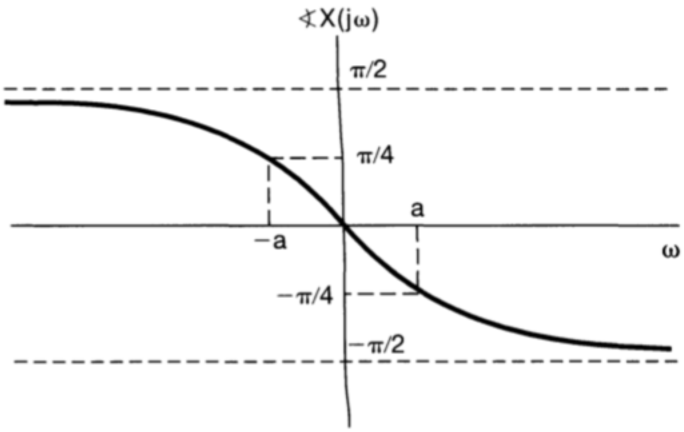
Important/Common CTFT Pairs

We will next take a look at some common CT signals and their Fourier transforms. A comprehensive table can be found in Table 4.2 of the textbook.

Decaying real exponential. One important signal to understand the Fourier transform for is $x(t) = e^{-at}u(t)$, $a > 0$. Applying the transform



(a) Magnitude plot $|X(j\omega)|$



(b) Phase plot $\theta_{X(j\omega)}$

Figure 2: Spectrum plot for a decaying real exponential signal.

equation, we get:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_{0}^{\infty} \\ &= \frac{1}{a+j\omega} \end{aligned}$$

To plot the spectrum of $x(t)$, we typically consider the magnitude and phase form of $X(j\omega)$. When $X(j\omega)$ is a fraction, the numerator and denominator divide for magnitude and subtract for phase:

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \qquad \theta_{X(j\omega)} = 0 - \arctan(\omega/a) = -\arctan(\omega/a)$$

These are plotted in Figure 2. The value $\omega = a$ is the point at which the spectrum has dropped in magnitude by $1/\sqrt{2}$ of its maximum value, or the half-power point: when a is larger, this happens at a higher frequency. This makes sense as a larger exponent causes a faster decay, and thus will induce higher frequency components.

We actually saw the frequency spectrum for a decaying exponential when we studied LTI system responses to periodic complex exponentials: $h(t) = e^{-at}u(t)$ it is the impulse response of a lowpass filter.

Unit impulse. Another important signal to understand the spectral composition of is the unit impulse $x(t) = \delta(t)$. From the transform equation,

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1$$

That is, the unit impulse has a Fourier transform consisting of equal contributions at all frequencies.

Rectangular pulse. The rectangular pulse signal

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases}$$

is another which is commonly encountered. It is the aperiodic version of the square wave we looked at in Fourier series. Applying the transform equation,

$$\begin{aligned} X(j\omega) &= \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1} = \frac{1}{\omega}\left(\frac{e^{jT_1\omega} - e^{-jT_1\omega}}{j}\right) \\ &= \frac{2\sin(T_1\omega)}{\omega} \end{aligned}$$

which is a sinusoid which has its amplitude decreasing as the frequency increases. A plot of $x(t)$ and $X(j\omega)$ can be seen in the top half of Figure 3 (marked as x_1); as $X(j\omega)$ is purely real, we only need one plot for it. Note that the value $X(0) = 2T_1$ is obtained from l'Hopital's rule:

$$X(0) = \lim_{\omega \to 0} \frac{f(\omega)}{g(\omega)} = \lim_{\omega \to 0} \frac{f'(\omega)}{g'(\omega)} = \lim_{\omega \to 0} \frac{2T_1 \cos(T_1\omega)}{1} = 2T_1$$

Sinc waveform. Let's look at the opposite form of the previous case, and find the signal $x(t)$ whose Fourier transform is rectangular:

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

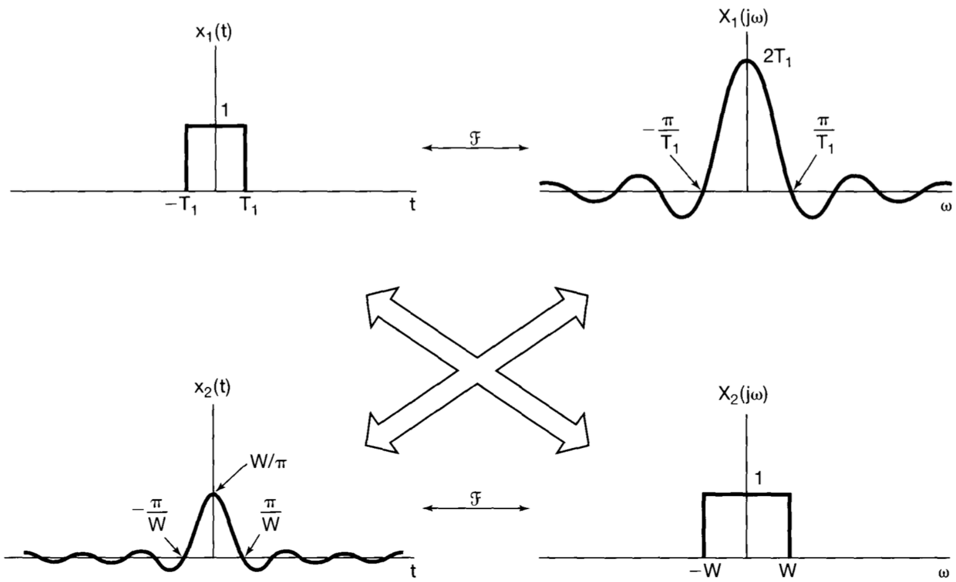


Figure 3: A rectangular wave in time ($x_1(t)$) has a sinc shape in frequency ($X_1(j\omega)$), and a sinc wave in time ($x_2(t)$) has a rectangular shape in frequency ($X_2(j\omega)$).

Applying the synthesis equation, we have

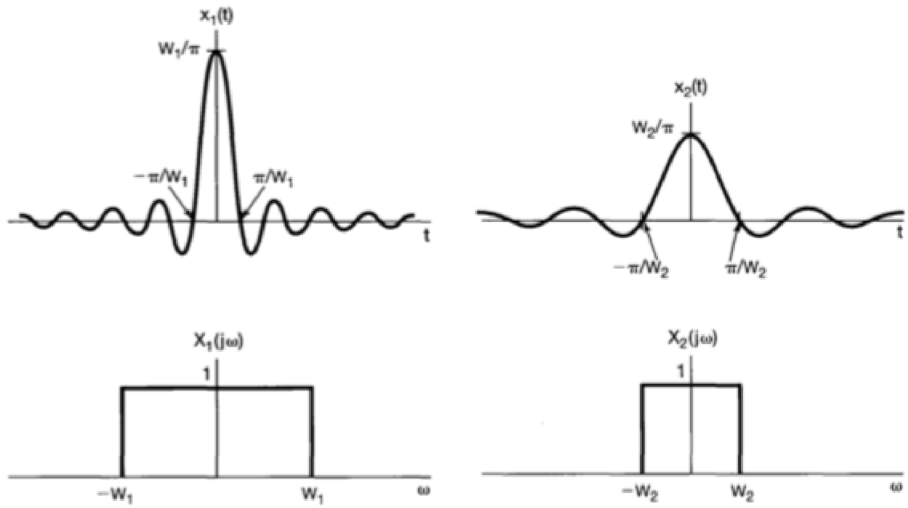
$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{1}{2\pi j t}e^{j\omega t}\Big|_{-W}^{W} \\ &= \frac{1}{\pi t}\left(\frac{e^{jWt}-e^{-jWt}}{2j}\right) = \frac{\sin Wt}{\pi t} \end{aligned}$$

which gives us a signal of the form $\sin(x)/x$, but now in the time domain. The function form $\sin(x)/x$ arises frequently in Fourier analysis and the study of LTI systems, and is given the special name **sinc functions** as a result, with a common definition being

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$$

With this definition, we can write

$$x(t) = \frac{\sin(Wt)}{\pi t} = \frac{\sin(\pi W t/\pi)}{\pi W t/W} = \frac{W}{\pi}\frac{\sin(\pi W t/\pi)}{\pi W t/\pi} = \frac{W}{\pi}\mathrm{sinc}\left(\frac{Wt}{\pi}\right)$$



(a) Larger W (b) Smaller W

Figure 4: Effect of varying W in the sinc waveform.

This waveform $x(t)$ and its spectrum are plotted in the bottom half of Figure 3 (marked as x_2).

We can gain some more insight into the relationship between the time domain and the frequency domain by considering what happens to $x(t)$ here and its frequency spectrum as W changes. In Figure 4, we plot two cases, one for a larger $W = W_1$ and one for a smaller $W = W_2$. As W increases, the spectrum $X(j\omega)$ becomes broader, and thus $x(t)$ has more high frequency components: it has a larger value W/π at $t = 0$, and is narrower. As W decreases, the spectrum becomes narrower while the signal becomes broader.

The last two Fourier transform pairs we studied – the rectangular and sinc waves – form an important *duality* in Fourier analysis, depicted in Figure 3. A rectangular pulse in time gives a sinc shape in frequency, while a sinc wave in time gives a rectangular shape in frequency. We will formalize duality as a more general property of the CTFT later.

The Fourier Transform for Periodic Signals

It is possible to take the Fourier transform of a periodic signal. Doing so, not surprisingly, gives us the Fourier series coefficients located at different frequencies. To see this, consider a signal $x(t)$ whose Fourier transform is a linear combination of equally spaced, weighted impulses, i.e.,

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Applying the synthesis equation to this, we look at each impulse separately. The inverse transform of $\delta(\omega - k\omega_0)$ is $e^{jk\omega_0 t}/2\pi$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - k\omega_0) e^{j\omega t} d\omega = \frac{e^{jk\omega_0 t}}{2\pi}$$

It follows that

$$x(t) = \sum_{k=-\infty}^{\infty} 2\pi a_k \mathcal{F}^{-1}\{\delta(\omega - k\omega_0)\} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

which is exactly the Fourier series representation of a periodic signal. Thus, the Fourier transform of a periodic signal with Fourier series coefficients a_k can be interpreted as a train of impulses occurring at the harmonically related frequencies.

With this, we can now look at some other important/common CTFT pairs.

Sinusoids. Let

$$x(t) = \sin \omega_0 t$$

The Fourier series coefficients for this signal are

$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}, \quad a_k = 0, k \neq 1 \text{ or } -1$$

Thus, the Fourier transform of $x(t) = \sin \omega_0 t$ is

$$X(j\omega) = 2\pi \left(a_1 \delta(\omega - \omega_0) + a_{-1}\delta(\omega + \omega_0)\right) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

Similarly, for

$$x(t) = \cos \omega_0 t$$

the Fourier series coefficients are

$$a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0, k \neq 1 \text{ or } -1$$

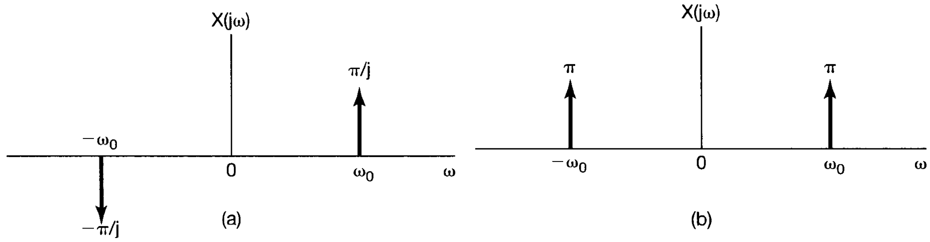


Figure 5: Fourier transforms of (a) $x(t) = \sin \omega_0 t$, (b) $x(t) = \cos \omega_0 t$.

The Fourier transform of $x(t) = \cos \omega_0 t$ is:

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

These two Fourier transforms are depicted in Figure 5. They will be of considerable importance when we consider sinusoidal modulation in communication systems systems later on.

Impulse train. A signal that we will find extremely useful in our analysis of sampling systems later is the impulse train

$$x(t) = \sum_{k=\infty}^{\infty} \delta(t - kT)$$

which is periodic with period T, as indicated in Figure 6(a). The Fourier series coefficients for this signal are given by

$$a_k = \frac{1}{T} \int_{-T/2}^{+T/2} \delta(t) e^{-jk\omega t} dt = \frac{1}{T}$$

That is, every Fourier coefficient of the periodic impulse train has the same value, $1/T$. Substituting this value for a_k in the general equation for Fourier series of periodic signals yields:

$$X(j\omega) = \frac{2\pi}{T} \sum_{k=\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$

as $X(j\omega)$ is nonzero only when $\omega = \omega_0 k = 2\pi k/T$. Thus, the Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$, as sketched in Figure 6(b).

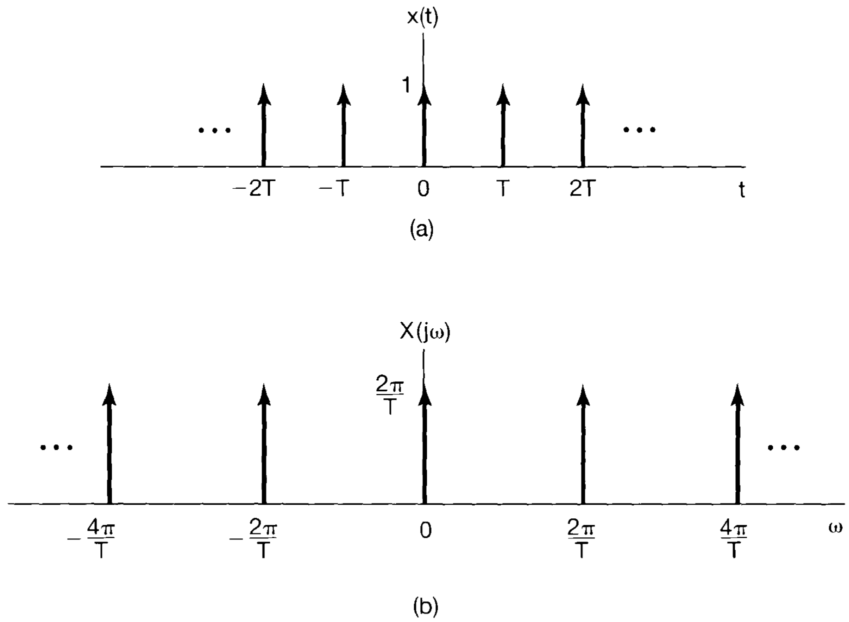


Figure 6: (a) Periodic impulse train; (b) its Fourier transform.

So, an impulse train in time becomes an impulse train in frequency. But here again, we see an illustration of the inverse relationship between the time and the frequency axes. As the spacing between the impulses in the time domain (i.e., the period) gets longer, the spacing between the impulses in the frequency domain (namely, the fundamental frequency) gets smaller.

Note that when we plot the Fourier transform $X(j\omega)$ of a periodic signal $x(t)$, it looks very similar to the plot of the Fourier series coefficients $\{a_k\}$. The difference in the plot is that the x-axis of $X(j\omega)$ is frequency ω, with the impulses spaced out at a width of ω_0. With the Fourier series, we plotted over integers k without an indication of ω_0 on the plot.

Example 1. *Consider the signal $x(t) = \cos(t) + \sin(\pi t)$.*

Q1: Is $x(t)$ periodic?

Ans: LCM $(\frac{2\pi}{1}, \frac{2\pi}{\pi})$ *does not exist. So* *not periodic.*

Q2: Find $X(j\omega)$.

Ans: Using Euler's formula,

$$x(t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} + \frac{1}{2j}e^{j\pi t} - \frac{1}{2j}e^{-j\pi t}$$

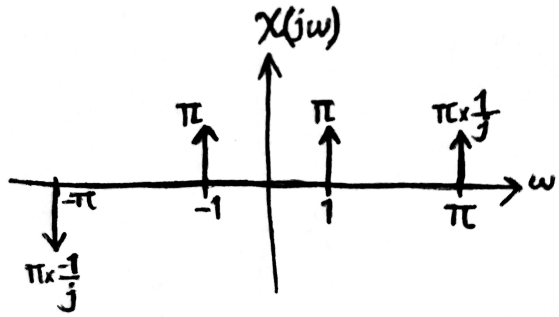
From our discussion on the Fourier transform of periodic complex exponentials,

$$\frac{1}{2\pi}e^{jat} \overset{\mathcal{F}}{\longleftrightarrow} \delta(\omega - a)$$

$$\Rightarrow X(j\omega) = \pi\delta(\omega - 1) + \pi\delta(\omega + 1) + \frac{\pi}{j}\delta(\omega - \pi) - \frac{\pi}{j}\delta(\omega + \pi)$$

Q3: Plot $X(j\omega)$.

Ans: We can do this in a single graph, with imaginary weights on the impulses as needed:



Example 1 shows us that while not all linear combinations of CT sinusoids will have a Fourier series (since they can be aperiodic), their Fourier transform will nonetheless exist (since it is meant for aperiodic waveforms).

Properties of the CTFT

As with the Fourier series, there are a number of important properties of the Fourier transform that are useful in analysis: linearity, time scaling, and so forth. The notation $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ will be useful in denoting a time-frequency Fourier transform pair.

Linearity. Like Fourier series, the Fourier transform is a linear operation. In particular, if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$ and $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$, then

$$ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$$

Time shifting. A time shift in the time domain corresponds to a phase shift in the frequency domain, with the magnitude remaining unchanged. In particular, if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then

$$x(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Conjugation. Taking the complex conjugate of a signal amounts to taking the conjugate of the transform and reversing the sign of ω. In particular, if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then

$$x^\star(t) \overset{\mathcal{F}}{\longleftrightarrow} X^\star(-j\omega)$$

As a result, we also have that

$$X(-j\omega) = X^\star(j\omega) \qquad \text{when } x(t) \text{ is real}$$

exhibiting conjugate symmetry. For a real signal, then, it follows that the spectrum magnitude $|X(j\omega)|$ must be an even function while the spectral phase $\theta_{X(j\omega)}$ must be an odd function. It also follows that the even and odd components of a real signal must take the real and imaginary components of the spectrum, respectively:

$$\mathcal{EV}\{x(t)\} \overset{\mathcal{F}}{\longleftrightarrow} \text{Re}\{X(j\omega)\}, \quad \mathcal{OD}\{x(t)\} \overset{\mathcal{F}}{\longleftrightarrow} j\text{Im}\{X(j\omega)\} \quad \text{when } x(t) \text{ is real}$$

From this, we can also deduce that a real and even signal $x(t)$ will have a purely real spectrum (with a phase of 0), while a real and odd signal $x(t)$ will have a purely imaginary spectrum (with a phase of $\pm\pi/2$).

Differentiation and integration. Differentiation and integration in the time domain reduce to multiplication and division by a factor of $j\omega$ in the frequency domain. In particular, if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) \qquad \int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

These properties will be extremely useful when we consider LTI systems described by differential equations. With integration, the extra factor $X(0)\delta(\omega)$ reflects the DC or average constant value that can result from integration.

Time and frequency scaling. Compressing a signal in time has the effect of expanding its frequency components, and vice versa. Formally, if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

The inverse relationship between the time and frequency domains is of great importance in a variety of signal and systems contexts, including filtering and filter design.

Now, consider $x(\alpha t - \beta)$, i.e., a composite time shift and scaling. By applying the shifting and scaling properties sequentially, we have that:

$$x(\alpha t - \beta) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|\alpha|} X\left(\frac{j\omega}{\alpha}\right) e^{-j\beta(\omega/\alpha)}$$

Duality. We mentioned previously in Figure 3 that the rect and sinc functions form a *dual pair* in CTFT analysis. More generally, because of the symmetry between the Fourier transform and the inverse Fourier transform, for any time-frequency transform pair, there will be a dual pair with the time and frequency variables interchanged. In particular, if $\mathcal{F}\{x(t)\} = X(j\omega)$, then

$$\mathcal{F}\{X(jt)\} = 2\pi \cdot x(-\omega)$$

This property is called **duality**.

Differentiation in frequency. One important case of duality arises from the differentiation property. If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, then

$$tx(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(j\omega)}{d\omega}.$$

Proof: If $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then $\mathcal{F}\{X(jt)\} = 2\pi x(-\omega)$ according to duality. Using the differentiation property in time

$$\mathcal{F}\{\frac{dX(jt)}{dt}\} = j\omega 2\pi x(-\omega).$$

Then, using the duality property again, we have

$$\mathcal{F}\{jt2\pi x(-t)\} = 2\pi \cdot \frac{dX(j(-\omega))}{d(-\omega)}.$$

Finally, use the time scaling property and simplify both sides, we have

$$\mathcal{F}\{tx(t)\} = j\frac{dX(j\omega)}{d\omega}.$$

Thus, multiplication by t in time time domain has the effect of differentiation in the frequency domain.

Frequency shifting. Another important case of duality comes from the time shifting property. If $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then

$$x(t)e^{j\omega_0 t} \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0))$$

Thus, multiplication by a periodic complex exponential in time has the effect of frequency shifting.

Parseval's relation. The total energy in a signal $x(t)$ can be computed either in the time domain or the frequency domain. In particular, Parseval's relation tells us that if $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

For this reason, $|X(j\omega)|^2$ is often referred to as the **energy-density spectrum** of the signal $x(t)$. Note how this relation, for finite-energy signals, differs from Parseval's relation for periodic signals: here, we obtain total energy, whereas in that case it was average power over a period.

In the rest of this module, we will go over several examples illustrating the use of these properties.

Example 2. *What is the Fourier transform of the signal $x(t)$ in Figure 7?*

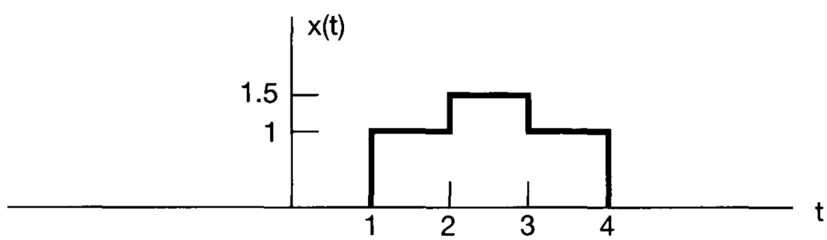


Figure 7: Signal $x(t)$ for Example 2.

Ans: We could apply the Fourier transform equation directly, but it is perhaps easier to leverage some of the properties we have just studied. In particular, consider the signal

$$x_r(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \textit{otherwise} \end{cases}$$

We can write

$$x(t) = x_r\left(\frac{t - 2.5}{3}\right) + 0.5 x_r\,(t - 2.5)$$

So if we can determine the transform of $x_r(t)$, we can get the one for $x(t)$. From our study of the rectangular pulse signal, we know that

$$x_1(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(j\omega) = \frac{2\sin(T_1\omega)}{\omega}$$

Thus, with $T_1 = 1/2$,

$$x_r(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{2\sin(\omega/2)}{\omega}$$

From linearity and time shifting, the Fourier transform is obtained as

$$\begin{aligned} X(j\omega) &= 3X_r(3j\omega)e^{-j2.5\omega} + 0.5X_r(j\omega)e^{-j2.5\omega} \\ &= e^{-j2.5\omega}\left(\frac{2\sin(3\omega/2) + \sin(\omega/2)}{\omega}\right) \end{aligned}$$

Example 3. *What is the Fourier transform of the signal $x(t)$ given below? How about of $g(t) = x(t) - \frac{1}{2}$?*

$$x(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ t + \frac{1}{2} & -\frac{1}{2} \le t \le \frac{1}{2} \\ 1 & t > \frac{1}{2} \end{cases}$$

Ans: $x(t)$ is a linear ramp from $t = -\frac{1}{2}$ to $t = \frac{1}{2}$, and then stays constant. It can be written as the integral of the rectangular pulse $x_r(t)$ from Example 2:

$$x(t) = \int_{-\infty}^{t} x_r(\tau)d\tau$$

By the integration property, then,

$$X(j\omega) = \frac{1}{j\omega}X_r(j\omega) + \pi X_r(0)\delta(\omega) = \frac{2\sin(\omega/2)}{j\omega^2} + \pi\delta(\omega)$$

since $X_r(0) = 1$ by l'Hopital's rule. As for $g(t)$, we need to subtract the Fourier transform of $1/2$. Determining this is easiest through duality: since $\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 1$, it follows that

$$1 \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

Thus

$$G(j\omega) = \frac{2\sin(\omega/2)}{j\omega^2}$$

which should make sense since $g(t)$ subtracts out the dc component of $x(t)$, which is captured in $\delta(\omega)$.

Example 4. *Let*

$$a(t) = \begin{cases} e^{-t} & 0 \leq t \leq 1 \\ 0 & otherwise \end{cases}$$

and let $x(t) = a(t) - a(-t)$. What is the Fourier transform of $x(t)$?

Ans: $x(t)$ is (twice) the odd component of $a(t)$. In other words,

$$x(t) = 2 \cdot \mathcal{OD}\{a(t)\}$$

Since one of our properties tells us

$$2 \cdot \mathcal{OD}\{a(t)\} \stackrel{\mathcal{F}}{\longleftrightarrow} 2j\,Im\{A(j\omega)\}$$

we just need to find $A(j\omega)$. Using the definition of the Fourier transform,

$$\mathcal{F}\{a(t)\} = \int_{-\infty}^{\infty} a(t)e^{-j\omega t}dt = \int_{0}^{1} e^{-t}e^{-j\omega t}dt = -\frac{1}{1+j\omega}e^{-(1+j\omega)t}\Big|_0^1$$

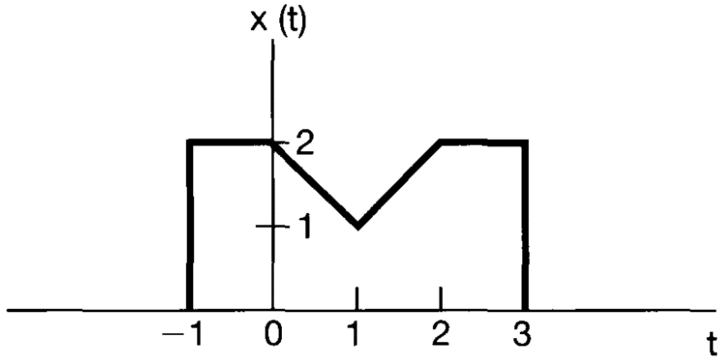
Now we apply Euler's formula and multiply by the complex conjugate of the denominator:

$$\begin{aligned} A(j\omega) &= \frac{1-e^{-(1+j\omega)}}{1+j\omega} = \frac{1-e^{-1}\cos(\omega)+je^{-1}\sin(\omega)}{1+j\omega} \\ &= \frac{1-e^{-1}\cos(\omega)+je^{-1}\sin(\omega)-j\omega+j\omega e^{-1}\cos(\omega)+\omega e^{-1}\sin(\omega)}{1+\omega^2} \\ &= \frac{1-e^{-1}\cos(\omega)+\omega e^{-1}\sin(\omega)+j\left(e^{-1}\sin(\omega)-\omega+\omega e^{-1}\cos(\omega)\right)}{1+\omega^2} \end{aligned}$$

Taking the imaginary component, it follows that

$$X(j\omega) = 2j\frac{e^{-1}\sin(\omega)-\omega+\omega e^{-1}\cos(\omega)}{1+\omega^2}$$

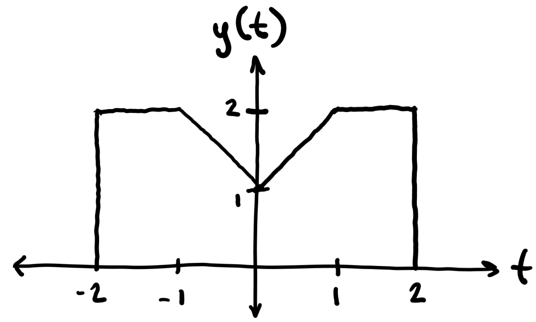
Example 5. *Consider $x(t)$ shown in the figure below. Find (a) $\angle X(j\omega)$, (b) $X(j \cdot 0)$, (c) $\int_{-\infty}^{\infty} X(j\omega)d\omega$, and (d) $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$.*



(a). *Even and odd signals have very simple phase relationships. While $x(t)$ is not exactly even, it is a time-delayed version of an even signal: if we define $y(t)$ as shown below, then $x(t) = y(t-1)$. By the time shift property of the CTFT, we know that $X(j\omega) = Y(j\omega) \cdot e^{-j\omega \cdot 1}$. Next, we make use of the property that the CTFT of an even signal has zero imaginary part; since $y(t)$ is even, then this means that $Y(j\omega)$ is real and that $\angle Y(j\omega) = 0$.*

Thus,

$$X(j\omega) = Y(j\omega)e^{-j\omega}$$
$$\Longrightarrow \angle X(j\omega) = \angle Y(j\omega) + \angle e^{-j\omega} = -\omega$$



(b). *To find $X(j \cdot 0)$, we can apply the CTFT formula:*

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ \Longrightarrow X(j \cdot 0) &= \int_{-\infty}^{\infty} x(t) \cdot e^{-j \cdot 0 \cdot t}dt = \int_{-\infty}^{\infty} x(t)dt = 7 \end{aligned}$$

This is the DC component of $x(t)$.

(c). *In order to find* $\int_{-\infty}^{\infty} X(j\omega)d\omega$, *recall the inverse CTFT formula:*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot e^{j\omega t}d\omega$$

Observe that if we plug in $t = 0$ *on both sides of this equation, then:*

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot e^{j\omega\cdot 0}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$$

$$\implies \int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0) = 2\pi\cdot 2 = 4\pi$$

(d). *To find* $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$, *we apply Parseval's relation:*

$$\begin{aligned}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega &= 2\pi\int_{-\infty}^{\infty} |x(t)|^2 dt = 2\pi\int_{-\infty}^{\infty} |y(t)|^2 dt \\ &= 2\cdot 2\pi\int_0^2 y^2(t)dt = \frac{76\pi}{3}\end{aligned}$$

To get the last step, we note that

$$y^2(t) = \begin{cases} (t+1)^2 & 0 < t < 1 \\ 4 & 1 < t < 2 \\ 0 & \textit{otherwise} \end{cases}$$