Signals and Systems: Module 10

Suggested Reading: S&S 8.1-8.4, 4.7

In this module, we will first investigate some of the fundamentals behind communication systems, which is one of the principle applications of Fourier analysis. Much of our discussion will focus on amplitude modulation (AM), which arises from the multiplication property of the Fourier transform. AM is one of the building blocks for many contemporary communication systems.

Then, at the end of the module, we will close off our discussion of the CTFT with one final application: analyzing systems characterized by differential equations.

Amplitude Modulation

Communication systems play a key role in our modem world in transmitting information between people, systems, and computers. In general terms, in all communication systems:

- The information at the source is processed by a transmitter or modulator to change it into a form suitable for transmission over the communication channel.
- At the receiver, the signal is then recovered through appropriate processing.

This processing at the receiver is required for a variety of reasons. In particular, quite typically, any specific communication channel has associated with it a *frequency range* over which it is best suited for transmitting a signal. Outside of this range, communication may be severely degraded or even impossible.

For example, the atmosphere will rapidly attenuate signals in the audible frequency range (10 Hz to 20 kHz), whereas it will propagate signals at a higher frequency range over longer distances. Thus, in transmitting audio signals such as speech or music over a communication channel that relies on propagation through the atmosphere, the transmitter first embeds the signal through an appropriate process into another, higher frequency signal. We aim to understand some of the basics behind how these transmit processes work, as well as their counterparts at the receiver side.

Complex Exponential and Sinusoidal AM

Many communication systems rely on the concept of **sinusoidal amplitude modulation (AM)**, which we considered at the end of the last module. In sinusoidal AM, a complex exponential or sinusoidal signal $c(t)$ has its amplitude multiplied (modulated) by the information-bearing signal $x(t)$. The signal $x(t)$ is typically referred to as the **modulating signal** and the signal $c(t)$ as the **carrier signal**. The modulated signal $y(t)$ is then the product of these two signals:

$$y(t) = x(t)c(t).$$

There are two common forms of sinusoidal amplitude modulation, one in which the carrier signal is a complex exponential of the form:

$$c(t) = e^{j\omega_c t + \theta_c}$$

and the second in which the carrier signal is sinusoidal and of the form

$$c(t) = \cos(\omega_c t + \theta_c)$$

In both cases, the frequency ω_c is referred to as the **carrier frequency**.

Complex exponential carrier. Let us consider first the case of a complex exponential carrier. For convenience, we choose $\theta_c = 0$. The modulated signal is

$$y(t) = x(t)e^{j\omega_c t}$$

From the multiplication property of CTFT, and with $X(j\omega)$, $Y(j\omega)$, and $C(j\omega)$ denoting the Fourier transforms of $x(t)$, $y(t)$, and $c(t)$, respectively,

$$Y(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)C(j(\omega - \theta))d\theta$$

For $c(t) = e^{j\omega_c t}$,

$$C(j\omega) = 2\pi\delta(\omega - \omega_c),$$

and hence,

$$Y(j\omega) = X(j(\omega - \omega_c)).$$

Thus, the spectrum of the modulated output $y(t)$ is simply that of the input shifted in frequency by an amount equal to the carrier frequency ω_c. For example, consider Figure 1: with $X(j\omega)$ band limited with highest frequency ω_M (and bandwidth $2\omega_M$), as depicted in Fig. 1a, the output spectrum $Y(j\omega)$ is that shown in Fig. 1c.

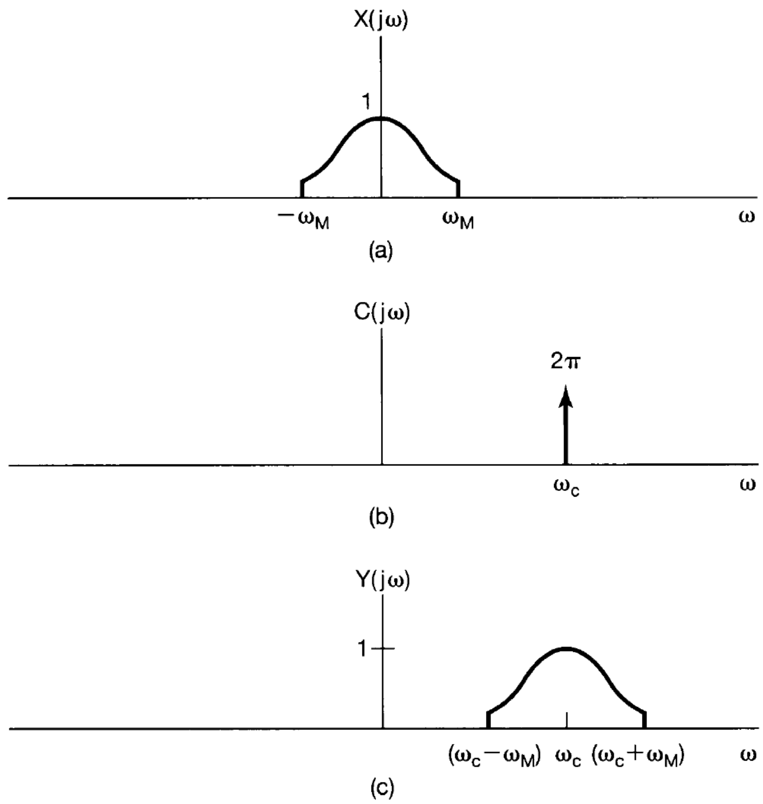

Figure 1: Effect in the frequency domain of amplitude modulation with a complex exponential carrier: (a) spectrum of modulating signal $x(t)$; (b) spectrum of carrier $c(t) = e^{j\omega_c t}$; (c) spectrum of amplitude-modulated signal $y(t) = x(t)e^{j\omega_c t}$.

$x(t)$ can be recovered from the modulated signal $y(t)$ by multiplying by the

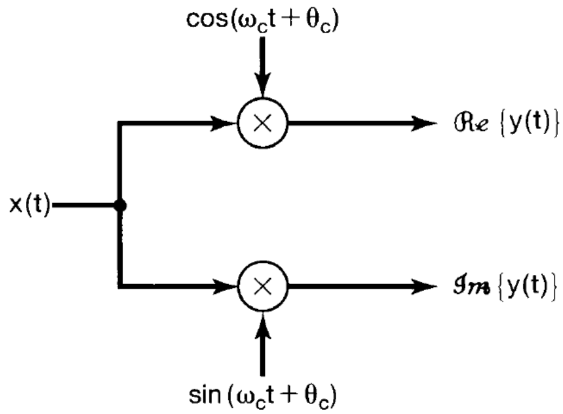

Figure 2: Implementation of amplitude modulation with a complex exponential carrier $c(t) = e^{j(\omega_c t+\theta_c)}$.

complex exponential $e^{-j\omega_c t}$; that is

$$x(t) = y(t)e^{-j\omega_c t}$$

In the frequency domain, this has the effect of shifting the spectrum of the modulated signal back to its original position on the frequency axis. The process of recovering the original signal from the modulated signal is referred to as **demodulation**.

Since $e^{j\omega_c t}$ is a complex signal, $y(t)$ can be rewritten as:

$$y(t) = x(t)\cos\omega_c t + jx(t)\sin\omega_c t$$

Therefore, one way to implement such an amplitude modulation is to have two separate multipliers and two sinusoidal carrier signals with a phase difference of $\pi/2$ to separately generate the real and imaginary part. This is depicted in Fig. 2.

Sinusoidal carrier. In many situations, using a sinusoidal carrier in the form

$$y(t) = x(t)\cos(\omega_c t + \theta_c)$$

is often simpler than and equally as effective as using a complex exponential carrier. In effect, using a sinusoidal carrier corresponds to retaining only the real or imaginary part of the output of Fig. 2. A system that uses a sinusoidal carrier is depicted in Fig. 3.

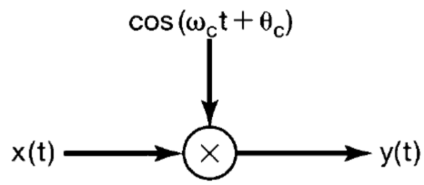

Figure 3: Amplitude modulation with a sinusoidal carrier.

Again, for convenience, we choose $\theta_c = 0$. In this case, the spectrum of the carrier signal is

$$C(j\omega) = \pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)],$$

and thus, the frequency response of the output signal is

$$Y(j\omega) = \frac{1}{2}\left[X(j\omega - j\omega_c) + X(j\omega + j\omega_c)\right].$$

With $X(j\omega)$ as depicted in Fig. 4a, the spectrum of $y(t)$ is that shown in Fig. 4c, which corresponds exactly to what we saw in Fig. 5 of the last module. Note that there is now a replication of the spectrum of the original signal, centered around both $+\omega_c$ and $-\omega_c$. As a consequence, $x(t)$ is recoverable from $y(t)$ only if $\omega_c > \omega_M$, since otherwise, the two replications will overlap in frequency. For example, Fig. 5 depicts a case of modulation for which $\omega_c = \omega_M/2$, i.e., the carrier frequency is one half of the signal frequency. Clearly, the spectrum of $x(t)$ is no longer replicated in $Y(jw)$, and thus, it may no longer be possible to recover $x(t)$ from $y(t)$.

Demodulation for Sinusoidal AM

At the receiver, the information-bearing signal $x(t)$ must be recovered through demodulation. Here we examine the process of demodulation for sinusoidal amplitude modulation. There are two potential schemes, each with their own advantages: synchronous and asynchronous.

Synchronous demodulation. In synchronous demodulation, we assume that the demodulating signal is synchronized in phase with the modulating signal. In this case, assuming that $\omega_c > \omega_M$, demodulation of a signal that was modulated with a sinusoidal carrier is relatively straightforward. Specifically, consider the signal

$$y(t) = x(t)\cos\omega_c t$$

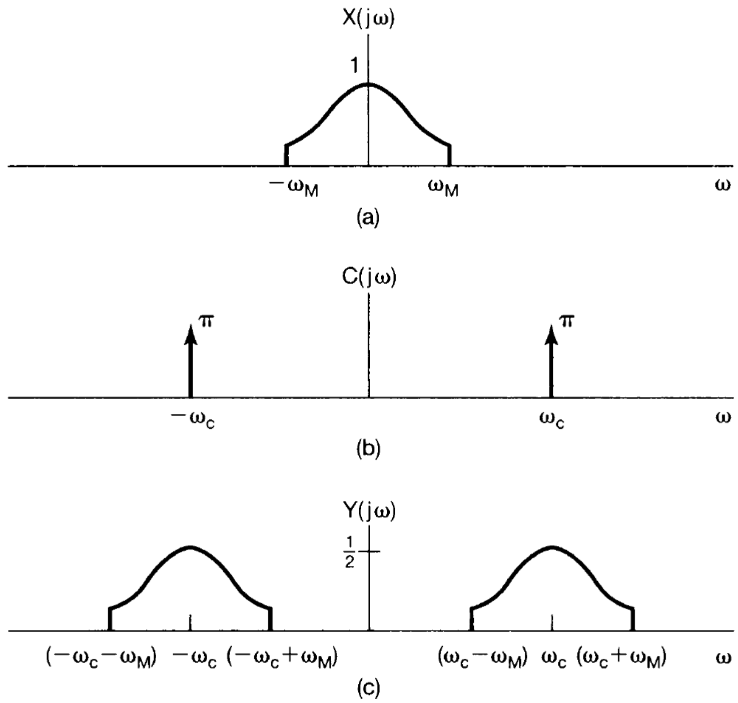

Figure 4: Effect in the frequency domain of amplitude modulation with a sinusoidal carrier: (a) spectrum of modulating signal $x(t)$; (b) spectrum of carrier $c(t) = \cos \omega_c t$; (c) spectrum of amplitude-modulated signal.

The original signal can be recovered by modulating $y(t)$ with the same sinusoidal carrier and applying a lowpass filter to the result. To see this, consider

$$\omega(t) = y(t) \cos \omega_c t$$

where we assume the $\cos \omega_c t$ multiplier has no phase shift relative to the transmitter. Fig. 6 shows the spectra of $y(t)$ and $w(t)$. $x(t)$ can be recovered from $w(t)$ by applying an ideal lowpass filter with a gain of 2 and a cutoff frequency that is greater than ω_M but less than $2\omega_c - \omega_M$. The frequency response of the lowpass filter is indicated by the dashed line in Fig. 6c.

The overall system for amplitude modulation and demodulation using a sinusoidal carrier is depicted in Fig. 7. This can be compared to the overall system for a complex exponential carrier depicted in Fig. 8. In Fig. 7, we can see there are three design parameters:

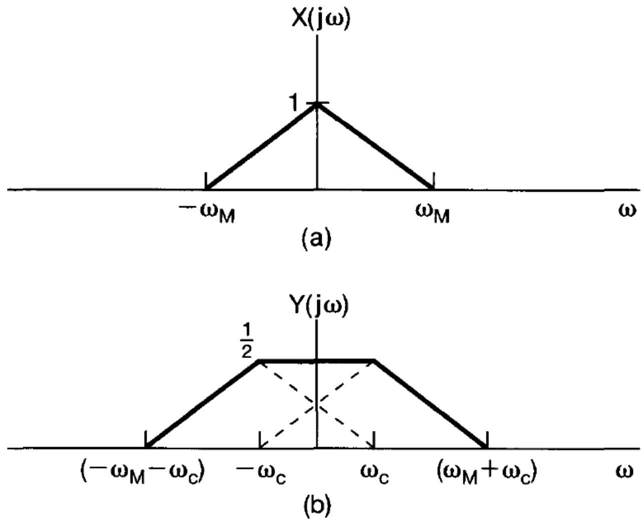

Figure 5: Sinusoidal amplitude modulation with carrier $\cos \omega_c t$ for which $\omega_c = \omega_M/2$: (a) spectrum of modulating signal; (b) spectrum of modulated signal.

- ω_M, the bandwidth of the signal $x(t)$ to be transmitted.
- ω_c, the carrier frequency, which must satisfy $\omega_c > \omega_M$.
- ω_{co}, the lowpass filter cutoff frequency at the receiver to recover the original signal. We must have $\omega_M < \omega_{co} < 2\omega_c - \omega_M$.

We analyzed this sinusoidal demodulation system mathematically using the frequency domain in Fig. 5 of the last module. In the time domain, the demodulation system $w(t)$ is expressed as

$$w(t) = x(t)\cos^2 \omega_c t$$

or, using the trigonometric identity

$$\cos^2 \omega_c t = \frac{1}{2} + \frac{1}{2}\cos 2\omega_c t$$

we can rewrite $w(t)$ as

$$w(t) = \frac{1}{2}x(t) + \frac{1}{2}x(t)\cos 2\omega_c t$$

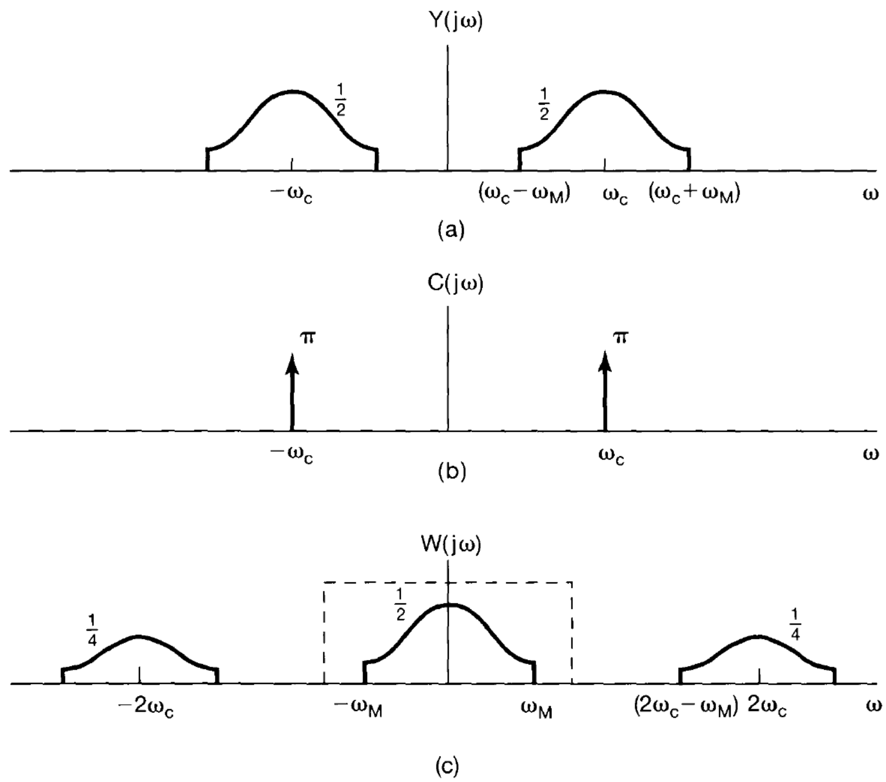

Figure 6: Demodulation of an amplitude-modulated signal with a sinusoidal carrier: (a) spectrum of modulated signal; (b) spectrum of carrier signal; (c) spectrum of modulated signal multiplied by the carrier. The dashed line indicates the frequency response of a lowpass filter used to extract the demodulated signal.

Thus, $w(t)$ consists of the sum of two terms, namely one-half the original signal and one-half the original signal modulated with a sinusoidal carrier at twice the original carrier frequency ω_c, which can also be seen from Fig. 6(c). Applying the lowpass filter to $w(t)$ corresponds to retaining the first term on the right-hand side and eliminating the second term.

Suppose, however, that the modulating and demodulating signals are not synchronized in phase. Formally, consider a transmission $y(t) = x(t)\cos(\omega_c t + \theta_c)$ and a reception of $w(t) = y(t)\cos(\omega_c t + \phi_c)$, where θ_c and ϕ_c are the phases for the modulating and demodulating signals, respectively. The input to the lowpass filter at the receiver will then be

$$w(t) = x(t)\cos(\omega_c t + \theta_c)\cos(\omega_c t + \phi_c)$$

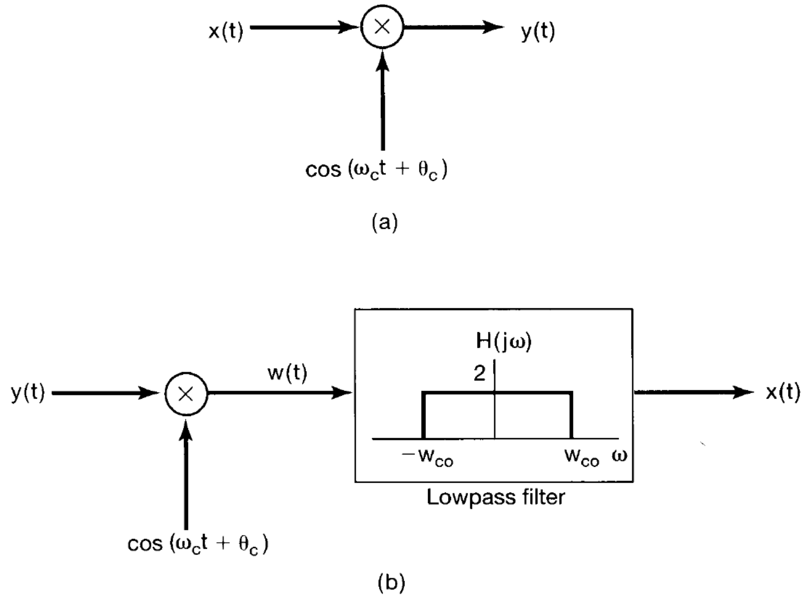

Figure 7: Amplitude modulation and demodulation with a sinusoidal carrier: (a) modulation system; (b) demodulation system. The lowpass filter cutoff frequency ω_{co} is greater than ω_M and less than $2\omega_c - \omega_M$.

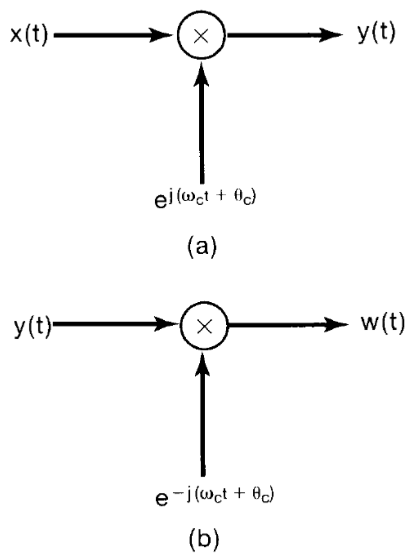

Figure 8: System for amplitude modulation and demodulation using a complex exponential carrier: (a) modulation; (b) demodulation.

which can be written using the more general trigonometric identity as

$$w(t) = \frac{1}{2}x(t)\cos(\theta_c - \phi_c) + \frac{1}{2}x(t)\cos(2\omega_c t + \theta_c + \phi_c)$$

The output of the lowpass filter in Fig. 7(b) will then be $x(t)$ multiplied by $\cos(\theta_c - \phi_c)$. If θ_c and ϕ_c are out of phase by a full 90°, the output will be completely nullified! The ideal case is completely in-sync, where $\theta_c = \phi_c$, which gives the maximum scale $\cos 0 = 1$ on the output signal. More importantly, the relationship between θ_c and ϕ_c must be maintained over time, so the amplitude factor $\cos(\theta_c - \phi_c)$ does not vary. This is often challenging as the modulator and demodulator are likely separated over a large geographic region. A **phase-locked loop (PLL)** circuit at the receiver can help with this. The PLL will adapt the demodulator phase according to a reference signal – namely, the received waveform.

Asynchronous. Asynchronous demodulation, by contrast, avoids the need for the modulator and demodulator to be synchronized. For sinusoidal AM, consider the modulated signal $y(t) = x(t)\cos(\omega_c t)$ in Fig. 9. The envelope of $y(t)$ – a smooth curve connecting the peaks – will be a good approximation to $x(t)$ if two conditions are met:

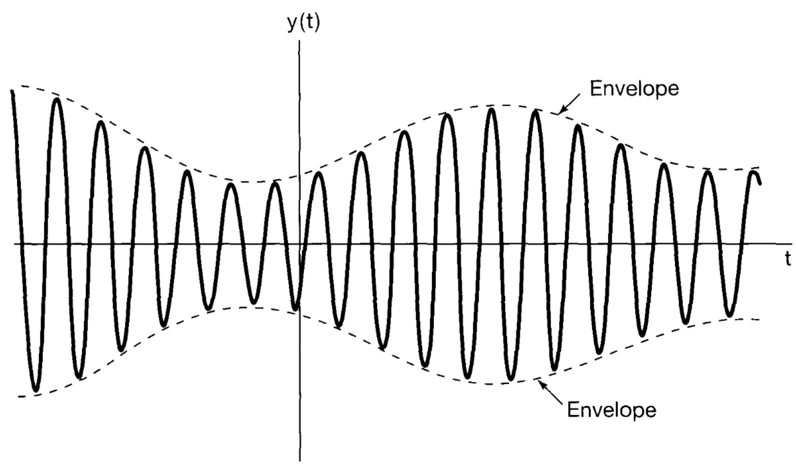

Figure 9: Envelope of the transmitted signal $y(t)$.

1. ω_c is much higher than ω_M, e.g., a few orders of magnitude. Having ω_c just slightly larger than ω_M will not be sufficient for the envelope to provide a good approximation.

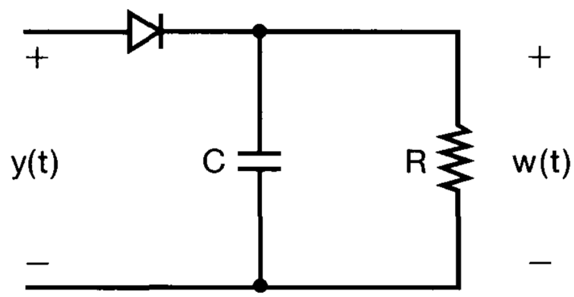

Figure 10: Half-wave rectifier for envelope detection.

2. $x(t)$ is positive for all time. Otherwise the upper envelope would be switching between $x(t)$ and $-x(t)$, and we could not figure out when.

If these two conditions are satisfied, then we can use an **envelope detector** system at the receiver to recover $x(t)$. One system we could use for this is a **half-wave rectifier**, depicted in Fig. 10: the diode is connected in series with an RC lowpass filter to reduce the variations at the carrier frequency.

With this in place, the recovered signal is shown in Fig. 11. The RC filter helps improve the approximation from $r(t)$ to $w(t)$ since it gives a smoother decay from the peak. As ω_c increases, the oscillations become more rapid, which will reduce the estimation error that occurs between the peaks.

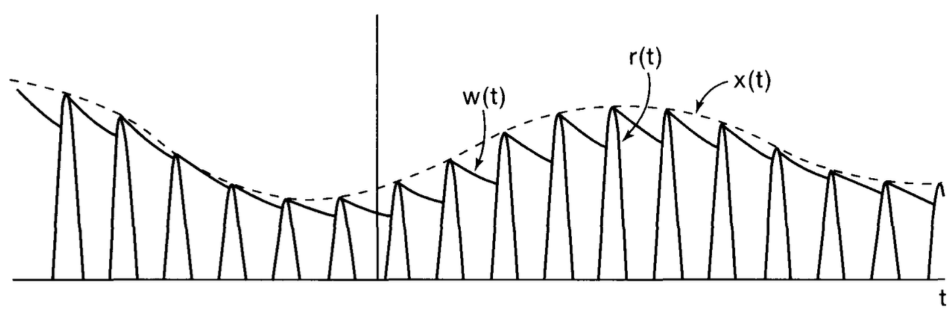

Figure 11: Recovered signal $w(t)$ from the circuit in Fig. 10.

How do we satisfy these two conditions? The first one comes from proper choice of the carrier frequency. The Federal Communications Commission (FCC) regulates which frequency ranges are allocated to different communication systems in the United States. If we assume ω_c is provided to us, then, satisfying the first condition corresponds to restricting the bandwidth

ω_M that we wish to transmit.

The second condition can be satisfied by simply adding a constant DC component, say A, to $x(t)$. Then, the transmitted signal will be

$$y(t) = (x(t) + A)\cos(\omega_c t) = x(t)\cos(\omega_c t) + A\cos(\omega_c t)$$

The envelope detector at the receiver will approximate $x(t) + A$, which can easily be translated to $x(t)$ by subtracting A at the end.

If K is the maximum amplitude of $x(t)$, i.e., $|x(t)| < K$, then to have $x(t) + A > 0$, we require $A > K$. The ratio K/A is often called the **modulation index**, and we therefore need $K/A \leq 1$. As K/A gets smaller, we are adding a larger DC component, which requires more power/energy at the transmitter. As is the case in almost all engineering problems, there is a tradeoff between efficiency and quality:

- A smaller modulation index requires more energy, but also improves the ability of the envelope detector to follow the peaks in the envelope, and thus improves reconstruction quality at the output.
- A larger modulation index (towards 1) requires less added energy at the transmitter, but also makes it harder for the envelope detector to follow the peaks.

In today's day and age, synchronous demodulation is much more commonly used than asynchronous demodulation techniques.

Frequency-Division Multiplexing

In practical communication systems, we will need to support multiple transmissions simultaneously, i.e., multiple pairs of transmitters and receivers. Luckily, most systems used for transmitting signals provide more bandwidth than is required for any one signal.

When different users/signal sources would like to "share" the same media with minimal quality degradation, this leads to the concept of **multiplexing**. Multiplexing is a method used by networks to consolidate multiple signals – digital or analog – into a single composite signal. **Frequency-division multiplexing** (FDM) is a special type of multiplexing where this consolidation takes place by dividing the usage of the media by different frequencies.

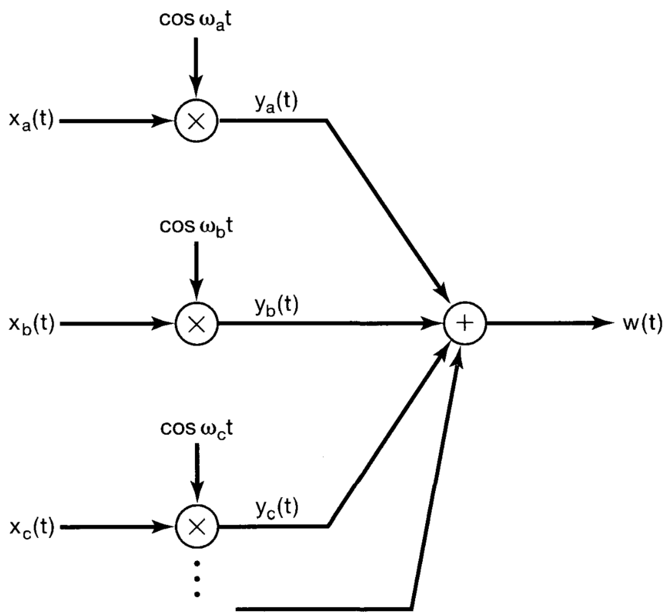

Figure 12: FDM for sinusoidal AM.

Fig. 12 gives a potential transmitter design for FDM when each input is using sinusoidal amplitude modulation. Each of the input signals $x_a(t), x_b(t), x_c(t), ...$ are placed on a different carrier frequency $\omega_a, \omega_b, \omega_c, ...$ The inputs are assumed to be bandlimited (if they are not, they will need to be passed through lowpass filters before modulation).

An example of this is given in Fig. 13, where each signal has bandwidth $-\omega_M < \omega < \omega_M$. The spectrum of the resulting multiplexed signal $w(t)$ is depicted at the bottom. When determining what carrier frequencies ω_a, ω_b, and ω_c should be used, we must choose these frequencies such that there will be no overlap in the frequency spectrum of the modulated $X_a(j\omega)$, $X_b(j\omega)$, and $X_c(j\omega)$. In the Fig. 13 example, we must at the very least choose these frequencies such that $\omega_a > 2\omega_M$, $\omega_b > \omega_a + 2\omega_M$, and $\omega_c > \omega_b + 2\omega_M$.

However, notice in Fig. 13 that there is some space along the frequency axis between the different modulated signals. This space is called a **guard band**, as illustrated in Fig. 14. Under ideal circumstances, we might be tempted to squeeze the signal components as tightly together as possible without overlapping. However, in practical circumstances, we must consider imperfections in transmission systems, such as imperfect low-pass or band-pass filters. Because of such imperfections, it is common practice to include

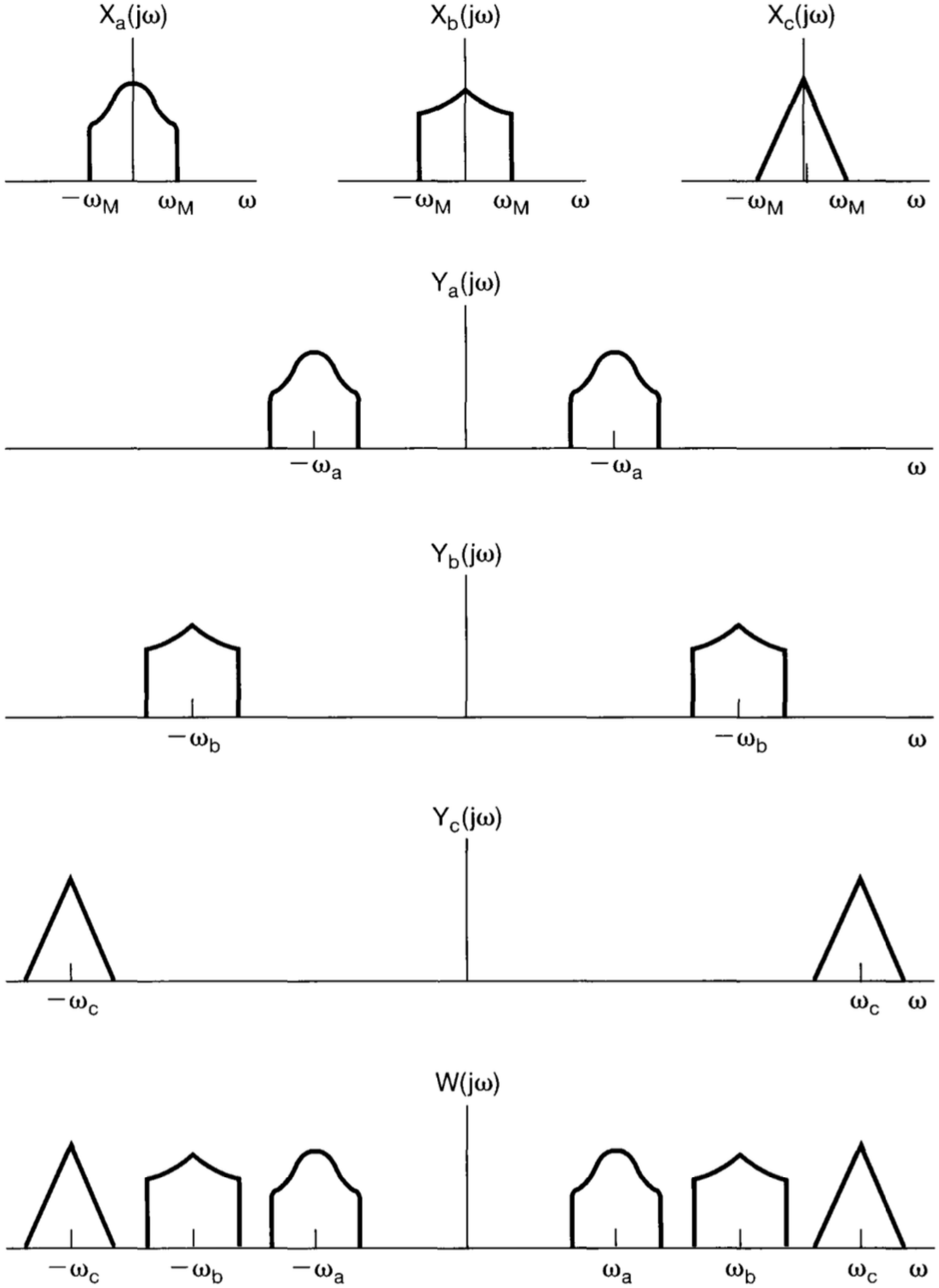

Figure 13: Example of the FDM system in Fig. 12 with three signals.

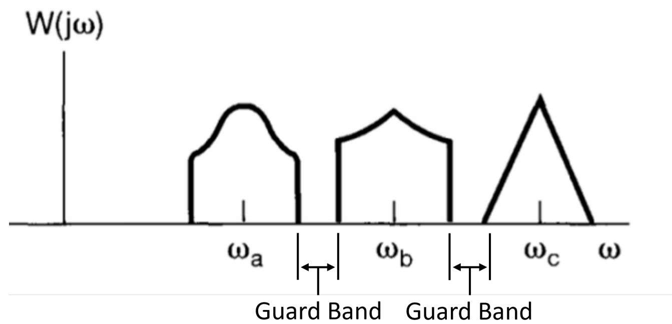

Figure 14: Illustration of a guard band separating different signal components for FDM.

a guard band which provides a buffer between the different signals in the frequency spectrum. This reduces the likelihood of interference between the different signals.

At the receiver side, we need to recover the transmitted signal of interest. This **demultiplexing** process requires two basic steps:

1. Bandpass filtering to extract the modulated signal corresponding to a specific channel.
2. Demodulation to recover the original signal.

This is illustrated in Fig. 15 in the case of a receiver trying to recover channel a. For purposes of illustration, we assume a synchronous demodulation scheme.

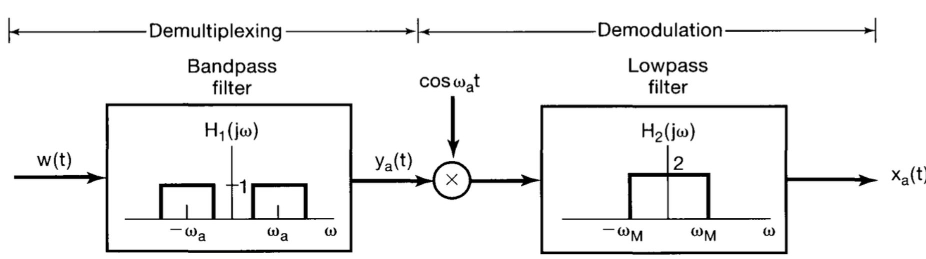

Figure 15: Demultiplexing and demodulation of channel a in Fig. 13.

Systems Characterized by Differential Equations

You have seen differential equations many times before, both in and outside of engineering. A particularly important class of continuous-time LTI systems are those for which the input $x(t)$ and output $y(t)$ satisfy a linear constant-coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

For example, you may recall that an RLC circuit (resistor, inductor, and capacitor) can be described as

$$\frac{d^2 y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC} y(t) = \frac{1}{L}\frac{dx(t)}{dt}$$

where $x(t)$ is the input voltage applied to the circuit and $y(t)$ is the output current that flows through the circuit. This is an example of a second order, ordinary, linear, differential equation, with constant coefficients.

We can use what we know about the CTFT to find the frequency response $H(j\omega)$ of such a system. In doing so, we will always assume that the LTI system is stable, so that we are guaranteed its frequency response exists. Note that the converse of this is not true in general: frequency responses can exist for unstable systems.

Taking the Fourier transform. Suppose we apply the CTFT to both sides of the differential equation:

$$\mathcal{F}\Big\{ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} \Big\} = \mathcal{F}\Big\{ \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \Big\}$$

By the linearity property, we can write

$$\sum_{k=0}^{N} a_k \mathcal{F}\Big\{ \frac{d^k y(t)}{dt^k} \Big\} = \sum_{k=0}^{M} b_k \mathcal{F}\Big\{ \frac{d^k x(t)}{dt^k} \Big\}$$

Now, from the differentiation property, we know $\mathcal{F}\{d^k y(t)/dt^k\} = (j\omega)^k Y(j\omega)$ and $\mathcal{F}\{d^k x(t)/dt^k\} = (j\omega)^k X(j\omega)$. Therefore, the transformed equation simplifies to

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

So, we can write the Fourier transform of the output in terms of the input as

$$Y(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} X(j\omega)$$

Obtaining the frequency response. Now, how do we find the frequency response of the system? The convolution property tells us that for an LTI system, the convolution $y(t) = x(t) * h(t)$ becomes $Y(j\omega) = X(j\omega)H(j\omega)$ in the frequency domain. Therefore, $H(j\omega) = Y(j\omega)/X(j\omega)$ for any input-output pair. It follows, then, that

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

And the impulse response can be obtained as $h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$, though a closed-form solution may be difficult depending on the nature of $H(j\omega)$.

Notice that $H(j\omega)$ is a rational function, i.e., a ratio of polynomials in $(j\omega)$. The terms for Y go in the denominator, and those for X go in the numerator, which can actually be done by inspection. The order of the differential equation – N – translates to an Nth order polynomial in the denominator.

Example 1. *Consider a stable LTI system characterized by the differential equation*

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

What is the frequency response of the system? How about the impulse response? What will be the output if the input $x(t) = e^{-t}u(t)$ is applied?

Ans: The frequency response is

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}$$

To obtain the impulse response, we need $H(j\omega)$ in a form for which the inverse transform can readily be applied. We can do this by factoring the denominator and writing out the partial fraction expansion:

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 3)(j\omega + 1)} = \frac{A}{j\omega + 3} + \frac{B}{j\omega + 1}$$

To solve for A, we can plug $j\omega = -3$ in to $H(j\omega)(j\omega + 3)$, and to solve for B, we plug $j\omega = -1$ into $H(j\omega)(j\omega + 1)$. We obtain

$$A = \frac{j\omega + 2}{j\omega + 1}\Big|_{j\omega=-3} = \frac{1}{2} \qquad\qquad B = \frac{j\omega + 2}{j\omega + 3}\Big|_{j\omega=-1} = \frac{1}{2}$$

Thus,

$$H(j\omega) = \frac{1/2}{j\omega + 3} + \frac{1/2}{j\omega + 1}$$

From this, the impulse response can be readily obtained as

$$h(t) = \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)$$

To determine $y(t)$ when $x(t) = e^{-t}u(t)$, we could perform convolution, but it is much easier to do this in the frequency domain:

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega + 2}{(j\omega + 3)(j\omega + 1)} \cdot \frac{1}{j\omega + 1} = \frac{j\omega + 2}{(j\omega + 3)(j\omega + 1)^2}$$

We need to do a partial fraction expansion again. Notice that there is a squared term in the denominator; this is handled by adding a term for both the square and its root, as follows:

$$Y(j\omega) = \frac{j\omega + 2}{(j\omega + 3)(j\omega + 1)^2} = \frac{A}{j\omega + 3} + \frac{B}{j\omega + 1} + \frac{C}{(j\omega + 1)^2}$$

We can solve for A and C using the same procedure as before:

$$A = \frac{j\omega + 2}{(j\omega + 1)^2}\Big|_{j\omega=-3} = -\frac{1}{4} \qquad\qquad C = \frac{j\omega + 2}{j\omega + 3}\Big|_{j\omega=-1} = \frac{1}{2}$$

For B, however, we need a slightly different method. Suppose we multiply the whole expression by $(j\omega + 1)^2$, as we did to find C. We get

$$\frac{j\omega + 2}{j\omega + 3} = A\frac{(j\omega + 1)^2}{j\omega + 3} + B(j\omega + 1) + C$$

Rather than evaluate this expression at $j\omega = -1$, we can evaluate the derivative at this point. C becomes 0, and we can ignore the term for A since that derivative will go to 0 with $j\omega = -1$; therefore,

$$B = \frac{d}{dj\omega}\left[\frac{j\omega + 2}{j\omega + 3}\right]\Big|_{j\omega=-1} = \frac{j\omega + 3 - j\omega - 2}{(j\omega + 3)^2}\Big|_{j\omega=-1} = \frac{1}{(j\omega + 3)^2}\Big|_{j\omega=-1} = \frac{1}{4}$$

Alternatively, in this case, since we have already solved for A and C, we could plug in another value for $j\omega$ (e.g., $j\omega = 0$) into the initial expression, and determine that $B = 1/4$.

Thus, we have

$$Y(j\omega) = \frac{-1/4}{j\omega + 3} + \frac{1/4}{j\omega + 1} + \frac{1/2}{(j\omega + 1)^2}$$

Applying our inverse transform identifies, we have

$$y(t) = -\frac{1}{4}e^{-3t}u(t) + \frac{1}{4}e^{-t}u(t) + \frac{1}{2}te^{-t}u(t)$$

This example shows how Fourier analysis allows us to reduce problems regarding LTI systems characterized by differential equations to straightforward algebraic manipulations. The caveat, as always, is that we need to be able to transform back to the time domain, which is not always easy or even possible analytically.