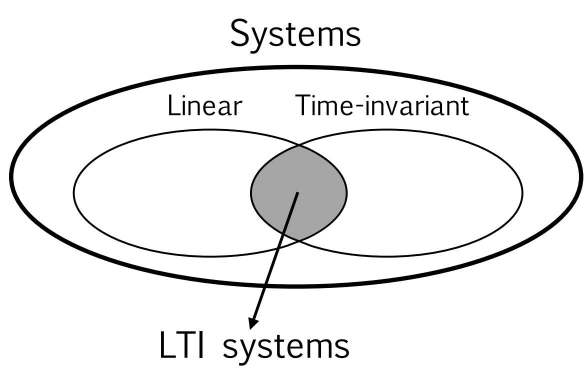
Signals and Systems: Module 5

Suggested Reading: S&S 2.1, 2.2, 2.3.1-3, 2.3.8

Linear Time Invariant (LTI) Systems

One of the first topics we looked at in this course was what it means for a system to be linear. When we combine this property with time-invariance, one of the others we just discussed, we get a very special class of systems: **Linear Time-Invariant**, or **LTI** systems.

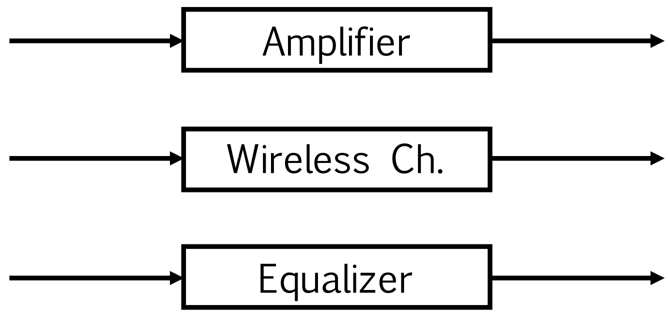


Why do we care about LTI systems?

1. Relative to other systems, their analysis is simple and elegant, as we will see.

2. Moreover, a lot of systems can be well approximated by an LTI system.

The following are examples of a few systems that are often approximated as, or designed to be, LTI systems:



Recall what the "L" and "TI" properties tell us:

- *Linearity:* If we scale the input, the output will be scaled by the same factor. Also, if we combine two inputs, the output will be the combination of the individual outputs.
- *Time-invariance:* The point in time that we apply the input signal to the system does not matter.

In an LTI system, then, we can figure out how the system will react to a "new" input if we know how it has reacted to "test" inputs that are related to the "new" input by superposition (i.e., linearity) and/or time shifting (i.e., time invariance). For this reason, the unit impulse – $\delta[n]$ in discrete-time and $\delta(t)$ in continuous-time – plays a particularly important role in LTI system analysis. As we will see, the fact that many signals can be represented as linear combinations of delayed impulses allows us to develop a complete characterization of any LTI system in terms of its response to a unit impulse. In other words, we use these impulses as our "test" signals.

Discrete-time LTI Systems

Recall that we can think of any DT signal as a sequence of individual impulses. In particular, since $x[k]\delta[n-k] = x[k]$ when $n = k$ and $x[k]\delta[n-k] = 0$ when $n \neq k$, by summing over all possible values of k, we can write

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Now, suppose we pass $x[n]$ through an LTI system $\mathcal{S}$. Considering the shifted impulses $x_k[n] = \delta[n-k]$ as test signals, by the *linearity* property,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h_k[n]$$

where $h_k[n] = \mathcal{S}\{\delta[n-k]\}$, i.e., the response to test signal k.

For a general linear system, the $h_k[n]$ may not be related for different k. However, when the system is *time-invariant*, we know that if $x[n] \to y[n]$, then $x[n-k] \to y[n-k]$. Since $h_k[n]$ is the response to $\delta[n-k]$, and $\delta[n-k]$ is a time-shifted version of $\delta[n]$, then we must have that

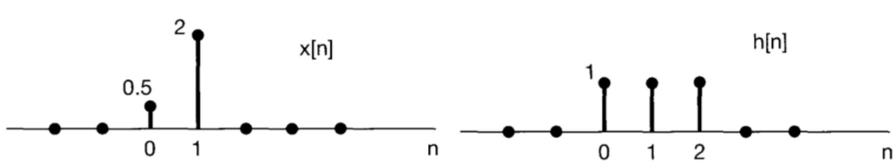
$$h_k[n] = h_0[n-k]$$

The response $h_0[n] = \mathcal{S}\{\delta[n]\}$ thus has special significance for LTI systems. Denoted by $h[n]$ for simplicity, $h[n]$ $(= h_0[n])$ is the **discrete-time unit impulse (sample) response**, i.e., the output when the input is an impulse $\delta[n]$. The output of a DT LTI system is then

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[n] * h[n]$$

This is referred to as the **convolution sum**, and the corresponding operation $x[n]*h[n]$ is known as the **convolution** of sequences $x[n]$ and $h[n]$. Since $x[n]$ can be any arbitrary input, we see that an LTI system is completely characterized by its response to a single test signal: the unit impulse.

Example 1. *Consider an LTI system with impulse response $h[n]$ given in Figure 1(b). What will be the output of the system to the input $x[n]$ in (a)?*



(a) Input signal (b) Impulse response

Figure 1: Signals $x[n]$ and $h[n]$ for the LTI system in Example 1.

Ans: We need to evaluate $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$. Since $x[k]$ is non-zero only for $k = 0$ and $k = 1$, the convolution sum reduces to

$$y[n] = x[0]h[n] + x[1]h[n-1] = 0.5h[n] + 2h[n-1]$$

This will be non-zero for $n = 0, 1, ..., 3$; in particular, we have $y[0] = 0.5h[0] + 2h[-1] = 0.5$, $y[1] = 0.5h[1] + 2h[0] = 2.5$, $y[2] = 0.5h[2] + 2h[1] = 2.5$, and $y[3] = 0.5h[3] + 2h[2] = 2$. The result is plotted in Figure 2.

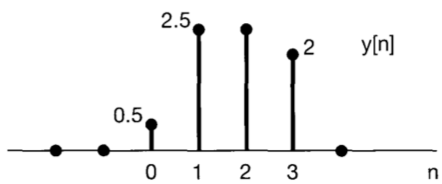


Figure 2: Output signal $y[n]$ for the LTI system in Example 1.

The process applied in Example 1 works fine for short, finite sequences. But, how can we perform convolution for large, possibly infinitely long sequences?

Consider $y[n]$ for a specific time n. There's a particularly convenient way of visualizing the calculation of $y[n]$ graphically, by viewing the two signals $x[k]$ and $h[n-k]$ as functions of k. Multiplying them, we obtain $g[k] = x[k]h[n-k]$, which is the contribution of $x[k]$ to the output at time n. By summing over the $g[k]$, then, we obtain $y[n]$ for a particular n, treating n as a constant.

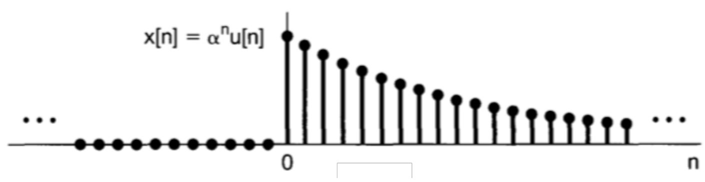
Still, doing this for each value of n separately would be tedious. Fortunately, changing the value of n has a very simple graphical interpretation in terms of $x[k]$ and $h[n-k]$, which allows us to group ranges of n in which the calculation will be consistent. This is best illustrated through examples.

Example 2. *Consider an LTI system that has $h[n] = u[n]$. What is the output $y[n]$ for an input $x[n] = \alpha^n u[n]$, with $0 < \alpha < 1$?*

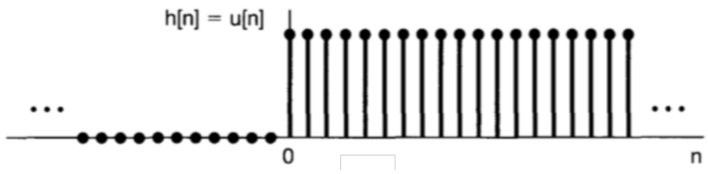
Ans: We start by graphing $x[n]$ (assuming $0 < \alpha < 1$) and $h[n]$, which are shown in Figure 3.

To visualize convolution, we will switch the variable in $x[n]$ to k, and consider $h[n-k]$ for different values of n. With k as the variable, $h[n-k]$ corresponds to a time reversal and then a shift by n.

In Figure 4, we graph $x[k]$ in (a) and $h[-k]$ in (b). In the convolution operation, we consider the product $x[k]h[n-k]$ for different n. Comparing Figure 4(a)&(b), we see that there will be two intervals of n where the form of the product will change: $n < 0$, where $h[n-k]$ is shifted to the left, and $n \geq 0$, where it shifts to the right. These cases are plotted in Figure 4(c)&(d).

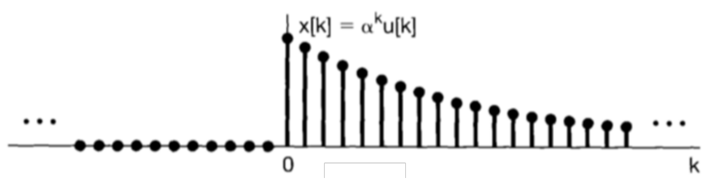


(a) Input signal

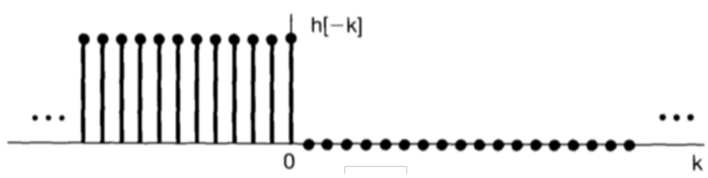


(b) Impulse response

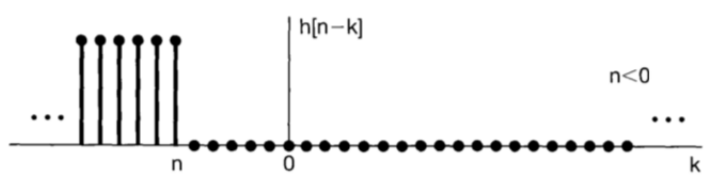
Figure 3: Graphs of (a) the input $x[n]$ (assuming $0 < \alpha < 1$) and (b) the impulse response $h[n]$ for the LTI System in Example 2.



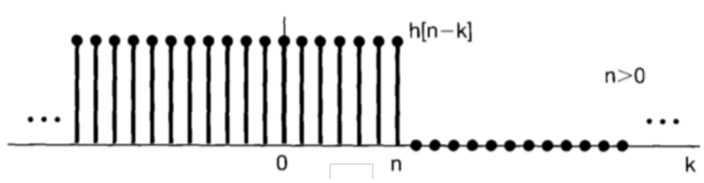
(a) Input signal



(b) Time-reversed impulse response



(c) Time-reversed and shifted impulse response for $n < 0$



(d) Time-reversed and shifted impulse response for $n > 0$

Figure 4: Graphs of the input signal and time-shifted impulse responses to visualize the convolution calculation.

Considering these two intervals:

- $n < 0$*: Comparing Figure 4(a)&(c), we see that* $y[n] = 0$ *since* $x[k] = 0$ *for* $k < 0$ *and* $h[n-k] = 0$ *for* $k > 0$.

- $n \geq 0$*: Comparing Figure 4(a)&(d), the product is still* 0 *for* $k < 0$*, but for* $k \geq 0$*, there is an overlap for* $0 \leq k \leq n$*. In particular,*

  $$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} \alpha^k u[k]u[n-k] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha} \quad n \geq 0$$

  since $u[k] = u[n-k] = 1$ *when* $0 \leq k \leq n$.

Thus, we have the following compact formula for the expression over all n*:*

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$

This is plotted in Figure 5 below.

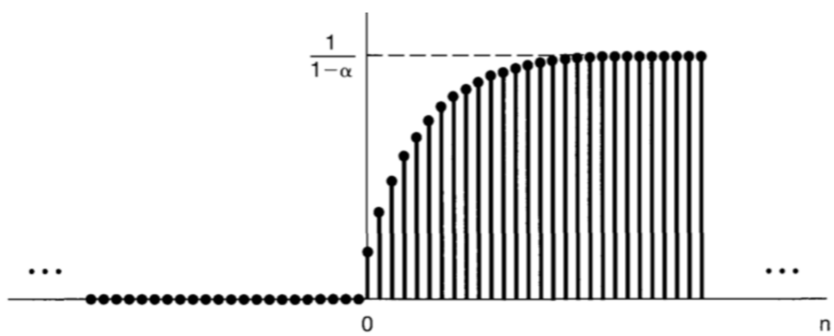


Figure 5: Graph of the output $y[n]$ for Example 2, assuming $1 > \alpha > 0$.

This interval technique will be useful in solving mostly all convolution problems that we encounter. The key is to find the "changepoints" of n where the sum $x[n] * h[n]$ changes form. When we calculate the convolution result, it is critical to always remember that our variable is k, not n – the latter is held constant in each summation.

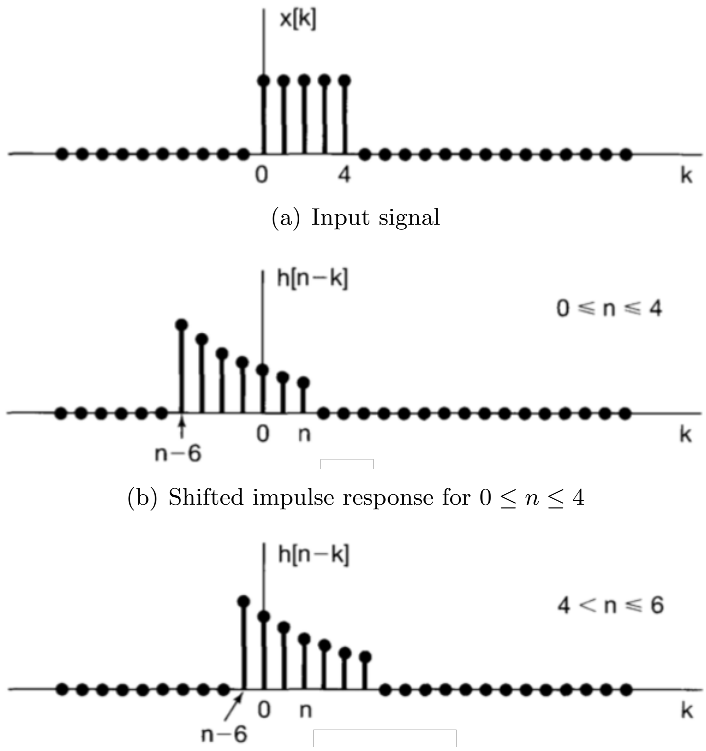
The following example shows how the interval technique is useful even for finite sequences.

Example 3. *Consider the two sequences*

$$x[n] = \begin{cases} 1 & 0 \leq n \leq 4 \\ 0 & otherwise \end{cases} \qquad h[n] = \begin{cases} \alpha^n & 0 \leq n \leq 6 \\ 0 & otherwise \end{cases}$$

What is $x[n] * h[n]$?

Ans: In Figure 6, we plot $x[k]$ in (a).



(a) Input signal

(b) Shifted impulse response for $0 \leq n \leq 4$

(c) Shifted impulse response for $4 < n \leq 6$

Figure 6: Graphs of the input signal and time-shifted impulse responses to visualize the convolution calculation.

$h[n-k]$ will only overlap with $x[k]$ when (i) $n \geq 0$, so that the rightmost of $h[n-k]$ has come beyond the leftmost of $x[k]$, and (ii) $n - 6 \leq 4$, so that the leftmost of $h[n-k]$ is still to the left of the rightmost of $x[k]$. Thus, $y[n] = 0$ when $n < 0$ or $n > 10$. Between these points, we must consider three intervals:

- $0 \leq n \leq 4$: *$h[n-k]$ for this case is plotted in Figure 6(b). The rightmost portion of $h[n-k]$ overlaps with $x[k]$. The convolution sum thus has a lower bound of $k=0$ (leftmost point of $x[k]$) and an upper bound of $k=n$ (rightmost point of $h[n-k]$):*

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} 1 \cdot \alpha^{n-k} = \sum_{m=n}^{0} \alpha^m = \frac{1-\alpha^{n+1}}{1-\alpha}$$

Here, we have used a change of variables $m = n-k$ to evaluate the summation.

- $4 < n \leq 6$: *$h[n-k]$ for this case is plotted in Figure 6(c). The middle portion of $h[n-k]$ overlaps with $x[k]$. The top bound in the convolution sum is replaced with 4 (rightmost point of $x[k]$):*

$$\begin{aligned} y[n] &= \sum_{k=0}^{4} x[k]h[n-k] = \sum_{k=0}^{4} 1 \cdot \alpha^{n-k} = \alpha^n \frac{1-(1/\alpha)^5}{1-(1/\alpha)} \\ &= \frac{\alpha^n - \alpha^{n-5}}{1-\alpha^{-1}} = \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} \end{aligned}$$

This shows an alternative (yet more tedious) way to evaluate the geometric summation instead of using a change of variables.

- $6 < n \leq 10$: *Now, the leftmost portion of $h[n-k]$ overlaps with $x[k]$. The lower bound in the sum changes to the leftmost point of $h[n-k]$ since $n-6 \geq 0$:*

$$\begin{aligned} y[n] &= \sum_{k=n-6}^{4} x[k]h[n-k] = \sum_{k=n-6}^{4} 1 \cdot \alpha^{n-k} = \sum_{k=0}^{10-n} \alpha^{6-k} \\ &= \alpha^6 \frac{1-(1/\alpha)^{11-n}}{1-(1/\alpha)} = \frac{\alpha^{n-4}-\alpha^7}{1-\alpha} \end{aligned}$$

Summarizing, then, the output is

$$y[n] = \begin{cases} 0 & n < 0 \\ \frac{1-\alpha^{n+1}}{1-\alpha} & 0 \leq n < 4 \\ \frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} & 4 \leq n < 6 \\ \frac{\alpha^{n-4}-\alpha^7}{1-\alpha} & 6 \leq n \leq 10 \\ 0 & n > 10 \end{cases}$$

Which is plotted in Figure 7 below (assuming $\alpha > 0$).

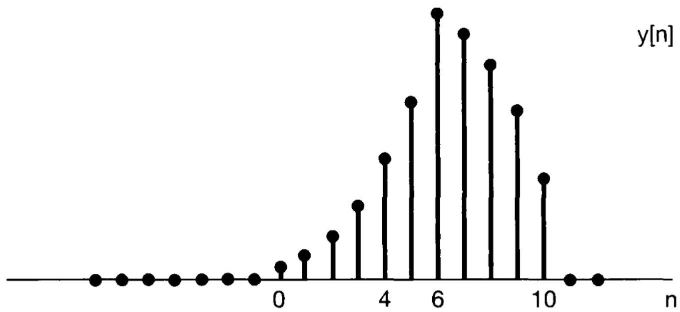


Figure 7: Graph of the output $y[n]$ for Example 3, assuming $\alpha > 0$.

Identifying these different intervals of n, and the corresponding summation bounds of k in each case, is crucial to every convolution problem. A similar technique is used for continuous-time LTI systems, as we will see next.

Continuous-time LTI Systems

The treatment of CT LTI systems is in many ways analogous to the DT case. Recall that we can write any CT signal $x(t)$ in terms of time-shifted versions of $\delta(t)$, but through integration instead of summation:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Now, suppose we pass $x(t)$ through an LTI system $\mathcal{S}$. With some hand-waving, using $x_\tau(t) = \delta(t-\tau)$ as our test signals, then by the linearity property,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau,$$

where $h_\tau(t) = \mathcal{S}\{\delta(t-\tau)\}$, i.e., the response to test signal τ (since these test signals are uncountable set, we technically should write these first few integrals in limit form, but the end result will be the same).

By the time-invariance property, we have that

$$h_\tau(t) = h_0(t-\tau).$$

Similar to in the DT case, we denote $h(t) = h_0(t)$ for brevity. $h(t)$ gives the **continuous-time unit impulse (sample) response**. The output of a CT LTI system can then be expressed as

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

This equation is referred to as the **convolution interval**. As in DT, the operation $x(t) * h(t)$ is known as the **convolution** of signals $x(t)$ and $h(t)$. We see again that the LTI system is completely characterized by $h(t)$, the response to a single test signal $\delta(t)$.

We will now walk through several examples of CT convolution. The procedure in CT – finding the intervals in which the integral has a different form – is very similar to that in DT.

Example 4. *Consider an LTI system with impulse response $h(t) = u(t)$. What is the response of the system to the input $x(t) = e^{-at}u(t), a > 0$?*

Ans: We need to evaluate the convolution integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau$$

for every value of t. In Figure 8(a)&(b), we plot $x(\tau)$ and $h(\tau)$ as functions of τ. Then, we apply time-reversal to $h(\tau)$ and shift it by t to obtain $h(t-\tau)$; the result is shown in 8(c) for $t < 0$.

Comparing Figures 8(b)&(c), we see that there are two intervals of t for which the convolution integral changes: $t \leq 0$ and $t > 0$. This problem is quite similar in structure to Example 2 (but now in CT):

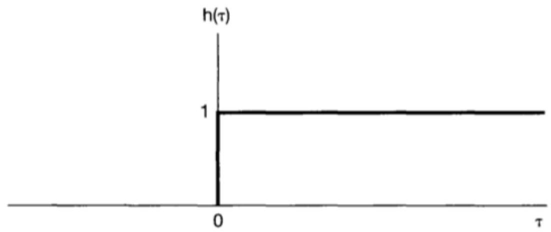
- *For $t \leq 0$, $x(\tau)$ and $h(t-\tau)$ have no overlap, i.e., $e^{-a\tau}u(\tau)u(t-\tau) = 0$. Thus, $y(t) = 0$.*
- *For $t > 0$, $x(\tau)$ and $h(t-\tau)$ overlap when both of the signals are non-zero. Specifically, the unit steps are both 1 when $0 < \tau \leq t$:*

$$y(t) = \int_{0}^{t} e^{-a\tau}d\tau = -\frac{1}{a}e^{-a\tau}\Big|_0^t = \frac{1}{a}\left(1 - e^{-at}\right)$$

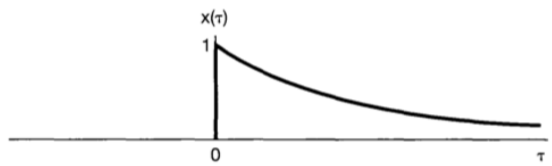
Thus, over all values of t, we can express the output compactly as

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t)$$

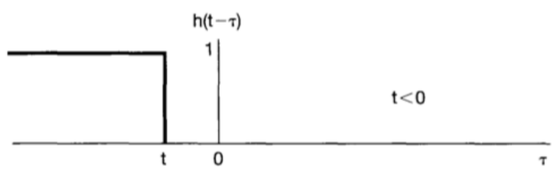
which is plotted in Figure 9.



(a) $h(\tau)$



(b) $x(\tau)$



(c) $h(t-\tau), t < 0$

Figure 8: Visualization of signals for the convolution integral in Example 4.

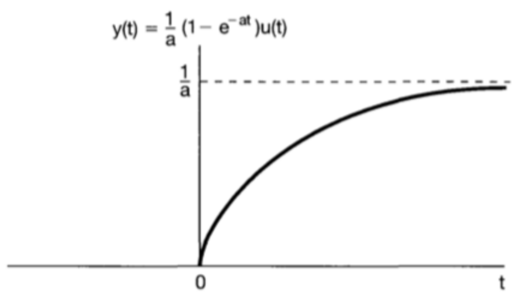
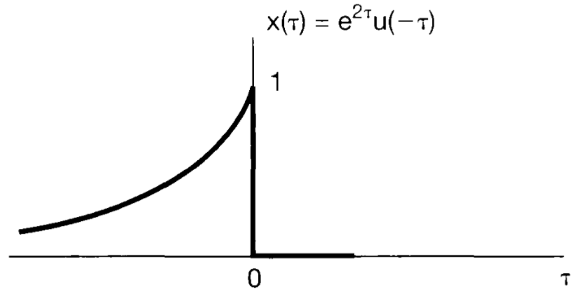


Figure 9: Output $y(t)$ for Example 4.

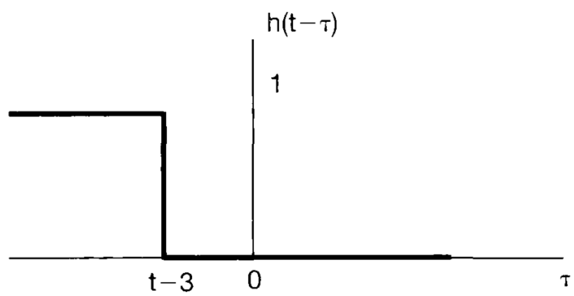
It is also important for us to get comfortable evaluating convolution integrals with signals that are defined for $t < 0$.

Example 5. *Consider an LTI system described by $h(t) = u(t-3)$. What is $y(t)$ when $x(t) = e^{2t}u(-t)$?*

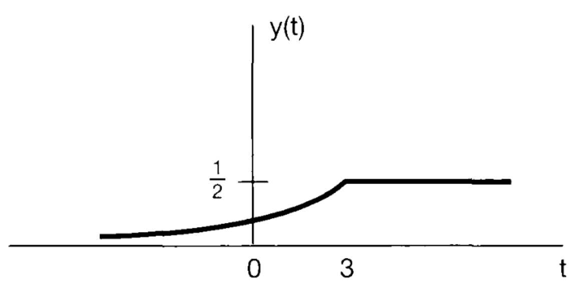
Ans: In Figure 10(a)&(b), we plot $x(\tau)$ and $h(t-\tau)$, respectively. In this case, $x(\tau)$ and $h(t-\tau)$ are overlapping for all values of t. The convolution integral will be different to the left and right of $t-3=0$, as $x(\tau)$ is zero for $\tau > 0$. Thus, we consider two intervals for t:



(a) $x(\tau)$



(b) $h(t-\tau), t < 3$



(c) $y(t)$

Figure 10: Visualization of the convolution integral for Example 5.

- *For $t < 3$, the upper limit on the integral will be $t - 3$:*

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} u(-\tau)u(-\tau+t-3)d\tau = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2}e^{2\tau}\Big|_{-\infty}^{t-3} = \frac{1}{2}e^{2(t-3)}$$

- *For $t \geq 3$, the upper limit is 0:*

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}e^{2\tau}\Big|_{-\infty}^{0} = \frac{1}{2}$$

  This result can also be obtained by plugging in $t = 3$ to the previous case.

Thus, the solution is

$$y(t) = \begin{cases} \frac{1}{2}e^{2(t-3)} & t < 3 \\ 1/2 & t \geq 3 \end{cases}$$

which is plotted in Figure 10(c).

Example 6. *What is the output of an LTI system with the following input and impulse response?*

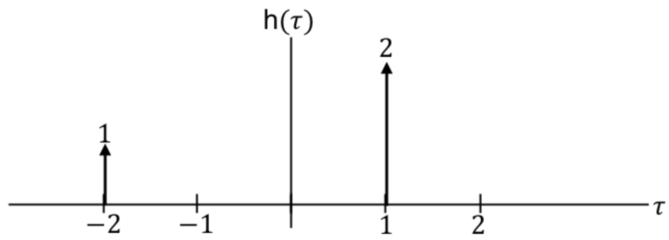
$$x(t) = \begin{cases} t+1 & 0 \leq t \leq 1 \\ 2-t & 1 < t \leq 2 \\ 0 & \textit{otherwise} \end{cases} \qquad h(t) = \delta(t+2) + 2\delta(t-1)$$

Ans: In Figure 11(a)&(b), $x(\tau)$ and $h(\tau)$ are plotted as functions of τ, and in (c), $h(t-\tau) = 2\delta(\tau - (t-1)) + \delta(\tau - (t+2))$ is plotted for $t < 0$. In this case, there are 7 intervals of t in which the integral changes, dictated by the overlap of the impulses in $h(t-\tau)$ with $x(\tau)$:

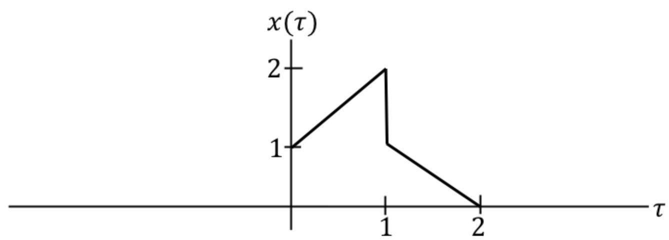
- *$t < -2$: In this case, there is no overlap between non-zero values of $x(\tau)$ and $h(t-\tau)$, since $\delta(\tau - (t+2))$ is to the left of $\tau = 0$. Therefore, $y(t) = 0$.*

- *$-2 \leq t < -1$: Here, $\delta(\tau - (t+2))$ overlaps with the portion of $x(\tau)$ in the interval $0 \leq \tau \leq 1$. Thus, $y(t) = \int_0^1 (\tau+1)\delta(\tau - (t+2))d\tau$. The result of this integral is $\tau + 1$ evaluated at $\tau = t+2$, i.e., $y(t) = t+2+1 = t+3$.*

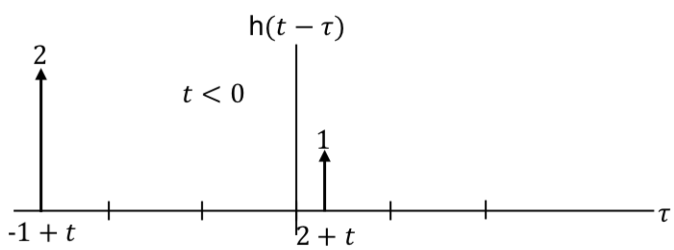
- *$-1 \leq t < 0$: Now $\delta(\tau - (t+2))$ overlaps with the portion of $x(\tau)$ between $1 \leq \tau \leq 2$. Thus, $y(t) = \int_1^2 (2-\tau)\delta(\tau - (t+2))d\tau = 2 - t - 2 = -t$.*



(a) Impulse response



(b) Input signal



(c) Reversed and shifted impulse response for $t < 0$

Figure 11: Visualization of the convolution integral for Example 6.

- $0 \leq t < 1$: *Here there is no overlap between non-zero values, since* $\delta(\tau - (t-1))$ *is still to the left of* $\tau = 0$, *and* $\delta(\tau - (t+2))$ *is to the right of* $\tau = 2$. *So* $y(t) = 0$.

- $1 \leq t < 2$: *Now we switch to the other impulse.* $2\delta(\tau - (t-1))$ *overlaps with* $x(\tau)$ *in the range* $0 \leq \tau \leq 1$. *Thus* $y(t) = \int_0^1 2(\tau+1)\delta(\tau-(t-1))d\tau = 2(t-1+1) = 2t$.

- $2 \leq t < 3$: *Now* $2\delta(\tau-(t-1))$ *overlaps with* $x(\tau)$ *in the range* $1 \leq \tau \leq 2$. *Thus* $y(t) = \int_1^2 2(2-\tau)\delta(\tau-(t-1))d\tau = 2(2-t+1) = 2(3-t)$.

- $t \geq 3$: *There again is no overlap between non-zero values, so* $y(t) = 0$.

The final output $y(t)$ is plotted in Figure 12 below.

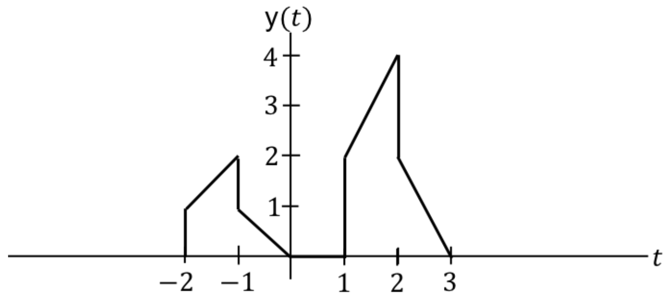


Figure 12: Output signal for Example 6.

The previous examples illustrate the utility of graphical interpretations to visualizing the evaluation of the convolution operation. We will next investigate several properties of LTI systems that in some cases will help simplify the operation further. In later modules, we will find ways to simplify convolution further by transforming signals to the frequency domain.

Properties of LTI Systems

LTI Systems have a number of special properties. One is the fact that the impulse response $h[n]$ or $h(t)$ completely characterizes the behavior of an LTI system, i.e., its input-output relationship. In the non-LTI category, on the other hand, there can be many systems that have the same impulse response.

In what follows, we will investigate a number of other important properties, all of which apply equally to both DT and CT LTI systems.

- **Commutative property.** Convolution is a **commutative** operation in both continuous and discrete time. That is,

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

This can be easily verified by performing a substitution of variables in the summation or integral, e.g., $r = n - k$.

Practically speaking, the commutative property tells us that the input x and impulse response h of an LTI system are interchangeable. This is particularly convenient because it allows us to choose which of these signals will be time-reversed and shifted in the convolution operation.

In Example 6, for instance, we could have evaluated the convolution integral in a single step by shifting and reversing $x(t)$ instead:

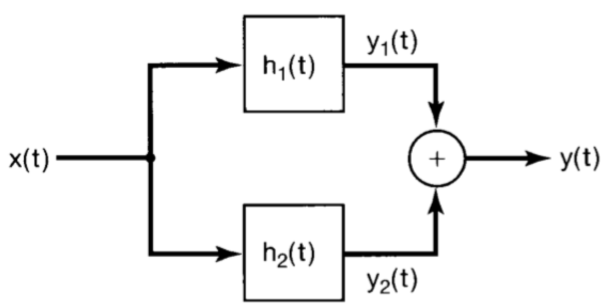
$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} \left(\delta(\tau+2) + 2\delta(\tau-1)\right) x(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} \delta(\tau+2)x(t-\tau)d\tau + \int_{-\infty}^{\infty} 2\delta(\tau-1)x(t-\tau)d\tau = x(t+2) + 2x(t-1)
\end{aligned}
$$

- **Distributive property.** Convolution is also distributive over addition, meaning

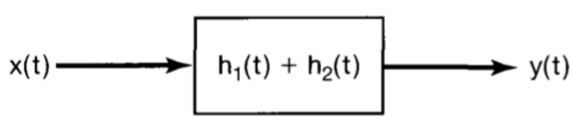
$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

This follows simply by the fact that integration and summation are both distributive operations.



(a) Parallel interconnection



(b) Additive impulse responses

Figure 13: Interpretation of the distributive property of convolution for parallel LTI systems. The systems in (a) and (b) are equivalent.

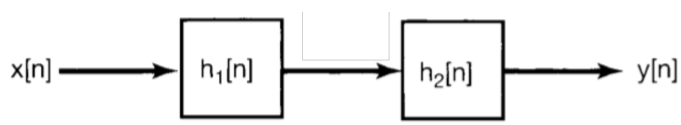
One of the implications of this is a method for simplifying LTI systems connected in parallel. Consider two such systems, as in Figure 13(a). The output of this system is $y(t) = x(t) * h_1(t) + x(t)h_2(t)$. By the distributive property, we also have $y(t) = x(t) * (h_1(t) + h_2(t))$, which means that we can represent this as a single LTI system with impulse response $h(t) = h_1(t) + h_2(t)$. In other words, the impulse response of a parallel interconnection is the sum of the individual impulse responses. This is illustrated in Figure 13(b).

- **Associative property.** Convolution is also an **associative** operation. That is,

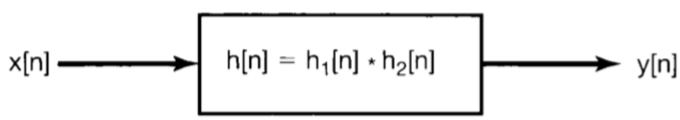
$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$$

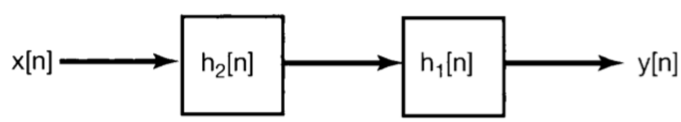
This means that expressions like $x[n]*h_1[n]*h_2[n]$ and $x(t)*h_1(t)*h_2(t)$ are unambiguous: the order in which we convolve the signals does not matter.



(a) Series interconnection



(b) Associative property



(c) Commutative property

Figure 14: Implication of the associative and commutative properties of convolution to series LTI systems. (a), (b), and (c) are all equivalent systems.

While the distributive property allows us to simplify parallel LTI interconnections, the associative and commutative properties have implications to series LTI systems. Consider Figure 14(a). The output is $y[n] = (x[n] * h_1[n]) * h_2[n]$, which by the associative property can be written as $y[n] =$

$x[n] * (h_1[n] * h_2[n])$. Thus, we can represent this as the single system $h[n] = h_1[n] * h_2[n]$ shown in Figure 14(b). Further, by the commutative property, we can reverse the order of h_1 and h_2 in the series interconnection, i.e., $y[n] = x[n] * h_2[n] * h_1[n]$, as in Figure 14(c).

It is important to emphasize that the capability of re-arranging the order of systems in cascade is special to LTI systems. As a simple counter-example for non-LTI, suppose we have two systems $S_1 : y[n] = x^2[n]$ and $S_2 : y[n] = 2x[n]$. The first is clearly non-linear. If we apply S_1 first, the final output is $y[n] = 2x^2[n]$. If we apply S_2 first, though, then it is $y[n] = 4x^2[n]$.

Step Response

Our characterization of LTI systems has focused on the impulse response, i.e., the response when a single unit impulse is applied to the system. Sometimes it is also convenient to look at the **unit step response**, which is the response when the input is a unit step signal. We denote this $s(t)$ in CT and $s[n]$ in DT, which are the responses when $x(t) = u(t)$ and $x[n] = \delta[n]$, respectively.

The responses s and h have convenient relationships for LTI systems. We can see these by noting that $s = u * h = h * u$, which gives an accumulator/first difference relationship in discrete-time:

$$s[n] = \sum_{k=-\infty}^{n} h[k], \qquad h[n] = s[n] - s[n-1]$$

and a running integral/derivative relationship in continuous-time:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \qquad h(t) = \frac{ds(t)}{dt} = s'(t)$$

Classifying LTI Systems

In the last module, we looked at several system classifications: memory/memoryless, invertible/non-invertible, and so-on. In the case of LTI systems, the impulse response provides us a more systematic way to infer these classifications.

- **Memory.** Recall that a DT system is memoryless if $y[n]$ depends only on $x[n]$ at time n, and no other points in time. Writing the output for an LTI system as $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, it follows that we must have $h[k] = 0$ for $k \neq 0$ in order for it to be memoryless. Therefore, for a memoryless DT LTI system, the impulse response and output must have the forms

$$h[n] = K\delta[n], \qquad y[n] = Kx[n]$$

where $K = h[0]$ is a constant. We can deduce similar properties for a memoryless CT LTI system:

$$h(t) = K\delta(t), \qquad y(t) = Kx(t)$$

- **Invertibility.** For a CT system S to be invertible, there must exist an inverse system S_I such that when cascaded, the final output $y(t) = x(t)$. In the LTI case, then, we need $x(t) * h(t) * h_I(t) = x(t)$, where $h_I(t)$ is the impulse response of the inverse system. For this to be true, we need the overall impulse response of the cascaded system to be $\delta(t)$. In other words, for continuous-time and discrete-time, we need

  $$h(t) * h_I(t) = \delta(t), \qquad h[n] * h_I[n] = \delta[n]$$

  The system is invertible if and only if such an h_I exists.

  Example 7. *Is the LTI system $h[n] = u[n]$ invertible? If so, what is the inverse system $h_I[n]$?*

  Ans: For the system to be invertible, we must have $h[n] * h_I[n] = \delta[n]$. Therefore, we need h_I such that

  $$\begin{aligned} u[n] * h_I[n] &= \sum_{k=-\infty}^{\infty} u[k]h_I[n-k] = \sum_{k=0}^{\infty} h_I[n-k] \\ &= h_I[n] + h_I[n-1] + h_I[n-2] + \cdots = \delta[n] \end{aligned}$$

  By inspection, we can see that this will be satisfied for

  $$h_I[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \end{cases}$$

  So, the system is invertible, and the impulse response of the inverse system is $h_I[n] = \delta[n] - \delta[n-1]$. This can be verified by noting $u[n] * (\delta[n] - \delta[n-1]) = u[n] - u[n-1] = \delta[n]$. The impulse response $h[n] = u[n]$ actually corresponds to the accumulator system $y[n] = \sum_{k=-\infty}^{n} x[k]$.

In general, for any given h, how do we construct h_I? This is a problem we will answer in the second half of the semester by looking at signal transforms.

- **Causality.** Recall that for a DT system to be causal, $y[n]$ must only depend on $x[k]$ for $k \leq n$. For an LTI system, $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, so for causality to hold we must have $h[k] = 0$ when $k < 0$, which is less strict than for memoryless. This gives us the following conditions for causality in the DT and CT LTI system cases:

$$h[n] = 0, n < 0, \qquad h(t) = 0, t < 0$$

So, the impulse response of a causal LTI system must be zero before the impulse occurs at $n = 0$ or $t = 0$, meaning h will not anticipate the arrival of δ. The convolution operations in this case also simplify to

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k], \qquad y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

- **Stability.** For a system to be stable, every bounded input must produce a bounded output, often referred to as BIBO stability. For a discrete-time LTI system, the magnitude of the output is

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} \big|h[k]\big| \cdot \big|x[n-k]\big|$$

from the triangle inequality. If the system is to be stable, we must have a bounded $|y[n]|$ when $|x[n]| < B$ for all n. In other words, we need to ensure $|y[n]| \leq B \cdot \sum_{k=-\infty}^{\infty} |h[k]|$ is bounded, which is entirely dependent on the impulse response. The condition for stability, in CT and DT, is then

$$\int_{-\infty}^{\infty} \big|h(\tau)\big| d\tau < \infty, \qquad \sum_{k=-\infty}^{\infty} \big|h[k]\big| < \infty$$

In other words, for an LTI system to be stable, the impulse response must be absolutely integrable and absolutely summable, respectively. Intuitively, a single impulse should not generate "too much" output: the effect should wear off after a while.

One result of this is that any LTI system which has a **finite-length impulse response** – one that is non-zero only on a finite range (and is "well-behaved") – will be stable, since the sum/integral will be finite.

Example 8. *Is the LTI system $h(t) = te^{-t}u(t+1)$ causal? Is it stable?*

Ans: Since $h(t)$ is non-zero for $-1 \le t \le 0$, we conclude that the system is not causal. For stability, we must evaluate

$$\int_{-\infty}^{\infty} \left|\tau e^{-\tau} u(\tau+1)\right| d\tau = \int_{-1}^{\infty} \left|\tau e^{-\tau}\right| d\tau = \int_{-1}^{0} \left|\tau e^{-\tau}\right| d\tau + \int_{0}^{\infty} \left|\tau e^{-\tau}\right| d\tau$$

Since $\left|\tau e^{-\tau}\right| = -\tau e^{-\tau}$ for $-1 \le \tau \le 0$, this becomes

$$\int_{-1}^{0} -\tau e^{-\tau} d\tau + \int_{0}^{\infty} \tau e^{-\tau} d\tau$$

Integrating by parts with $u = \tau$ and $dv = e^{-\tau} d\tau$, this becomes

$$\begin{aligned} \tau e^{-\tau}\Big|_{-1}^{0} \quad &- \quad \int_{-1}^{0} e^{-\tau} d\tau - \tau e^{-\tau}\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-\tau} d\tau \\ &= \quad \tau e^{-\tau}\Big|_{-1}^{0} + e^{-\tau}\Big|_{-1}^{0} - \tau e^{-\tau}\Big|_{0}^{\infty} - e^{-\tau}\Big|_{0}^{\infty} \\ &= \quad (0+e) + (1-e) - \lim_{\tau\to\infty} \tau e^{-\tau} - (0-1) = 2 \end{aligned}$$

The last step follows since $\lim_{\tau\to\infty} \tau e^{-\tau} = 0$, i.e., $\int_{-\infty}^{\infty} |h(\tau)| d\tau = 2$. Therefore, since the impulse response is absolutely integrable, the system is stable.

As practice, in addition to the homework, verify the classifications of the following LTI systems in terms of memory, causality, and stability:

Impulse response	Memory	Causality	Stability
$h_1[n] = \left(\frac{1}{2}\right)^n$	W.M.	NC	NS
$h_2[n] = \left(\frac{1}{2}\right)^n u[n]$	W.M.	C	S
$h_3(t) = e^{-\|t\|}$	W.M.	NC	S
$h_4(t) = e^{-t}u(t)$	W.M.	C	S