Signals and Systems: Module 11

Suggested Reading: S&S 5.1-5.3, 5.4-5.6, 5.8

We will now complete our study of the essential tools in Fourier analysis by formalizing the discrete-time Fourier transform. This is the transform used to obtain the frequency spectrum of discrete-time aperiodic signals. In doing so, we will draw several analogies to our development of the CTFT, and in fact will follow a parallel treatment. We will also see a number of salient differences between continuous-time and discrete-time: for example, the discrete-time Fourier transform is periodic, as are the Fourier series coefficients for discrete-time signals.

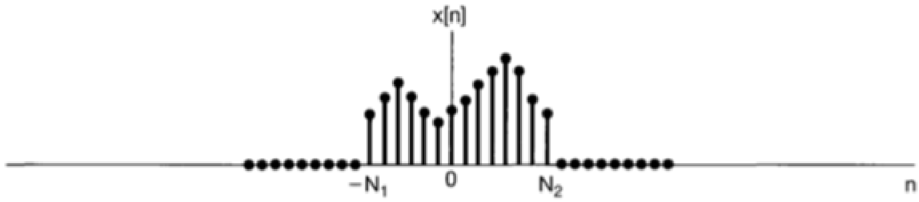
Discrete Time Fourier Transform

Consider a sequence $x[n]$ that has a finite duration, say $x[n] = 0$ outside $-N_1 \leq n \leq N_2$, as shown in Figure 1(a). The tool that we have for analyzing frequency contents in discrete time, the Fourier series, is not directly applicable as $x[n]$ is aperiodic. As we did in the derivation of the CTFT, suppose we construct a periodic sequence $\tilde{x}[n]$ for which $x[n]$ is a single period, as in Figure 1(b). As the period N is chosen to be larger, $\tilde{x}[n]$ will become closer to $x[n]$ over a longer interval, and in the limit as $N \to \infty$ they will be equivalent.

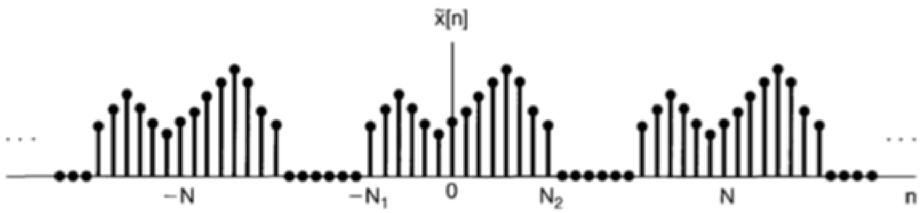
Fourier series of $\tilde{x}[n]$. The Fourier series of $\tilde{x}[n]$ is obtained as

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$

where $\omega_0 = 2\pi/N$ is the fundamental frequency. Since $\tilde{x}[n] = x[n]$ over the period $\langle N \rangle$ that includes $-N_1 \leq n \leq N_2$ and $x[n] = 0$ otherwise, the



(a) Finite-duration signal $x[n]$



(b) Periodic $\tilde{x}[n]$ constructed from $x[n]$

Figure 1: Constructing a periodic version $\tilde{x}[n]$ of a finite-duration signal $x[n]$ that is equal to $x[n]$ over one period.

equation for a_k can be written as

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]e^{-jk\omega_0 n}$$

Deriving the DTFT. Now, define the function $X(e^{j\omega})$ such that the coefficients a_k are its samples:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad a_k = \frac{1}{N}X(e^{jk\omega_0})$$

Note that $\omega_0 = 2\pi/N$ are the spacing of the samples in the frequency domain. With these substitutions, we can write the synthesis equation as

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n} = \frac{1}{2\pi}\sum_{k=<N>} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0$$

As N increases, ω_0 decreases, and the summation approaches an integral. Since the summation is carried out over N consecutive rectangles each of width $\omega_0 = 2\pi/N$, the interval of integration has width 2π. Therefore, in the limit as $\tilde{x}[n]$ converges to $x[n]$, we have the following pair of equations:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

where this integral can be taken over any interval of length 2π.

$X(e^{j\omega})$ is the **discrete-time Fourier transform** (or **DTFT**) of $x[n]$, and the two form a Fourier transform pair. The equation for $X(e^{j\omega})$ is analysis, while that for $x[n]$ is synthesis.

DTFT Interpretation

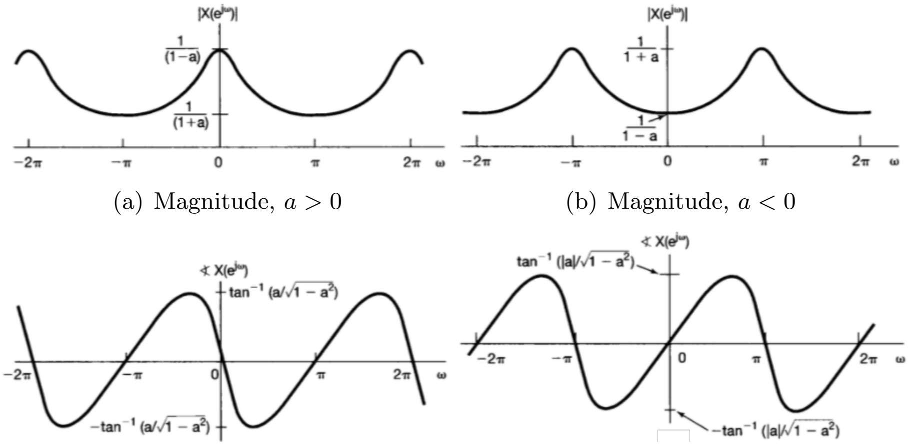
The DTFT is, in effect, breaking down an aperiodic DT signal into a linear combination of complex exponentials that are infinitesimally close in frequency and have amplitudes $X(e^{j\omega})(d\omega/2\pi)$. For this reason, as in continuous time, we also refer to $X(e^{j\omega})$ as the frequency spectrum of $x[n]$, because it tells us what $x[n]$ is composed of at different frequencies.

These equations are the discrete-time counterparts of what we saw for the CTFT. Clearly, they share many similarities. What are the key differences?

- The DTFT analysis equation $X(e^{j\omega})$ is periodic, with period 2π, while the CTFT $X(j\omega)$ is in general aperiodic. (Recall that the DTFS coefficients are also periodic, but with period N.)
- The DTFT synthesis equation only has a finite interval of integration, while the CTFT is an infinite interval. (Recall that the DTFS synthesis has a finite interval of *summation*.)

Both of these differences stem from the fact that the discrete-time complex exponential $e^{j\omega n}$ is identical every 2π, i.e., $e^{j\omega n} = e^{j(\omega+2\pi k)n}$ for any integer k. The same is not true for the continuous-time complex exponential $e^{j\omega t}$.

When we first discussed the periodic complex exponential $e^{j\omega_0 n}$, we saw that its rate of oscillation is highest at odd multiples of π $(\pi, 3\pi, 5\pi, ...)$ and lowest near even multiples of π $(0, 2\pi, 4\pi, ...)$. $e^{j\pi n}$, for example, has a higher frequency of variation than $e^{j2\pi n}$ since $e^{j2\pi n} = e^{j0n} = 1$ has no variation at all. Thus, discrete-time signals with DTFTs concentrated near frequencies that are even multiples of π are considered low-frequency signals, while those concentrated near odd multiples are high-frequency signals. In continuous-time, by contrast, higher variations are monotonic in ω.



(a) Magnitude, $a > 0$ (b) Magnitude, $a < 0$

(c) Phase, $a > 0$ (d) Phase, $a < 0$

Figure 2: Magnitude and phase plots of the Fourier transform of the geometric sequence $x[n]$. The sign of a changes the shape of both components.

Common DTFT Pairs

Geometric sequence. One common discrete-time signal that we need to understand the DTFT for is the infinite geometric sequence

$$x[n] = a^n u[n], \quad |a| < 1$$

Applying the equation for $X(e^{j\omega})$, we have

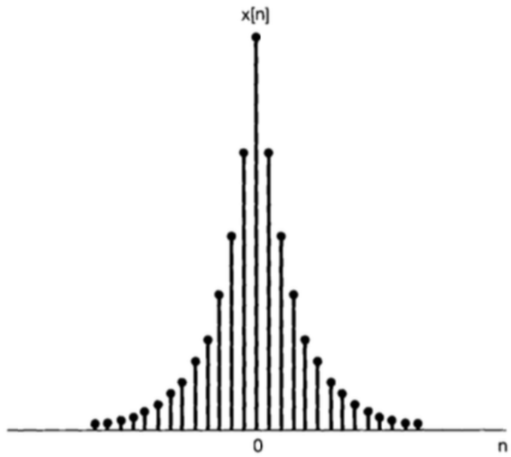
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

The magnitude $|X(e^{j\omega})|$ and phase $\angle X(e^{j\omega})$ are plotted in Figure 2 for two different cases of a: $a > 0$ (in (a) and (c)) and $a < 0$ (in (b) and (d)). We see that the sign of a shifts the concentration of the magnitude around even multiples of π (for $a > 0$) to odd multiples of π (for $a < 0$), which is consistent with $a < 0$ causing a higher rate of variation in $x[n]$. In each case, note that both the magnitude and phase are periodic in ω with period 2π.

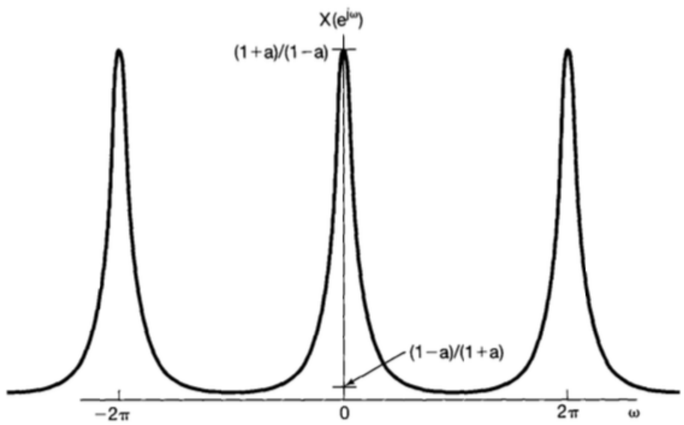
Example 1. *What is the Fourier transform of $x[n] = a^{|n|}$, $|a| < 1$? Plot the frequency spectrum for $0 < a < 1$.*

Ans: We have

$$X(e^{j\omega}) \quad = \quad \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$



(a) Signal $x[n]$



(b) Spectrum $X(e^{j\omega})$

Figure 3: Plot of the signal $x[n]$ and its Fourier transform $X(e^{j\omega})$ in Example 1, for $0 < a < 1$. The transform is purely real.

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} a^n e^{j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \left(\frac{1}{1-ae^{j\omega}} - 1\right) + \frac{1}{1-ae^{-j\omega}} \\
&= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{ae^{j\omega}\left(1-ae^{-j\omega}\right)}{(1-ae^{j\omega})(1-ae^{-j\omega})} + \frac{1-ae^{j\omega}}{(1-ae^{j\omega})(1-ae^{-j\omega})} \\
&= \frac{ae^{j\omega} - a^2 + 1 - ae^{j\omega}}{1+a^2-a(e^{j\omega}+e^{-j\omega})} = \frac{1-a^2}{1-2a\cos(\omega)+a^2}
\end{aligned}
$$

In this case, $X(e^{j\omega})$ is purely real; it is no coincidence that $x[n]$ is also even. The spectrum is plotted in Figure 3(b), next to the signal in Figure 3(a), for $0 < a < 1$. The maximum and minimum values occur when $\cos(\omega)$ is maximized and minimized, respectively:

$$\omega = 0 : X(e^{j\omega}) = \frac{(1-a)(1+a)}{(1-a^2)^2} = \frac{1+a}{1-a}$$

$$\omega = \pi : X(e^{j\omega}) = \frac{(1-a)(1+a)}{(1+a)^2} = \frac{1-a}{1+a}$$

Rectangular pulse. Another common discrete-time signal is the rectangular pulse:

$$x[n] = \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases}$$

We studied its continuous-time and periodic versions previously. We have

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{n=0}^{2N_1} e^{-j\omega(n-N_1)} = e^{j\omega N_1} \sum_{n=0}^{2N_1} e^{-j\omega n} \\ &= e^{j\omega N_1} \left(\frac{1 - e^{-j\omega(2N_1+1)}}{1 - e^{-j\omega}} \right) = \frac{e^{-j\omega/2}}{e^{-j\omega/2}} \left(\frac{e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)}}{e^{j\omega/2} - e^{-j\omega/2}} \right) \\ &= \frac{\sin\left(\omega(N_1+1/2)\right)}{\sin\left(\omega/2\right)} \end{aligned}$$

$x[n]$ and $X(e^{j\omega})$ for the rectangular pulse are shown in Figure 4 for $N_1 = 2$. $X(e^{j\omega})$ here is the discrete-time counterpart of the sinc function in continuous-time.

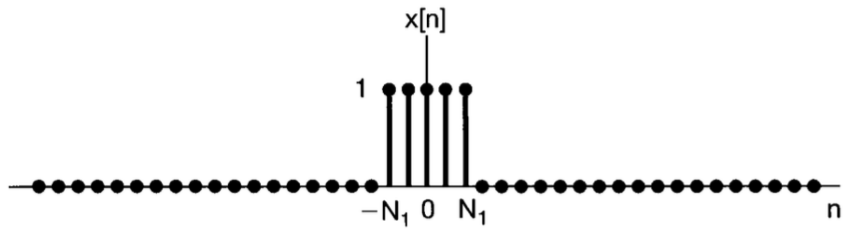
Unit impulse. For the unit impulse $x[n] = \delta[n]$, we can easily evaluate

$$X(e^{j\omega}) = 1$$

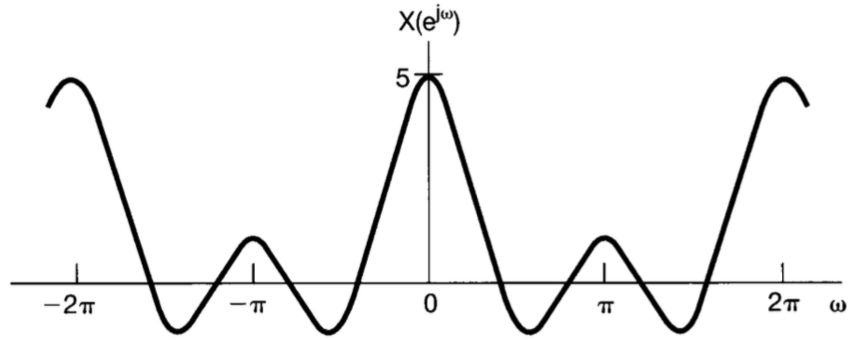
which is the same as the CTFT pair $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$.

DTFT Convergence

Though our derivation of the DTFT assumed a finite duration signal $x[n]$, it remains valid for an extremely broad class of signals with infinite duration



(a) Signal $x[n]$



(b) Spectrum $X(e^{j\omega})$

Figure 4: Plot of the signal $x[n]$ and its Fourier transform $X(e^{j\omega})$ for the rectangular pulse with $N_1 = 2$.

(such as the previous two examples). The question is, under what conditions will the analysis $X(e^{j\omega})$ and synthesis $x[n]$ equations converge? Similar to the CTFT, we can show that the analysis $X(e^{j\omega})$ will converge either if $x[n]$ is absolutely summable or the sequence has finite energy, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad \text{or} \quad \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

In contrast, the synthesis $x[n]$ actually generally has no convergence issues, because the integral is over a finite interval (2π).

DTFT for Periodic Signals

In continuous-time, we saw that the Fourier transform of $e^{j\omega_0 t}$ was the impulse $2\pi\delta(\omega - \omega_0)$, i.e., an impulse at ω_0. Will the transform of $e^{j\omega_0 n}$ be similar? Yes, but we need to account for periodicity of the DTFT, i.e., we need impulses at ω_0, $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$, and so on. In particular,

$$x[n] = e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

which we can verify by noting that the synthesis integral will always include exactly one of the $\delta(\omega - \omega_0 - 2\pi l)$ terms over any period 2π. This is because the inverse Fourier Transform of $X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$ is:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)e^{-j\omega n} = e^{j\omega_0 n}.$$

Now, consider a general periodic sequence $x[n]$ expressed in Fourier series form as

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

If we apply the transform to each exponential separately, we have

$$\begin{aligned} X(e^{j\omega}) &= \sum_{k=<N>} \sum_{l=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N} - 2\pi l\right) \\ &= \sum_{k=<N>} \sum_{l=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi(k - Nl)}{N}\right) \end{aligned}$$

If we consider $k = 0, ..., N-1$, the values $k - Nl$ for all integers l will also cover all integers, without duplication. Thus, the double summation is equivalent to summing over all integers:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Note that this will be periodic in 2π, since the Fourier series coefficients a_k are periodic in N.

Sinusoids. We can consider the DTFT of the sinusoid $x[n] = \cos\omega_0 n$, $0 < \omega_0 < \pi$. From Euler's formula, we can write

$$x[n] = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi\delta(\omega - \omega_0 - 2\pi k) + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + \omega_0 - 2\pi k)$$

So, in the interval $-\pi < \omega < \pi$, $X(e^{j\omega}) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$, and this repeats every 2π. The result is plotted in Figure 5.

The sinusoid $x[n] = \sin \omega_0 n$ gives a very similar result:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} -\frac{\pi}{j}\delta(\omega - \omega_0 - 2\pi k) + \sum_{k=-\infty}^{\infty} \frac{\pi}{j}\delta(\omega + \omega_0 - 2\pi k)$$

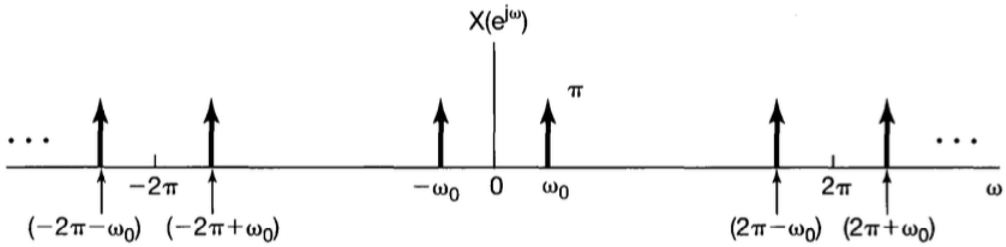


Figure 5: Frequency spectrum $X(e^{j\omega})$ of $x[n] = \cos(\omega_0 n)$, $0 < \omega < \pi$.

Complex exponential. Recall that the complex exponential $x[n] = e^{j\omega_0 n}$ is only periodic if the lowest common multiple $\mathrm{LCM}(2\pi/\omega_0, 1)$ exists. However, even if $x[n]$ is aperiodic, $X(e^{j\omega})$ still exists:

$$x[n] = e^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi k) \qquad \text{for any } \omega_0$$

This holds for the sinusoids discussed above too. The difference is that when $x[n]$ is aperiodic, the Fourier *series* does not exist.

Properties of the DTFT

As with the CTFT, there are a number of properties of the DTFT that aid in analysis. Many of these are either similar or identical to a corresponding CTFT counterpart. In presenting these, we will adopt the terminology $X(e^{j\omega}) = \mathcal{F}\{x[n]\}$, $x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\}$, and $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$.

Periodicity. The DTFT is always periodic in ω with period 2π, i.e.,

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

The CTFT, on the other hand, is not periodic in general.

Linearity. If $x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(e^{j\omega})$ and $x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(e^{j\omega})$, then

$$ax_1[n] + bx_2[n] \overset{\mathcal{F}}{\longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

Time and frequency shifting. If $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$, then

$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}X(e^{j\omega}) \quad \text{and} \quad e^{j\omega_0 n}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)})$$

Conjugation and conjugate symmetry. If $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$, then

$$x^\star[n] \overset{\mathcal{F}}{\longleftrightarrow} X^\star(e^{-j\omega})$$

Further, if $x[n]$ is real, it follows that $X(e^{j\omega}) = X^\star(e^{-j\omega})$.

Differencing and accumulation. With linearity and time shifting, it follows that the Fourier transform of the first-difference signal is

$$x[n] - x[n-1] \overset{\mathcal{F}}{\longleftrightarrow} (1 - e^{-j\omega})X(e^{j\omega})$$

Further, the accumulation signal has the transform

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

The $\pi X(e^{j0})$ term represents the DC value that can result from summation. Different from the CTFT, it weights an impulse train rather than a single impulse, by virtue of the fact that the DTFT is periodic.

Time reversal. Reversing the sign of n in the equation for $X(e^{j\omega})$, it follows that

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega})$$

Differentiation in frequency. By taking the derivative of the analysis equation, we can find that

$$nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's relation. If $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$, then

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Whereas the Fourier coefficients a_k of a periodic signal can be used to determine its average power, the Fourier transform $X(e^{j\omega})$ of an aperiodic signal can be used to determine its total energy. In analogy with the continuous-time case, $|X(e^{j\omega})|^2$ is referred to as the energy-density spectrum of $x[n]$.

The Convolution Property

Recall that with the CTFT, there were two properties of particular importance to the analysis of LTI systems: convolution and multiplication. We start here by presenting their discrete-time equivalents. Then, we will discuss how to use the DTFT to find the frequency response of systems described by difference equations.

Consider an LTI system with impulse response $h[n]$. If $x[n]$ is the input to the system, then the input-output relationship is given by

$$y[n] = x[n] * h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

where $X(e^{j\omega})$, $H(e^{j\omega})$, and $Y(e^{j\omega})$ are the DTFTs of $x[n]$, $h[n]$, and $y[n]$, respectively. $H(e^{j\omega})$ is the frequency response of the LTI system.

As in continuous time, the convolution property maps the convolution of two signals in the time domain to the simple algebraic operation of multiplication in the frequency domain. This is an extension of the result for periodic signals that the each of the spectral coefficients of the input are simply multiplied by the frequency response at the corresponding harmonic frequencies. This property also gives us insight into filter design: we want $|H(e^{j\omega})| \approx 1$ at frequencies which should be passed, and $|H(e^{j\omega})| \approx 0$ at those which should be rejected.

Example 2. *Consider an LTI system with impulse response $h[n] = \alpha^n u[n]$, $|\alpha| < 1$. What is the output when the input $x[n] = \beta^n u[n]$, $|\beta| < 1$ is applied?*

Ans: The Fourier transforms of the geometric sequences $x[n]$ and $h[n]$ are

$$X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}} \qquad H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

The transform of the output is therefore

$$Y(e^{j\omega}) = \frac{1}{(1 - \beta e^{-j\omega})(1 - \alpha e^{-j\omega})}$$

To inverse transform this, we first apply partial fraction expansion:

$$\frac{1}{(1 - \beta e^{-j\omega})(1 - \alpha e^{-j\omega})} = \frac{B}{1 - \beta e^{-j\omega}} + \frac{A}{1 - \alpha e^{-j\omega}}$$

To solve for A and B, we only slightly adjust our method from the CTFT, this time setting $e^{-j\omega}$ (rather than $j\omega$ such that the denominators are zero:

$$B = \frac{1}{1 - \alpha e^{-j\omega}}\bigg|_{e^{-j\omega}=1/\beta} = \frac{\beta}{\beta - \alpha}$$

$$A = \frac{1}{1-\beta e^{-j\omega}}\Big|_{e^{-j\omega}=1/\alpha} = \frac{\alpha}{\alpha-\beta}$$

Now we can readily obtain $y[n]$ *by the inverse transform:*

$$y[n] = \mathcal{F}^{-1}\left\{\frac{\alpha/(\alpha-\beta)}{1-\alpha e^{-j\omega}}\right\} + \mathcal{F}^{-1}\left\{\frac{\beta/(\beta-\alpha)}{1-\beta e^{-j\omega}}\right\} = \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n]$$

Simplifying slightly, we arrive at

$$y[n] = \frac{1}{\alpha-\beta}\left(\alpha^{n+1}u[n] - \beta^{n+1}u[n]\right)$$

for the case $\alpha \neq \beta$*. When* $\alpha = \beta$*, we need a slightly different approach; in particular, we have*

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

By the frequency differentiation property, we know that for any given signal $z[n]$*,*

$$nz[n] \stackrel{F}{\longleftrightarrow} j\frac{dZ(e^{j\omega})}{d\omega}$$

If we let $z[n] = \alpha^n u[n]$*, then*

$$a[n] = nz[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{1-\alpha e^{-j\omega}}\right] = j\frac{-1\cdot -\alpha \cdot -je^{-j\omega}}{(1-\alpha e^{-j\omega})^2} = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} = A(e^{j\omega})$$

We can express $Y(e^{j\omega}) = \frac{1}{\alpha}A(e^{j\omega})e^{j\omega}$*, which corresponds to a scale and time advance in the time domain:* $Y(e^{j\omega}) \stackrel{\mathcal{F}}{\longleftrightarrow} y[n] = \frac{1}{\alpha}a[n+1]$*. Thus,*

$$y[n] = (n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]$$

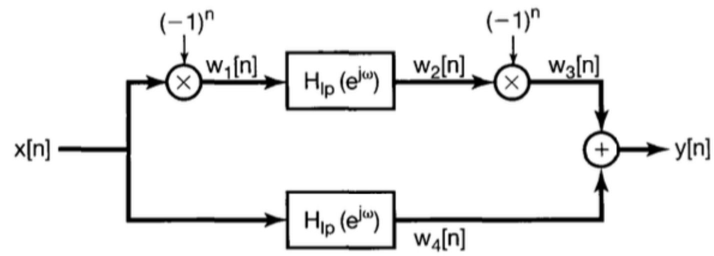
since $n+1 = 0$ *at* $n = -1$*.*

To summarize,

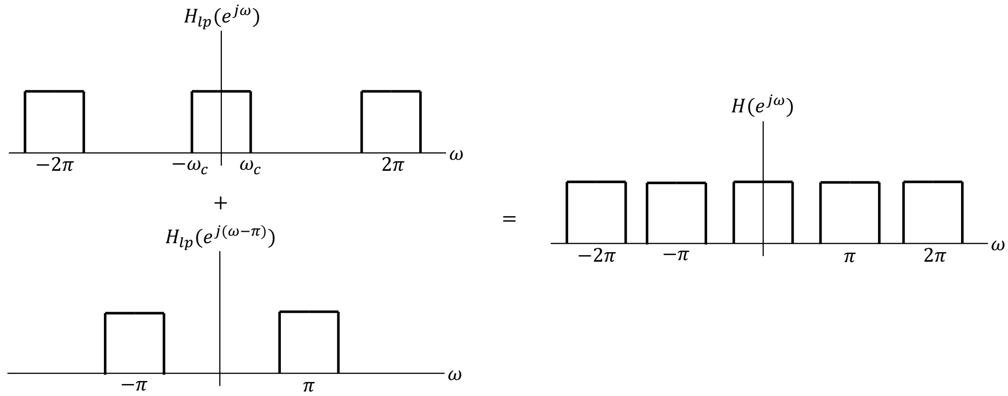
$$y[n] = \begin{cases} \frac{1}{\alpha-\beta}\left(\alpha^{n+1}u[n] - \beta^{n+1}u[n]\right) & \alpha \neq \beta \\ (n+1)\alpha^n u[n] & \alpha = \beta \end{cases}$$

Example 3. *What is the overall frequency response of the interconnection system in Figure 6(a)? Assume the systems with frequency response* $H_{lp}(e^{j\omega})$ *are ideal lowpass filters with cutoff frequency* ω_c*.*

Ans: The upper path is a combination of multiplication and convolution in the time domain, while the lower path is just convolution. Focusing on the



(a) System interconnection



(b) Frequency response $H(e^{j\omega})$

Figure 6: System interconnection for Example 2 and corresponding frequency response $H(e^{j\omega})$, which is a bandpass filter.

upper path first, we can trace the evolution of $x[n]$ step-by-step. After the first multiplication, $w_1[n] = x[n](-1)^n$. Since $(-1)^n = e^{j\pi n}$, by the frequency shifting property,

$$w_1[n] = x[n]e^{j\pi n} \overset{\mathcal{F}}{\longleftrightarrow} W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

Then, $w_2[n] = w_1[n] * h_{lp}[n]$ is the output of a lowpass filter, so by the convolution property,

$$W_2(e^{j\omega}) = W_1(e^{j\omega})H_{lp}(e^{j\omega}) = X(e^{j(\omega-\pi)})H_{lp}(e^{j\omega})$$

Then, $w_3[n] = w_2[n]e^{j\pi n}$, so we again apply the frequency shifting property:

$$w_3[n] = w_2[n]e^{j\pi n} \overset{\mathcal{F}}{\longleftrightarrow} W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)}) = X(e^{j(\omega-2\pi)})H_{lp}(e^{j(\omega-\pi)})$$

On the lower path, $x[n]$ goes through a lowpass filter. So, $y[n] = w_3[n] + x[n] * h_{lp}[n]$ has a transform

$$Y(e^{j\omega}) = X(e^{j(\omega-2\pi)})H_{lp}(e^{j(\omega-\pi)}) + X(e^{j\omega})H_{lp}(e^{j\omega})$$

By periodicity of the DTFT, $X(e^{j(\omega-2\pi)}) = X(e^{j\omega})$. Therefore,

$$Y(e^{j\omega}) = \left[H_{lp}(e^{j\omega}) + H_{lp}(e^{j(\omega-\pi)})\right] X(e^{j\omega})$$

And

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = H_{lp}(e^{j\omega}) + H_{lp}(e^{j(\omega-\pi)})$$

This frequency response is plotted in Figure 6(b).

Recall that with the DTFT, high frequencies are concentrated near π (and odd multiples). Thus, in Example 2, $H_{lp}(e^{j(\omega-\pi)})$ is actually an ideal high-pass filter. The combination of $H_{lp}(e^{j\omega})$ and $H_{lp}(e^{j(\omega-\pi)})$ passes both low and high frequencies and stops those between these two passbands. A filter with such characteristics is called an ideal **band-stop filter**.

Stability. Not every discrete-time LTI system has a frequency response. Recall though that if an LTI system is stable, then its impulse response is absolutely summable. We said earlier that absolute summability of a sequence is sufficient to guarantee convergence of the DTFT analysis equation. In other words, if an LTI system is stable, then its frequency response exists.

The Multiplication Property

As in continuous-time, when we multiply two discrete-time signals in the time domain, it becomes convolution in the frequency domain. Formally,

$$y[n] = x_1[n]x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

where the integral can be taken over any interval of length 2π. As this is not our standard convolution from $-\infty$ to ∞, it instead is often referred to as the **periodic convolution** of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$. Periodic convolution may be denoted with the symbol $\circledast$, i.e., $X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$.

Example 4. *Find the Fourier transform $X(e^{j\omega})$ of a signal $x[n]$ defined as*

$$x[n] = x_1[n]x_2[n]$$

where x_1 and x_2 are given as

$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$$

Ans: Using the multiplication property, we know that $X(e^{j\omega})$ can be found based on periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ in the frequency domain:

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

In order to complete this computation, we need $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$. Using Table 5.2 from the textbook, we determine that

$$X_1(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le 3\pi/4 \\ 0 & 3\pi/4 < |\omega| < \pi \\ \textit{periodic with period } 2\pi \end{cases}$$

$$X_2(e^{j\omega}) = \begin{cases} 1 & 0 \le |\omega| \le \pi/2 \\ 0 & \pi/2 < |\omega| < \pi \\ \textit{periodic with period } 2\pi \end{cases}$$

Plots of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ are shown in the top and bottom images of Figure 7. In order to compute the necessary periodic convolution, one helpful approach is to consider $\hat{X}_1(e^{j\omega})$ (illustrated in Figure 7), where

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & \textit{for } -\pi \le \omega \le \pi \\ 0 & \textit{otherwise} \end{cases}$$

Thus, we can rewrite the periodic convolution formula as follows,

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta, \end{aligned}$$

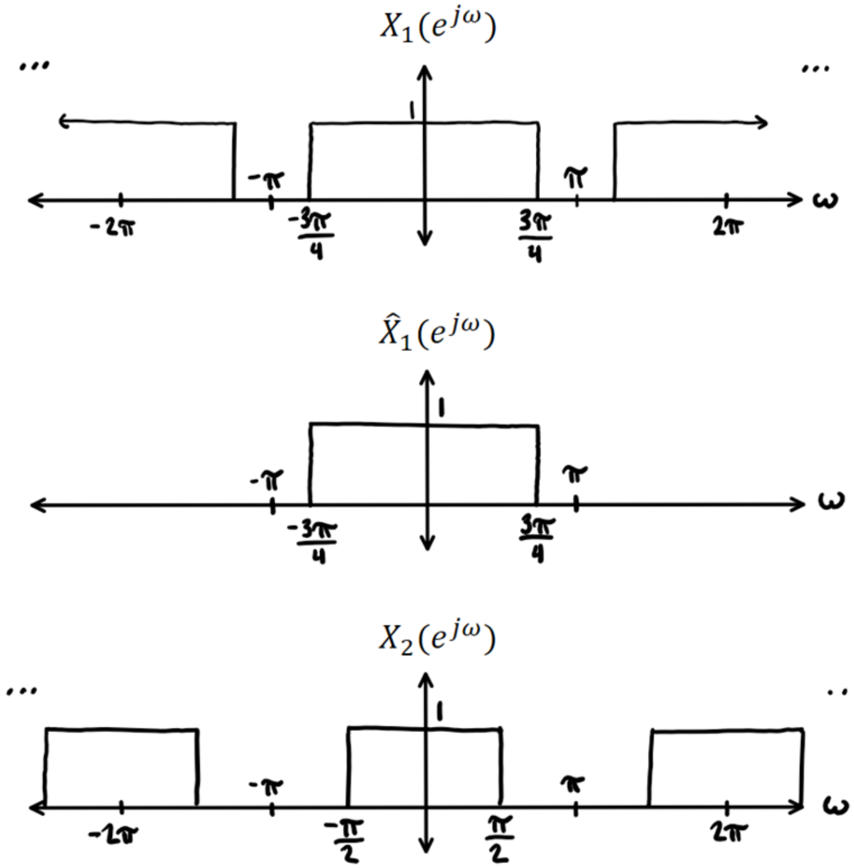


Figure 7: Plots of $X_1(e^{j\omega})$, $\hat{X}_1(e^{j\omega})$ (which represents isolating a single period of $X_1(e^{j\omega})$), and $X_2(e^{j\omega})$ for Example 4.

which takes the form of aperiodic convolution of $\hat{X}_1(e^{j\omega})$ and $X_2(e^{j\omega})$. A plot of this convolution is shown in the top of Figure 8, and a final plot of $X(e^{j\omega})$ is shown in the bottom of Figure 8.

Systems Characterized by Difference Equations

A particularly important class of discrete-time LTI systems are those for which the input $x[n]$ and output $y[n]$ satisfy a linear constant-coefficient

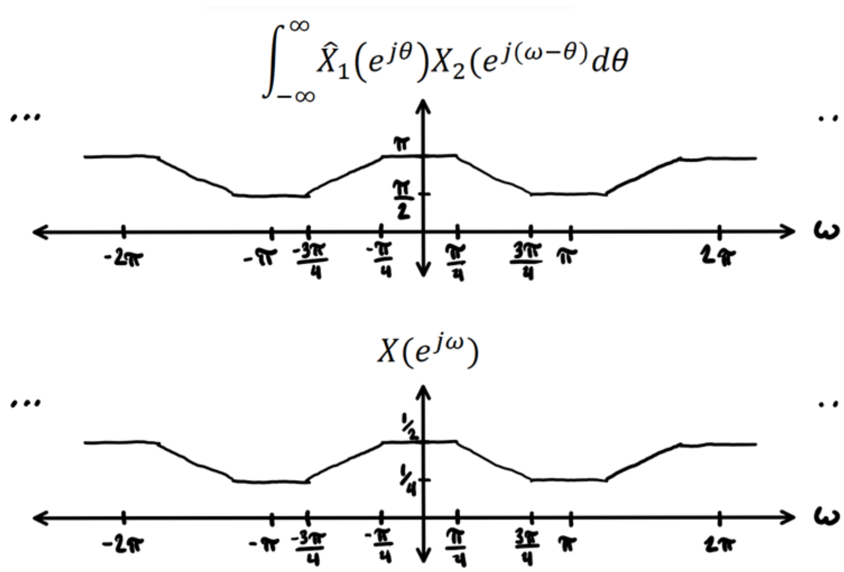


Figure 8: (Top) Plot of aperiodic convolution of $\hat{X}_1(e^{j\omega})$ and $X_2(e^{j\omega})$; (Bottom) Plot of $X(e^{j\omega})$ (the top plot scaled by $\frac{1}{2\pi}$).

difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

We wish to find the frequency response $H(e^{j\omega})$ of such systems. When we studied the CTFT, we saw its utility in analyzing LTI systems described by linear constant-coefficient differential equations. It should not come as a surprise, then, that the DTFT can be used to analyze the discrete-time counterpart of these systems. In doing so, we will always assume the LTI system is stable, so we are guaranteed its frequency response exists.

Taking the DTFT of both sides and applying the time-shifting property, we get

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

Now, how do we find the frequency response of the system? The convolution property tells us that for an LTI system, the convolution $y(t) = x(t) * h(t)$ becomes $Y(j\omega) = X(j\omega)H(j\omega)$ in the frequency domain. Therefore,

$H(j\omega) = Y(j\omega)/X(j\omega)$ for any input-output pair. Thus, we can obtain the frequency response as

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

And the impulse response can be obtained as $h(t) = F^{-1}\{H(e^{j\omega})\}$, though a closed-form solution may be difficult depending on the nature of $H(e^{j\omega})$.

Notice that $H(e^{j\omega})$ is a rational function, i.e., a ratio of polynomials in $e^{j\omega}$. The terms for Y go in the denominator, and those for X go in the numerator, which can actually be done by inspection. The order of the difference equation – N – translates to an Nth order polynomial in the denominator.

Example 5. *Consider a stable LTI system characterized by the difference equation*

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

What is the frequency response of the system? How about the impulse response? What is the output when an input $x[n] = \left(\frac{1}{4}\right)^n u[n]$ is applied?

Ans: The frequency response is obtained by taking the Fourier transform:

$$Y(e^{j\omega}) - \frac{3}{4}Y(e^{j\omega})e^{-j\omega} + \frac{1}{8}Y(e^{j\omega})e^{-2j\omega} = 2X(e^{j\omega})$$

So

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}$$

Now, to obtain the impulse response $h[n]$, we need to apply partial fraction expansion to be able to inverse transform $H(e^{j\omega})$. Since $1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega} = (1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})$, we have

$$H(e^{j\omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}}$$

By which we can obtain

$$A = \frac{2}{1-\frac{1}{4}e^{-j\omega}}\bigg|_{e^{-j\omega}=2} = \frac{2}{1-\frac{1}{2}} = 4, \qquad B = \frac{2}{1-\frac{1}{2}e^{-j\omega}}\bigg|_{e^{-j\omega}=4} = \frac{2}{1-2} = -2$$

So

$$H(e^{-j\omega}) = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

This can readily be inverse transformed using the transform pair $a^n u[n] \stackrel{F}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$*:*

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

With an input $x[n] = \left(\frac{1}{4}\right)^n u[n]$*, by the convolution property,*

$$\begin{aligned} Y(e^{j\omega}) &= H(e^{j\omega})X(e^{j\omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} \cdot \frac{1}{1-\frac{1}{4}e^{-j\omega}} \\ &= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})^2} \end{aligned}$$

To inverse transform this, we apply partial fraction expansion again:

$$\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})^2} = \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}} + \frac{C}{(1-\frac{1}{4}e^{-j\omega})^2}$$

Solving for A *and* C *is the same procedure as before:*

$$A = \left.\frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}\right|_{e^{-j\omega}=2} = \frac{2}{(1-1/2)^2} = 8, \qquad C = \left.\frac{2}{1-\frac{1}{2}e^{-j\omega}}\right|_{e^{-j\omega}=4} = \frac{2}{-1} = -2$$

For B*, we can multiply through by* $(1-\frac{1}{4}e^{-j\omega})^2$ *to find an expression. We get:*

$$\frac{2}{1-\frac{1}{2}e^{-j\omega}} = A \cdot \frac{(1-\frac{1}{4}e^{-j\omega})^2}{1-\frac{1}{2}e^{-j\omega}} + B \cdot \left(1-\frac{1}{4}e^{-j\omega}\right) + C$$

Now, take the derivative of both sides with respect to $e^{-j\omega}$ *(rather than* $j\omega$ *as we did in continuous-time), and evaluate it as* $e^{-j\omega} = 4$*:*

$$\left.\frac{-2 \cdot -\frac{1}{2}}{(1-\frac{1}{2}e^{-j\omega})^2}\right|_{e^{-j\omega}=4} = A \cdot 0 + B \cdot -\frac{1}{4} \to B = -\frac{4}{(1-2)^2} = -4$$

So,

$$Y(e^{j\omega}) = \frac{8}{1-\frac{1}{2}e^{-j\omega}} - \frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}$$

Which inverse transforms to

$$y[n] = 8\left(\frac{1}{2}\right)^n u[n] - 4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n]$$

Duality

With the CTFT, we saw a duality relationship between the analysis and synthesis equations. One key difference with the DTFT is that no such duality exists. However, there *is* a duality relationship between the DTF**S** analysis and synthesis equations. Additionally, and perhaps surprisingly, a duality can be shown between the **DTFT** and the **CTFS**. More details can be found in Chapter 5.7 of the textbook.

The last few modules have focused on Fourier analysis for different classes of signals. The following table is a useful summary of how the characteristics of a signal in time impact its characteristics in frequency, and vice versa. Each row corresponds to one of the four series/transform pairs we have studied.

Time			Frequency		
Class	Periodicity		Class	Periodicity	
Continuous	Periodic		Discrete	Aperiodic	(CTFS)
Continuous	Aperiodic	$\overset{\mathcal{F}}{\longleftrightarrow}$	Continuous	Aperiodic	(CTFT)
Discrete	Periodic		Discrete	Periodic	(DTFS)
Discrete	Aperiodic		Continuous	Periodic	(DTFT)