Signals and Systems: Module 7

Suggested Reading: S&S 3.6-3.8

The last module was our first foray into frequency representations of signals and systems, where we looked at the Fourier series representation of continuous-time periodic signals. We will continue with that theme in this module, considering instead discrete-time periodic signals.

Fourier Series in Discrete-Time

The development of the Fourier series in discrete time closely parallels its development in continuous time. However, there are some notable differences we need to keep in mind, starting with the set of signals used in the representation.

In earlier modules, we showed that the discrete-time signal $x[n] = e^{j\omega_0 n}$ is periodic if ω_0 can be written as $2\pi/N$ for an integer N; in this case, the period is N. We also found that the set of all discrete-time complex exponentials that are periodic with period N can be expressed as the following set of harmonically related complex exponentials (HRCEs):

$$x_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}, k = 0, 1, 2, ..., N-1$$

Importantly, there are only N signals in this set, since those differing in frequency by a multiple of 2π are identical, i.e., $x_k[n] = x_{k+rN}[n]$ for any integer r. This is very different from the HRCEs in continuous time where each $x_k(t)$ is distinct.

Fourier series expansion. Let $x[n]$ be an arbitrary discrete-time signal/sequence with fundamental period N. Representing $x[n]$ in terms of the HRCEs $x_k[n]$ with period N as

$$x[n] = \sum_{k=<N>} a_k x_k[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

is called the **discrete-time Fourier series (DTFS)** expansion of $x[n]$. The notation $k =< N >$ here denotes taking the summation over N consecutive integers, e.g., $k = 0, 1, ..., N-1$ or $k = 2, 3, ..., N+1$. Different ranges $< N >$ will produce the same results since there are only N unique HRCEs. The a_k are the corresponding Fourier series coefficients. Without loss of generality, we will often assume we are taking the summation over $k = 0, ..., N-1$, and index the coefficients accordingly.

Determining the coefficients. One way to determine the coefficients for a periodic signal $x[n]$ would be to set it up as a system of N equations: $x[0] = \sum_{k=<N>} a_k$, $x[1] = \sum_{k=<N>} a_k e^{jk\omega_0}$, $x[2] = \sum_{k=<N>} a_k e^{j2k\omega_0}$, and so on. In this case, $a_0, ..., a_{N-1}$ would be the N unknowns, and we could solve a linear system to determine them. However, this can become a rather large system.

Instead, as in continuous time, we can come up with a closed-form expression for the coefficients. Proceeding as we did there, we first multiply both sides by $e^{-jr(2\pi/N)n}$ for $r \in < N >$, sum over N terms (rather than integrate), and then interchange the order of the summations:

$$\begin{aligned} \sum_{n=<N>} x[n] e^{-jr(2\pi/N)n} &= \sum_{n=<N>} \sum_{k=<N>} a_k e^{j(k-r)(2\pi/N)n} \\ &= \sum_{k=<N>} a_k \sum_{n=<N>} e^{j(k-r)(2\pi/N)n} \end{aligned}$$

This appears complicated, but it simplifies dramatically by the fact that

$$\sum_{n=<N>} e^{j(k-r)(2\pi/N)n} = \begin{cases} N & k = r, r \pm N, r \pm 2N, ... \\ 0 & \text{otherwise} \end{cases}$$

What this says is that the sum over one period of the values of a periodic complex exponential is zero unless the exponential is a constant (which is true when $k = r, r \pm N, ...$). Thus, the double summation becomes zero except in a single case, since $|k - r| < N$, which we can assume is $k = r$. The right hand side of the summation therefore reduces to $a_r N$, so we can solve for the coefficients as

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk(2\pi/N)n}$$

As in continuous-time, solving for a_k is sometimes called the analysis step, while the resulting expansion for $x[n]$ is called the synthesis step.

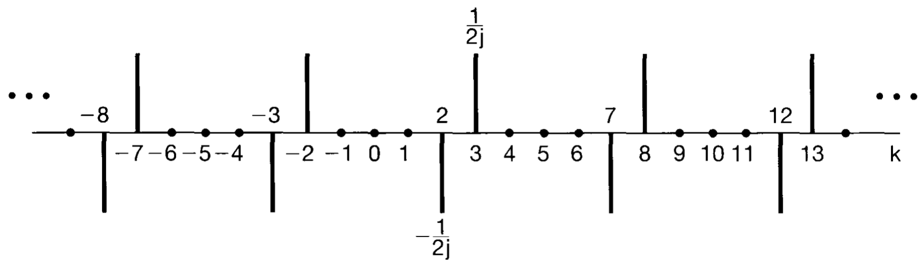


Figure 1: Coefficients a_k for Example 1 with $N = 5$ and $M = 3$.

Since we can choose any set of N successive values in building the Fourier series, we can conclude that $a_k = a_{k+iN}$ for integers i, i.e., the values of a_k also repeat periodically with period N. So, in discrete time, both the signal $x[n]$ and its Fourier series coefficients a_k are periodic with period N.

Example 1. *Consider the signal $x[n] = \sin \omega n$, where $\omega = 2\pi M/N$ for integers M and N, $M \leq N$. What are its Fourier series coefficients? What is its Fourier series expansion?*

Ans: The fundamental period of $x[n]$ is $LCM(2\pi/\omega, 1) = N$. Therefore the Fourier series representation of $x[n]$ has the form:

$$x[n] = \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn}$$

Rather than applying the formula for a_k, it is easier to expand the sinusoid using Euler's identity:

$$x[n] = \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) = -\frac{j}{2}e^{j\omega n} + \frac{j}{2}e^{-j\omega n} = \frac{1}{2j}e^{j(2\pi/N)Mn} - \frac{1}{2j}e^{-j(2\pi/N)Mn}$$

By comparing the above equation with the Fourier series representation of $x[n]$, we find $a_M = -j/2$ and $a_{-M} = j/2$, while all the remaining coefficients in the interval of summation are 0. By periodicity of the coefficients, we also have $a_{N+M} = -j/2$, $a_{N-M} = j/2$, and so on.

Assuming $M = 3$ and $N = 5$, the coefficients are plotted in Figure 1. One graph suffices here as the a_k are purely imaginary. The periodicity of the coefficients in N is incorporated here, but notice that over any period of length 5 there are only two nonzero Fourier coefficients, and therefore only two nonzero terms in the synthesis equation. The two non-zero coefficients *included in any summation will be $Nk+M$ and $N(k+1)-M$ for an integer k. For example, if we write $x[n]$ as the summation of Fourier series from 0 to $N-1$, the non-zero Fourier series coefficients will be a_2 and a_3:*

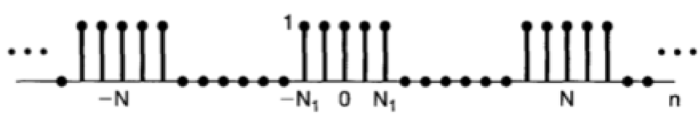
$$x[n] = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn} = a_2 e^{j\frac{4\pi}{N}n} + a_3 e^{j\frac{6\pi}{N}n}.$$

Next, we will consider the popular square wave signal, but this time in discrete time.

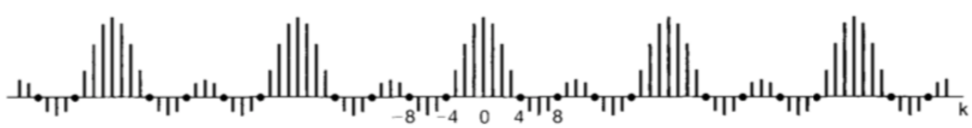
Example 2. *Consider the discrete-time periodic square wave shown in Figure 2(a). In a manner analogous to the continuous-time square wave, it is defined mathematically over a single period as*

$$x[n] = \begin{cases} 1 & 0 \le |n| \le N_1 \\ 0 & \textit{otherwise} \end{cases}$$

. What is its Fourier series representation?



(a) Signal $x[n]$



(b) Coefficients a_k for $N_1 = 2$ and $N = 20$

Figure 2: Periodic square wave $x[n]$ in Example 2 and its Fourier series coefficients for specific N_1 and N.

Ans: Choosing a length-N interval of summation that includes $-N_1 \le n \le N_1$, we have

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1 \cdot e^{-jk(2\pi/N)n}$$

This can be solved as a geometric series. To do this, we use a change of *variables* $m = n + N_1$ *to obtain*

$$\begin{aligned} a_k &= \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} = \frac{1}{N}e^{jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m} \\ &= \frac{1}{N}e^{jk2\pi N_1/N}\left(\frac{1-e^{-jk2\pi(2N_1+1)/N}}{1-e^{-jk(2\pi/N)}}\right) = \frac{1}{N}\left(\frac{e^{jk2\pi N_1/N}-e^{-jk2\pi(N_1+1)/N}}{1-e^{-jk(2\pi/N)}}\right) \end{aligned}$$

This can be further simplified by Euler's formula. In particular, if we factor out $e^{-jk2\pi/(2N)}$ *from the numerator and denominator, we get*

$$a_k = \frac{1}{N}\frac{e^{-jk2\pi/(2N)}}{e^{-jk2\pi/(2N)}}\left(\frac{e^{jk2\pi(N_1+1/2)/N}-e^{-jk2\pi(N_1+1/2)/N}}{e^{jk2\pi/(2N)}-e^{-jk2\pi/(2N)}}\right) = \frac{1}{N}\left(\frac{\sin\left[2\pi k(N_1+1/2)/N\right]}{\sin\left(\pi k/N\right)}\right)$$

for $k \neq 0, \pm N, \pm 2N, ...$ *At these other values of* k*, we get*

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} 1 = \frac{2N_1+1}{N}, \quad k = 0, \pm N, \pm 2N, ...$$

Now, suppose $N_1 = 2$ *and* $N = 20$*. The coefficients become*

$$a_k = \frac{1}{20}\left(\frac{\sin(\pi k/4)}{\sin(\pi k/20)}\right), k \neq 0, \pm 20, \pm 40, ... \qquad a_k = \frac{1}{4}, k = 0, \pm 20, \pm 40, ...$$

These are plotted in Figure 2(b); since they are purely real, we only need one graph. a_k *repeats after every* $N = 20$ *values of* k*, as we expect. For synthesis, we can sum over any interval of* 20*, e.g.,*

$$x[n] = \frac{1}{4} + \sum_{k=1}^{19}\frac{1}{20}\left(\frac{\sin(\pi k/4)}{\sin(\pi k/20)}\right)e^{jk\pi n/5}$$

Convergence. For the continuous-time Fourier series, we noted some convergence issues: in particular, as the number of terms in the partial sum approaches infinity, Gibbs phenomenon will be observed at points of discontinuity. With the discrete-time Fourier series, by contrast, there are no convergence issues in general. The reason for this stems from the fact that any discrete-time periodic sequence $x[n]$ only takes a *finite* number of values over any given period N, i.e., we only need the coefficients to represent a finite set through the synthesis equation. In the continuous case, by contrast, $x(t)$ takes on a full continuum of values over any given period T, and we need an infinite number of coefficients to synthesize the signal.

Properties of Discrete-time Fourier Series

Like the CTFS, the DTFS has many properties that can help us simplify the computations of the Fourier coefficients for discrete-time signals. A full list of these properties can be found in Table 3.2 of the textbook. We point out some of the important ones here, and stress those that are different from the CTFS case, as many of them are practically the same. In the CT case, we showed how these properties can be proved using the analysis or synthesis equation; for DT, we will leave most of the proofs of these properties to you.

In what follows, assume $x[n]$ and $y[n]$ are both periodic DT signals with period N, and that $x[n] \xleftrightarrow{FS} a_k$ and $y[n] \xleftrightarrow{FS} b_k$. Summarizing the equations:

$$\text{Analysis: } a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-j(2\pi/N)kn}$$

$$\text{Synthesis: } x[n] = \sum_{k=<N>} a_k e^{j(2\pi/N)kn}$$

These first three properties are identical to the CT case:

Linearity. $Ax[n] + By[n] \xleftrightarrow{FS} Aa_k + Bb_k$

Time shifting. $x[n - n_0] \xleftrightarrow{FS} a_k e^{-jk(2\pi/N)n_0}$

Time reversal. $x[-n] \xleftrightarrow{FS} a_{-k}$

Further, as the Fourier series coefficient a_k is periodic with N, $a_{-k} = a_{N-k}$. Thus, if you use $a_0, ..., a_{N-1}$ to synthesize the signal $x[n]$, the time reversal property can also be written as a symmetry around $k = N$:

$$x[-n] \xleftrightarrow{FS} c_k, \text{ where}$$

$$c_k = \begin{cases} a_0, & k = 0 \\ a_{N-k} & k = 1, ..., N-1 \end{cases}$$

Multiplication. Multiplication is a property that is different, albeit slightly, from the CT case. The Fourier coefficients of the product $x[n]y[n]$ are given by

$$x[n]y[n] \xleftrightarrow{FS} d_k = \sum_{l=<N>} a_l b_{k-l}$$

Different than continuous-time, the sum of products is taken only over an interval of N consecutive samples.

The summation is similar in form to the convolution sum, but the summation interval in this case is over any consecutive period of length N. This form is also called periodic/cycling convolution.

Example 3. *Consider two DT signals $x[n]$ and $y[n]$ that are periodic with fundamental period $N = 5$. The Fourier series coefficients of both signals are:*

$$x[n] \overset{FS}{\longleftrightarrow} a_k,$$

$$y[n] \overset{FS}{\longleftrightarrow} b_k.$$

The signal $z[n] = x[n]y[n]$ has the Fourier series coefficient d_k. Write d_2 using a_k and b_k.

Ans: As $N = 5$, let's consider writing d_k using $a_0, ..., a_4$ and $b_0, ..., b_4$. Based on the multiplication property,

$$d_2 = \sum_{l=0}^{4} a_l b_{2-l} = a_0 b_2 + a_1 b_1 + a_2 b_0 + a_3 b_{-1} + a_4 b_{-2}.$$

As $b_{-1} = b_4$ and $b_{-2} = b_3$, c_2 can be written as

$$d_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 + a_3 b_4 + a_4 b_3.$$

First difference. The first difference operation, $x[n]-x[n-1]$, has Fourier series coefficients

$$x[n] - x[n-1] \overset{FS}{\longleftrightarrow} \left(1 - e^{-jk(2\pi/N)}\right) a_k$$

The analogous operation in continuous time is differentiation; recall that the Fourier series of $dx(t)/dt$ for a periodic signal $x(t)$ is

$$\frac{dx(t)}{dt} \overset{FS}{\longleftrightarrow} jk\omega_0 a_k$$

The CT and DT versions of this property are thus not the same. However, they do both enhance high-frequency components, and we can see some mathematical similarity by taking the Taylor expansion of $1 - e^{-jk\omega_0}$.

Parseval's relation. In discrete-time, the average power of a periodic signal over one period can be obtained as

$$\frac{1}{N}\sum_{n=<N>}|x[n]|^2 = \sum_{k=<N>}|a_k|^2$$

Like the multiplication property, in the discrete-time version of Parseval's relation we only sum over the N distinct harmonic components.

Example 4. *Let*

$$x[n] = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & 8 \le n \le 9 \end{cases}$$

be a periodic signal with fundamental period $N = 10$, and let $g[n] = x[n] - x[n-1]$. What are the Fourier series coefficients of $g[n]$? How about $x[n]$?

Ans: The signals $x[n]$ and $g[n]$ are plotted in Figure 3. Clearly, they are both periodic with $N = 10$, and thus $\omega_0 = 2\pi/N = \pi/5$. It will be easiest to calculate the Fourier series coefficients of $g[n]$ and then use the first difference relationship to find those for $x[n]$.

Considering the interval $n = -4, -3, ..., 5$, the coefficients b_k for $g[n]$ can be calculated as

$$\begin{aligned} b_k &= \frac{1}{N}\sum_{n=<N>} g[n]e^{-jk\omega_0 n} = \frac{1}{10}g[n]e^{-jk\pi n/5}\Big|_{n=-2} + \frac{1}{10}g[n]e^{-jk\pi n/5}\Big|_{n=0} \\ &= -\frac{1}{10}e^{j2\pi n/5} + \frac{1}{10} = \frac{1}{10}\left(1 - e^{j2\pi k/5}\right) \end{aligned}$$

Then we can write the DFTS as

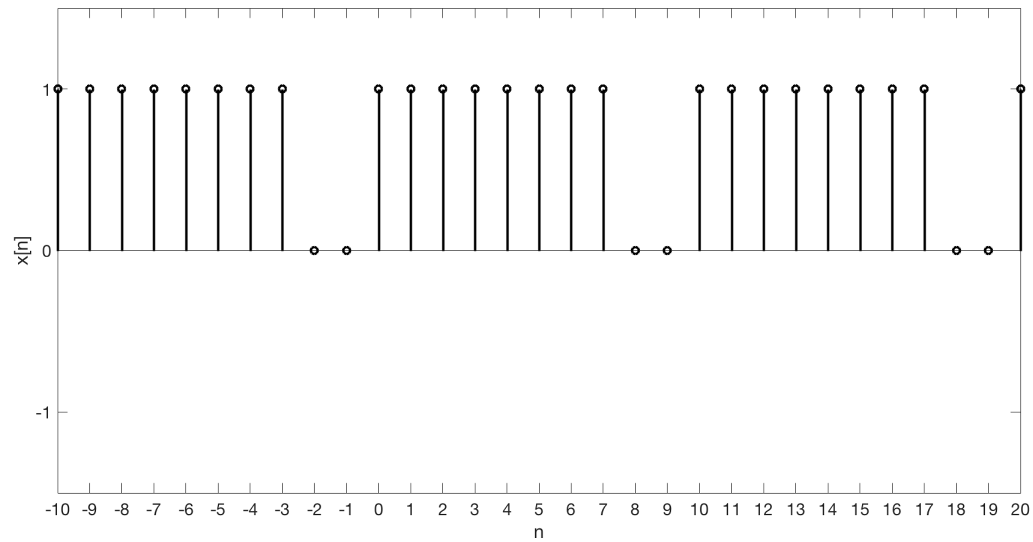
$$g[n] = \sum_{k=-4}^{5}\frac{1}{10}\left(1 - e^{j2\pi k/5}\right)e^{jk\pi n/5}$$

Now, let a_k be the coefficients of $x[n]$. Since $g[n]$ is the first difference of $x[n]$, a_k and b_k must be related by

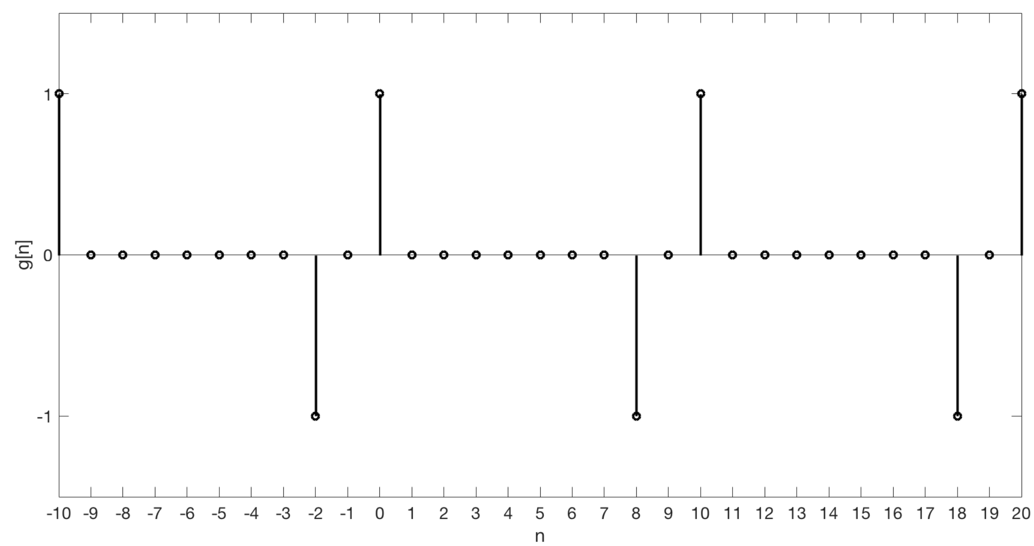
$$b_k = \left(1 - e^{-jk(2\pi/N)}\right)a_k = \left(1 - e^{-jk\pi/5}\right)a_k$$

Thus,

$$a_k = \frac{1}{10}\cdot\frac{1 - e^{j2\pi k/5}}{1 - e^{-jk\pi/5}}, \quad k \neq 0$$



(a) Signal $x[n]$



(b) Signal $g[n] = x[n] - x[n-1]$

Figure 3: Signals $x[n]$ and $g[n]$ in Example 4.

And

$$a_0 = \frac{1}{N} \sum_{n=<N>} x[n] = \frac{4}{5}$$

Therefore, we can write the DTFS as

$$x[n] = \frac{4}{5} + \sum_{\substack{k=-4 \\ k\neq 0}}^{5} \frac{1}{10} \cdot \frac{1-e^{j2\pi k/5}}{1-e^{-jk\pi/5}} \cdot e^{jk\pi n/5}$$

Fourier Series and LTI Systems

We will now formally tie together Fourier series and LTI systems for both continuous and discrete time. Recall from the previous module that if the input to a continuous-time LTI system with impulse response $h(t)$ is $x(t) = e^{j\omega t}$, then the output is

$$y(t) = H(j\omega)e^{j\omega t} \qquad \text{where} \qquad H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

We can extend this to discrete-time LTI systems, too. If the input to a discrete-time LTI system with impulse response $h[n]$ is $x[n] = e^{j\omega n}$, then the output is

$$y[n] = \sum_{k=-\infty}^{\infty} h[n]e^{j\omega(n-k)} = e^{j\omega n} \sum_{k=-\infty}^{\infty} h[n]e^{-j\omega k}$$

Thus, for simplicity

$$y[n] = H(e^{j\omega})e^{j\omega n} \qquad \text{where} \qquad H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

$H(j\omega)$ and $H(e^{j\omega})$ are referred to as the **frequency responses** of the systems, i.e., the responses to the systems as ω changes. So, the response of an LTI system to a complex exponential input is particularly simple to express in terms of the frequency response. By the superposition property, then, so will be a linear combination of such inputs, by which we can represent any periodic signal of practical importance.

Response based on Fourier series. If $x(t)$ is a periodic signal with a Fourier series representation, then the following is the input-output relationship:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \rightarrow \qquad y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

Thus, $y(t)$ is also periodic with the same ω_0, and $a_k H(jk\omega_0)$ are its coefficients. The LTI system simply modifies each of the Fourier coefficients by the value of the frequency response.

In discrete time, the relationships are analogous; if $x[n]$ is a periodic signal, then

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \rightarrow \qquad y[n] = \sum_{k=<N>} a_k H(e^{jk\omega_0}) e^{jk\omega_0 n}$$

Note that in DT, calculating the frequency response $H(e^{j\omega})$ takes the summation over all n, whereas the calculation of the Fourier series coefficients a_k only takes the summation over a single period.

Convolutional Properties of CT Fourier Series

The above conclusions are especially useful as they greatly simplify the convolution operation for LTI systems in the case of periodic input signals. They also suggest another important Fourier series property. If $x(t) \xleftrightarrow{FS} a_k$ is input to an LTI system to produce an output $y(t) \xleftrightarrow{FS} b_k$, then b_k is given by the product of a_k and the frequency response:

$$y(t) = x(t) * h(t) \xleftrightarrow{FS} b_k = a_k H(jk\omega_0)$$

This shows one important reason why we want to study Fourier series. Given a periodic input signal $x(t)$ and an LTI system with impulse response $h(t)$, there are two ways to calculate the output signal $y(t)$, depicted in Figure 4:

1. Via the convolution integral.
2. Calculate the Fourier series of $x(t)$, a_k, and the frequency response of $h(t)$, $H(j\omega)$. Then, use the multiplication property to find the Fourier series of $y(t)$, b_k, via $b_k = a_k H(jk\omega_0)$. The output $y(t)$ can be finally calculated via the Fourier synthesis equation.

Later in the semester, we will show how this is generalized to non-periodic signals $x(t)$ through the Fourier transform.

Example 5. *Suppose the signal*

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t) + \sin(2\pi t) + 2\cos(3\pi t)$$

$$
\begin{tikzcd}
x(t) \arrow[r, leftrightarrow, "* h(t)"] \arrow[d, leftrightarrow, "\text{F.S.}"'] & y(t) \arrow[d, leftrightarrow, "\text{F.S.}"] \\
a_k \arrow[r, leftrightarrow, "H(j\omega) = \int_{-\infty}^{\infty} h(s) e^{-j\omega s} ds"] & b_k = a_k H(jk\omega_0)
\end{tikzcd}
$$

Figure 4: Two ways to calculate the output of a CT LTI system when the input $x(t)$ is periodic.

is applied to an LTI system that has impulse response $h(t) = e^{-t}u(t)$. What is the output $y(t)$?

Ans: We first find the frequency response:

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(1+j\omega)t}dt = \frac{1}{1+j\omega}$$

Then, we obtain the Fourier series of the input through Euler's formula:

$$\begin{aligned} x(t) &= 1 + \frac{1}{2}\left(\frac{e^{j2\pi t} + e^{-j2\pi t}}{2}\right) + \left(\frac{e^{j2\pi t} - e^{-j2\pi t}}{2j}\right) + 2\left(\frac{e^{j3\pi t} + e^{-j3\pi t}}{2}\right) \\ &= 1 + e^{-j3\pi t} + \left(\frac{1}{4} + \frac{j}{2}\right) e^{-j2\pi t} + \left(\frac{1}{4} - \frac{j}{2}\right) e^{j2\pi t} + e^{j3\pi t} \end{aligned}$$

The output is thus

$$\begin{aligned} y(t) = 1 \cdot H(j0) + H(-j3\pi)e^{-j3\pi t} + \left(\frac{j}{2} + \frac{1}{4}\right) H(-j2\pi)e^{-j2\pi t} \\ + \left(\frac{1}{4} - \frac{j}{2}\right) H(j2\pi)e^{j2\pi t} + H(j3\pi)e^{j3\pi t} \end{aligned}$$

Which becomes

$$y(t) = 1 + \frac{e^{-j3\pi t}}{1 - j3\pi} + \frac{\left(\frac{1}{4} + \frac{j}{2}\right) e^{-j2\pi t}}{1 - j2\pi} + \frac{\left(\frac{1}{4} - \frac{j}{2}\right) e^{j2\pi t}}{1 + j2\pi} + \frac{e^{j3\pi t}}{1 + j3\pi}$$

This must be a real-valued output, since both $x(t)$ and $h(t)$ are real. The imaginary components will cancel out with some algebra. To see how, let's jump back a few steps and look to combine $k = \pm 2$ and $k = \pm 3$ into real signals noting that $H(-j\omega) = H^\star(j\omega)$:

$$
\begin{aligned}
y(t) = H(j0) &+ \left[H(j3\pi)e^{j3\pi t} + H^\star(j3\pi)e^{-j3\pi t}\right] \\
&+ \left[\left(\frac{1}{4} - \frac{j}{2}\right) H(j2\pi)e^{j2\pi t} + \left(\frac{1}{4} + \frac{j}{2}\right) H^\star(j2\pi)e^{-j2\pi t}\right] \\
&= H(0) + 2Re\left\{H(j3\pi)e^{j3\pi t}\right\} + 2Re\left\{\left(\frac{1}{4} - \frac{j}{2}\right) H(j2\pi)e^{j2\pi t}\right\} \\
&= 1 + 2Re\left\{\frac{1}{1+j3\pi}e^{j3\pi t}\right\} + 2Re\left\{\frac{\left(\frac{1}{4} - \frac{j}{2}\right)}{1+j2\pi}e^{j2\pi t}\right\} \\
&= 1 + 2Re\left\{\frac{1-j3\pi}{1+(3\pi)^2}e^{j3\pi t}\right\} + 2Re\left\{\frac{\left(\frac{1}{4} - \frac{j}{2}\right)(1-j2\pi)}{1+(2\pi)^2}e^{j2\pi t}\right\} \\
&= 1 + 2\cdot\left(\frac{\cos(3\pi t) + 3\pi\sin(3\pi t)}{1+(3\pi)^2}\right) + 2Re\left\{\frac{\left(\frac{1}{4} - \frac{j}{2}\right)(1-j2\pi)}{1+(2\pi)^2}e^{j2\pi t}\right\} \\
&= 1 + 2\cdot\left(\frac{\cos(3\pi t) + 3\pi\sin(3\pi t)}{1+(3\pi)^2}\right) + 2Re\left\{\frac{\left(\frac{1}{4} - \pi\right) - \left(\frac{1}{2} + 2\pi\right)j}{1+(2\pi)^2}e^{j2\pi t}\right\} \\
&= 1 + \frac{2\cos(3\pi t)}{1+(3\pi)^2} + \frac{6\pi\sin(3\pi t)}{1+(3\pi)^2} + \frac{2\left(\frac{1}{4} - \pi\right)\cos(2\pi t)}{1+(2\pi)^2} + \frac{2\left(\frac{1}{2} + 2\pi\right)\sin(2\pi t)}{1+(2\pi)^2}
\end{aligned}
$$

Convolutional Properties of DT Fourier Series

In DT, we have an analogous convolutional property:

$$y[n] = x[n] * h[n] \overset{FS}{\longleftrightarrow} b_k = a_k H(e^{jk\omega_0})$$

The two approaches to calculate the output of a DT LTI system are depicted in Figure 5, and closely parallel that in Figure 4:

1. Via the convolution sum.
2. Calculate the Fourier series of $x[n]$, a_k, and the frequency response of $h[n]$, $H(e^{j\omega})$. Then, use the multiplication property to find the Fourier

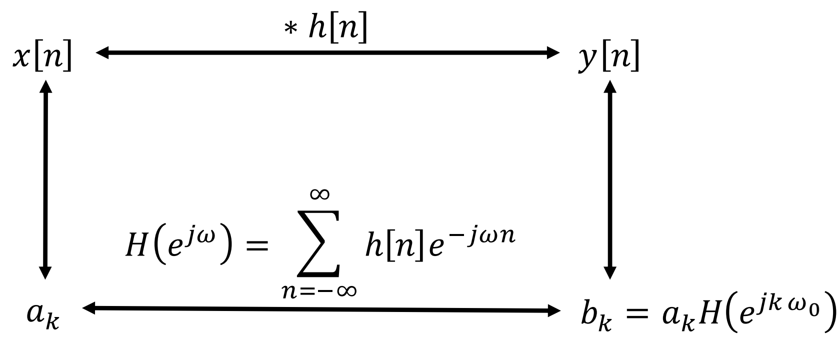


Figure 5: Two ways to calculate the output of a DT LTI system when the input $x[n]$ is periodic.

series of $y[n]$, b_k, via $b_k = a_k H(e^{jk\omega_0})$. The output $y[n]$ can finally be calculated via the Fourier synthesis equation.

Similar to CT, we will show later in the semester that this conclusion can be generalized to when $x[n]$ is not periodic.

Example 6. *Consider an LTI system with impulse response* $h[n] = \alpha^{|n|}$, $-1 < \alpha < 1$, *and input*

$$x[n] = \cos\left(\frac{2\pi n}{N}\right)$$

What is the output $y[n]$*?*

Ans: As in the previous example, we first calculate the frequency response:

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n + \sum_{n=0}^{\infty} (\alpha e^{j\omega})^n - h[0] \\ &= \frac{1}{1 - \alpha e^{-j\omega}} + \frac{1}{1 - \alpha e^{j\omega}} - 1 \end{aligned}$$

Then, we calculate the Fourier series of the input, by applying Euler's formula in this case:

$$x[n] = \frac{1}{2} e^{j2\pi n/N} + \frac{1}{2} e^{-j2\pi n/N}$$

Then, with $\omega_0 = 2\pi/N$,

$$\begin{aligned} y[n] &= \frac{1}{2}e^{-j\omega_0 n}H(e^{-j\omega_0}) + \frac{1}{2}e^{j\omega_0 n}H(e^{j\omega_0}) \\ &= \frac{1}{2}\left(\frac{e^{-j\omega_0 n}}{1-\alpha e^{j\omega_0}} + \frac{e^{-j\omega_0 n}}{1-\alpha e^{-j\omega_0}} - e^{-j\omega_0 n} + \frac{e^{j\omega_0 n}}{1-\alpha e^{-j\omega_0}} + \frac{e^{j\omega_0 n}}{1-\alpha e^{j\omega_0}} - e^{j\omega_0 n}\right) \\ &= \frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2(1-\alpha e^{j\omega_0})} + \frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2(1-\alpha e^{-j\omega_0})} - \frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2} \end{aligned}$$

To simplify this, we could proceed similarly to Example 5. As a simpler approach, note that

$$\frac{1}{1-\alpha e^{j\omega_0}} = \frac{1-\alpha e^{-j\omega_0}}{1+\alpha^2-\alpha e^{j\omega_0}-\alpha e^{-j\omega_0}} = \frac{1-\alpha e^{-j\omega_0}}{1+\alpha^2-2\alpha\cos(\alpha)}$$

is the complex conjugate of

$$\frac{1}{1-\alpha e^{-j\omega_0}} = \frac{1-\alpha e^{j\omega_0}}{1+\alpha^2-\alpha e^{j\omega_0}-\alpha e^{-j\omega_0}} = \frac{1-\alpha e^{j\omega_0}}{1+\alpha^2-2\alpha\cos(\alpha)}$$

Thus, we can express

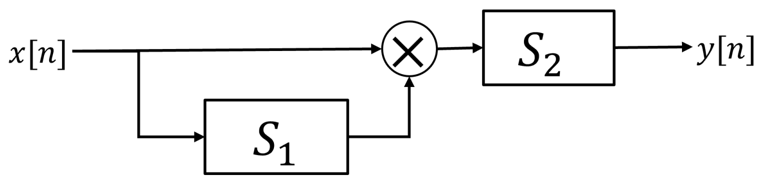
$$\frac{1}{1-\alpha e^{j\omega_0}} = re^{j\phi}, \qquad \frac{1}{1-\alpha e^{-j\omega_0}} = re^{-j\phi}$$

where r *is the magnitude and* $\pm\phi$ *is the phase for specific values of* ω_0 *and* ϕ. *With this, we can write*

$$\begin{aligned} y[n] &= \frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2}\left(re^{j\phi}+re^{-j\phi}\right) - \cos(\omega_0 n) \\ &= 2r\cos(\omega_0 n)\cos(\phi) - \cos(\omega_0 n) \\ &= (2r\cos(\phi)-1)\cos\left(\frac{2\pi}{N}n\right) \end{aligned}$$

Finally, we will consider an example of how the analysis of composite systems can be simplified using Fourier series properties.

Example 7. *Consider the system shown in the diagram below, where the circled "$\times$" symbol represents multiplication of the two inputs. Let* S_1 *represent an LTI system with impulse response* $h_1[n] = \delta[n-1]$, *and let* S_2 *represent an LTI system with impulse response* $h_2[n] = e^{-n}u[n]$. *If the input to the system is* $x[n] = \sin(\pi n/4)$, *determine the DTFS coefficients of the output* $y[n]$.



Ans: We will approach this problem by finding the DTFS coefficients at each stage throughout the system. First, let us determine the DTFS coefficients of $x[n]$. Note that $x[n] = \sin(\pi n/4)$ is periodic with period $N = 8$ (and fundamental frequency $\omega_0 = \pi/4$). We can express $x[n]$ as a sum of exponentials:

$$x[n] = \sin(\pi n/4) = \frac{1}{2j}(e^{j\pi n/4} - e^{-j\pi n/4})$$

Hence by inspection we determine the coefficients of $x[n]$ are

$$a_0 = 0, \quad a_1 = \frac{1}{2j}, \quad a_2 = a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{2j}$$

and a_k is periodic with period $N = 8$.

Next, we will find the DTFS coefficients of the output of S_1. The output of S_1 is $x[n] * h_1[n]$; convolution of $x[n]$ with $\delta[n-1]$ gives the output $x[n-1]$. By the time-shift property of the DTFS, we know that if a_k are the DTFS coefficients of $x[n]$, then the DTFS coefficients of $x[n-1]$ are given by $b_k = a_k e^{-jk\omega_0} = a_k e^{-jk\pi/4}$. Hence we determine the coefficients of $x[n-1]$ are

$$b_0 = 0, \quad b_1 = \frac{1}{2j}e^{-j\pi/4}, \quad b_2 = b_3 = b_4 = b_5 = b_6 = 0, \quad b_7 = -\frac{1}{2j}e^{-j7\pi/4}$$

and b_k is periodic with period $N = 8$.

Next, let us consider the input to S_2, which is the product $x[n]x[n-1]$. We can determine the DTFS coefficients of this input by applying the multiplication property of the DTFS. This property tells us that if a_k represents the FS coefficients of $x[n]$ and b_k represents the FS coefficients of $x[n-1]$, then the coefficients of the product $x[n]x[n-1]$ can be computed by:

$$c_k = \sum_{l=<8>} a_l b_{k-l} = \sum_{l=0}^{7} a_l b_{k-l} = a_1 b_{k-1} + a_7 b_{k-7}$$

We can find c_0 through c_7 as follows. We show the calculations here for the non-zero ones:

$$\begin{aligned} c_0 &= a_1 b_{0-1} + a_7 b_{0-7} = a_1 b_{-1} + a_7 b_{-7} = a_1 b_7 + a_7 b_1 \\ &= \frac{1}{2j} \cdot -\frac{1}{2j} e^{j7\pi/4} + -\frac{1}{2j} \cdot \frac{1}{2j} e^{j\pi/4} \\ &= \frac{1}{4} e^{j7\pi/4} + \frac{1}{4} e^{j\pi/4} = \frac{\sqrt{2}}{4} \end{aligned}$$

$$\begin{aligned} c_2 &= a_1 b_{2-1} + a_7 b_{2-7} = a_1 b_1 + a_7 b_3 = \frac{1}{2j} \cdot \frac{1}{2j} e^{j\pi/4} \\ &= -\frac{1}{4} e^{j\pi/4} = -\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8} j \end{aligned}$$

$$\begin{aligned} c_6 &= a_1 b_{6-1} + a_7 b_{6-7} = a_1 b_5 + a_7 b_7 = -\frac{1}{2j} \cdot -\frac{1}{2j} e^{j7\pi/4} = \\ &= -\frac{1}{4} e^{j7\pi/4} = -\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{8} j \end{aligned}$$

$$c_1 = c_3 = c_4 = c_5 = c_7 = 0$$

The FS coefficients c_k will also be periodic with period $N = 8$. However, notice that only even values of k have non-zero c_k. Whenever this happens, the fundamental frequency ω_0 has doubled. To see this, note

$$x[n]x[n-1] = c_0 + c_2 e^{j\frac{\pi}{2}n} + c_6 e^{j\frac{3\pi}{2}n}$$

for which $\omega_0 = \pi/2$, and thus $N = 4$. With re-indexing, then, we have a simpler Fourier series expression summing over $k = 0, ..., 3$:

$$x[n]x[n-1] = d_0 + d_1 e^{j\frac{\pi}{2}n} + d_3 e^{j\frac{3\pi}{2}n}$$

where

$$d_0 = \frac{\sqrt{2}}{4}, \;\; d_1 = -\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8} j, \;\; d_2 = 0, \;\; d_3 = -\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{8} j$$

For the final step, we observe that $x[n]x[n-1]$ is passed through S_2, which means that it is convolved with $h_2[n] = e^{-n}u[n]$. By the convolution property *of the DTFS, the FS coefficients of $y[n]$ (i.e., the output of S_2) will be given as*

$$e_k = d_k H_2(e^{jk\omega_0})$$

where

$$\begin{aligned}
H_2(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h_2[n]e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-n}e^{-j\omega n} = \sum_{n=0}^{\infty}(e^{-1-j\omega})^n \\
&= \frac{1}{1-e^{-1-j\omega}} \qquad \textit{by geometric sum formula, noting } |e^{-1-j\omega}| < 1
\end{aligned}$$

Thus we determine that $y[n]$ will have DTFS coefficients e_k which are periodic with period $N = 4$ and values given as:

$$\begin{aligned}
e_0 &= d_0 H_2(e^{j0}) = \frac{\sqrt{2}}{4} \cdot \frac{1}{1-e^{-1}} \\
e_1 &= d_1 H_2(e^{j\pi/2}) = (-\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}j)\frac{1}{1-e^{-1}e^{-j\pi/2}} = \frac{-\sqrt{2}-\sqrt{2}j}{8+8je^{-1}} \\
e_2 &= 0 \\
e_3 &= d_3 H_2(e^{j3\pi/2}) = (-\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{8}j)\frac{1}{1-e^{-1}e^{-j3\pi/2}} = \frac{-\sqrt{2}+\sqrt{2}j}{8-8je^{-1}}
\end{aligned}$$